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Overdetermined PDEs in general relativity

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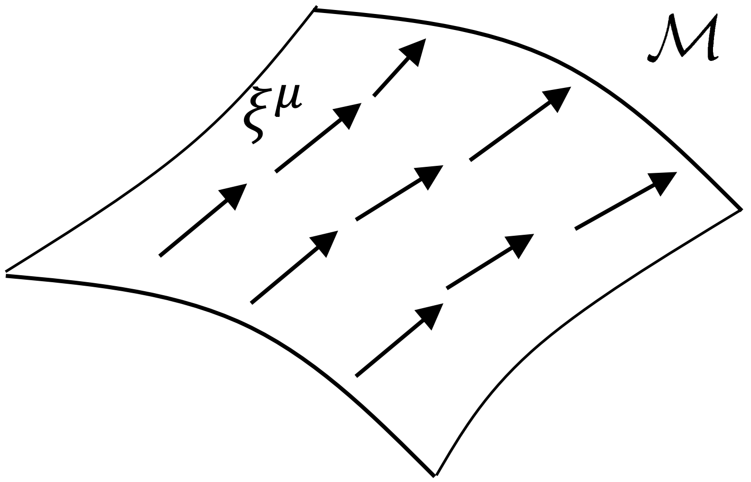
Spacetime symmetry

- Killing vector fields:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$



Killing symmetries

vector fields	Killing	Conformal Killing
symmetric	Killing-Stackel Stackel 1895	Conformal Killing-Stackel
anti-symmetric	Killing-Yano Yano 1952	Conformal Killing-Yano Tachibana 1969, Kashiwada 1968

Hidden symmetry of spacetime

- Killing-Stackel tensors

$$\nabla_{(\mu}K_{\nu_1\nu_2\ldots\nu_n)} = 0 \qquad K_{(\mu_1\mu_2\ldots\mu_n)} = K_{\mu_1\mu_2\ldots\mu_n}$$

- Killing-Yano tensors

$$\nabla_{(\mu}\xi_{\nu_1)\nu_2\ldots\nu_n} = 0 \qquad \xi_{[\mu_1\mu_2\ldots\mu_n]} = \xi_{\mu_1\mu_2\ldots\mu_n}$$

Why Killing symmetry?

- Conserved quantities along geodesics
- Separability

  Hamilton-Jacobi equations for geodesics, Klein-Gordon and Dirac equations
- Exact solutions

  Stationary, axially symmetric black holes with spherical horizon topology

The purpose of this talk

To show a simple method for finding Killing symmetries for a given metric.

Key words: Overdetermined PDEs, integrability condition, prolongation

Plan

Introduction

Review I: Integrability conditions for systems of first order PDEs

Review II: Prolongation of PDEs and jet space

Prolongation of Killing equation

Prolongation of Killing-Yano equation

Prolongation of Killing-Stackel equation

Summary

Review I:
Integrability conditions for systems of first order PDEs

A system of first order PDEs

$$\frac{\partial u^\alpha}{\partial x^i} = \psi_i^\alpha(x, u) \qquad i = 1, \cdots, n \qquad \alpha = 1, \cdots, N$$

x; variables $\quad x = (x^1, x^2, \cdots, x^n)$

u; unknown functions $\quad u = (u^1, u^2, \cdots, u^N) \quad u^\alpha = u^\alpha(x)$

<u>Question:</u>

Does solution exist?

How many constants does the solution depend on?

Explicit expressions?

Integrability condition

(also called curvature condition, consistency condition)

$$\frac{\partial u^\alpha}{\partial x^i} = \psi_i^\alpha(x,u)$$

$$\frac{\partial}{\partial x^j}\frac{\partial u^\alpha}{\partial x^i} = \frac{\partial \psi_i^\alpha}{\partial x^j} + \sum_\beta \frac{\partial \psi_i^\alpha}{\partial u^\beta}\frac{\partial u^\beta}{\partial x^j} = \frac{\partial \psi_i^\alpha}{\partial x^j} + \sum_\beta \frac{\partial \psi_i^\alpha}{\partial u^\beta}\psi_j^\beta$$

$$\frac{\partial}{\partial x^i}\frac{\partial u^\alpha}{\partial x^j} = \frac{\partial \psi_j^\alpha}{\partial x^i} + \sum_\beta \frac{\partial \psi_j^\alpha}{\partial u^\beta}\frac{\partial u^\beta}{\partial x^i} = \frac{\partial \psi_j^\alpha}{\partial x^i} + \sum_\beta \frac{\partial \psi_j^\alpha}{\partial u^\beta}\psi_i^\beta$$

$$\frac{\partial \psi_i^\alpha}{\partial x^j} - \frac{\partial \psi_j^\alpha}{\partial x^i} + \sum_\beta \left(\frac{\partial \psi_i^\alpha}{\partial u^\beta}\psi_j^\beta - \frac{\partial \psi_j^\alpha}{\partial u^\beta}\psi_i^\beta\right) = 0$$

Frobenius theorem

The necessary and sufficient conditions for the unique solution $u^\alpha = u^\alpha(x)$ to the system

$$\frac{\partial u^\alpha}{\partial x^i} = \psi_i^\alpha(x, u) \qquad i = 1, \cdots, n \qquad \alpha = 1, \cdots, N$$

such that $u(x_0) = u_0$ to exist for any initial data (x_0, u_0) is that the relation

$$\frac{\partial \psi_i^\alpha}{\partial x^j} - \frac{\partial \psi_j^\alpha}{\partial x^i} + \sum_\beta \left(\frac{\partial \psi_i^\alpha}{\partial u^\beta} \psi_j^\beta - \frac{\partial \psi_j^\alpha}{\partial u^\beta} \psi_i^\beta \right) = 0$$

hold.

Discussion

- If the Frobenius integrability conditions hold, the general solution depends on N arbitrary constants.
- If not, they give a set of algebraic equations

$$F_1(x, u) = 0$$

- Differentiating these equations and eliminating the derivatives of u using the original equation leads to a new set of equations

$$F_2(x, u) = 0$$

- Proceeding in this way we get a sequence of sets of equations

$$F_1(x, u) = 0, \quad F_2(x, u) = 0, \quad F_3(x, u) = 0, \quad \cdots$$

Theorem

The system

$$\frac{\partial u^\alpha}{\partial x^i} = \psi_i^\alpha(x,u) \qquad i = 1,\cdots,n \qquad \alpha = 1,\cdots,N$$

admits solution if and only if there exists a positive integer $K \leq N$ such that the set of algebraic equations

$$F_1 = F_2 = F_3 = \cdots = F_K = 0$$

is compatible and that the set $F_{K+1} = 0$ is satisfied identically.

If p is the number of independent equations in the first K sets, then the general solution depends on $\mathrm{N} - \mathrm{p}$ arbitrary constants.

Particular case

$$\frac{\partial u^\alpha}{\partial x^i} = \psi_i^\alpha(x, u) \qquad i = 1, \cdots, n \qquad \alpha = 1, \cdots, N$$

In particular, if ψ_i^α are homogeneous linear functions of u^β, the system is written as

$$\frac{\partial u^\alpha}{\partial x^i} = \psi_{i\beta}^\alpha(x) u^\beta$$

This system and its integrability condition can be expressed in terms of geometry.

Parallel equation

$$\frac{\partial u^\alpha}{\partial x^i} = \psi^\alpha_{i\beta}(x) u^\beta \qquad i = 1, \cdots, n \qquad \alpha = 1, \cdots, N$$

$$\Leftrightarrow \quad \frac{\partial u^\alpha}{\partial x^i} - \psi^\alpha_{i\beta}(x) u^\beta = 0$$

$$\Leftrightarrow \quad \boxed{D_i u^\alpha = 0} \qquad D_i u^\alpha := \frac{\partial u^\alpha}{\partial x^i} - \psi^\alpha_{i\beta}(x) u^\beta$$

The system can be expressed as parallel equation for a section u^α of a vector bundle of rank N.

Curvature condition

(also called integrability condition, consistency condition)

For a connection D_i

$$D_i u^\alpha := \frac{\partial u^\alpha}{\partial x^i} - \psi^\alpha_{i\beta}(x) u^\beta$$

the curvature of D_i is defined by $(D_i D_j - D_j D_i) u^\alpha = -F_{ij\beta}{}^\alpha u^\beta$.

$$D_i u^\alpha = 0 \quad \Rightarrow \quad \boxed{F_{ij\beta}{}^\alpha u^\beta = 0}$$

This is equivalent to the Frobenius integrability condition

Frobenius theorem

The necessary and sufficient conditions for the unique solution $u^\alpha = u^\alpha(x)$ to the system

$$D_i u^\alpha = 0 \qquad i = 1, \cdots, n \qquad \alpha = 1, \cdots, N$$

where

$$D_i u^\alpha := \frac{\partial u^\alpha}{\partial x^i} - \psi^\alpha_{i\beta}(x) u^\beta$$

such that $u(x_0) = u_0$ to exist for any initial data (x_0, u_0) is that the relation

$$F_{ij\beta}{}^{\alpha} u^\beta = 0$$

hold.

Discussion

- If the curvature conditions hold, the general solution depends on N arbitrary constants.
- If not, they give a set of algebraic equations

$$F_{ij\beta}{}^{\alpha} u^{\beta} = 0$$

- Differentiating these equations and eliminating the derivatives of u using the original equation leads to a new set of equations

$$(D_k F_{ij\beta}{}^{\alpha}) u^{\beta} = 0 \qquad D_k F_{ij\beta}{}^{\alpha} := \partial_k F_{ij\beta}{}^{\alpha} - \psi^{\alpha}_{k\gamma} F_{ij\beta}{}^{\gamma} + F_{ij\gamma}{}^{\alpha} \psi^{\gamma}_{k\beta}$$

- Proceeding in this way we get a sequence of sets of equations

$$F_{ij\beta}{}^{\alpha} u^{\beta} = 0, \quad (D_k F_{ij\beta}{}^{\alpha}) u^{\beta} = 0, \quad (D_\ell D_k F_{ij\beta}{}^{\alpha}) u^{\beta} = 0, \quad \cdots$$

Theorem

The system

$$D_i u^\alpha = 0 \qquad i = 1, \cdots, n \qquad \alpha = 1, \cdots, N$$

admits solution if and only if there exists a positive integer $K \leq N$ such that the set of algebraic equations

$$F_{ij\beta}{}^{\alpha} u^\beta = 0, \quad (D_k F_{ij\beta}{}^{\alpha}) u^\beta = 0, \quad (D_\ell D_k F_{ij\beta}{}^{\alpha}) u^\beta = 0, \quad \cdots$$

is compatible and that the set $\left(D^{(K+1)} F\right) u = 0$ is satisfied identically.

If p is the number of independent equations in the first K sets, then the general solution depends on $\mathrm{N} - \mathrm{p}$ arbitrary constants.

Review II: Prolongation of PDEs and jet space

Prolongation

$$F\left(x^i, f^a, \partial_i f^a, \partial_j \partial_i f^a, \cdots\right) = 0$$

↓

Introduce new functions

$$u^\alpha := \partial \cdots \partial f^a$$

$$\frac{\partial u^\alpha}{\partial x^i} = \psi_i^\alpha(x, u) \qquad i = 1, \cdots, n \qquad \alpha = 1, \cdots, N$$

Example

$$u_x = au + bv$$
$$u_y + v_x = cu + dv$$
$$v_y = eu + fv$$

Introduce $w = u_y - v_x$

$$u_x = au + bv$$
$$u_y = \frac{1}{2}(cu + dv + w)$$
$$v_x = \frac{1}{2}(cu + dv - w)$$
$$v_y = eu + fv$$
$$w_x = w_x(u, v, w)$$
$$w_y = w_y(u, v, w)$$

Prolongation

$$F(x, f, \partial f, \partial\partial f, \cdots) = 0$$

Introduce new functions

$$u^\alpha := \partial \cdots \partial f^a$$

Not always possible

$$\frac{\partial u^\alpha}{\partial x^i} = \psi_i^\alpha(x, u) \qquad i = 1, \cdots, n \qquad \alpha = 1, \cdots, N$$

Example

Cauchy-Riemann equation

$$u_x = v_y$$

$$u_y = -v_x$$

Impossible to make a prolongation!

In fact, solution of this system depends on one holomorphic function.

Prolongation

$$F(x, f, \partial f, \partial\partial f, \cdots) = 0$$

Introduce new functions

$$u^\alpha := \partial \cdots \partial f^a$$

Not always possible

$$\frac{\partial u^\alpha}{\partial x^i} = \psi_i^\alpha(x, u) \qquad i = 1, \cdots, n \qquad \alpha = 1, \cdots, N$$

When can we make a prolongation successfully?

Prolongation of Killing equation

Killing symmetries

vector fields	Killing	Conformal Killing
symmetric	Killing-Stackel Stackel 1895	Conformal Killing-Stackel
anti-symmetric	Killing-Yano Yano 1952	Conformal Killing-Yano Tachibana 1969, Kashiwada 1968

Killing equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$

- $\nabla_\mu \xi_\nu = L_{\mu\nu}\,, \quad L_{\mu\nu} = \nabla_{[\mu}\xi_{\nu]}$
- $\nabla_\mu L_{\nu\rho} = -R_{\nu\rho\mu}{}^{\sigma}\xi_\sigma$

- $\nabla_\mu \xi_\nu = L_{\mu\nu}\,, \quad L_{\mu\nu} = \nabla_{[\mu}\xi_{\nu]}$
- $\nabla_\mu L_{\nu\rho} = -R_{\nu\rho\mu}{}^{\sigma}\xi_\sigma$

- Killing connection

$$D_\mu \hat{\xi}_A \equiv \nabla_\mu \begin{pmatrix} \xi_\nu \\ L_{\nu\rho} \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -R_{\nu\rho\mu}{}^{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \xi_\sigma \\ L_{\mu\nu} \end{pmatrix}$$

- $\hat{\xi}_A = (\xi_\mu, L_{\mu\nu})$: a section of $E^1 \equiv \Lambda^1(M) \oplus \Lambda^2(M)$
- D_μ : a connection on E^1

$$D_\mu \hat{\xi}_A = 0$$

The key

Killing vector fields $\Longleftrightarrow$ Parallel sections of E^1

- The number of parallel sections of $\boldsymbol{E^1}$ is bound by the rank of $\boldsymbol{E^1}$, which is given by

$$\boldsymbol{N = \binom{n}{1} + \binom{n}{2} = n(n+1)/2} \,.$$

Hence, the maximum number of Killing vector fields is given by $\boldsymbol{n(n+1)/2}$.

Curvature condition

$$D_\mu \hat{\xi}_A = 0$$

$$\Downarrow$$

$$R_{\mu\nu A}{}^{B}\, \hat{\xi}_B \equiv (D_\mu D_\nu - D_\nu D_\mu)\hat{\xi}_A = 0$$

- The number of the solutions provides an upper bound on the number of Killing vector fields.
- The solutions themselves can be used as an ansatz for solving Killing equation.

Prolongation of Killing-Yano equation

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vector fields	Killing	Conformal Killing
symmetric	Killing-Stackel Stackel 1895	Conformal Killing-Stackel
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Killing-Yano equation

$$\nabla_{(\mu}\xi_{\nu_1)\nu_2\ldots\nu_n} = 0$$

$$\xi_{[\mu_1\mu_2\ldots\mu_n]} = \xi_{\mu_1\mu_2\ldots\mu_n}$$

$$\nabla_X\xi = \frac{1}{p+1}i(X)d\xi$$

$$\nabla_X(d\xi) = \frac{p+1}{p}R^+(X)\xi$$

$$R^+(X) := e^a \wedge R(X, X_a)$$

Prolongation of Killing-Yano equation

$$\mathcal{D}_X\hat{\psi} = 0$$

$$\mathcal{D}_X\hat{\psi} := \nabla_X\begin{pmatrix}\psi_p \\ \psi_{p+1}\end{pmatrix} + \begin{pmatrix} 0 & -\frac{1}{p+1}i(X) \\ -\frac{p+1}{p}R^+(X) & 0 \end{pmatrix}\begin{pmatrix}\psi_p \\ \psi_{p+1}\end{pmatrix}$$

$$\hat{\psi} = (\psi_p, \psi_{p+1}) \in \Gamma(E^p) \qquad E^p = \Lambda^p(M) \oplus \Lambda^{p+1}(M)$$

Killing-Yano tensors of rank p

- Killing connection [Semmelmann 2002]

Rank-p KY tensors $\Leftrightarrow$ Parallel sections of $E^p = \Lambda^p(M) \oplus \Lambda^{p+1}(M)$

$$D_\mu \hat{\xi}_A = 0 \qquad \hat{\xi}_A = (\xi_{\mu_1 \dots \mu_p}, L_{\mu_1 \dots \mu_{p+1}})$$

- The maximal number

$$N = \binom{n}{p} + \binom{n}{p+1} = \binom{n+1}{p+1}$$

- Curvature conditions [TH-Yasui 2014]

$$R_{\mu\nu A}{}^{B}\, \hat{\xi}_B \equiv (D_\mu D_\nu - D_\nu D_\mu)\hat{\xi}_A = 0$$

Curvature condition

[TH-Yasui 2014]

- $\mathcal{R}(X,Y)\colon \Gamma(E^p) \to \Gamma(E^p)$, $E^p = \Lambda^p(M) \oplus \Lambda^{p+1}(M)$

$$\boldsymbol{\mathcal{R}(X,Y)} = \begin{pmatrix} \boldsymbol{N_{11}(X,Y)} & \boldsymbol{0} \\ \boldsymbol{N_{21}(X,Y)} & \boldsymbol{N_{22}(X,Y)} \end{pmatrix}$$

- $N_{11}(X,Y)\colon \Lambda^{\mathrm{p}}(\mathrm{M}) \to \Lambda^p(M)$

$$N_{11}(X,Y) = R(X,Y) + \frac{1}{p}\left(i(X) \wedge R^+(Y) - i(Y) \wedge R^+(X)\right)$$

- $N_{21}(X,Y)\colon \Lambda^{\mathrm{p}}(\mathrm{M}) \to \Lambda^{p+1}(M)$

$$N_{21}(X,Y) = -\frac{p+1}{p}\left((\nabla_X R)^+(Y) - (\nabla_Y R)^+(X)\right)$$

- $N_{22}(X,Y)\colon \Lambda^{\mathrm{p+1}}(\mathrm{M}) \to \Lambda^{p+1}(M)$

$$N_{22}(X,Y) = R(X,Y) + \frac{1}{p}\left(R^+(X)(i(Y)) - R^+(Y)(i(X))\right)$$

The number of KY tensors in maximally symmetric space

$$N = \binom{n+1}{p+1}$$

U Semmelmann 2002

	P=1	P=2	P=3	P=4
3D	6	4		
4D	10	10	5	
5D	15	20	15	6

Symmetry of Kerr spacetime

Kerr metric

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta\, d\phi\right)^2 + \frac{\sin^2\theta}{\Sigma}\left(a\, dt - \left(r^2 + a^2\right)d\phi\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma\, d\theta^2$$

$$\Delta = r^2 - 2Mr + a^2\,, \quad \Sigma = r^2 + a^2\cos^2\theta$$

- Two Killing vector fields: $\partial/\partial t$ and $\partial/\partial\phi$
- One rank-2 Killing-Yano tensor:

$$f = a\cos\theta\, dr \wedge \left(dt - a\sin^2\theta\, d\phi\right) + r\sin\theta\, d\theta \wedge \left(a\, dt - \left(r^2 + a^2\right)d\phi\right)$$

Our result:

Kerr metric admits exactly two Killing vector fields, one rank-2 and no rank-3 KY tensors.

The number of rank-p KY tensors

4D metrics	$p = 1$	$p = 2$	$p = 3$
Maximally symmetric	10	10	5
Plebanski-Demianski	2	0	0
Kerr	2	1	0
Schwazschild	4	1	0
FLRW	6	4	1
Self-dual Taub-NUT	4	4	0
Eguchi-Hanson	4	3	0

The number of rank-p KY tensors

5D metrics	$p = 1$	$p = 2$	$p = 3$	$p = 4$
Maximally symmetric	15	20	15	6
Myers-Perry	3	0	1	0
Emparan-Reall	3	0	0	0
Kerr string	3	1	0	1

Prolongation of Killing-Stackel equation

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The number of KS tensors in maximally symmetric space

$$\boldsymbol{N} = \frac{\boldsymbol{1}}{\boldsymbol{n}}\binom{\boldsymbol{n}+\boldsymbol{p}}{\boldsymbol{p}+\boldsymbol{1}}\binom{\boldsymbol{n}+\boldsymbol{p}-\boldsymbol{1}}{\boldsymbol{p}}$$

C Barbance 1973

	P=1	P=2	P=3	P=4	
3D	6	20	50	105	…
4D	10	50	175	490	…
5D	15	105	490	1764	…

Work in progress

Summary

We have shown a prolongation of Killing, Killing-Yano equations.

*Prolongation of Killing-Stackel equation is in progress.

Once one make a prolongation successfully, one can discuss properties of solution to the system.