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P01

Gauge-invariant perturbation theory on the Schwarzschild spacetime including l=0,1 modes

--- Realizations of exact solutions ---

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References :

K.N. PTP **110** (2003), 723; K.N. PTP **113** (2005), 413.
K.N. PRD **74** (2006), 101301R; K.N. PTP **117** (2007), 17; K.N. Bulg. J. Phys. **35** (2008), 489;
K.N. PRD **80** (2009), 124021; K.N. PTP **121** (2009), 1321; K.N. Adv. in Astron. **2010** (2010), 576273;
A.J. Christopherson, et al, CQG **28** (2011), 225024;
K.N. CQG **28** (2011), 122001; K.N. PTEP **2013** (2013), 043E02;
K.N. IJMPD **21** (2012), 1242004;
K.N. CQG **31** (2014), 135013;
K.N. Book Chapter *"Theory and Applications of Physical Science Vol.3"* Chapter I (2020);
K.N. CQG **38** (2021), 145010; [arXiv:2102.00830v3 [gr-qc]];
K.N. LHEP **2021** (2021), 215; [arXiv:2102.10650v3 [gr-qc]];
K.N. arXiv:2110.13508v6 [gr-qc] : Full paper Part I : --- Formulation and odd-mode perturbations ---
K.N. arXiv:2110.13512v4 [gr-qc] : Full paper Part II : --- Even-mode perturbations ---
K.N. arXiv:2110.13519v4 [gr-qc] : Full paper Part III: --- Realization of exact solutions ---

I. Introduction

The higher order perturbation theory in general relativity has very wide physical motivation.

- **Cosmological perturbation theory**
  - Expansion law of universe : "Back-reaction?" (ΛCDM ?, inhomogeneous cosmology ?, or modified gravity ?)
  - Non-Gaussianity in CMB (beyond Planck)
  - Hubble tension, σ_8-tension, ...
- **Gravitational-wave physics**
  - **Black hole perturbations**
    - Radiation reaction effects due to the gravitational wave emission.
      - » GW from SgA* EMRI. --> LISA target.
  - **Binary coalescence through the post-Minkowski expansion**
    - LIGO-Virgo detected GW from BH-BH binary coalescence !!!
- **Perturbation of a star (Neutron star)**
  - Rotation – pulsation coupling (Kojima 1997)

Gravitational physics is now toward a precise science.

There are many physical situations to which higher order perturbation theory should be applied.

However, general relativistic perturbation theory requires very delicate treatments of "gauges".

It is worthwhile to formulate the higher-order gauge-invariant perturbation theory from general point of view.

- According to this motivation, from 2003, we have been formulating a general-relativistic higher-order perturbation theory in a gauge-invariant manner.
  - **General formulation :**
    - General framework of higher-order gauge-invariant perturbation theory : K.N. PTP**110** (2003),723; *ibid.* **113** (2005), 413.
    - Construction of gauge-invariant variables for the linear-order metric perturbation and the proposal of the "zero-mode problem" : K.N. CQG**28** (2011),122001; PTEP**2013** (2013),043E02; IJMPD**21** (2012), 1242004.
    - The nth-order extension of the definitions of gauge-invariant variables : K.N. CQG **31** (2014), 135013.
  - **Application to cosmological perturbation theory** :
    - Einstein equations : K.N. PRD**74** (2006), 101301R; PTP**117** (2007), 17.
    - Equations of motion for matter fields : K.N. PRD**80** (2009), 124021.
    - Consistency of the 2nd order Einstein equations : K.N. PTP**121** (2009), 1321.
    - Summary of current status of this formulation : K.N. Adv. in Astron. **2010** (2010), 576273.
    - Comparison with a different formulation : A.J. Christopherson, et al., CQG**28** (2011), 225024.
    - Summary of current status updated 2019 : K.N. Book *"Theory and Applications of Physical Science Vol.3"*, Chapter I, (2020); arXiv:1912.1280v2 [gr-qc].
  - **Application to Black Hole perturbation theory** :
    - Proposal of a gauge-invariant treatment of l=0,1-mode perturbations on Schwarzschild background spacetime: K.N. CQG **38** (2021), 145010. [arXiv:2102.00830v3[gr-qc]].
    - Formal solutions of the any-order mass, angular-momentum, dipole perturbations on the Schwarzschild background spacetime: K.N., LHEP 2021 (2021), 215. [arXiv:2102.10650v3[gr-qc]].
    - **Full Paper-Series of these short papers:**
      Part I : --- Formulation and odd-mode perturbation --- K.N., arXiv:2110.13508v6 [gr-qc];
      Part II : --- Even-mode perturbations --- K.N., arXiv:2110.13512v4 [gr-qc];
      Part III : --- Realization of exact solutions -- K.N. arXiv:2110.13519v4 [gr-qc].

Our general formulations for the higher-order gauge-invariant perturbation theory is based on the following conjecture:

Conjecture (Decomposition conjecture):

If the gauge-transformation rule for a tensor field h_{ab} is given by ${}_{\mathscr{Y}}h_{ab} - {}_{\mathscr{X}}h_{ab} = \pounds_{\sigma} g_{ab}$ with the background metric g_{ab}, then, there exist a tensor field $\mathscr{F}_{ab}$ and a vector field Y^a such that h_{ab} is decomposed as $h_{ab} = \mathscr{F}_{ab} + \pounds_Y g_{ab}$, where $\mathscr{F}_{ab}$ and Y^a are transformed as ${}_{\mathscr{Y}}\mathscr{F}_{ab} - {}_{\mathscr{X}}\mathscr{F}_{ab} = 0$, ${}_{\mathscr{Y}}Y^a - {}_{\mathscr{X}}Y^a = \sigma^a$ under the gauge-transformation $\Phi_\varepsilon = \mathscr{X}_\varepsilon^{-1} \circ \mathscr{Y}_\varepsilon$, respectively.

In [K.N.(2011); K.N.(2013).], a proof of this conjecture was discussed, but the existence of Green functions for some elliptic differential operators was assumed. For this reason, the kernel modes (**zero modes**) of these elliptic differential operators were ignored (**zero-mode problem**).

In the perturbations on the Schwarzschild background spacetime, l=0,1 modes correspond to these kernel modes!!

We proposed a gauge-invariant treatments of l=0,1 modes [K.N. CQG vol.38 (2021), 145010.] and develop a higher-order gauge-invariant perturbation theory on the Schwarzschild background spacetime [K.N. LHEP vol.2021 (2021), 215.].

In this poster, ...

I briefly explain our proposal of a gauge-invariant treatment of l=0,1-mode perturbations on the Schwarzschild background spacetime.

K.N., CQG38 (2021), 145010; [arXiv:2102.00830v3 [gr-qc]];
K.N., arXiv:2110.13508v6 [gr-qc] (Full Paper Part I);
K.N., arXiv:2110.13512v4 [gr-qc] (Full Paper Part II).

I mainly show that our derived linear solutions realize two linearized exact solutions, the Lemaitre-Tolman-Bondi solution, non-rotating C-metric.

K.N., arXiv:2110.13519v4 [gr-qc] (Full Paper Part III).

II. "Gauge" in general relativity

(R.K. Sachs (1964).)

- There are two kinds of "gauge" in general relativity.
  - The concepts of these two "gauges" are closely related to the general covariance.
  - "General covariance" :
    There is no preferred coordinate system in nature.

- **The first kind "gauge"** is a coordinate system on a single spacetime manifold.

- **The second kind "gauge"** appears in the perturbation theory.
  This is a point identification between the physical spacetime and the background spacetime.
  - To explain this second kind gauge, we have to remind what we are doing in perturbation theories.
  - **Our gauge-invariant formulation exclude this second kind "gauge".**

III. Construction of gauge-invariant variables

- $\bar{g}_{ab}$: metric on PS, g_{ab} : metric on BGS.

  metric expansion : $\mathscr{X}\bar{g}_{ab} := \mathscr{X}_\varepsilon^* \bar{g}_{ab} = \sum_{l=0}^{n} \frac{\varepsilon^l}{l!} {}^{(l)}g_{ab} + o(\varepsilon^n), \quad {}^{(0)}g_{ab} := g_{ab}.$

Our general framework of the higher-order gauge invariant perturbation theory is based on a single assumption.

○ **linear order (decomposition conjecture) :** $h_{ab} := {}^{(1)}g_{ab}$

If there is a symmetric tensor field h_{ab} of the second rank, whose gauge-transformation rule is $\mathscr{Y}h_{ab} - \mathscr{X}h_{ab} = \pounds_\sigma g_{ab}$, then there exist a tensor field $\mathscr{F}_{ab}$ and a vector field Y^a such that h_{ab} is decomposed as

$$\boxed{h_{ab} = \mathscr{F}_{ab} + \pounds_Y g_{ab}}$$

where $\mathscr{F}_{ab}$ and Y^a are transformed as

$$\mathscr{Y}\mathscr{F}_{ab} - \mathscr{X}\mathscr{F}_{ab} = 0, \quad \mathscr{Y}Y^a - \mathscr{X}Y^a = \sigma^a$$

under the gauge-transformation $\Phi_\varepsilon = \mathscr{X}_\varepsilon^{-1} \circ \mathscr{Y}_\varepsilon$, respectively.

This conjecture is almost proved but is still a conjecture due to the "zero-mode problem" !!

K.N. CQG**28** (2011), 122001; PTEP 2013 (2013), 043E02; IJMPD**21** (2012), 1242004.

Proposal to solve this l=0,1-mode problem

- To resolve the problem of l=0,1-mode perturbations, **we use the decomposition of the perturbations by $S = S_\delta$ instead of by $S = Y_{lm}$.**

$$S_\delta = \begin{cases} Y_{lm} & \text{for} \quad l \geq 2; \\ k_{(\hat{\Delta}+2)m} & \text{for} \quad l = 1; \quad k_{(\hat{\Delta}+2)m} \in \mathscr{K}_{(\hat{\Delta}+2)} := \{f \in \mathscr{F}(S^2) | [\hat{\Delta}+2] f = 0\}. \\ k_{(\hat{\Delta})} & \text{for} \quad l = 0. \quad k_{(\hat{\Delta})} \in \mathscr{K}_{(\hat{\Delta})} := \{f \in \mathscr{F}(S^2) | \hat{\Delta} f = 0\}. \end{cases} \quad \text{– (I)}$$

$$\begin{aligned} h_{AB} &= \sum_{l,m} \tilde{h}_{AB} S_\delta, \quad h_{Ap} = r\hat{D}_p \sum_{l,m} \tilde{h}_{(e1)A} S_\delta + r\varepsilon_{pq}\hat{D}^q \sum_{l,m} \tilde{h}_{(o1)A} S_\delta, \\ h_{pq} &= \frac{1}{2}\gamma_{pq} r^2 \sum_{l,m} \tilde{h}_{(e0)} S_\delta + r^2 \left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\Delta\right) \sum_{l,m} \tilde{h}_{(e2)} S_\delta + r^2 2\varepsilon_{r(p}\hat{D}_{q)}\hat{D}^r \sum_{l,m} \tilde{h}_{(o2)} S_\delta, \end{aligned} \quad \text{– (II)}$$

- This decomposition is invertible if we choose

$$\begin{aligned} & k_{(\hat{\Delta})} = 1 + \delta \ln\left(\frac{1-z}{1+z}\right), \quad \delta \in \mathbb{R}, \quad z = \cos\theta \\ & k_{(\hat{\Delta}+2)m=0} = z\left\{1 + \delta\left(\frac{1}{2}\ln\frac{1+z}{1-z} - \frac{1}{z}\right)\right\}, \quad k_{(\hat{\Delta}+2)m=\pm 1} = (1-z^2)^{1/2}\left\{1 + \delta\left(\frac{1}{2}\ln\frac{1+z}{1-z} + \frac{z}{1-z^2}\right)\right\} e^{\pm i\phi}. \end{aligned} \quad \text{– (III)}$$

- When $\delta = 0$, we have $k_{(\hat{\Delta})} \propto Y_{00}, \quad k_{(\hat{\Delta}+2)m} \propto Y_{1m}$.
- These functions are singular except for the case $\delta = 0$.

Proposal :

We decompose perturbations h_{ab} on a spherically symmetric background space-time through Eqs. (I), (II), and (III). Then, the decomposition (II) becomes invertible including $l = 0, 1$ modes. After deriving the field equations such as linearized Einstein equations by using the harmonic function S_δ, we choose $\delta = 0$ when we solve these field equations as regularity of solutions.

Owing to this proposal, we can show the decomposition conjecture

$$\boxed{{}^{(1)}g_{ab} = {}^{(1)}\mathscr{F}_{ab} + \pounds_{(1)Y} g_{ab}}$$

for the perturbations on the Schwarzschild background spacetime including l=0,1 modes.

IV. The solution to Linearized Einstein equations.

The solutions are "gauge invariant" in the sense of **"the second kind"**.

- l=1 mode solution (we concentrate only on m=0 mode):

$$a_1(t,r) = -\frac{16\pi}{3M} r^3 f \int dt \tilde{T}_{(o1)r} + a_{10} = -\frac{16\pi}{3M} \int dr \frac{r^3}{f} \tilde{T}_{(o1)t} + \underline{a_{10}}.$$ **Kerr parameter perturbation**

$$2\mathscr{F}_{Ap}(dx^A)_{(a}(dx^p)_{b)} = \left(6Mr^2 \int dr \frac{a_1(t,r)}{r^4}\right) \sin^2\theta (dt)_{(a}(d\phi)_{b)} + \pounds_{V_{(o1)}} g_{ab}.$$

$$V_{(o1)a} := \left(\beta(t) + W_{(o)}(t,r)\right) r^2 \sin^2\theta (d\phi)_a, \quad \beta(t)$$ **: arbitrary function of t.**

- l=0 even mode solution:

$$\mathscr{F}_{ab} = \frac{2}{r}\left(M_1 + 4\pi \int dr \left[\frac{r^2}{f}\tilde{T}_{tt}\right]\right)\left((dt)_a(dt)_b + \frac{1}{f^2}(dr)_a(dr)_b\right) + 2\left[4\pi r \int dt \left(\frac{1}{f}\tilde{T}_{tt} + f\tilde{T}_{rr}\right)\right](dt)_{(a}(dr)_{b)} + \pounds_{V_{(e0)}} g_{ab},$$

where $V_{(e0)a} := \left(\frac{1}{4} f\Upsilon + \frac{1}{4} r f \partial_r \Upsilon + \gamma(r)\right)(dt)_a + \frac{r}{4f}\partial_t \Upsilon (dr)_a, \quad \gamma(r) := f \int dr \left[\frac{1}{4f} r\zeta(r) - \frac{1}{4f^2}\int dr (1-3f)\zeta(r) + \frac{\xi}{f^2}\right].$

- l=1 even mode solution:

$$\begin{aligned}\mathscr{F}_{ab} = & -\frac{16\pi r^2 f^2}{3(1-f)}\left[\frac{1+f}{2}\tilde{T}_{rr} + rf\partial_r \tilde{T}_{rr} - \tilde{T}_{(e0)} - 4\tilde{T}_{(e1)r}\right]\cos\theta (dt)_a(dt)_b + 16\pi r^2\left[\tilde{T}_{tr} - \frac{2r}{3f(1-f)}\partial_t \tilde{T}_{tt}\right]\cos\theta (dt)_{(a}(dr)_{b)} \\ & + \frac{8\pi r^2(1-3f)}{f^2(1-f)}\left[\tilde{T}_{rr} - \frac{2rf}{3(1-3f)}\partial_r \tilde{T}_{tt}\right]\cos\theta (dr)_a(dr)_b - \frac{16\pi r^4}{3(1-f)}\tilde{T}_{tt}\cos\theta \gamma_{ab} \\ & + \pounds_{V_{(e1)}} g_{ab},\end{aligned}$$

where $V_{(e1)a} := -r\partial_t \Phi_{(e)} \cos\theta (dt)_a + \left(\Phi_{(e)} - r\partial_r \Phi_{(e)}\right)\cos\theta (dr)_a - r\Phi_{(e)} \sin\theta (d\theta)_a.$

These terms are regarded as **"the first kind gauge"**.

V. Realizations of exact solutions

5-1. Realization of Lemaitre-Tolman-Bondi (LTB) solution (1)

- Energy-momentum tensor : $T_{ab} = \rho u_a u_b, \quad u_a = -(d\tau)_a.$

- Exact LTB solution : $g_{ab} = -(d\tau)_a(d\tau)_b + \frac{(\partial_R r)^2}{1+f(R)}(dR)_a(dR)_b + r^2\gamma_{ab}, \quad r = r(\tau, R),$

  Einstein eqs. : $(\partial_\tau r)^2 = \frac{F(R)}{r} + f(R), \quad 8\pi\rho = \frac{\partial_R F(R)}{(\partial_R r) r^2}.$

$$\begin{cases} r = \frac{F(R)}{2f(R)}(\cosh\eta - 1), \quad \tau_0(R) - \tau = \frac{F(R)}{2(f(R))^{3/2}}(\sinh\eta - \eta), \quad f(R) > 0. \\ r = \frac{F(R)}{-2f(R)}(1 - \cos\eta), \quad \tau_0(R) - \tau = \frac{F(R)}{2(-f(R))^{3/2}}(\eta - \sin\eta), \quad f(R) < 0. \\ r = \left(\frac{9F(R)}{4}\right)^{1/3}[\tau_0(R) - \tau]^{2/3}, \quad f(R) = 0. \end{cases}$$

- Vacuum case : $\rho = 0 \longrightarrow \partial_R F(R) = 0.$ We choose $\tau_0(R) = R, \quad f(R) = 0.$

$$(dr)_a = \left(\frac{F}{r}\right)^{1/2}[(dR)_a - (d\tau)_a], \quad (dt)_a = (d\tau)_a - \left(1 - \frac{F}{r}\right)^{-1}\left(\frac{F}{r}\right)^{1/2}(dr)_a.$$

$$g_{ab} = -f(dt)_a(dt)_b + f^{-1}(dr)_a(dr)_b + r^2\gamma_{ab}, \quad f = 1 - \frac{F}{r} = 1 - \frac{2M}{r}.$$

(Schwarzschild spacetime)

5-1. Realization of LTB solution (2)

Linear perturbations with the Schwarzschild background :

$$\begin{cases} F(R) &= 2M+\varepsilon 2m_1(R)+O(\varepsilon^2), \\ f(R) &= 0+\varepsilon f_1(R)+O(\varepsilon^2), \\ \tau_0(R) &= R+\varepsilon\tau_1(R)+O(\varepsilon^2). \end{cases} \Longrightarrow \begin{cases} r(\tau,R)=r_s(\tau,R)+\varepsilon r_1(\tau,R)+O(\varepsilon^2). \\ r_s(\tau,R)=\left(\frac{9M}{2}\right)^{1/3}[R-\tau]^{2/3}, \quad (\partial_\tau r_s)^2-\frac{2M}{r_s}=0. \end{cases}$$

$$\begin{cases} (\partial_\tau r)^2 &= \frac{F(R)}{r}+f(R), \\ 8\pi\rho &= \frac{\partial_R F(R)}{(\partial_R r)r^2}. \end{cases} \xrightarrow{\text{Linearize}} \begin{cases} (1-f)^{1/2}(\partial_\tau r_1)+\frac{m_1(R)}{r}-\frac{M}{r^2}r_1+\frac{1}{2}f_1(R)=0, \\ 8\pi\rho_1=\frac{2\partial_R m_1(R)}{(\partial_R r)r^2}, \quad (r_s\to r). \end{cases}$$

$$\begin{aligned} \mathscr{X}_\varepsilon^* \bar{g}_{ab} &= -(d\tau)_a(d\tau)_b+\frac{(\partial_R r)^2}{1+f(R)}(dR)_a(dR)_b+r^2\gamma_{ab} \\ &= \underline{-(d\tau)_a(d\tau)_b+(\partial_R r)^2(dR)_a(dR)_b+r^2\gamma_{ab}} \text{ Background} \\ &\quad \underline{+\varepsilon\left[(2(\partial_R r_1)-(\partial_R r)f_1)(\partial_R r)(dR)_a(dR)_b+2rr_1\gamma_{ab}\right]} \text{ Linear perturbation} \\ &\quad +O(\varepsilon^2) \\ &=: \boxed{g^{(0)}_{ab}+\varepsilon\,{}_{\mathscr{X}}h_{ab}}+O(\varepsilon^2). \end{aligned}$$

Gauge choice $\mathscr{X}_\varepsilon$: $\{\tau,R,\theta,\phi\}\in\mathscr{M}_{ph}\longrightarrow\{\tau,R,\theta,\phi\}\in\mathscr{M}$

Linear metric perturbation :

$$\boxed{{}_{\mathscr{X}}h_{ab}=(2(\partial_R r_1)-(\partial_R r)f_1)(dR)_a(dR)_b+2rr_1\gamma_{ab}.} \qquad \boxed{{}_{\mathscr{Y}}h_{ab}={}_{\mathscr{X}}h_{ab}+\pounds_\xi g_{ab}.}$$

5-1. Realization of LTB solution (3)

Our linearized l=0 solution : $\boxed{{}_{\mathscr{X}}h_{ab}=\mathscr{F}_{ab}+\pounds_{\mathscr{X}Y}g_{ab}.}$

$$\mathscr{F}_{ab}=\frac{2}{r}\left(M_1+4\pi\int dr\left[\frac{r^2}{f}\tilde{T}_{tt}\right]\right)\left((dt)_a(dt)_b+\frac{1}{f^2}(dr)_a(dr)_b\right)+2\left[4\pi r\int dt\left(\frac{1}{f}\tilde{T}_{tt}+f\tilde{T}_{rr}\right)\right](dt)_{(a}(dr)_{b)}+\pounds_{V_{(e0)}}g_{ab},$$

Traceless in the tangent space $\{(dt)_a,(dr)_a\}$

The term γ_{ab} is included in this term.

$$V_{(e0)a}:=\left(\frac{1}{4}f\Upsilon+\frac{1}{4}rf\partial_r\Upsilon+\gamma(r)\right)(dt)_a+\frac{r}{4f}\partial_t\Upsilon(dr)_a, \quad \gamma(r):=f\int dr\left[\frac{1}{4f}r\zeta(r)-\frac{1}{4f^2}\int dr(1-3f)\zeta(r)+\frac{\xi}{f^2}\right].$$

Linearized LTB solution : ${}_{\mathscr{X}}h_{ab}=(2(\partial_R r_1)-(\partial_R r)f_1)(dR)_a(dR)_b+2rr_1\gamma_{ab}.$

$$(dR)_a = (dt)_a+f^{-1}(1-f)^{-1/2}(dr)_a, \quad f=1-\frac{2M}{r}, \quad V_{(LTB)a}:=\left[(1-f)^{1/2}r_1+\frac{f}{2}\int dt f_1(R(t,r))\right](dt)_a+\frac{r_1}{f}(dr)_a.$$

$$\boxed{{}_{\mathscr{X}}h_{ab}=\frac{2m_1(R)}{r}\left[(dt)_a(dt)_b+\frac{1}{f^2}(dr)_a(dr)_b\right]+\frac{2-f}{f(1-f)^{1/2}}\frac{m_1(R)}{r}2(dt)_{(a}(dr)_{b)}+\pounds_{V_{(LTB)}}g_{ab}.}$$

Realization from our l=0 solution :

- We consider the case $M_1=0$, because we can always add this term with an appropriate term of $\pounds_V g_{ab}$.

$$\begin{cases} 4\pi\int dr\left[\frac{r^2}{f}\tilde{T}_{tt}\right]=4\pi\int dr\left[\frac{r^2}{f}\rho_1\right]=\int dr\left[\frac{\partial_R m_1(R)}{f(1-f)^{1/2}}\right]=\int dr\partial_r m_1(R)=m_1(R). \\ 4\pi r\int dt\left(\frac{1}{f}\tilde{T}_{tt}+f\tilde{T}_{rr}\right)=\frac{(2-f)}{rf(1-f)^{1/2}}\int dt\partial_R m_1(R)=\frac{(2-f)}{rf(1-f)^{1/2}}\int dt\partial_t m_1(R)=\frac{2-f}{f(1-f)^{1/2}}\frac{m_1(R)}{r}. \end{cases}$$

5-2. Realization of non-rotating C-metric (1)

- To compare with the Schwarzschild metric, we consider the C-metric:

$$g_{ab} = \frac{1}{(1+\alpha r\cos\theta)^2}\left[-Q(dt)_a(dt)_b + \frac{1}{Q}(dr)_a(dr)_b + \frac{r^2}{P}\left\{(d\theta)_a(d\theta)_b + P^2C^2\sin^2\theta(d\phi)_a(d\phi)_b\right\}\right].$$

$$P = 1+2\alpha m\cos\theta, \quad Q = (1-\alpha^2r^2)\left(1-\frac{2m}{r}\right).$$

We consider the situation m is finite and the acceleration parameter α and the deficit/excess angle $C-1$ is infinitesimally small.

- Perturbative expansion of C-metric : $\bar{g}_{ab}(\bar{M},\bar{\alpha},\bar{C};\bar{x}) = \bar{g}_{\mu\nu}(\bar{M},\bar{\alpha},\bar{C};\bar{x})(d\bar{x}^\mu)_a(d\bar{x}^\nu)_b$
  - We regard the metric $\bar{g}_{ab}$ on the physical spacetime $\mathscr{M}_\varepsilon$ is obtained by the replacement $m\to\bar{M}$, $\alpha\to\bar{\alpha}$, $C\to\bar{C}$, $x^\mu\to\bar{x}^\mu$ in the above metric.
  - As the gauge choice $\mathscr{X}_\varepsilon$ of the second kind, we identify the point $\{x^\mu\}=\{t,r,\theta,\phi\}$ on the background spacetime $\mathscr{M}$ and the point $\{\bar{x}^\mu\}=\{\bar{t},\bar{r},\bar{\theta},\bar{\phi}\}$ on the physical spacetime $\mathscr{M}_\varepsilon$: $\mathscr{X}_\varepsilon : \{\bar{x}^\mu\} \overset{\leftarrow}{=} \{x^\mu\}$.
  - We consider the parameter expansion

$$\bar{M} =: M+\varepsilon M_1+O(\varepsilon^2), \quad \bar{\alpha} =: \alpha+\varepsilon\alpha_1+O(\varepsilon^2), \quad \bar{C} =: C+\varepsilon C_1+O(\varepsilon^2).$$

  Then, the linear perturbation $\mathscr{x}h_{ab}$ in the gauge choice $\mathscr{X}_\varepsilon$ is given by

$$\boxed{\mathscr{x}h_{ab} = M_1\partial_M\bar{g}_{ab}(M,\alpha=0,C=1;x)+\alpha_1\partial_\alpha\bar{g}_{ab}(M,\alpha=0,C=1;x)+C_1\partial_C\bar{g}_{ab}(M,\alpha=0,C=1;x).}$$

  - Of course, we may a different gauge choice $\mathscr{Y}_\varepsilon$: $\mathscr{Y}_\varepsilon : \{\bar{x}^\mu\} \overset{\leftarrow}{=} \{x'^\mu\}$, $x'^\mu = x^\mu+\varepsilon\xi^\mu+O(\varepsilon^2)$
    In this gauge, the linear perturbation $\mathscr{y}h_{ab}$ is given by

$$\boxed{\mathscr{y}h_{ab} = M_1\partial_M\bar{g}_{ab}(M,\alpha=0,C=1,x)+\alpha_1\partial_\alpha\bar{g}_{ab}(M,\alpha=0,C=1,x)+C_1\partial_C\bar{g}_{ab}(M,\alpha=0,C=1,x)+\underline{\pounds_\xi g_{ab}}.}$$

We may always have such term due to the different choice of the second-kind gauge.

5-2. Realization of non-rotating C-metric (2)

- Straightforward calculations leads to the linear perturbation $\mathscr{x}h_{ab}$ as

$$\begin{aligned}\mathscr{x}h_{ab} \quad = \quad & \frac{2M_1}{r}\left((dt)_a(dt)_b+f^{-2}(dr)_a(dr)_b\right)\\ & +2\alpha_1\left[-r\cos\theta g_{ab}+Mr^2\left(-(d\theta)_a(d\theta)_b+\sin^2\theta(d\phi)_a(d\phi)_b\right)\right]\\ & +2C_1r^2\sin^2\theta(d\phi)_a(d\phi)_b.\end{aligned}$$

- Components of the linearized C-metric :
  - The (t-r)-part of h_{ab} is given by

$$h_{AB} = \frac{2M_1}{r}\left[(dt)_A(dt)_B+f^{-2}(dr)_A(dr)_B\right]-2\alpha_1 r\cos\theta\left[-f(dt)_A(dt)_B+f^{-1}(dr)_A(dr)_B\right]$$

  - The h_{Ap} components are absent.
  - The h_{pq} components :

$$h_{pq} = 2\alpha_1\left[-r\cos\theta r^2\gamma_{pq}+Mr^2\cos\theta\left(-(d\theta)_p(d\theta)_q+\sin^2\theta(d\phi)_p(d\phi)_q\right)\right]+2C_1r^2\sin^2\theta(d\phi)_a(d\phi)_b$$

    Trace part : $\frac{1}{r^2}\gamma^{pq}h_{pq} = -4\alpha_1 r\cos\theta+2C_1.$

    Traceless part : $\frac{1}{r^2}h_{pq}-\frac{1}{2}\gamma_{pq}\frac{1}{r^2}\gamma^{rs}h_{rs} = (2\alpha_1M\cos\theta+C_1)\left(-(d\theta)_p(d\theta)_q+\sin^2\theta(d\phi)_p(d\phi)_q\right).$

5-2. Realization of non-rotating C-metric (3)

Summary of the mode decomposition of the C-metric perturbation with he Schwarzschild background :

- $l=0$ mode :
$$\begin{cases} \tilde{h}_{AB} = \dfrac{2M_1}{r}\left\{(dt)_a(dt)_b + f^{-2}(dr)_a(dr)_b\right\}, \quad \tilde{h}_{(e1)A} = \tilde{h}_{(o1)A} = 0, \\ \tilde{h}_{(e0)} = 2C_1, \quad \tilde{h}_{(e2)} = \tilde{h}_{(o2)} = 0. \end{cases}$$
  gauge-variant part : $\boxed{\mathscr{L}Y_a = 0}$
  gauge-invariant part :
$$\boxed{\begin{aligned} &\tilde{F}_A = 0, \\ &\tilde{F} = 2C_1, \quad \tilde{F}_{AB} = \frac{2M_1}{r}\left((dt)_a(dt)_b + f^{-2}(dr)_a(dr)_b\right). \end{aligned}}$$

- $l=1$ mode :
$$\begin{cases} \tilde{h}_{AB} = -2\alpha_1 r\left\{-f(dt)_a(dt)_b + f^{-1}(dr)_a(dr)_b\right\}, \quad \tilde{h}_{(e1)A} = \tilde{h}_{(o1)A} = 0, \\ \tilde{h}_{(e0)} = -4\alpha_1 r, \quad \tilde{h}_{(e2)} = \tilde{h}_{(o2)} = 0. \end{cases}$$
  gauge-variant part : $\boxed{\mathscr{L}Y_a = 0}$
  gauge-invariant part :
$$\boxed{\begin{aligned} &\tilde{F}_A = 0, \\ &\tilde{F} = -4\alpha_1 r, \quad \tilde{F}_{AB} = -2\alpha_1 r\left(-f(dt)_a(dt)_b + f^{-1}(dr)_a(dr)_b\right). \end{aligned}}$$

- $l \geq 2$ mode :
$$\begin{cases} \tilde{h}_{AB} = 0,, \quad \tilde{h}_{(e1)A} = \tilde{h}_{(o1)A} = 0, \\ \tilde{h}_{(e0)} = -\dfrac{2(2l+1)}{(l-1)l(l+1)(l+2)}\left[(+2\alpha_1 M + C_1) + (-2\alpha_1 M + C_1)(-1)^l\right], \quad \tilde{h}_{(e2)} = \tilde{h}_{(o2)} = 0. \end{cases}$$
  gauge-variant part :
$$\boxed{\begin{aligned} &\mathscr{L}\tilde{Y}_A = 0, \quad \mathscr{L}\tilde{Y}_{(o)} = 0, \\ &\mathscr{L}\tilde{Y}_{(e)} = -\frac{r^2(2l+1)}{(l-1)l(l+1)(l+2)}\left[(+2\alpha_1 M + C_1) + (-2\alpha_1 M + C_1)(-1)^l\right]. \end{aligned}}$$
  gauge-invariant part :
$$\boxed{\begin{aligned} &\tilde{F}_A = 0, \quad \tilde{F}_{AB} = 0 \\ &\tilde{F} = -\frac{2(2l+1)}{(l-1)(l+2)}\left[(+2\alpha_1 M + C_1) + (-2\alpha_1 M + C_1)(-1)^l\right]. \end{aligned}}$$

5-2. Realization of non-rotating C-metric (4)

Realization of $l \geq 2$ mode perturbations :

- $l \geq 2$ mode gauge-invariant metric perturbations :
$$\tilde{F}_A = 0, \quad \tilde{F}_{AB} = 0 \quad \Longrightarrow \quad \tilde{F}^D_D = 0, \quad \tilde{\mathbb{F}}_{AB} := \tilde{F}_{AB} - \frac{1}{2}y_{AB}\tilde{F}_C{}^C = 0.$$
$$\tilde{F} = -\frac{2(2l+1)}{(l-1)(l+2)}\left[(+2\alpha_1 M + C_1) + (-2\alpha_1 M + C_1)(-1)^l\right] = const.$$
- Linearized Einstein eq.
  - Constraints : $\tilde{F}_D{}^D = -16\pi r^2 \tilde{T}_{(e2)}, \quad \bar{D}^D\tilde{\mathbb{F}}_{AD} - \frac{1}{2}\bar{D}_A\tilde{F} = 16\pi\left[r\tilde{T}_{(e1)A} - \frac{1}{2}r^2\bar{D}_A\tilde{T}_{(e2)}\right]. \Longrightarrow \boxed{\tilde{T}_{(e2)} = 0, \quad \tilde{T}_{(e1)A} = 0.}$
  - Evolution equation (1) :
$$\left[-\bar{D}_D\bar{D}^D - \frac{2}{r}(\bar{D}_D r)\bar{D}^D + \frac{4}{r}(\bar{D}^D\bar{D}_D r) + \frac{l(l+1)}{r^2}\right]\tilde{\mathbb{F}}_{AB} + \frac{4}{r}(\bar{D}^D r)\bar{D}_{(A}\tilde{\mathbb{F}}_{B)D} - \frac{2}{r}(\bar{D}_{(A}r)\bar{D}_{B)}\tilde{F} = 16\pi S_{(\mathbb{F})AB} \Longrightarrow S_{(\mathbb{F})AB} = 0.$$
$$\begin{aligned} S_{(\mathbb{F})AB} \;:=\; & \tilde{T}_{AB} - \frac{1}{2}y_{AB}\tilde{T}_C{}^C - 2\left(\bar{D}_{(A}(r\tilde{T}_{(e1)B)}) - \frac{1}{2}y_{AB}\bar{D}^D(r\tilde{T}_{(e1)D})\right) + 2\left((\bar{D}_{(A}r)\bar{D}_{B)} - \frac{1}{2}y_{AB}(\bar{D}^D r)\bar{D}_D\right)(r\tilde{T}_{(e2)}) \\ & + r\left(\bar{D}_A\bar{D}_B - \frac{1}{2}y_{AB}\bar{D}^D\bar{D}_D\right)(r\tilde{T}_{(e2)}) + 2\left((\bar{D}_A r)(\bar{D}_B r) - \frac{1}{2}y_{AB}(\bar{D}^C r)(\bar{D}_C r)\right)\tilde{T}_{(e2)} + 2y_{AB}(\bar{D}^C r)\tilde{T}_{(e1)C} - ry_{AB}(\bar{D}^C r)\bar{D}_C\tilde{T}_{(e2)}, \end{aligned}$$
$$\Longrightarrow \tilde{T}_{AB} - \frac{1}{2}y_{AB}\tilde{T}_C{}^C = 0 \Longrightarrow \boxed{\tilde{T}_{tr} = 0, \quad \tilde{T}_{tt} + f^2\tilde{T}_{rr} = 0}$$
  - Evolution equation (2) :
$$\left(\bar{D}_D\bar{D}^D + \frac{2}{r}(\bar{D}^D r)\bar{D}_D - \frac{(l-1)(l+2)}{r^2}\right)\tilde{F} - \frac{4}{r^2}(\bar{D}_C r)(\bar{D}_D r)\tilde{\mathbb{F}}^{CD} = 16\pi S_{(F)},$$
$$S_{(F)} := \tilde{T}_C{}^C + 4(\bar{D}_D r)\tilde{T}^D_{(e1)} - 2r(\bar{D}_D r)\bar{D}^D\tilde{T}_{(e2)} - (l(l+1)+2)\tilde{T}_{(e2)}.$$
$$\Longrightarrow \boxed{\tilde{F} = \frac{32\pi}{(l-1)(l+2)}\frac{r^2}{f}\tilde{T}_{tt} = const.}$$

- Linearized divergence of the energy-momentum tensor:
$$\bar{D}^C\tilde{T}_C{}^B + \frac{2}{r}(\bar{D}^D r)\tilde{T}_D{}^B - \frac{1}{r}l(l+1)\tilde{T}^B_{(e1)} - \frac{1}{r}(\bar{D}^B r)\tilde{T}_{(e0)} = 0, \Longrightarrow \text{Trivial!}$$
$$\bar{D}^C\tilde{T}_{(e1)C} + \frac{3}{r}(\bar{D}^C r)\tilde{T}_{(e1)C} + \frac{1}{2r}\tilde{T}_{(e0)} - \frac{1}{2r}(l-1)(l+2)\tilde{T}_{(e2)} = 0, \Longrightarrow \boxed{\tilde{T}_{(e0)} = 0.}$$

- Summary of energy-momentum tensor for $l \geq 2$ modes :
$$\boxed{{}^{(1)}\mathscr{T}_{ac}(l \geq 2) =: -\frac{1}{r^2}y_{ab}\sum_{l=2}^{\infty}\lambda_l P_l(\cos\theta), \quad 16\pi\lambda_l := 16\pi\frac{r^2}{f}\tilde{T}_{tt} = -(2l+1)\left[(+2\alpha_1 M + C_1) + (-2\alpha_1 M + C_1)(-1)^l\right].}$$

5-2. Realization of non-rotating C-metric (5)

Realization of $l=0$ mode perturbations :

- l=0 mode (gauge-invariant) metric perturbations :

$$\tilde{h}_{ab}(l=0)=\frac{2M_1}{r}\left((dt)_a(dt)_b+f^{-2}(dr)_a(dr)_b\right)+r^2C_1\gamma_{ab},$$

- Since the vector- and tensor-harmonics vanish for l=0 modes, we may choose

$$\boxed{\tilde{T}_{(e2)}=0,\quad \tilde{T}_{(e1)A}=0,\quad (\tilde{T}_{(o2)}=0,\quad \tilde{T}_{(o1)A}=0).}$$

From the component of the divergence of the energy-momentum tensor

$$\bar{D}^C\tilde{T}_{(e1)C}+\frac{3}{r}(\bar{D}^Cr)\tilde{T}_{(e1)C}+\frac{1}{2r}\tilde{T}_{(e0)}+\frac{1}{r}\tilde{T}_{(e2)}=0, \Longrightarrow \boxed{\tilde{T}_{(e0)}=0.}$$

- Since the mass parameter perturbation M_1 is always introduced through the addition of the appropriate $\pounds_V g_{ab}$ term, we choose $M_1=0$, here :

$$\tilde{h}_{ab}(l=0)=r^2C_1\gamma_{ab}=\frac{1-3f}{2}C_1\left\{(dt)_a(dt)_b+f^{-2}(dr)_a(dr)_b\right\}+\pounds_{V_{(C_1)}}g_{ab},\qquad V_{(C_1)a}=\frac{1}{2}ftC_1(dt)_a+\frac{r}{2f}C_1(dr)_a.$$

- Our derived l=0 solutions to the linearized Einstein eq. :

$$\mathscr{F}_{ab}(l=0)=\frac{2}{r}\left(M_1+4\pi\int dr\left[\frac{r^2}{f}\tilde{T}_{tt}\right]\right)\left((dt)_a(dt)_b+\frac{1}{f^2}(dr)_a(dr)_b\right)+2\left[4\pi r\int dt\left(\frac{1}{f}\tilde{T}_{tt}+f\tilde{T}_{rr}\right)\right](dt)_{(a}(dr)_{b)}+\pounds_{V_{(e0)}}g_{ab},$$

$$\Longrightarrow 4\pi\int dr\left[\frac{r^2}{f}\tilde{T}_{tt}\right]=\frac{1-3f}{4}rC_1,\quad 4\pi r\int dt\left(\frac{1}{f}\tilde{T}_{tt}+f\tilde{T}_{rr}\right)=0. \Longrightarrow \boxed{\frac{r^2}{f}\tilde{T}_{tt}=-\frac{1}{8\pi}C_1,\quad \tilde{T}_{tt}+f^2\tilde{T}_{rr}=0.}$$

From the component of the divergence of the energy-momentum tensor, we may choose $\boxed{\tilde{T}_{tr}=0.}$

$$\bar{D}^C\tilde{T}_C{}^B+\frac{2}{r}(\bar{D}^Dr)\tilde{T}_D{}^B=0, \Longrightarrow \begin{cases}-\partial_t\tilde{T}_{tt}+f^2\partial_r\tilde{T}_{rt}+\frac{(1+f)f}{r}\tilde{T}_{rt}=0\\ -\partial_t\tilde{T}_{rt}+\frac{1-f}{2rf}\tilde{T}_{tt}+f^2\partial_r\tilde{T}_{rr}+\frac{(3+f)f}{2r}\tilde{T}_{rr}=0.\end{cases} \Longrightarrow r^2f\tilde{T}_{tr}=const.$$

- Summary of the energy-momentum tensor for l=0 modes:

$$\boxed{{}^{(1)}\mathscr{T}_{ac}(l=0)=:-\frac{1}{r^2}y_{ab}\lambda_{l=0},\quad \lambda_{l=0}=-\frac{C_1}{8\pi},\quad 16\pi\lambda_l:=16\pi\frac{r^2}{f}\tilde{T}_{tt}=-(2l+1)\left[(+2\alpha_1M+C_1)+(-2\alpha_1M+C_1)(-1)^l\right]}.$$

5-2. Realization of non-rotating C-metric (6)

Realization of $l=1$ mode perturbations :

- l=1 mode (gauge-invariant) metric perturbations : $\tilde{h}_{ab}(l=1)=-2\alpha_1r\cos\theta g_{ab}$.
- Inspecting the $l\geq 2$ and l=0 cases, we assume

$$\boxed{\tilde{T}_{(e2)}=0,\quad \tilde{T}_{(e1)A}=0,\quad (\tilde{T}_{(o2)}=0,\quad \tilde{T}_{(o1)A}=0).}$$

From the component of the divergence of the energy-momentum tensor

$$\bar{D}^C\tilde{T}_{(e1)C}+\frac{3}{r}(\bar{D}^Cr)\tilde{T}_{(e1)C}+\frac{1}{2r}\tilde{T}_{(e0)}=0, \Longrightarrow \boxed{\tilde{T}_{(e0)}=0.}$$

Furthermore, we assume $\boxed{\tilde{T}_{tr}=0,\quad \tilde{T}_{tt}+f^2\tilde{T}_{rr}=0,\quad \lambda_{l=1}:=\frac{r^2}{f}\tilde{T}_{tt}=const.}$

Then, $\bar{D}^C\tilde{T}_C{}^B+\frac{2}{r}(\bar{D}^Dr)\tilde{T}_D{}^B=0, \Longrightarrow \begin{cases}-\partial_t\tilde{T}_{tt}=0,\\ \partial_r\left(\tilde{T}_{tt}+f^2\tilde{T}_{rr}\right)-\frac{f}{r^2}\partial_r\left(\frac{r^2}{f}\tilde{T}_{tt}\right)+\frac{5f-1}{2rf}\left(\tilde{T}_{tt}+f^2\tilde{T}_{rr}\right)=0.\end{cases}$ are trivial.

- Owing to these assumptions, our derived l=1 solutions to the linearized Einstein equations is given by

$$\begin{aligned}\mathscr{F}_{ab}(l=1) &= \pounds_{V_{(vac)}}g_{ab}-\frac{16\pi\lambda_{l=1}f}{3(1-f)}\left[\frac{1+f}{2}(dt)_a(dt)_b-\frac{1-3f}{2f^2}(dr)_a(dr)_b+r^2\gamma_{ab}\right]\cos\theta\\ &= \pounds_{V_{(vac)}}g_{ab}-\frac{16\pi\lambda_{l=1}f}{3(1-f)}\left[-\frac{1-f}{2}(dt)_a(dt)_b+\frac{1+3f}{2f^2}(dr)_a(dr)_b+\frac{1+f}{f}r^2\gamma_{ab}\right]+\frac{16\pi\lambda_{l=1}}{3(1-f)}\cos\theta g_{ab}\\ &= \pounds_{V_{(l=1)}}g_{ab}+\frac{16\pi\lambda_{l=1}}{3(1-f)}\cos\theta g_{ab},\quad V_{(l=1)a}=V_{(vac)a}+W_a,\quad W_a:=-\frac{16\pi\lambda_{l=1}r}{6(1-f)}\cos\theta(dr)_a-\frac{16\pi\lambda_{l=1}r^2}{6(1-f)}\sin\theta(d\theta)_a\end{aligned}$$

- If $16\pi\lambda_{l=1}=-12\alpha_1M$, $\frac{16\pi\lambda_{l=1}}{3(1-f)}\cos\theta g_{ab}=-2\alpha_1r\cos\theta g_{ab}$. This coincides with the above $\tilde{h}_{ab}(l=1)$.

- Summary of the energy-momentum tensor for l=1 modes:

$$\boxed{{}^{(1)}\mathscr{T}_{ac}(l=1)=:-\frac{1}{r^2}y_{ab}\lambda_{l=1},\quad 16\pi\lambda_{l=1}=-12\alpha_1M,\quad 16\pi\lambda_l:=16\pi\frac{r^2}{f}\tilde{T}_{tt}=-(2l+1)\left[(+2\alpha_1M+C_1)+(-2\alpha_1M+C_1)(-1)^l\right]}.$$

5-3. Source term of realized of non-rotating C-metric

Summary of the realized energy-momentum tensor for linearized C-metric :

$$\boxed{{}^{(1)}\mathcal{T}_{ac} =: -\frac{1}{r^2}y_{ab}\sum_{l=0}^{\infty}\lambda_l P_l(\cos\theta), \quad 16\pi\lambda_l := 16\pi\frac{r^2}{f}\tilde{T}_{tt} = -(2l+1)\left[(+2\alpha_1 M + C_1) + (-2\alpha_1 M + C_1)(-1)^l\right].}$$

- δ-function on S² : $\delta^{(2)}(\mathbf{n}-\mathbf{n}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=l} Y_{lm}(\mathbf{n})Y^*_{lm}(\mathbf{n}'), \quad \sum_{m=-l}^{m=l} Y_{lm}(\mathbf{n})Y^*_{lm}(\mathbf{n}') = \frac{2l+1}{4\pi}\underbrace{C_l^{1/2}}_{\text{Gegenbauer polynomial}}(\mathbf{n}\cdot\mathbf{n}'),$

  $\mathbf{n}$ and $\mathbf{n}'$ are the position vectors which point to the points on S² in embedded in $\mathbb{R}^3$, respectively.
- The orthogonal coordinate system (x,y,z) in $\mathbb{R}^3$.

  Any point on S² is described by $(x,y,z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$,

  South pole on S² : $(x,y,z) = (0,0,-1)$, North pole on S² : $(x,y,z) = (0,0,+1)$.

  $\Longrightarrow \quad \mathbf{n}\cdot\mathbf{n}_{\text{north}} = \cos\theta, \quad \mathbf{n}\cdot\mathbf{n}_{\text{south}} = -\cos\theta,$

$$\boxed{\begin{aligned}\sum_{m=-l}^{m=l} Y_{lm}(\mathbf{n})Y^*_{lm}(\mathbf{n}_{\text{north}}) &= \frac{2l+1}{4\pi}C_l^{1/2}(\cos\theta) = \frac{2l+1}{4\pi}P_l(\cos\theta),\\ \sum_{m=-l}^{m=l} Y_{lm}(\mathbf{n})Y^*_{lm}(\mathbf{n}_{\text{south}}) &= \frac{2l+1}{4\pi}C_l^{1/2}(-\cos\theta) = \frac{2l+1}{4\pi}P_l(-\cos\theta) = \frac{2l+1}{4\pi}(-1)^l P_l(\cos\theta).\end{aligned}} \Longrightarrow \boxed{\begin{aligned}\delta^{(2)}(\mathbf{n}-\mathbf{n}_{\text{north}}) &= \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta),\\ \delta^{(2)}(\mathbf{n}-\mathbf{n}_{\text{south}}) &= \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)(-1)^l P_l(\cos\theta).\end{aligned}}$$

$$\boxed{\begin{aligned}{}^{(1)}\mathcal{T}_{ac} &= -\frac{1}{r^2}y_{ab}\sum_{l=0}^{\infty}\left\{-\frac{2l+1}{16\pi}\left[(+2\alpha_1 M + C_1) + (-2\alpha_1 M + C_1)(-1)^l\right]\right\}P_l(\cos\theta)\\ &= \frac{1}{4r^2}y_{ab}\left[(+2\alpha_1 M + C_1)\frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta) + (-2\alpha_1 M + C_1)\frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)(-1)^l P_l(\cos\theta)\right]\\ &= -\frac{1}{r^2}y_{ab}\left[\mu_n\delta^{(2)}(\mathbf{n}-\mathbf{n}_{\text{north}}) + \mu_s\delta^{(2)}(\mathbf{n}-\mathbf{n}_{\text{south}})\right], \quad \mu_n := -\frac{1}{4}(2\alpha_1 M + C_1), \quad \mu_s := -\frac{1}{4}(2\alpha_1 M - C_1).\end{aligned}}$$

- cf. H. Kodama, PTP **120** (2008), 371.

  Treatments of l=1 modes are essentially different from this work.

VI. Summary

- We proposed a gauge-invariant treatment of the l=0,1 mode perturbations on the Schwarzschild background spacetime.
  - Harmonics:

    (I)
    $$S_\delta = \begin{cases} Y_{lm} & \text{for } l \geq 2; \\ k_{(\hat{\Delta}+2)m} & \text{for } l = 1; \quad k_{(\hat{\Delta}+2)m} \in \mathscr{K}_{(\hat{\Delta}+2)} := \{f \in \mathscr{F}(S^2) | [\hat{\Delta}+2]f = 0\}. \\ k_{(\hat{\Delta})} & \text{for } l = 0. \quad k_{(\hat{\Delta})} \in \mathscr{K}_{(\hat{\Delta})} := \{f \in \mathscr{F}(S^2) | \hat{\Delta}f = 0\}. \end{cases}$$

    (III)
    $$k_{(\hat{\Delta})} = 1 + \delta\ln\left(\frac{1-z}{1+z}\right), \quad \delta \in \mathbb{R}, \quad z = \cos\theta$$
    $$k_{(\hat{\Delta}+2)m=0} = z\left\{1 + \delta\left(\frac{1}{2}\ln\frac{1+z}{1-z} - \frac{1}{z}\right)\right\},$$
    $$k_{(\hat{\Delta}+2)m=\pm1} = (1-z^2)^{1/2}\left\{1 + \delta\left(\frac{1}{2}\ln\frac{1+z}{1-z} + \frac{z}{1-z^2}\right)\right\}e^{\pm i\phi}.$$

  - Decomposition:

    (II)
    $$\begin{aligned} h_{AB} &= \sum_{l,m}\tilde{h}_{AB}S_\delta, \quad h_{Ap} = r\hat{D}_p\sum_{l,m}\tilde{h}_{(e1)A}S_\delta + r\varepsilon_{pq}\hat{D}^q\sum_{l,m}\tilde{h}_{(o1)A}S_\delta,\\ h_{pq} &= \frac{1}{2}\gamma_{pq}r^2\sum_{l,m}\tilde{h}_{(e0)}S_\delta + r^2\left(\hat{D}_p\hat{D}_q - \frac{1}{2}\gamma_{pq}\Delta\right)\sum_{l,m}\tilde{h}_{(e2)}S_\delta + r^2 2\varepsilon_{r(p}\hat{D}_{q)}\hat{D}^r\sum_{l,m}\tilde{h}_{(o2)}S_\delta,\end{aligned}$$

  - **Proposal :**

    We decompose perturbations h_{ab} on a spherically symmetric background spacetime through Eqs. (I), (II), and (III). Then, the decomposition (II) becomes invertible including $l = 0, 1$ modes. After deriving the field equations such as linearized Einstein equations by using the harmonic function S_δ, we choose $\delta = 0$ when we solve these field equations as regularity of solutions.
  - Following our proposal, we derive the l=0,1 mode perturbation solution to the linearized Einstein equation with a generic linearized energy-momentum tensor for the matter field.
  - **Our derived solutions realize the linearized LTB solution and linearized C-metric.**
  - **In these realizations, the terms of the Lie derivative of the background metric play essential roles.**

  ➡ **Our proposal is reasonable!!!**

- **Thus, we have resolved the zero-mode problem in the perturbations on the Schwarzschild background spacetime.**
- **Then, we can apply our general formulation of higher-order perturbation theory to this background spacetime and developed it to the any-order perturbations.**

Generalization of Riemannian Penrose inequality in weak gravity region

Area of Static BH

$$A \leq 4\pi(2Gm)^2$$

PTEP 2017 3, 033E03
PTEP 2021 8, 083E02
arXiv:2209.14124

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with

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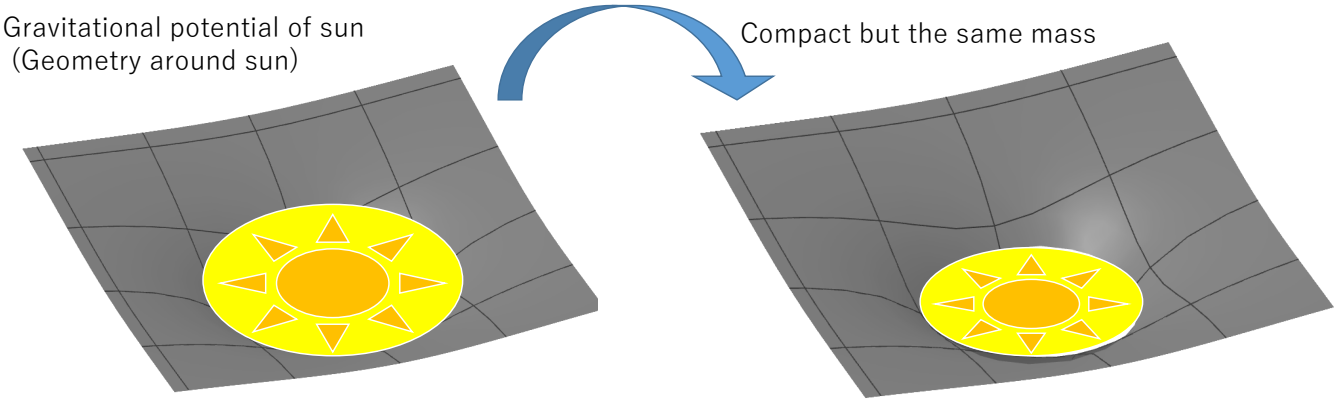
素粒子宇宙起源研究所

Plans of talk

- Basics of Black hole physics (Theoretical)
- Area inequality for horizon
- Motivation of generalization of inequality
- Area inequality for "Photon Sphere"
- Area inequality for a surface in weak gravity region
- Summary

Spherically symmetric static solutions

General Relativity:
gravitational force is described by **geometry**



Spherically symmetric static solutions

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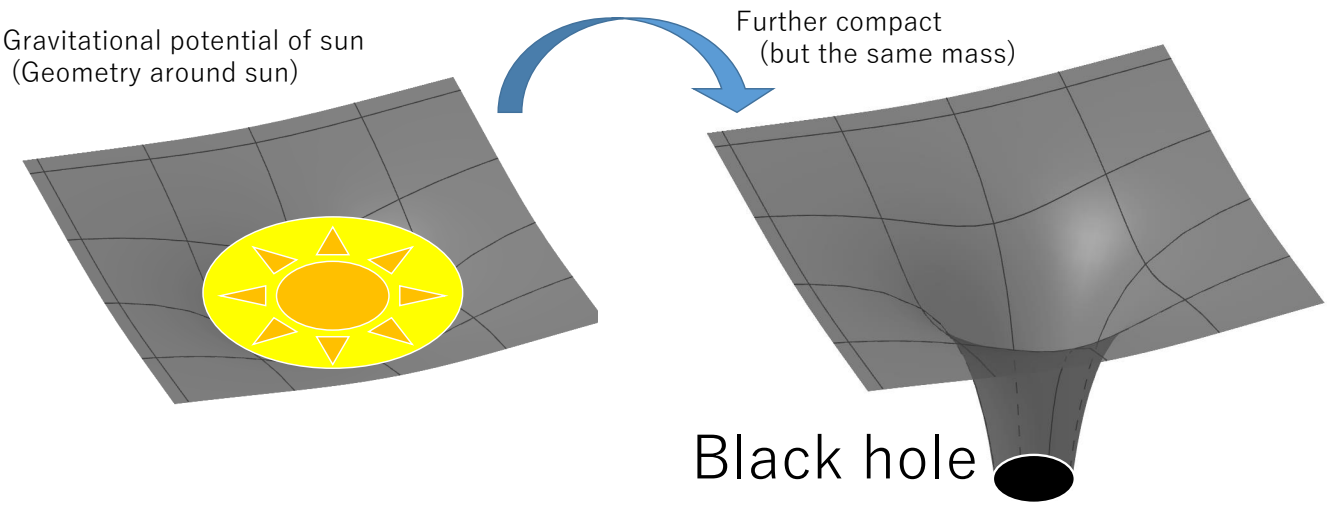


Schwarzschild solution

Spherically symmetric solution (Schwarzschild 1916)

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2$$

G : Newton constant

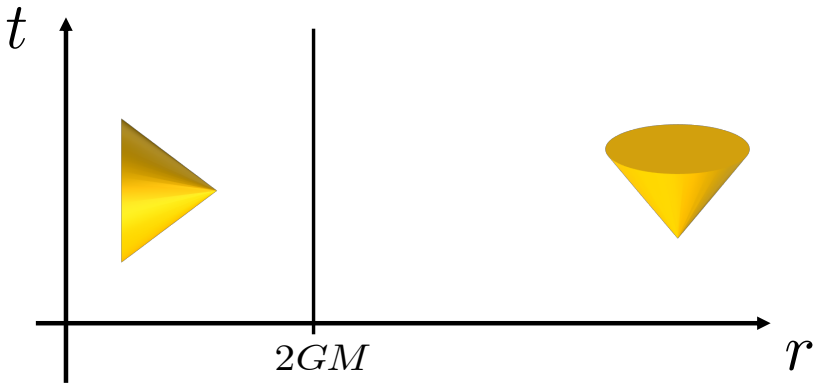
M : Parameter of solution (mass)

(Karl Schwarzschild, picture from Wikipedia)

For $1 - \frac{2GM}{r} = 0$

this metric is singular
but not physical singularity

but causal structure becomes



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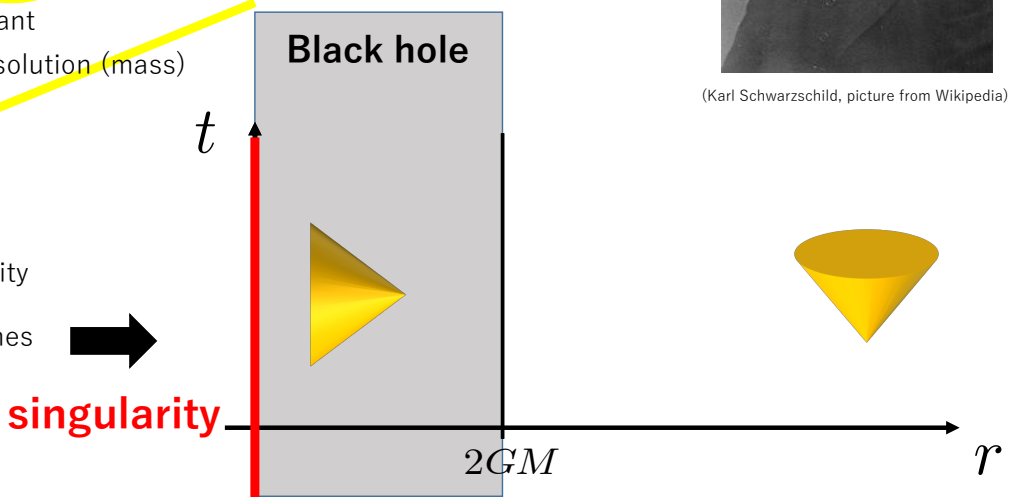


Singularity Theorem

In 1960's

Is a black hole a real object?

Does singularity occurs in physical process?

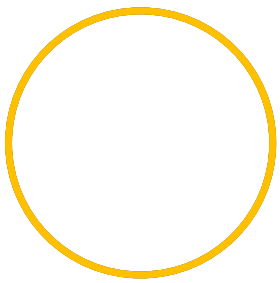
Maybe due to the symmetry??

Penrose: If trapped surface is formed, singularity exists in its future.

(Penrose, 1965)

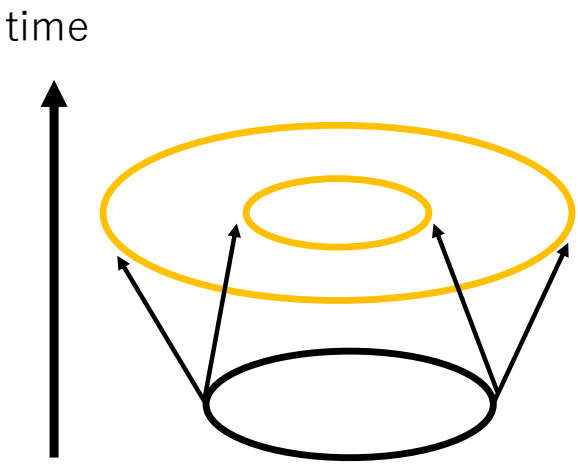
Strong gravity region

Trapped surface



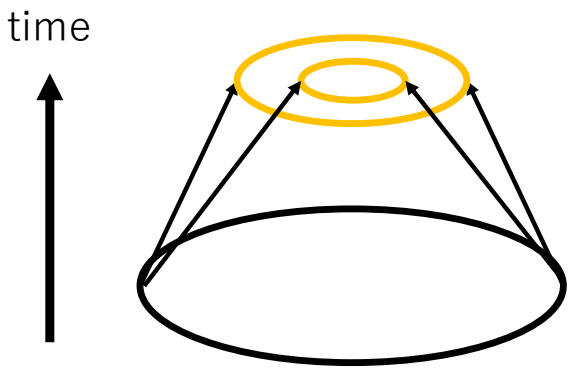
Wave fronts on flat space time

(weak gravity)

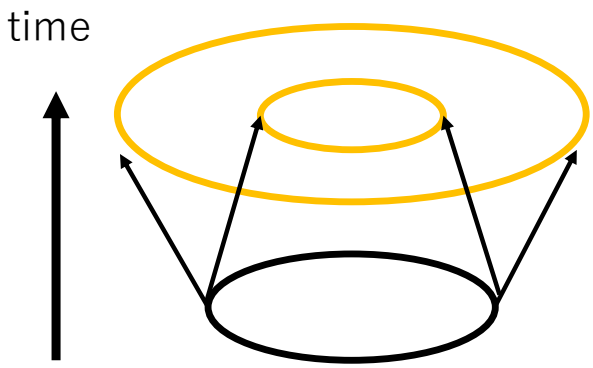


Trapped surface

Wave fronts in strong gravity



Wave fronts on flat space time
(weak gravity)



Area of outgoing photon wave front decreases

➡ Trapped region **Boundary** Trapped surface

Singularity Theorem

In 1960's

~~Is a black hole a real object?~~
~~Does singularity occurs in physical process?~~

Maybe due to the symmetry??

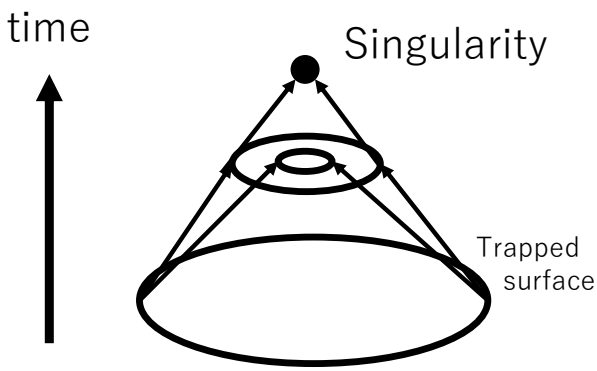
Penrose: If trapped surface is formed, singularity exists in its future.

(Penrose, 1965)

Strong gravity region

Outermost trapped surface
is called

Apparent horizon



Event horizon

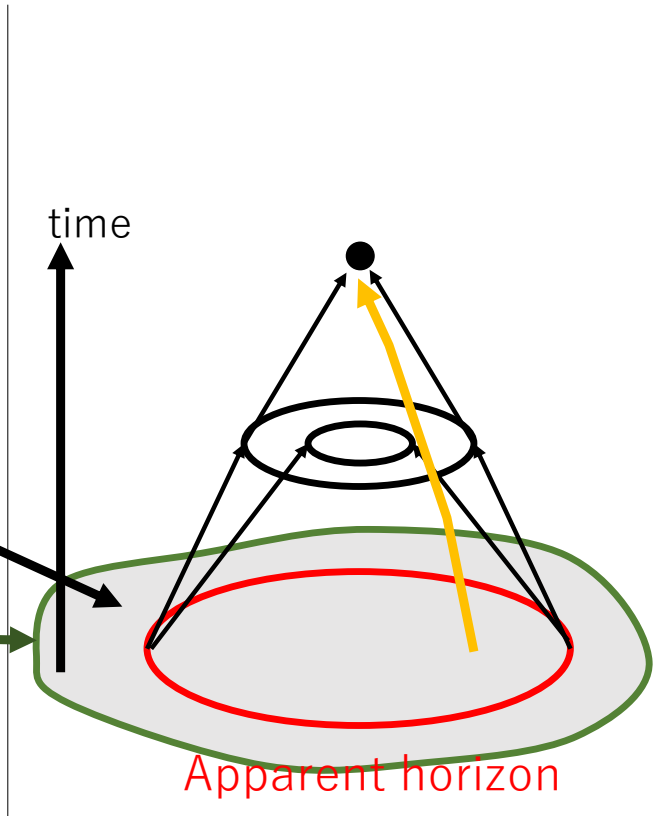
Any photons emitted in trapped region would not escape to infinity

A region from which no photon reaches to infinity

➡ **Black hole**

Boundary of black hole:

Event Horizon



Cosmic Censorship Conjecture

If the intuition of left figure is true

All singularities are hidden by **Event horizon**

This is called

Cosmic Censorship Conjecture (Penrose, 1969)

Any singularity created dynamically is hidden by Event horizon.

(no naked (no visible) singularity)

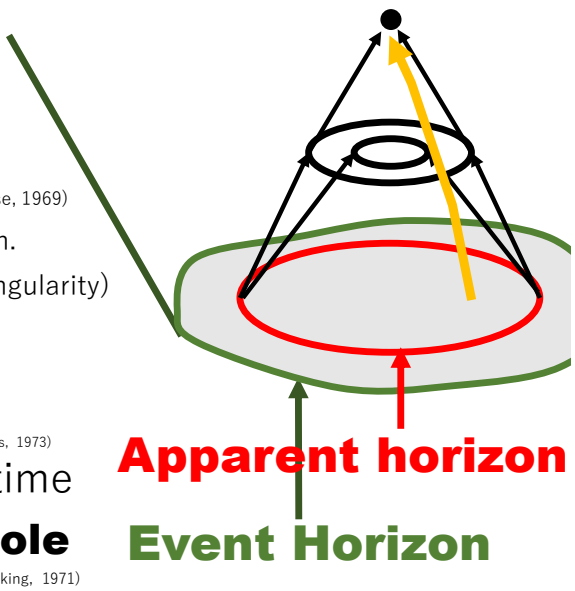
➡ ・Apparent horizon is hidden by Event horizon

(Hawking Ellis, 1973)

・Area of Event horizon increase with time

Area increasing law of Black hole

(Hawking, 1971)



BH thermodynamics

(Bekenstein, 1973)

$$T = \frac{\kappa}{2\pi}$$: Hawking temperature

(Hawking, 1974)

$$S = \frac{A}{4}$$: Bekenstein entropy

(Bekenstein, 1973)

A : Area of BH

M : Mass of BH

κ :surface gravity on event horizon

・1st law

$$dM = TdS + \Omega dJ + \Phi dQ$$

・2nd law

$$\frac{dS}{dt} \geq 0 \qquad \frac{dA}{dt} \geq 0$$

Plans of talk

- Basics of Black hole physics (Theoretical)
- Area inequality for horizon
- Motivation of generalization of inequality
- Area inequality for “Photon Sphere”
- Area inequality for a surface in weak gravity region
- Summary

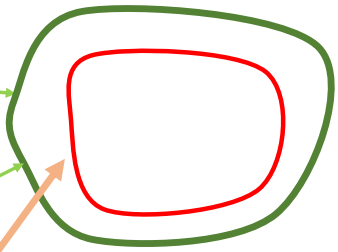
Penrose Inequality (conjecture) ⟺ Riemannian Penrose Inequality (Theorem)

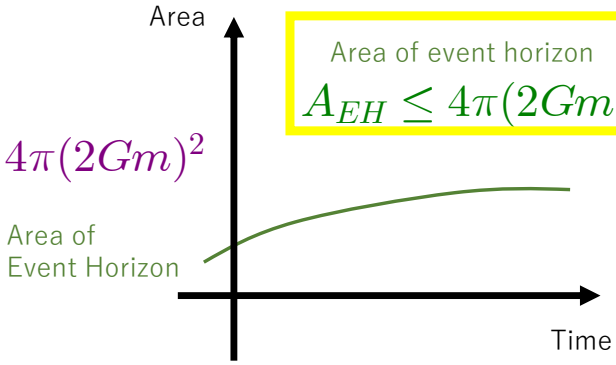
Under Cosmic Censorship Conjecture

Area increasing law of Event horizon (Hawking, 1971)

- Area of event horizon is non-decreasing function of time
- Schwarzschild or Kerr BH is final state.

Its area is equal to or less than $4\pi(2Gm)^2$ (Area of Sch. BH)





Area of event horizon

$$A_{EH} \leq 4\pi(2Gm)^2$$

Under Cosmic Censorship Conjecture

Apparent horizon is hidden by Event horizon.

It's area would be smaller than event horizon's. (Hawking Ellis, 1973)

$$A_{AH} \leq A_{EH} \leq 4\pi(2Gm)^2$$

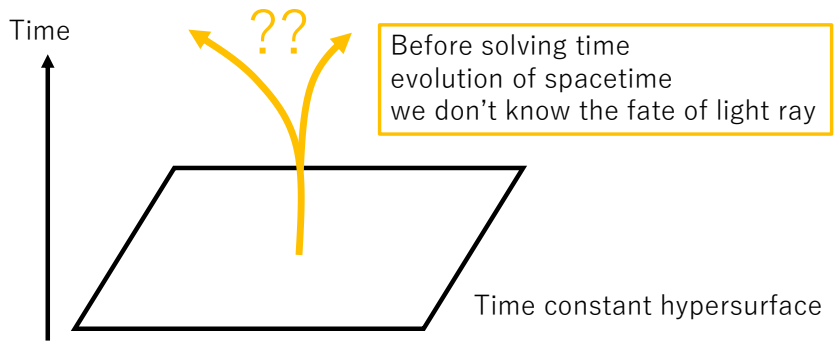
Penrose Inequality (conjecture) ⟺ Riemannian Penrose Inequality (Theorem)

Evidence for Cosmic Censorship Conjecture

- Why apparent horizon, not event horizon?

Event horizon (Black hole)

A region which is not causally related to future null infinity.



Apparent horizon (Trapped region)

A region where area of wave front of outgoing null is increasing

Fixed by metric and its derivative on time constant hypersurface

Don't need to solve time evolution

Penrose Inequality (conjecture) ⟺ Riemannian Penrose Inequality (Theorem)

Area inequality for apparent horizon

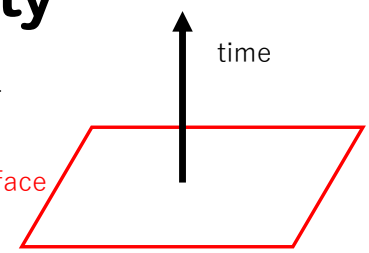
- Asymptotically flat spacetime with ADM mass m
- Dominant energy condition + General relativity

➡ Area of apparent horizon : $A_{AH} \leq 4\pi(2Gm)^2$

Penrose inequality would imply upper bound of Entropy.

But Penrose inequality is still a conjecture.

Riemannian Penrose Inequality



Penrose inequality (Lorentzian or pseudo-Riemannian) [conjecture]

- Asymptotically flat spacetime with ADM mass m
  space
- Dominant energy condition + GR

On time-symmetric surface

Non-negative Ricci scalar ${}^{(3)}R \geq 0$

➡ Area of apparent horizon $A \leq 4\pi(2Gm)^2$

outermost minimal surface

Riemannian Penrose inequality (Euclidian) [theorem]

Asymptotically flat space
Non-negative Ricci scalar ➡ Area of outermost minimal surface $A \leq 4\pi(2Gm)^2$

Jang & Wald (1977) under assumption of existence of inverse mean curvature flow (IMCF)
Huisken & Ilmanen (2001) resolving the singularity of IMCF
Bray (2001) using Conformal flow

Riemannian Penrose Inequality

Riemannian Penrose inequality (Euclidian) [theorem]

Asymptotically flat space
Non-negative Ricci scalar ➡ Area of outermost minimal surface $A \leq 4\pi(2Gm)^2$

This theorem is applicable to any asympt. flat solution in GR in following sense.

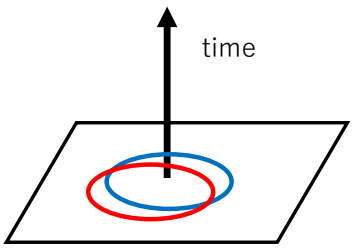
・Asymptotically flat spacetime with ADM mass m ➡ We can always take maximal slice as a time constant hypersurface

Dominant energy condition + GR

Non-negative Ricci scalar ${}^{(3)}R \geq 0$

➡ outermost minimal surface $A_{MS} \leq 4\pi(2Gm)^2$

~~Apparent horizon~~



Plans of talk

- Basics of Black hole physics (Theoretical)
- Area inequality for horizon
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- Summary

Observation VS Theory in BH physics

Observation

GW from BH collision

Direct observation of BH shadow

It's a good time
to explore BH physics!!

Theory

Uniqueness theorem

Riemannian Penrose Inequality

⋮

Solid but .....

Horizon is assumed,
which can never be observed

Theorem not relied on Horizon is required!!

BH evaporation

Event horizon (Black hole)

A region which is not causally related to future null infinity.

⬇

Time evolution problem

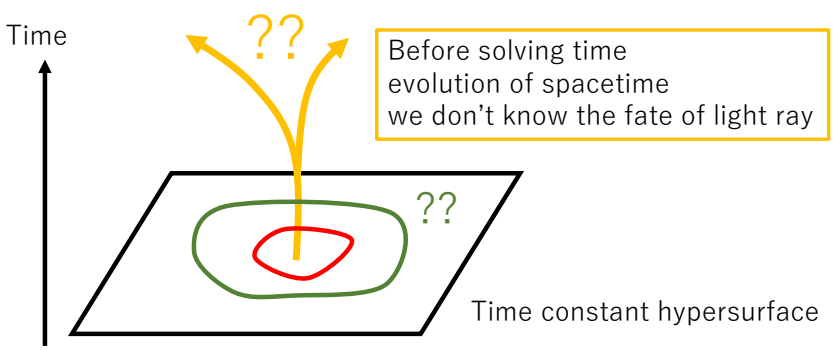
⬇

We don't know
geometry in future

⬇

Unable to know where is
event horizon

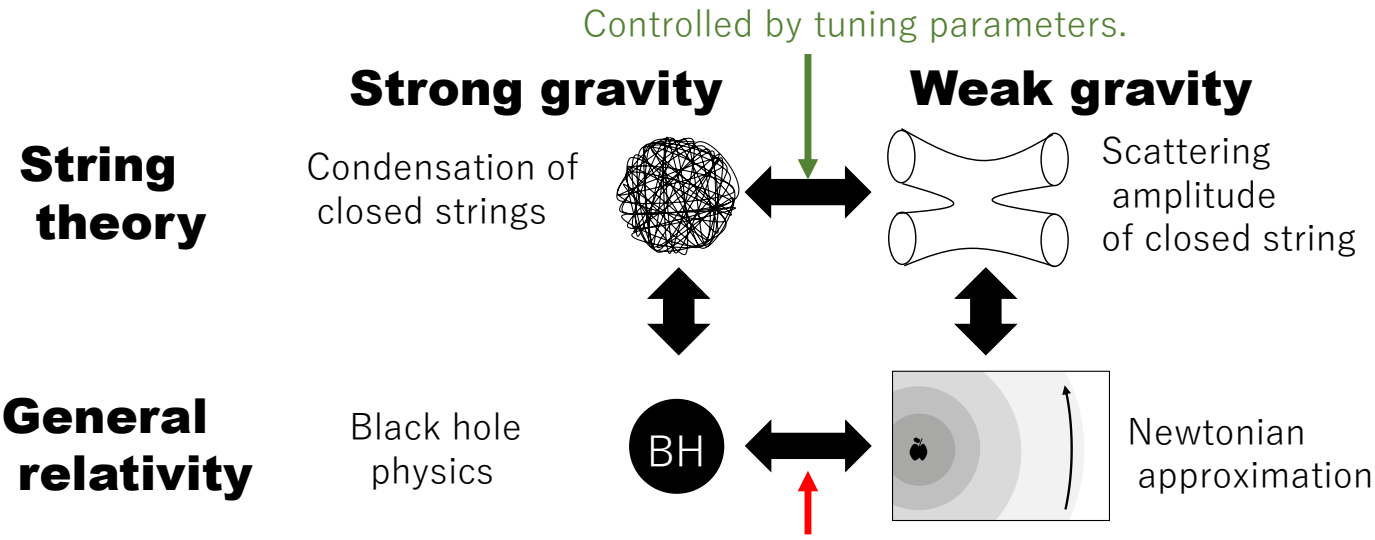
⬇



How about apparent horizon? Apparent horizon can exist far from event horizon.

Event horizon does not have any characteristic property in terms of t-const. surface.
If "event horizon" is important, we need a theorem based on general surfaces.

Form Strong gravity to weak gravity

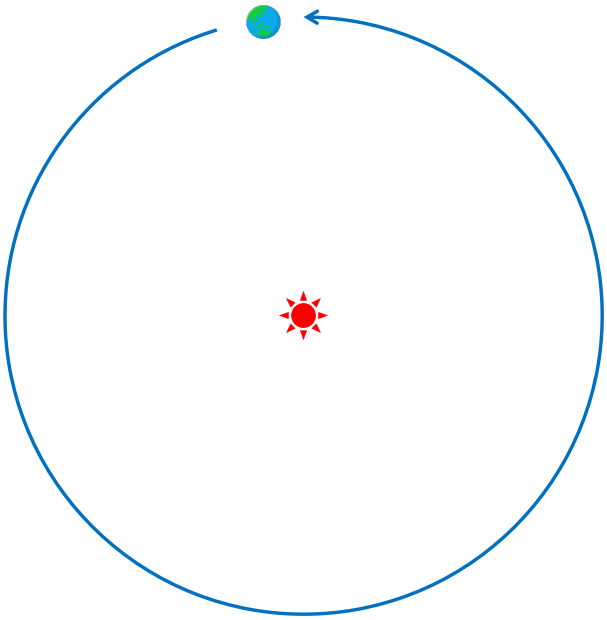


Try to understand this relation

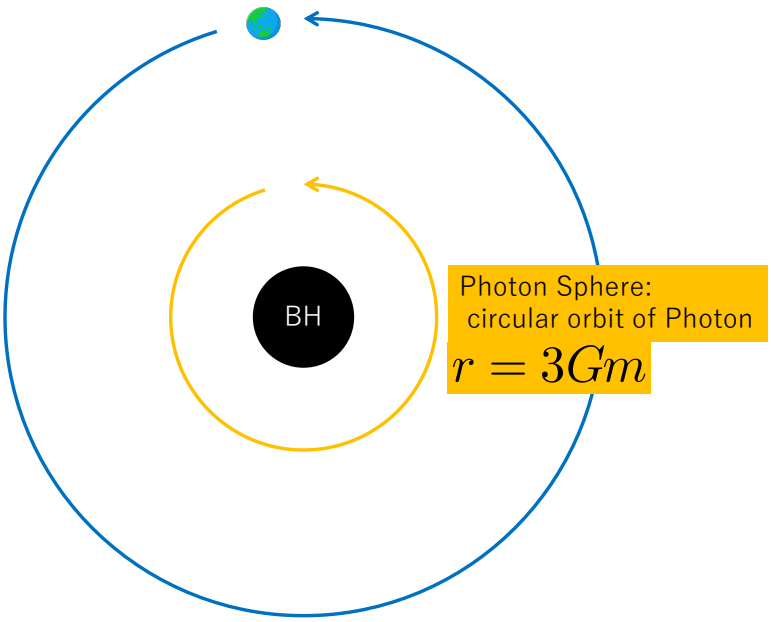
Plans of talk

- Basics of Black hole physics (Theoretical)
- Area inequality for horizon
- Motivation of generalization of inequality
- Area inequality for "Photon Sphere"
- Area inequality for a surface in weak gravity region
- Summary

Photon sphere as Strong-gravity region



Photon sphere as Strong-gravity region



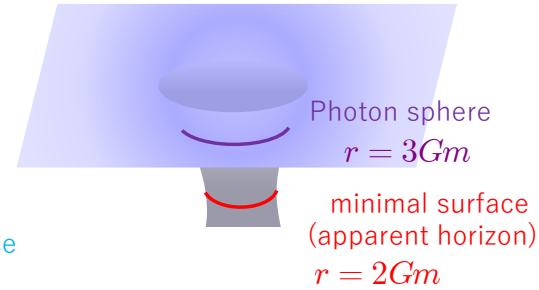
Area bound for a surface in a strong gravity region

T. Shiromizu, Y. Tomikawa, **K. I.**, H. Yoshino (2017)

Strong gravity region but outside of event horizon?

Learn from Sch BH.

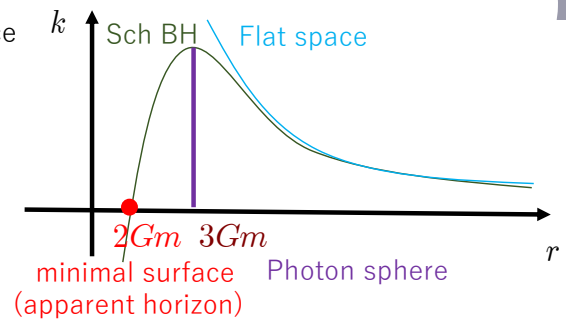
$$ds^2 = -f(r)dt^2 + \underline{f^{-1}(r)dr^2 + r^2 d\Omega_2^2} \qquad f(r) := 1 - \frac{2Gm}{r}$$



Take spatial (time-constant) slice

Take r=const. foliation on the spatial slice.

k :mean curvature on r= const.



Area bound for a surface in a strong gravity region

T. Shiromizu, Y. Tomikawa, **K. I.**, H. Yoshino (2017)

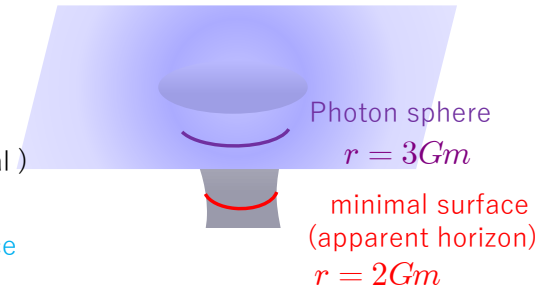
Strong gravity region but outside of event horizon?

generalization

Loosely trapped surface

$k > 0 \quad \& \quad n^a D_a k \geq 0$

(n^a :outward unit normal)

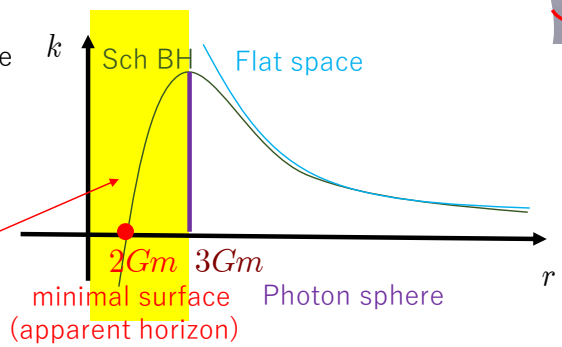


Take spatial (time-constant) slice

Take r=const. foliation on the spatial slice.

k :mean curvature on r= const.

$$\frac{dk}{dr} \geq 0$$



Area bound for Loosely trapped surface

loosely trapped surface(LTS):

Shiromizu, Tomikawa, **K.I.**, Yoshino (2017)

$$k > 0, \quad r^a D_a k > 0$$

$$A_{LTS} \leq 4\pi(3Gm)^2$$

Riemannian Penrose Inequality

minimal surface (MS):

$$k = 0$$

$$A_{MS} \leq 4\pi(2Gm)^2$$

Asymptotically flat
3-dim space with ${}^{(3)}R \geq 0$

minimal surface
(apparent horizon)

Plans of talk

- Basics of Black hole physics (Theoretical)
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- Summary

Area bound for a surface in a waek gravity region

K. I., Y. Tomikawa, T. Shiromizu, H. Yoshino (2021)

Strong gravity region but outside of event horizon?

Learn from Sch BH.

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Take spatial (time-constant) slice

Take r=const. foliation on the spatial slice.

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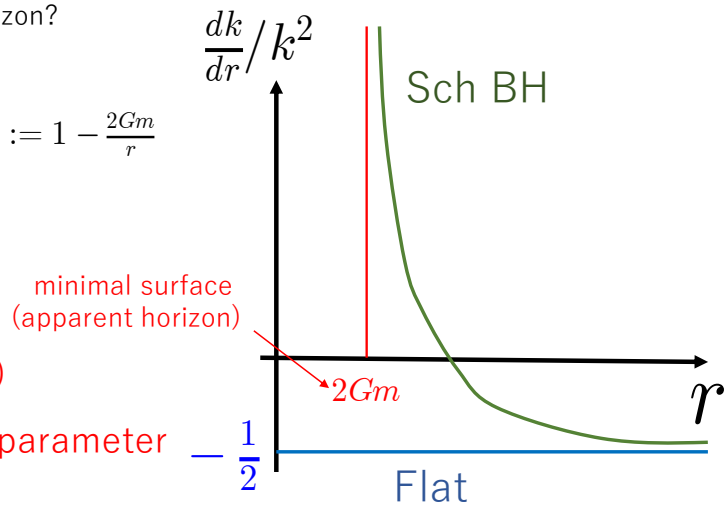


Attractive gravity probe surface (AGPS)

$$(r^a D_a k)/k^2 > \alpha \qquad \left(\begin{array}{l}\alpha \text{ : constant parameter} \\ \alpha > -\frac{1}{2}\end{array}\right.$$

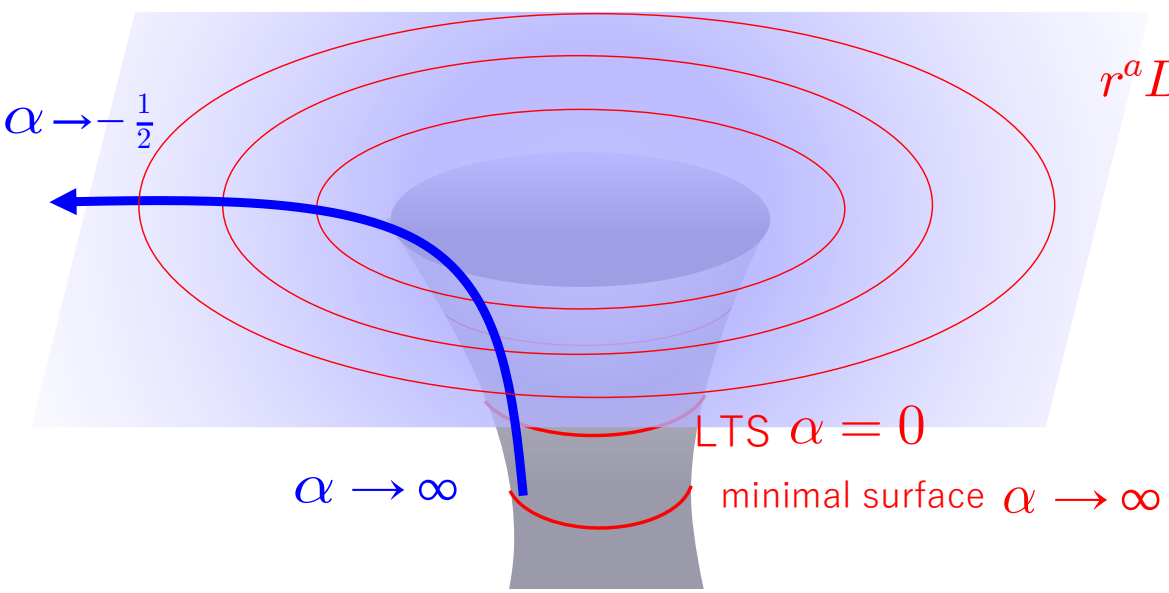
$$k > 0$$

Applicability in weak gravity region

$(\alpha > -\frac{1}{2})$

$r^a D_a k > \alpha k^2$

$k > 0$



Further generalization

K.I., Tomikawa, Shiromizu, Yoshino (2021)

minimal surface (MS):

$$k = 0$$

$\alpha \to \infty$

loosely trapped surface(LTS):

Shiromizu, Tomikawa, K.I., Yoshino (2017)

$$k > 0, \quad r^a D_a k > 0$$

$\alpha = 0$

Spatial infinity $(r \to \infty)$

$\alpha \to -\frac{1}{2}$

Generalise them with a non-dimensional parameter α

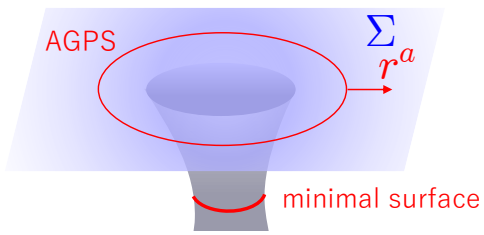
$$r^a D_a k > \alpha k^2 \qquad k > 0$$

Our theorem

K.I., Tomikawa, Shiromizu, Yoshino (2021)

3-dim space: Σ

- Asymptotically flat with ADM mass m
- $R \geq 0$ (R : 3-dim Ricci scalar)



Riemanninan Penrose Inequality

Area of minimal surface:

$$k = 0$$

(k :Tr. of extrinsic curvature)

$$A_{MS} \leq 4\pi(2Gm)^2$$

$\alpha \to \infty$

Area of **attractive gravity probe surface** (AGPS):

$$k > 0, \quad r^a D_a k > \alpha k^2 \quad (\alpha > -\tfrac{1}{2})$$

$$A_{AGPS} \leq 4\pi\left(\frac{3+4\alpha}{1+2\alpha}Gm\right)^2$$

Generalization of RPI

Our theorem

K.I., Tomikawa, Shiromizu, Yoshino (2021)

3-dim space: Σ

- Asymptotically flat with ADM mass m
- $R \geq 0$ (R : 3-dim Ricci scalar)

Riemanninan Penrose Inequality

Area of minimal surface:

$$k = 0$$

(k :Tr. of extrinsic curvature)

$$A_{MS} \leq 4\pi(2Gm)^2$$

$\alpha \longrightarrow \infty$

Area of **loosely trapped surface**(LTS):

Shiromizu, Tomikawa, K.I., Yoshino (2017)

$$k > 0, \quad r^a D_a k > 0$$

$$A_{LTS} \leq 4\pi(3Gm)^2$$

$\alpha = 0$

Area of **attractive gravity probe surface** (AGPS):

$$k > 0, \quad r^a D_a k > \alpha k^2 \quad (\alpha > -\tfrac{1}{2})$$

$$A_{AGPS} \leq 4\pi\left(\frac{3+4\alpha}{1+2\alpha}Gm\right)^2$$

Generalization of RPI

Recent progress

(K.I., Tomikawa, Shiromizu, Yoshino, arXiv:2209.14124)

- Higher dimension $3 \leq d < 8$
- refine conditions

attractive gravity probe surface (AGPS):

$$k > 0, \quad r^a D_a k > \alpha k^2 \quad (\alpha > -\tfrac{1}{d-1})$$

$\cap$

Refined AGPS

$$k > 0, \quad {}^{(d-1)}R + 2k\mathcal{D}^2 k^{-1} \geq \left(2\alpha + \frac{d}{d-1}\right)k^2 \quad (\alpha > -\tfrac{1}{d-1})$$

$$\Rightarrow \quad A \leq \omega_{d-1}\left[\frac{d+2(d-1)\alpha}{1+(d-1)\alpha}Gm\right]^{\frac{d-1}{d-2}}$$

ω_{d-1} :Area of (d-1)-dim unit sphere

Summary

- Theorem of BH is required to be extended without assumption of horizon.
- Properties of surface are important in gravitational physics.

We generalise Riemannian Penrose inequality applicable in weak gravity region.

- Theorem is also mathematically interesting.

Riemannian Penrose inequality

Area of minimal surface:

$$k = 0$$

(k :Tr. of extrinsic curvature)

$$A_{MS} \leq 4\pi(2Gm)^2$$

$\alpha \longrightarrow \infty$

Our theorem

Area of **attractive gravity probe surface** (AGPS):

$$k > 0, \quad r^a D_a k > \alpha k^2 \quad \left(\alpha > -\tfrac{1}{2}\right)$$

$$A_{AGPS} \leq 4\pi\left(\frac{3+4\alpha}{1+2\alpha}Gm\right)^2$$

Generalization of RPI

Other plans

Higher dimension

done (appear on arXiv yesterday!) (**K.I.**, Tomikawa, Shiromizu, Yoshino, arXiv:2209.14124)

Including electric charge and angular momentum

Partially done (Lee, Shiromizu, **K.I.** Phys.Rev.D 105 (2022) 4, 044037),
(Lee, Shiromizu, **K.I.**, Yoshino, Tomikawa Phys.Rev.D 106 (2022) 6, 064028)

AdS/CFT

(Fischetti, Wiseman, Class.Quant.Grav. 34 (2017) 12, 125005)

Meaning in Newtonian limit

Work in progress

P03

P03 The Einstein-Vlasov system with an R×SU(2)×U(1) symmetry

○ Hiroki Asami, Chulmoon Yoo, Ryo Kitaku and Keiya Uemichi
QG lab, Nagoya University

ABSTRACT

The Einstein-Vlasov (EV) system is a collisionless many-particle system in general relativity. For the static and spherically symmetric case, the existence and stability of the solutions are well-known. However, for the stationary rotating case, these properties have not been fully revealed due to the less symmetry. In this talk, we investigate the rotating EV system by considering the five-dimensional spacetime with a R×SU(2)×U(1) symmetry.

Introduction

What is the EV system?

Distribution function determines the system

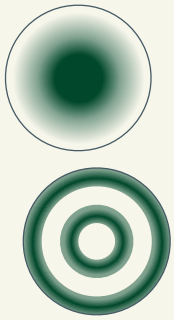
$f(\boldsymbol{x}, \boldsymbol{p})$

$f = e^{-\beta\varepsilon}$

$f = \left(1 - \frac{\varepsilon}{\varepsilon_0}\right)^k \left(\frac{J}{J_0} - 1\right)^l$

$T_{\mu\nu} = \int dV_p\, p_\mu p_\nu f$

$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$

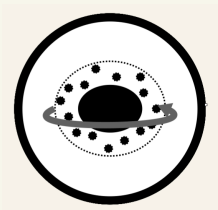


Rotating EV system

Less symmetry
Hard to solve (PDE in general)

Can we keep the high symmetry with a finite angular momentum?

Self-gravitating systems in AdS



- **Asymptotic structure of AdS**
  Particles are confined by the potential
  → Thermal equilibrium states can exist
- **Macroscopic model of matter in AdS**
  Turbulent instability may causes collapses
  → What is the conditions for the collapse?
  → General final state?

Set-up

Metric Ansatz

Free functions $\mu(r),\ \nu(r),\ a(r),\ h(r),$

$$g = -e^{2\mu(r)}\,dt^2 + e^{2\nu(r)}\,dr^2 + \frac{r^2}{4}\left[(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\right] + h(r)\left(dt - \frac{a(r)}{2}\sigma^3\right)^2$$

Killing vectors:

$$\begin{cases} \xi_1 = -\sin\psi\partial_\theta + \csc\theta\cos\psi\partial_\phi - \cot\theta\cos\psi\partial_\psi, \\ \xi_2 = -\cos\psi\partial_\theta - \csc\theta\sin\psi\partial_\phi + \cot\theta\sin\psi\partial_\psi, \\ \xi_3 = \partial_\psi. \\ \sigma_3 = \partial_\phi, \\ \eta = \partial_t, \end{cases}$$

$SU(2)_\sigma$ breaks into $U(1)_\sigma$

→ $R_t \times SU(2)_\xi \times U(1)_\sigma$

Distribution Function Ansatz

$$f(\varepsilon, j_\sigma) = \exp\left[\alpha - \beta(\varepsilon - \Omega j_\sigma)\right].$$

It realizes the thermal equilibrium states for $\Omega = 0$

$\varepsilon = -\boldsymbol{p}\cdot\boldsymbol{\eta}$ Energy of the particle
$j_\sigma = \boldsymbol{p}\cdot\boldsymbol{\sigma}_3$ Angular momentum

Conserved quantities

Vlasov equation

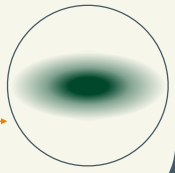
→ $p^\mu \nabla_\mu f = 0$ is automatically satisfied

Einstein equations

$$G_{\mu\nu} - \frac{6}{L^2} g_{\mu\nu} = 8\pi T_{\mu\nu}$$

Numerically solve

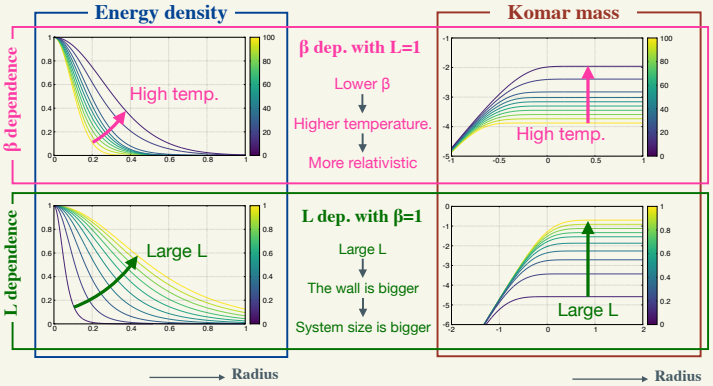
The system contains three parameters: (β, Ω, L)



Results

Confinement by the AdS potential

Sph. sym. cases: $\Omega = 0$
Normalization: $8\pi\rho(0) = 1$



Frame dragging and squashing functions

Co-rotating frame at infinity

$s(r)$ Squashing function $\omega(r)$ Dragging function

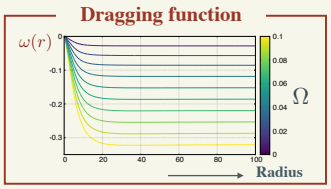
$$g = -\left(e^{2\mu} - \frac{r^2 h}{r^2 + a^2 h}\right)dt^2 + e^{2\nu}\,dr^2 + \frac{r^2}{4}\left[(\sigma^1)^2 + (\sigma^2)^2 + \boxed{\left(1 + \frac{a^2 h}{r^2}\right)}\left(\sigma^3 - \boxed{\frac{2ah}{r^2 + a^2 h}}\,dt\right)^2\right]$$

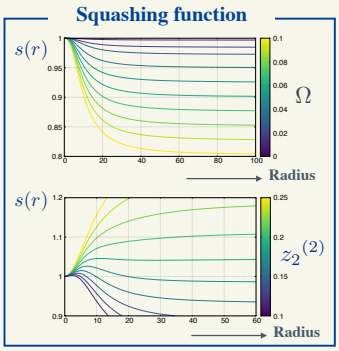
Vlasov particles squash S^3

→ In general, $s(r) \neq 1$ at infinity

→ Asymp. AdS sol. are realized by tuning

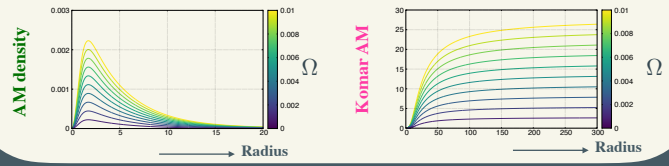
$$z_2^{(2)} := \frac{d^2}{dr^2}\left(\frac{a^2 h}{r^2}\right)\Big|_{r=0} \longrightarrow s(r) \to 1$$





Angular momentum (AM)

Ω dependence of density and Komar integral



Summary

- Rotating Einstein-Vlasov system with $R_t \times SU(2)_\xi \times U(1)_\sigma$
- Confinement by the AdS potential
- Solutions with a finite angular momentum
- Squashing function becomes non-trivial due to the Vlasov matter

Gravitational Wave from Axion-SU(2) Gauge Fields in Kinetically Driven Inflation

Yuki Watanabe（NIT, Gunma College）

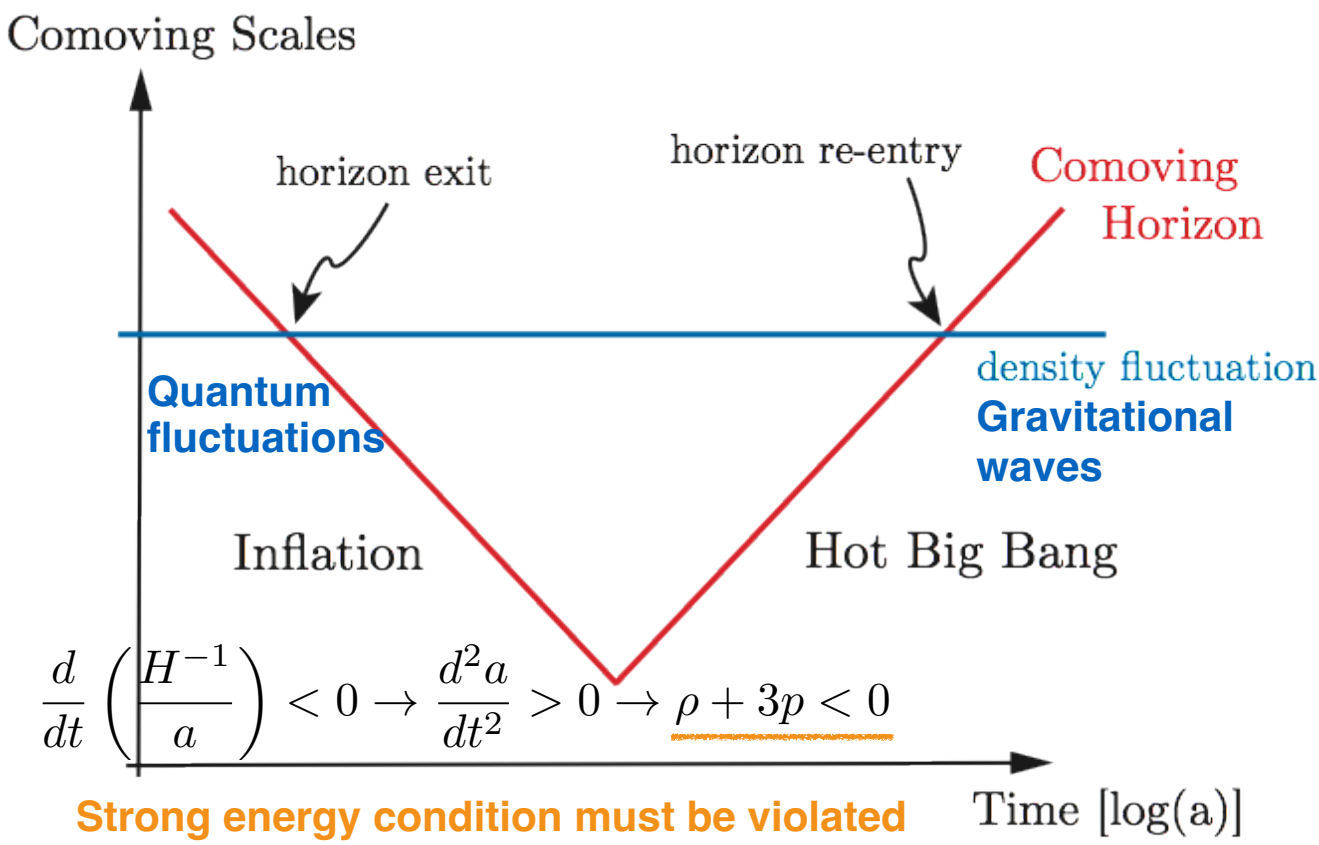
Collaborator：E. Komatsu (MPA, Kavli IPMU)

Based mainly on 2004.04350 [hep-th]

The 13th RESCEU International Symposium
JGRG31 Workshop

Koshiba Hall, The University of Tokyo, Tokyo (Hybrid style)
Oct. 24-28, 2022

Inflation: Quantum fluctuations => Seeds



Natural inflation: Axion as Inflaton

In natural inflation, the field ϕ is a pseudo-Nambu-Goldstone Boson with decay constant f and periodicity $2\pi f$ [Freese et al 1990].

$$\mathcal{L} = \sqrt{-g}\left[\frac{M_p^2}{2}R - \frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - me^{i\frac{\phi}{f}}\bar{\psi}(1+\gamma_5)\psi - \bar{\psi}\not{\mathcal{D}}\psi - \frac{1}{2}\mathrm{Tr}F_{\alpha\beta}F^{\alpha\beta}\right],$$

- where ψ is a fermion charged under the (non-abelian) gauge field with field strength $F_{\alpha\beta}$, $\not{\mathcal{D}} = \gamma^\alpha\mathcal{D}_\alpha$ is the gauge invariant derivative and $m \sim f$ is the fermion mass scale after spontaneous symmetry breaking.
- The action is invariant under the chiral (global) symmetry $\psi \to e^{i\gamma_5\alpha/2}\psi$, where α is a constant.
- This symmetry is related to the invariance under shift symmetry of ϕ, i.e. $\phi \to \phi - \alpha\, f$.

Natural inflation: Axion as Inflaton

- Suppose the chiral symmetry is broken at energies $f >$TeV (like in the QCD axion case) [chiral anomaly induces $(\phi/f)F\tilde{F}$]
- a potential of pNGB ϕ is produced by the instanton effect:

$$V(\phi) \sim \Lambda^4\left[1 \pm \cos\frac{\phi}{f}\right]$$

  which is protected from QG UV corrections by the restoration of global shift symmetry $\phi \to \phi + c$ as $\Lambda/M_p \to 0$
- If $\Lambda \sim 10^{16}$ GeV (GUT scale), inflation is produced with

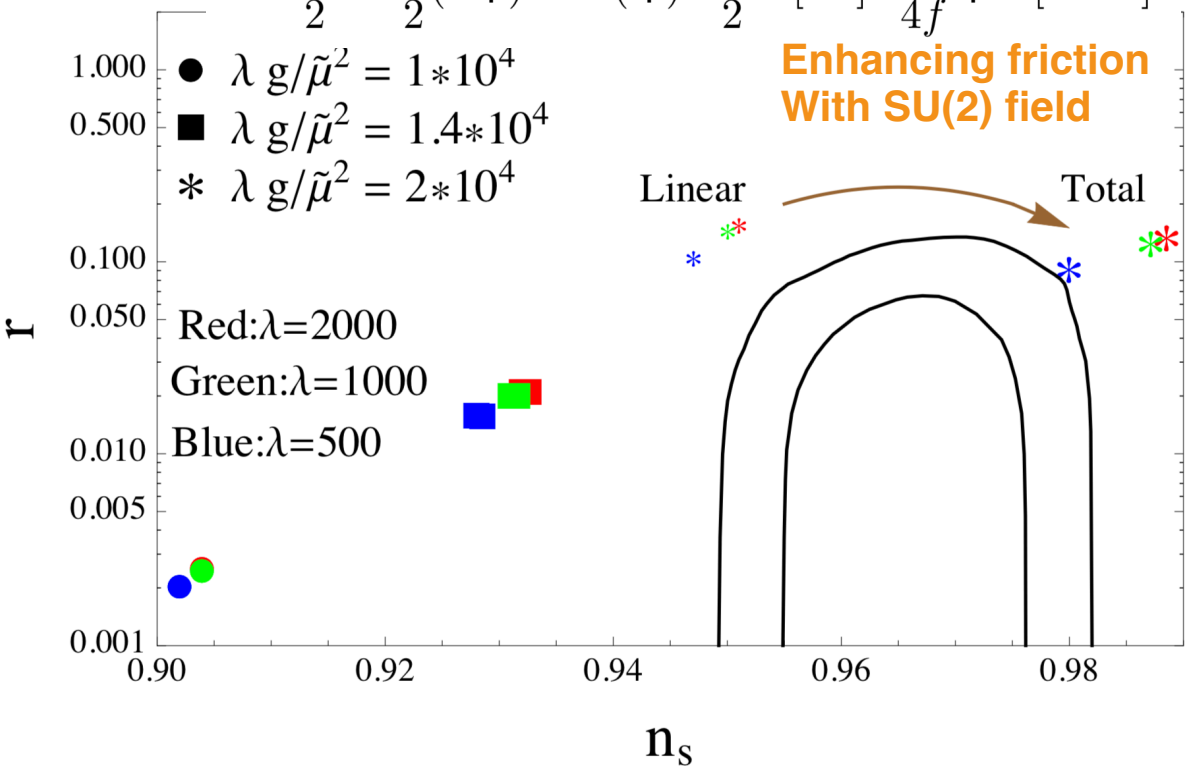
$$\boxed{n_s - 1\,(\propto \epsilon) \simeq -\frac{M_p^2}{8\pi f^2}}$$

- so $n_s - 1 \simeq -0.04 \to f > M_p$
  $\Rightarrow$ The potential may not be protected from QG UV corrections.

Chromo-Natural Inflaton

[Maleknejad, Sheikh-Jabbari 2011; Adshead, Wyman 2012; Papageorgiou, Peloso, Unal 2018]

$$\mathcal{L} = \frac{1}{2}R - \frac{1}{2}(\partial\Phi)^2 - V(\Phi) - \frac{1}{2}\mathrm{Tr}[F^2] - \frac{\lambda}{4f}\Phi\mathrm{Tr}[F\wedge F].$$



EFT of Axion below the PQ scale *f_a*

[Georgi, Kaplan, Randall 1986]

- Assuming $H < f$, $H < M$, the most general Lagrangian for the axion ϕ is (up to higher derivative terms):

$$\mathcal{L}_a = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) - \frac{\lambda}{2f_a}\phi\mathrm{Tr}(F^{\mu\nu}\tilde{F}_{\mu\nu})$$

+ deriv. coupled SM + *"desert"* or BSM

- Invariant under $\phi \to \phi + const.$

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[Georgi, Kaplan, Randall 1986]

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$$X = -\frac{1}{2}(\partial\phi)^2$$

$$\mathcal{L}_a = c_1 X + \frac{c_2}{M^4}X^2 + \frac{c_3}{M^3}X\Box\phi + \frac{c_4}{M^2}G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$
$$-\frac{1}{2}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) - \frac{\lambda}{2f_a}\phi\mathrm{Tr}(F^{\mu\nu}\tilde{F}_{\mu\nu})$$

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- Field eqs. contain derivatives only up to 2nd order, i.e. no Ostrogradski ghost. [Deffayet et al 2011; Kobayashi et al 2011]

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k-essense

Galileon

Einstein tensor

$$\mathcal{L}_a = c_1 X + \frac{c_2}{M^4}X^2 + \frac{c_3}{M^3}X\Box\phi + \frac{c_4}{M^2}G^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$$
$$-\frac{1}{2}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) - \frac{\lambda}{2f_a}\phi\mathrm{Tr}(F^{\mu\nu}\tilde{F}_{\mu\nu})$$

SU(2) Gauge fields

+ deriv. coupled SM + *"desert"* or BSM

- Invariant under $\phi \to \phi + const.$
- Field eqs. contain derivatives only up to 2nd order, i.e. no Ostrogradski ghost. [Deffayet et al 2011; Kobayashi et al 2011]

Kinetically driven Axion Inflation

- In an FLRW background, EoM for axion ϕ & SU(2) gauge field A are given by [Maleknejad, Sheikh-Jabbari 2011]

$$A_0^a = 0, \quad A_i^a = \delta_i^a a(t) Q(t)$$

Gauge field

$$\ddot{Q} + 3H\dot{Q} + (\dot{H} + 2H^2)Q + 2g^2 Q^3 = \frac{g\lambda}{f}\dot{\phi}Q^2$$

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Axion

$$\dot{J} + 3HJ = -\frac{3g\lambda}{f}Q^2(\dot{Q} + HQ)$$

$$\phi = \phi(t) \qquad J \equiv c_1\dot{\phi} + \frac{c_2}{M^4}\dot{\phi}^3 - 3\frac{c_3 H}{M^3}\dot{\phi}^2 + 6\frac{c_4 H^2}{M^2}\dot{\phi}$$

- In the absence of non-trivial gauge VEV, an attractor solution:

J ~ a^(-3) → 0 with

H = const,

$$\dot{\phi} = const \sim M^2 \qquad (c_1 < 0, c_2 > 0)$$

$$\rho + 3p \sim -M^4$$

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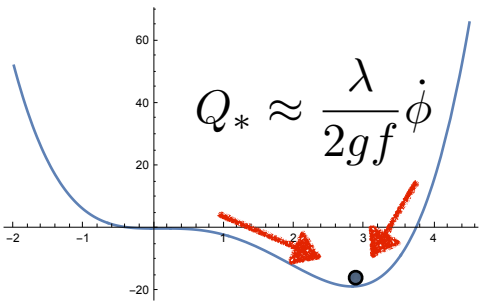
Gauge field

$$\ddot{Q} + 3H\dot{Q} + (\cancel{\dot{H}} + 2H^2)Q + 2g^2Q^3 = \frac{g\lambda}{f}\dot{\phi}Q^2$$

$$V_{\rm eff}(Q) = H^2Q^2 + \frac{1}{2}g^2Q^4 - \frac{g\lambda}{3f}\dot{\phi}Q^3$$

$$\boxed{J_* = -\frac{g\lambda}{f}Q_*^3}$$



$$Q_* \approx \frac{\lambda}{2gf}\dot{\phi}$$

Condition:

$$\left|\frac{\lambda}{f}\dot{\phi}\right| > 4H$$

Scale dependence? How to end inflation?

- Invoke a shift symmetry breaking term in potential as in tilted ghost inflation [Arkani-Hamed et al 2004; Senatore 2005]

- Invoke a shift symmetry breaking term in the kinetic sector as in k-Inflation, Galileon Inflation [Armendaritz-Picon et al 1999; Kobayashi et al 2010; Burrage et al 2010]

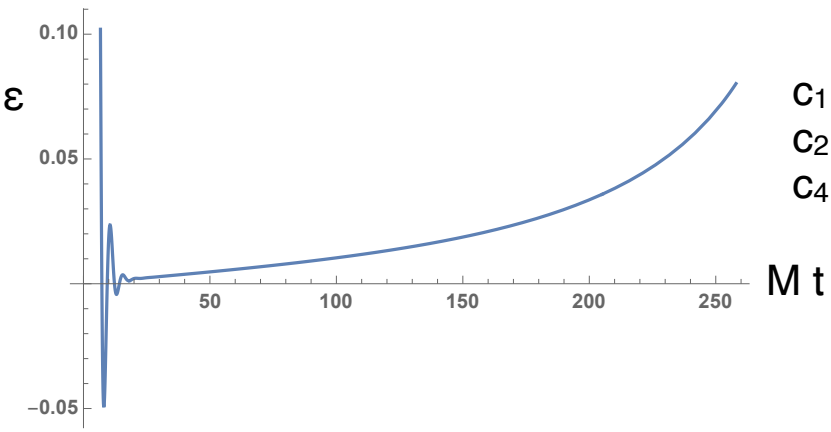
- Back-reaction from particle production may become important. [Anber, Sorbo 2009; Barnaby, Peloso 2011; Maleknejad, Komatsu 2018; Domcke, Ema, Mukaida, Sato 2018; Domcke, Sander 2019; Lozanov, Maleknejad, Komatsu 2018; Mirzagholi, Maleknejad, Lozanov 2019]

- Invoke a shift symmetry breaking term in potential as in tilted ghost inflation [Arkani-Hamed et al 2004; Senatore 2005]

$$J \simeq -\frac{g\lambda}{f}Q^3 - \frac{V_{,\phi}}{3H} + \frac{C}{a^3}$$

$$\left(1 - \frac{2a_4 X}{M_{\rm p}^2} - \frac{Q^2}{2M_{\rm p}^2}\right)\epsilon_* = -\frac{\dot{\phi}V_{,\phi}}{6H^3M_{\rm p}^2} + \frac{a_3 X\ddot{\phi}}{H^2M_{\rm p}^2} - \frac{2a_4\dot{X}}{HM_{\rm p}^2} + \frac{\dot{Q}^2}{H^2M_{\rm p}^2} + \frac{2Q\dot{Q}}{HM_{\rm p}^2}$$

- Invoke a shift symmetry breaking term in the kinetic sector as in k-Inflation, Galileon Inflation [Armendaritz-Picon et al 1999; Kobayashi et al 2010; Burrage et al 2010]



$c_1 \sim -(1-10^{-5}\phi^2/M^2)$
$c_2 = 1/2$, $c_3 = 0$
$c_4 = 0$

Cosmological perturbations

- With SU(2) gauge field, one chirality of **tensor modes** grows large due to **tachyonic instability**. [Adshead et al 2012; Dimastrogiovanni, Peloso 2012; Dimastrogiovanni, Fasiello, Fujita 2016; Domcke, Sander 2019]

$$\left[\frac{\partial^2}{\partial\tau^2} + k^2 + \frac{2k\xi}{\tau} + 2\left(\frac{\xi}{\tau^2} + \frac{k}{\tau}\right)\frac{gQ}{H}\right] t_R = \mathcal{O}(h_R)$$

$$\xi \equiv \frac{\lambda\phi}{2fH}$$

- **Non-Gaussian signals** become interesting! [Agrawal, Fujita, Komatsu 2018]

- Instabilities in scalar modes? If *Q < 2H/g*, there is an instability in scalar modes with canonical kinetic term.

- Higher derivative terms induce ghost or gradient instabilities? They can be avoided by choosing parameters, *c_i*.

Cosmological perturbations

- Scalar perturbations in the unitary gauge (δϕ = 0 at all orders) are given by (after using Gauss, Hamiltonian, momentum constraints and assuming heavy mass of SU(2) field)

$$S_{\zeta^2} = \int d^3x dt\, a^3 \mathcal{G}_{\rm s}\left[\dot{\zeta}^2 - \frac{c_{\rm s}^2}{a^2}(\partial_i\zeta)^2\right] ,$$

$$\mathcal{G}_{\rm s} = \frac{\dot{\phi}}{2H^2}\left[J - 6a_3HX + 4a_2X\dot{\phi}\left(1 - \frac{2a_3X\dot{\phi}}{HM_{\rm p}^2} + \frac{8a_4X}{M_{\rm p}^2}\right)\right]$$

$$c_{\rm s}^2 = \frac{M_{\rm p}^2\epsilon_{\rm s}}{\mathcal{G}_{\rm s}} , \quad \epsilon_{\rm s} = \epsilon - \frac{a_3X\dot{\phi}}{HM_{\rm p}^2} + \frac{\xi^2Q^2}{M_{\rm p}^2} , \quad \xi \equiv \frac{\lambda\dot{\phi}}{2fH} ,$$

$$\mathcal{P}_\zeta \equiv \frac{k^3}{2\pi^2}|\zeta_{\rm k}|^2 \simeq \frac{H^2}{8\pi^2\mathcal{G}_{\rm s}c_{\rm s}^3} \qquad n_{\rm s} - 1 \equiv \left.\frac{d\ln\mathcal{P}_\zeta}{d\ln k}\right|_{c_{\rm s}k=aH} \simeq -2\epsilon - g_{\rm s} - 3\delta_{\rm s} ,$$

$$g_{\rm s} \equiv \frac{\dot{\mathcal{G}}_{\rm s}}{H\mathcal{G}_{\rm s}} , \quad \delta_{\rm s} \equiv \frac{\dot{c}_{\rm s}}{Hc_{\rm s}} .$$

- Tensor perturbations are from both metric and gauge fields, and they are gauge-invariant (under both coordinate and SU(2) gauge transformations) at linear order.

$$S_{h^2} = \int d^3x dt\ a^3 \frac{\mathcal{G}_t}{8}\left[\dot{h}_{ij}^2 - \frac{c_t^2}{a^2}(\partial_k h_{ij})^2\right] ,$$

$$\mathcal{G}_t = M_p^2 - 2a_4 X , \quad c_t^2 = \frac{1}{\mathcal{G}_t}\left(M_p^2 + 2a_4 X\right) ,$$

$$S_{hT} = \int d^3x dt\ a^3 HQ h_{ij}\left(\dot{T}_{ij} + \frac{m_Q}{a}\epsilon^{ikl}\partial_k T_{jl}\right) ,$$

$$S_{T^2} = \int d^3x dt\ a^3 \frac{1}{2}\left[\dot{T}_{ij}^2 - \frac{1}{a^2}(\partial_k T_{ij})^2 - 2\xi m_Q H^2 T_{ij}^2 + 2(\xi + m_Q)\frac{H}{a}\epsilon^{ijk}T_{kl}\partial_i T_{jl}\right]$$

$$m_Q = \frac{g_A Q}{H}, \quad \xi = \frac{\lambda\dot{\phi}}{2fH}$$

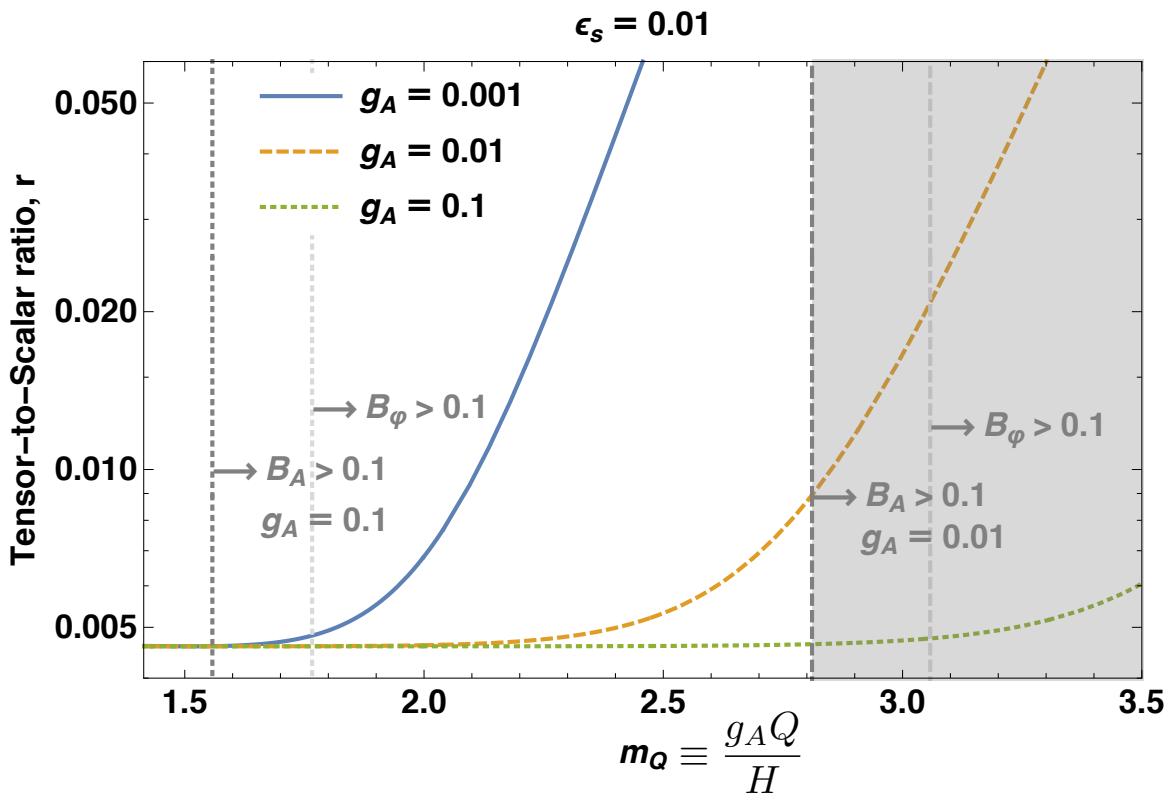
$$m_Q \simeq \xi$$

Only +2 helicity mode is amplified: Chiral GW signal

$$r = 16\frac{\mathcal{G}_s c_s^3}{\mathcal{G}_t c_t^3}\left[1 + \frac{Q^2 e^{\pi(\xi+m_Q)}|G_+|^2}{2M_p^2}\right] \simeq 4.6\times 10^{-3}\left(\frac{\epsilon_s}{10^{-2}}\right)^{3/2}\left[1 + \frac{H^2 m_Q^2 e^{\pi(\xi+m_Q)}|G_+|^2}{2g_A^2 M_p^2}\right]$$

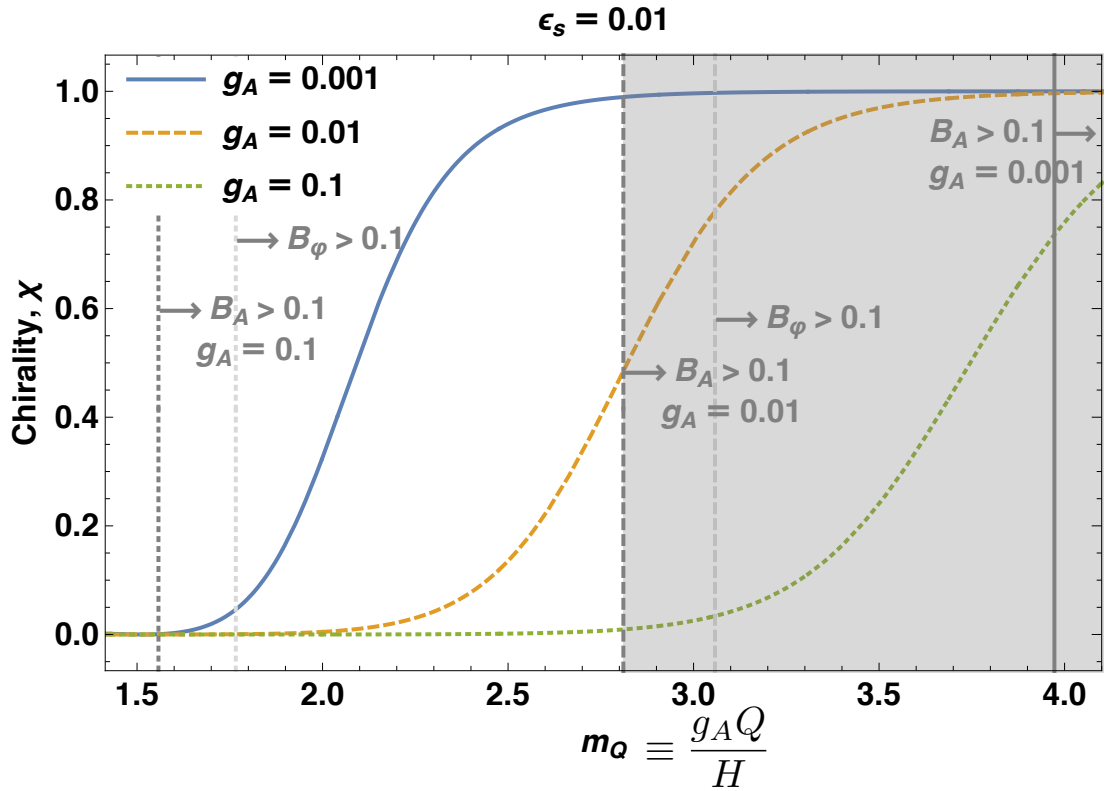
for $\mathcal{G}_s \simeq 12M_p^2$, $c_s \simeq \sqrt{\epsilon_s/12}$, $\mathcal{G}_t \simeq M_p^2$, and $c_t \simeq 1$. Note that $|G_+|^2 = |G_+(\xi_\phi, \xi_Q)|^2 \lesssim \mathcal{O}(10^{-3})$.

$$f_{NL}^{eq} \sim 1/\epsilon_s$$



$$\chi = \frac{r_{\rm sourced}}{r_{\rm vacuum} + r_{\rm sourced}}$$

$$f_{NL}^{eq} \sim 1/\epsilon_s$$



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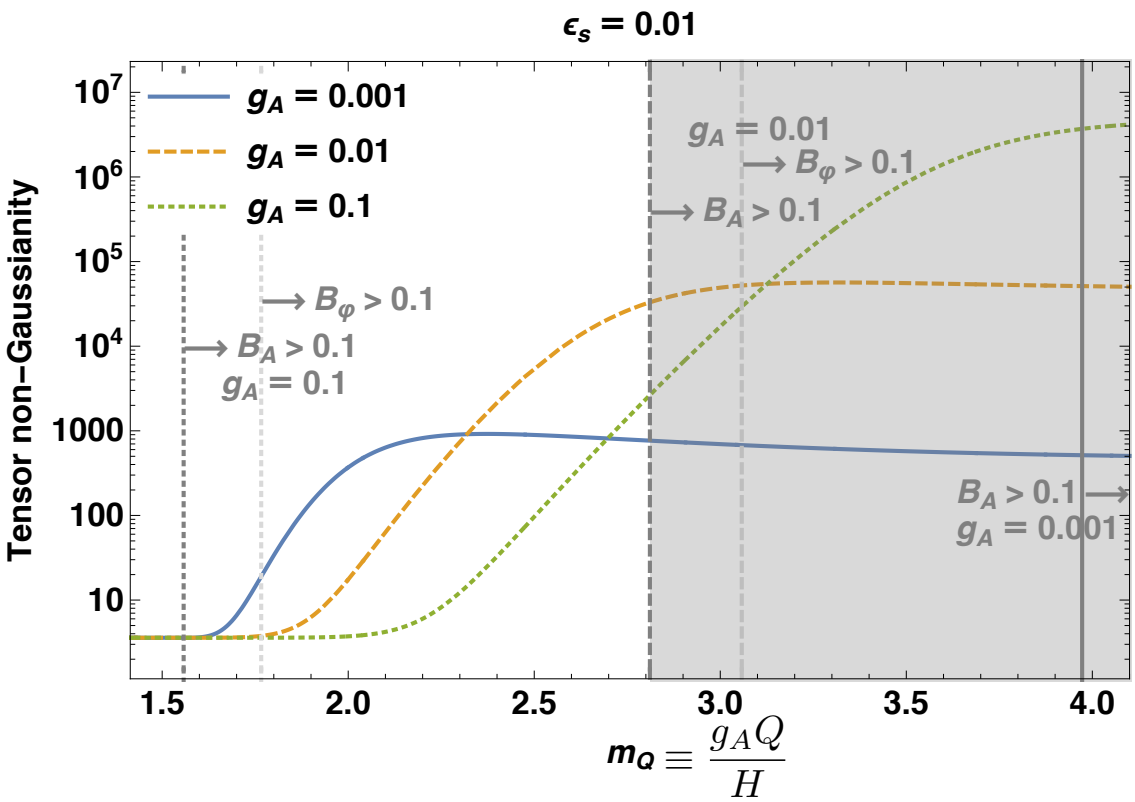
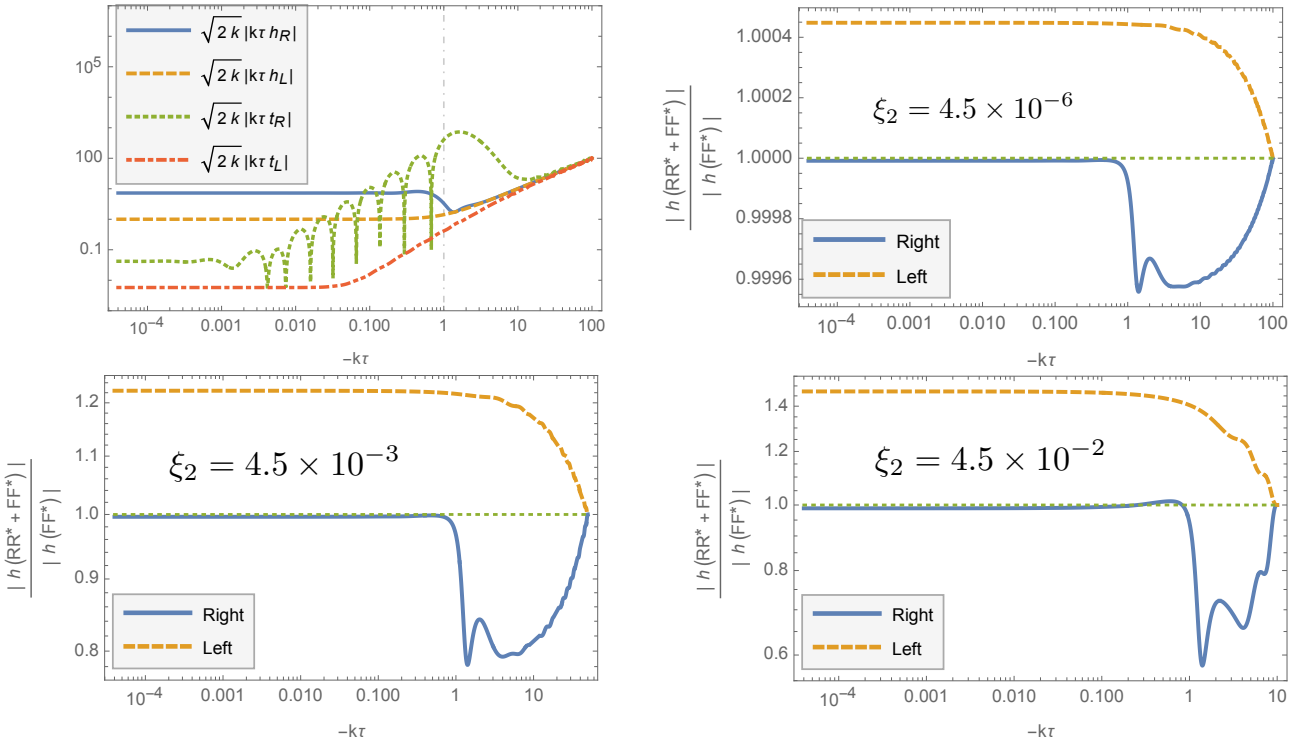


Effects of gravitational Chern-Simons term ϕRR*

- The impact of gCS term on GWs can be as large as 50% enhancement for non-sourced (left) helicity modes.



Summary [YW, Komatsu 2004.04350]

- The axion without potential can drive inflation with the non-canonical kinetic structure.
- A non-trivial VEV of SU(2) gauge field can be acquired during inflation.
- r ~ 5 x 10^{-3} or greater, partially chiral and non-Gaussian gravitational waves can be obtained.

$$1.1\times10^{-5}\left(\frac{\epsilon_s}{10^{-2}}\right)^{3/4} < \frac{\lambda}{\xi} < 0.56 \qquad m_Q \simeq \xi$$

- Inclusion of gravitational CS term can affect the tensor spectrum. [Mirzagholi, Komatsu, Lozanov, YW 2003.05931]

2022 Oct. / Poster Session JGRG31
(Koshiba Hall, Univ. of Tokyo)

P05

Mode conversion phenomena of the Einstein-Maxwell system in the cylindrically symmetric spacetime by full nonlinearity

T. Mishima (Nihon Univ.)
S. Tomizawa (Toyota Tech. Inst.)

- **Previously we gave a simple and convenient harmonic method to construct cylindrically symmetric wave solutions of the Einstein-Maxwell system, and using the solution obtained, pointed out the possibility that the conversion phenomena occur between the gravitational and electromagnetic waves even without any background field.**
- **This place, as a subsequent, we examine in more detail how and to what extent the conversion phenomena occur. In particular, we show how the conversion between gravitational and electromagnetic modes changes with time.**
- **This contribution is mainly based on the previous work(arXiv: 2202.12060 [gr-qc]) and the subsequent work newly being prepared for publication elsewhere.**

I. Purpose

Construction of exact solutions of Einstein-Maxwell system with full modes(4 modes)

⇩

Full nonlinear analysis of conversion between gravitational and electro-magnetic modes without any background field

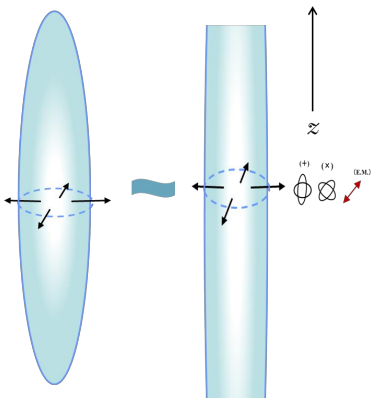
Gravit-elemag conversion in the background electromagnetic field: perturbative approaches (many)

cf. Gerlach[1974], Olson & Unruh[1974], … , Saito, Soda & Yoshino[2021], Hadj & Dolan[2022]

＜Assumptions and features＞

- **Cylindrical symmetry**
- **No conical singularity on the axis**
- **Locally asymptotic flatness**
- **Following Einstein-Maxwell equations (*G* = *c* = 1)**

$$\begin{cases} R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R = 8\pi T^{\mu}{}_{\nu}, \\ \nabla_{\mu}F^{\mu}{}_{\nu} = 0 \end{cases} \qquad T^{\mu}{}_{\nu} = \frac{1}{4\pi}\left(F^{\mu\rho}F_{\nu\rho} - \frac{1}{4}\delta^{\mu}{}_{\nu}F^{\rho\delta}F_{\rho\delta}\right)$$



II. Summary of settings and solution-generating method

II.1 Settings and basic equations

<Kompaneets - Jordan - Ehlers metric and elemag. gauge field > *** $(f = e^{2\psi},\ w,\ A_\phi,\ A_z)$

(1)
$$\begin{cases} ds^2 = f(dz - w d\phi)^2 + f^{-1}\rho^2 d\phi^2 + f^{-1}e^{2\gamma}(-dt^2 + d\rho^2), & \left[f := e^{2\psi}\right] \\ \mathbf{A} = A_\phi d\phi + A_z dz & \text{(gauge field)} \end{cases}$$

< Two kinds of Ernst eq. as harmonic map equations (from Einstein-Maxwell eq.) >

$$1^{\text{st}}\begin{cases} (\Re[\mathbf{E}] - |\mathbf{F}|^2)\nabla^2\mathbf{F} = (\nabla\mathbf{E} - 2\bar{\mathbf{F}}\nabla\mathbf{F})\cdot\nabla\mathbf{F} \\ (\Re[\mathbf{E}] - |\mathbf{F}|^2)\nabla^2\mathbf{E} = (\nabla\mathbf{E} - 2\bar{\mathbf{F}}\nabla\mathbf{F})\cdot\nabla\mathbf{E} \end{cases} \Leftrightarrow 2^{\text{nd}}\begin{cases} (\xi\bar{\xi} + \eta\bar{\eta} - 1)\nabla^2\xi = 2(\bar{\xi}\nabla\xi + \bar{\eta}\nabla\eta)\cdot\nabla\xi \\ (\xi\bar{\xi} + \eta\bar{\eta} - 1)\nabla^2\eta = 2(\bar{\xi}\nabla\xi + \bar{\eta}\nabla\eta)\cdot\nabla\eta \end{cases}$$

[Ernst '68] [Kinnersley 1973], [Mazur 1981]

(Ernst potentials) (A_ϕ) (w)

(2) $\boxed{\mathbf{F} := A_z + i\chi \quad \mathbf{E} := f + |\mathbf{F}|^2 - i\Phi}$ **(3)** $\boxed{\mathbf{E} = \dfrac{1+\xi}{1-\xi} \quad \mathbf{F} = \dfrac{\eta}{1-\xi}}$

- **Base space:** $\mathbf{M}^{1,2}$ ⇒ **target space / potential space:** $\mathrm{H}^2_{\mathrm{C}}$ **(Ball model)** [Parker'03]

II.2 Simple solutions to the Ernst eq. / harmonic map eq.

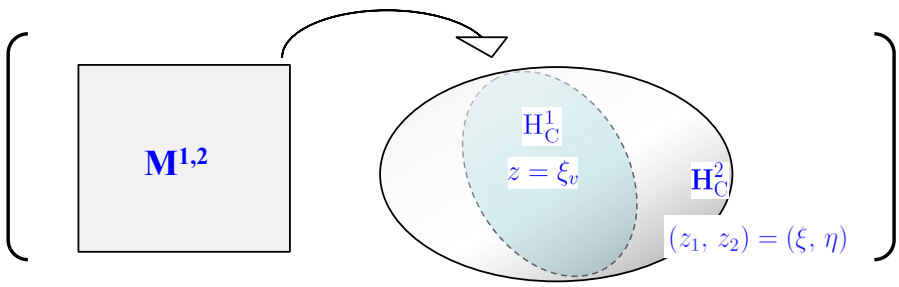
- ***Harmonic map*** **from the base to** ***a totally geodesic surface*** **embedded in the target** ⇒ **desired harmonic map** [Eells & Sampson '63]

< Application to 2nd Ernst equation>

Embedding map **of** $\mathrm{H}^1_{\mathrm{C}}$ **in** $\mathrm{H}^2_{\mathrm{C}}$ ✚ $\xi_v(t,\rho)$ ***harmonic map*** **from** $\mathbf{M}^{1,2}$ **to** $\mathrm{H}^1_{\mathrm{C}}$

(Sol. of vacuum Ernst eq.)

(4) $\boxed{(\xi, \eta) = (\cos 2\theta\, \xi_v, \sin 2\theta\, \xi_v)}$



＜Final form of the solutions＞

(5) $$(\xi, \eta) = (\cos 2\theta\, \xi_v(t,\rho), \sin 2\theta\, \xi_v(t,\rho))$$

Halilsoy[1989], Griffith['91]

Once given $\xi_v(t,\rho)$, reversing the above steps gives the metric and the electromagnetic field !

(3) $\mathbf{E} = \dfrac{1+\xi}{1-\xi}$ $\quad \mathbf{F} = \dfrac{\eta}{1-\xi}$ ⇒ (2) $\mathbf{F} := A_z + i\chi$ $\quad \mathbf{E} := f + |\mathbf{F}|^2 - i\Phi$

⇓

(1) $$\begin{cases} ds^2 = f(dz - w d\phi)^2 + f^{-1}\rho^2 d\phi^2 + f^{-1}e^{2\gamma}(-dt^2 + d\rho^2), \\ \mathbf{A} = A_\phi d\phi + A_z dz \end{cases}$$

＜Specific solutions easy to handle ＞

$\xi_v(t,\rho)$: Reuse of the vacuum solutions considered before [MT 2017, 2022]

(6) $$\xi_v = \frac{1 - e^{-2\tau} + iA}{1 + e^{-2\tau} - iA} \qquad (\nabla^2 \tau = 0)$$

τ: seed function

wave function on $\mathbf{M}^{1,2}$

(*geodesics* in $\mathrm{H}^1_{\mathrm{C}}$)

＋

(WWB's wave packet solution)

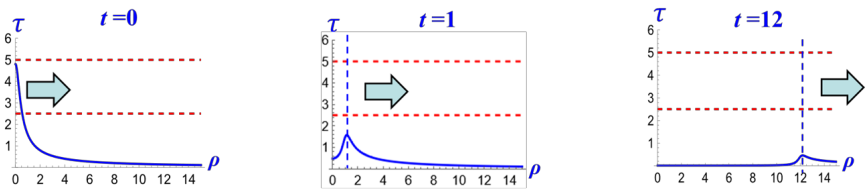
(7) $$\tau(t,\rho) = \frac{c}{\sqrt{2}}\left[\frac{\sqrt{4a^2t^2 + (a^2+\rho^2-t^2)^2} + a^2 + \rho^2 - t^2}{4a^2t^2 + (a^2+\rho^2-t^2)^2}\right]^{1/2}$$

(a :width, c/a : height)

[Weber-Wheeler '57], [Bonnor '57]

e.g.

(a, c)=(1/3, 8/5)



III. Nonlinear phenomena of Einstein-Maxwell system (mode conversion)

III.1 metric, electromagnetic potential derived form the method mentioned above

$$
(1)\quad \begin{cases}
e^{2\psi} = \dfrac{1}{A^2 e^{2\tau}\cos^4\theta + e^{-2\tau}(\cos^2\theta + e^{2\tau}\sin^2\theta)^2}, \qquad \Phi = -Ae^{2\tau}\cos 2\theta\, e^{2\psi}, \\
\chi = \dfrac{1}{2}Ae^{2\tau}\sin 2\theta\, e^{2\psi}, \quad A_z = -\dfrac{1}{2}[A^2 e^{2\tau}\cos^2\theta + (e^{-2\tau}-1)(\cos^2\theta + e^{2\tau}\sin^2\theta)]\sin 2\theta\, e^{2\psi} \\
w = -4A\cos^4\theta \displaystyle\int^{(t,\rho)} \rho(\,\partial_\rho\tau dt + \partial_t\tau d\rho) \quad \Rightarrow \quad \text{same as vacuum solution treated in [TM 2017]} \\
A_\phi = \displaystyle\int^{(t,\rho)} [(\rho e^{-2\psi}\,\partial_\rho\chi - w\,\partial_t A_z)dt + (\rho e^{-2\psi}\,\partial_t\chi - w\,\partial_\rho A_z)d\rho] \\
\gamma = \displaystyle\int^{(t,\rho)} \rho\,\{\,[(\partial_t\tau)^2 + (\partial_\rho\tau)^2]d\rho + 2\partial_t\tau\partial_\rho\tau dt\,\} \quad \Rightarrow \quad \text{same as vacuum solution treated in [TM 2017]}
\end{cases}
$$

- **Space-time regularity may be considered the same as the vacuum solution[TM 2017].**
- **Parameters:** $\left\{\begin{array}{l} \theta \Rightarrow \text{mixing degree of the electromagnetic modes} \\ A \Rightarrow \text{turn on/off the gravitational + mode} \end{array}\right\}$

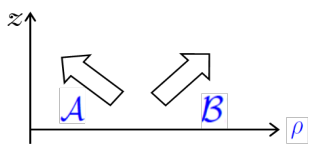
III.2 Preparation: useful quantities related to C-energy γ Thorne[1965] Piran, Safier and Stark [1985]

(Amplitudes)

The waves split into gravitational modes $(+,\times)$ and elemag. modes (ϕ, z); also ingoing waves $\mathcal{A}$ and outgoing waves $\mathcal{B}$

$$
(2)\quad \boxed{\begin{array}{l}
\mathcal{A}_+ := 2\,\partial_v\psi,\ \mathcal{A}_\times := \dfrac{1}{\rho}e^{2\psi}\partial_v w = e^{-2\psi}[\partial_v\Phi + 2(A_z\,\partial_v\chi - \chi\,\partial_v A_z)],\ \mathcal{A}_z := 2e^{-\psi}\partial_v A_z,\ \mathcal{A}_\phi := \dfrac{2}{\rho}e^{\psi}(\partial_v A_\phi + w\,\partial_v A_z) = 2e^{-\psi}\partial_v\chi, \\
\mathcal{B}_+ := 2\,\partial_u\psi,\ \mathcal{B}_\times := \dfrac{1}{\rho}e^{2\psi}\partial_u w = -e^{-2\psi}[\partial_u\Phi + 2(A_z\,\partial_u\chi - \chi\,\partial_u A_z)],\ \mathcal{B}_z := 2e^{-\psi}\partial_u A_z,\ \mathcal{B}_\phi := \dfrac{2}{\rho}e^{\psi}(\partial_u A_\phi + w\,\partial_u A_z) = -2e^{-\psi}\partial_u\chi
\end{array}}
$$

$\left(\, u = \frac{1}{2}(t-r) \text{ and } v = \frac{1}{2}(t+r) \,\right)$



(Generalized C-energy density)

$$
(3)\quad \boxed{\begin{array}{l|l}
\partial_\rho\gamma = \dfrac{\rho}{8}(\mathcal{A}_+^2 + B_+^2 + \mathcal{A}_\times^2 + B_\times^2 & + \mathcal{A}_z^2 + \mathcal{B}_z^2 + \mathcal{A}_\phi^2 + \mathcal{B}_\phi^2) \\
\mathcal{E} = \quad \mathcal{E}_g & + \quad \mathcal{E}_{em} \\
\quad \text{(gravitational part)} & \quad \text{(electromagnetic part)}
\end{array}}
$$

Each mode contribution

$$
\left(\begin{array}{c} \mathcal{E}_I := \dfrac{\rho}{8}(\mathcal{A}_I^2 + \mathcal{B}_I^2) \\ (I = g,\ em,\ +,\ \times,\ z,\ \phi) \end{array}\right)
$$

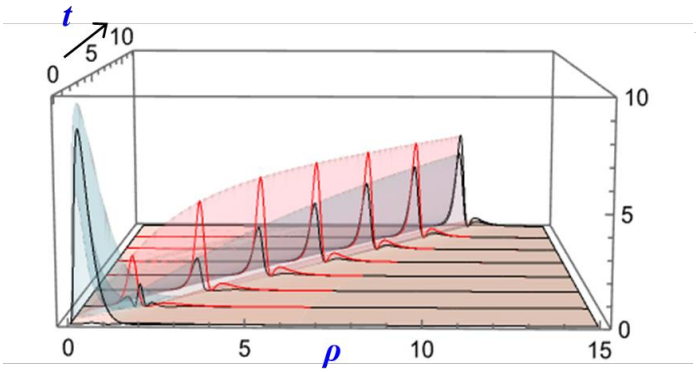
- **These quantities are useful to analyze mode conversion.**

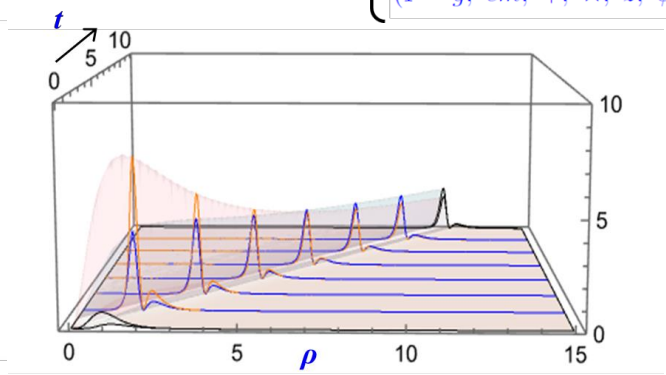
<Demonstration of temporal behavior of each mode part of C energy density>

Each mode contribution

$$\left(\mathcal{E}_I := \frac{\rho}{8}(\mathcal{A}_I^2 + \mathcal{B}_I^2) \quad (I = g,\ em,\ +,\ \times,\ z,\ \phi) \right)$$

e.g. $(A, \theta, a, c) = (1/6, \pi/10, 1/3, 8/5)$





$\{ \mathcal{E}_+ ,\ \mathcal{E}_z \}$ $\{ \mathcal{E}_\times ,\ \mathcal{E}_\phi \}$

- **The initially dominant gravit. + mode quickly converts to other modes and bounces back a bit.** → **Introduction of convenient physical quantities for analysis**

next slide

III.3 Introduction of occupancy ratios from a clearer perspective

- **Evaluation of local contribution of each mode to C-energy density by the seed function $\tau(t, \rho)$** [MT 2022]

$$R_I := \mathcal{E}_I / \mathcal{E} \qquad \left(\mathcal{E} = \rho\,[\,(\partial_t \tau)^2 + (\partial_\rho \tau)^2\,] \right)$$

<Expression of occupancy>

$$\left[D := A^2 e^{4\tau} \cos^4\theta + (\cos^2\theta + e^{2\tau}\sin^2\theta)^2 \right]$$

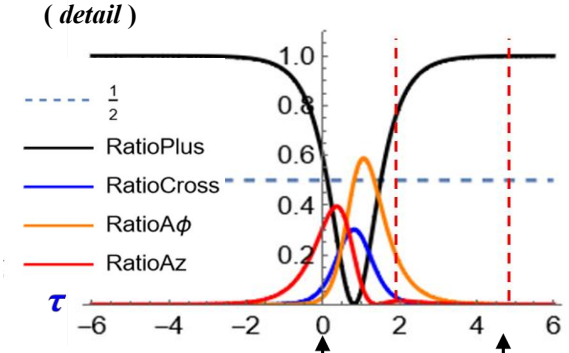
$$(4)\ \begin{cases} R_g = D^{-1}[\,A^2 e^{4\tau}\cos^4\theta + (\cos^2\theta - e^{2\tau}\sin^2\theta)^2\,], \quad R_{em} = D^{-1} e^{2\tau}\sin^2(2\theta) & \text{gravity/elemag.} \\ R_+ = D^{-2}[\,(A^2 e^{4\tau} - 1)\cos^4\theta + e^{4\tau}\sin^4\theta\,]^2, \quad R_\times := 4A^2 D^{-2} e^{4\tau}\cos^8\theta & \\ R_\phi = 4A^2 D^{-3} e^{6\tau}(\cos^2\theta + e^{2\tau}\sin^2\theta)^2\cos^4\theta\sin^2(2\theta) & \text{detail} \\ R_z = D^{-3} e^{2\tau}[\,A^2 e^{4\tau}\cos^4\theta - (\cos^2\theta + e^{2\tau}\sin^2\theta)^2\,]^2 \sin^2(2\theta) & \end{cases}$$

- **The occupancy gives a rough estimate of mode conversion from the behavior of τ.**

e.g. When τ becomes enough large, gravitational + mode becomes overwhelmingly dominant.

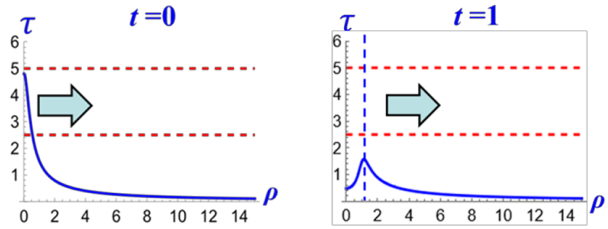
- **Demonstration of the occupancy: <u>e.g.</u> $(A, \theta, a, c) = (1/6, \pi/10, 1/3, 8/5)$**

(*detail*)

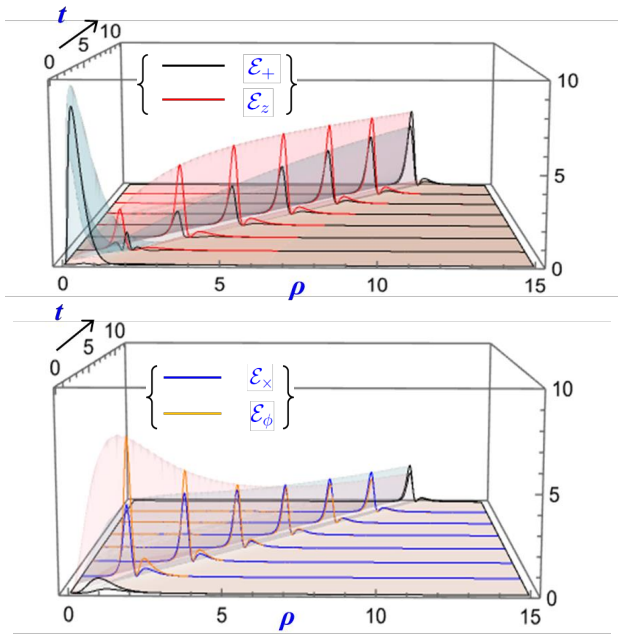


Position of τ-peak ⟶ **infinity** ⟸ **axis**

(**seed function τ**)



<Behavior of each mode part of C energy density>



III.4 Evaluation of electromagnetic-gravitational mode conversion

(*em-modes*) (*g-modes*) **[MT 2022]**

- **Time evolution of the ratio of *electromagnetic modes* to C energy**

$$T_{\rm em}(t) := \frac{\gamma_{\rm em}(t,\rho=\infty)}{\gamma(t,\rho=\infty)} \approx \frac{\gamma_{\rm em}(t,\rho=1000)}{(c/2a)^2} \qquad \left(\gamma_\bullet(t,\rho) := \int_0^\rho \mathcal{E}_\bullet(t,r)\,dr\right)$$

$\gamma(t,\rho=\infty) = (c/2a)^2$

(Integral numerically evaluated)

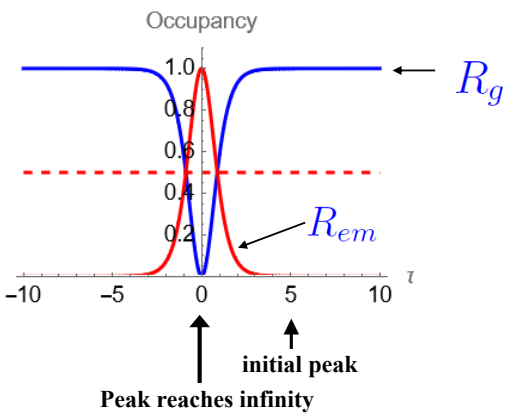
<Example of large conversion: *g-modes* ⇒ *em-modes*>

<u>e.g.</u> $(A, \theta, a, c) = (0, \pi/4, 1/20, 1/4)$

C-energy $\gamma(t,\rho=\infty) = 5$ ⟶ **Deficit angle** $\Delta\phi/2\pi \approx 99\%$

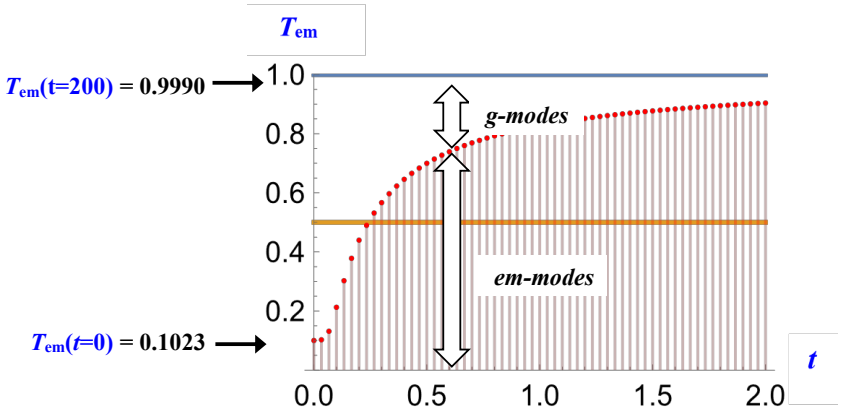
$\Delta\phi = 2\pi\left(1 - e^{-\gamma}\right)$

- **The large conversion may be expected.**



$T_{em}(t=0) = 0.1023$, $T_{em}(t=200) = 0.9990$ (numerically evaluated)

$T_{em}(t=200) / T_{em}(t=0) = 9.8$



- **About 10 times amplification of *em-modes***

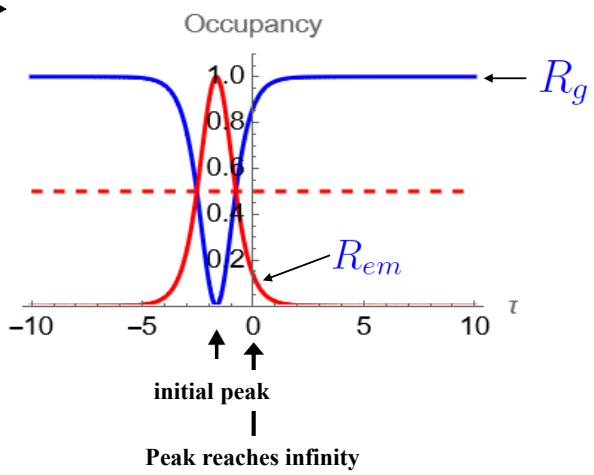
- **Time evolution of the ratio of *gravitational modes* to C energy**

$$T_g(t) := \frac{\gamma_g(t, \rho = \infty)}{\gamma(t, \rho = \infty)} \approx \frac{\gamma_g(t, \rho = 1000)}{(c/2a)^2}$$

<Example of large conversion: *em-modes* ⇒ *g-modes*>

e.g. $(A, \theta, a, c) = (1/6, 11\pi/25, 1/18, -2/15)$

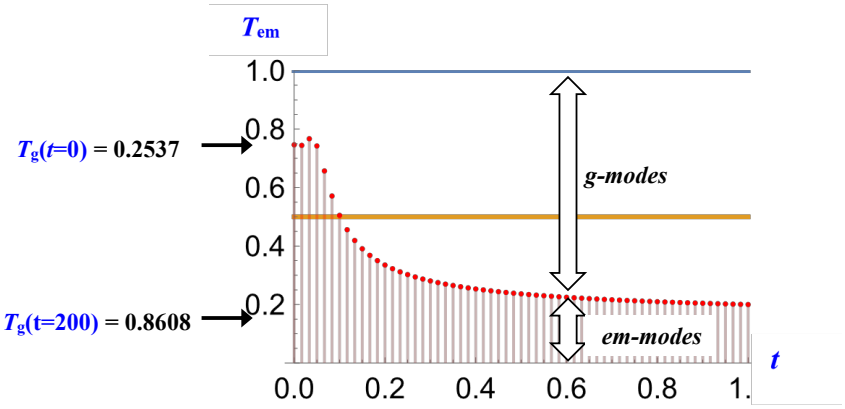
$$\left(\begin{array}{c} \gamma(t, \rho = \infty) = 1.44 \longrightarrow \Delta\phi/2\pi \approx 76\% \\ \Delta\phi = 2\pi\left(1 - e^{-\gamma}\right) \end{array} \right)$$



- **The large conversion may be expected.**

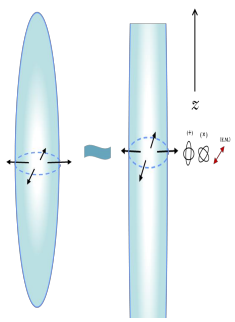
$T_g(t=0) = 0.2537$, $T_g(t=200) = 0.8608$ (numerically evaluated)

$T_g(t=200) / T_g(t=0) = 3.4$



- **About 10 times amplification of *g-modes***

III.5 On the treatment of elongated finite energy objects [MT 2022]



- **Combine the following two formulas of deficit angle: μ is *the line energy density* $(\Delta\phi)$**

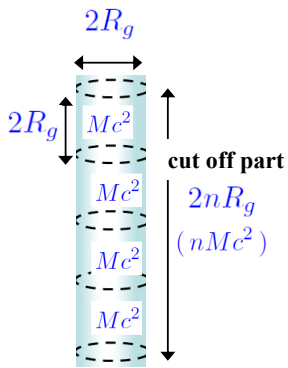
[Vielenkin 1985, Got III 1985] $\Delta\phi = \frac{8\pi G}{c^4}\mu$ ✚ $\Delta\phi = 2\pi\left(1 - e^{-\gamma}\right)$ [MT 2017]

Arbitrary mass scale : M

$$\mu = \frac{c^4}{4G}\left(1 - e^{-\gamma}\right) = \frac{1}{2}\frac{Mc^2}{2GM/c^2}\left(1 - e^{-\gamma}\right)$$

$$= \frac{Mc^2}{2R_g}\left(1 - e^{-\gamma}\right) \qquad \left(R_g := 2GM/c^2\right)$$

Gravitational radius



From several examples such as the above cases, large conversions seem to require 2π-deficit angle:

Hoop conjecture

$$\Delta\phi \approx 2\pi \longleftrightarrow \gamma \gtrapprox 1 \longrightarrow \mu \approx \frac{Mc^2}{2R_g} = \frac{nMc^2}{2nR_g}$$

- ***Elongated finite energy distributions of the above μ may collapse into a BH, if so the large conversions can be prevented by the BH formation, or not ?***

V. Summary

- **We introduced simple but useful exact solutions for the analysis of the conversion phenomenoa between gravitational waves and electromagnetic waves, and also observed the temporal behavior of the mode conversion phenomena.**

- **We found that, depending on the parameters, large conversions between electromagnetic and gravitational modes occur.**

- **It can be said that the conversion phenomena exhibited by these solutions are caused by the nonlinear interaction between the wave modes without the need for an external field.**

- **We discussed the possibility of black hole formation to prevent the occurrence of large conversions in the case of Elongated finite energy distributions, according to "the Hoop Conjecture".**

JGRG31

Threshold of Primordial Black Hole Formation against Velocity Dispersion in Matter-Dominated Era

Chulmoon Yoo (Nagoya Univ.)

with Tomohiro Harada, Kazunori Kohri, Misao Sasaki, Takahiro Terada

PBH formation in early MD

PBH in MD

◎Early **MD** era (e.g., oscillation of inflation soon after the inflation ~ effectively no pressure)

◎PBH formation would be easier than radiation dominated era

◎What does prevent PBH formation?

- ➢ Ellipticity [1609.01588, Harada, CY+]
- ➢ Angular momentum [1707.03595, Harada, CY+]
- ➢ Effect of inhomogeneity [1810.03490, Kokubu+]
- ➢ Velocity dispersion

◎Smaller scale inhomogeneities ⇒ velocity dispersion ⇒ prevent PBH formation?

Summary of Previous Works

◎Effect of ellipticity [1609.01588, Harada, CY+]

- ➢ Zel'dovich approximation
- ➢ Doroshkevich's probability distribution (Gaussian random field) for the initial profile
- ➢ Hoop conjecture for BH formation criterion
- ➢ PBH production probability

$$\beta \sim 0.0556\sigma^5$$

σ: standard deviation of density fluctuation

◎Effect of angular momentum [1707.03595, Harada, CY+]

- ➢ Newtonian perturbation theory
- ➢ 2nd order(1st^2) contribution to angular momentum [Peebles (1969)]
- ➢ 1st order contribution assuming elliptic collapsing region [Catelan, Theuns (1996)]
- ➢ PBH formation criterion: the spin parameter $a < 1$

$$\delta_{\rm th} = \frac{3\cdot 2^2}{5^3} q^2 \quad \textbf{1st}$$

$$\delta_{\rm th} = \left(\frac{2}{3}\mathcal{I}\right)^{2/3} \sigma^{2/3} \quad \textbf{2nd}$$

⇒ PS(Carr's) formula

$$\beta \sim \sigma \exp[-\sigma^{-2}] \quad \textbf{1st}$$

$$\beta \sim \sigma^{1/3} \exp[-\sigma^{-2/3}] \quad \textbf{2nd}$$

Summary of the previous results

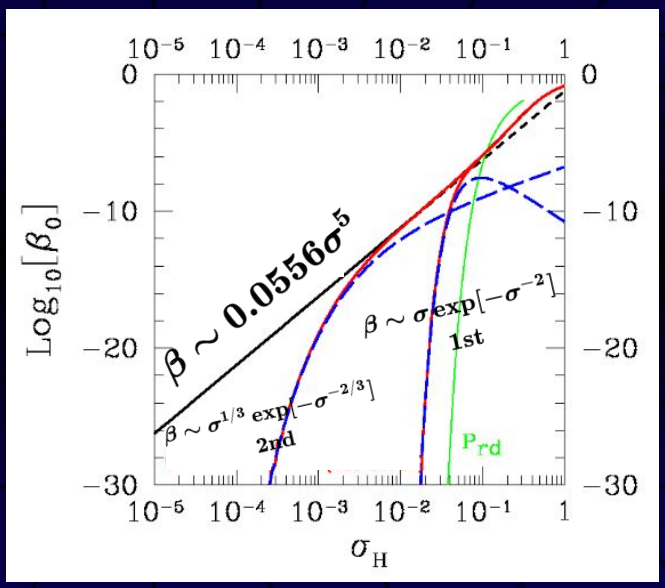
◎Effect of ellipticity

$$\beta \sim 0.0556\sigma^5$$

◎Effect of angular momentum

$$\beta \sim \sigma \exp[-\sigma^{-2}] \quad \text{1st}$$
$$\beta \sim \sigma^{1/3} \exp[-\sigma^{-2/3}] \quad \text{2nd}$$

◎Effect of velocity dispersion?



Effect of velocity dispersion against collapse

Velocity dispersion shared in a collapsing sphere

◎Adiabatic evolution of the velocity dispersion σ_v

➢ Fiducial time t_*, density $\rho(t)$, radius $R(t)$

$$\sigma_v(t) = \sigma_v(t_*)\left(\frac{\rho(t)}{\rho(t_*)}\right)^{1/3} = \sigma_v(t_*)\frac{R(t_*)}{R(t)}$$

◎Crossing time scale

$$t_{\rm cross} \sim R(t)/\sigma_v(t) \sim \frac{R^2(t)}{\sigma_v(t_*)R(t_*)} \propto R^2$$

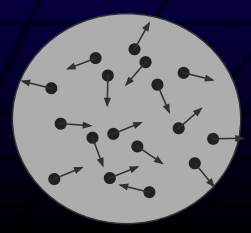
◎Gravitational free fall time scale

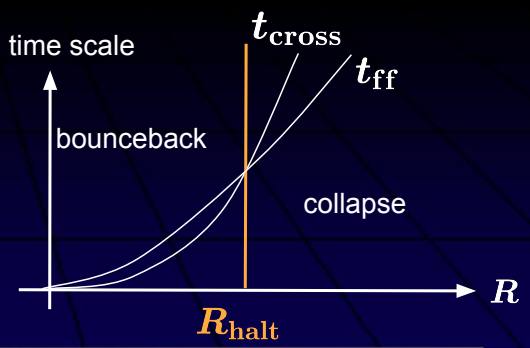
$$t_{\rm ff} \sim \frac{1}{\sqrt{G\rho(t)}} \sim t_*\left(\frac{R(t)}{R(t_*)}\right)^{3/2} \propto R^{3/2}$$

◎ $t_{\rm ff} = t_{\rm cross} \Leftrightarrow R_{\rm halt} \simeq \sigma^2(t_*)\frac{R^2(t_*)}{R_{\rm ent}^2}R_{\rm ent}$

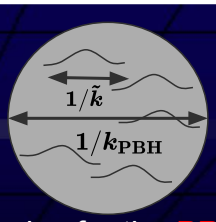
◎Collapse halts if $R_{\rm halt} > R_{\rm Sch} = 2GM = R_{\rm ent}$

$\Rightarrow$ PBH non-formation condition : $\sigma_v(t_*)R(t_*) > 2GM$





Generation of Velocity Dispersion



◎Two scales $1/k_{\rm PBH}$ and $1/\tilde{k} < 1/k_{\rm PBH}$

◎Hereafter quantities with "~" are those for the smaller scale $(1/\tilde{k})$ otherwise for the PBH scale

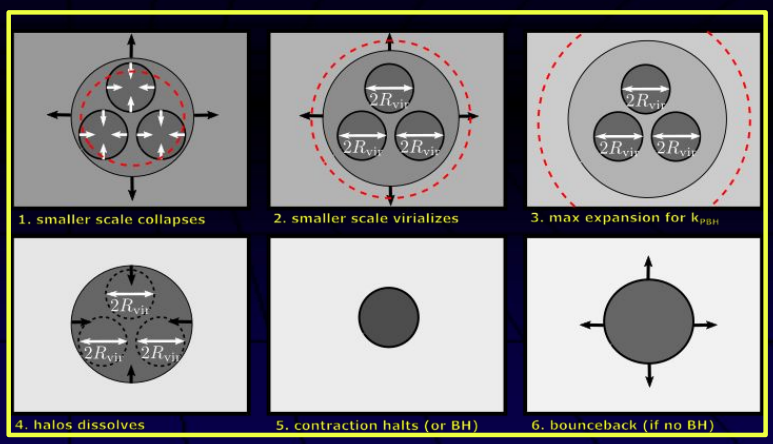
◎Characteristic times

➢ Horizon entry time: $\tilde{t}_{\rm ent}$, $t_{\rm ent}$
➢ Collapsing time: $\tilde{t}_{\rm coll}$, $t_{\rm coll}$
➢ Virialization time: $\tilde{t}_{\rm vir} \gtrsim \tilde{t}_{\rm coll}$

◎Case I : $\tilde{t}_{\rm vir} < t_{\rm coll}$

◎Case II: $\tilde{t}_{\rm vir} > t_{\rm coll}$

➢ No time for virialization
➢ $\tilde{\delta}$ and v (velocity perturbation) grow
➢ become non-linear
➢ velocity dispersion released and shared in the PBH scale



Derivation of non-PBH formation condition

◎Closed FLRW collapsing model: $R_{\rm max} = R_{\rm ent}\frac{a_{\rm max}}{a_{\rm ent}} \sim R_{\rm ent}\frac{1}{\delta_{\rm ent}}$

◎Case I : $\tilde{t}_{\rm vir} < t_{\rm coll} \Leftrightarrow \tilde{\delta}_{\rm ent} > \delta_{\rm ent}\frac{k_{\rm ent}^2}{\tilde{k}^2}$

- ➢ $\sigma_v^2 \simeq \frac{2\tilde{M}}{\tilde{R}_{\rm max}} \simeq \frac{2\tilde{M}}{\tilde{R}_{\rm ent}}\frac{\tilde{R}_{\rm ent}}{\tilde{R}_{\rm max}} \simeq \tilde{\delta}_{\rm ent}$
- ➢ $\rho(t_*) = \tilde{\rho}_{\rm vir} = 8\tilde{\delta}_{\rm ent}^3\tilde{\rho}_{\rm ent} \Leftrightarrow 2\tilde{\delta}_{\rm ent} = \left(\frac{\rho(t_*)}{\tilde{\rho}_{\rm ent}}\right)^{1/3} = \frac{R(\tilde{t}_{\rm ent})}{R(t_*)} = \frac{R_{\rm ent}}{R(t_*)}\frac{\tilde{a}_{\rm ent}}{a_{\rm ent}} = \frac{R_{\rm ent}}{R(t_*)}\frac{k_{\rm PBH}^2}{\tilde{k}^2} \Leftrightarrow R(t_*) \simeq \frac{k_{\rm PBH}^2}{\tilde{k}^2}\frac{R_{\rm ent}}{\tilde{\delta}_{\rm ent}}$

PBH non-formation condition for case I: $\sigma_v(t_*)R(t_*) > 2GM \Leftrightarrow \tilde{\delta}_{\rm ent} < \left(\frac{k_{\rm PBH}}{\tilde{k}}\right)^4$

◎Case II: $\tilde{t}_{\rm vir} > t_{\rm coll} \Leftrightarrow \tilde{\delta}_{\rm ent} < \delta_{\rm ent}\frac{k_{\rm ent}^2}{\tilde{k}^2}$

- ➢ No time for virialization
- ➢ $\tilde{\delta}$ and $\tilde{v}$ (velocity perturbation) grow
- ➢ become non-linear : $\tilde{\delta}_{\rm co}(t_*) = 1$

$$(1-\cos(\sqrt{\mathcal{K}}\eta))\frac{d^2}{d\eta^2}\tilde{\delta}_{\rm co} + \sqrt{\mathcal{K}}\sin(\sqrt{\mathcal{K}}\eta)\frac{d}{d\eta}\tilde{\delta}_{\rm co} - 3\mathcal{K}\tilde{\delta}_{\rm co} = 0$$

$$\tilde{v} = -(\triangle + 3\mathcal{K})^{-1}\frac{d}{d\eta}\tilde{\delta}_{\rm co} \simeq \tilde{k}^{-2}\frac{d}{d\eta}\tilde{\delta}_{\rm co}$$

$$\Rightarrow \sigma_v \sim \tilde{k}\tilde{v} \simeq \tilde{k}^{-1}\frac{d}{d\eta}\tilde{\delta}_{\rm co}$$

PBH non-formation condition for case II: $\sigma_v(t_*)R(t_*) > 2GM \Leftrightarrow \tilde{\delta}_{\rm ent} > \frac{\tilde{k}}{k_{\rm PBH}}\delta_{\rm ent}^{5/2}$

Summary of inequalities

◎For simplicity, we ignore O(1) factors

◎We consider $\tilde{\delta}_{\rm ent} \sim \sigma_{\rm ent}(\tilde{k})$ (standard deviation) and introduce $\kappa := \frac{\tilde{k}}{k_{\rm PBH}} > \kappa_{\rm min} > 1$

◎Threshold dividing the two cases

$$\tilde{t}_{\rm vir} = t_{\rm coll} \Leftrightarrow \sigma_{\rm ent} = \frac{k_{\rm PBH}^2}{\tilde{k}^2}\delta_{\rm ent} = \delta_{\rm ent}\kappa^{-2}$$

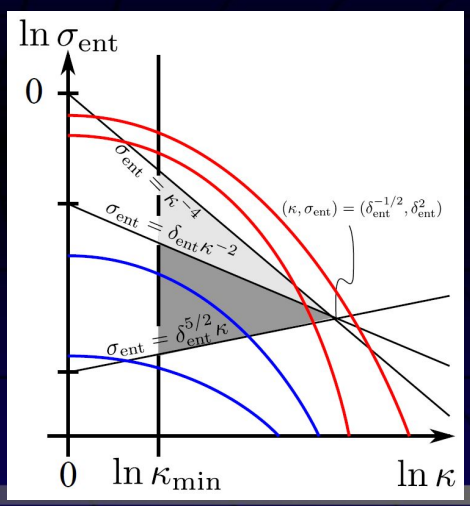
◎Case I $\sigma_{\rm ent} > \delta_{\rm ent}\kappa^{-2}$

Condition for no PBH formation: $\sigma_{\rm ent} < \kappa^{-4}$

◎Case II $\sigma_{\rm ent} < \delta_{\rm ent}\kappa^{-2}$

Condition for no PBH formation: $\sigma_{\rm ent} > \kappa\delta_{\rm ent}^{5/2}$

◎If the spectral line passes through the shaded region, there exist perturbations halt the collapse



Threshold for log-normal spectrum

Threshold for a log-normal spectrum

◎Log-normal spectrum

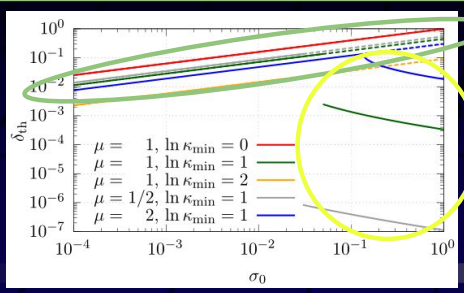
$$\ln \sigma_{\rm ent}(\kappa) = -\mu(\ln\kappa)^2 + \ln\sigma_0$$

◎For $\ln \sigma_{\rm ent}(\kappa_{\rm min}) > -4\ln\kappa_{\rm min}$

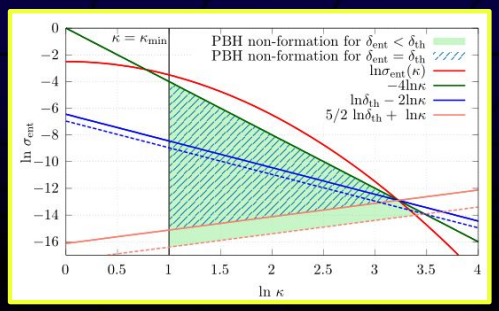
PBH formation if $\delta_{\rm ent} > \exp\left[-\frac{2}{\mu}\left(2+\sqrt{4+\mu\ln\sigma_0}\right)\right]$

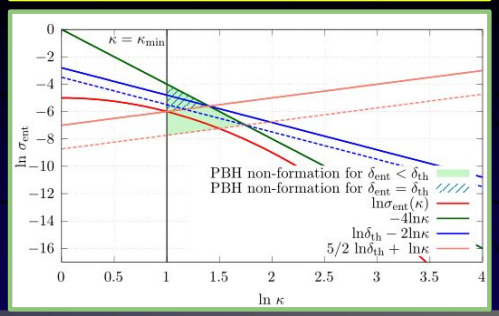
◎For $\ln \sigma_{\rm ent}(\kappa_{\rm min}) < -4\ln\kappa_{\rm min}$

PBH formation if $\delta_{\rm ent} > \exp\left[-\frac{2}{5}\mu(\ln\kappa_{\rm min})^2\right]\left(\frac{\sigma_0}{\kappa_{\rm min}}\right)^{2/5}$



too small
to be effective





PBH production probability

◎Effect of ellipticity

$$\beta \sim 0.0556\sigma^5$$

◎Effect of angular momentum

$$\delta_{\rm th} = \frac{3\cdot 2^2}{5^3} q^2 \qquad \text{1st}$$

$$\delta_{\rm th} = \left(\frac{2}{3}\mathcal{I}\right)^{2/3} \sigma^{2/3} \qquad \text{2nd}$$

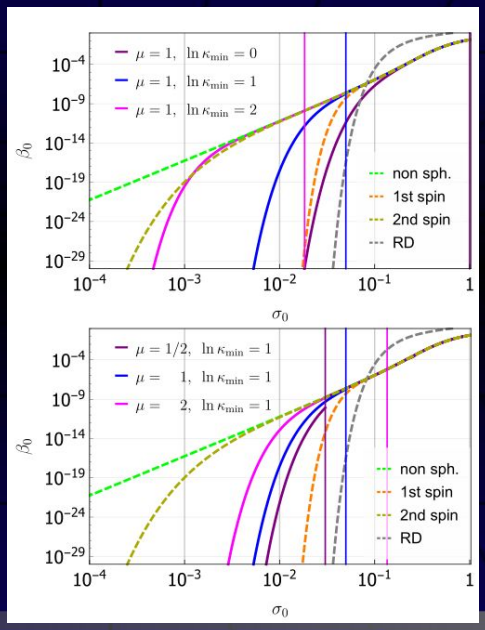
$$\beta \sim \sigma \exp[-\sigma^{-2}] \qquad \text{1st}$$

$$\beta \sim \sigma^{1/3} \exp[-\sigma^{-2/3}] \qquad \text{2nd}$$

◎Effect of velocity dispersion?

$$\delta_{\rm th} \sim \sigma^{2/5}$$

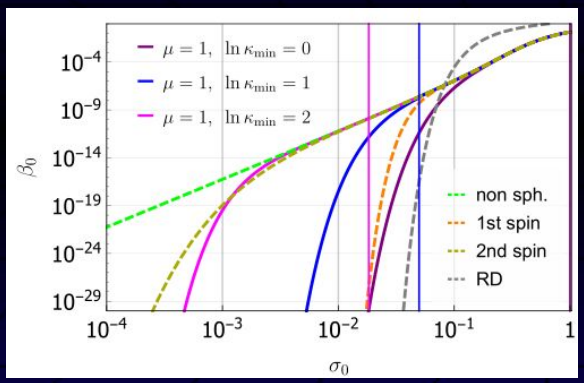
$$\beta \sim \sigma^{3/5} \exp[-\sigma^{-6/5}]$$

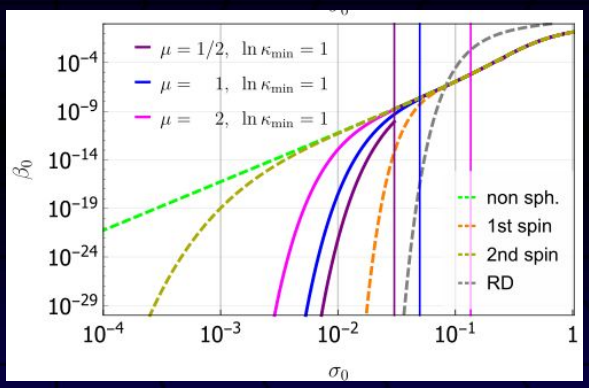


Summary

◎Effects of velocity dispersion on PBH formation is discussed

◎It may give a comparable contribution to the spin effect





Thank you for your attention!

Soft Graviton Theory and Infrared Triangle

P07 Hideo Furugori (Nagoya University) (In preparation)

Abstract

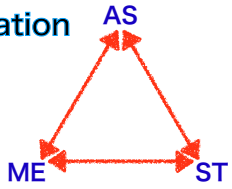
IR triangle relations appear in many places, Yang-Mills, Gravity, SUSY,⋯. I review the IR triangle in four-dim asymptotic-flat spacetime. The soft theorem shows that soft particles play an important role in QFT, but the theorem is not enough to confirm unitarity. I explain a soft graviton theory in the dressed state formalism proposed before (PRD 104.125004). I show that the IR relations vividly appear in the formalism, and discuss the role of soft gravitons.

Introduction

Infrared Triangle Relation

IR triangle relation connects three corners

- Asymptotic Symmetry
- Memory Effect
- Soft Theorem



Strominger et.al. (2015)

It may give us a clue to understand quantum aspect of gravity

Asymptotic Symmetry (BMS)

AS is transformation preserving gauge conditions / asymptotic conditions

Four-dimensional asym-flat spacetime possesses BMS symmetry

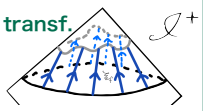
$$\xi^\mu\partial_\mu = \alpha\partial_u - \frac{1}{r}\eth^A\alpha\partial_A + \frac{1}{2}\eth^A\eth_A\alpha\partial_r\,,\ \alpha = \alpha(x^A),\ x^A \in \mathbb{S}^2$$

Supertranslation ⊃ Translation Functional d.o.f.

Angular depend transf.

→ Can infinite many charges exist?

Idea of the soft hair



(Gravitational) Memory Effect

ME is permanent displacement due to a radiation

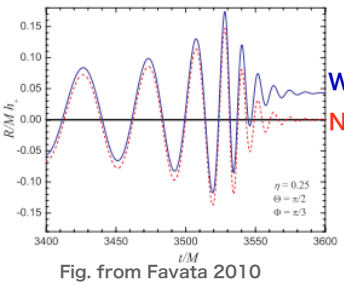


Fig. from Favata 2010

With memory

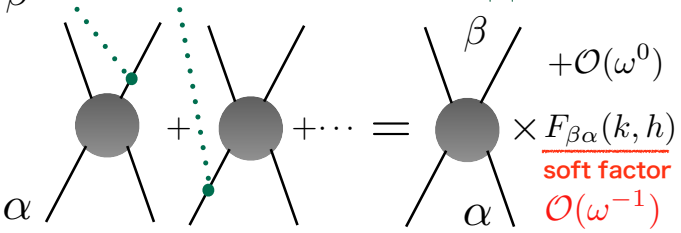
No memory

Fig. from Strominger's note

Soft (Graviton) Theorem

ST is relation bet. transition amplitudes with soft gravitons / without soft graviton

β + graviton with helicity h, momentum k, $|\vec{k}| := \omega$:small



ST is essential for dealing with the IR div problem in QFT

We cannot get a unitary time evolution

Soft factor is proportional to displacement memory above

→ Can soft gravitons carry information on a process?

Some hint on the information loss paradox?

Dressed State Formalism

Actually, ST cannot resolve the IR div problem itself

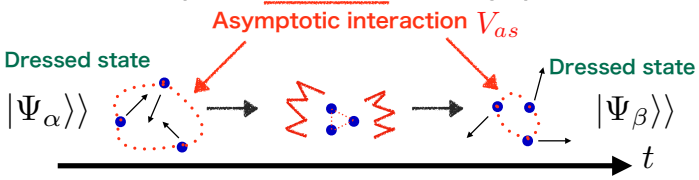
Conventional S-matrix is NOT well-defined

We should consider alternative definition of S-matrix

Basic Concepts of Dressed States

IR div. tells us the breakdown of the particle picture

We should incorporate interaction to the asymptotic states

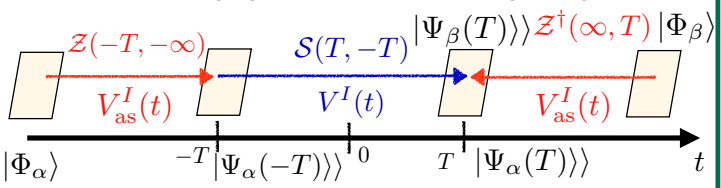


Our Dressed State Formalism

Fix the frame by defining the time scale T of scat problem

and define softness by T : $\omega_{\rm soft} < 1/T$

then we derive asymptotic states dressed by soft particles



S-matrix: $S^{\rm as}_{\beta\alpha}(T) = \langle\langle\Psi_\beta(T)|\mathcal{S}(T,-T)|\Psi_\alpha(-T)\rangle\rangle$

Soft Graviton Theory

Consider adding gravitons to a theory

How to add gravitons: ref. Weinberg '65

$$\mathscr{V}^{\rm I} = \mathscr{V}^{\rm I}_\phi - \frac{\kappa}{2}h_{\mu\nu}T^{\mu\nu}_\phi + \mathscr{V}^{\rm I}_{\rm Newton}$$

Non-covariant term cancelling a non-cov. propagator effect

!: Lorentz sym is broken due to graviton energy-momentum

Soft gravitons in Our Formalism

Consider adding only soft gravitons to ϕ^4 theory

$$V^{\rm I}_{\rm as}(t) = -\frac{\kappa}{2}\int d^3x h^{\rm soft}_{\mu\nu}(x)T^{\mu\nu}_{\rm cl}(x) \qquad T^{\mu\nu}_{\rm cl}(x) = \int d^3p\frac{p^\mu p^\nu}{E_p}\hat{N}(p)\delta^3(\vec{x}-\vec{p}t/E_p)$$

$$|\Psi_\alpha(T)\rangle\rangle = \mathcal{T}\exp\left[-i\int_T^\infty d\tau V^{\rm I}_{\rm as}(\tau)\right]|\Phi_\alpha\rangle$$

S-matrix $S^{\rm as}_{\beta\alpha}(T) = S^{\phi^4}_{\beta\alpha}(1/T)$ S-matrix in ϕ^4 with IR cut-off $\lambda = 1/T$

Gauge sym $S^{\rm as}_{\beta\alpha}(T) \to \Delta_(\varepsilon^{\rm in},\varepsilon^{\rm out})S^{\rm as}_{\beta\alpha} = \begin{cases} S^{\rm as}_{\beta\alpha} & \varepsilon^\mu_{\rm in} = \varepsilon^\mu_{\rm out} \\ 0 & \text{otherwise}\end{cases}$

Asymptotic symmetry condition

Lorentz transf. causes gauge transf. $\varepsilon^\mu_{\rm in} = \varepsilon^\mu_\Lambda = \varepsilon^\mu_{\rm out}$

Asymptotic symmetry recovers Lorentz symmetry!

Super-selection sectors of different frames $\mathcal{H} = \bigoplus_{\varepsilon_\Lambda}\mathcal{H}_{\varepsilon_\Lambda}$

We can calculate the memory

$\delta\langle h_{\mu\nu}(k,h)\rangle := \langle h_{\mu\nu}(k,h)\rangle|_{\Sigma_+} - \langle h_{\mu\nu}(k,h)\rangle|_{\Sigma_-}$ Well-known formula $+\alpha$

More predictable memory: $\delta\langle h_{\mu\nu}(\vec{x})\rangle$

Discussion and future works

I will speak orally in person !

【P08】 Generation of helical magnetic fields and baryogenesis from the coupling of electromagnetic fields with a higher curvature term in inflationary cosmology

Reference: **Phys. Dark Univ. 36, 101025 (2022)**

The 13th RESCEU International Symposium
The 31st Workshop on General Relativity and Gravitation in Japan (JGRG31)

Fukushima University October 24 - 28, 2022

Presenter: **Kazuharu Bamba (*Fukushima University*)**

Collaboration with: S. D. Odintsov（IEEC-CSIS）
Tanmoy Paul（Chandernagore College）
Debaprasad Maity（IIT Guwahati）

Baryon Asymmetry of the Universe (BAU)

There exists a net excess of baryons over anti-baryons in the universe.

⟶ **Baryon asymmetry of the Universe (BAU)**

$$\frac{n_{\rm B}}{s} = 0.92 \times 10^{-10}$$

[Spergel et al., Astrophys. J. Suppl. Ser. 148, 175 (2003)]

$n_{\rm B}$: Baryon number density, s : Entropy density

< Sakharov's three conditions > [Sakharov, JETP Lett. 5, 24 (1967)]

(1) Baryon number nonconservation

(2) C and CP violation

(3) A departure from thermal equilibrium

Models

Reviews of the BAU: [Dolgov Phys. Rep. 222, 309 (1992)]
[Dine and Kusenko, Rev. Mod. Phys. 76, 1 (2004)]

(1) GUT baryogenesis ← Decay of heavy particles

[Yoshimura, Phys. Rev. Lett. 41, 281 (1978); 42, 746E (1978)] $(M_{\rm GUT} \approx 10^{16}{\rm GeV})$

(2) Electro Weak (EW) baryogenesis ← $\triangle(B+L) = N(\neq 0)$, $\triangle(B-L) = 0$

[Kuzmin, Rubakov, and Shaposhnikov, Phys. Lett. 155B, 36 (1985)]

Anomalous process (Sphaleron process)

(3) Leptogenesis ← $\triangle(B-L) = \tilde{N}(\neq 0)$

[Fukugita and Yanagida, Phys. Lett. B 174, 45 (1986)]

B : Baryon number
L : Lepton number

(4) Affleck-Dine baryogenesis ← Coherent motion of scalar fields in supersymmetric theory

[Affleck and Dine, Nucl. Phys. B249, 361 (1985)]

(5) Spontaneous baryogenesis ← $\mathcal{L} \sim (\partial_\mu \theta) J_{\rm B}$

[Cohen and Kaplan, Phys. Lett. B 199, 251 (1987); Nucl. Phys. B308, 913 (1988)]

θ : Scalar field (the resultant Goldstone boson)
$J_{\rm B}$: Baryon current

(6) Black hole evaporation

[Zel'dovich, Pis'ma Zh. Eksp. Teor. Fiz. 24, 29 (1976)]
[Dolgov, Zh. Eksp. Teor. Fiz. 79, 337 (1980); Phys. Rev. D 24, 1042 (1981)]

Background of this study

- The hypermagnetic helicity, which is the volume integration of the inner product between a magnetic field and a vector potential, is regarded as the Chern-Simons number stored in the hypercharge electromagnetic fields.
- If the hypermagnetic fileds with their hypermagnetic helicity are generatd from inflation before the electroweak phase transition (EWPT), the hypermagnetic helicity (i.e., the Chern-Simons number) is converted into fermions (baryon number) at the EWPT owing to the Abelian anomaly.

[Giovannini & Shaposhnikov, PRL 80, 22 (1998); PRD 57, 2186 (1998)]

Cf. [Brustein & Oaknin, PRL 82, 2628 (1999); PRD 60, 023508 (1999)]

[Giovannini, PRD 61, 063004 (2000); PRD 61, 063502 (2000)]

Purpose of this study

- We explore the generation of magnetic fields from inflation due to the breaking of the conformal invariance of electromagnetic fields for the case that the electromagnetic fields couple with the Ricci scalar and the Gauss-Bonnet invariant. [KB, Elizalde, Odintsov, Paul, JCAP 04 (2021) 009]
- When the magnetic fields originated from the quantum fluctuations of the electromagnetic fields during inflation have their magnetic helicity, the baryon number can be generated due to the quantum anomaly.

  Cf. [KB, PRD 74, 123504 (2006)]
  [Fujita and Kamada, PRD 93, 083520 (2016)]
- We discuss that the magnetic fields at the galactic scale with enough strength can lead to the resultant value of the ratio of the baryonic number density to the entropy density consistent with the observational data. Cf. [Kamada, Uchida and Yokoyama, JCAP 04 (2021) 034]

II. Model

Action: $S = S_{grav} + S_{em}^{(can)} + S_{CB}$

(1) $$S_{grav} = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi - V(\Phi) - \xi(\Phi)\mathcal{G}\right]$$

$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$: Gauss-Bonnet invariant

$V(\Phi)$: Potential of the inflaton Φ

$\xi(\Phi)$: Coupling term between Φ and $\mathcal{G}$

R : Ricci scalar

(2) $$S_{em}^{(can)} = \int d^4x\sqrt{-g}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]$$: Action of electromagnetism

$\widetilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$

$\epsilon^{\mu\nu\alpha\beta} = \frac{1}{\sqrt{-g}}[\mu\nu\alpha\beta]$

: Levi-Civita tensor $[0123] = 1$

(3) Non-minimal coupling between electromagnetic fields and gravity

$$S_{CB} = \int d^4x\sqrt{-g}f(R,\mathcal{G})\left[\lambda F_{\mu\nu}\widetilde{F}^{\mu\nu}\right]$$

Coupling constant

$$f(R,\mathcal{G}) = \kappa^{2q}\left(R^q + \mathcal{G}^{q/2}\right)$$

q : Model parameter (constant)

⇨ **Breaking of the conformal invariance of electromagnetic fields (CB)**

- Flat FLRW metric

$$ds^2 = a^2(\eta)\left[-d\eta^2 + d\vec{x}^2\right]$$

$a(\eta)$: Scale factor

η : Conformal time

$(dt = a(\eta)d\eta\ )$

- **Field eqution**

$$\partial_\alpha\left[\sqrt{-g}\left\{g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu} - 8\lambda f(R,\mathcal{G})\ \epsilon^{\mu\nu\alpha\beta}\partial_\mu A_\nu\right\}\right] = 0$$

Coulomb gauge $\partial_i A^i = 0$, $A_0 = 0$

$$\longrightarrow \quad A_i''(\eta,\vec{x}) - \partial_l\partial^l A_i - 8\lambda f'(R,\mathcal{G})\ \epsilon_{ijk}\partial_j A_k = 0$$

- **Quasi de Sitter Inflation**

η_0 : Initial time of inflation

$$N = \int^\eta aH\ d\eta$$

: e-folding number

$$a(\eta) = \left(\frac{-\eta}{\eta_0}\right)^{\beta+1} \qquad \beta = -2 - \epsilon = -3 + \frac{\mathcal{H}'}{\mathcal{H}^2}$$

$$\epsilon = -\frac{\mathcal{H}'}{\mathcal{H}^2} + 1$$

: Slow-roll parameter

$$' = \frac{d}{d\eta}$$

$\mathcal{H} = \frac{a'}{a}$: Conformal Hubble parameter

$$\longrightarrow \quad f(R,\mathcal{G}) = \kappa^{2q}\left\{\frac{\left[6\beta(\beta+1)\right]^q + \left[-24(\beta+1)^3\right]^{q/2}}{\eta_0^{2q}}\right\}\left(\frac{-\eta}{\eta_0}\right)^{2\epsilon q}$$

III. Spectrum of electric and magnetic fields and magnetic helicity

- **Energy-momentum tensor of electromagnetic fields**

$[\nu\nu\rho\sigma]$: antisymmetric permutation

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\alpha\beta}}\left[\sqrt{-g}\left\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \lambda\ f(R,\mathcal{G})\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\right\}\right]$$

$$= -\frac{1}{4}\left\{g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu} - 4g^{\mu\nu}F_{\mu\alpha}F_{\nu\beta}\right\} + \frac{2\lambda}{\sqrt{-g}}\ [\mu\nu\rho\sigma]\ F_{\mu\nu}F_{\rho\sigma}\left(\frac{\delta f(R,\mathcal{G})}{\delta g^{\alpha\beta}}\right)$$

⇨ **Energy density of electric and magnetic fields and the interaction between these fields**

$$T_0^0 = -\frac{1}{2a^4}(A_i')^2 - \frac{1}{4a^4}F_{ij}F_{ij} + \frac{2\lambda q}{a^4}\ \epsilon_{ijk}A_i'F_{jk}\left(\frac{\kappa}{\eta_0}\right)^{2q}\left\{[6\beta(\beta+1)]^q + [-24(\beta+1)^3]^{q/2}\right\}\left(\frac{-\eta}{\eta_0}\right)^{2\epsilon q}$$

$$\rho(\vec{E}) = -\frac{1}{2a^4}(A_i')^2\ , \qquad \rho(\vec{B}) = -\frac{1}{4a^4}F_{ij}F_{ij}$$

$$\rho_{\rm int}(\vec{E},\vec{B}) = \frac{2\lambda q}{a^4}\ \epsilon_{ijk}A_i'F_{jk}\left(\frac{\kappa}{\eta_0}\right)^{2q}\left\{[6\beta(\beta+1)]^q + [-24(\beta+1)^3]^{q/2}\right\}\left(\frac{-\eta}{\eta_0}\right)^{2\epsilon q}$$

$\propto$ non-minimal coupling parameter λ

<Quantization of U(1) gauge fields>

- **Mode expansion** $\hat{A}_i(\eta,\vec{x}) = \int \frac{d\vec{k}}{(2\pi)^3} \sum_{r=1,2} \epsilon_{ri} \left[\hat{b}_r(\vec{k}) A_r(k,\eta) e^{i\vec{k}.\vec{x}} + \hat{b}_r^+(\vec{k}) A_r^*(k,\eta) e^{-i\vec{k}.\vec{x}}\right]$

$$\left[\hat{b}_p(\vec{k}), \hat{b}_r^+(\vec{k}')\right] = \delta_{pr}\delta(\vec{k}-\vec{k}')$$

$\vec{k}$: Comoving wave number

ϵ_{ri} : Polarization vector

$\hat{b}_r(\vec{k})$: Annihilation operator $\quad \hat{b}_r^+(\vec{k})$: Creation operator

- **Energy density** **※Initial condition: Bunch-Davies vacuum**

$$\langle\rho(\vec{E})\rangle = \sum_{r=1,2}\int \frac{dk}{2\pi^2}\frac{k^2}{a^4}\left|A_r'(k,\eta)\right|^2, \qquad \langle\rho(\vec{B})\rangle = \sum_{r=1,2}\int \frac{dk}{2\pi^2}\frac{k^4}{a^4}\left|A_r(k,\eta)\right|^2$$

$$\langle\rho_{\rm int}(\vec{E},\vec{B})\rangle = -4\lambda q\left(\frac{\kappa}{\eta_0}\right)^{2q}\left\{[6\beta(\beta+1)]^q + [-24(\beta+1)^3]^{q/2}\right\}\left(\frac{-\eta}{\eta_0}\right)^{2\epsilon q}\sum_{r=1,2}\int\frac{dk}{2\pi^2}\frac{k^3}{a^4}\left|A_r(k,\eta)A_r'(k,\eta)\right|$$

Energy density of mangnetic helicity

$$\Rightarrow \langle\rho_h\rangle = \int\frac{dk}{2\pi^2}\frac{k^3}{a^3}\left\{\left|A_+(k,\eta)\right|^2 - \left|A_-(k,\eta)\right|^2\right\}$$

$\vec{\epsilon}_+ = \frac{1}{\sqrt{2}}(1,i,0)$, $\vec{\epsilon}_- = \frac{1}{\sqrt{2}}(1,-i,0)$

: Polarization vectors

- **Power spectrum**

$$\mathcal{P}(\vec{E}) = \frac{\partial\langle\rho(\vec{E})\rangle}{\partial\ln k} = \sum_{r=1,2}\frac{k}{2\pi^2}\frac{k^2}{a^4}\left|A_r'(k,\eta)\right|^2, \qquad \mathcal{P}(\vec{B}) = \frac{\partial\langle\rho(\vec{B})\rangle}{\partial\ln k} = \sum_{r=1,2}\frac{k}{2\pi^2}\frac{k^4}{a^4}\left|A_r(k,\eta)\right|^2$$

$$\mathcal{P}_h = \frac{\partial\langle\rho_h\rangle}{\partial\ln k} = \frac{k}{2\pi^2}\frac{k^3}{a^3}\left\{\left|A_+(k,\eta)\right|^2 - \left|A_-(k,\eta)\right|^2\right\}$$

- **Equations of motion of U(1) gauge fields**

$$A_\pm''(k,\eta) + \left(k^2 \mp \frac{Bk}{\eta}\right)A_\pm(k,\eta) = 0, \qquad B = (16\epsilon q\lambda)\left(\frac{\kappa}{\eta_0}\right)^{2q}\left\{[6\beta(\beta+1)]^q + [-24(\beta+1)^3]^{q/2}\right\}$$

<Solutions>

$$A_+(k,\eta) = e^{ik\eta}(-k\eta)\left\{C_1\, U\left(1+\frac{iB}{2},2,-2ik\eta\right) + C_2\, 1F1\left(1+\frac{iB}{2},2,-2ik\eta\right)\right\}$$

$$A_-(k,\eta) = e^{ik\eta}(-k\eta)\left\{C_3\, U\left(1-\frac{iB}{2},2,-2ik\eta\right) + C_4\, 1F1\left(1-\frac{iB}{2},2,-2ik\eta\right)\right\}$$

$C_i\ (i=1,2,3,4)$: integration constants

$U(a,b,z)$: confluent hypergeometric function

$1F1(a,b,z)$:
Kummer confluent hypergeometric function

- **Super-horizon mode:** $|k\eta| \ll 1$

$$\frac{\partial\rho(\vec{E})}{\partial \ln k} = \frac{2H_0^4\cosh(\pi B/2)}{\pi^2|\Gamma(iB/2)|^2}(-k\eta)^4\left[\left(\gamma+\ln(-k\eta)+ln2\right)^2+\frac{B^2}{4}\right], \quad \frac{\partial\rho(\vec{B})}{\partial \ln k} = \frac{H_0^4\cosh(\pi B/2)}{2\pi^2|\Gamma(1+iB/2)|^2}(-k\eta)^4$$

$$\frac{\partial\rho_{\rm int}}{\partial \ln k} = -\frac{1}{B}\left(\frac{\lambda q}{2\pi^2}\right)\left(\frac{\kappa}{\eta_0}\right)^{2q}\left\{[6\beta(\beta+1)]^q+[-24(\beta+1)^3]^{q/2}\right\}\left(\frac{-\eta}{\eta_0}\right)^{2\epsilon q}\left(\frac{H_0^4\cosh(\pi B/2)}{|\Gamma(iB/2)|^2}\right)(-k\eta)^4[\ln(-k\eta)]$$

$$\frac{\partial\langle\rho_h\rangle}{\partial \ln k} = -\frac{1}{2\pi^2}\left(\frac{k^3}{a^3}\right)\left(\frac{\sinh(\pi B/2)}{|\Gamma(1+iB/2)|^2}\right)$$

※Energy density of electromagnetic fields during inflation ($\eta = \eta_c$)

$\rho_{em}(\eta_c) \sim H_0^4$ $\qquad$ $H_0 = 10^{-5}M_{Pl}$: Energy scale of inflation

$\longrightarrow \rho_{em} \ll M_{Pl}^2 H^2$ ⇨ Back reaction at the inflationary stage can be neblected.

$$B_0 = \frac{1}{\sqrt{\pi}}\left\{\frac{\cosh(\pi B/2)}{|\Gamma(1+iB/2)|^2}\right\}^{\frac{1}{2}}\left(\frac{a_f}{a_0}\right)^2 H_0^2(-k\eta_f)^2$$

η_f : Final time of inflation

※Subscription "0" means the present time.

$$\left.\frac{\partial\rho_h}{\partial \ln k}\right|_0 = -\frac{1}{2\pi^2}\left(\frac{a_f}{a_0}\right)^3\left(\frac{\sinh(\pi B/2)}{|\Gamma(1+iB/2)|^2}\right)H_0^3(-k\eta_f)^3$$

$$B = (16\epsilon q\lambda)\left(\frac{\kappa}{\eta_0}\right)^{2q}\left\{[6\beta(\beta+1)]^q+[-24(\beta+1)^3]^{q/2}\right\}$$

⇨ **Present magnetic strength for the case of instantaneous reheating**

$$\left.B_0\right|_{CMB} \approx 10^{-51}\mathrm{G}$$

※Observation: $10^{-22}\mathrm{G} \lesssim B_0 \lesssim 10^{-10}\mathrm{G}$

$k_{CMB} \approx 10^{-40}\mathrm{GeV} \approx 0.02\mathrm{Mpc}^{-1}$

$q = 0.5\,,\ \lambda \sim 1$

$N_f = 58\,,\quad 1\mathrm{G} = 1.95\times10^{-20}\mathrm{GeV}^2$

$H_f = 5.1\times10^{-8}M_{Pl}$

$\frac{a_0}{a_f} \approx 10^{30}\left(H_f/10^{-5}M_{Pl}\right)^{1/2}$

Electric fields are screened because the value of the electric conductivity becomes very large instantaneously.

IV. Present magnetic strength for the case of the reheating phase with a non-zero e-folding number

※Growth of magnetic strength at the reheating stage: [Kobayashi and Sloth, Phys.Rev.D 100, 023524 (2019)]

e-folds number at the reheating stage

[Dai, Kamionkowski and Wang, Phys. Rev. Lett 133, 041302 (2014)]
[Cook, Dimastrogiovanni, Easson and Krauss, JCAP 04(2015), 047]

$$N_{\rm re} = \frac{4}{(1-3\omega_{\rm eff})}\left[-\frac{1}{4}\ln\left(\frac{45}{\pi^2 g_{re}}\right) - \frac{1}{3}\ln\left(\frac{11 g_{s,re}}{43}\right) - \ln\left(\frac{k}{a_0 T_0}\right) - \ln\left(\frac{(3H_f^2 M_{\rm Pl}^2)^{1/4}}{H_0}\right) - N_f\right]$$

$$T_{\rm re} = H_0\left(\frac{43}{11 g_{s,re}}\right)^{\frac{1}{3}}\left(\frac{a_0 T_0}{k}\right)\exp\left[-(N_f + N_{\rm re})\right]$$: Reheating temperature

$T_0 = 2.725{\rm K}$: Present temperature of the cosmic microwave background (CMB) radiation

$\frac{k}{a_0} \approx 0.02{\rm Mpc}^{-1}$ $\quad g_{s,re} = g_{re} \approx 100$

a_0 : Present value of a

$\omega_{\rm eff}$: **Effective equation of state**

※the suffix 're' denotes the end instance of reheating

- **Equations of motion of U(1) gauge fields**

$$A_\pm''^{(re)}(k,\eta) + k^2 A_\pm^{(re)}(k,\eta) = 0$$

$$\longrightarrow \quad A^{(re)}(k,\eta) = \frac{1}{\sqrt{2k}}\left[c_\pm\, e^{-ik(\eta-\eta_f)} + d_\pm\, e^{ik(\eta-\eta_f)}\right]$$

$A^{(re)}(k,\eta)$: Mode function of U(1) gauge fields at the reheating stage

$$c_\pm = \sqrt{\frac{k}{2}}\, A_\pm(k,\eta_f) + \frac{i}{\sqrt{2k}}\, A'_\pm(k,\eta_f), \quad d_\pm = \sqrt{\frac{k}{2}}\, A_\pm(k,\eta_f) - \frac{i}{\sqrt{2k}}\, A'_\pm(k,\eta_f)$$

※Connection condition at the end of inflation

$$A_\pm^{(re)}(k,\eta_f) = A_\pm(k,\eta_f), \quad A_\pm'^{(re)}(k,\eta_f) = A'_\pm(k,\eta_f)$$

- **Bogoliubov coefficients at the reheating stage**

※ **The vacuum of the electromagnetic fields change from the Bunch-Davies vacuum owing to the particle creation at the reheating stage.**

$$\alpha_\pm(k,\eta) = \sqrt{\frac{k}{2}}\, A_\pm^{(re)}(k,\eta) + \frac{i}{\sqrt{2k}}\, A_\pm^{\prime(re)}(k,\eta)$$

$$\beta_\pm(k,\eta) = \sqrt{\frac{k}{2}}\, A_\pm^{(re)}(k,\eta) - \frac{i}{\sqrt{2k}}\, A_\pm^{\prime(re)}(k,\eta)$$

$$\Rightarrow \quad \alpha_\pm(k,\eta) = c_\pm\, e^{-ik(\eta-\eta_f)}, \qquad \beta_\pm(k,\eta) = d_\pm\, e^{ik(\eta-\eta_f)}$$

$\longrightarrow$ Solution: $A_\pm^{(re)}(k,\eta) = \frac{1}{\sqrt{2k}}\left[\alpha_\pm(k,\eta_f)\, e^{-ik(\eta-\eta_f)} + \beta_\pm(k,\eta_f)\, e^{ik(\eta-\eta_f)}\right]$

$$c_+ = -\frac{i}{2}\sqrt{\frac{k}{2}}e^{ik\eta_f}\Big\{2C_2\ 1F1\left(1+\frac{iB}{2},2,-2ik\eta_f\right) - C_2\ k\eta_f\,(2i-B)\ 1F1\left(2+\frac{iB}{2},3,-2ik\eta_f\right)$$
$$+\ 2C_1\left[U\left(\frac{iB}{2},0,-2ik\eta_f\right) - U\left(\frac{iB}{2},1,-2ik\eta_f\right)\right]\Big\}$$

$$c_- = -\frac{i}{2}\sqrt{\frac{k}{2}}e^{ik\eta_f}\Big\{2C_4\ 1F1\left(1-\frac{iB}{2},2,-2ik\eta_f\right) - C_4\ k\eta_f\,(2i+B)\ 1F1\left(2-\frac{iB}{2},3,-2ik\eta_f\right)$$
$$+\ 2C_3\left[U\left(-\frac{iB}{2},0,-2ik\eta_f\right) - U\left(-\frac{iB}{2},1,-2ik\eta_f\right)\right]\Big\}$$

$$d_+ = \frac{i}{2}\sqrt{\frac{k}{2}}e^{ik\eta_f}\Big\{2C_2\ 1F1\left(1+\frac{iB}{2},2,-2ik\eta_f\right) + C_2\ k\eta_f\,(2i+B)\ 1F1\left(1+\frac{iB}{2},3,-2ik\eta_f\right)$$
$$-\ 2C_1\ U\left(\frac{iB}{2},1,-2ik\eta_f\right)\Big\}$$

$$d_- = \frac{i}{2}\sqrt{\frac{k}{2}}e^{ik\eta_f}\Big\{2C_4\ 1F1\left(1-\frac{iB}{2}\quad -2ik\eta_f\right) + C_4\ k\eta_f\,(2i-B)\ 1F1\left(1-\frac{iB}{2},3,-2ik\eta_f\right)$$
$$-\ 2C_3\ U\left(-\frac{iB}{2},1,-2ik\eta_f\right)\Big\}$$

$$C_1 = \frac{2^{1-iB/2}\ i}{\sqrt{2k}}\left(\frac{\Gamma(1+iB/2)}{\Gamma(1-iB/2)}\right)e^{-\pi B/4} \qquad C_2 = \frac{2^{1-iB/2}\ i}{\sqrt{2k}}\Gamma(1+iB/2)\,e^{\pi B/4}$$

$$C_3 = \frac{2^{1+iB/2}\ i}{\sqrt{2k}}\left(\frac{\Gamma(1-iB/2)}{\Gamma(1+iB/2)}\right)e^{\pi B/4} \qquad C_4 = \frac{2^{1+iB/2}\ i}{\sqrt{2k}}\Gamma(1-iB/2)\,e^{-\pi B/4}$$

- **Spectrum of magnetic fields and magnetic helicity at the reheating stage**

$$\left.\frac{\partial\rho(\vec{B})}{\partial\ln k}\right|_{re} = \frac{1}{\pi^2}\left(\frac{k^4}{a_{re}^4}\right)\sum_{r=+,-}|\beta_r(k,\eta_f)|^2\left\{\frac{4}{3\omega_{\rm eff}+1}\left(\frac{k}{a_f H_f}\right)\left[\left(\frac{H_f}{H_{re}}\right)^{\frac{3\omega_{\rm eff}+1}{3\omega_{\rm eff}+3}}-1\right]\right\}^2$$

$$\left.\frac{\partial\rho_h}{\partial\ln k}\right|_{re} = \frac{1}{\pi^2}\left(\frac{k^3}{a_{re}^3}\right)\left(|\beta_+(k,\eta_f)|^2-|\beta_-(k,\eta_f)|^2\right)\left\{\frac{4}{3\omega_{\rm eff}+1}\left(\frac{k}{a_f H_f}\right)\left[\left(\frac{H_f}{H_{re}}\right)^{\frac{3\omega_{\rm eff}+1}{3\omega_{\rm eff}+3}}-1\right]\right\}^2$$

$$H_{re} = H_f\left(\frac{a_{re}}{a_f}\right)^{-\frac{3}{2}(1+\omega_{eff})}, \qquad \ln\left(\frac{a_{re}}{a_f}\right) = N_{\rm re}$$

- **Spectrum of magnetic fields and magnetic helicity at the present time**

$$B_0 = \frac{\sqrt{2}}{\pi}\left(\frac{k}{a_0}\right)^2\sqrt{|\beta_+(k,\eta_f)|^2+|\beta_-(k,\eta_f)|^2}\left\{\frac{4}{3\omega_{\rm eff}+1}\left(\frac{k}{a_f H_f}\right)\left[\left(\frac{H_f}{H_{re}}\right)^{\frac{3\omega_{\rm eff}+1}{3\omega_{\rm eff}+3}}-1\right]\right\}$$

$$\left.\frac{\partial\rho_h}{\partial\ln k}\right|_0 = -\frac{2}{\pi^2}\left(\frac{k^3}{a_0^3}\right)\sinh\left(\frac{\pi B}{2}\right)\left\{\frac{4}{3\omega_{\rm eff}+1}\left(\frac{k}{a_f H_f}\right)\left[\left(\frac{H_f}{H_{re}}\right)^{\frac{3\omega_{\rm eff}+1}{3\omega_{\rm eff}+3}}-1\right]\right\}^2$$
$$\times\left\{[\gamma+\ln(-2k\eta_f)]^2+\left(\frac{B}{2}\right)^2-\left|\Gamma\left(\frac{iB}{2}\right)\Gamma\left(1+\frac{iB}{2}\right)\right|^2\right\}$$

$$\left.\frac{\partial\rho_h}{\partial\ln k}\right|_0 = \left(\frac{a_{re}}{a_0}\right)^3\left.\frac{\partial\rho_h}{\partial\ln k}\right|_{re}, \qquad \left.\frac{\partial\rho(\vec{B})}{\partial\ln k}\right|_0 = \left(\frac{a_{re}}{a_0}\right)^4\left.\frac{\partial\rho(\vec{B})}{\partial\ln k}\right|_{re}, \qquad \left.\frac{\partial\rho(\vec{B})}{\partial\ln k}\right|_0 = \frac{1}{2}B_0^2$$

- **Baryon number density**

$$n_{\rm B} = \frac{e^2}{4\pi^2}\int_{k_0}^{k_{\rm f}} d\ln k \left(\frac{\partial \rho_h}{\partial \ln k}\right)$$

Cf. $n_{\rm B} = -\frac{n_{\rm f}}{2}\Delta n_{\rm CS}\,, \qquad n_{\rm f} = 3$

$$\Delta n_{\rm CS} = -\frac{g'^2}{4\pi^2}\int^t \vec{E}.\vec{B}\; dt$$

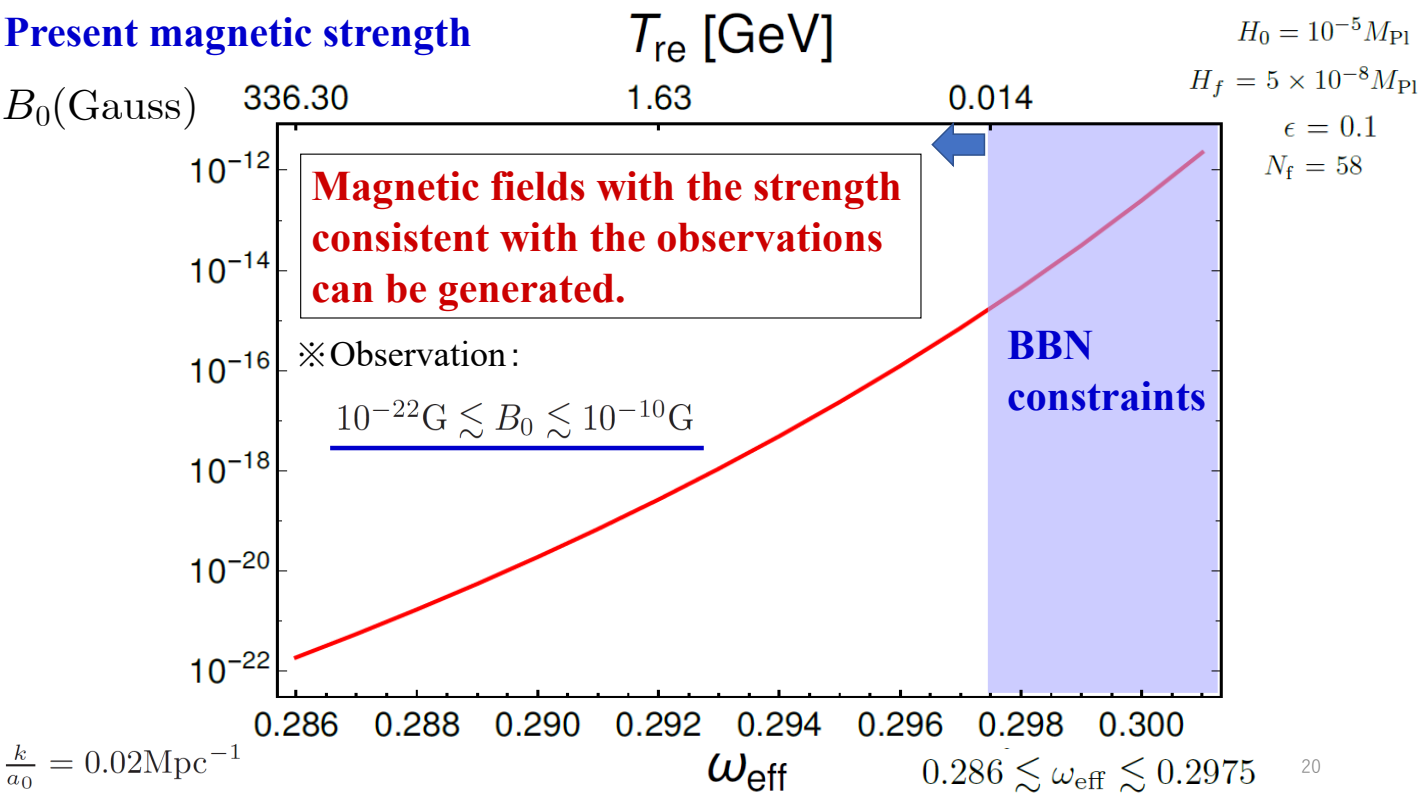
$n_{\rm f}$ is the number of fermionic generations

$g'^2/(4\pi) = \alpha_{\rm EM}$

$$\Rightarrow \quad n_{\rm B} = \frac{e^2}{10^3\pi^4}\left(\frac{k_0}{a_0}\right)^3 \sinh\left(\frac{\pi B}{2}\right)\left\{\frac{4}{3\omega_{\rm eff}+1}\left(\frac{k_0}{a_f H_f}\right)\left[\left(\frac{H_f}{H_{re}}\right)^{\frac{3\omega_{\rm eff}+1}{3\omega_{\rm eff}+3}} - 1\right]\right\}^2 \times \left[20\ln(-k_0\eta_f)\Big(-2+10\gamma+5\ln(-4k_0\eta_f)\Big)\right]$$

γ : Euler's constant $\simeq 0.577216$

Present magnetic strength



$H_0 = 10^{-5} M_{\rm Pl}$

$H_f = 5\times 10^{-8} M_{\rm Pl}$

$\epsilon = 0.1$

$N_{\rm f} = 58$

$\frac{k}{a_0} = 0.02\mathrm{Mpc}^{-1}$

$0.286 \lesssim \omega_{\rm eff} \lesssim 0.2975$

Baryon asymmetry at the present time (Baryon number density/ entropy density)

Baryon asymmetry compatible with the observations can be generated.

Current magnetic field: $B_0^{\rm gal}$ [Gauss]	$n_{\rm B}/s_0$ (Dimensionless)
3.27×10^{-13}	6.09×10^{-10}
3.26×10^{-13}	6.05×10^{-10}
3.28×10^{-13}	6.12×10^{-10}
3.29×10^{-13}	6.14×10^{-10}

$$s_0 = 2.97 \times 10^3 \left(\frac{T_0}{2.75\,[K]}\right)^3 \ {\rm cm}^{-3}$$: Entropy density at the present time

V. Conclusions

- **We have investigated the generation of magnetic fields from inflation for the case that the electromagnetic fields couple with the Ricci scalar and the Gauss-Bonnet invariant.**
- **We have analyzed the evolution of the quantum fluctuations of the electromagnetic fields during inflation due to the breaking of the conformal invariance of the electromagnetic fields.**
- **When the magnetic fields originated from the quantum fluctuations of the electromagnetic fields during inflation have their magnetic helicity, we estimated the baryon number density generated due to the quantum anomaly**
- **It has been shown that the magnetic fields at the galactic scale with strength about 10^{-13} G can lead to the resultant value of the ratio of the baryonic number density to the entropy density as large as $\sim 10^{-10}$, which is consistent with the observational data.**

Cf. Recent important related work

"Baryon isocurvature constraints on the primordial hypermagnetic fields"

[K. Kamada, F. Uchida and J. Yokoyama, JCAP 04 (2021) 034]

Abstract. It has been pointed out that hypermagnetic helicity decay at the electroweak symmetry breaking may have produced the observed baryon asymmetry of the Universe through the chiral anomaly in the standard model of particle physics. Although fully helical magnetic field that can adequately produce the observed baryon asymmetry is not strong enough to explain the origin of the intergalactic magnetic field inferred by the Fermi satellite, the mixture of helical and nonhelical primordial magnetic fields may explain both baryogenesis and the intergalactic magnetic fields simultaneously. We first show that such a scenario is ruled out by the constraint on the amplitude of baryon isocurvature perturbations produced by the primordial magnetic fields to avoid overproduction of deuterium at the big bang nucleosynthesis. Then we show that any attempt to explain the origin of intergalactic magnetic field by primordial magnetogenesis before the electroweak symmetry breaking does not work due to the above constraint irrespective of the helicity and baryogenesis mechanism.

P09

Cartan F(R) gravity and Inflation

This slide is based on *Symmetry* **2022**, *14*(9), 1830【arXiv:2204.01255】

2022.10.24-28　JGRG31

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Masahiko Taniguchi, Tomohiro Inagaki, Hiroki Sakamoto

- **Overview**
- Introduction
- Cartan $F(R)$ gravity
- Cartan $F(R)$ model and Inflation

Overview

Cartan $F(R)$ gravity

$$S = \int d^4x e \left(\frac{{M_{\rm Pl}}^2}{2} F(R) + \mathcal{L}_{\rm m} \right)$$

with Vierbein e^{i}_{μ}

Flat

curved

Torsion

$$T^k{}_{ij} = \frac{1}{2}(\delta^k{}_j e_i{}^\lambda - \delta^k{}_i e_j{}^\lambda)\partial_\lambda \ln F'(R(\Sigma))$$

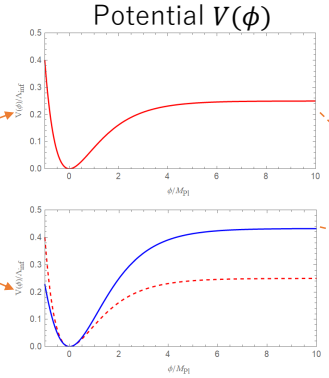
$$S = \int d^4x e \left(\frac{{M_{\rm Pl}}^2}{2} R_E - \frac{1}{2}\partial_\lambda \phi \partial^\lambda \phi - V(\phi) \right)$$

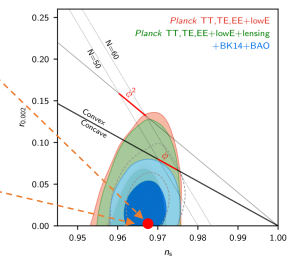
...Derive scalar-tensor theory

$R - R^2$

$R - R\mathrm{Log}(R+1)$

Potential $V(\phi)$





Explain Inflation!

(We think...) **Cartan $F(R)$ gravity is a new theory of $F(R)$ modified theory!!**

Overview

Cartan $F(R)$	VS	Usual $F(R)$
Easy!	Potential $V(\phi)$	A little complicated
$V(\phi) = -\frac{{M_{\rm Pl}}^2}{2} f(R)\|_{R=R(\phi)}$		$V(\phi) = \frac{{M_{\rm Pl}}^2}{2} \frac{AF' - F}{F'^2}\Big\|_{A=A(\phi)}$
One Frame	Scalar-tensor theory	Jordan vs Einstein Frame
From Torsion We are in Jordan Frame		From Conformal transformation We are in Jordan or Einstein ?
Much work remains	Future work	Much work was done
Fresh Theory Please join us		Famous Theory Elite joined it

- Overview
- **Introduction**
- Cartan $F(R)$ gravity
- Cartan $F(R)$ model and Inflation

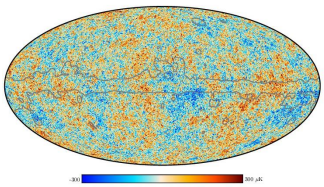
Introduction

☆Horizon problem

CMB $\quad cH_0^{-1} \sim 10^{26}\mathrm{m}, \quad T_0 = 2.715\mathrm{K}, \ |\delta \mathrm{T}/\mathrm{T}_0| \sim 10^{-5}$

↑

Causality



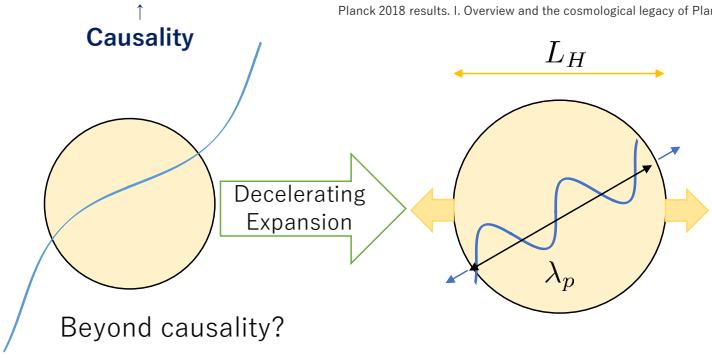
Planck 2018 results. I. Overview and the cosmological legacy of Planck

Decelerating expansion

$$a(t) = a_i t^p \ (0 < p < 1)$$ $a(t)$:scale factor

$$\frac{L_H}{\lambda_p} = \frac{a^{\frac{1}{p}}}{a} > 1$$

L_H ...Hubble radius (Region of Causality)
λ_p ...Physical wavelength



☆Flatness Problem

Density parameter for spatial curvature K

$$|\Omega_K| = \frac{|K|c^2}{a^2 H^2} = \frac{|K|c^2}{{a_i}^2 p^2} t^{2(1-p)}$$ ⋯ Increasing with time

$|\Omega_K^{(0)}| < 0.003$ → Must be 0 with high accuracy in the past

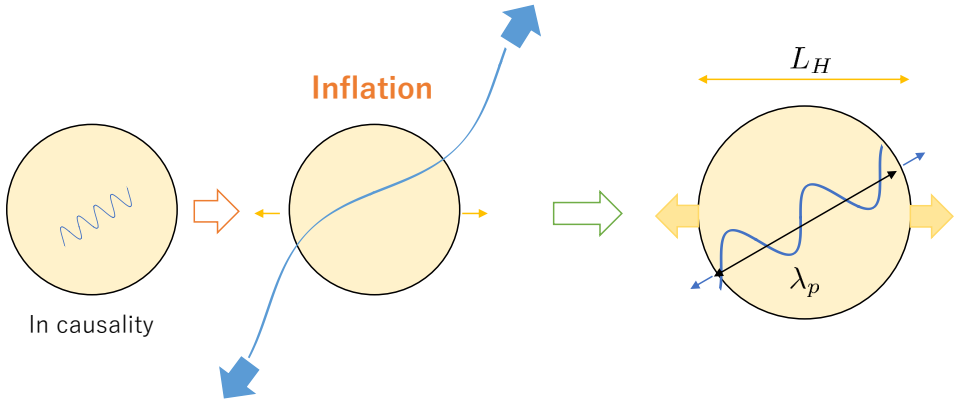
Inflation

Accelerating expansion with $p \gg 1$ in the early universe=Inflation!

☆Horizon problem

$$a(t) = a_i t^p$$

$$\frac{L_H}{\lambda_p} = \frac{a^{\frac{1}{p}}}{a} \ll 1$$



☆Flatness problem

Density parameter for spatial curvature K

$$|\Omega_K| = \frac{|K|c^2}{a^2 H^2} = \frac{|K|c^2}{{a_i}^2 p^2} t^{2(1-p)} \sim t^{-2p}$$ … "Equalized" during inflation

Inflation can solve these problems!

Inflation

The origin of Inflation?

Scalar-Tensor theory

$$S = \int d^4x\sqrt{-g}\left(\frac{{M_{\rm Pl}}^2}{2}R - \frac{1}{2}\partial_\lambda\phi\partial^\lambda\phi - V(\phi)\right) \simeq \int d^4x\sqrt{-g}\left(\frac{{M_{\rm Pl}}^2}{2}R - \Lambda_{\rm inf}\right) \quad \leftarrow V(\phi) \simeq \Lambda_{\rm inf}$$

Friedmann Eq

$$3H^2 \approx V(\phi) = \Lambda_{\rm inf}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \Lambda_{\rm inf}$$

$$a(t) \propto e^{\sqrt{\frac{\Lambda_{\rm inf}}{3}}t}$$

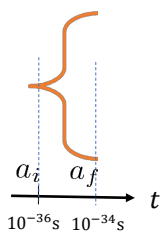
Exponential accelerating expansion

e-folding number = the scale of expansion

$$N = \ln\frac{a_f}{a_i}$$

To solve…
Horizon problem
Flatness problem

$$N \sim 50 - 65 \quad \rightarrow e^{60} \sim 10^{26}$$



Cell(10μm)
Credit:いらすとや

M31: The Andromeda Galaxy(10^{21}m)
Image Credit & Copyright: Robert Gendler

Inflation

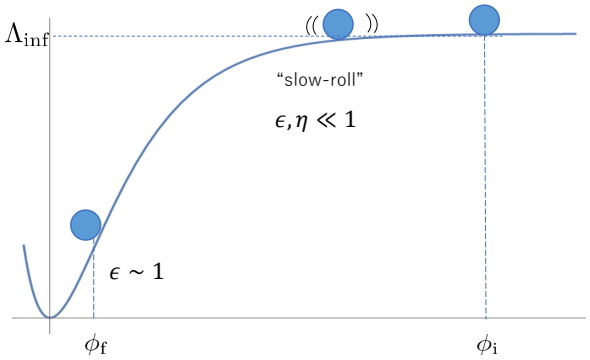
○Slow-roll parameter

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi),\ \dot{\phi}^2 \ll V(\phi)$$ $V(\Phi)$ behaves constant, $\Lambda_{\rm inf}$

$$\epsilon = \frac{{M_{\rm Pl}}^2}{2}\left(\frac{V_\phi}{V}\right)^2,\ \eta = \frac{{M_{\rm Pl}}^2 V_{\phi\phi}}{V}$$

During inflation $\epsilon, \eta \ll 1$

Finish inflation $\epsilon, \eta \sim 1$



○Fluctuation

$$\phi + \delta\phi \rightarrow V(\phi) \rightarrow \delta N = N - N_0 = \mathcal{R}(t_f)$$

Curvature fluctuation

$$\delta\phi = \dot{\phi}\delta t$$

$$\mathcal{R}(t_f) = H\delta t = \frac{H}{\dot{\phi}}\delta\phi$$

Power spectrum

$$\mathcal{P}_{\mathcal{R}}(k)$$

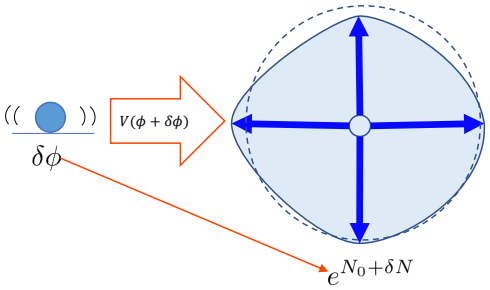
Spectrum index

$$n_s - 1 = \left.\frac{d\ln\mathcal{P}_{\mathcal{R}}}{d\ln k}\right|_{k=aH} = -6\epsilon + 2\eta$$

Scalar-tensor ratio

$$r = \frac{\mathcal{P}_h}{\mathcal{P}_{\mathcal{R}}} = 16\epsilon$$

expressed by ϵ, η



Inflation

What potential is good?

$$V(\phi) \propto \phi^2,\ \phi^{4/3},\ \phi,\ \phi^{2/3}$$

Natural inflation

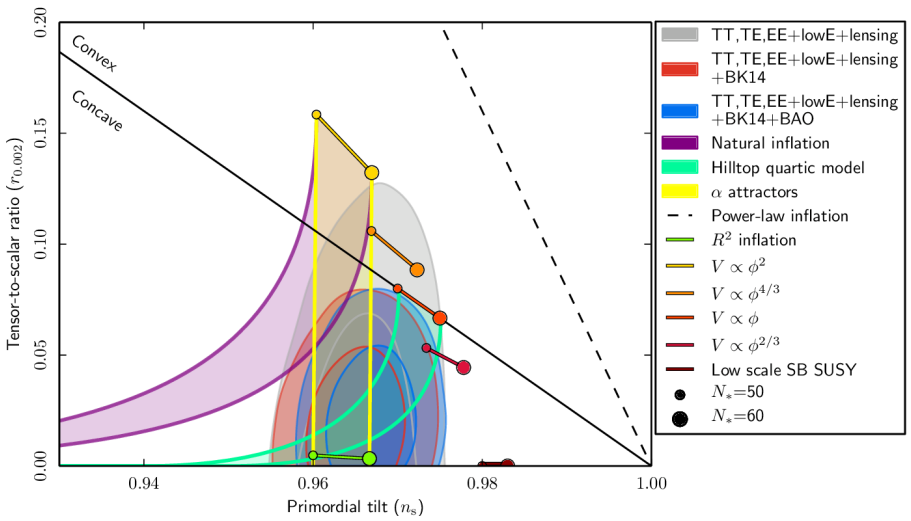
$$V(\phi) \propto \left(1 - \cos\left(\frac{\phi}{f}\right)\right)$$

Hilltop inflation

$$V(\phi) \propto \left[1 - \lambda\left(\frac{\phi}{v}\right)^q\right]^2$$

R^2(Starobinsky) inflation

$$V(\phi) \propto \left(1 - e^{-\sqrt{\frac{2}{3}}\phi}\right)^2$$



Planck 2018 results. X. Constraints on inflation

↑ Most consistent with observation ⇒ What is the origin?

$F(R)$ modified gravity

$R \to F(R)$ In Einstein-Hilbert action, $S = \int d^4x\sqrt{-g}\frac{{M_{\rm Pl}}^2}{2}R$

$R \to F(R)$

Starobinsky model A. A. Starobinsky (1979)

$$F(R) = R + \frac{R^2}{M^2}$$

Conformal transformation

$$g'_{\mu\nu} = F' g_{\mu\nu}$$

$\longrightarrow$ Equivalent scalar-tensor theory

$$S = \int d^4x\sqrt{-g}\left(\frac{{M_{\rm Pl}}^2}{2}R - \frac{1}{2}\partial_\lambda\phi\partial^\lambda\phi - V(\phi)\right)$$

$$\phi \equiv \sqrt{\frac{3}{2}}M_{\rm Pl}\ln F'(A)$$

$$V(\phi) = \left.\frac{AF' - F}{F'^2}\right|_{A=A(\phi)}$$

$\longrightarrow$

$$F'(A) = e^{\sqrt{\frac{2}{3}}\frac{\phi}{M_{\rm Pl}}},\ A = M^2\left(e^{\sqrt{\frac{2}{3}}\frac{\phi}{M_{\rm Pl}}} - 1\right)$$

$$V(\phi) = \left.\frac{{M_{\rm Pl}}^2}{2}\frac{AF' - F}{F'^2}\right|_{A=A(\phi)} = \underline{\frac{M^2{M_{\rm Pl}}^2}{8}\left(1 - e^{-\sqrt{\frac{2}{3}}\phi}\right)^2}$$

Starobinsky potential

$F(R)$ modified gravity

$R \to F(R)$

Conformal transformation = Local scale transformation S. Nojiri, S. Odintsov, and V. Oikonomou, Phys. Rept. 692, 1 (2017).

Jordan Frame

$g_{\mu\nu}$

$F(R)$

Conformal trans $\Longrightarrow$

Einstein Frame

$g'_{\mu\nu} = F' g_{\mu\nu}$

R(Tensor) +Scalar field

$$S = \int d^4x\sqrt{-g}\frac{{M_{\rm Pl}}^2}{2}F(R)$$

H. A. Buchdahl (1970)

$$S = \int d^4x\sqrt{-g'}\left(\frac{{M_{\rm Pl}}^2}{2}R - \frac{1}{2}\partial_\lambda\phi\partial^\lambda\phi - V(\phi)\right)$$

$$\phi \equiv \sqrt{\frac{3}{2}}M_{\rm Pl}\ln F'(A)$$

$$V(\phi) = \left.\frac{AF' - F}{F'^2}\right|_{A=A(\phi)}$$

Are the two frames equivalent?

Some studies discussed...

R. Catena, M. Pietroni, and L. Scarabel (2007)
C. F. Steinwachs and A. Y. Kamens (2011)
A. Y. Kamenshchik and C. F. Steinw (2015)
Yuta Hamada, Hikaru Kawai, Yukari Nakanishi, and Kin-ya Oda (2017)

- Overview
- Introduction
- **Cartan $F(R)$ gravity**
- Cartan $F(R)$ model and Inflation

Cartan formalism

Cartan formalism : $g_{\mu\nu} \to e^{i}{}_{\mu}$ E. Cartan (1923)

$$e^{i}{}_{\mu}$$

i ← Flat(local Lorentz frame)

μ ← Curved(general coordinate frame)

$$\underbrace{g_{\mu\nu}}_{\text{curved}} = \underbrace{\eta_{ij}}_{\text{flat}} e^{i}{}_{\mu} e^{j}{}_{\nu}$$

Vierbein: a "bridge" between curved and flat space-time

Gravity ⇄ Quantum filed

Torsion

$$\boxed{T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu}}$$

Connection : $\Gamma^{\rho}{}_{\mu\nu} = e_{a}{}^{\rho} D_{\nu} e^{a}{}_{\mu},$

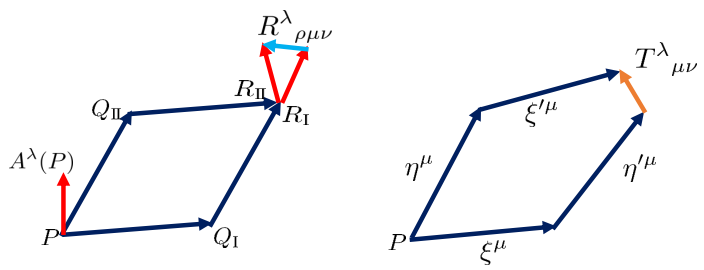
Covariant Derivative

for local Lorentz trans

$$D_{\nu} e^{k}{}_{\mu} = \partial_{\nu} e^{a}{}_{\mu} + \omega^{k}{}_{l\nu} e^{l}{}_{\mu}$$

↑ spin connection

Geometric image



Cartan $F(R)$ gravity

Action

$$S = \int d^4x e \left(\frac{{M_{\rm Pl}}^2}{2} F(R) + \mathcal{L}_{\rm m} \right)$$

$$e = \det[e^i{}_\mu] = \sqrt{-g}$$

○**Modified Einstein Eq** ⇐ Varying with respect to $e^i{}_\mu$

$$F' R^i{}_\mu - \frac{1}{2} e^i{}_\mu F(R) = {M_{\rm Pl}}^{-2} \Sigma^i{}_\mu$$

$$\to R(\Sigma)$$

○**Cartan Eq** ⇐ Varying with respect to $e^i{}_\mu$ ω

※$\mathcal{L}_m$ does not depend on matter

$$(T^\mu{}_{kl} - e_l{}^\mu T_k + e_k{}^\mu T_l) F'(R) + (e_k{}^\alpha e_l{}^\mu - e_k{}^\mu e_l{}^\alpha) \partial_\alpha F'(R) = 0$$

$$\to T^k{}_{ij} = \frac{1}{2} (\delta^k{}_j e_i{}^\lambda - \delta^k{}_i e_j{}^\lambda) \partial_\lambda \ln F'(R)$$

Torsion is related to $F(R)$ modification

$$T^k{}_{ij} = \frac{1}{2} (\delta^k{}_j e_i{}^\lambda - \delta^k{}_i e_j{}^\lambda) \partial_\lambda \ln F'(R(\Sigma))$$

※ $F(R) = R \to T^k{}_{ij} = 0$

Cartan $F(R)$ gravity

Extract Torsion effect

$$\begin{cases} \Gamma^\lambda{}_{\mu\nu} = (\Gamma_E)^\lambda{}_{\mu\nu} + \frac{1}{2} \left(T^\lambda{}_{\mu\nu} + T_{\mu\nu}{}^\lambda + T_{\nu\mu}{}^\lambda \right) \\ \omega_{ijk} = \underbrace{\omega_{\diamond ijk}}_{\text{Non-torsion}} + \underbrace{\frac{1}{2} (T_{ijk} + T_{jki} + T_{kji})}_{\text{Torsion}} \end{cases}$$

Levi-Civita connection : $(\Gamma_E)^\lambda{}_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$

$$\omega_{\diamond ijk} = \frac{1}{2} (\Delta_{kij} - \Delta_{ijk} + \Delta_{jik}), \quad \Delta_{kij} = (e_i{}^\mu e_j{}^\nu - e_j{}^\mu e_i{}^\nu) \partial_\nu e_{k\mu}$$

Ricci scalar

$$R = e_i{}^\mu e_j{}^\nu R^{ij}{}_{\mu\nu}(\omega, \partial\omega), \quad R^{ij}{}_{\mu\nu} = \partial_\mu \omega^{ij}{}_\nu - \partial_\nu \omega^{ij}{}_\mu + \omega^i{}_{k\mu} \omega^{kj}{}_\nu - \omega^i{}_{k\nu} \omega^{kj}{}_\mu$$

$$= R_E + T - 2\nabla_{E\mu} T^\mu$$

$$T = \frac{1}{4} T^{\rho\mu\nu} T_{\rho\mu\nu} - \frac{1}{4} T^{\rho\mu\nu} T_{\mu\nu\rho} - \frac{1}{4} T^{\rho\mu\nu} T_{\nu\rho\mu} - T^\mu T_\mu$$

$$= R_E - \frac{3}{2} \partial_\lambda \ln F'(R) \partial^\lambda \ln F'(R) - 3\Box \ln F'(R) \longleftarrow T^k{}_{ij} = \frac{1}{2} (\delta^k{}_j e_i{}^\lambda - \delta^k{}_i e_j{}^\lambda) \partial_\lambda \ln F'(R(\Sigma))$$

Cartan $F(R)$ gravity

$$\Gamma^{\lambda}{}_{\mu\nu} = (\Gamma_E)^{\lambda}{}_{\mu\nu} + \frac{1}{2}\left(T^{\lambda}{}_{\mu\nu} + T_{\mu\nu}{}^{\lambda} + T_{\nu\mu}{}^{\lambda}\right)$$

$$\omega_{ijk} = \omega_{\diamond ijk} \quad + \frac{1}{2}\left(T_{ijk} + T_{jki} + T_{kji}\right)$$

$$R = R_E - \frac{3}{2}\partial_\lambda \ln F'(R)\partial^\lambda \ln F'(R) - 3\Box \ln F'(R)$$

Apply $F(R) = R + f(R)$

$$S = \int d^4x e \frac{{M_{\rm Pl}}^2}{2}\left(R + f(R)\right)$$

$$= \int d^4x e \left(\frac{{M_{\rm Pl}}^2}{2} R_E - \frac{1}{2}\partial_\lambda \phi \partial^\lambda \phi - V(\phi)\right)$$

$$V(\phi) = -\frac{{M_{\rm Pl}}^2}{2} f(R)|_{R=R(\phi)}$$

Define the scalar field

$$\phi \equiv -\sqrt{\frac{3}{2}} M_{\rm Pl} \ln F'(R)$$

Derive scalar-tensor theory from Torsion

⇒**Without Conformal transformation**

×**Jordan vs Einstein**

Cartan $F(R)$ gravity…A new form of $F(R)$ gravity

- Overview
- Introduction
- Cartan $F(R)$ gravity
- **Cartan $F(R)$ model and Inflation**

R^2 model

An Example of Cartan $F(R)$ model

$$F(R) = R - \frac{R^2}{M^2}, \quad f(R) = -\frac{R^2}{M^2}$$

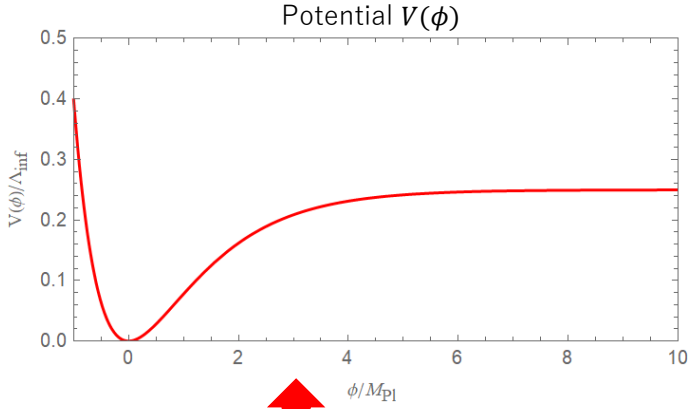
STEP.1

$$\begin{cases} \phi = -\sqrt{\frac{3}{2}} M_{\rm Pl} \ln F'(R) & \text{From scalar definition} \\ F'(R) = e^{-\sqrt{\frac{2}{3}}\phi} \\ 1 - \frac{2R}{M^2} = e^{-\sqrt{\frac{2}{3}}\phi} \\ R(\phi) = \frac{M^2}{2}\left(1 - e^{-\sqrt{\frac{2}{3}}\phi}\right) \end{cases}$$

STEP.2

$$\begin{cases} V(\phi) = -\frac{{M_{\rm Pl}}^2}{2} f(R)|_{R=R(\phi)} & \text{From potential definition} \\ \quad = \frac{{M_{\rm Pl}}^2}{2} \frac{R^2}{M^2}\Big|_{R=R(\phi)} = \underline{\frac{M^2 {M_{\rm Pl}}^2}{8}\left(1 - e^{-\sqrt{\frac{2}{3}}\phi}\right)^2} \end{cases}$$

Potential $V(\phi)$



Starobinsky potential again!

Logarithmic model

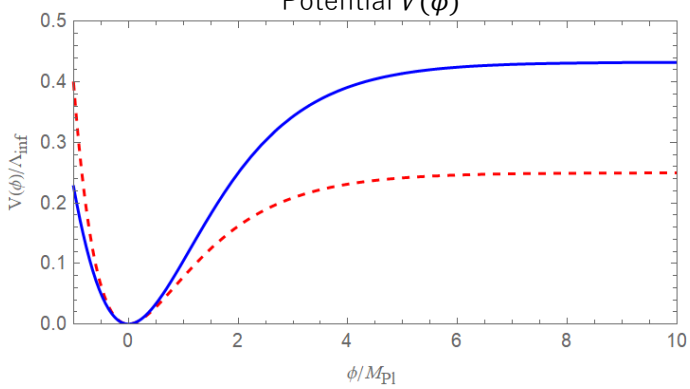
$$F(R) = R\left(1 - \ln\left[\frac{R}{R_c} + 1\right]\right), \quad f(R) = -R \ln\left[\frac{R}{R_c} + 1\right]$$

Inspired by quantum correction

$$\phi(R) = -\sqrt{\frac{3}{2}} M_{\rm Pl} \ln F'(R)$$

$$V(R) = -\frac{{M_{\rm Pl}}^2}{2} f(R)$$

Potential $V(\phi)$



Plotting the potential with R as a parameter

Logarithmic model

R^2 model

Root model

$$F(R) = \frac{M^2}{2}\left(\sqrt{\frac{4R}{M^2}+1}-1\right)$$

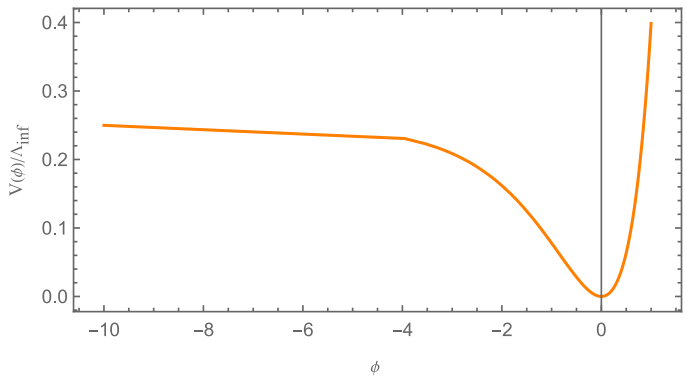
$$= R - \frac{R^2}{M^2} + \frac{2R^3}{M^4} - \frac{5R^4}{M^6} + \ldots$$

The form of $R + f(R)$

$$V(\phi) = -\frac{{M_{\rm Pl}}^2}{2} f(R)|_{R=R(\phi)}$$
$$= \frac{M^2 {M_{\rm Pl}}^2}{8}\left(1 - e^{\sqrt{\frac{2}{3}}\phi}\right)^2$$

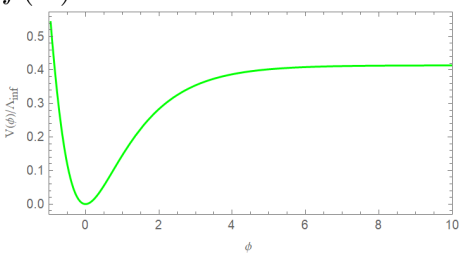
Left-right symmetrical Starobinsky potential

Potential $V(\phi)$

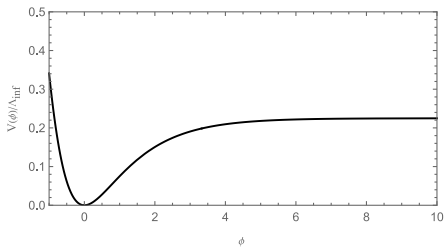


Others

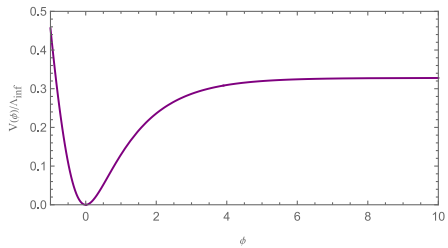
$f(R) = 1 - \cosh R$



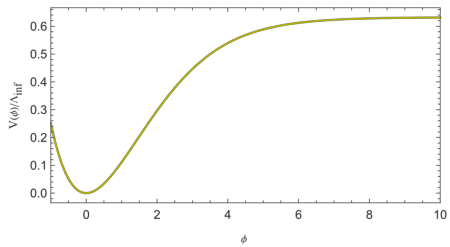
$f(R) = -R\sinh R$



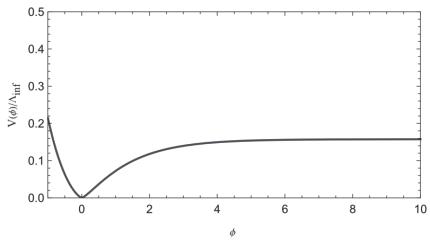
$f(R) = 1 - e^{R^2/2}$



$f(R) = Re^{-R} - R$



$f(R) = -R^4$



The potential is easy,

$$V(\phi) = -\frac{{M_{\rm Pl}}^2}{2} f(R)|_{R=R(\phi)}$$

Easy to find models

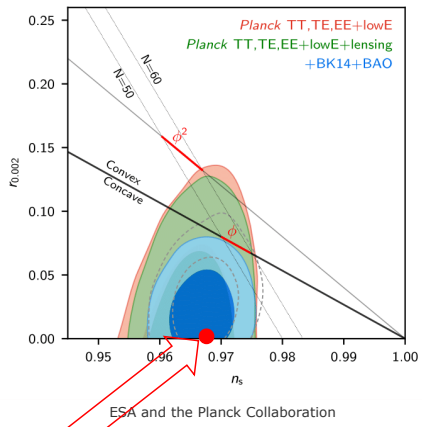
Observation

Translation to $F(R)$ gravity

$$N = \int \frac{V}{V_\phi} d\phi$$
$$\epsilon = \frac{{M_{\rm Pl}}^2}{2}\left(\frac{V_\phi}{V}\right)^2$$
$$\eta = \frac{{M_{\rm Pl}}^2 V_{\phi\phi}}{V}$$

$V(R) = -\frac{{M_{\rm Pl}}^2}{2} f(R)$

$$N = \frac{3}{2}\int_{R_f}^{R} \frac{f}{f'}\left(\frac{F''}{F'}\right)^2 dR$$
$$\epsilon = \frac{1}{3}\left(\frac{f'F'}{fF''}\right)^2$$
$$\eta = \frac{2}{3f}\left[\frac{f''F'^2}{F''^2} + \frac{f'F'}{F''}\left(1 - \frac{F'F'''}{F''^2}\right)\right]$$



ESA and the Planck Collaboration

Set the e-folding number $N \sim 60$
Calculation of observables such as n_s, r

Model	e-folding N	n_s	$r(\times 10^{-3})$	$\alpha_s(\times 10^{-3})$
$-R^2$	60.3	0.96797	2.94	-0.519
$-R\ln(R+1)$	60.2	0.96758	3.04	-0.529
$1-\cosh R$	60.3	0.96825	2.86	-0.512
Observation	-	0.9625(48)	<66	2 ± 10

Summary

Cartan $F(R)$ gravity

Derive scalar-tensor theory by torsion

$$S = \int d^4x e\left(\frac{{M_{\rm Pl}}^2}{2}F(R) + \mathcal{L}_{\rm m}\right) \xrightarrow[\text{Torsion}]{} S = \int d^4x e\left(\frac{{M_{\rm Pl}}^2}{2}R_E - \frac{1}{2}\partial_\lambda\phi\partial^\lambda\phi - V(\phi) + \mathcal{L}_m\right)$$

$$T^k{}_{ij} = \frac{1}{2}(\delta^k{}_j e_i{}^\lambda - \delta^k{}_i e_j{}^\lambda)\partial_\lambda \ln F'(R(\Sigma))$$

× Conformal transformation

Thank you for taking the time to look at the last slide!

We will study

- Model exploration
- Reheating (Matter interaction) etc.

We would appreciate your advice.

Please check our paper

Poster session [P10]

Black hole perturbations in modified gravity with two tensorial degrees of freedom

Rikkyo University Jin Saito
Collaborator: Tsutomu Kobayashi

Work in progress

2021 10/24-10/28 JGRG31 in Tokyo University

Outline

I. Introduction

II. Modified gravity
with two tensorial degrees of freedom

III. Static and spherically symmetric black hole solution

IV. Black hole perturbations

- ✦Odd mode $(l \geq 2)$
- ✦Even mode $(l = 0)$
- ✦Even mode $(l = 1)$

Our work

V. Summary and future work

Introduction

* General relativity (GR) is standard theory of gravity
  - This is consistent with observations such as deflection of light, Mercury's perihelion shift.

Why modified gravity?

* Late time accelerated expansion of the Universe
  - ΛCDM or modified gravity?
* Verification of gravitational theory
  - To what extent is GR correct?

Modified gravity with two tensorial DoFs

Gravitational theories which have two tensorial DoFs are not limited on GR!

* Spatially covariant theory

Extrinsic curvature

$$K_{ij} := \frac{1}{2N}(\dot{\gamma}_{ij} - D_i N_j - D_j N_i)$$

3-dim induced metric

$$S = \int \mathrm{d}t\mathrm{d}x^3 N\sqrt{\gamma}\mathcal{L}(N, \gamma_{ij}, K_{ij}, R_{ij}, D_i, t)$$

lapse function

3-dim (intrinsic)curvature

3-dim covariant derivative

* Written in terms of ADM variables, which are geometrical quantity of 3-dim space-like hypersurface.
* In general, this theory has 3 DoFs (2 tensors and 1 scalar).
* This is equivalent to scalar-tensor theory under unitary gauge.

Modified gravity with two tensorial DoFs

(Gao, Yao 2020)

Hamiltonian analysis yields the conditions for a theory to have only two tensorial DoFs,

$$S = \frac{1}{2}\int \mathrm{d}t\mathrm{d}^3x N\sqrt{\gamma}\left[\frac{\beta_0 N}{\beta_2+N}K_{ij}K^{ij} - \frac{\beta_0}{3}\left(\frac{2N}{\beta_1+N}+\frac{N}{\beta_2+N}\right)K^2 + \alpha_1 + \alpha_2 R + \frac{1}{N}(\alpha_3+\alpha_4 R)\right] \qquad \alpha_a, \beta_a = \alpha_a(t), \beta_a(t)$$

- $\beta_0 = \alpha_2 = 1, \alpha_1 = \mathrm{Const}, \beta_1 = \alpha_3 = \alpha_4 = 0 \longrightarrow$ GR
- $\beta_1 = \beta_2 \longrightarrow$ Extended cuscuton

Evading solar-system test and propagation speed of GWs,

$$S = \frac{1}{2}\int \mathrm{d}t\,\mathrm{d}x^3\, N\sqrt{\gamma}\left[K_{ij}K^{ij} - \frac{1}{3}\left(\frac{2N}{\beta+N}+1\right)K^2 + R^{(3)} + \alpha_1 + \frac{\alpha_3}{N}\right]$$

If ignoring α_1, α_3 as contributing only cosmological scale, the difference between two theories is only one parameter β.

Static and spherically symmetric BH solution

(Iyonaga, Kobayashi 2021)

- Metric ansatz

$$\mathrm{d}s^2 = -(N^2 - B^2)\mathrm{d}t^2 + 2BF\mathrm{d}t\mathrm{d}r + F^2\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

$$N = N(r), \quad N_i\mathrm{d}x^i = B(r)F(r)dr, \quad \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j = F^2(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

- Particular solution

$$N = N_0\sqrt{f(r)}, \quad F = \frac{1}{\sqrt{f(r)}}, \quad B = \frac{N_0 b_0}{r^2}$$

$$f(r) := 1 - \frac{\mu_0}{r} + \frac{b_0^2}{r^4} \qquad \mu_0, N_0, b_0\text{:Constants}$$

If we diagonalized this metric by introducing Stuckerberg scalar,

$$\mathrm{d}s^2 = -h(r)\mathrm{d}T^2 + \frac{\mathrm{d}r^2}{h(r)} + r^2\mathrm{d}\Omega^2 \qquad h(r) = 1 - \frac{\mu_0}{r}$$

$$\phi = \frac{1}{N_0}\left(T + \int^r \frac{b_0/r^2}{h(r)\sqrt{h(r)+b_0^2/r^4}}\mathrm{d}r\right)$$

This is a stealth Schwarzschild solution.

Black hole perturbations

* Remark:
  - Background metric:

$$\mathrm{d}s^2 = -(N^2 - B^2)\mathrm{d}t^2 + 2BF\mathrm{d}t\mathrm{d}r + F^2\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

  - Action:

$$S = \frac{1}{2}\int \mathrm{d}t\,\mathrm{d}x^3\, N\sqrt{\gamma}\left[K_{ij}K^{ij} - \frac{1}{3}\left(\frac{2N}{\beta + N} + 1\right)K^2 + R^{(3)} + \underbrace{\cancel{\alpha_1} + \cancel{\frac{\alpha_2}{N}}}\right]$$

Ignoring those terms

By analyzing phenomena at black hole, we would like to know whether this theory is distinguishable from GR or not.

Odd parity sector $(l \geq 2)$

ϵ_{ab} :Levi-Civita tensor $(\epsilon_{\theta\theta} = \sin\theta)$
∇_a :Covariant derivative of σ_{ab}
$\sigma_{ab}\mathrm{d}x^a\mathrm{d}x^b = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$

* Perturbations

$$N \longrightarrow N$$

$$N_a \longrightarrow N_a + \sum_{l,m} h_0^{(lm)}(t,r)\epsilon^b{}_a\partial_b Y_{lm}(\theta,\phi)$$

$$\gamma_{ra} \longrightarrow \gamma_{ra} + \sum_{l,m} h_1^{(lm)}(t,r)\epsilon^b{}_a\partial_b Y_{lm}(\theta,\phi)$$

$$\gamma_{ab} \longrightarrow \gamma_{ab} + \frac{1}{2}\sum_{l,m} h_2^{(lm)}(t,r)(\epsilon_a{}^c\nabla_c\nabla_b + \epsilon_b{}^c\nabla_c\nabla_a)Y_{lm}(\theta,\phi)$$

* Gauge transformation

Under $x_a \to x_a + \sum_{l,m}\epsilon^b{}_a\partial_b Y_{lm}(\theta,\phi)\,, \qquad h_2^{(lm)} \to h_2^{(lm)} - 2\xi^{(lm)}$

$\longrightarrow$ We can always eliminate $h_2^{(lm)}$.

We consider only $h_0^{(lm)}, h_1^{(lm)}$.

Odd parity sector $(l \geq 2)$

$c_l := (l-1)l(l+1)(l+2)$
$h_0, h_1 := h_0^{(lm)}, h_1^{(lm)}$

* 2nd-order Lagrangian:

$$\mathcal{L}^{(2)}_{\rm odd} = \frac{1}{2}\left[\frac{2l(l+1)}{r^2FN} + \frac{c_l}{2}\frac{F}{r^2N}\right]|h_0|^2 - \frac{1}{2}\frac{l(l+1)}{rFN}(|h_0|^2)' + \frac{c_l}{4}\frac{B^2}{r^2FN}|h_1|^2$$
$$- \frac{c_l}{2}\frac{B}{r^2N}h_1h_0^* + \frac{1}{2}\frac{l(l+1)}{2}\frac{1}{FN}\left[\left|\dot{h}_1 - h_0'\right|^2 + \frac{4}{r}h_0h_1^*\right] + c.c.$$

→ β -dependence disappears in $\mathcal{L}^{(2)}_{\rm odd}$.

Master equation is identical to Regge-Wheeler equation in GR.

With odd parity sector, we can't distinguish between this theory and GR.

Even parity sector $(l = 0)$

$Y_{00}(\theta, \phi) = \frac{1}{\sqrt{4\pi}}$
$\sigma_{ab}\mathrm{d}x^a\mathrm{d}x^b = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$

* Perturbations

$$N \to N + N\sum_{l,m}\frac{\mathsf{H}_0^{(00)}(t,r)}{\sqrt{4\pi}}, \qquad N_r \to N_r + \sum_{l,m}\frac{\mathsf{H}_1^{(00)}(t,r)}{\sqrt{4\pi}}$$

$$N_a \to N_a, \qquad \gamma_{rr} \to \gamma_{rr} + \sum_{l,m}\frac{\mathsf{H}_2^{(00)}(t,r)}{\sqrt{4\pi}}$$

$$\gamma_{ra} \to \gamma_{ra}, \qquad \gamma_{ab} \to \gamma_{ab} + r^2\sum_{l,m}\sigma_{ab}\frac{\mathsf{K}^{(00)}(t,r)}{\sqrt{4\pi}}$$

* Gauge transformation

Under $r \to r + \frac{\xi_r^{(00)}}{\sqrt{4\pi}}$, $\qquad \mathsf{K}^{(00)} \to \mathsf{K}^{(00)} - \frac{2}{r}\xi_r^{(00)}$

→ We can always eliminate $\mathsf{K}^{(00)}$.

We consider only $\mathsf{H}_0^{(00)}, \mathsf{H}_1^{(00)}, \mathsf{H}_2^{(00)}$

Even parity sector $(l=0)$

* 2nd-order Lagrangian: $(\mathsf{H}_0, \mathsf{H}_1, \mathsf{H}_2 = \mathsf{H}_0^{(00)}, \mathsf{H}_1^{(00)}, \mathsf{H}_2^{(00)})$

$$\mathcal{L}_{l=0}^{(2)} = \frac{\beta r^2}{12N_0(\beta + N_0\sqrt{f})}\left[\frac{\dot{\mathsf{H}}_2}{\sqrt{f}} - \frac{2N_0 b_0}{r^2}\widetilde{\mathsf{H}}_0' - \frac{2}{r^2}\left(r^2\widetilde{\mathsf{H}}_1\right)'\right]^2 - \frac{2b_0}{r}\widetilde{\mathsf{H}}_0'\widetilde{\mathsf{H}}_1 - N_0 r f \widetilde{\mathsf{H}}_0'\mathsf{H}_2 - \frac{r}{N_0\sqrt{f}}\mathsf{H}_2\dot{\widetilde{\mathsf{H}}}_1$$

$$\widetilde{\mathsf{H}}_0 := \mathsf{H}_0 + \frac{1}{2}\mathsf{H}_2 \ , \ \widetilde{\mathsf{H}}_1 := \sqrt{f}\mathsf{H}_1 - \frac{N_0 b_0}{r^2}(\mathsf{H}_0 + \mathsf{H}_2)$$

Solve the equations for $\widetilde{\mathsf{H}}_0, \widetilde{\mathsf{H}}_1, \widetilde{\mathsf{H}}_2$ and rewrite the solutions in terms of $\mathsf{H}_0, \mathsf{H}_1, \mathsf{H}_2$,

$$\mathsf{H}_0 = \frac{b_0 C}{N_0 r^4 f} - \frac{\dot{C}}{N_0^2}\int\frac{\mathrm{d}r}{r^2 f^{3/2}}, \ \mathsf{H}_1 = \frac{C}{r^2 f^{1/2}} - \frac{b_0^2 C}{r^6 f^{3/2}} - \frac{b_0\dot{C}}{N_0 r^4 f^{1/2}}\int\frac{\mathrm{d}r}{r^2 f^{3/2}}, \ \mathsf{H}_2 = -\frac{\delta\mu}{rf} - \frac{2b_0 C}{N_0 r^4 f}$$

where $\delta\mu$ is constant and C is function depending time.

Even parity sector $(l=0)$

* $\delta\mu$ ⟶ Shift of the black hole mass $\mu_0 \to \mu_0 + \delta\mu$
* C ⟶ Change the configuration of Stuckerberg scalar without changing geometrical structures

If considering the coordinate transformation

$$T \to \widetilde{T} = T + \delta T \ , \ \delta T = -\frac{C}{\sqrt{4\pi}N_0}\int\frac{\mathrm{d}r}{r^2 f(r)^{3/2}} \ ,$$

we obtain the metric and scalar field

$$\mathrm{d}s^2 = -h(r)\mathrm{d}\widetilde{T}^2 + \frac{\mathrm{d}r^2}{h(r)} + r^2\mathrm{d}\Omega^2$$

$$\widetilde{\phi} = \underbrace{T - \frac{C}{\sqrt{4\pi}N_0}\int\frac{\mathrm{d}r}{r^2 f(r)^{3/2}}}_{=\widetilde{T}} + \int\frac{b_0/r^2}{h(r)\sqrt{h(r) + b_0/r^4}}\mathrm{d}r$$

Even parity sector$(l=1)$

* Perturbations

$$N \to N + N\sum_m \mathsf{H}_0^{(1m)}(t,r)Y_{1m}(\theta,\phi) \quad , \quad N_r \to N_r + \sum_m \mathsf{H}_1^{(1m)}(t,r)Y_{1m}(\theta,\phi)$$

$$\gamma_{rr} \to \gamma_{rr} + F^2\sum_m \mathsf{H}_2^{(1m)}(t,r)Y_{1m}(\theta,\phi) \quad , \quad N_a \to N_a + \sum_m \mathsf{b}^{(1m)}(t,r)\partial_a Y_{1m}(\theta,\phi)$$

$$\gamma_{ra} \to \gamma_{ra} + \sum_m \mathsf{a}^{(1m)}(t,r)\partial_a Y_{1m}(\theta,\phi)$$

$$\gamma_{ab} \to \gamma_{ab} + r^2\sum_m \widetilde{\mathsf{K}}^{(1m)}\sigma_{ab}Y_{1m}(\theta,\phi) \qquad \left(\widetilde{\mathsf{K}}^{(1m)} := \mathsf{K}^{(1m)}(t,r) - \mathsf{G}^{(1m)}(t,r)\right)$$

* Gauge transformation

$$\begin{cases} \mathsf{H}_0^{(1m)} \to \mathsf{H}_0^{(1m)} - \dfrac{N'}{N}\xi_r^{(1m)} \\ \widetilde{\mathsf{K}}^{(1m)} \to \widetilde{\mathsf{K}}^{(1m)} - \dfrac{2}{r}\xi_r^{(1m)} + 2\xi_\Omega^{(1m)} \end{cases} \quad \text{under} \begin{cases} r \to r + \sum_m \xi_r^{(1m)}(t,r)Y_{1m}(\theta,\phi) \\ x^a \to x^a + \sum_m \xi_\Omega^{(1m)}(t,r)\nabla^a Y_{1m}(\theta,\phi) \end{cases}$$

$\longrightarrow$ We can always eliminate $\mathsf{H}_0^{(1m)}, \widetilde{\mathsf{K}}^{(1m)}$.

We consider only $\mathsf{H}_1^{(1m)}, \mathsf{H}_2^{(1m)}, \mathsf{a}^{(1m)}, \mathsf{b}^{(1m)}$

Even parity sector$(l=1)$

* 2nd-order Lagrangian$(m=0)$:

$$\begin{aligned}\mathcal{L}^{(2)}_{l=1,m=0} = {} & \frac{\beta r^2}{12N_0(\beta+N_0\sqrt{f})}\left[\frac{\dot{\mathsf{H}}_2}{\sqrt{f}} - \frac{2}{r^2}\left(r^2\widetilde{\mathsf{H}}_1\right)' - \frac{N_0 b_0}{r^2}\mathsf{H}_2' + \frac{4}{r}\widetilde{\mathsf{b}} + \frac{6(\beta+N_0\sqrt{f})}{\beta r}\left(\widetilde{\mathsf{H}}_1 - \widetilde{\mathsf{b}}\right)\right]^2 \\ & + \frac{r^2}{2N_0}\left[\dot{\widetilde{\mathsf{a}}} - r\left(\frac{\sqrt{f}}{r}\widetilde{\mathsf{b}}\right)' - N_0 b_0 r\left(\frac{\sqrt{f}}{r^3}\widetilde{\mathsf{a}}\right)' - \frac{\widetilde{\mathsf{H}}_1}{r\sqrt{f}}\right]^2 - \frac{3\sqrt{f}}{\beta}\left(\widetilde{\mathsf{H}}_1 - \widetilde{\mathsf{b}}\right)^2 + \frac{6b_0}{r^2}\widetilde{\mathsf{H}}_1\widetilde{\mathsf{a}} \\ & - \frac{3b_0}{r^2}\mathsf{H}_2\widetilde{\mathsf{b}} + N_0 f\widetilde{\mathsf{a}}^2 - \frac{N_0}{2}\{f + (rf)'\}\widetilde{\mathsf{a}}\mathsf{H}_2 + \frac{N_0}{4}(rf)'\mathsf{H}_2^2\end{aligned}$$

$$\widetilde{\mathsf{H}}_1 := \sqrt{f}\mathsf{H}_1 - \frac{N_0 b_0}{r^2}\mathsf{H}_2 \quad , \quad \widetilde{\mathsf{b}} := \frac{\mathsf{b}}{r\sqrt{f}} - \frac{N_0 b_0}{r^3}\mathsf{a} \quad , \quad \widetilde{\mathsf{a}} := \frac{\mathsf{a}}{r}$$

After very long calculation, we obtain

$$\widetilde{\mathcal{L}}^{(2)}_{l=1,m=0} = a_1(r)(\partial_r^2\chi)^2 + a_2(r)(\partial_r\chi)^2 + a_3(r)\chi^2$$

where $\chi = \chi(t,r)$ is master variable making from $\widetilde{\mathsf{H}}_1, \mathsf{H}_2, \widetilde{\mathsf{b}}, \widetilde{\mathsf{a}}$.

Even parity sector$(l=1)$

$$\widetilde{\mathcal{L}}^{(2)}_{l=1,m=0} = a_1(r)(\partial_r^2\chi)^2 + a_2(r)(\partial_r\chi)^2 + a_3(r)\chi^2$$

Derive the equation of motion for master variable $\chi(t,r)$,

$$b_1(r)\partial_r^4\chi + b_2(r)\partial_r^3\chi + b_3(r)\partial_r^2\chi + b_4(r)\partial_r\chi + b_5(r)\chi = 0$$

- This is the 4th-order differential equation involving only spatial derivatives.
- Coefficients $b_i(r), \quad (i=1,2,3,4,5)$ is very complicated.

Even parity sector$(l=1)$

$$b_1(r)\partial_r^4\chi + b_2(r)\partial_r^3\chi + b_3(r)\partial_r^2\chi + b_4(r)\partial_r\chi + b_5(r)\chi = 0$$

Assume the asymptotic behavior the solutions as,

$$\chi(t,r) \to \left[c_0(t) + \frac{c_1(t)}{r} + \frac{c_2(t)}{r^2} + \cdots\right] + \ln r\left[d_0(t) + \frac{d_1(t)}{r} + \frac{d_2(t)}{r^2}\cdots\right]$$

where only 4 parameters is independent; the others is written by these parameters.

Impose the boundary condition $\mathsf{H}_1, \mathsf{H}_2, \mathsf{a}, \mathsf{b} \to 0$ as $r \to \infty$

Only one parameter $c_5(t) = \mathrm{Const}$ is allowed.

As far as we investigate the asymptotic behavior at $r \to \infty$, we find the solution that distorts black hole geometry is allowed.

Summary and future work

* Odd parity sector is identical to GR.

* From the analysis of monopole perturbations of even parity sector, the solution which changes the configuration of scalar field without changing the geometrical structure exists.

* We will calculate the even parity GWs $(l \geq 2)$ and analyze QNMs (Quasi-Normal Modes) of BH.

P12

弘前大学 Hirosaki University

Poster(P12), JGRG31, Tokyo, October 26, 2022

Gravitational Wave Physics and Astronomy Genesis

Time-domain reconstruction of pulsar GW polarizations

Hideki Asada (Hirosaki University)

Kuwahara and HA arXiv:2202.00171 (PRD 106, 024051, 2022), 桒原 (2021年度修士論文)

We prove that waveforms for each polarization can be uniquely reconstructed in a time domain for a given set of the strain outputs at a single detector

1 Introduction

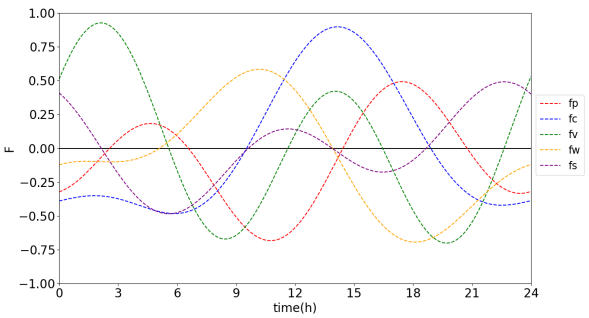
A strain output at a GW interferometer

$$S(t) = F^S(t)h_S(t) + F^V(t)h_V(t) + F^W(t)h_W(t) + F^+(t)h_+(t) + F^\times(t)h_\times(t) + n(t) = \sum_{I=S,V,W,+,\times} F^I(t)h_I(t) + n(t),$$

for a general class of metric theory of gravity.

Antenna pattern changes with Earth rotation:

$$F^I(\theta(t), \phi(t), \psi(t))$$



Daily varition: LIGO-H and Crab pulsar

2 Basic Formulation

Periodic GW: $h_I(t) = h_I(t + nT_P)$

$$\begin{aligned}
S_1(t) &\equiv S(t) \\
&= \sum_I F^I(t)h_I(t) + n(t), \\
S_2(t) &\equiv S(t+T_P) \\
&= \sum_I F^I(t+T_P)h_I(t+T_P) + n(t+T_P), \\
&\cdots \\
S_N(t) &\equiv S(t+(N-1)T_P) \\
&= \sum_I F^I(t+(N-1)T_P)h_I(t+(N-1)T_P) \\
&\quad + n(t+(N-1)T_P).
\end{aligned}$$

Least square

$$\begin{aligned}
A(t) &\equiv \left(S_1(t) - \sum_I F_1^I(t)h_I(t)\right)^2 \\
&\quad + \cdots + \left(S_N(t) - \sum_I F_N^I(t)h_I(t)\right)^2 \\
&= \sum_{a=1}^N \left(S_a(t) - F_a^I(t)h_I(t)\right)^2,
\end{aligned}$$

The minimizer provides the basic equation as

$$M_N(t)\vec{H}_N(t) = \vec{L}_N(t)$$

$$\vec{H}_N(t) = \begin{pmatrix} h_{+N}(t) \\ h_{\times N}(t) \\ h_{VN}(t) \\ h_{WN}(t) \\ h_{SN}(t) \end{pmatrix},$$

$$\vec{L}_N(t) = \begin{pmatrix} \sum_a F_a^+(t)S_a(t) \\ \sum_a F_a^\times(t)S_a(t) \\ \sum_a F_a^V(t)S_a(t) \\ \sum_a F_a^W(t)S_a(t) \\ \sum_a F_a^S(t)S_a(t) \end{pmatrix},$$

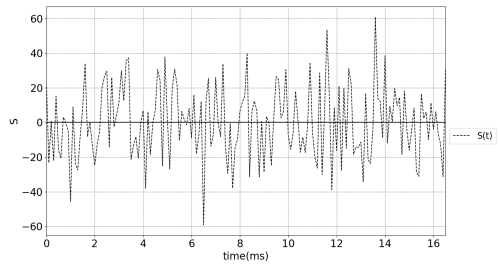
$$M_N(t) = \begin{pmatrix} \sum_a |F_a^+(t)|^2 & \sum_a F_a^+(t)F_a^\times(t) & \sum_a F_a^+(t)F_a^V(t) & \sum_a F_a^+(t)F_a^W(t) & \sum_a F_a^+(t)F_a^S(t) \\ \sum_a F_a^\times(t)F_a^+(t) & \sum_a |F_a^\times(t)|^2 & \sum_a F_a^\times(t)F_a^V(t) & \sum_a F_a^\times(t)F_a^W(t) & \sum_a F_a^\times(t)F_a^S(t) \\ \sum_a F_a^V(t)F_a^+(t) & \sum_a F_a^V(t)F_a^\times(t) & \sum_a |F_a^V(t)|^2 & \sum_a F_a^V(t)F_a^W(t) & \sum_a F_a^V(t)F_a^S(t) \\ \sum_a F_a^W(t)F_a^+(t) & \sum_a F_a^W(t)F_a^\times(t) & \sum_a F_a^W(t)F_a^V(t) & \sum_a |F_a^W(t)|^2 & \sum_a F_a^W(t)F_a^S(t) \\ \sum_a F_a^S(t)F_a^+(t) & \sum_a F_a^S(t)F_a^\times(t) & \sum_a F_a^S(t)F_a^V(t) & \sum_a F_a^S(t)F_a^W(t) & \sum_a |F_a^S(t)|^2 \end{pmatrix}$$

The existence and uniqueness of the solution

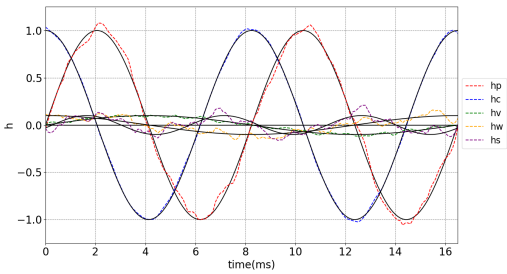
$$\vec{H}_N(t) = M_N^{-1}(t)\vec{L}_N(t)$$

3 Numerical Example

Mock data for one period



Reconstruction in time-domain



4 Summary

Time-domain GW reconstruction is formulated.

This reconstruction will allow us to test speed of extra polarizations.

Pulsar spin modulation, LOD (Length-of-Day) modulation are also addressed.

Estimation of computational costs:
A core-day of a high-performance PC for 12 year data

JGRG 31, Oct.24-28, 2022

High-Precision Simulations for Collisional Self-Gravitating Systems Incorporating Relativistic Effects

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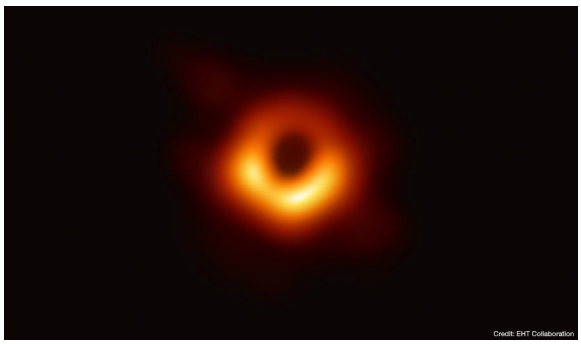
Introduction

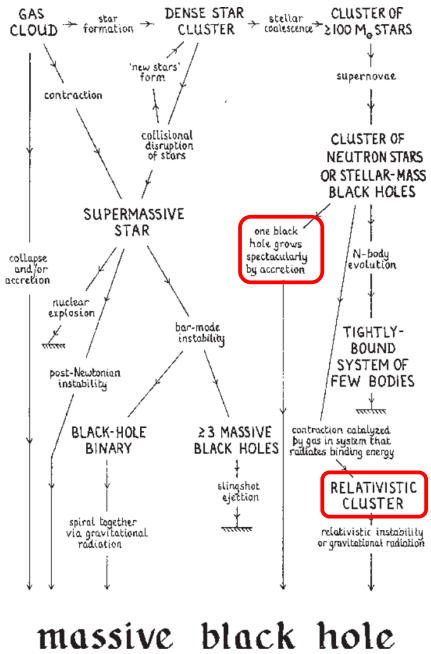
- Observations have confirmed the existence of SMBHs in M87, the center of the Milky Way galaxy.
- The relationship between SMBHs and velocity dispersion is also known for other galaxies, indirectly suggesting the existence of SMBHs.
- How did these objects evolve?



Shadow of SMBH in M87
(by EHT Collaboration)

Scenarios for formation of SMBH

- Rees (1984)
  Many scenarios for formation of SMBH were proposed.
- Our study focuses on the red box on the right figure. In addition, we analyze the evolution of the clusters around SMBH.



Effect of general relativity

- Consider a scenario in which SMBHs and galaxies evolved together.
- In this model, the gravity between the central heavy object and the surrounding stars plays an important role.
- In particular, the gravity imposed by the SMBH is so strong that the effects of general relativity must be taken into account.
- Therefore, we consider the evolution of a self-gravitating system with a correction for the general relativistic effects.

Basic Equations

- We consider many-body gravitating system.
  To incorporate relativistic effects, 1PN correction is applied to the Newton gravity.
- The equation is known as Einstein-Infeld-Hoffman (EIH) equation (Einstein, Infeld, Hoffmann 1938).
- The acceleration is described as follows:

$$
\begin{aligned}
\boldsymbol{a}_i = &-\sum_{j\neq i}\frac{Gm_j\boldsymbol{x}_{ij}}{r_{ij}^3} \\
&+\frac{1}{c^2}\sum_{j\neq i}\frac{Gm_j\boldsymbol{x}_{ij}}{r_{ij}^3}\left[4\frac{Gm_j}{r_{ij}}+5\frac{Gm_i}{r_{ij}}+\sum_{k\neq i,j}\frac{Gm_k}{r_{jk}}+4\sum_{k\neq i,j}\frac{Gm_k}{r_{ik}}\right. \\
&\qquad\left.-\frac{1}{2}\sum_{k\neq i,j}\frac{Gm_k}{r_{jk}^3}\left(\boldsymbol{x}_{ij}\cdot\boldsymbol{x}_{jk}\right)-v_i^2+4\left(\boldsymbol{v}_i\cdot\boldsymbol{v}_j\right)-2v_j^2+\frac{3}{2r_{ij}^2}\left(\boldsymbol{v}_j\cdot\boldsymbol{x}_{ij}\right)^2\right] \\
&-\frac{7}{2c^2}\sum_{j\neq i}\frac{Gm_j}{r_{ij}}\sum_{k\neq i,j}\frac{Gm_k\boldsymbol{x}_{jk}}{r_{jk}^3}+\frac{1}{c^2}\sum_{j\neq i}\frac{Gm_j}{r_{ij}^3}\boldsymbol{x}_{ij}\cdot\left(4\boldsymbol{v}_i-3\boldsymbol{v}_j\right)\boldsymbol{v}_{ij}\,.
\end{aligned}
$$

Conserved quantities

- In 1PN-order approximation, the total energy and the total momentum are conserved. (Because emission of gravitational wave is derived by 2.5PN-order approximation.)

$$
\begin{aligned}
E = &\frac{1}{2}\sum_i m_i\left(v_i^2-\sum_{j\neq i}\frac{Gm_i}{r_{ij}}\right) \\
&+\frac{1}{c^2}\sum_i m_i\left[\frac{3}{8}v_i^4+\frac{3}{2}v_i^2\sum_{j\neq i}\frac{Gm_j}{r_{ij}}+\frac{1}{2}\sum_{j\neq i}\sum_{k\neq i}\frac{G^2m_jm_k}{r_{ij}r_{ik}}\right. \\
&\qquad\left.-\frac{1}{4}\sum_{j\neq i}\frac{Gm_j}{r_{ij}}\left(7\boldsymbol{v}_i\cdot\boldsymbol{v}_j+\frac{(\boldsymbol{v}_i\cdot\boldsymbol{x}_{ij})(\boldsymbol{v}_j\cdot\boldsymbol{x}_{ij})}{r_{ij}^2}\right)\right]\,, \\
\boldsymbol{P} = &\sum_i m_i\boldsymbol{v}_i+\frac{1}{2c^2}\sum_i m_i\boldsymbol{v}_i\left(v_i^2-\sum_{j\neq i}\frac{Gm_j}{r_{ij}}\right)-\frac{G}{2c^2}\sum_i\sum_{j\neq i}\frac{m_im_j}{r_{ij}^3}\left(\boldsymbol{v}_i\cdot\boldsymbol{x}_{ij}\right)\boldsymbol{x}_{ij}\,.
\end{aligned}
$$

Time evolution

- For Hamiltonian system, symplectic integrator is known as accuracy formula about conservation of total energy.
- However, if Hamiltonian cannot be separated to kinetic terms and potential terms, the integrator is described by implicit form. (H(p, q) = K(p)+V(q) (separatable))
- Explicit Form:
  To obtain present quantities, past quantities are required.
- Implicit Form:
  To obtain present quantities, past and present quantities are required.
- Implicit Form is known to have a smaller error, but it is difficult to compute.
- **Hermite Integrator** described below is considered to partially incorporate Implicit Form.

Collisionless and collisional system

- Collisionless system
  The two-body relaxation time is sufficiently long compared to the evolutionary time scale of the system that the effect of the two-body relaxation is negligible.
- Collisional system
  A system in which the effect of a two-body relaxation on the evolution of the system is significant.
- For a system consisting of N particles, the two-body relaxation time can be estimated as

$$t_{\mathrm{relax}} \simeq \frac{N}{\ln N} t_{\mathrm{cross}}$$

- From the virial theorem, for each individual particle mass m and radius R of the object, t_{cross} is given by

$$t_{\mathrm{cross}} = \sqrt{\frac{R^3}{GNm}}\,.$$

Collisionless and collisional system (cont.)

- Age of Universe: $t_{\mathrm{Univ}} \cong 1.38 \times 10^{10}[\mathrm{yr}]$
- Typical value of two-body relaxation time.
  Galaxy ($N = 10^{11}$): $t_{\mathrm{cross}} = 10^{8}[\mathrm{yr}], t_{\mathrm{relax}} = 10^{18}[\mathrm{yr}]$
  -> Collisionless system ($t_{\mathrm{relax}} \gg t_{\mathrm{Univ}}$)
  Globular Cluster ($N = 10^{5}$): $t_{\mathrm{cross}} = 10^{5}[\mathrm{yr}], t_{\mathrm{relax}} = 10^{9}[\mathrm{yr}]$
  -> Collisional system ($t_{\mathrm{relax}} < t_{\mathrm{Univ}}$)

- Even if we consider the evolution of galaxy, the density near the center is so high that it must be considered collisional system.
- Numerical methods with high accuracy are required for the evolution of collisional systems. Therefore, Hermite integrator is applied.

Hermite integrator

- For accuracy formula, we apply Hermite integrator (Makino 1991), which is one of predictor-corrector method.
- For 4th-order Hermite integrator, the formula requires time derivative of accelerations.
- Here <u>subscripts mean time, predictor, or corrector.</u>

predictor

$$\boldsymbol{x}_p = \boldsymbol{x}_0 + \boldsymbol{v}_0 \Delta t + \frac{(\Delta t)^2}{2}\boldsymbol{a}_0 + \frac{(\Delta t)^3}{6}\dot{\boldsymbol{a}}_0 \,,$$

$$\boldsymbol{v}_p = \boldsymbol{v}_0 + \boldsymbol{a}_0 \Delta t + \frac{(\Delta t)^2}{2}\dot{\boldsymbol{a}}_0 \,,$$

corrector

$$\boldsymbol{x}_c = \boldsymbol{x}_p + \frac{(\Delta t)^4}{24}\boldsymbol{a}_0^{(2)} + \frac{(\Delta t)^5}{120}\boldsymbol{a}_0^{(3)} \,,$$

$$\boldsymbol{v}_c = \boldsymbol{v}_p + \frac{(\Delta t)^3}{6}\boldsymbol{a}_0^{(2)} + \frac{(\Delta t)^4}{24}\boldsymbol{a}_0^{(3)} \,.$$

Hermite formula (cont.)

$$\boldsymbol{a}_0^{(2)} = \frac{-6(\boldsymbol{a}_0 - \boldsymbol{a}_1) - \Delta t(4\dot{\boldsymbol{a}}_0 + 2\dot{\boldsymbol{a}}_1)}{(\Delta t)^2} ,$$
$$\boldsymbol{a}_0^{(3)} = \frac{12(\boldsymbol{a}_0 - \boldsymbol{a}_1) + 6\Delta t(\dot{\boldsymbol{a}}_0 + \dot{\boldsymbol{a}}_1)}{(\Delta t)^3} .$$

- $\boldsymbol{a}_1$and $\dot{\boldsymbol{a}}_1$ is calculated using predictor $\boldsymbol{x}_p$ and $\boldsymbol{v}_p$.
- To apply Hermite method, the time derivative of acceleration is derived from EIH equation.

Example for Hermite formula

- Here we suppose harmonic oscillator.

$$\ddot{x} = -x, x(0) = x_0, v(0) = v_0 ,$$

1. Predictor is derived.

$$x_p = x_0 + v_0\Delta t - \frac{x_0}{2}(\Delta t)^2 - \frac{v_0}{6}(\Delta t)^3 ,$$
$$v_p = v_0 - x_0\Delta t - \frac{v_0}{2}(\Delta t)^2 .$$

2. Correction terms are calculated.

$$a_0 = -x_0 , a_1 = -x_p ,$$
$$\dot{a}_0 = -v_0 , \dot{a}_1 = -v_p ,$$
$$\rightarrow a_0^{(2)} = x_0 , a_0^{(3)} = v_0$$

3. Corrector is derived. Then corrector is considered as the position and velocity at the next time.

$$x_c = x_p + \frac{(\Delta t)^4}{24}x_0 + \frac{(\Delta t)^5}{120}v_0 ,$$
$$v_c = v_p + \frac{(\Delta t)^3}{6}x_0 + \frac{(\Delta t)^4}{24}v_0 .$$

Time differential of EIH equation (1)

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{a}_i}{\mathrm{d}t} = & \sum_{j\neq i}\left\{\frac{3Gm_j}{r_{ij}^5}\left(\boldsymbol{x}_{ij}\cdot\boldsymbol{v}_{ij}\right)\boldsymbol{x}_{ij} - \frac{Gm_j}{r_{ij}^3}\boldsymbol{v}_{ij}\right\} \\
& + \frac{1}{c^2}\sum_{j\neq i}\left[-\frac{16G^2m_j^2}{r_{ij}^6}\left(\boldsymbol{x}_{ij}\cdot\boldsymbol{v}_{ij}\right)\boldsymbol{x}_{ij} + \frac{4G^2m_j^2}{r_{ij}^4}\boldsymbol{v}_{ij} - \frac{20G^2m_im_j}{r_{ij}^6}\left(\boldsymbol{x}_{ij}\cdot\boldsymbol{v}_{ij}\right)\boldsymbol{x}_{ij} + \frac{5G^2m_im_j}{r_{ij}^4}\boldsymbol{v}_{ij}\right. \\
& \quad + \sum_{k\neq i,j}\left\{-\frac{3G^2m_jm_k}{r_{ij}^5r_{jk}}\left(\boldsymbol{x}_{ij}\cdot\boldsymbol{v}_{ij}\right)\boldsymbol{x}_{ij} - \frac{G^2m_jm_k}{r_{ij}^3r_{jk}^3}\left(\boldsymbol{x}_{jk}\cdot\boldsymbol{v}_{jk}\right)\boldsymbol{x}_{ij} + \frac{G^2m_jm_k}{r_{ij}^3r_{jk}}\boldsymbol{v}_{ij}\right\} \\
& \quad + \sum_{k\neq i,j}\left\{-\frac{12G^2m_jm_k}{r_{ij}^5r_{ik}}\left(\boldsymbol{x}_{ij}\cdot\boldsymbol{v}_{ij}\right)\boldsymbol{x}_{ij} - \frac{4G^2m_jm_k}{r_{ij}^3r_{ik}^3}\left(\boldsymbol{x}_{ik}\cdot\boldsymbol{v}_{ik}\right)\boldsymbol{x}_{ij} + \frac{4G^2m_jm_k}{r_{ij}^3r_{ik}}\boldsymbol{v}_{ij}\right\} \\
& \quad + \sum_{k\neq i,j}\left\{\frac{3G^2m_jm_k}{2r_{ij}^5r_{jk}^3}\left(\boldsymbol{x}_{ij}\cdot\boldsymbol{v}_{ij}\right)\left(\boldsymbol{x}_{ij}\cdot\boldsymbol{x}_{jk}\right)\boldsymbol{x}_{ij} + \frac{3G^2m_jm_k}{2r_{ij}^3r_{jk}^5}\left(\boldsymbol{x}_{jk}\cdot\boldsymbol{v}_{jk}\right)\left(\boldsymbol{x}_{ij}\cdot\boldsymbol{x}_{jk}\right)\boldsymbol{x}_{ij}\right. \\
& \qquad \left.\left. - \frac{G^2m_jm_k}{2r_{ij}^3r_{jk}^3}\left(\left(\boldsymbol{v}_{ij}\cdot\boldsymbol{x}_{jk}\right) + \left(\boldsymbol{x}_{ij}\cdot\boldsymbol{v}_{jk}\right)\right)\boldsymbol{x}_{ij} - \frac{G^2m_jm_k}{2r_{ij}^3r_{jk}^3}\left(\boldsymbol{x}_{ij}\cdot\boldsymbol{x}_{jk}\right)\boldsymbol{v}_{ij}\right\}\right.
\end{aligned}
$$

Continue to next page

Time differential of EIH equation (2)

$$
\begin{aligned}
& + \frac{3Gm_j}{r_{ij}^5}v_i^2\left(\boldsymbol{x}_{ij}\cdot\boldsymbol{v}_{ij}\right)\boldsymbol{x}_{ij} - \frac{2Gm_j}{r_{ij}^3}\left(\boldsymbol{v}_i\cdot\boldsymbol{a}_i\right)\boldsymbol{x}_{ij} - \frac{Gm_j}{r_{ij}^3}v_i^2\boldsymbol{v}_{ij} \\
& - \frac{12Gm_j}{r_{ij}^5}\left(\boldsymbol{x}_{ij}\cdot\boldsymbol{v}_{ij}\right)\left(\boldsymbol{v}_i\cdot\boldsymbol{v}_j\right)\boldsymbol{x}_{ij} + \frac{4Gm_j}{r_{ij}^3}\left(\left(\boldsymbol{a}_i\cdot\boldsymbol{v}_j\right) + \left(\boldsymbol{v}_i\cdot\boldsymbol{a}_j\right)\right)\boldsymbol{x}_{ij} + \frac{4Gm_j}{r_{ij}^3}\left(\boldsymbol{v}_i\cdot\boldsymbol{v}_j\right)\boldsymbol{v}_{ij} \\
& + \frac{6Gm_j}{r_{ij}^5}v_j^2\left(\boldsymbol{x}_{ij}\cdot\boldsymbol{v}_{ij}\right)\boldsymbol{x}_{ij} - \frac{4Gm_j}{r_{ij}^3}\left(\boldsymbol{v}_j\cdot\boldsymbol{a}_j\right)\boldsymbol{x}_{ij} - \frac{2Gm_j}{r_{ij}^3}v_j^2\boldsymbol{v}_{ij} \\
& - \frac{15Gm_j}{2r_{ij}^7}\left(\boldsymbol{x}_{ij}\cdot\boldsymbol{v}_{ij}\right)\left(\boldsymbol{v}_j\cdot\boldsymbol{x}_{ij}\right)^2\boldsymbol{x}_{ij} + \frac{3Gm_j}{r_{ij}^3}\left(\left(\boldsymbol{a}_j\cdot\boldsymbol{x}_{ij}\right) + \left(\boldsymbol{v}_j\cdot\boldsymbol{v}_{ij}\right)\right)\left(\boldsymbol{v}_j\cdot\boldsymbol{x}_{ij}\right)\boldsymbol{x}_{ij} \\
& + \frac{3Gm_j}{2r_{ij}^5}\left(\boldsymbol{v}_j\cdot\boldsymbol{x}_{ij}\right)^2\boldsymbol{v}_{ij} \\
& \quad + \sum_{k\neq i,j}\left\{\frac{7G^2m_jm_k}{2r_{ij}^3r_{jk}^3}\left(\boldsymbol{x}_{ij}\cdot\boldsymbol{v}_{ij}\right)\boldsymbol{x}_{jk} + \frac{21G^2m_jm_k}{2r_{ij}r_{jk}^5}\left(\boldsymbol{x}_{jk}\cdot\boldsymbol{v}_{jk}\right)\boldsymbol{x}_{jk} - \frac{7G^2m_jm_k}{2r_{ij}r_{jk}^3}\boldsymbol{v}_{jk}\right\} \\
& - \frac{3Gm_j}{r_{ij}^5}\left(\boldsymbol{x}_{ij}\cdot\boldsymbol{v}_{ij}\right)\left(\boldsymbol{x}_{ij}\cdot\left(4\boldsymbol{v}_i - 3\boldsymbol{v}_j\right)\right)\boldsymbol{v}_{ij} \\
& + \frac{Gm_j}{r_{ij}^3}\left(\left(\boldsymbol{v}_{ij}\cdot\left(4\boldsymbol{v}_i - 3\boldsymbol{v}_j\right)\right) + \left(\boldsymbol{x}_{ij}\cdot\left(4\boldsymbol{a}_i - 3\boldsymbol{a}_j\right)\right)\right)\boldsymbol{v}_{ij} \\
& \left. + \frac{Gm_j}{r_{ij}^3}\left(\boldsymbol{x}_{ij}\cdot\left(4\boldsymbol{v}_i - 3\boldsymbol{v}_j\right)\right)\boldsymbol{a}_{ij}\right] .
\end{aligned}
$$

Implementation in Code

- We are developing C-language code that applies the EIH equation to Hermite integrator.
  The time increments are not independent.
- The code is parallelized using OpenMP for fast execution on shared-memory computers.

- The evolution of the system by this code is under testing, and we will analyze the long-time evolution of a realistic system in the future.

Summary

- We are analyzing the co-evolution of SMBH and a galaxy, with emphasis on self-gravity.
- In the interaction by SMBHs, the effect of general relativity cannot be negligible.
- Therefore, we focus on the EIH equation, which is the equation of the N-body problem with 1PN relativistic correction, and develop a code using the H-integral method that can perform the calculation with high accuracy.
- The code is almost developed, and we are in the process of analyzing a realistic model in the future.

Future work

- In general, 1PN approximation (EIH equation) for N-body system requires $O(N^3)$ computation.

- When we consider the system with one SMBH and (main-sequence) stars, the interaction between stars can be calculated in Newtonian gravity, which is expected to reduce the computation (Will 2014). ($O(N^3) \rightarrow O(N^2)$)
- On the case of reduced calculations for collisionless system, we developed simulation code (Tatekawa 2019).
- We will develop simulation code for collisional system on the case of reduced calculations.

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Hawking radiation of scalar particles from 4D Einstein-Gauss-Bonnet black holes based on a generalized uncertainty principle

Ken Matsuno

（Osaka Metropolitan University）

Introduction

Hawking radiation is interesting phenomenon where both of general relativity and quantum theory play a role.

- ➢ During its final stages, semiclassical approach would be expected to break down due to dominance of quantum gravity effects.
- ✓ Some quantum gravity models suggest that there exists a minimal measurable length which would be of order Planck length.

Introduction

Hawking radiation is interesting phenomenon where both of general relativity and quantum theory play a role.

- ➢ During its final stages, semiclassical approach would be expected to break down due to dominance of quantum gravity effects.
- ✓ Some quantum gravity models suggest that there exists a minimal measurable length which would be of order Planck length.
- ✓ From string theories, such minimal length would be obtained by generalized uncertainty principle, which is quantum gravity inspired modification to Heisenberg uncertainty principle.
- ✓ Some generalized uncertainty principles, derived from thought experiments, have been applied to some different systems and played an important role to consider its corrections by supposed quantum theories of gravity.

Introduction

- Hawking radiation emitted by black holes may contain several kinds of particles.
- ✓ Choice of emitted particles is based on prediction that Hawking radiation would favor lower spin, lighter particles and it would be expected to be dominated by scalar particles.

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- Hawking radiation emitted by black holes may contain several kinds of particles.

✓ Choice of emitted particles is based on prediction that Hawking radiation would favor lower spin, lighter particles and it would be expected to be dominated by scalar particles.

➢ We consider **Hawking radiation** from four-dimensional charged Einstein-Gauss-Bonnet black holes by **tunneling of charged scalar particles**, including **quantum gravity effects predicted by quadratic generalized uncertainty principle with minimal measurable length**.

➢ We investigate evaporation process of 4D EGB BHs with quantum gravity effects by tunneling of scalar particles.

Black holes in 4D Einstein-Gauss-Bonnet gravity

- *D*-dim. Einstein-Maxwell-Gauss-Bonnet theory with cosmological constant Λ:

$$\begin{cases} S = \dfrac{1}{16\pi}\displaystyle\int d^D x\sqrt{-g}\left(R - 2\Lambda + \dfrac{\alpha}{D-4}R^2_{\mathrm{GB}} - F_{\mu\nu}F^{\mu\nu}\right) \\ R^2_{\mathrm{GB}} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \quad \text{: Gauss-Bonnet term} \end{cases}$$

✓ Rescaling coupling constant as $\alpha \to \alpha/(D-4)$ and then taking limit $D \to 4$ in *D*-dim. solutions, one obtains 4D charged (A)dS black holes [Fernandes]:

$$\begin{cases} ds^2 = -f dt^2 + \dfrac{dr^2}{f} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \quad A_\mu dx^\mu = \dfrac{Q}{r}dt \\ f = 1 + \dfrac{r^2}{2\alpha}\left(1 \pm \sqrt{1 + 4\alpha\left(\dfrac{2M}{r^3} - \dfrac{Q^2}{r^4} + \dfrac{\Lambda}{3}\right)}\right) \end{cases}$$

M : BH mass
Q : BH charge

➢ We restrict ourselves to solutions with minus signs as one recovers general relativity solutions in appropriate limit:

$$f = 1 - \frac{2\tilde{M}}{r} + \frac{\tilde{Q}^2}{r^2} - \frac{\tilde{\Lambda}}{3}r^2 + O\left(r^{-4}\right), \quad f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3}r^2 + O(\alpha)$$

$$\left(\sqrt{1 + 4\alpha\Lambda/3} = 1 + 2\alpha\tilde{\Lambda}/3,\ (\tilde{M}, \tilde{Q}^2) = 3(M, Q^2)/(3 + 2\alpha\tilde{\Lambda})\right)$$

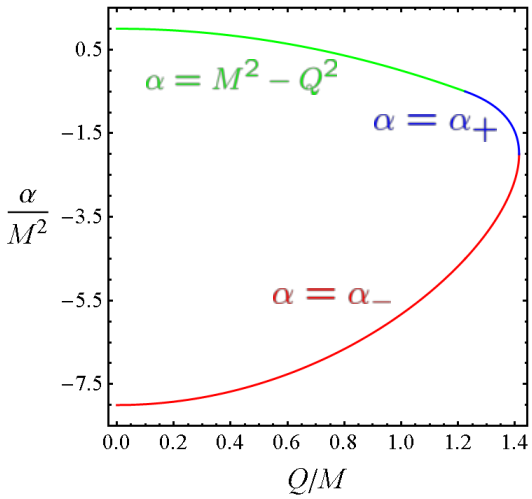
4D charged Einstein-Gauss-Bonnet black holes

$$\begin{cases} ds^2 = -f dt^2 + \dfrac{dr^2}{f} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \\ f = 1 + \dfrac{r^2}{2\alpha}\left(1 - \sqrt{1 + 4\alpha\left(\dfrac{2M}{r^3} - \dfrac{Q^2}{r^4} + \dfrac{\Lambda}{3}\right)}\right) \end{cases}$$

✓ Hawking temperature associated with surface gravity κ of BH horizon r_+ :

$$T_+ = \frac{\kappa}{2\pi} = \frac{r_+^2 - \alpha - Q^2 - \Lambda r_+^4}{4\pi r_+\left(r_+^2 + 2\alpha\right)}$$

- Λ=0 limit: Asymptotically flat black holes



✓ Outer and inner horizons:

$$r_\pm = M \pm \sqrt{M^2 - Q^2 - \alpha}$$

✓ Parameter ranges:

$$\begin{cases} \alpha_- < \alpha \le M^2 - Q^2 & \left(0 \le \dfrac{Q}{M} < \sqrt{\dfrac{3}{2}}\right) \\ \alpha_- < \alpha < \alpha_+ & \left(\sqrt{\dfrac{3}{2}} < \dfrac{Q}{M} < \sqrt{2}\right) \\ \alpha_\pm = Q^2 - 4M^2 \pm 2M\sqrt{4M^2 - 2Q^2} \end{cases}$$

Heisenberg Uncertainty Principle

- For operator A, $\Delta A = A - \langle A\rangle$, $(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2$

➢ Robertson inequality: $(\Delta A)^2(\Delta B)^2 \ge \frac{1}{4}|\langle[A,B]\rangle|^2$

✓ Canonical commutation relation: $[x,p] = i\hbar$

➢ Heisenberg uncertainty relation: $\Delta x \Delta p \ge \frac{\hbar}{2}$

⇒ Δx can be made arbitrarily small by letting Δp grow correspondingly.

✓ Resolution of small distances requires test particles of short wavelengths and thus of high energies.

➢ At such small scales, gravitational effects by high energies of test particles would significantly disturb spacetime structure which was tried.

Generalized Uncertainty Principle (GUP)

[Kempf, Mangano, Mann]

✓ Some quantum gravity theories, including string theory and loop quantum gravity, suggest that there would exist a finite limit to possible resolution of distances, which would be of order Planck length and obtained by modified Heisenberg uncertainty principle.

➢ Modified commutation relation with correction term:

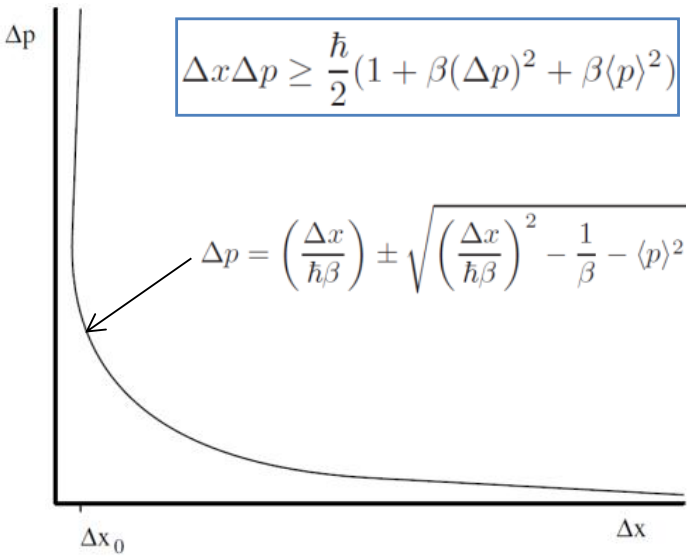
$$[x,p] = i\hbar\left(1+\beta p^2\right) \qquad \begin{cases} \beta = \beta_0 l_p^2 = \beta_0 / m_p^2, \\ l_p = \sqrt{\hbar G_4}, \quad m_p = \sqrt{\hbar / G_4} \end{cases}$$

➢ Modified uncertainty relation from Robertson inequality:

$$\boxed{\Delta x \Delta p \geq \frac{\hbar}{2}(1+\beta(\Delta p)^2+\beta\langle p\rangle^2)}$$

Generalized Uncertainty Principle (GUP)

[Kempf, Mangano, Mann]



$$\boxed{\Delta x \Delta p \geq \frac{\hbar}{2}(1+\beta(\Delta p)^2+\beta\langle p\rangle^2)}$$

$$\Delta p = \left(\frac{\Delta x}{\hbar\beta}\right) \pm \sqrt{\left(\frac{\Delta x}{\hbar\beta}\right)^2 - \frac{1}{\beta} - \langle p\rangle^2}$$

✓ Minimal position uncertainty:

$$\Delta x_{\min}(\langle p\rangle) = \hbar\sqrt{\beta}\sqrt{1+\beta\langle p\rangle^2}$$

➢ Absolutely smallest uncertainty in position with mirror symmetry $\langle p\rangle = 0$: $\Delta x_0 = \hbar\sqrt{\beta}$

➢ Generalized uncertainty principle with minimal measurable length:

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left(1+\beta\left(\Delta p\right)^2\right), \qquad \beta = \beta_0 l_p^2 = \frac{\beta_0}{m_p^2}$$

(β_0 : deformation parameter)

Generalized Uncertainty Principle (GUP)

[Kempf, Mangano, Mann]

$$\begin{cases} \Delta x \Delta p \geq \frac{1}{2}\left[1+\beta\left(\Delta p\right)^2\right] \\ \beta = \beta_0 l_p^2 = \beta_0 / m_p^2, \quad l_p = \sqrt{\hbar G_4}, \quad m_p = \sqrt{\hbar / G_4} \end{cases}$$

(β_0 : deformation parameter)

✓ Minimal measurable length : $\Delta x_0 = l_p \sqrt{\beta_0}$

➢ Modified commutation relation : $\left[x_i, p_j\right] = i\left(1+\beta g^{kl} p_k p_l\right)\delta_{ij}$

Generalized Uncertainty Principle (GUP)

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$$\begin{cases} \Delta x \Delta p \geq \frac{1}{2}\left[1+\beta\left(\Delta p\right)^2\right] \\ \beta = \beta_0 l_p^2 = \beta_0 / m_p^2, \quad l_p = \sqrt{\hbar G_4}, \quad m_p = \sqrt{\hbar / G_4} \end{cases}$$

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➢ Modified position and momentum operators :

$$x_i = x_{0i}, \; p_i = p_{0i}\left(1+\beta g^{jk} p_{0j} p_{0k}\right), \quad \left[x_{0i}, p_{0j}\right] = i\delta_{ij}$$

$$\Rightarrow \; p^2 = g^{ij} p_i p_j = -g^{ij}\partial_i \partial_j \left(1 - 2\beta g^{kl}\partial_k \partial_l\right) + O\left(\beta^2\right)$$

➢ Using energy mass shell condition $E^2 = p^2 + m^2$ (m : particle mass), modified energy operator up to first order in β :

$$\tilde{E} = E\left(1 - \beta m^2 - \beta g^{ij} p_i p_j\right), \quad E = i\left(\partial / \partial t\right)$$

Equation of motion for scalar particles

- Klein-Gordon eq. with Maxwell field up to first order in β [Feng *et al.*]:

$$\left[g^{t\mu}\left(\hbar\partial_t+ieA_t\right)\left(\hbar\partial_\mu+ieA_\mu\right)\right.$$
$$\left.+\left(g^{ij}\left(\hbar\partial_i+ieA_i\right)\left(\hbar\partial_j+ieA_j\right)-m^2\right)\left(1-2\beta m^2+2\beta\hbar^2 g^{kl}\partial_k\partial_l\right)\right]\Psi=0$$

✓ Wave function of Klein-Gordon equation: $\Psi=\exp\left(\frac{i}{\hbar}I\left(x^\mu\right)\right)$

- WKB approximation to leading order in $\hbar$ in 4D EGB BH geometry:

$$\begin{cases}\frac{1}{f}\left(\partial_t I+eA_t\right)^2+\sigma\left(2\beta\sigma-1\right)=0\\ \sigma=m^2+f\left(\partial_r I\right)^2+\frac{1}{r^2}\left(\partial_\theta I\right)^2+\frac{1}{r^2\sin^2\theta}\left(\partial_\phi I\right)^2\end{cases}$$

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$$\left.+\left(g^{ij}\left(\hbar\partial_i+ieA_i\right)\left(\hbar\partial_j+ieA_j\right)-m^2\right)\left(1-2\beta m^2+2\beta\hbar^2 g^{kl}\partial_k\partial_l\right)\right]\Psi=0$$

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✓ According to Killing vector fields, the action ansatz: $I=-\omega t+W(r,\theta)+J\phi$

(ω, J : scalar particle energy and angular momentum)

- Since tunneling is a quantum effect arising within Planck length near horizon region, we restrict ourselves to s-wave particles with θ =const, J=0.
- Equation of motion for charged scalar particles:

$$m^2\left(1-2\beta m^2\right)-\frac{1}{f}\left(\omega-eA_t\right)^2+f\left(1-4\beta m^2\right)\left(\frac{\partial W}{\partial r}\right)^2-2\beta f^2\left(\frac{\partial W}{\partial r}\right)^4=0$$

- Function W can be written as $W(r,\theta)=R(r)+\Theta(\theta)$

Tunneling of scalar particles

- Action for outgoing and ingoing modes for classically forbidden trajectory:

$$\begin{cases} \mathrm{Im}R_{\mathrm{out}} = -\mathrm{Im}R_{\mathrm{in}} = \dfrac{\pi r_+\left(r_+^2+2\alpha\right)\left(\omega - eA_+\right)}{r_+^2-\alpha-Q^2-\Lambda r_+^4}\left(1+\beta\Xi\right)+O\left(\beta^2\right) \\ \Xi = \dfrac{m^2}{2} + \dfrac{3eA_+\left(r_+^2+2\alpha\right)\left(\omega-eA_+\right)}{r_+^2-\alpha-Q^2-\Lambda r_+^4} - \dfrac{2Q^2\left(\omega-eA_+\right)^2\left(\alpha\left(2\alpha+Q^2\right)+2\alpha r_+^2+2r_+^4(1+\alpha\Lambda)\right)}{\left(r_+^2+2\alpha\right)\left(r_+^2-\alpha-Q^2-\Lambda r_+^4\right)^2} \\ \quad - \dfrac{2\left(\omega-eA_+\right)^2\left(\alpha^3+6\alpha^2r_+^2+3\alpha r_+^4(1-2\alpha\Lambda)-r_+^6(1+6\alpha\Lambda)+\alpha\Lambda^2r_+^8\right)}{\left(r_+^2+2\alpha\right)\left(r_+^2-\alpha-Q^2-\Lambda r_+^4\right)^2} \end{cases}$$

- Tunneling probability amplitude of charged s-wave scalar particles:

$$\Gamma \simeq \frac{\exp\left(-2\mathrm{Im}R_{\mathrm{out}}\right)}{\exp\left(-2\mathrm{Im}R_{\mathrm{in}}\right)} \simeq \exp\left(-\frac{4\pi r_+\left(r_+^2+2\alpha\right)\left(1+\beta\Xi\right)}{r_+^2-\alpha-Q^2-\Lambda r_+^4}\left(\omega-eA_+\right)\right)$$

- Comparing tunneling probability amplitude with Boltzmann factor $\Gamma = \exp\left(-\left(\omega - eA_+\right)/T\right)$ at temperature *T*, we obtain **modified Hawking temperature of Einstein-Gauss-Bonnet black hole**:

$$T = T_+\left(1-\beta\Xi\right)+O\left(\beta^2\right)$$

Modified Hawking temperature

$$T = T_+\left(1-\beta\Xi\right)+O\left(\beta^2\right)$$

- ✓ Hawking temperature depends upon energy *ω*, mass *m* and charge *e* of emitted scalar particle, and is modified by black hole geometry, Maxwell field and generalized uncertainty principle through parameters *α*, Λ, *Q* and β.
- ✓ Taking limits, we obtain some known quantum-corrected Hawking temperatures of four-dimensional black holes:

α=0:	4D Reissner-Nordstrom-(A)dS black hole
α=0, Λ=0:	4D Reissner-Nordstrom black hole
α=0, Λ=0, *Q*=0:	4D Schwarzschild black hole

Modified Hawking temperature

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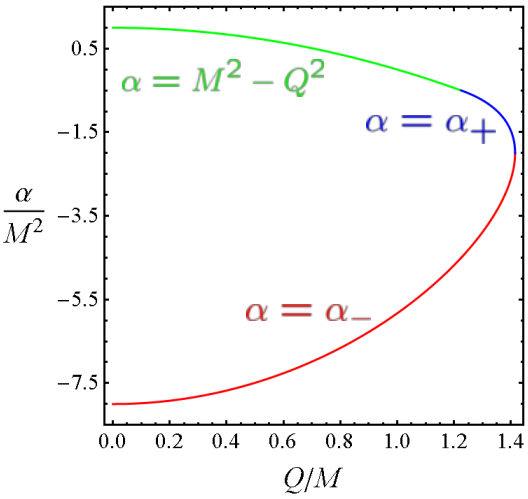
- ➢ We consider evaporation process of asymptotically flat Einstein-Gauss-Bonnet black hole by tunneling of scalar particles as one of quantum gravity effects in Hawking radiation, i.e., Λ=0.
- ✓ Since all particles emitted by Hawking radiation near horizon region are effectively massless, mass of emitted scalar particle is not taken into account in the following discussion, i.e., m=0.

Evaporation of black holes

$$T = T_+ (1 - \beta\Xi) + O\left(\beta^2\right)$$

- Mass of Einstein-Gauss-Bonnet BH would decrease due to radiation.
- ➢ When black hole mass approaches order of Planck mass, quantum gravity effect could be considered and we may discuss the value of correction Ξ in T:
- Ξ=0: Effects related to energy of scalar particle, black hole geometry and Maxwell field are canceled.
  ⇒ Hawking temperature without correction T_+ appears.
- Ξ<0: Temperature T is higher than temperature T_+ and black hole accelerates evaporation.
- Ξ>0: Combination of effects related to energy of scalar particle, black hole geometry, Maxwell field and generalized uncertainty principle slows down increase of temperature due to radiation.
  ⇒ Evaporation may cease at particular mass of BH and BH may be in a stable balanced state, leading to remnant mass.

Parameter regions of correction Ξ



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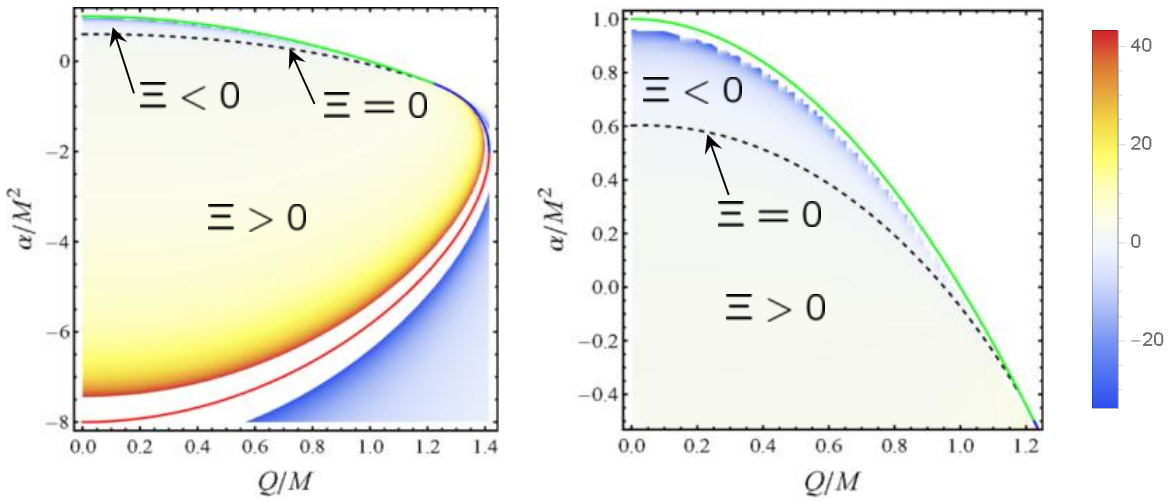
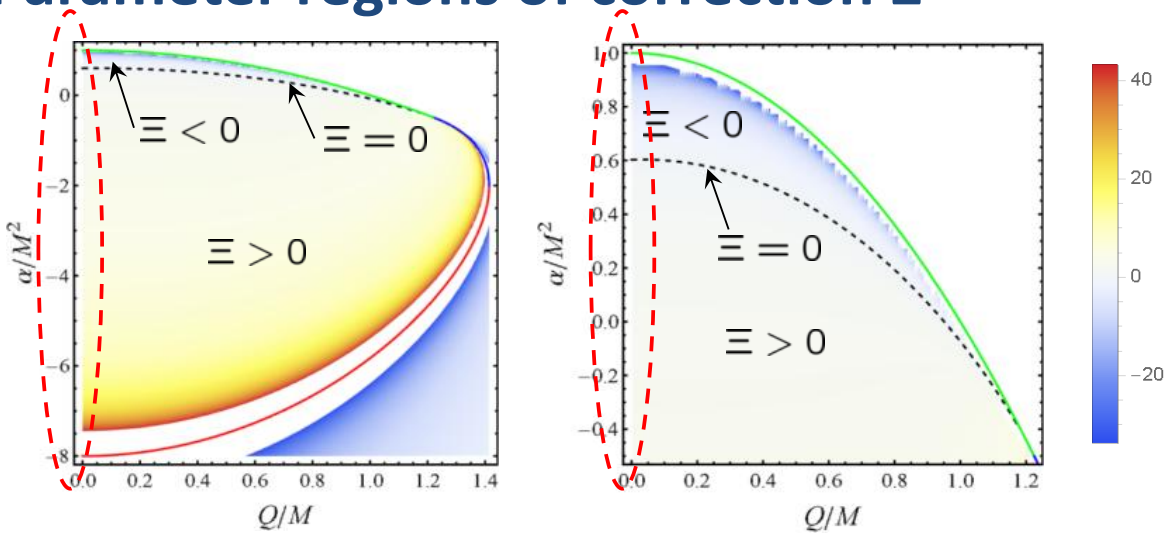


- ✓ Positive Ξ region increases with increasing e/ω.
- ✓ Black holes with large α and small Q would favor negative Ξ and accelerate evaporation.
- ✓ Black holes with small α and large Q would favor positive Ξ and turn into remnants at the final stage of evaporation.

Parameter regions of correction Ξ



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- ✓ Black holes with large α and small Q would favor negative Ξ and accelerate evaporation.
- ✓ Black holes with small α and large Q would favor positive Ξ and turn into remnants at the final stage of evaporation.

Modified thermodynamics

- ✓ We consider uncharged scalar particle radiation from uncharged EGB BH, i.e., Q=0.
- ➢ Using lower bound on emitted particle energy $\omega \geq 1/\Delta x$ and uncertainty in position x for events near black hole horizon $\Delta x \simeq 2r_+$,

  Hawking temperature and entropy with quantum corrections: $\left(\mu = M/m_p,\ \tilde{\alpha} = \alpha/l_p^2\right)$

$$\begin{cases} \dfrac{T}{m_p} \simeq \dfrac{\left(2\mu^2+\tilde{\alpha}\right)\sqrt{\mu^2-\tilde{\alpha}} - 2\mu\left(\mu^2-\tilde{\alpha}\right)}{2\pi\tilde{\alpha}\left(8\mu^2+\tilde{\alpha}\right)}\left(1-\beta_0\dfrac{\tilde{\alpha}^3+20\tilde{\alpha}^2\mu^2-16\tilde{\alpha}\mu^4-32\mu^6+8\mu\left(2\mu^2+\tilde{\alpha}\right)^2\sqrt{\mu^2-\tilde{\alpha}}}{8\tilde{\alpha}^2\left(\mu^2-\tilde{\alpha}\right)\left(8\mu^2+\tilde{\alpha}\right)}\right) \\ S \simeq 2\pi\mu\left(2\mu-\sqrt{\mu^2-\tilde{\alpha}}\right)+\beta_0\dfrac{\pi\mu\left(\tilde{\alpha}^2-2\mu^2\left(4\mu^2+\tilde{\alpha}\right)+2\mu\left(4\mu^2+5\tilde{\alpha}\right)\sqrt{\mu^2-\tilde{\alpha}}\right)}{4\tilde{\alpha}\left(\mu^2-\tilde{\alpha}\right)^{3/2}} \end{cases}$$

- ➢ Heat capacity with quantum correction: $C = T\left(\partial S/\partial T\right)$

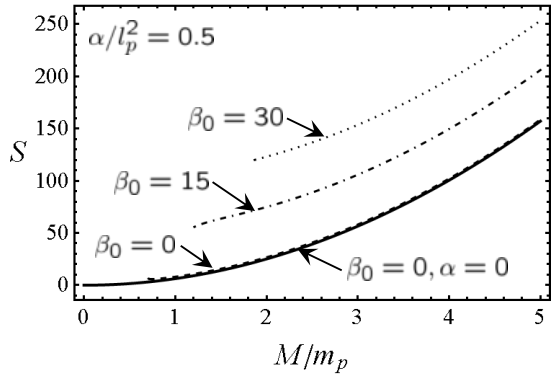
Modified thermodynamics

- ✓ We consider uncharged scalar particle radiation from uncharged EGB BH, i.e., Q=0.
- ➢ Using lower bound on emitted particle energy $\omega \geq 1/\Delta x$ and uncertainty in position x for events near black hole horizon $\Delta x \simeq 2r_+$,

  Hawking temperature and entropy with quantum corrections: $\left(\mu = M/m_p,\ \tilde{\alpha} = \alpha/l_p^2\right)$

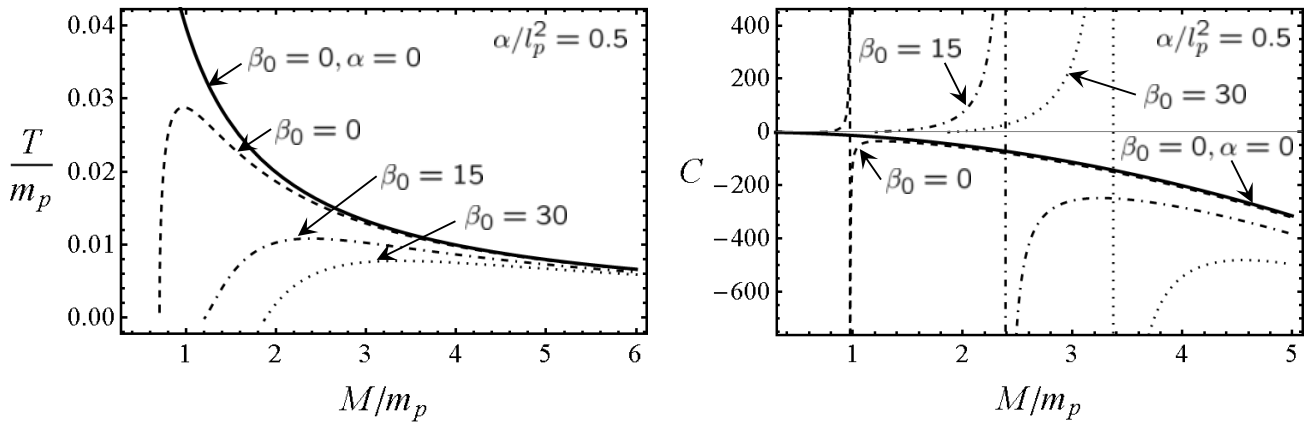
$$\begin{cases} \dfrac{T}{m_p} \simeq \dfrac{\left(2\mu^2+\tilde{\alpha}\right)\sqrt{\mu^2-\tilde{\alpha}} - 2\mu\left(\mu^2-\tilde{\alpha}\right)}{2\pi\tilde{\alpha}\left(8\mu^2+\tilde{\alpha}\right)}\left(1-\beta_0\dfrac{\tilde{\alpha}^3+20\tilde{\alpha}^2\mu^2-16\tilde{\alpha}\mu^4-32\mu^6+8\mu\left(2\mu^2+\tilde{\alpha}\right)^2\sqrt{\mu^2-\tilde{\alpha}}}{8\tilde{\alpha}^2\left(\mu^2-\tilde{\alpha}\right)\left(8\mu^2+\tilde{\alpha}\right)}\right) \\ S \simeq 2\pi\mu\left(2\mu-\sqrt{\mu^2-\tilde{\alpha}}\right)+\beta_0\dfrac{\pi\mu\left(\tilde{\alpha}^2-2\mu^2\left(4\mu^2+\tilde{\alpha}\right)+2\mu\left(4\mu^2+5\tilde{\alpha}\right)\sqrt{\mu^2-\tilde{\alpha}}\right)}{4\tilde{\alpha}\left(\mu^2-\tilde{\alpha}\right)^{3/2}} \end{cases}$$

- ➢ Heat capacity with quantum correction: $C = T\left(\partial S/\partial T\right)$



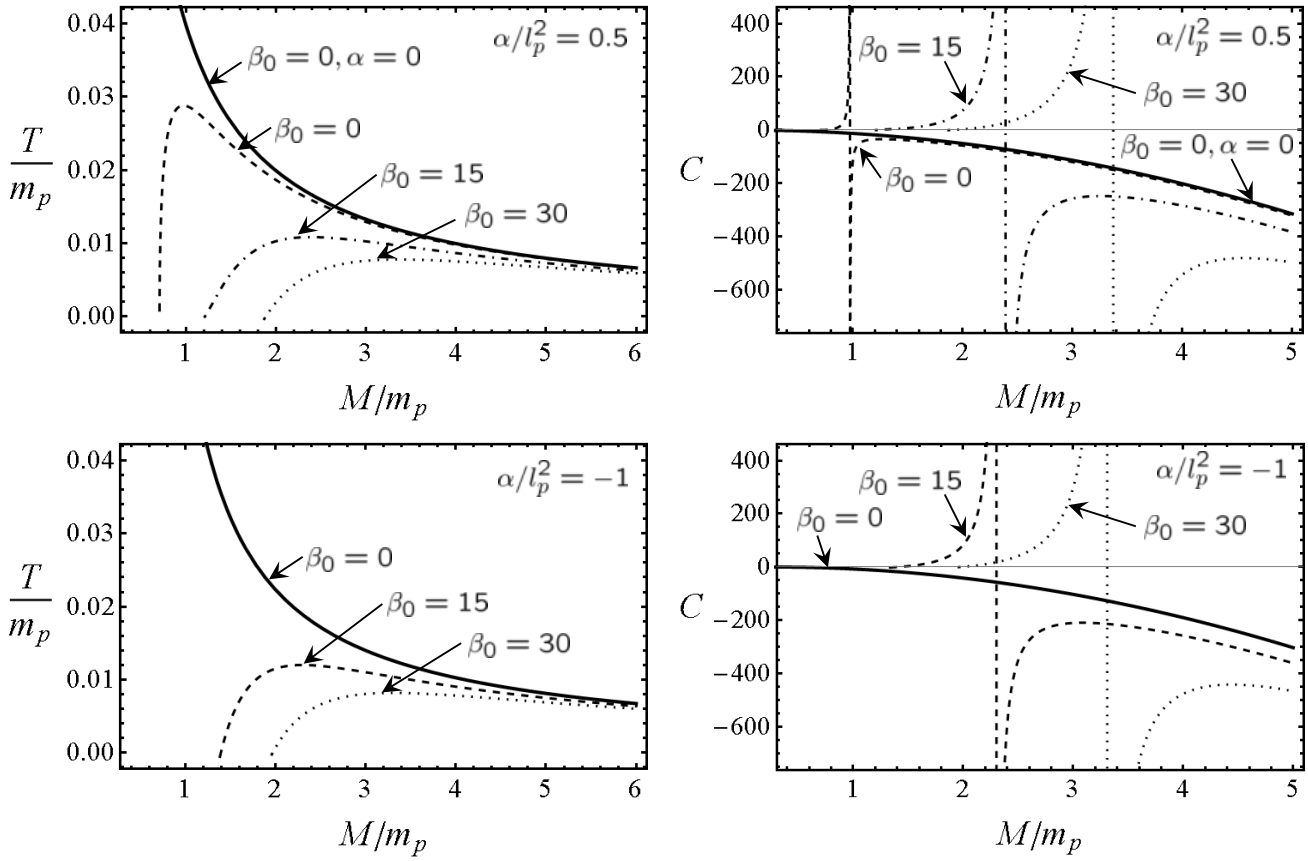
- ✓ Final value of entropy S after BH evaporation increases with increasing β_0 for fixed α/l_p^2.
- ➢ Black hole evaporation modified by GUP: quantum gravity effect in Hawking radiation

Modified evaporation process of black hole



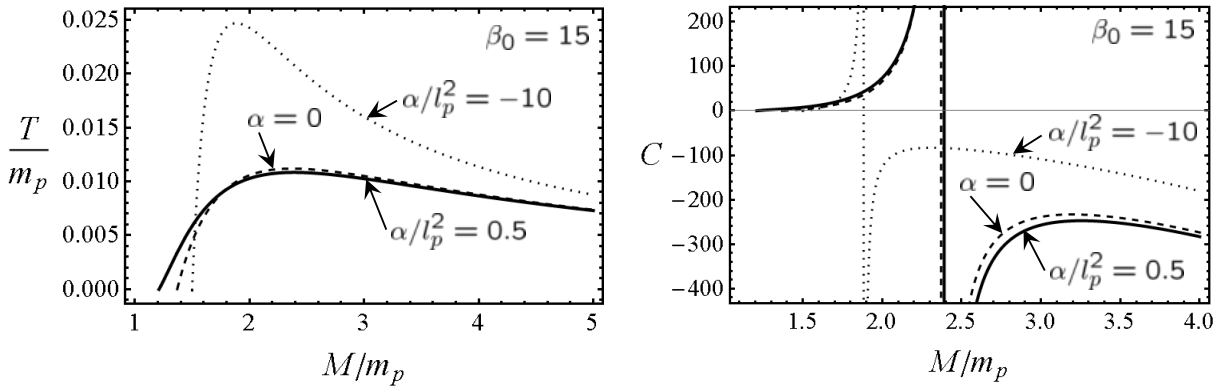
- ✓ As black hole mass decreases, temperature with $\alpha \neq 0$ reaches local maximum value and then decreases to zero at minimum value of mass M_{rm}.
- ✓ At local maximum temperature, the system undergoes transition from unstable negative heat capacity phase to stable positive heat capacity cooling down towards cold extremal configuration with mass M_{rm}.
- ✓ At minimum mass M_{rm}, since both Hawking temperature and heat capacity vanish, BH may not exchange its energy with surrounding environment.
- ➢ **Generalized uncertainty principle prevents Einstein-Gauss-Bonnet black hole to completely evaporate and results in thermodynamic stable remnant.**

Modified evaporation process of black hole



✓ **There also exists black hole remnant in α<0, β_0≠0 case.**

Einstein-Gauss-Bonnet black hole remnant



- ✓ Mass of black hole remnant M_{rm} increases with increasing β_0 for fixed α/l_p^2, while it decreases with increasing α/l_p^2 for fixed β_0.
- ✓ If deformation parameter is $\beta_0 \fallingdotseq 1$ and Gauss-Bonnet coupling constant is $\alpha \fallingdotseq -l_p^2$ m^2, mass of black hole remnant is $M_{rm} \fallingdotseq 10^{-8}$ kg, which is of order Planck mass like black hole remnants in minimally geometric deformation model, quadratic gravity and asymptotically safe gravity.
- ✓ Since such a Planck mass remnant would be a ground state mass of black hole, Einstein-Gauss-Bonnet black hole remnant would have no hair.
  ⇒ This would be a desirable property as dark matter.

Summary

We study Hawking radiation from 4D charged Einstein-Gauss-Bonnet black hole by tunneling of charged scalar particles.

- We consider phenomenological quantum gravity effects predicted by quadratic generalized uncertainty principle with minimal measurable length.

✓ We derive corrections of Hawking temperature to general relativity, which are related to energy of emitted particle, Gauss-Bonnet coupling constant, charge of black hole and existence of minimal length in the black hole geometry.

✓ We obtain some known Hawking temperatures in 4D black hole spacetimes by taking limits in modified temperature.

✓ We show that generalized uncertainty principle may slow down the increase of Hawking temperature due to radiation, which may lead to thermodynamic stable remnant of the order of Planck mass after evaporation of 4D Einstein-Gauss-Bonnet black hole.

Discussion (1)

✓ Since a Planck mass remnant would be a ground state mass of black hole, Einstein-Gauss-Bonnet black hole remnant would have no hair. This would be a desirable property as dark matter.

➢ If Einstein-Gauss-Bonnet metric would describe geometry around a primordial black hole and generalized uncertainty principle considered in this work would play an important role in the quantum nature of black hole, Einstein-Gauss-Bonnet black hole remnants would be a dark matter candidate.

✓ Since black hole remnant would not radiate and its gravitational interaction would be very weak, it would be difficult to observe remnants in our Universe directly.

➢ It would be expected that one possible indirect signature of black hole remnant might be associated with cosmic gravitational wave background.

Discussion (2)

- We consider Hawking radiation with quantum gravity effects inspired from modification on commutation relation of matter field in fixed Einstein-Gauss-Bonnet background geometry.

➢ When black hole mass approaches the order of Planck mass due to radiation, it would be expected that some quantum gravity effects would lead to some quantum fluctuations in background metric.

✓ Such background geometries would be given by some noncommutative black holes and asymptotically safe gravity black holes. Since these black holes are analogous to Reissner-Nordstrom solution, quantum corrections in these frameworks affect things in the same way as black hole charge.

➢ If there would exist a correspondence between charged EGB black holes and some quantum-corrected black holes, black hole charge in EGB spacetime might have something to do with the very structure of spacetime manifold and result in some quantum fluctuations in background geometry as a quantum gravity effect.

Appendix

Linear and quadratic generalized uncertainty principle

[Ali, Das, Vagenas]

- Modified commutation relation : $\alpha = \alpha_0 / M_{\mathrm{Pl}} c = \alpha_0 \ell_{\mathrm{Pl}} / \hbar$

$$[x_i, p_j] = i\hbar\left(\delta_{ij} - \alpha\left(p\delta_{ij} + \frac{p_i p_j}{p}\right) + \alpha^2(p^2\delta_{ij} + 3p_i p_j)\right)$$

- Modified uncertainty relation :

$$\Delta x \Delta p \geq \frac{\hbar}{2}\left[1 + \left(\frac{\alpha}{\sqrt{\langle p^2\rangle}} + 4\alpha^2\right)\Delta p^2 + 4\alpha^2\langle p\rangle^2 - 2\alpha\sqrt{\langle p^2\rangle}\right]$$

✓ Minimum measurable length & maximum measurable momentum :

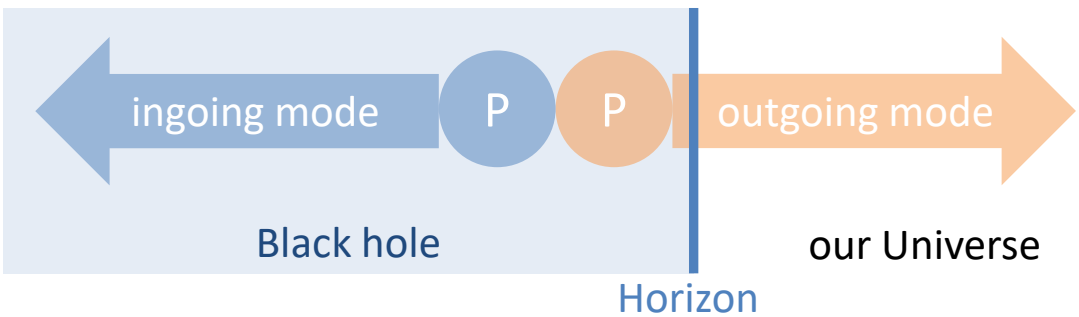
$$\Delta x \geq (\Delta x)_{\min} \approx \alpha_0 \ell_{\mathrm{Pl}}, \quad \Delta p \leq (\Delta p)_{\max} \approx \frac{M_{\mathrm{Pl}} c}{\alpha_0}$$

- Modified position and momentum operators :

$$x_i = x_{0i}, \quad p_i = p_{0i}(1 - \alpha p_0 + 2\alpha^2 p_0^2), \quad [x_{0i}, p_{0j}] = i\hbar\delta_{ij}$$

Derivation of Hawking radiation on the basis of tunneling mechanism [Srinivasan, Padmanabhan]

- Forming of a particle-antiparticle pair close to horizon inside black hole



ingoing mode: trapped inside horizon

outgoing mode: escape from black hole by tunneling effect

⇔ radiation from black hole

Derivation of Hawking temperature [Parikh, Wilczek]

- WKB tunneling probability amplitude of outgoing particle escapes from black hole for classically forbidden trajectory

$$\Gamma = |\phi|^2 \simeq \left|e^{iI}\right|^2 \simeq \exp\left(-\frac{2\pi\omega}{\kappa}\right)$$

(ω : energy of particle, κ : surface gravity on black hole horizon)

- Comparing Γ with Boltzmann factor in thermal equilibrium state at temperature T :

$$\Gamma = \exp\left(-\frac{\omega}{T}\right)$$

$$T = \frac{\kappa}{2\pi}$$: Hawking temperature

- Using the method, we evaluate backreaction of radiation

Classical limit of minimal length uncertainty relation

(Benczik *et al.*)

✓ Generalized uncertainty principle in D dimensions:

$$\begin{cases} \left[x_i, p_j\right] = i\hbar\left(1+\beta p^2\right)\delta_{ij} \\ \left[p_i, p_j\right] = 0, \quad \left[x_i, x_j\right] = 2i\hbar\beta\left(p_i x_j - p_j x_i\right) \end{cases}$$

➢ In classical limit: $\frac{1}{i\hbar}[A,B] \Rightarrow \{A,B\}$

$$\begin{cases} \boxed{\{x_i, p_j\} = (1+\beta p^2)\,\delta_{ij}} \\ \{p_i, p_j\} = 0, \quad \{x_i, x_j\} = 2\beta\,(p_i x_j - p_j x_i) \end{cases}$$

➢ Deformed Hamilton's equations:

$$\begin{cases} \boxed{\dot{x}_i = \{x_i, H\} = \{x_i, p_j\}\frac{\partial H}{\partial p_j} + \{x_i, x_j\}\frac{\partial H}{\partial x_j}} \\ \dot{p}_i = \{p_i, H\} = -\{x_i, p_j\}\frac{\partial H}{\partial x_j} \end{cases}$$

Hawking radiation as tunneling process

- Tunneling mechanism: quantum effect near black hole horizon
  ⇒ derivation of Hawking radiation as tunneling process
  by using technique of dimensional reduction is justified

- To describe across-horizon phenomena, defining Painlevé-like coordinate:

$$dt_p = dt + \frac{K}{F}\sqrt{1-\frac{F}{K}}d\rho$$

➢ 2D effective metric as background metric near black hole horizon

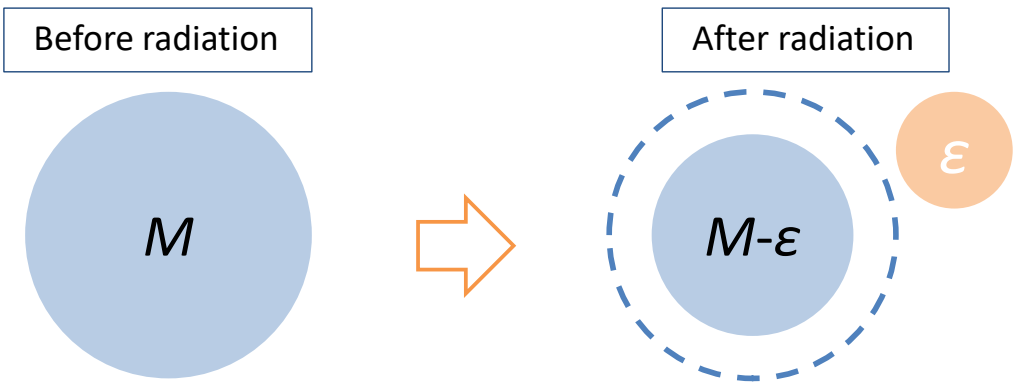
$$ds^2_{2\mathrm{D}} = -\frac{\rho^2 - 2M\rho + Q^2}{\rho^2 K}dt_p^2 + 2\sqrt{\frac{K-1}{K} + \frac{2M}{\rho K} - \frac{Q^2}{\rho^2 K}}dt_p d\rho + d\rho^2$$

➢ outgoing radial null geodesic

$$\dot{\rho} = \frac{d\rho}{dt_p} = 1 - \sqrt{\frac{K-1}{K} + \frac{2M}{\rho K} - \frac{Q^2}{\rho^2 K}}$$

- We can ignore mass of matter field near black hole horizon
  by discussion of dimensional reduction
  ⇒ derivation of Hawking radiation by using null geodesic is justified

- Restricting ourselves to uncharged radiations from BH and taking into account backreaction of radiation
  (ε : energy of particle escapes from BH by tunneling mechanism)



$$\rho_{\mathrm{in}} = \rho_+ = M + \sqrt{M^2 - Q^2}$$

$$\rho_{\mathrm{out}} = M - \mathcal{E} + \sqrt{(M-\mathcal{E})^2 - Q^2}$$
$$\mathcal{E} = E\left(1 + \beta E^2\right)$$

WKB Tunneling probability amplitude for classically forbidden trajectory: Γ≃exp(-2Im*I*)

- Imaginary part of action for outgoing positive energy particle from ρ_{in} to ρ_{out} by using deformed Hamilton's equation $\dot{\rho} = \{\rho, H\} = \{\rho, p_\rho\}\, dH/dp_\rho|_\rho$ in classical limit of commutation relation $\{\rho, p_\rho\} = 1 + \beta p_\rho^2,\ \ p_\rho \simeq \mathcal{E}$

$$
\begin{aligned}
\mathrm{Im}I &= \mathrm{Im}\int_{\rho_{\text{in}}}^{\rho_{\text{out}}} p_\rho d\rho = \mathrm{Im}\int_{\rho_{\text{in}}}^{\rho_{\text{out}}}\int_0^{p_\rho} dp'_\rho d\rho \\
&= \mathrm{Im}\int_{\rho_{\text{in}}}^{\rho_{\text{out}}}\int_0^{H} \frac{1+\beta\left(\mathcal{E}'\right)^2}{\dot{\rho}} dH' d\rho = \mathrm{Im}\int_M^{M-\mathcal{E}}\int_{\rho_{\text{in}}}^{\rho_{\text{out}}} \frac{1+\beta\left(\mathcal{E}'\right)^2}{\dot{\rho}} d\rho dH' \\
&= \mathrm{Im}\int_0^{\mathcal{E}}\int_{\rho_{\text{in}}}^{\rho_{\text{out}}} \frac{1+\beta\left(\mathcal{E}'\right)^2}{1-\sqrt{\frac{K-1}{K}+\frac{2(M-\mathcal{E}')}{\rho K}-\frac{Q^2}{\rho^2 K}}} d\rho\left(-d\mathcal{E}'\right) \\
&= \mathrm{Im}\int_0^{\mathcal{E}} -\pi i \frac{2\tilde{\rho}_+\sqrt{\tilde{\rho}_+\left(\tilde{\rho}_+ + \rho_0\right)}\left(1+\beta\left(\mathcal{E}'\right)^2\right)}{\tilde{\rho}_+ - \tilde{\rho}_-}\left(-d\mathcal{E}'\right) \\
&\left(\tilde{\rho}_\pm = M - \mathcal{E}' \pm \sqrt{\left(M-\mathcal{E}'\right)^2 - Q^2}\right)
\end{aligned}
$$

WKB Tunneling probability amplitude for classically forbidden trajectory: Γ≃exp(-2Im*I*)

$$
\mathrm{Im}I = \frac{2\pi\rho_+\sqrt{\rho_+\left(\rho_+ + \rho_0\right)}}{\rho_+ - \rho_-} E\left(1+\Xi_m\right) + O\left(E^5\right)
$$

$$
\left\{
\begin{aligned}
&\Xi_m = -\delta_1 E - \delta_2 E^2 + \frac{4}{3}\beta E^2 - \delta_3 E^3 - \frac{5}{2}\beta\delta_1 E^3, \quad \delta_1 = \frac{2\rho_+\left(\rho_+ - 3\rho_-\right) + \rho_0\left(\rho_+ - 5\rho_-\right)}{2\left(\rho_+ + \rho_0\right)\left(\rho_+ - \rho_-\right)^2} \\
&\delta_2 = \frac{\rho_0^2\left(\rho_+^2 - 14\rho_+\rho_- - 35\rho_-^2\right) - 48\rho_+^2\rho_-^2 - 12\rho_+\rho_-\rho_0\left(\rho_+ + 7\rho_-\right)}{6\left(\rho_+ + \rho_0\right)^2\left(\rho_+ - \rho_-\right)^4} \\
&\delta_3 = \frac{1}{8(\rho_+ + \rho_0)^3(\rho_+ - \rho_-)^6}\left[-105\rho_-^3\rho_0^3 - 189\rho_+\rho_-^2\rho_0^2(2\rho_- + \rho_0) + 2\rho_+^4(4\rho_- + \rho_0)(\rho_0 - 20\rho_-)\right. \\
&\quad \left. -9\rho_+^2\rho_-\rho_0\left(48\rho_-^2 + 58\rho_-\rho_0 + 3\rho_0^2\right) + \rho_+^3\left(\rho_0^3 - 160\rho_-^3 - 496\rho_-^2\rho_0 - 62\rho_-\rho_0^2\right)\right]
\end{aligned}
\right.
$$

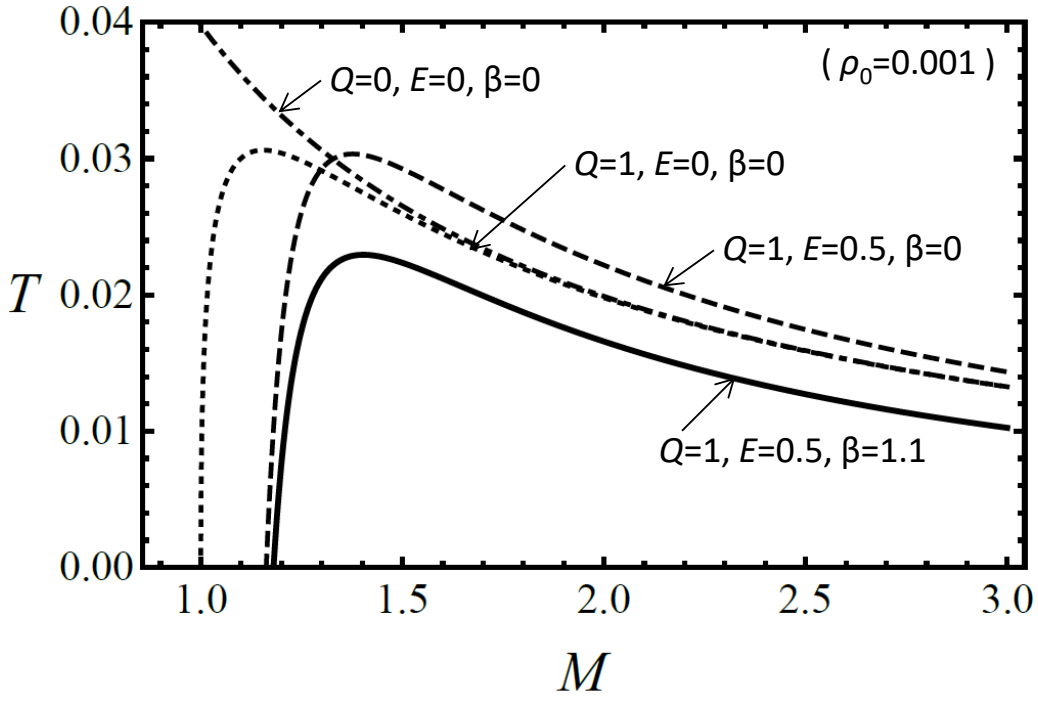
- ➢ WKB tunneling probability amplitude of outgoing particle

$$
\Gamma \simeq \exp\left(-2\mathrm{Im}I\right) \simeq \exp\left(-\frac{4\pi\rho_+\sqrt{\rho_+\left(\rho_+ + \rho_0\right)}\left(1+\Xi_m\right)}{\rho_+ - \rho_-} E\right)
$$

Modified temperature and black hole evaporation

- Comparing Γ with Boltzmann factor, we obtain BH temperature with corrections

$$T = T_{\mathrm{KK}}\left(1 - \Xi_m\right) + O\left(E^4\right), \quad T_{\mathrm{KK}} = \frac{\rho_+ - \rho_-}{4\pi\rho_+\sqrt{\rho_+\left(\rho_+ + \rho_0\right)}}$$



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- Modified temperature T may become lower than original Hawking temperature T_{KK}, which implies that quantum gravity effect may slow down increase of black hole temperature and stop black hole from completely evaporating

  ⇒ Black hole remnant would be left at the end of evaporation

ρ_0=0, β =0 limit

- 2D effective metric near black hole horizon

$$ds_{2D}^2 = -\frac{\rho^2 - 2M\rho + Q^2}{\rho^2}dt^2 + \frac{\rho^2}{\rho^2 - 2M\rho + Q^2}d\rho^2$$

- WKB tunneling probability amplitude

$$\Gamma \simeq \exp\left[\pi\left(\rho_{\text{out}}^2 - \rho_{\text{in}}^2\right)\right]$$
$$= \exp\left[-4\pi\omega\left(M - \frac{\omega}{2}\right) + 2\pi\left(M-\omega\right)\sqrt{(M-\omega)^2 - Q^2} - 2\pi M\sqrt{M^2 - Q^2}\right]$$

1st order in ω : Hawking temperature of 4D Reissner-Nordström BH

$$T_{\text{H}} = \frac{\sqrt{M^2 - Q^2}}{2\pi\left(M + \sqrt{M^2 - Q^2}\right)^2}$$

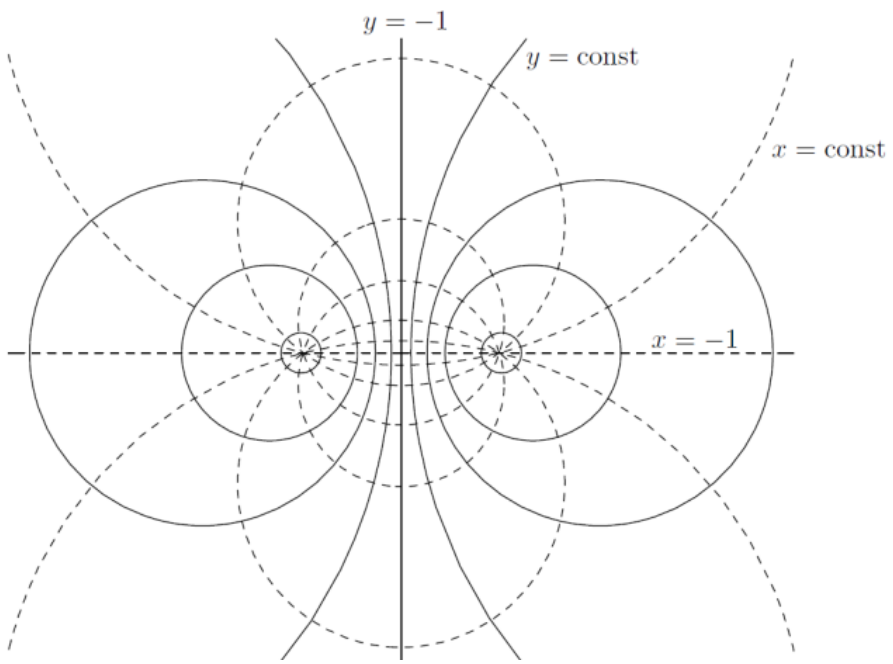
2nd order in ω : correction by backreaction of radiation

(coincides with that by Parikh, Wilczek)

5D vacuum black rings

$$\mathrm{d}s^2 = -\frac{F(y)}{F(x)}\left(\mathrm{d}t - CR\frac{1+y}{F(y)}\mathrm{d}\psi\right)^2 + \frac{R^2}{(x-y)^2}F(x)\left[-\frac{G(y)}{F(y)}\mathrm{d}\psi^2 - \frac{\mathrm{d}y^2}{G(y)} + \frac{\mathrm{d}x^2}{G(x)} + \frac{G(x)}{F(x)}\mathrm{d}\phi^2\right]$$

$$F(\xi) = 1 + \lambda\xi \qquad G(\xi) = (1-\xi^2)(1+\nu\xi) \qquad C = \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}}$$
$$-1 \leqslant x \leqslant 1, \qquad -\infty \leqslant y \leqslant -1 \qquad 0 < \nu \leqslant \lambda < 1$$



5D vacuum black rings

$$\mathrm{d}s^2 = -\frac{F(y)}{F(x)}\left(\mathrm{d}t - CR\frac{1+y}{F(y)}\mathrm{d}\psi\right)^2 + \frac{R^2}{(x-y)^2}F(x)\left[-\frac{G(y)}{F(y)}\mathrm{d}\psi^2 - \frac{\mathrm{d}y^2}{G(y)} + \frac{\mathrm{d}x^2}{G(x)} + \frac{G(x)}{F(x)}\mathrm{d}\phi^2\right]$$

$$\begin{cases} F(\xi) = 1+\lambda\xi \qquad G(\xi) = (1-\xi^2)(1+\nu\xi) \qquad C = \sqrt{\lambda(\lambda-\nu)\dfrac{1+\lambda}{1-\lambda}} \\ -1 \leqslant x \leqslant 1, \qquad -\infty \leqslant y \leqslant -1 \qquad 0 < \nu \leqslant \lambda < 1 \end{cases}$$

$$\begin{cases} x \to -1,\ y \to -1 : \text{asymptotically flat} \\ y = -1/\nu \qquad : \text{horizon with } S^2 \times S^1 \text{ topology} \end{cases}$$

- mass M, angular momentum J, Hawking temperature T_{br} :

$$\begin{cases} M = \dfrac{3\pi R^2}{4G}\dfrac{\lambda}{1-\nu} \qquad J = \dfrac{\pi R^3}{2G}\dfrac{\sqrt{\lambda(\lambda-\nu)(1+\lambda)}}{(1-\nu)^2} \\ T_{\mathrm{br}} = \dfrac{1}{4\pi R}\dfrac{1+\nu}{\sqrt{\lambda\nu}}\sqrt{\dfrac{1-\lambda}{1+\lambda}} \end{cases}$$

Dimensional reduction near black ring horizon

$$S[\varphi] = \int \mathrm{d}^5x\left(\varphi\partial_i\sqrt{-g}g^{ij}\partial_j\varphi + \sqrt{-g}V_{\mathrm{int}}(\varphi)\right)$$

- Partial wave decomposition:

$$\begin{cases} \varphi = \displaystyle\sum_{k,l}\frac{1}{2\pi}\varphi^{(kl)}(t,x,y)\,\mathrm{e}^{\mathrm{i}k\phi}\,\mathrm{e}^{il\psi} \\ \varphi^{(kl)}(t,x,y) = \displaystyle\sum_n \varphi^{(kln)}(t,y)P_n(x) \qquad \left(\int_{-1}^{1}\mathrm{d}x\frac{P_m(x)P_n(x)}{(x-y)^2} = a_{mn}(y)\right) \end{cases}$$

- 2D effective field near black ring horizon

$$S[\varphi] = \int \mathrm{d}t\,\mathrm{d}y\sum_{k,l,n}\tilde{\varphi}^{(kln)}(t,y)\left[\frac{CR(1+y)}{\sqrt{-F(y)}G(y)}\left(\partial_t + il\frac{F(y)}{CR(1+y)}\right)^2 - \partial_y\frac{\sqrt{-F(y)}}{CR(1+y)}G(y)\partial_y\right]\tilde{\varphi}^{(kln)}(t,y)$$

$$\begin{cases} \mathrm{d}s^2 = -f(y)\,\mathrm{d}t^2 + \dfrac{1}{f(y)}\mathrm{d}y^2 \qquad \left(A_t(y) = -\dfrac{F(y)}{CR(1+y)}\right) \\ f(y) = \dfrac{\sqrt{-F(y)}}{CR(1+y)}G(y) \end{cases}$$

WKB tunneling probability amplitude

- To describe across-horizon phenomena, defining Painlevé-like coordinate:

$$t \to t - \int \frac{CR(1+y)}{\sqrt{-F(y)}G(y)}\sqrt{1-G(y)}\,\mathrm{d}y$$

$$\Rightarrow \quad \mathrm{d}s^2 = -\frac{\sqrt{-F(y)}G(y)}{CR(1+y)}\mathrm{d}t^2 + 2\sqrt{1-G(y)}\,\mathrm{d}t\,\mathrm{d}y + \frac{CR(1+y)}{\sqrt{-F(y)}}\mathrm{d}y^2$$

➢ "Radial" null geodesics: $\dot{y} = \frac{\sqrt{-F(y)}}{CR(1+y)}(\pm 1 - \sqrt{1-G(y)})$

➢ WKB tunneling probability amplitude of outgoing particle

$$\Gamma \simeq \exp\left(-2\mathrm{Im}I\right) \simeq \exp\left(-\frac{4\pi R}{1+\nu}\sqrt{\frac{\lambda\nu(1+\lambda)}{1-\lambda}}\left(1+\Xi_{\mathrm{br}}\right)E\right)$$

$$\begin{cases} \Xi_{\mathrm{br}} = -\delta_1 E - \delta_2 E^2 + \frac{4}{3}\beta E^2 - \delta_3 E^3 - \frac{5}{2}\beta\delta_1 E^3, \quad \delta_1 = \frac{(1-\nu)^3}{3\pi R^2\lambda\nu(1+\nu)}, \\ \delta_2 = \frac{2(1-\nu)^4(1+10\nu+\nu^2)}{27\pi^2R^4\lambda^2\nu^2(1+\nu)^2}, \quad \delta_3 = \frac{(1-\nu)^5(1+8\nu+46\nu^2+8\nu^3+\nu^4)}{27\pi^3R^6\lambda^3\nu^3(1+\nu)^3} \end{cases}$$

Modified temperature and black ring evaporation

➢ Comparing Γ with Boltzmann factor, we obtain modified temperature

$$\begin{cases} T = T_{\mathrm{br}}\left(1 + \delta_1 E + \delta_2 E^2 - \frac{4}{3}\beta E^2 + \delta_3 E^3 + \frac{5}{2}\beta\delta_1 E^3\right) \\ T_{\mathrm{br}} = \frac{1}{4\pi R}\frac{1+\nu}{\sqrt{\lambda\nu}}\sqrt{\frac{1-\lambda}{1+\lambda}} \end{cases}$$

➢ modified temperature T may become lower than original Hawking temperature T_{br}, which would imply that quantum gravity effect may slow down increase of black ring temperature and stop black ring from completely evaporating

⇒ Black ring remnant would be left at the end of evaporation

Summary

- We study Hawking radiation from 5D squashed Kaluza-Klein black holes and 5D vacuum black rings by tunneling of massless scalar particles, including quantum gravity effects predicted by quadratic generalized uncertainty principle with minimal measurable length
- We consider evaporation process of 5D black holes with quantum gravity effects by tunneling of particles

<u>Future plans</u>:

✓Other particle tunneling with other uncertainty relations

✓Sparisity of Hawking radiation

✓Statistical correlations between Hawking modes emitted

Cosmological evolution of gravitational leptogenesis with right-handed neutrinos

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COLLABORATOR : BAMBA KAZUHARU

Background

The mechanism that causes the asymmetry between particles and antiparticles is called baryogenesis.

Variation of baryon and lepton numbers

$B - L$ is saved but $B + L$ is not saved.

It is essential that $B - L$ doesn't become 0 to get B.

Model of leptogenesis

Leptogenesis

In this theory, we use right-handed Majorana neutrinos.

$$N \to l_L + \bar{\phi}, \qquad N \to \bar{l_L} + \phi$$

By violating of CP, a difference occurs in the decay rate and lepton number is generated.

The popular theory is gravitational leptogenesis.

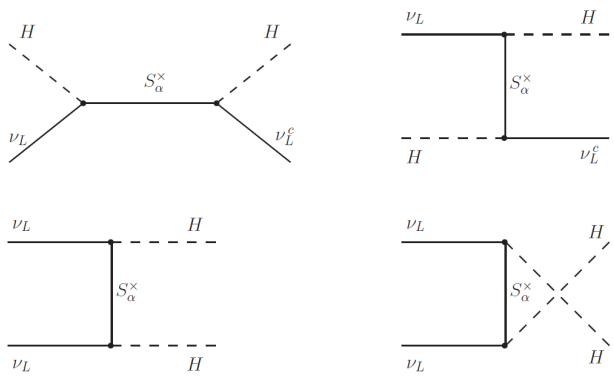
l_L:light left-handed neutrino

ϕ:Higgs field

Feynman diagrams

Feynman diagrams for the lepton number violating reactions $\nu_L H \leftrightarrow \nu_L^c H$ and $\nu_L \nu_L \leftrightarrow HH$, modified by the `charge violating' ν_R propagator $S_\alpha^\times$.

H:Higgs boson



[Jamie I. McDonald and Graham M. Shore, Dynamical Evolution of Gravitational Leptogenesis , arXiv:2006.09425v2 [hep-ph] 2020]

Model of RIGL

Radiatively-induced gravitational leptogenesis (RIGL)

By tidal gravitational effects, an asymmetry of lepton and anti-lepton is generated.

This theory has the potential to explain the presently observed baryon-to-photon ratio $\eta \simeq 6.0 \times 10^{-10}$.

$$\eta \simeq 0.02|N_L|$$
$$|N_L| \simeq 10^{-8}$$

N_L:lepton-to-photon ratio

[Jamie I. McDonald and Graham M. Shore, Dynamical Evolution of Gravitational Leptogenesis , arXiv:2006.09425v2 [hep-ph] 2020]

Derivation of equation(1)

By operators of CP even and CP odd, a Boltzmann equation is derived.

The following graph is represented by the equation.

$\tilde{a}, c, \tilde{d}$:CP even operator

b:CP odd operator

$\tilde{a} = a - \frac{1}{2}d$

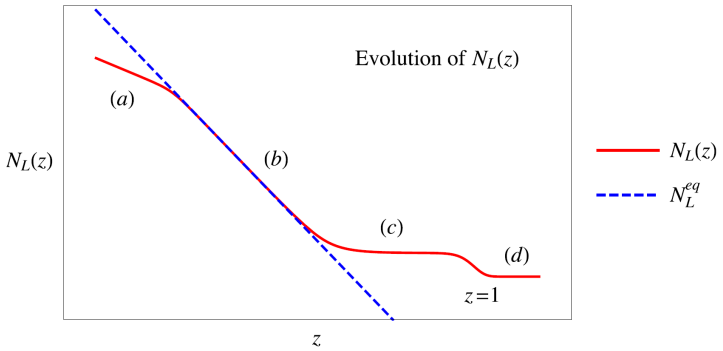
$\tilde{d} = -d$

$\hat{b} = \frac{1}{2}a + c + \frac{1}{4}d$

[Jamie I. McDonald and Graham M. Shore, Dynamical Evolution of Gravitational Leptogenesis , arXiv:2006.09425v2 [hep-ph] 2020]

Derivation of equation(2)

Graph showing inverse of time and lepton-to-photon ratio derived from the Boltzmann equation,



[Jamie I. McDonald and Graham M. Shore, Dynamical Evolution of Gravitational Leptogenesis , arXiv:2006.09425v2 [hep-ph] 2020]

Derivation of equation(3)

$z = M/T\,,\ N_L = n_L/n_\gamma\,,$

$$\boxed{\frac{dN_L}{dz} = -W(z)\left(N_L - N_L^{eq}(z)\right) - \widetilde{W}(z)N_L,}$$

$$W = \frac{\Gamma}{zH},\ \widetilde{W} = -3(1+w)\left[2a - 2\hat{b}(1-3w)\right]\frac{\rho}{M_p^2}\frac{1}{z}.$$

W:ratio of the reaction rate to the expansion rate of the universe

$\widetilde{W}$:newly introduced evolution term

Identification of N_L

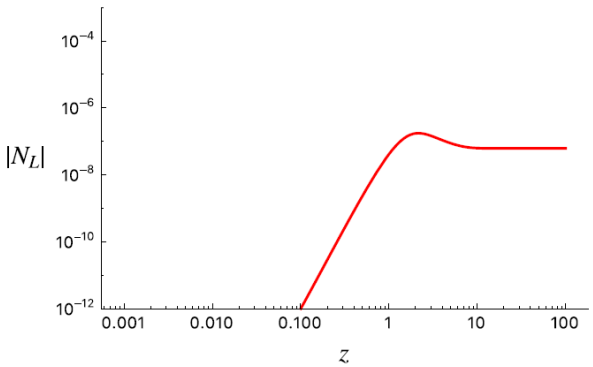
Graph showing the corresponding absolute values of the lepton asymmetry induced by the out-of-equilibrium decay of ν_R^1,

$M_1 = 5.0 \times 10^{10}[\mathrm{GeV}],\ K = 5,\ \varepsilon_1 = 10^{-6}$.

M_1:the lightest sterile neutrino mass

K:ratio of the zero-temperature decay rate to the Hubble parameter at $\mathrm{T} = M_1$

ε_1:CP violating decay parameter controlling the contribution to the lepton asymmetry from the out-of-equilibrium ν_R^1 decays



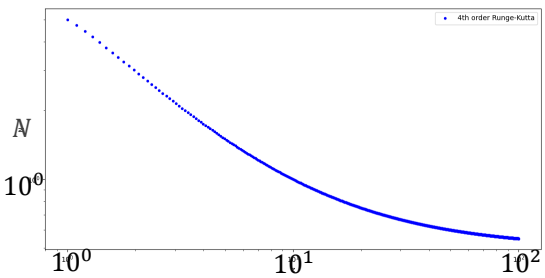
[Jamie I. McDonald and Graham M. Shore, Dynamical Evolution of Gravitational Leptogenesis , arXiv:2006.09425v2 [hep-ph] 2020]

Result(1)

From the fact that a new evolution term induces the dependence of the lepton asymmetry $N_L \sim 1/z^2$ in the ultra-high temperature region and moderates the sharp rise of the equilibrium value $N_L^{eq} \sim 1/z^5$,

$$\frac{dN_L}{dz} = -W\left(N_L - N_L^{eq}\right) - \widetilde{W}N_L = -WN_L + WN_L^{eq} - \widetilde{W}N_L = -W/z^2 + W/z^5 - \widetilde{W}/z^2 ,$$

$W = 2,\ \widetilde{W} = 3.$



Result(2)

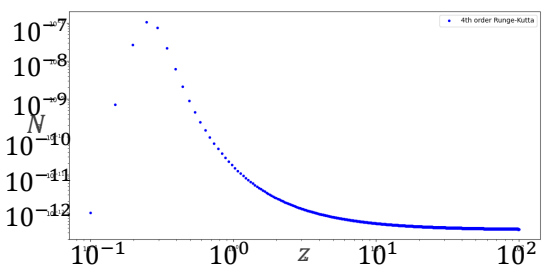
In the Boltzmann equation,

$$W \sim \lambda^4 \left(\frac{M_1}{M_p}\right)^{-1} \frac{1}{z^2},\ N_L^{eq} \sim \lambda^4 \left(\frac{M_1}{M_p}\right)^{3} \frac{1}{z^5},\ \widetilde{W} \sim \lambda^2 \left(\frac{M_1}{M_p}\right)^{2} \frac{1}{z^5},$$

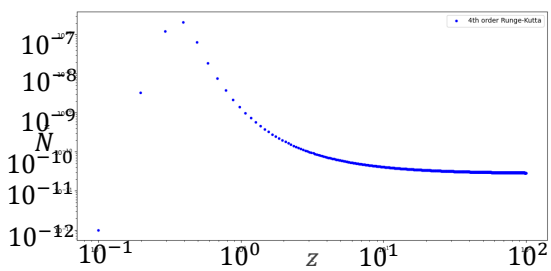
$$\frac{dN_L}{dz} = -\left(\lambda^4 \left(\frac{M_1}{M_p}\right)^{-1} \frac{1}{z^2}\right) N_L + \lambda^4 \left(\frac{M_1}{M_p}\right)^{-1} \frac{1}{z^2} \times \lambda^4 \left(\frac{M_1}{M_p}\right)^{3} \frac{1}{z^5} - \left(\lambda^2 \left(\frac{M_1}{M_p}\right)^{2} \frac{1}{z^5}\right) N_L,$$

$\lambda = 0.02,$
$M_p = 2.4 \times 10^{18},$
$M_1 = 10^{11}.$

Result(3)



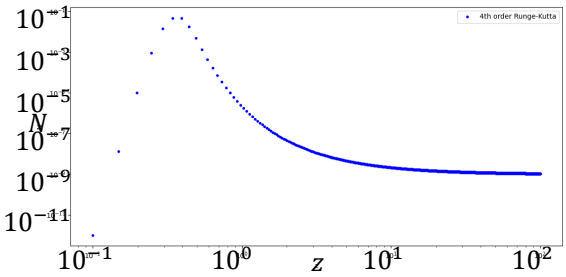
computation width:2^{11}



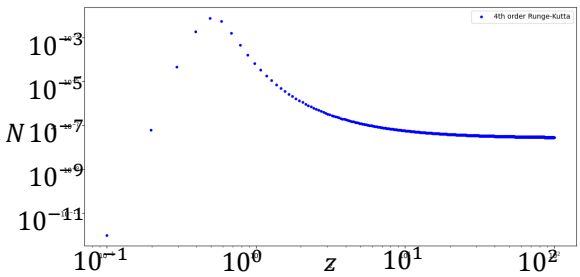
computation width: 2^{10}

Result(4)

Since a suitable M_1 value is given as 5.0×10^{10}[GeV],



computation width: 2^{11}



computation width: 2^{10}

Future work

The final N_L leading to η in this case is close to the paper, but there are differences in the maximum values of the graphs.

The goal is to derive the Yukawa coupling constant λ and computational width to derive approximate graphs.

In addition, there are papers in which the mass of the right-handed Majorana neutrinos that causes baryogenesis is expressed as 10^{11}[GeV], so I aim to confirm the consistently of the theory in that case.

JGRG31(2022)

Spectrum of gravitational waves due to the axion-gravity Chern-Simons coupling

Presenter: Yuta Murakoshi (Fukushima University)

Collaborator: Kazuharu Bamba (Fukushima University)

Introduction

Gravitational waves were discovered in 2015 and active research has begun.
Gravitational waves can be used to further develop fundamental physics.

If there is an axion, it is allowed to be coupled with the gravitational Chern-Simon term as a pseudoscalar.
We explore spectrum of gravitational waves propagating in axion.

Introduction

The parametric resonantly amplification of the amplitude of the gravitational waves propagating the axion dark matter filed can occur if the gravitational wave frequency matches with that of the coherent oscillation of the axion.

Model

Action: $S = S_{EH} + S_{CS} + S_{\Phi}$

The Einstein-Hilbert action: $S_{EH} \equiv \kappa \int_{\nu} dx^4 \sqrt{-g} R$ $\kappa = (16\pi G)^{-1}$

The Chern-Simon terms: $S_{CS} = \frac{1}{4}\alpha \int_{\nu} dx^4 \sqrt{-g} \Phi \tilde{R} R$ R:ricci scalar

The Pontryagin density:

$\tilde{R}R \equiv \tilde{R}^{\alpha\ \gamma\delta}_{\ \beta} R^{\beta}_{\ \alpha\gamma\delta}$, $\tilde{R}^{\alpha\ \gamma\delta}_{\ \beta} \equiv \frac{1}{2}\epsilon^{\gamma\delta\rho\sigma} R^{\alpha}_{\ \beta\rho\sigma}$ Φ:axion field

The coupling constant: $\alpha = \sqrt{\frac{\kappa}{2}}\ell^2$

The action of the axion field $\boldsymbol{\Phi}$: $S_{\Phi} \equiv -\frac{1}{2}\int_{\nu} dx^4 \sqrt{-g}[g^{\mu\nu}(\nabla_{\sigma}\Phi)(\nabla^{\sigma}\Phi) - g_{\mu\nu}V(\Phi)]$

Daisuke Yoshida, Jiro Soda, *Int.J.Mod.Phys.D* 27 (2018) 09

The potential of the axion field: $V(\Phi) \equiv \frac{1}{2} m^2 \Phi^2$ m :axion mass

We solve the equations of motion in the homogeneous background spacetime:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2(\eta)(-d\eta^2 + \delta_{ij} dx^i dx^j)$$

The modified Klein-Gordon equation

$$\Phi'' + 2\frac{a'}{a}\Phi' + a^2 m^2 \Phi = 0$$

$$\Phi = \Phi_0 \cos(m\eta)$$

$$\Phi_0 = \frac{\sqrt{2\rho}}{m} \simeq 2.1 \times 10^7 \mathrm{eV} \times \left(\frac{10^{-10}\mathrm{eV}}{m}\right) \sqrt{\frac{\rho}{0.3\ \mathrm{GeV/cm^3}}}$$

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The diagonalized gravitational wave equation:

$$h_A'' + \frac{\epsilon_A \delta \cos(m\eta)}{1 + \epsilon_A \frac{k}{m} \delta \sin(m\eta)} k h_A' + k^2 h_A = 0$$

k: wave number

The dimensionless parameter δ: $\delta \equiv \frac{\alpha}{\kappa} m^2 \Phi_0$

The ϵ_A is determined by each parity state. $\epsilon_A \equiv \begin{cases} 1 & A = R \\ -1 & A = L \end{cases}$

We solve the differential equation for this Fourier transformed gravitational wave.

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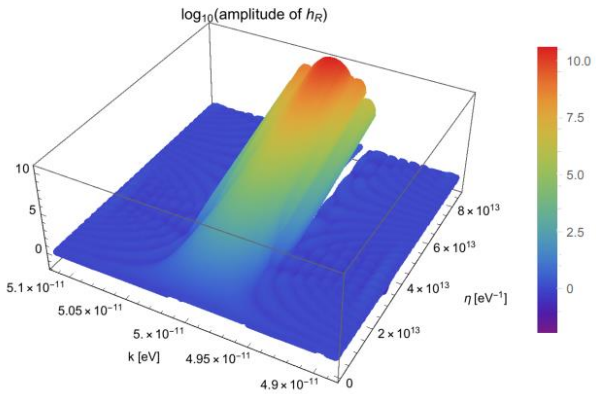
Previous study



This is a plot of the gravitational waves formed by intense astronomical events as they pass through our galaxy, and how they are modified by the coherent oscillations of the axion.

In previous study, the figure shows that there is a distinct narrow peak.

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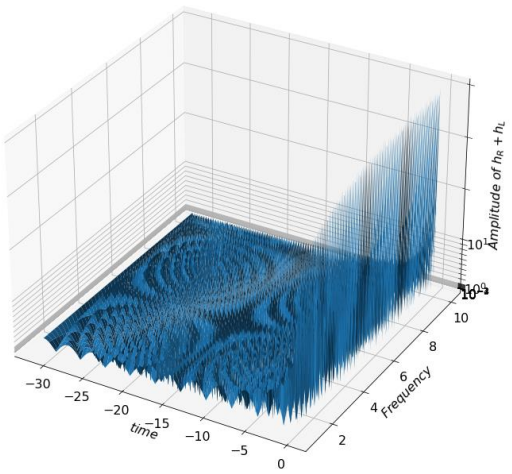
Results and discussion

$$\mathrm{m} = 10^{-10}\ \mathrm{eV}$$
$$\ell = 10^{8}\ \mathrm{km}$$
$$h_{R0} = \cos(m\eta_0)$$
$$\dot{h}_{R0} = -\sin(m\eta_0)$$

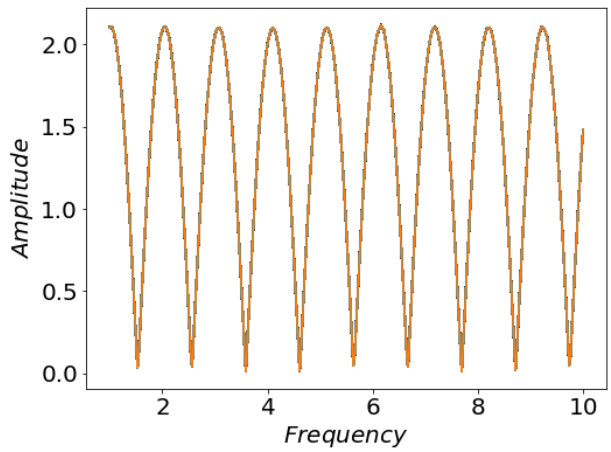


We plotted our numerical results.

We could not find a narrow peak of gravitational wave resonance.
Small waves were found in four separate areas.

Gravitational wave amplitudes for later times could not be plotted in the figure because they would overflow.

Results and discussion



This figure is amplitude of gravitational wave by frequency at a specific time.

Gravitational waves propagating in axion is a very common waveform.

The calculation method may need to be reviewed for further accurate numerical calculations.

Summary

We did not find any noticeable peaks in the gravitational waves propagating in the axion like previous study showed.

Small waves were found in four separate areas, and the calculated gravity waves were not different from those of a typical gravity wave.

Future work

The calculation method may need to be reviewed for further accurate numerical calculations.

We will investigate how gravitational waves by making arrangements such as changing the potential of the axion field (ex.$V_0[1 + cos\,\frac{\phi}{f}]$).

P17

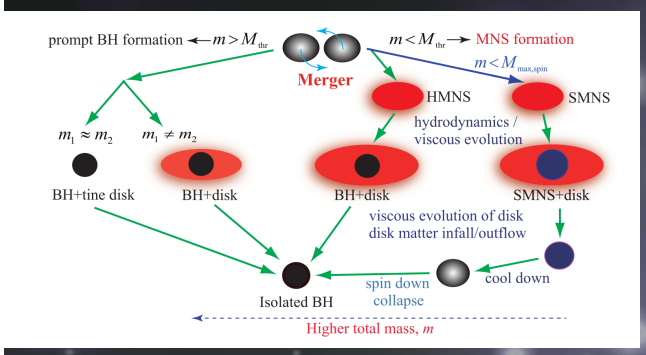
General relativistic rotating stars with arbitrary differential rotation

Kotaro Fujisawa (U. of Tokyo)

Collaborators

Hirotada Okawa (Waseda), Misa Ogata (Waseda), Shoichi Yamada (Waseda), Nobutoshi Yasutake (CIT)

A massive neutron star with differential rotation



Shibata & Hotokezaka (2019)

- A massive neutron star formed after the neutron star merger has a differential rotation
- Many previous works obtained a NS with differential rotation in GR, but the types of differential rotation are limited

We have developed a new numerical method to calculate rotating stars with arbitrary differential rotation in GR

Basic equations for General relativistic rotating star

Komatsu, Eriguchi, Hachisu (1989)(KEH)

Metric

$$ds^2 = -e^{\gamma+\rho}dt^2 + e^{2\alpha}(dr^2 + r^2 d\theta^2) + e^{\gamma-\rho}r^2\sin^2\theta(d\varphi - \omega dt)^2,$$

Euler eq. (r component)

$$\frac{1}{\varepsilon+p}\frac{\partial p}{\partial r} + \frac{\partial}{\partial r}\left(\frac{\gamma+\rho}{2}\right) - \frac{v}{1-v^2}\frac{\partial v}{\partial r} + j\frac{\partial \Omega}{\partial r} = 0,$$

Euler eq. (θ component)

$$\frac{1}{\varepsilon+p}\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\gamma+\rho}{2}\right) - \frac{v}{1-v^2}\frac{1}{r}\frac{\partial v}{\partial \theta} + j\frac{1}{r}\frac{\partial \Omega}{\partial \theta} = 0,$$

+ Einstein's equation

$$\nabla^2[\rho e^{\gamma/2}] = S_\rho(r,\mu),$$
$$\left(\nabla^2 + \frac{1}{r}\partial_r - \frac{\mu}{r^2}\partial_\mu\right)[\gamma e^{\gamma/2}] = S_\gamma(r,\mu),$$
$$\left(\nabla^2 + \frac{2}{r}\partial_r - \frac{2\mu}{r^2}\partial_\mu\right)[\omega e^{(\gamma-2\rho)/2}] = S_\omega(r,\mu),$$

$$\begin{aligned}
S_\rho(r,\mu) &= e^{\gamma/2}\Big\{8\pi e^{2\alpha}(\varepsilon+p)\frac{1+v^2}{1-v^2} + r^2(1-\mu^2)e^{-2\rho}\left(\omega_{,r}^2 + \frac{1-\mu^2}{r^2}\omega_{,\mu}^2\right) \\
&\quad + \frac{1}{r}\gamma_{,r} - \frac{\mu}{r^2}\gamma_{,\mu} + \frac{\rho}{2}\left[16\pi e^{2\alpha}p - \gamma_{,r}\left(\frac{1}{2}\gamma_{,r} + \frac{1}{r}\right) - \frac{1}{r^2}\gamma_{,\mu}\left(\frac{1-\mu^2}{2}\gamma_{,\mu} - \mu\right)\right]\Big\}, \\
S_\gamma(r,\mu) &= e^{\gamma/2}\left\{16\pi e^{2\alpha}p + \frac{\gamma}{2}\left[16\pi e^{2\alpha}p - \frac{1}{2}\gamma_{,r}^2 - \frac{1-\mu^2}{2r^2}\gamma_{,\mu}^2\right]\right\}, \\
S_\omega(r,\mu) &= e^{(\gamma-2\rho)/2}\Big\{-16\pi e^{2\alpha}\frac{(\Omega-\omega)(\varepsilon+p)}{1-v^2} \\
&\quad + \omega\left[-8\pi e^{2\alpha}\left(\frac{(\varepsilon+p)(1+v^2)}{1-v^2} - p\right) - \frac{1}{r}\left(2\rho_{,r} + \frac{1}{2}\gamma_{,r}\right) + \frac{\mu}{r^2}\left(2\rho_{,\mu} + \frac{1}{2}\gamma_{,\mu}\right)\right. \\
&\quad \left. + \frac{1}{4}(4\rho_{,r}^2 - \gamma_{,r}^2) + \frac{1-\mu^2}{4r^2}(4\rho_{,\mu}^2 - \gamma_{,\mu}^2) - r^2(1-\mu^2)e^{-2\rho}\left(\omega_{,r}^2 + \frac{1-\mu^2}{r^2}\omega_{,\mu}^2\right)\right]\Big\},
\end{aligned}$$

Energy-momentum tensor

$$T^{\mu\nu} = (\varepsilon+p)u^\mu u^\nu + pg^{\mu\nu},$$

Proper velocity

$$v = (\Omega - \omega)r\sin\theta e^{-\rho}$$

Angular momentum

$$j = u^t u_\varphi = \frac{v^2}{(1-v^2)(\Omega-\omega)}.$$

Equation of state

$$p = p(\rho)$$

$$\begin{aligned}
\alpha_{,\mu} &= -\frac{1}{2}(\rho_{,\mu} + \gamma_{,\mu}) - \{(1-\mu^2)(1+r\gamma_{,r})^2 + [\mu - (1-\mu^2)\gamma_{,\mu}]^2\}^{-1}\Big[\\
&\quad \frac{1}{2}[r^2(\gamma_{,rr} + \gamma_{,r}^2) - (1-\mu^2)(\gamma_{,\mu\mu} + \gamma_{,\mu}^2)][-\mu + (1-\mu^2)\gamma_{,\mu}] + r\gamma_{,r}\left[\frac{1}{2}\mu + \mu r\gamma_{,r} + \frac{1}{2}(1-\mu^2)\gamma_{,\mu}\right] \\
&\quad + \frac{3}{2}\gamma_{,\mu}[-\mu^2 + \mu(1-\mu^2)\gamma_{,\mu}] - r(1-\mu^2)(\gamma_{,r\mu} + \gamma_{,r}\gamma_{,\mu})(1+r\gamma_{,r}) - \frac{1}{4}\mu r^2(\rho_{,r} + \gamma_{,r})^2 \\
&\quad - \frac{r}{2}(1-\mu^2)(\rho_{,r} + \gamma_{,r})(\rho_{,\mu} + \gamma_{,\mu}) + \frac{1}{4}\mu(1-\mu^2)(\rho_{,\mu} + \gamma_{,\mu})^2 - \frac{r^2}{2}(1-\mu^2)\gamma_{,r}(\rho_{,r} + \gamma_{,r})(\rho_{,\mu} + \gamma_{,\mu}) \\
&\quad + \frac{1}{4}(1-\mu^2)\gamma_{,\mu}[r^2(\rho_{,r} + \gamma_{,r})^2 - (1-\mu^2)(\rho_{,\mu} + \gamma_{,\mu})^2] + (1-\mu^2)e^{-2\rho}\Big\{\frac{1}{4}r^4\mu\omega_{,r}^2 + \frac{1}{2}r^3(1-\mu^2)\omega_{,r}\omega_{,\mu} \\
&\quad - \frac{1}{4}r^2\mu(1-\mu^2)\omega_{,\mu}^2 + \frac{1}{2}r^4(1-\mu^2)\gamma_{,r}\omega_{,r}\omega_{,\mu} - \frac{1}{4}r^2(1-\mu^2)\gamma_{,\mu}[r^2\omega_{,r}^2 - (1-\mu^2)\omega_{,\mu}^2]\Big\}\Big].
\end{aligned}$$

First integral of the Euler equation

Komatsu, Eriguchi, Hachisu (1989)(KEH)
Cook, Shapiro, Teukolsky (1992)

If a star is stationary, there is a first integral of the Euler equation as

Euler eq.

$$\frac{1}{\varepsilon+p}\frac{\partial p}{\partial r} + \frac{\partial}{\partial r}\left(\frac{\gamma+\rho}{2}\right) - \frac{v}{1-v^2}\frac{\partial v}{\partial r} + j\frac{\partial \Omega}{\partial r} = 0,$$
$$\frac{1}{\varepsilon+p}\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\gamma+\rho}{2}\right) - \frac{v}{1-v^2}\frac{1}{r}\frac{\partial v}{\partial \theta} + j\frac{1}{r}\frac{\partial \Omega}{\partial \theta} = 0,$$

→

First integral

$$\int \frac{dp}{\varepsilon+p} + \frac{\gamma+\rho}{2} + \frac{1}{2}\ln(1-v^2) + \int j(\Omega)d\Omega = 0,$$

Rotation law (e.g. Komatsu et al. 1989)

$$j(\Omega) = A^2(\Omega_c - \Omega),$$

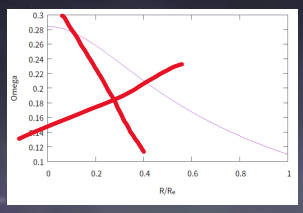
$$\int j(\Omega)d\Omega = -\frac{1}{2}A^2(\Omega_c - \Omega)^2.$$

Integrability condition: barotropic p = p(ε) (or ε(p)) & rotation law j = j(Ω) (or Ω=Ω(j))

Rotation law in GR star Ω(R)

Komatsu, Eriguchi, Hachisu (1989)
Uryu et al. (2017)
Iosif & Stergioulas (2021)

1. In general, we cannot set the angular velocity profile Ω(R) directly



2. Instead, we set the rotation law (Ω = Ω(j) or j = j(Ω))

$$j(\Omega) = A^2(\Omega_c - \Omega),$$

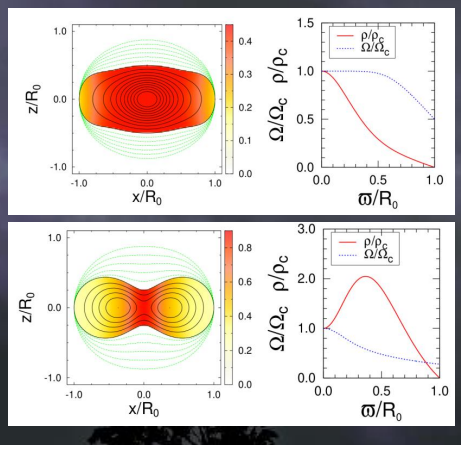
$$\Omega(j; p, q, A, B) = \Omega_c \frac{1 + (j/B^2\Omega_c)^p}{1 + (j/A^2\Omega_c)^{q+p}}. \quad (8)$$

$$\Omega(j; p, A, B) = \Omega_c \left[1 + \left(\frac{j}{B^2\Omega_c}\right)^p\right]\left(1 - \frac{j}{A^2\Omega_c}\right). \quad (9)$$

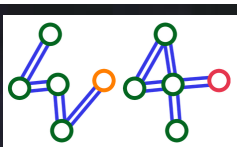
3. Calculate the first integral & Einstein's eq. iteratively

→

4. After we get a solution, we obtain an angular velocity profile Ω(R)



5. But, the physical meaning of the rotation law is unclear



Top Demo Docs

Nonlinear Solver Lab

Demonstration of the W4SV method

W4 method

Okawa, Fujisawa et al. Applied Numerical Mathematics, 183, 157 (2023)
Fujisawa, Okawa et al. ApJ. (2019)
Okawa, Fujisawa et al. arXiv:2204.09943, 2204.09941

We use the W4 method to converge the iteration

A new method for solving nonlinear eqs.

Newton-Raphson

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n + \delta\boldsymbol{x},$$

$$\delta x_i = -\sum_{j=1}^{N} J_{ij}^{-1} F_j,$$

W4

$$\frac{d^2\boldsymbol{x}}{d\tau^2} + 2\frac{d\boldsymbol{x}}{d\tau} + M\boldsymbol{F} = 0,$$

$$\boldsymbol{x}^{n+1} = \boldsymbol{x} + \frac{1}{2}L^{-1}\boldsymbol{p}^n, \quad \boldsymbol{p}^{n+1} = \frac{1}{2}U^{-1}\boldsymbol{F}(\boldsymbol{x}$$

$$J \equiv UL$$

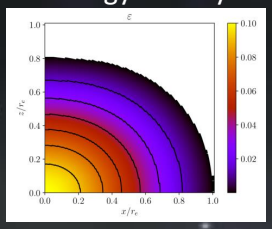
$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \ell_{21} & 1 & 0 & 0 \\ \ell_{31} & \ell_{42} & 1 & 0 \\ \ell_{41} & \ell_{42} & \ell_{43} & 1 \end{pmatrix} \quad U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix}$$

Comparison with the previous method (KEH)

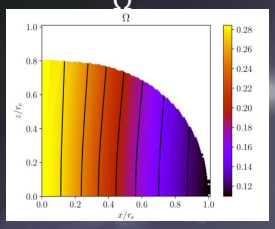
M/R∼0.12

New method (GR Virial GRV2, GRV3 ∼ 10^{-4})

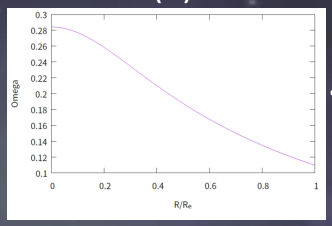
Energy density



Ω

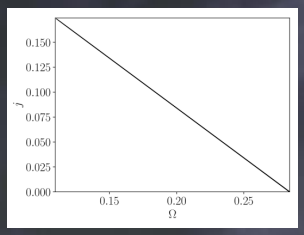


Ω(R)



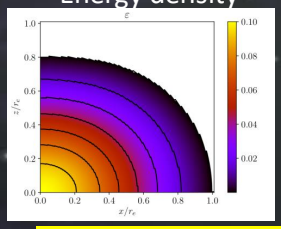
→

Rotation law

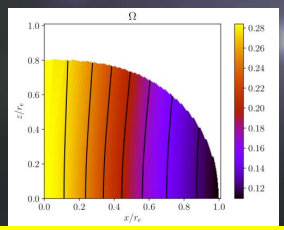


Previous method (KEH)

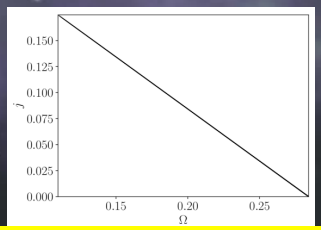
Energy density



Ω

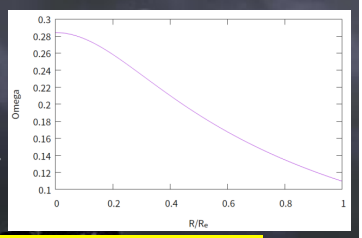


Rotation law



→

Ω(R)



Our new method reproduces well results of the previous method !

Summary

- Developed a new numerical method for obtaining differential rotating NS in GR
- Calculated rotating star with arbitrary rotation using the new method
- Obtained new solutions

Future works:

Calculate solution sequences of hot & massive NS or proto NSs by using the new numerical method

Hokkai-Gakuen University

@JGRG31
26th October 2022, University of Tokyo, Tokyo, Japan

Criteria for energy conditions

Hideki Maeda

(Hokkai-Gakuen University, Japan)

This talk is based on
HM and T. Harada, Class. Quant. Grav. 39 (2022) 19, 195002
e-Print:2205.12993 [gr-qc]

Role of Energy Conditions (ECs) in GR

- Einstein equations $G_{\mu\nu}=T_{\mu\nu}$
  - For a given matter field ($T_{\mu\nu}$), one can solve the Einstein equations
- Fundamental properties of gravity should be derived independent of the form of $T_{\mu\nu}$
- In GR, important theorems have been proven under certain energy conditions
  - Black-hole area theorem (Hawking `72)
  - Positive mass theorem (Schoen-Yau `79, Nester `81, Witten `81)
  - Black-hole positive mass theorem (Gibbons-Hawking-Horowitz-Perry `83)

Standard energy conditions (ECs)

- Null (NEC): $T_{\mu\nu}k^\mu k^\nu \geq 0$ for any null k^μ
- Weak (WEC): $T_{\mu\nu}v^\mu v^\nu \geq 0$ for any timelike v^μ

} Non-negative energy density is observed

- Dominant (DEC): WEC + $J_\mu J^\mu \leq 0$ for any timelike v^μ, where $J^\mu := -T_{\mu\nu}v^\nu$
  - Energy flux does not propagates faster than the speed of light
- Strong (SEC): $(T_{\mu\nu} - Tg_{\mu\nu}/2)v^\mu v^\nu \geq 0$ for any timelike v^μ
  - Gravity is attractive: Violation of SEC is not so pathological (ex. Inflationary Universe)
- Relations: DEC ⊂ WEC ⊂ NEC and SEC ⊂ NEC
  - NEC violation → All energy conditions are violated
  - DEC holds → WEC & NEC are respected as well (SEC is independent)

To check these conditions for a given matter field is important to exclude unphysical ones
But how? Answer: To compute components of $T_{\mu\nu}$ in a frame with orthonormal basis vectors

Hawking-Ellis type of $T_{\mu\nu}$

- Hawking-Ellis classification: Depending on the signature of eigenvectors n^μ of $T_{\mu\nu}$
  - Eigenvalue equations $$T^{(a)(b)}n_{(b)} = \lambda\eta^{(a)(b)}n_{(b)} \quad \Leftrightarrow \quad T^{\mu\nu}n_\nu = \lambda g^{\mu\nu}n_\nu,$$
  - Components in an orthonormal frame $$T^{(a)(b)} = T^{\mu\nu}E^{(a)}_\mu E^{(b)}_\nu$$
  - Basis vectors in an orthonormal frame $$E^\mu_{(a)} = (E^\mu_{(0)}, E^\mu_{(1)}, \cdots, E^\mu_{(n-1)})$$

    where $$E^\mu_{(a)}E_{(b)\mu} = \eta_{(a)(b)} = \mathrm{diag}(-1, 1, \cdots, 1)$$
- Any $T^{(a)(b)}$ can be classified into 4 types in n dimensions (Hawking & Ellis for n=4)

Type	Eigenvectors
I	1 timelike, $n-1$ spacelike
II	1 null (doubly degenerated), $n-2$ spacelike
III	1 null (triply degenerated), $n-3$ spacelike
IV	2 complex, $n-2$ spacelike

Hawking-Ellis type I and ECs

- The simplest form of $T_{(a)(b)}$ can be found for each type by local Lorentz transformations

$$\tilde{E}^{\mu}_{(a)} := L_{(a)}{}^{(b)} E^{\mu}_{(b)} \quad \text{where} \quad L_{(a)}{}^{(c)} L_{(b)}{}^{(d)} \eta_{(c)(d)} = \eta_{(a)(b)}$$

- Equivalent expressions to ECs for each HE type are available (HM & Martinez '20)
- Type I: Ex. Perfect fluid, Cosmological constant (Maxwell & Scalar field can be of type I)

$$T^{(a)(b)} = \begin{pmatrix} \rho & 0 & 0 & 0 & \cdots & 0 \\ 0 & p_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & p_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & p_{n-1} \end{pmatrix}$$

Canonical form of type I

- NEC: $\rho + p_i \geq 0$.
- WEC: $\rho + p_i \geq 0$ and $\rho \geq 0$.
- SEC: $\rho + p_i \geq 0$ and $(n-3)\rho + \sum_{i=1}^{n-1} p_i \geq 0$.
- FEC: $\rho^2 \geq p_i^2$.
- DEC: $\rho \geq |p_i|$ and $\rho \geq 0$.

Equivalent expressions to ECs

Hawking-Ellis type II and ECs

- Type II: Ex. Null dust (Maxwell & Scalar field can be of type II)

$$T^{(a)(b)} = \begin{pmatrix} \rho+\nu & \nu & 0 & 0 & \cdots & 0 \\ \nu & -\rho+\nu & 0 & 0 & \cdots & 0 \\ 0 & 0 & p_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & p_{n-1} \end{pmatrix}$$

Canonical form of type II

- NEC: $\nu \geq 0$ and $\rho + p_i \geq 0$.
- WEC: $\nu \geq 0$, $\rho + p_i \geq 0$, and $\rho \geq 0$.
- SEC: $\nu \geq 0$, $p_i + \rho \geq 0$, and $(n-4)\rho + \sum_{j=2}^{n-1} p_j \geq 0$.
- FEC: $\rho\nu \geq 0$ and $\rho^2 \geq p_i^2$.
- DEC: $\nu \geq 0$, $\rho \geq |p_i|$, and $\rho \geq 0$.

Equivalent expressions to ECs

Hawking-Ellis type III & IV and ECs

- Type III:
  Ex. Gyraton (Null dust with angular momentum)

Canonical form of type III

$$T^{(a)(b)} = \begin{pmatrix} \rho+\nu & \nu & \zeta & 0 & 0 & \cdots & 0 \\ \nu & -\rho+\nu & \zeta & 0 & 0 & \cdots & 0 \\ \zeta & \zeta & -\rho & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & p_3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & p_{n-1} \end{pmatrix}$$

- Type IV:
  Ex. Quantum vacuum expectation value $\langle T_{\mu\nu} \rangle$ can be of type IV

$$T^{(a)(b)} = \begin{pmatrix} \rho & \nu & 0 & 0 & \cdots & 0 \\ \nu & -\rho & 0 & 0 & \cdots & 0 \\ 0 & 0 & p_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & p_{n-1} \end{pmatrix}$$

Canonical form of type IV

Type III & IV violate all the ECs

Our main result: Set up

- To find basis vectors to give a canonical form of $T^{(a)(b)}$ is not easy in general
- We often encounter this form:
  - If $T^{(0)(1)}=0$, it is of type I: Easy to check ECs
  - What about if $T^{(0)(1)}\neq 0$?
- Lemma 1: Hawking-Ellis type is

$$\begin{aligned} (T^{(0)(0)}+T^{(1)(1)})^2 > 4(T^{(0)(1)})^2 &\quad\Rightarrow\quad \text{Type I,} \\ (T^{(0)(0)}+T^{(1)(1)})^2 = 4(T^{(0)(1)})^2 &\quad\Rightarrow\quad \text{Type II,} \\ (T^{(0)(0)}+T^{(1)(1)})^2 < 4(T^{(0)(1)})^2 &\quad\Rightarrow\quad \text{Type IV.} \end{aligned}$$

$$T^{(a)(b)} = \begin{pmatrix} T^{(0)(0)} & T^{(0)(1)} & 0 & 0 & \cdots & 0 \\ T^{(0)(1)} & T^{(1)(1)} & 0 & 0 & \cdots & 0 \\ 0 & 0 & p_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & p_{n-1} \end{pmatrix}.$$

- Our main result = Energy-condition criteria for this $T^{(a)(b)}$

Main result: Energy-condition criteria

- Proposition 1: Equivalent expressions to the energy conditions are

$T^{(0)(0)} + T^{(1)(1)} \geq 0,$ & $(T^{(0)(0)} + T^{(1)(1)})^2 \geq 4(T^{(0)(1)})^2$

and

$$
\begin{aligned}
NEC:\ & T^{(0)(0)} - T^{(1)(1)} + 2p_i + \sqrt{(T^{(0)(0)} + T^{(1)(1)})^2 - 4(T^{(0)(1)})^2} \geq 0 \\
& \textit{for } i = 2, 3, \cdots, n-1, \\
WEC:\ & T^{(0)(0)} - T^{(1)(1)} + \sqrt{(T^{(0)(0)} + T^{(1)(1)})^2 - 4(T^{(0)(1)})^2} \geq 0 \\
& \textit{in addition to NEC}, \\
DEC:\ & T^{(0)(0)} - T^{(1)(1)} - 2p_i + \sqrt{(T^{(0)(0)} + T^{(1)(1)})^2 - 4(T^{(0)(1)})^2} \geq 0 \\
& \textit{for } i = 2, 3, \cdots, n-1 \ \textit{ and } \ T^{(0)(0)} - T^{(1)(1)} \geq 0 \ \textit{ in addition to WEC}, \\
SEC:\ & (n-4)(T^{(0)(0)} - T^{(1)(1)}) + 2\sum_{j=2}^{n-1} p_j \\
& + (n-2)\sqrt{(T^{(0)(0)} + T^{(1)(1)})^2 - 4(T^{(0)(1)})^2} \geq 0 \ \textit{ in addition to NEC.}
\end{aligned}
$$

Application 1: Symmetric spacetime

- Consider a spacetime $M_2 \times E^{n-2}$ $\mathrm{d}s^2 = g_{AB}(y)\mathrm{d}y^A\mathrm{d}y^B + R^2(y)\gamma_{ij}(z)\mathrm{d}z^i\mathrm{d}z^j,$ E^{n-2}: (n-2)-dim Einstein space

  with $T_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = T_{AB}(y)\mathrm{d}y^A\mathrm{d}y^B + p_t(y)R(y)^2\gamma_{ij}(z)\mathrm{d}z^i\mathrm{d}z^j.$

- In diagonal coordinates:

$$\mathrm{d}s^2 = -e^{2\Phi(t,x)}\mathrm{d}t^2 + e^{2\Psi(t,x)}\mathrm{d}x^2 + R(t,x)^2\gamma_{ij}(z)\mathrm{d}z^i\mathrm{d}z^j,$$

- Equivalent expressions:

$$\left.\begin{array}{c} e^{-2\Phi}T_{tt} + e^{-2\Psi}T_{xx} \geq 0 \\ + \\ \mathcal{D}_1 \geq 0 \end{array}\right\} \text{and}$$

$$
\begin{aligned}
NEC:\ & e^{-2\Phi}T_{tt} - e^{-2\Psi}T_{xx} + 2p_t + \sqrt{\mathcal{D}_1} \geq 0, \\
WEC:\ & e^{-2\Phi}T_{tt} - e^{-2\Psi}T_{xx} + \sqrt{\mathcal{D}_1} \geq 0 \ \textit{ in addition to NEC}, \\
DEC:\ & e^{-2\Phi}T_{tt} - e^{-2\Psi}T_{xx} - 2p_t + \sqrt{\mathcal{D}_1} \geq 0 \\
& \textit{and } \ e^{-2\Phi}T_{tt} - e^{-2\Psi}T_{xx} \geq 0 \ \textit{ in addition to WEC}, \\
SEC:\ & (n-4)(e^{-2\Phi}T_{tt} - e^{-2\Psi}T_{xx}) + 2(n-2)p_t + (n-2)\sqrt{\mathcal{D}_1} \geq 0 \\
& \textit{in addition to NEC.}
\end{aligned}
$$

where $\mathcal{D}_1 := (e^{-2\Phi}T_{tt} + e^{-2\Psi}T_{xx} + 2e^{-\Phi-\Psi}T_{tx})(e^{-2\Phi}T_{tt} + e^{-2\Psi}T_{xx} - 2e^{-\Phi-\Psi}T_{tx}).$

Application 1: Symmetric spacetime

- Consider a spacetime $M_2 \times E^{n-2}$ $\mathrm{d}s^2 = g_{AB}(y)\mathrm{d}y^A\mathrm{d}y^B + R^2(y)\gamma_{ij}(z)\mathrm{d}z^i\mathrm{d}z^j,$ E^{n-2}: (n-2)-dim Einstein space

  with $T_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = T_{AB}(y)\mathrm{d}y^A\mathrm{d}y^B + p_t(y)R(y)^2\gamma_{ij}(z)\mathrm{d}z^i\mathrm{d}z^j.$
- In single-null coordinates: $\mathrm{d}s^2 = -f(u,r)\mathrm{d}u^2 - 2\epsilon e^{-\delta(u,r)}\mathrm{d}u\mathrm{d}r + R(u,r)^2\gamma_{ij}(z)\mathrm{d}z^i\mathrm{d}z^j,$
- Equivalent expressions:

$$\left.\begin{array}{c} T_{uu} - \epsilon f e^{\delta} T_{ur} + (f^2/4+1)e^{2\delta}T_{rr} \geq 0 \\ + \\ \mathcal{D}_2 \geq 0 \end{array}\right\} \text{ and } \begin{array}{ll} NEC: & 2\epsilon e^{\delta}T_{ur} - fe^{2\delta}T_{rr} + 2p_t + \sqrt{\mathcal{D}_2} \geq 0, \\ WEC: & 2\epsilon e^{\delta}T_{ur} - fe^{2\delta}T_{rr} + \sqrt{\mathcal{D}_2} \geq 0 \ \ \textit{in addition to NEC}, \\ DEC: & 2\epsilon e^{\delta}T_{ur} - fe^{2\delta}T_{rr} - 2p_t + \sqrt{\mathcal{D}_2} \geq 0 \\ & \textit{and} \ \ 2\epsilon e^{\delta}T_{ur} - fe^{2\delta}T_{rr} \geq 0 \ \ \textit{in addition to WEC}, \\ SEC: & (n-4)(2\epsilon e^{\delta}T_{ur} - fe^{2\delta}T_{rr}) + 2(n-2)p_t + (n-2)\sqrt{\mathcal{D}_2} \geq 0 \\ & \textit{in addition to NEC}. \end{array}$$

where $\mathcal{D}_2 := e^{2\delta}T_{rr}(4T_{uu} - 4\epsilon f e^{\delta}T_{ur} + f^2 e^{2\delta}T_{rr}).$

Application 1: Symmetric spacetime

- Consider a spacetime $M_2 \times E^{n-2}$ $\mathrm{d}s^2 = g_{AB}(y)\mathrm{d}y^A\mathrm{d}y^B + R^2(y)\gamma_{ij}(z)\mathrm{d}z^i\mathrm{d}z^j,$ E^{n-2}: (n-2)-dim Einstein space

  with $T_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = T_{AB}(y)\mathrm{d}y^A\mathrm{d}y^B + p_t(y)R(y)^2\gamma_{ij}(z)\mathrm{d}z^i\mathrm{d}z^j.$
- In double-null coordinates: $\mathrm{d}s^2 = -2e^{-f(u,v)}\mathrm{d}u\mathrm{d}v + R(u,v)^2\gamma_{ij}(z)\mathrm{d}z^i\mathrm{d}z^j,$
- Equivalent expressions: $T_{uu}T_{vv} \geq 0$ + $T_{uu} + T_{vv} \geq 0$

  and

$$\begin{array}{ll} NEC: & T_{uv} + p_t e^{-f} + \sqrt{T_{uu}T_{vv}} \geq 0, \\ WEC: & T_{uv} + \sqrt{T_{uu}T_{vv}} \geq 0 \ \ \textit{in addition to NEC}, \\ DEC: & T_{uv} - p_t e^{-f} + \sqrt{T_{uu}T_{vv}} \geq 0 \ \ \textit{and} \ \ T_{uv} \geq 0 \ \textit{in addition to WEC}, \\ SEC: & (n-4)T_{uv} + (n-2)p_t e^{-f} + (n-2)\sqrt{T_{uu}T_{vv}} \geq 0 \ \ \textit{in addition to NEC}. \end{array}$$

Application 2: Generalized Kerr spacetime

- Gurses-Gursey spacetime ('75): Kerr if M(r) is constant

$$
\begin{aligned}
\mathrm{d}s^2 = &-\left(1-\frac{2M(r)r}{\Sigma(r,\theta)}\right)\mathrm{d}t^2 - \frac{4aM(r)r\sin^2\theta}{\Sigma(r,\theta)}\mathrm{d}t\mathrm{d}\phi \\
&+\frac{\Sigma(r,\theta)}{\Delta(r)}\mathrm{d}r^2 + \Sigma(r,\theta)\mathrm{d}\theta^2 + \left(r^2+a^2+\frac{2a^2M(r)r\sin^2\theta}{\Sigma(r,\theta)}\right)\sin^2\theta\mathrm{d}\phi^2, \\
&\Sigma(r,\theta) := r^2 + a^2\cos^2\theta, \qquad \Delta(r) := r^2 + a^2 - 2rM(r),
\end{aligned}
$$

  - This metric has been used to construct non-singular black-hole models
- Equivalent expressions to ECs in GR ($T_{\mu\nu} = G_{\mu\nu}$) :

$$
\begin{aligned}
&NEC: \ 2M'(r^2 - a^2\cos^2\theta) - rM''\Sigma \geq 0, \\
&WEC: \ M' \geq 0 \ \ \textit{in addition to NEC}, \\
&DEC: \ rM'' + 2M' \geq 0 \ \ \textit{in addition to WEC}, \\
&SEC: \ rM''\Sigma + 2M'a^2\cos^2\theta \leq 0 \ \ \textit{in addition to NEC}.
\end{aligned}
$$

Application 3 (without spacetime symmetry) : Imperfect fluid without shear viscosity

- Imperfect fluid: $T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} + (u_\mu q_\nu + q_\mu u_\nu) + \pi_{\mu\nu},$ where $u_\mu u^\mu = -1, \ h_{\mu\nu} := g_{\mu\nu} + u_\mu u_\nu$
  - q^μ = heat flux vector (spacelike) $u_\mu q^\mu = 0,$ Projection tensor $h_{\mu\nu}u^\nu = 0,$
  - Viscous shear tensor $\pi_{\mu\nu} = -\zeta\theta h_{\mu\nu} - 2\eta\sigma_{\mu\nu},$ where $\theta := \nabla_\mu u^\mu,$ Expansion scalar
    - Coefficient of bulk viscosity $\zeta(\geq 0)$
    - Coefficient of shear viscosity $\eta(\geq 0)$
    - Orthogonal to u^μ : $\pi_{\mu\nu}u^\mu = 0.$

$$\sigma_{\mu\nu} := \nabla_{(\mu}u_{\nu)} + a_{(\mu}u_{\nu)} - \frac{1}{n-1}\theta h_{\mu\nu}.$$

Shear tensor (symmetric)

$a^\mu := u^\nu\nabla_\nu u^\mu$ Acceleration vector

$a_\mu u^\mu = 0$ ➡ $\sigma^\mu{}_\mu = 0$ & $\sigma_{\mu\nu}u^\nu = 0.$

- Equivalent expressions in the case of η=0:
  - NEC: $(\rho + p - \zeta\theta)^2 \geq 4Q^2$ + $\rho + p - \zeta\theta \geq 0$ where $Q^2 := q_\mu q^\mu.$
  - Others:

$$
\begin{aligned}
&WEC: \ \rho - p + \zeta\theta + \sqrt{(\rho+p-\zeta\theta)^2 - 4Q^2} \geq 0, \text{ + NEC} \\
&DEC: \ \rho - 3(p-\zeta\theta) + \sqrt{(\rho+p-\zeta\theta)^2 - 4Q^2} \geq 0 \\
&\qquad \textit{and} \ \ \rho - p + \zeta\theta \geq 0 \ \ \textit{in addition to WEC}, \\
&SEC: \ (n-4)\rho + n(p-\zeta\theta) + (n-2)\sqrt{(\rho+p-\zeta\theta)^2 - 4Q^2} \geq 0. \text{ + NEC}
\end{aligned}
$$

Application 4 (without spacetime symmetry) : Minimally coupled scalar field

- Lagrangian density: $\mathcal{L}_{\rm m} = -\left(\frac{1}{2}\varepsilon(\nabla\phi)^2 + V(\phi)\right),$ ➡ $T_{\mu\nu} = \varepsilon(\nabla_\mu\phi)(\nabla_\nu\phi) - g_{\mu\nu}\left(\frac{1}{2}\varepsilon(\nabla\phi)^2 + V(\phi)\right).$
  - ε = 1: Real scalar field, ε = -1: Ghost scalar field
- Equivalent expressions to ECs :
  - If $\nabla_\mu\phi = 0$: NEC holds, WEC is $V \geq 0$, DEC is $V \geq 0$, SEC is $V \leq 0$
  - If $\nabla_\mu\phi \neq 0$: with ε = -1 : All ECs are violated
  - If $\nabla_\mu\phi \neq 0$: with ε = 1 : NEC holds, and others depend on signature of $\nabla_\mu\phi$ as

$\nabla_\mu\phi(\neq 0)$ *with* $\varepsilon = 1$	*WEC*	*DEC*	*SEC*
Timelike	$V \geq (\nabla\phi)^2/2$	$V \geq 0$	$V \leq -(n-2)(\nabla\phi)^2/2$
Spacelike	$V \geq -(\nabla\phi)^2/2$	$V \geq 0$	$V \leq 0$
Null	$V \geq 0$	$V \geq 0$	$V \leq 0$

All the ECs are satisfied if and only if the scalar field is real (ε = 1) and massless (V=0)

Summary

- Now we know how to identify Hawking-Ellis type & Energy conditions for

$$T^{(a)(b)} = \begin{pmatrix} T^{(0)(0)} & T^{(0)(1)} & 0 & 0 & \cdots & 0 \\ T^{(0)(1)} & T^{(1)(1)} & 0 & 0 & \cdots & 0 \\ 0 & 0 & p_2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & p_{n-1} \end{pmatrix}.$$

- We have adopted our main results to 4 different systems as demonstrations
  - We have obtained equivalent expressions to ECs without assuming any spacetime symmetry for
    - Imperfect fluid without shear viscosity $T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} + (u_\mu q_\nu + q_\mu u_\nu) + \pi_{\mu\nu},$ with $\pi_{\mu\nu} = -\zeta\theta h_{\mu\nu}$
    - Minimally coupled scalar field

$$T_{\mu\nu} = \varepsilon(\nabla_\mu\phi)(\nabla_\nu\phi) - g_{\mu\nu}\left(\frac{1}{2}\varepsilon(\nabla\phi)^2 + V(\phi)\right).$$

Hunting dark energy with pressure-dependent photon-photon scattering

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[3] Graduate School of Advanced Science and Engineering, Hiroshima University

Introduction

A variety of independent observations have confirmed accelerated expansion of the Universe, which indicates dark energy (DE). Modified gravity theory has been considered one of the solutions to DE problem as an alternative to cosmological constant to GR. It is known that a dilatonic scalar field shows up in modified gravity for DE. A class of such a dilaton is called chameleon field.

Recently, it was suggest that a stimulated radar-collider concept may be applicable to probe gravitationally coupled scalar field. We demonstrate feasibility analysis for hunting chameleons as DE in F(R) gravity with the stimulated photon-photon collider.

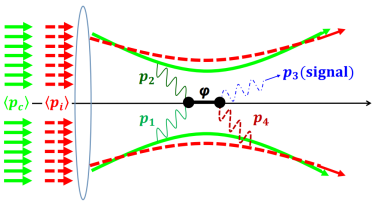

Fig.1 Concept of the stimulated resonant photon-photon scattering in quasiparallel collision geometry

Methods

Chameleon mechanism and coupling to photon

Chameleon field $\varphi(x)$ in generic scalar-tensor theory:

$$S = \int d^4x\sqrt{-g_*}\left[\frac{M_{\rm pl}^2}{2}R_* - \frac{1}{2}g_*^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi) - V(\varphi)\right] + \int d^4x\sqrt{-g_*}\mathcal{L}_{\rm M}\left[g_*^{\mu\nu},\varphi,\Phi_i\right]$$

Chameleon mass m_φ changes depending on other matters.

$$V_{\rm eff}(\varphi) \equiv V(\varphi) - \frac{1}{4}\sum_i e^{-\frac{4\beta_i\varphi}{M_{\rm pl}}}T^\mu_{\mu(i)}\,,\ m_\varphi^2 = V(\varphi_{\rm min})_{{\rm eff},\varphi\varphi}$$

Interaction to two-photons induced by trace anomaly:

$$\mathcal{L}_{\rm M} = \frac{1}{4}e^{-\frac{4\beta_\gamma\varphi}{M_{\rm pl}}}T^\mu_{\mu({\rm anomaly})}\,,\ \mathcal{L}_{\varphi\gamma\gamma} = \frac{1}{4}\frac{\tilde{\beta}_\gamma\varphi}{M_{\rm pl}}F_{\mu\nu}F^{\mu\nu}\,,\ \tilde{\beta}_\gamma \equiv \frac{b_{\rm em}\cdot\alpha_{\rm em}}{2\pi}\beta_\gamma$$

F(R) gravity and coupling constant

In F(R) gravity, bare potential is written in terms of F(R) function.

$$V(\varphi) \equiv \frac{M_{\rm pl}^2}{2}\frac{R(\varphi)F_R(R(\varphi)) - F(R(\varphi))}{F_R^2(R(\varphi))}\,,\ {\rm e}^{2\sqrt{1/6}\varphi/M_{\rm pl}} \equiv F_R(R)$$

$\beta_i = 1/\sqrt{6}$ for all matters, thus $\tilde{\beta}_\gamma = O(10^{-3})$ and $\beta_m = O(1)$.

Expected sensitivity in mass-coupling domain

Because the coupling constants are independent of F(R) function, F(R) gravity models are testable if they predict chameleon fields with masses around $m_\varphi = (0.1 - 1)\ [\mu{\rm eV}]$.

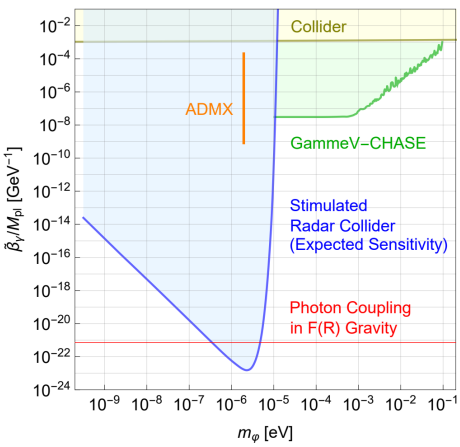

Fig.2 Expected detection sensitivity in proposed photon-photon scattering experiment (blue) compared with constraints from GammeV-CHASE (green), ADMX (orange), and collider experiments (yellow)

Red horizontal line corresponds to chameleon-photon coupling predicted in F(R) gravity.

Chameleon in F(R) gravity is hunted along the red line that overlaps with the detection sensitivity domain.

Results

Chameleon mechanism

Two ambient sources: background photon and gas in chamber.

$$T^\mu_\mu = T^\mu_{\mu({\rm photon})} + T^\mu_{\mu({\rm gas})}$$

The photon contribution is proportional to $F_{\mu\nu}F^{\mu\nu} = 2(\boldsymbol{B}^2 - \boldsymbol{E}^2)$. It turns out that square of field strength vanishes on focal plane in case of circularly polarized beams. Therefore, we can ignore the contribution from background photon density.

The gas contribution is approximated as EOS for the ideal gas.

$$T^\mu_{\mu({\rm gas})} = -(\rho - 3P)\,,\ -T^\mu_{\mu({\rm gas})} \approx \rho\,[{\rm kg/m^3}] = \frac{P\,[{\rm Pa}]}{\mathcal{R}T}$$

Evaluation of chameleon mass

We consider an R^2 corrected DE model to which Hu-Sawicki and Starobinsky models are reduced.

$$F(R) = R - \lambda R_c\left[1 - \left(\frac{R}{R_c}\right)^{-2n}\right] + \alpha R^2 \quad \lambda, \alpha, n \text{ are positive parameters}$$

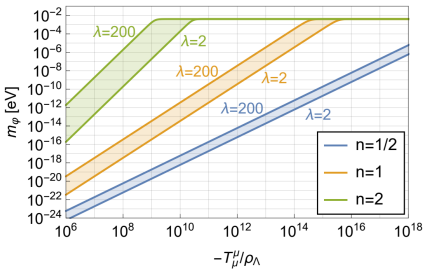

Fig.3 Chameleon mass vs energy-momentum tensor for $n = 1/2, 1, 2$ with $\Lambda = 4.2 \times 10^{-66}\ [{\rm eV}^2]$. As a referenced value, we choose $\alpha = 10^4\ [{\rm eV}^{-2}]$ to be consistent with current constraint from fifth-force experiment.

Feasibility to extract chameleon character

We apply the model to a low pressure interval $10^{-8}\ [{\rm Pa}] < P < 10^{-6}\ [{\rm Pa}]$ within the reach of current vacuum technology.
In such an ultrahigh vacuum chamber, residual gas consists mainly of hydrogen molecules, and for the hydrogen at $T = 300\ [{\rm K}]$ gives

$$1.4 \times 10^{12} \lesssim -\frac{T^\mu_\mu}{\rho_\Lambda} \lesssim 1.4 \times 10^{14}\,,\ \rho_\Lambda \sim 2.5 \times 10^{-11}\ [{\rm eV}^4]$$

For the range $\lambda = 2 - 200$, $n = 1$ case predicts chameleon in testable range. $n = 1/2$ case is also testable by raising gas pressure or changing gas species, although larger λ for $n = 1/2$ is testable even in the current setup.

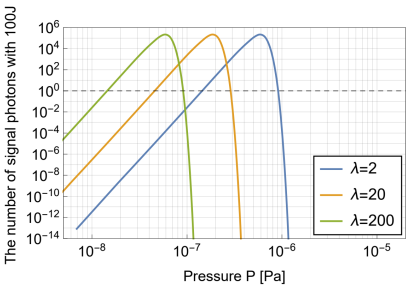

Fig.4 Expected number of stimulated signal photons as a function of gas pressure comprising hydrogen molecules at 300 [K] for $n = 1$ with $\lambda = 2, 20, 200$ cases. The dashed horizontal line shows unity above which experiments can test characteristic features of chameleon models.

Conclusion

- We found remarkable potential to probe chameleon particles in the stimulated photon-photon scattering experiment.
- Proposed method can provide a unique opportunity to strictly constrain viable F(R) gravity models for DE.
- This method will pave a way to directly unveil DE in completely controlled manner with future developments on technologies.

Determining parameters of a Black-Hole Accretion-Disk system by observing the shadow

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arXiv:2210.02164

The statements in this talk are those of the authors and do not represent the views of Sumitomo Mitsui Banking Corporation or its staffs.

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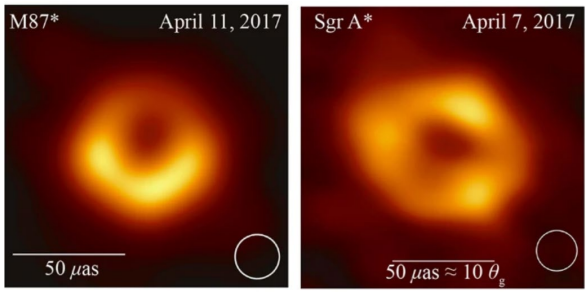
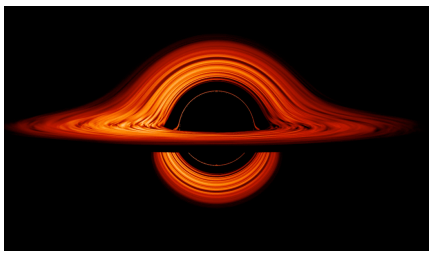
Oct. 26th, 2022

Introduction

- Shadows of BHs (M87 and Sgr A*) were observed by EHT!!
- We will (hopefully) observe many BH shadows near future.
- Shadows of BH w/ and w/t Accretion Disk (AD) contain much info.
  - M: mass
  - $J = Ma$: angular momentum $(0 \leq \frac{a}{M} < 1)$
  - i: inclination angle
  - r_o: distance between BH & observer
- Can we determine these parameters solely by observing shadows?



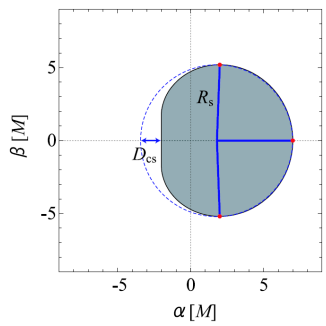
BH shadow of M87 and Sgr A*
(credit: Event horizon Telescope)



BH w/ Accretion Disk
(credit: NASA)

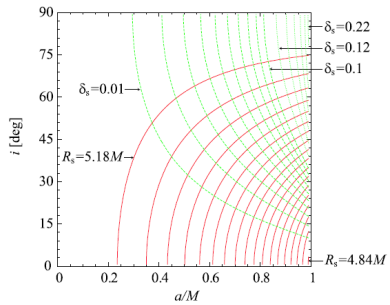
Works so far & Question

- Many papers study BH shadow
  - Bare BHs (Synge '66, Bardeen '73, …)
  - BHs with AD (Luminet '79, Falcke et al '00, Takahashi '04, …)
- Determining $(\frac{a}{M}, i)$ of Kerr BH from shadow (Hioki-Maeda '09)
- Most works assume $r_o \to +\infty$
  Info. of distance is lost.
  (Exceptions: Grenzenbach et al '14, Abdolrahimi-Mann-Tzounis '15)
- Question: If we allow r_o to be FINITE, can we determine all parameters (e.g. M) from shadow?
  ➔ Let's consider Sch. BH with AD as 1st step.



R: radius of an approx. circle
D: size of dent
$\delta := D/R$

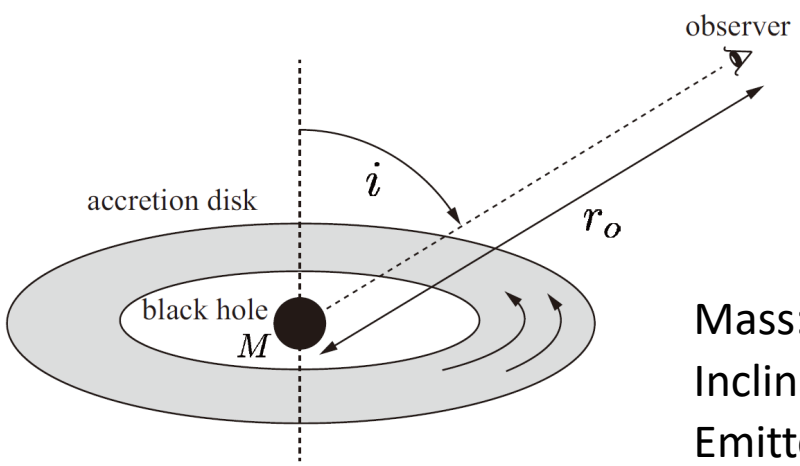
(Hioki-Maeda '09)



∃ 1-to-1 corresp. between $(\frac{a}{M}, i)$ and (R, δ)

Setup: 3 parameters of our system

(Schwarzschild BH + thin AD + observer)

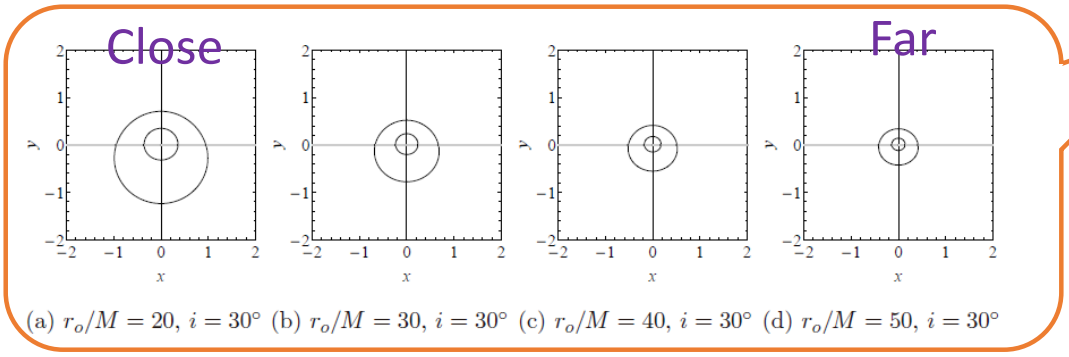


$(c = G = 1)$

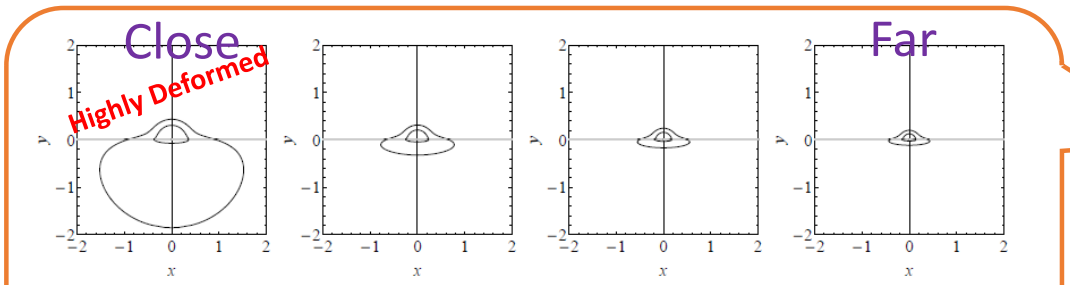
Mass: $M \in (0,+\infty)$
Inclination angle: $i \in [0^\circ, 90^\circ)$
Emitter on AD: $r_e \in [6M, 20M]$
Observer: $r_o \in [20M, +\infty)$

Results: Apparent shape of Accretion Disk

$i = 30^\circ$

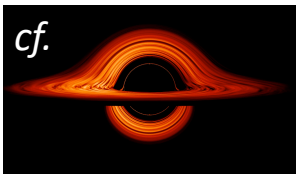


(a) $r_o/M = 20$, $i = 30°$ (b) $r_o/M = 30$, $i = 30°$ (c) $r_o/M = 40$, $i = 30°$ (d) $r_o/M = 50$, $i = 30°$

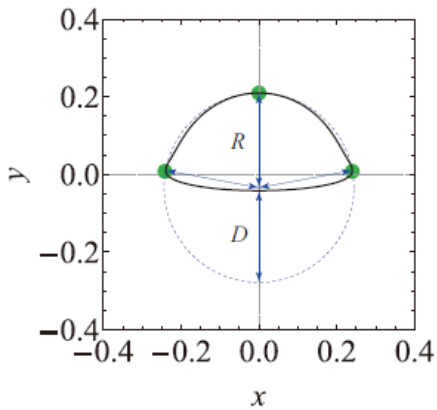


(e) $r_o/M = 20$, $i = 80°$ (f) $r_o/M = 30$, $i = 80°$ (g) $r_o/M = 40$, $i = 80°$ (h) $r_o/M = 50$, $i = 80°$

$i = 80^\circ$

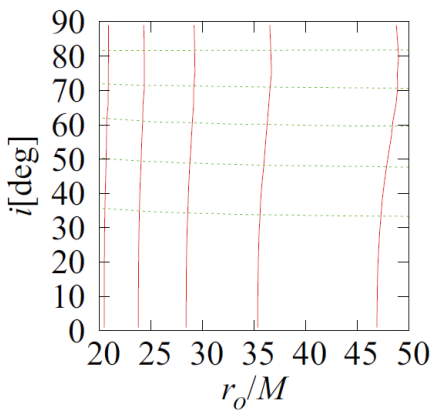


Determine $(\frac{r_o}{M}, i)$ from BH's Apparent Shape



Apparent shape of BH

R: radius of an approx. circle
D: size of dent
$\delta := D/R$



Contours of R & δ
Red: R=const.
Green: δ=const.

∃ 1-to-1 corresp. between
$(\frac{r_o}{M}, i)$ and (R, δ)

Determine (M, r_o, i) from flux & accretion rate

- Energy flux on AD $F_o(x,y)$ can be estimated (Page-Thorne '74)

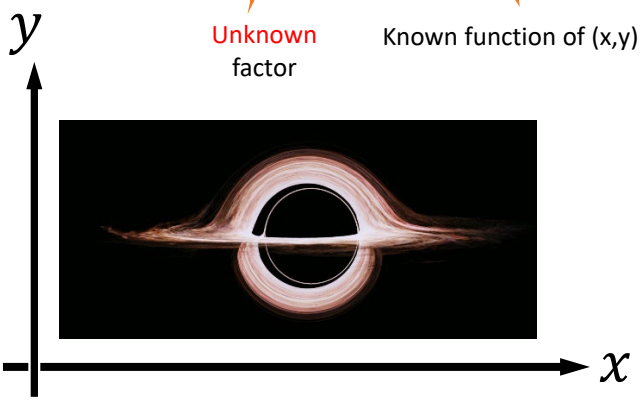
$$F_o(x,y) = \frac{\dot{M}}{M^2} F_{o*}(x,y)$$

Unknown factor

Known function of (x,y)

- Assuming that we know $\dot{M}$, we can estimate M by observing $F_o(x,y)$ and comparing it with $F_{o*}(x,y)$.

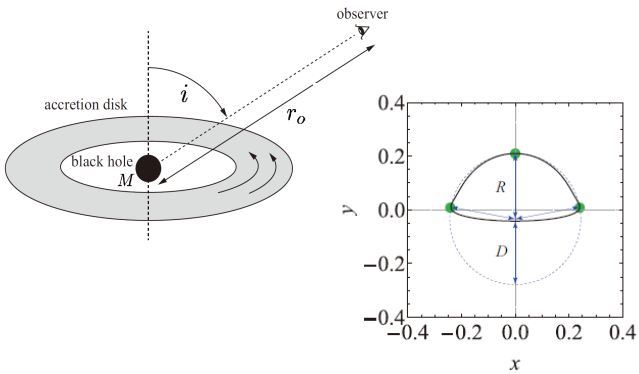
$$M = \sqrt{\dot{M}\frac{F_{o*}(x,y)}{F_o(x,y)}}$$

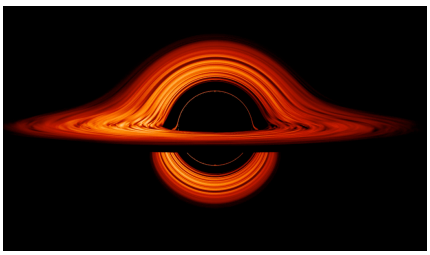


Conclusion & Prospects

- We considered system of (Sch. BH) + (thin AD) + (Observer) w/ 3 parameters (M, r_o, i).
- $\left(\frac{r_o}{M}, i\right)$ can be determined by (R, δ) (size & shape of shadow).
- (M, r_o, i) can be determined by (R, δ, F_o) and $\dot{M}$.

- Future works
  - Kerr BH w/ and w/t AD
  - Determining BH parameters from shadow's movie



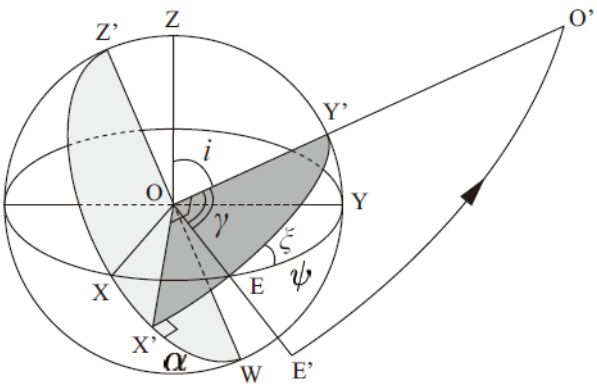


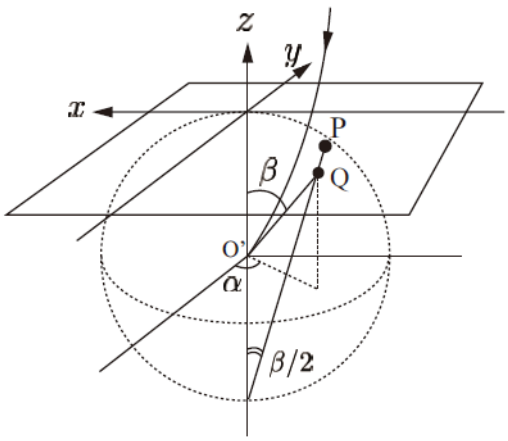
BH w/ Accretion Disk
(credit: NASA)

Appendix

How to draw 2d image:

Stereographic projection onto (x,y) plane

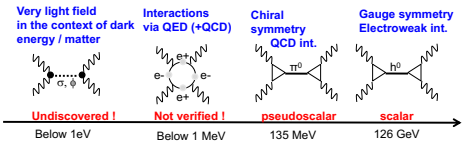


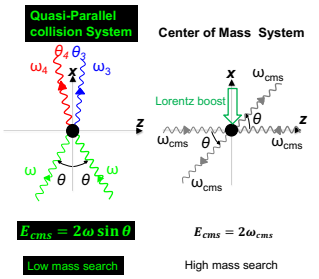


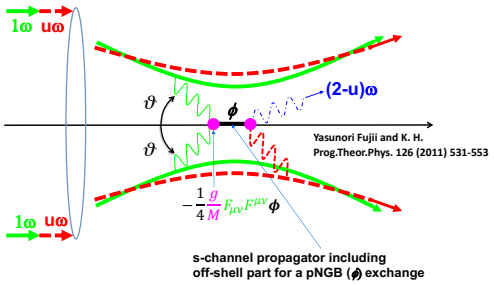
P21 Perspectives of a stimulated GHz-photon collider for probing gravitationally weak coupling scalar fields

Kensuke Homma, Hiroshima University

K. H. and Y. Kirita
Stimulated radar collider for probing gravitationally weak coupling pseudo Nambu-Goldstone bosons , JHEP 95 (2020)







Scattering amplitude for two-body interactions in stimulated resonant scattering

$$S^{(2)} = \frac{i^2}{2}\int d^4x \int d^4y T[F_{\mu\nu}(x)F^{\mu\nu}\phi(x)F_{\rho\sigma}(y)F^{\rho\sigma}(y)\phi(y)]$$
$$N[F_{\mu\nu}(x)F^{\mu\nu}(x)F_{\sigma\rho}(y)F^{\sigma\rho}(y)\langle 0|T[\phi(x)\phi(y)]|0\rangle]$$
$$\propto a_i^\dagger a_j^\dagger a_{\vec{k}}\, a_{\vec{l}}$$ pNGB-propagator

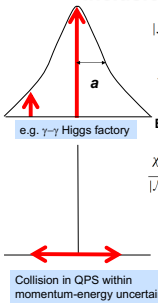
Coherent state:

$$|N\rangle \equiv \exp(-N/2)\sum_{n=0}^{\infty}\frac{N^{n/2}}{\sqrt{n!}}|n\rangle \quad |n\rangle = \frac{1}{\sqrt{n!}}\left(a^\dagger\right)^n|0\rangle$$
$$\langle\langle N|N\rangle\rangle = 1 \quad \langle\langle N|\left(a^\dagger a\right)|N\rangle\rangle = N \quad a|N\rangle\rangle = \sqrt{N}|N\rangle\rangle, \quad \text{and} \quad \langle\langle N|a^\dagger = \sqrt{N}\langle\langle N|$$

Transition amplitude: 1 + 1 → 3 + 4

$$\ll N_1| \ll N_4| \ll 1_3|S^{(2)}|N_1\gg|N_4\gg|0>$$
$$\propto \ll N_1| \ll N_4| \ll 1_3|a_i^\dagger a_j^\dagger a_{\vec{k}}\, a_{\vec{l}}\,|N_1\gg|N_4\gg|0>$$
$$\propto \sqrt{N_1}\sqrt{N_1}\sqrt{N_4} \ll N_1|N_1\gg\ll N_4|N_4\gg<0|0>$$

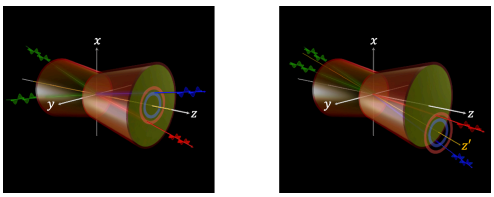
Inclusion of a pole is essential

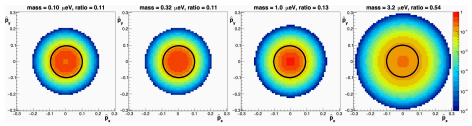


$$|\mathcal{M}|^2 \approx (4\pi)^2\frac{a^2}{\chi^2+a^2} \qquad a = \frac{\omega_r^2}{16\pi}\left(\frac{gm}{M}\right)^2$$
$$\chi = \omega^2 - \omega_r^2 \quad \omega_r^2 = \frac{m^2/2}{1-\cos 2\vartheta}$$
$$\chi \gg a \to |\mathcal{M}|^2 \propto a^2 \propto M^{-4}$$
$$\omega = \omega_r \to |\mathcal{M}|^2 \propto (4\pi)^2$$
$$\chi_\pm \equiv \pm\eta a \text{ with } \eta \gg 1$$
$$\overline{|\mathcal{M}|^2} = \frac{1}{\chi_+ - \chi_-}\int_{\chi_-}^{\chi_+}|\mathcal{M}|^2 d\chi$$
$$= \frac{(4\pi)^2}{2\eta a}2a\tan^{-1}(\eta) = (4\pi)^2\eta^{-1}\tan^{-1}(\eta)$$
$$\approx (4\pi)^2\eta^{-1}\frac{\pi}{2} = 8\pi^3\frac{a}{|\chi_\pm|}$$

Gain by M^2

Extension to asymmetric-incident and non-coaxial collisions in quasi-parallel collisions





Cubic dependence is essential

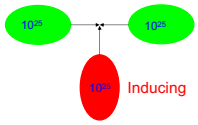
proton-proton collisions

Combination ~$(10^{11})^2$

100 W light bulb (100 J / s)

Visible light — *100J ~10^{20} photons*

GHz microwave — *100J ~10^{25} photons*

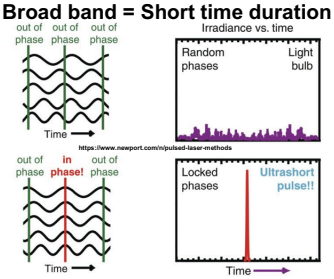


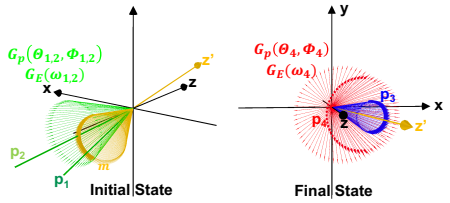
Combination $(10^{25})^3$

Possible to observe scattering via gravitational coupling strength

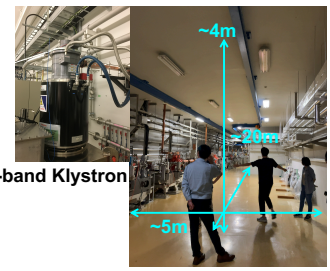
Fourier transform limit is essential

Broad band = Short time duration





Preparation at SPring-8

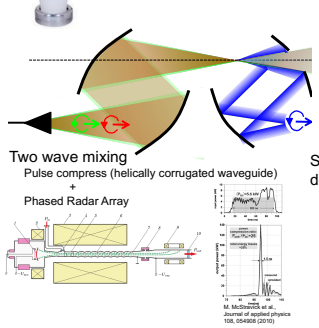


S-band Klystron

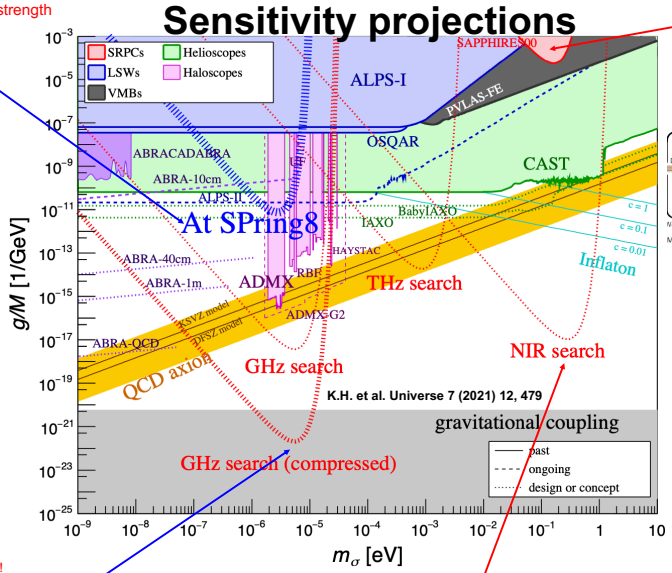
Klystron can provide ~10^{25} photons

Toshiba E3712

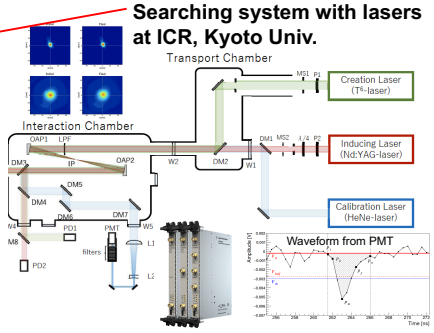
Central frequency: 2.856 GHz
Line width : ~10^{-4} too narrow !
Peak power : 100 MW (100 J / μs)
Repetition rate : 50 Hz too long !

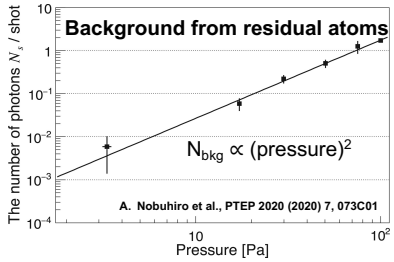


Sensitivity projections



Searching system with lasers at ICR, Kyoto Univ.

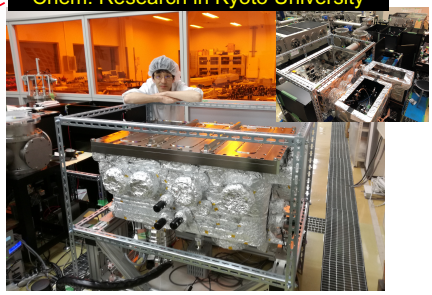




High-intensity laser 10 PW at Extreme-Light-Infrastructure (ELI-NP)

K. H. et al. (SAPPHIRES) JHEP 12 (2021) 108

High-intensity laser 10 TW at Inst. for Chem. Research in Kyoto University



[P22] JGRG31/ Univ. of Tokyo / 24-28 Oct. (2022)

Testing the equivalence principle using a gravitationally bound quantum system: with a short report of technological developments

International Center for Elementary Particle Physics,
The Univ. of Tokyo

Yoshio Kamiya
kamiya@icepp.s.u-tokyo.ac.jp
http://www.icepp.s.u-tokyo.ac.jp/~kamiya/

Special Things on Gravity

Gravity is the very common force experienced in everyday life, however the most unusual from the view of particle physics. → Very attractive!

- Extremely weak! compered to the other forces.

  Gravity between protons is weaker than Coulomb force by 10^{-36} .

Electroweak scale	~ 0.1 TeV (Vacuum Expectation Value of the Higgs)
Gravitational Interaction scale	~ 10^{16} TeV (the Planck mass)

Question 1

Is there any force with intermediate strength? — fifth force search experiment

Y. K., K. Itagaki, M. Tani *et al.*,
PRL 114, 161101 (2015)

Special Things on Gravity

Gravity is the very common force experienced in everyday life, however the most unusual from the view of particle physics. → Very attractive!

- Geometry! as a result of the equivalence between inertial and gravitational mass
  → Weak Equivalence Principle (WEP)
  → General Relativity

The WEP have been experimentally confirmed by several tests.

. . . mostly in the classical framework.

Question 3

Is the weak equivalence principle OK in the framework of quantum mechanics? — test of WEP in quantum regime

Question 2

Is there any observation of quantum signature due to the gravitational field? — test of quantum signature due to gravity

There were not so many.

G. Ichikawa, S. Komamiya, Y. K. *et al.*, PRL 112, 071101 (2014)

Storyline

Question 1

Is there any force with intermediate strength? — fifth force search experiment

Y. Kamiya, K. Itagaki, M. Tani *et al.*, PRL 114, 161101 (2015)
"Constraints on New Gravitylike Forces in the Nanometer Range"
(Fifth force might violate the WEP at the microscopic range!)

Question 2 ←

Is there any observation of quantum signature due to the gravitational field? — test of quantum signature due to gravity

There were not so many.

G. Ichikawa, S. Komamiya, Y. Kamiya *et al.*, PRL 112, 071101 (2014)
"Observation of the Spatial Distribution of Gravitationally Bound Quantum States of Ultracold Neutrons and its Derivation using the Wigner Function"

We got new quantum probe for gravity at the micron range!

Question 3

Is the weak equivalence principle OK in the framework of quantum mechanics? — test of WEP in quantum regime

Main topic of today's talk

Bouncing neutron: How is UCN trapped in Gravity?

When you spill UCN on the floor, they bounce like a ball because of their small kinetic energy.

Schrodinger equation for UCNs under the gravity

$$\{-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V(z)\}\psi_n(z) = E_n\psi_n(z),$$

neutron mass

where $V(z) = \begin{cases} mgz, & z \geq 0 \\ \infty, & z \leq 0 \end{cases}$

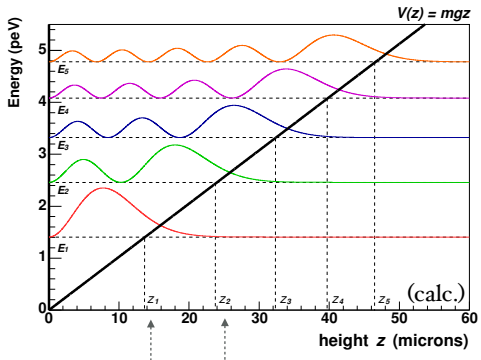
Dimensionless equation is written as a function of $\xi_n \equiv z/z_0 - E_n/E_0$

$$(\frac{\mathrm{d}^2}{\mathrm{d}\xi_n^2} - \xi_n)\psi_n(\xi_n) = 0$$

where the system's scales,

$$z_0 = (\frac{\hbar^2}{2m^2 g})^{1/3} \quad \sim 6\ \mu\mathrm{m}$$

$$E_0 = (\frac{mg^2\hbar^2}{2})^{1/3} \quad \sim 0.6\ \mathrm{peV}$$



Classical turning points
$z_n \equiv z_0 E_n/E_0 \quad (\xi_n = 0)$

Gravitationally Bound Quantum States of UCNs

V. V. Nesvizhevsky et al., Nature 415, 297 (2002)

Exp. Setup

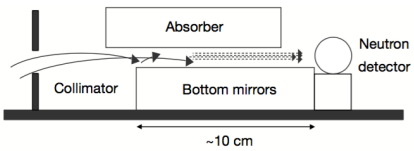


Figure 2 Layout of the experiment. The limitation of the vertical velocity component depends on the relative position of the absorber and mirror. To limit the horizontal velocity component we use an additional entry collimator. The relative height and size of the entry collimator can be adjusted.

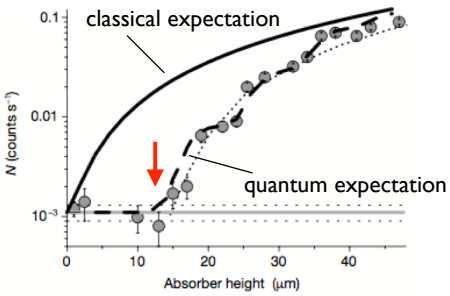


Figure 4 The neutron throughput versus the absorber height at low height values. The data points are summed up in intervals of 2 μm. The dashed curve corresponds to a fit using the quantum-mechanical calculation, in which all level populations and the height resolution are fitted from the experimental data. The solid curve is again the full classical treatment. The dotted line is a truncated fit in which it is assumed that only the lowest quantum state—which leads to the first step—exists.

Neutron transmission through this thin neutron guide was measured as a function of the height.

Neutron Guide:

length = 10 cm

height = ~ several tens microns

The ceiling scatters neutrons with higher energy (at higher energy states).

If the quantum states are formed in the guide,

When the height is equal to the spatial width of the lowest quantum state, the transmission would increase sharply.

The data shows a sharp edge around 13 micron.

Scale of the wave function for the first(grand) state.

It seems to be a quantum feature.

If neutron behave classically,

transmission would increase smoothly as $N \propto g^{1/2}h^{3/2}$

(h: height of the guide)

Gravitationally Bound Quantum States of UCNs

UCNs: Ultra-cold neutrons are

Very very slow neutrons:

Velocity : ~ 10 m/s → Carl Lewis might catch them!

Wavelength : ~ 1000 Å → UCN feels potentials from several atoms in materials simultaneously.

— pseudo fermi potentials

IAAF World Championships @ Tokyo (1991)

Kinetic energy : ~ 100 neV

comparable to

Potential of usual materials : ~ 100 neV (Al: 54 neV, Ni 244 neV)

Gravitational potential : ~ 100 neV/m

→ UCNs can be confined to a bottle or trapped in gravity.

Bouncing neutron: How is UCN trapped in Gravity?

When you spill UCN on the floor, they bounce like a ball because of their small kinetic energy.

Schrodinger equation for UCNs under the gravity

$$\{-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}z^2} + V(z)\}\psi_n(z) = E_n\psi_n(z),$$

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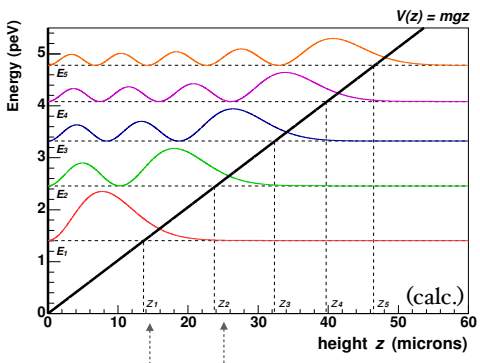

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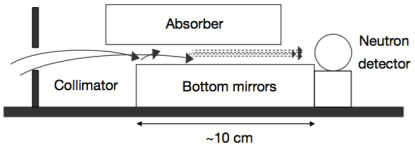

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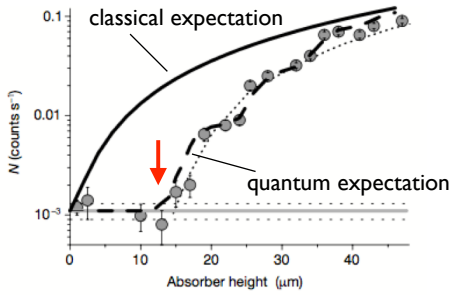

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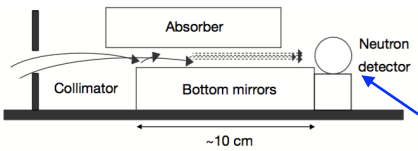

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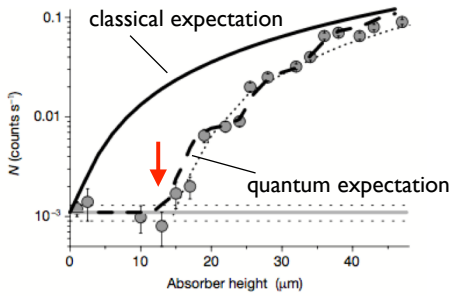

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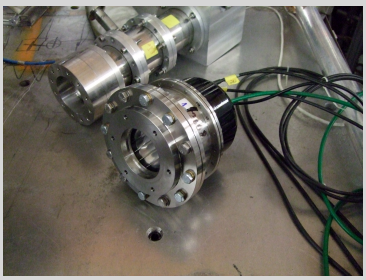
Detector:

probably ^{3}He gas filled proportional counter.



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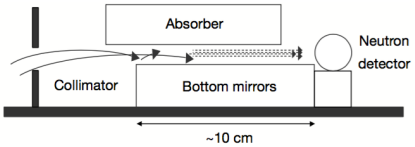

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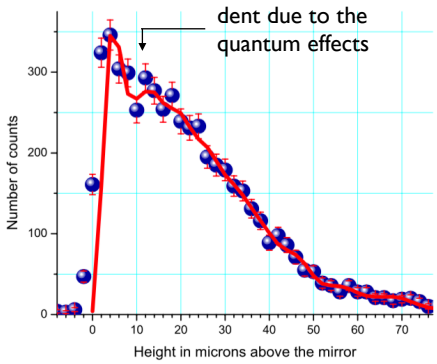


Neutron distribution was directly measured using a position sensitive neutron detector.

Fixed height of the guide is to avoid uncertainties which come from a modelling of the ceiling.

Detector: ^{235}U coated CR39 (plastic)

V.V. Nesvizhevsky et al., NIM A 440, 754 (2000)

<Principle>

Daughter nuclei from nuclear fission are make defects on the CR39.

The defects are enlarged by chemical etching.

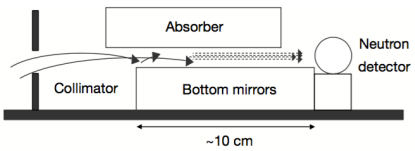
Measure the points using microscope.

Spatial resolution is reported to be about 1 micron.

Gravitationally Bound Quantum States of UCNs

V. V. Nesvizhevsky et al., Nature 415, 297 (2002)

Exp. Setup



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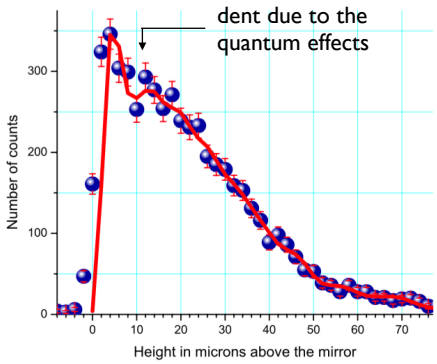

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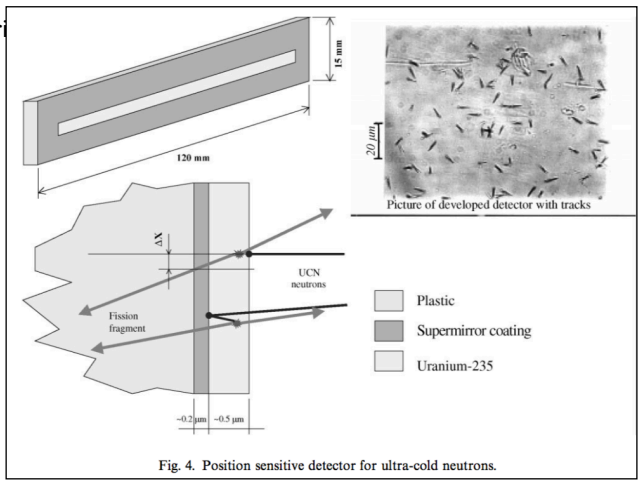

Fig. 4. Position sensitive detector for ultra-cold neutrons.

Measurement of CCD-based Imaging Detector

G. Ichikawa, S. Komamiya, Y. K. *et al.*,
PRL 112, 071101 (2014)

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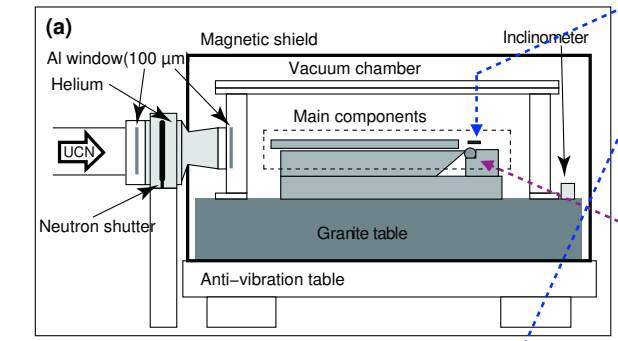
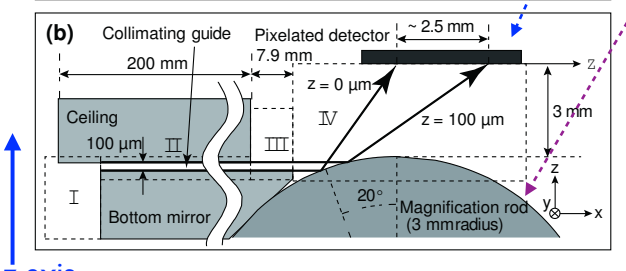


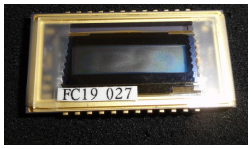


z-axis

(HAMAMATSU)

CCD-based Imaging Detector

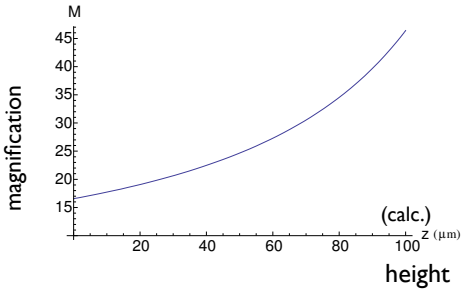
S. Kawasaki, G. Ishikawa, M. Hino, Y.K. et al.,
NIM A 615,42 (2010)



Position meas.: 3 microns
Timing meas.: ~ sec

Magnification Rod

magnify the neutron distribution using
a glass rod by 17～40 times



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G. Ichikawa, S. Komamiya, Y. K. *et al.*,
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Measurement
(a) expectations from quantum mechanics
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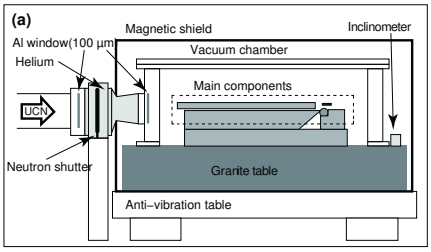
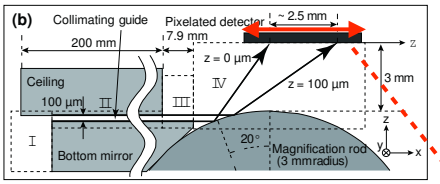
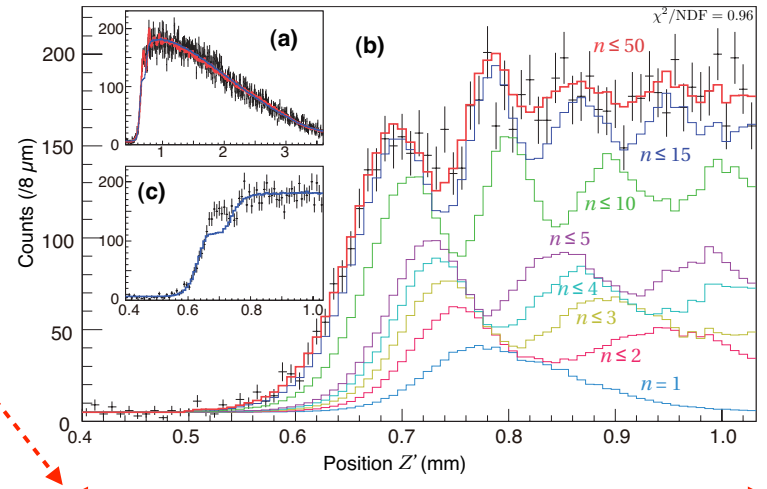
(c) expectations from classical mechanics
consistent with quantum mechanics







Modulated distribution due to
quantum effects is clearly measured!!

Crosses : data / Histograms : model fittings

6 fitting parameters:

+ origin of z' position
+ detector rotation
+ distance between rod to sensor
+ signal to total event ratio
+ γ: neutron loss on the ceiling
+ β: neutron loss on the floor

Measurement of CCD-based Imaging Detector

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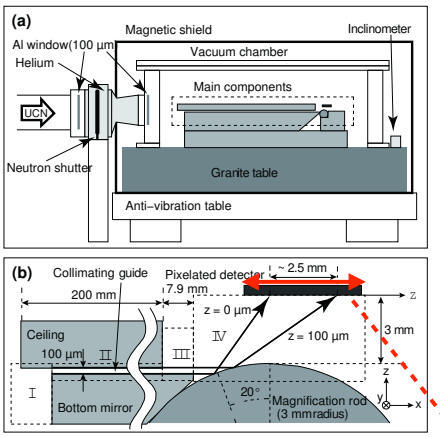

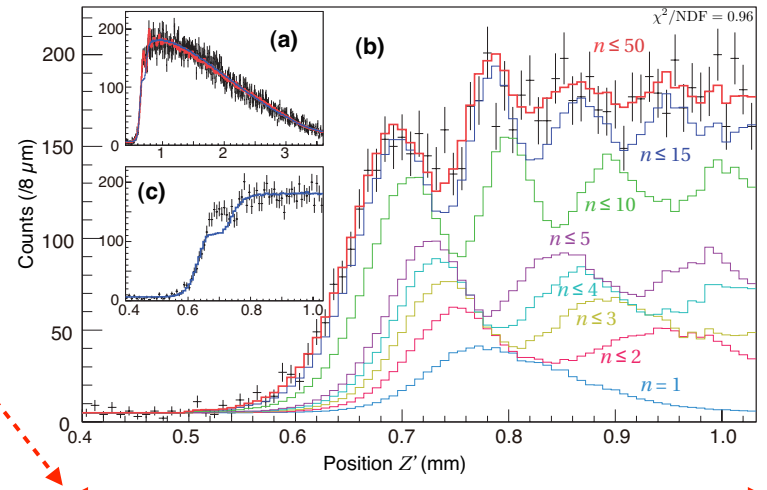

Modulated distribution due to quantum effects is clearly measured!!

Now, we got a quantum probe which couples to gravity!!!

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(Fifth force might violate the WEP at the microscopic range!)

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Question 3

Is the weak equivalence principle OK in the framework of quantum mechanics? — test of WEP in quantum regime

Main topic of today's talk

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G. Ichikawa, S. Komamiya, Y. K. *et al.*,
PRL 112, 071101 (2014)

Measurement
(a) expectations from quantum mechanics
(b) expectations from quantum mechanics (zoomed in)
(c) expectations from classical mechanics
consistent with quantum mechanics

Scales of the System

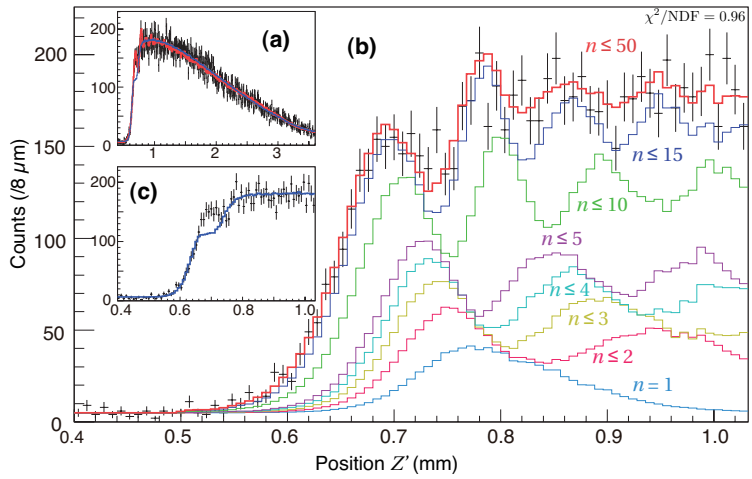
$$z_0 = \left(\frac{\hbar^2}{2m^2 g}\right)^{1/3} \sim 6\ \mu\mathrm{m}$$

$$E_0 = \left(\frac{mg^2\hbar^2}{2}\right)^{1/3} \sim 0.6\ \mathrm{peV}$$

$$z_0 = \left(\frac{\hbar^2}{2m_i m_g g}\right)^{1/3} \sim 6\ \mu\mathrm{m}$$

$$E_0 = \left(\frac{m_g^2 g^2 \hbar^2}{2m_i}\right)^{1/3} \sim 0.6\ \mathrm{peV}$$

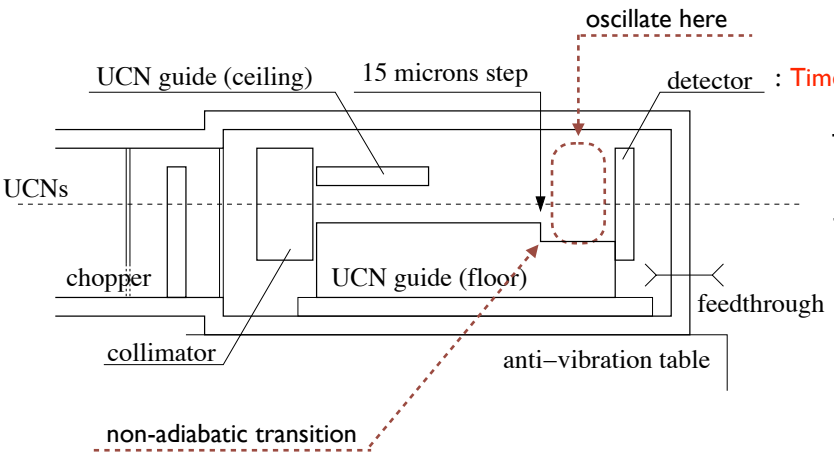
m_g : gravitational mass
m_i : inertial mass



Crosses : data / Histograms : model fittings

We can evaluate the mass ratio $\eta \equiv \frac{m_g}{m_i}$ by measuring length and energy scales simultaneously!!

How to Measure the Two Scales Simultaneously?



: Time-resolving imaging detector

- length scale is measured by seeing spacial distribution of neutrons directly
- energy scale is evaluated by measuring oscillation frequencies between the states

The step rotates the eigenstates of this quantum system.
Then, the object starts oscillation between the new eigenstates.

$$\psi_{(z,t=0)} = a_1\phi_{1(z)} + a_2\phi_{2(z)}$$

$$|\psi_{(z,t)}|^2 = |\psi_{(z,t=0)}|^2 - \underbrace{4a_1a_2\phi_{1(z)}\phi_{2(z)}\sin^2\frac{(\varepsilon_2-\varepsilon_1)}{2}t}_{\text{oscillating term}}$$

energy scale

Tests of the WEP in the World

<Classical Regime>

Lunar Laser Ranging Tests:	10^{-13} level	PRL 93, 261101 (2004)
Torsion Balance:	10^{-13} level	PRL 100, 041101 (2008)
Atom Interferometer:	10^{-11} level	PRL 125, 191101 (2020)
Neutron Interferometer (COW):	10^{-3} level	PRA 21, 1419 (1980) (Some bias in analysis was pointed out.) Physica B151, 22 (1988)

<Quantum Regime>

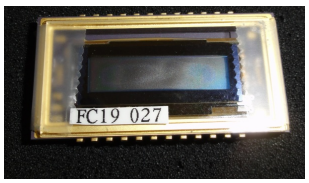
Our first target is 10^{-3} level

Time-resolving Imaging Detector for UCNs

Our previous detector : CCD based

Position resolution : 3 microns
Readout time : ~ sec

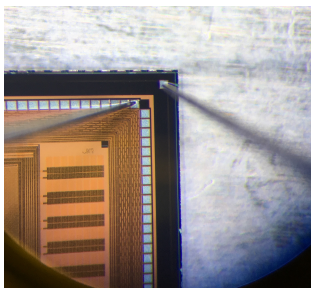
Not enough :-(



New detector design : CMOS based

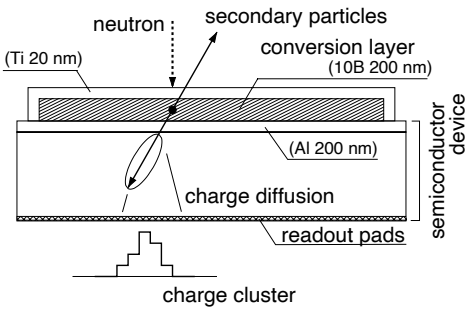
Position resolution : a few microns (expected.)
Readout time : ~ msec

Sounds OK! :-)



Charge sharing?
Charge overflow?

Schematic



^{10}B layer is used as a neutron to charge particle converter

$$\mathrm{n}+{}^{10}\mathrm{B} \rightarrow \begin{cases} \alpha(1.47\,\mathrm{MeV}) + {}^{7}\mathrm{Li}^{*}(0.84\,\mathrm{MeV}) & (93.9\,\%) \\ \quad \hookrightarrow {}^{7}\mathrm{Li}(0.84\,\mathrm{MeV}) + \gamma(0.48\,\mathrm{MeV}) & \\ \alpha(1.78\,\mathrm{MeV}) + {}^{7}\mathrm{Li}(1.01\,\mathrm{MeV}) & (6.1\,\%) \end{cases}$$

Development of ^{10}B-INTPIX4

Y. K., T. Miyoshi, H. Iwase *et al.*,
NIMA 979, 164400 (2020)

INTPIX4:

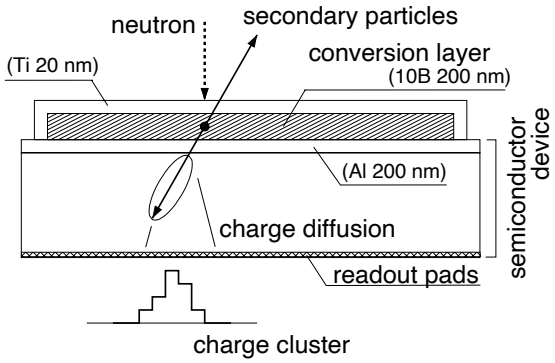
+ size： 512 x 832 pixels （17 μm-square / 10.2 x 15.4 mm^2）
+ thickness： ~ 300 μm
+ readout speed： 280 nsec/pixel (measured)
+ gain: 13.1 μV/e$^-$ (S. Mitsui, Y. Arai, T. Miyoshi, and A. Takeda, NIMA 953, 163106 (2019))

Forming ^{10}B layer:

using Argon RF sputtering technology

+ 400(100) W RF Power for Ti/B (pre-sputtering)
+ Sputtering rate for Ti(B): ~ 1.5(0.3)/10 [nm/sec]

sputtering machine in the clean room (class 100)



Neutron Irradiation Test

Y. K., T. Miyoshi, H. Iwase *et al.*,
NIMA 979, 164400 (2020)

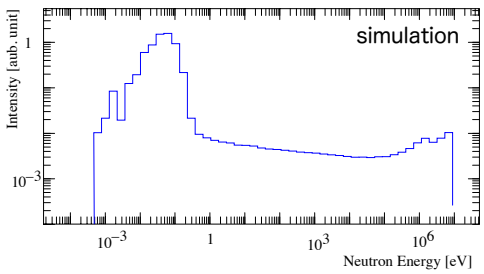
at

+ KEK - Thermal Neutron Calibration Lab.
+ KURNS (Institute for Integrated Radiation and Nuclear Science, Kyoto Univ.) - E3 port
+ J-PARC, MLF - Beam Line 10 (BL10)

Am-Be neutron source + carbon pile

https://www.kek.jp/ja/Facility/ARL/RSC/ThermalNeutron/

Energy distribution at the surface of the pile

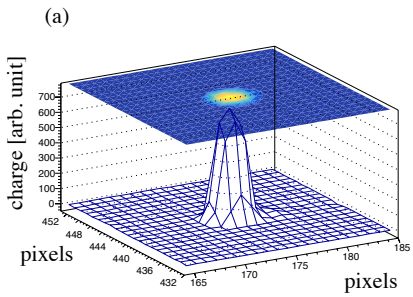


Neutron intensity at the surface is well determined to be 20 /cm^2 /s.

Detection efficiency is evaluated to be 1.5% for the thermal-range neutrons.

Event Shape

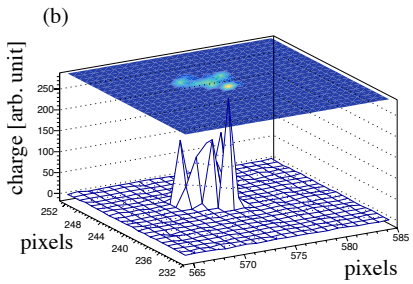
Y. K., T. Miyoshi, H. Iwase *et al.*,
NIMA 979, 164400 (2020)



< Isotropic and concentrated cluster shape >

Made by heavy charged particle, α or Li, from 10B (n, α) neutron capture event.

Neutron Signal!



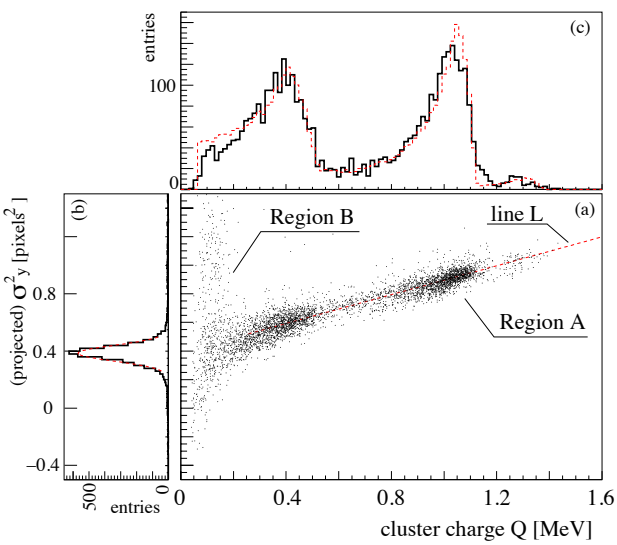
< somewhat moving cluster shape >

Made by running electron, which is scattered off γ-rays.

Background!

Clustering

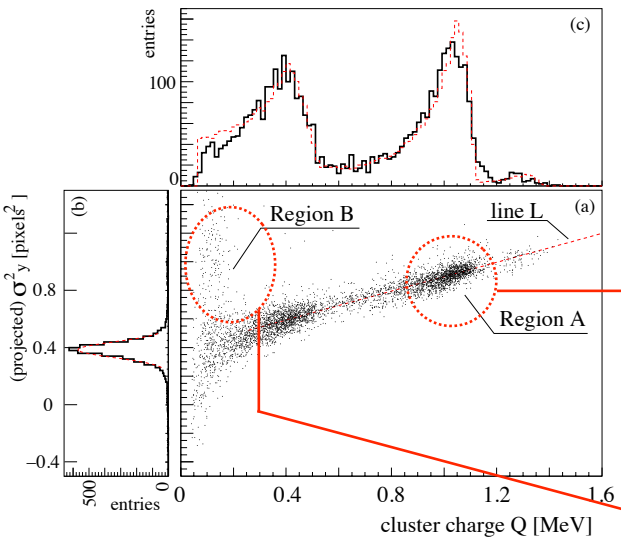
Y. K., T. Miyoshi, H. Iwase *et al.*,
NIMA 979, 164400 (2020)

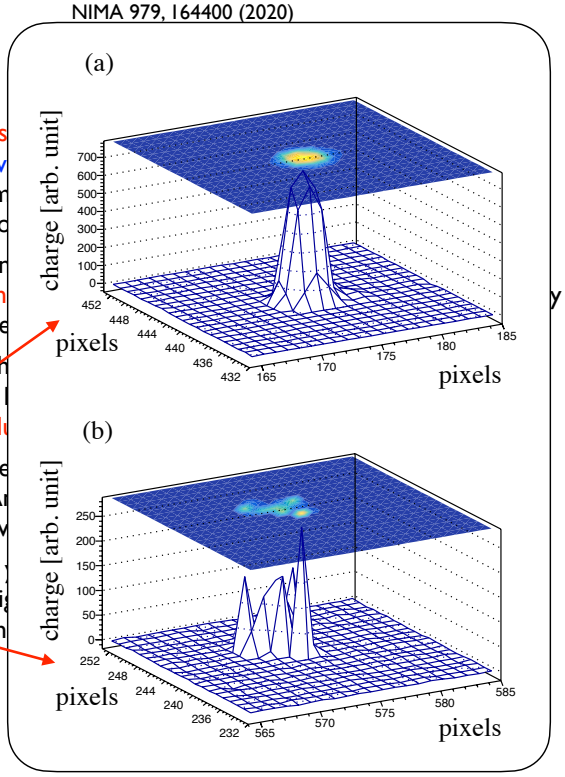


+ estimate pedestals of each pixel on each image by averaging successive 10 images before and after the image to be evaluated, and make the pedestal corrections.
+ find the pixel which has higher charge than the threshold level which corresponds to a 34 keV energy deposit.
+ check the adjacent pixels inside 7 x 7 distance to find a local maximum pixel, and define it as a seed of cluster.
+ determine a 7 x 7 pixels frame centered on the seed. Any structures inside the frame is treated as one event/cluster.
+ if you find the only one pixel, which have charge higher than 3.4 keV, in the frame, this event is due to the noisy pixel, then reject it.

Clustering

Y. K., T. Miyoshi, H. Iwase *et al.*,
NIMA 979, 164400 (2020)



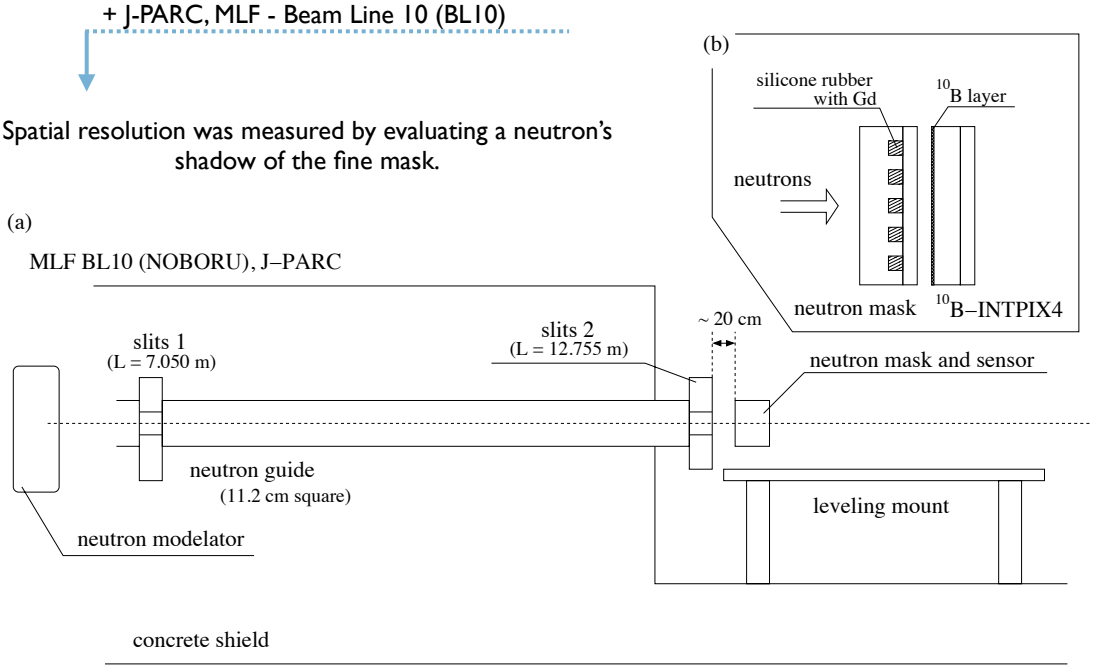


Neutron Irradiation Test

Y. K., T. Miyoshi, H. Iwase *et al.*,
NIMA 979, 164400 (2020)

at
+ KEK - Thermal Neutron Calibration Lab.
+ KURNS (Institute for Integrated Radiation and Nuclear Science, Kyoto Univ.) - E3 port
+ J-PARC, MLF - Beam Line 10 (BL10)

Spatial resolution was measured by evaluating a neutron's shadow of the fine mask.



Neutron Irradiation Test

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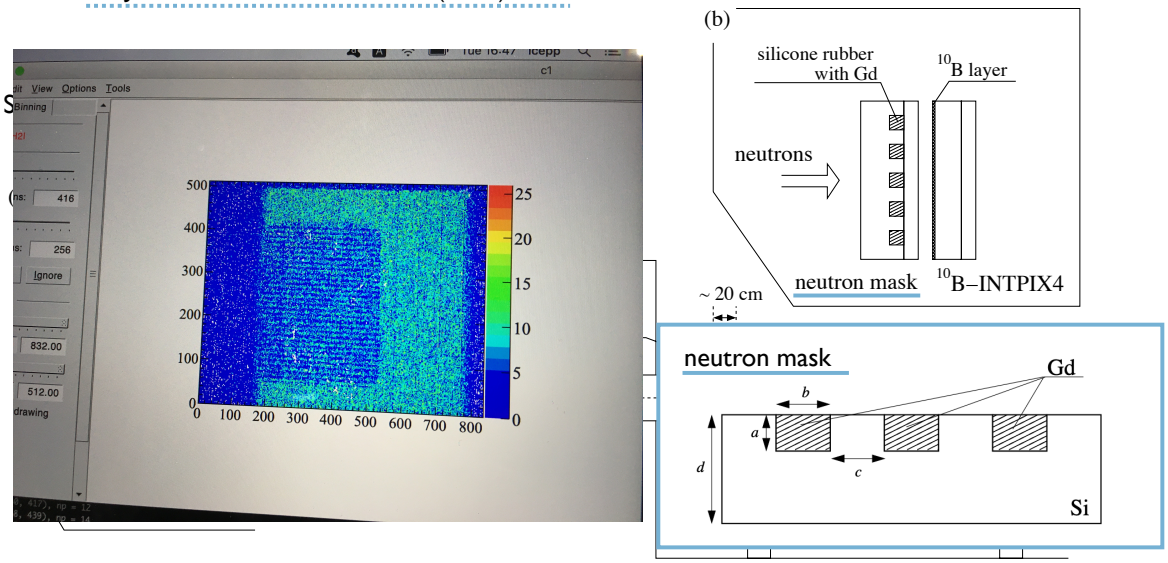
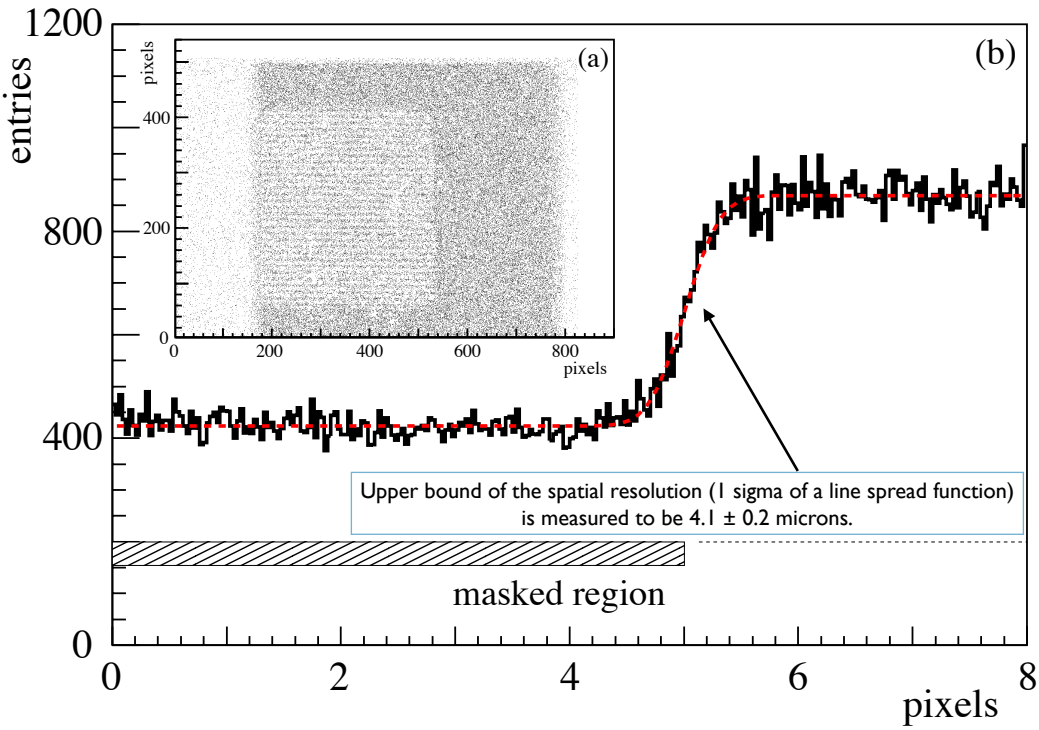


concrete shield

Evaluation of the Position Resolution

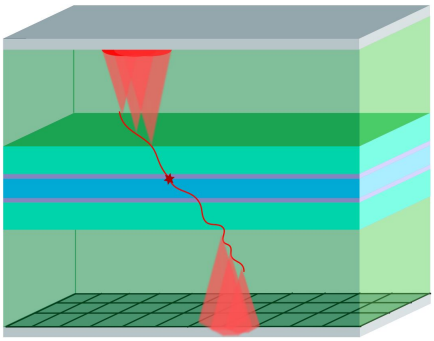
Y. K., T. Miyoshi, H. Iwase *et al.*,
NIMA 979, 164400 (2020)

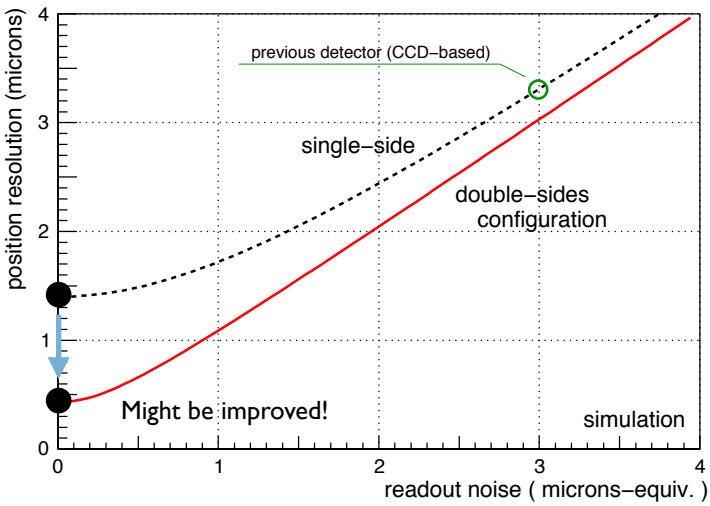


Sandwich Configuration

Two body decay

$$\mathrm{n}+{}^{10}\mathrm{B} \rightarrow \begin{cases} \alpha(1.47\,\mathrm{MeV}) + {}^{7}\mathrm{Li}^{*}(0.84\,\mathrm{MeV}) & (93.9\,\%) \\ \qquad\qquad \hookrightarrow {}^{7}\mathrm{Li}(0.84\,\mathrm{MeV}) + \gamma(0.48\,\mathrm{MeV}) & \\ \alpha(1.78\,\mathrm{MeV}) + {}^{7}\mathrm{Li}(1.01\,\mathrm{MeV}) & (6.1\,\%) \end{cases}$$

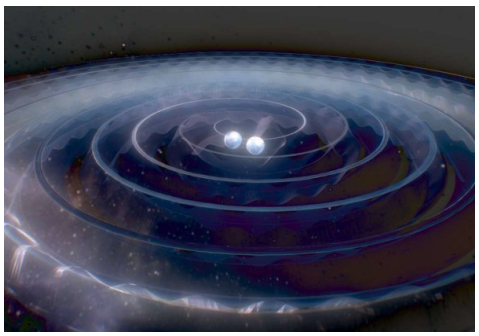




Summary

We have performed a series of experiments for testing gravity.

(- searched for new gravity-like interactions at the nanometer range.)
- measured wave functions of gravitationally bound ultracold neutrons with sub-micron resolution.
- preparing the quantum probe to test the weak equivalence principle in the Quantum regime.



Live as if you were to die tomorrow.
Learn as if you were to live forever.
- Mahatma Gandhi

P23

P23

Turbulence of Nambu-Goto string : Non-linear perturbation analysis

Ryo Kitaku , Yoo Chul-Moon :Graduate School of Science of Nagoya University

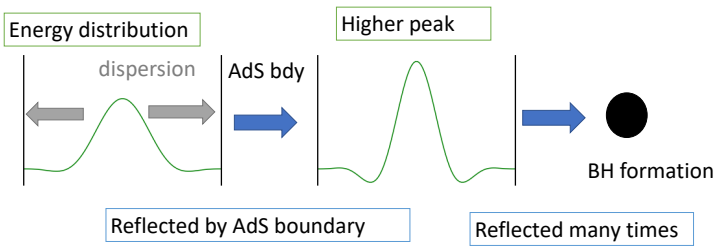
Abstract

In this talk, we analyze Nambu-Goto string in Global AdS spacetime by non-linear perturbation analysis. In our setup, eigen functions at linear order can be obtained analytically. Then we can perform analysis of higher order easily. Through this analysis, we see some similarity to AdS instability and get some suggestion.

1.Introduction

AdS Instability

Previous works:[Bizon, Rostworowski 2011] etc.



Essential factors	Nambu-Goto string whose endpoints are fixed on AdS bdy
AdS **confinement**	AdS **confinement**
Non linearlity of Einstein equation	**Non linearlity** of EoM of string

We can consider Nambu-Goto string as probe field of AdS instability !

We examine energy cascade of Nambu-Goto string by Non-Linear perturbation analysis.

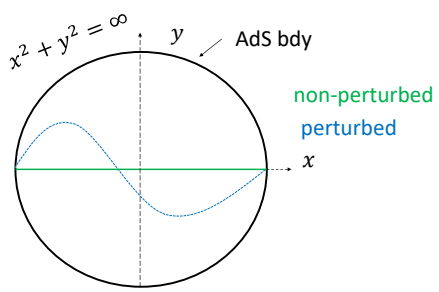
Previous work:[Ishii,Murata 2015], :[Vegh 2018]
Direct energy cascade occur then cusp occur

2.Our Setup

Non perturbed solution

$$\begin{cases} t = \ell\tau \\ x = \ell\tan(\sigma) \\ y = 0 \end{cases}$$

ℓ: AdS radius



Perturbed solution

$$y = y_{(1)}(\tau,\sigma)s + y_{(2)}(\tau,\sigma)s^2 + \cdots, \qquad |s| \ll 1$$

We solve EoM order by order

3.Results

Linear order

$$L\, y_{(1)} - \frac{\partial^2 y_{(1)}}{\partial\tau^2} = 0, L \equiv \frac{\partial^2}{\partial\sigma^2} - 2\sec^2(\sigma) \qquad e_j = \begin{cases} \sqrt{\frac{2}{\pi(j^2-1)}}\left(j\cos(j\sigma) + \sin(j\sigma)\tan(\sigma)\right), & (j \equiv 0 (\mathrm{mod}\ 2)) \\ \sqrt{\frac{2}{\pi(j^2-1)}}\left(-\cos(j\sigma)\tan(\sigma) + j\sin(j\sigma)\right), & (j \equiv 1 (\mathrm{mod}\ 2)) \end{cases}$$

Solution: $y_{(1)} = \sum_{j=2}^{\infty} c_{(1)}^j \cos(jt + \alpha^j) e_j$

Second order

Due to symmetry $y \to -y$ of EoM , $y_{(2)}$ satisfy the same equation as $y_{(1)}$
Then we can renormalize $y_{(2)}$ into $y_{(1)}$

$\therefore y_{(2)} = 0$

This is the same as AdS instability !

Third order

$$y_{(3)} = \sum_{j=2}^{\infty} c_{(3)}^j(t) e_j$$

$$\ddot{c}_{(3)}^j + j^2 c_{(3)}^j = \sum_{\{l,m,n\geq 2\cap l,m,n\in\mathbb{Z}\}} S_{l,\pm m,\pm n}^j A^l A^m A^n \cos(jt + \alpha^l \pm \alpha^m \pm \alpha^n) + \text{non resonant terms}, \qquad l \pm m \pm n = j$$

$$y = \sum_{j=2}^{\infty} c^j(t) e_j$$

$$c^j = sA^j\cos(jt + \alpha^j) + \frac{s^3}{3!} c_{(3)}^j(t) + O(s^4)$$
$$= sA^j\cos(jt + \alpha^j)$$
$$+ \frac{s^3}{12j} \sum_{\{l,m,n\geq 2\cap l,m,n\in\mathbb{Z}\}} \underbrace{(tS_{l,m,n}^j A^l A^m A^n \sin(jt + \alpha^l + \alpha^m + \alpha^n)}_{(+,+,+)} + \underbrace{tS_{l,-m,-n}^j A^l A^m A^n \sin(jt + \alpha^l + \alpha^m - \alpha^n)}_{(+,+,-)}$$
$$+ \underbrace{tS_{l,-m,-n}^j A^l A^m A^n \sin(jt + \alpha^l + \alpha^m + \alpha^n))}_{(+,-,-)}$$

+Non secular terms+$O(s^4)$, $l \pm m \pm n = j$

Colored term proportional to t, then these terms may excite j-mode(energy cascade)
Note that red-colored terms whose phase is α^j can renormalized into first order term by changing angular frequency (Poincaré Lindsted)

$$c^j = s\cos(jt + \alpha^j) + s^3\, t\sin(jt + \alpha^j) \sim s\cos((j - s^2)t + \alpha_j)$$

To analyze in general cases is difficult, so we analyze by putting $y_{(1)}$ specific form. As a consequence ,we find following possibilities:

$$y_{(1)} = \sum_{j=2}^{J} A_j(t)\cos(jt + \alpha_j)\, e_j$$

1.$(+,+,+)$, $(+,-,-)$ may not exist.

2. If l, m, n are all odd or even ,then mode excitation may not occur

3.Direct(indirect) energy cascade may occur

4.Conclusion

We analyzed open Nambu-Goto string whose endpoint are fixed on AdS bdy by non-linear perturbation analysis.

We found similarities to AdS instability and possibility of energy cascades.

5.Acknowledgement

We are grateful to Dr. T. Ishii and Dr. K. Murata for very useful insight.

P24

Propagation of tensor and scalar gravitational waves in Horndeski theory

arXiv:2209.00795

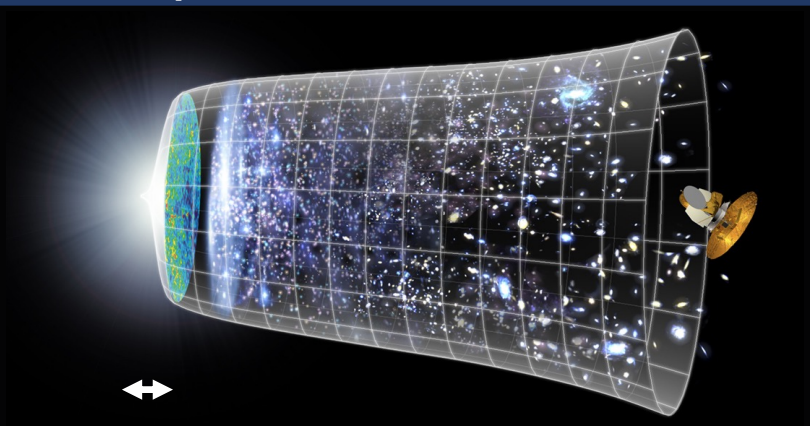
Kei-ichiro Kubota

Collaborator：Shinji Mukhoyama, Shun Arai

JGRG 2022/10/26

Accelerated expansion

Image Credit: WMAP



Inflation

GR + inflaton ϕ
Modified gravity ?

Late-time accelerated expansion

GR + cosmological const. Λ
GR + scalar field ϕ ?
Modified gravity ?

→ The origin of the accelerated expansion is an open question.

Horndeski theory [Hornderki 1974, Kobayashi et. al. 2011]

- Most general theory with $g_{\mu\nu}$ & ϕ and second-order differential EoMs in 4-D.
- In this talk, we consider the subclass with luminal tensor modes.
  ∵GW170817(NS-NS) and gamma-ray burst GW1700817A provide a constrain $c_{\rm T} \simeq c_{\rm light}$.

$$\mathcal{L} = \frac{1}{16\pi G_{\rm N}}\Big(G_2(\phi, X) + G_3(\phi, X)\Box\phi + G_4(\phi)R\Big) \quad X := -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$$

- Even this subclass include major dark energy models and not just the theories described by the scalar field.

cosmological const.
quintessence, k-essence

$G_2 = -2\Lambda$ or $K(\phi, X)$,
$G_3 = 0$,
$G_4 = 1$(GR),
$\rightarrow \mathcal{L} = \frac{1}{16\pi G_{\rm N}}[R - 2\Lambda]$
or $\frac{1}{16\pi G_{\rm N}}[R + K(\phi, X)]$

Brans-Dicke theory

$G_2 = G_{\rm N}\left[\frac{\omega(\phi)}{\phi}2X + V(\phi)\right]$,
$G_3 = 0$,
$G_4 = G_{\rm N}\phi$,
$\rightarrow \mathcal{L} = \frac{1}{16\pi}\left[\phi R + \frac{\omega(\phi)}{\phi}2X + V(\phi)\right]$

f(R)

$\mathcal{L} = \frac{1}{16\pi G_{\rm N}}f(R)$
$\rightarrow \mathcal{L} = \frac{1}{16\pi G_{\rm N}}[\phi R - U(\phi)]$
where
$U(\phi) = \chi(\phi)\phi - f(\chi(\phi))$
$\phi = f_{,\chi}(\chi)$

Modification of gravity in this subclass

- The term $G_4(\phi)R$ imply a modification of gravity.
  "Newton constant" is **no longer constant** and a function of ϕ.

$$\mathcal{L} = \frac{1}{16\pi G_{\rm N}}\Big(G_2(\phi, X) + G_3(\phi, X)\Box\phi + \boxed{G_4(\phi)R}\Big)$$

$$\mathcal{L}_{\text{Einstein-Hilbert}} = \frac{1}{16\pi G_{\rm N}}R \qquad \frac{G_4(\phi)}{16\pi G_{\rm N}} = \frac{1}{16\pi G_{\rm eff}(\phi)}$$

This coupling is strongly motivated by higher-dimensional theories like super string theory and Kaluza-Klein-like theory.

- GW amplitude is modified by this modification, which is known as "Running Planck mass".
  Equation of motion for plus mode in background FLRW spacetime

$$\partial_\eta^2 h_+ + \underbrace{(2\partial_\eta a/a + \partial_\eta G_4/G_4)}_{\text{Friction term is modified}}\partial_\eta h_+ - k^2 h_+ = 0$$

η: conformal time
a: scale factor
k: wave number

→GW can be a tool for this modification of gravity.

Running Planck mass

- "Running Planck mass" modify only friction term term of the GWs.

  Gravitational wave $\partial_\eta^2 h_+ + (2\partial_\eta a/a + \partial_\eta G_4/G_4)\partial_\eta h_+ - k^2 h_+ = 0$

  Electromagnetic wave $\partial_\eta^2 E + (2\partial_\eta a/a)\partial_\eta E - k^2 E = 0$

  Because the amplitude gives the luminosity distance from the source to the detector, "Running Planck mass" can be seen by measuring the luminosity distance with each of the electromagnetic waves and the gravitational wave. [Dalang & Lombriser 2019 arXiv:1906.12333]

- However, the current time variation of "Newton constant" is measured by the other experience, and tight constraints are placed.

  e.g. Lunar Laser Ranging experiment $\frac{\dot{G}_{\rm eff}}{G_{\rm N}} = (4 \pm 9) \times 10^{-13}\ {\rm yr}^{-1}$
  [Williams et. al. 2004 arXiv:gr-qc/0411113]

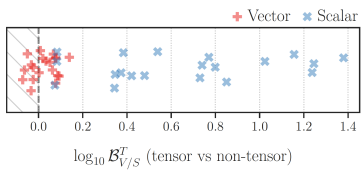
  So, the contribution of the Planck mass run on the propagation of the gravitational waves is expected to be small.

Knowledge of modified gravity given by GWs

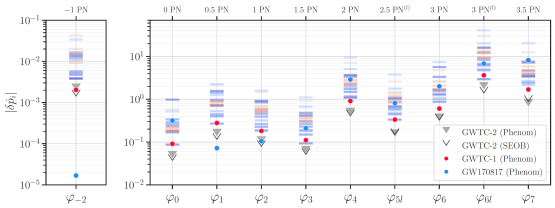
✓GWs propagate at light speed [LIGO & Virgo 2017 arXiv:1710.05834]

$$-3\times10^{-15} < c_{\rm T} - 1 < 7\times10^{-16}$$

✓Tensor mode is preferred to the other scalar and vector mode
[LIGO & Virgo 2021 arXiv:2010.14529 and Takeda & Nishizawa 2020 arXiv:2105.00253]



✓No deviation from GR in PN approximation
[LIGO & Virgo 2021 arXiv:2010.14529]
→Binary motion of the inspiral is consistent with GR



✓GW propagation in cosmological distance is also consistent with ΛCDM
[Arai & Nishizawa 2018 arXiv:1711.03776]
→No deviation from standard cosmology (ΛCDM)

Currently, no deviations from ΛCDM and GR have been seen by GW observations.

Propagation on FLRW + cosmological perturabtion

- The deviations from ΛCDM and GR are expected to be small.

- Many studies have examined the propagation assuming FLRW spacetime as background spacetime.
  However GWs propagate with a sense of the matter distribution.
  The amplitude of the GWs is also affected by the matter distribution in addition to the modification of the gravitational theory. [Garoffolo et. al. 2021 arXiv:2007.13722]

- So, we need the propagation equation of GWs taking into account both matter distribution and the modification of gravitational theory.

Matter distribution vs Modification of gravitational theory

- In our study, we derive the EoMs for tensor and scalar modes on general background spacetime.

Assumptions & Results

- Assumptions
  - (amplitudes of GWs) << 1
    →The back-reaction from the GWs to the background spacetime is small.
    → This allows the second order of GWs to be negligible
    and enables expansion with respect to the kinetic terms.
  - (wave length) << (typical variation scale of background spacetime)
    →This allow geometric optical approximations to be applied.

- Results
  - In Generalized Brans-Dicke theory ($G_3 = 0$),
    the EoMs for tensor and scalar modes are decoupled in leading and next-to-leading order.
  - Adding the G_3 term,
    the EoMs are decoupled in leading order, but are not decouple in next-to-leading order.
  - These results were consistent with similar study [Dalang et. al. 2021 arXiv:2009.11827].

Results

- When we take into account G_3, EoMs are too long and complicated to write here. (We wrote this equation in arXiv:2209.00795)
- However, for Generalized Brans-Dicke theory, we can rewritten the EoMs in the simple wave equations using the effective metric.

$$0 = \Box^{(\mathrm{eff})}\left(\mathcal{A}e^{i\theta}\right) + (\text{higher order})$$

$$= \frac{1}{\mathcal{A}}\Big[\underbrace{g^{\mu\nu}_{(\mathrm{eff})}k_\mu k_\nu}_{k^\mu \text{ is null in } g^{\mu\nu}_{(\mathrm{eff})}}\mathcal{A}^2 + i\underbrace{g^{\mu\nu}_{(\mathrm{eff})}\nabla^{(\mathrm{eff})}_\mu\left(k_\nu\mathcal{A}^2\right)}_{\text{Conservation eq. of number of particles}}\Big]e^{i\theta} + (\text{higher order})$$

$$g^{\mu\nu}_{(\mathrm{eff})} = \begin{cases} g^{ab}_{(\mathrm{E})} = \dfrac{1}{G_4}g^{ab} & \text{for } h^{(\mathrm{tensor})}_{\mu\nu} \\ \underbrace{\dfrac{G_4\sqrt{G_{2X}G_4+2XG_{2XX}G_4+3G_{4\phi}^2}}{\sqrt{\left(G_{2X}G_4+3G_{4\phi}^2\right)^3}}}_{\text{For conservation eq.}}\left(-n^\mu n^\nu + \underbrace{\dfrac{G_{2X}G_4+3G_{4\phi}^2}{G_{2X}G_4+2XG_{2XX}G_4+3G_{4\phi}^2}}_{\text{For } k^\mu \text{ being null}}\gamma^{\mu\nu}\right) & \text{for } \delta\phi \end{cases}$$

$$n_\mu := \nabla_\mu\phi/\sqrt{2X}$$

$$\gamma_{\mu\nu} := n_\mu n_\nu + g_{\mu\nu}$$

→ The effective metric for a tensor mode is the metric in Einstein frame.

Summary and Future direction

<u>Summary</u>

- We derive the EoMs for tensor and scalar modes.
- In Generalized Brans-Dicke theory, the EoMs for tensor and scalar modes are decoupled in leading and next-to-leading order. We simplify the EoMs.
- Adding the G_3 term, the EoMs are decoupled in leading order, but are not decouple in next-to-leading order.

<u>Future work</u>

- Using the derived equations, discuss which contribution is larger, matter distribution or modified gravity, within the range of currently allowed gravitational theories.