JGRG24

The 24th Workshop on General Relativity and Gravitation in Japan

10 (Mon) — 14 (Fri) November 2014

Kavli IPMU, the University of Tokyo

Kashiwa, Japan

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P13 Allan L. Alinea (Osaka) "Third Order Power Spectrum Using Uniform Approximation" [JGRG24(2014)P13]

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P17 Jonathan White (RESCEU) “Open inflation and scalar suppression on large scales” [JGRG24(2014)P17]

P18 Yuhei Miyamoto (RESCEU) “Effects of thermal fluctuations at the end of thermal inflation” [JGRG24(2014)P18]

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P20 Tomohito Suzuki (Hirosaki) “MSCO: Part 1 Formulation” [JGRG24(2014)P20]

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P23 Takahiro Tanaka (Kyoto) “On the creation of a baby universe” [JGRG24(2014)P23]

P24 Takashi Torii (Osaka Inst. of Tech.) “Stability of the wormholes in higher dimensional spacetime” [JGRG24(2014)P24]

P25 Hisaaki Shinkai (Osaka Inst. of Tech.) “Wormhole evolutions in n-dimensional Gauss-Bonnet gravity” [JGRG24(2014)P25]

P26 Yoshimune Tomikawa (Nagoya) “Wormhole on DGP brane” [JGRG24(2014)P26]

P27 Hideki Ishihara (Osaka City) “Charged multi-black strings in a five-dimensional Kaluza-Klein universe” [JGRG24(2014)P27]

P28 Daisuke Nitta (Nagoya) “Polarization of photons around black holes in non-minimally coupled Einstein-Maxwell theory” [JGRG24(2014)P28]

P29 Ken Matsuno (Osaka City) “Multi-black holes on Kerr-Taub-bolt space in five-dimensional Einstein-Maxwell theory” [JGRG24(2014)P29]

P30 Masaaki Takahashi (Aichi U. of Education) "Time Variability of an orbiting Hot Spot around a Black Hole" [JGRG24(2014)P30]

P31 Yasunari Kurita (Kanagawa Inst. of Tech.) "Hawking-Page phase transition in AdS 3 and extremal CFTs" [JGRG24(2014)P31]

P32 Kouichi Nomura (Kyoto) "Bimetric gravity and the AdS/CFT correspondence" [JGRG24(2014)P32]

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P34 Yosuke Misonoh (Waseda) "Black holes in non-projectable HoravaLifshitz gravity" [JGRG24(2014)P34]

P35 Daiki Kikuchi (Hirosaki) "Relativistic Sagnac effect by CS gravity" [JGRG24(2014)P35]

P36 Masato Minamitsuji (IST, Lisbon) "Disformal transformation of cosmological perturbations" [JGRG24(2014)P36]

P37 Yuichi Ohara (Nagoya) "New model of massive spin-2 on curved spacetime" [JGRG24(2014)P37]

P38 Seiju Ohashi (KEK) "Multi-scalar Extention of Horndeski Theory" [JGRG24(2014)P38]

“Gravitational radiation reaction to the Lagrange’s solution of the three-body problem I: Reaction force”

Kouta Iseki (Hirosaki)

[JGRG24(2014)P01]

Gravitational radiation reaction to the Lagrange's solution of the three-body problem I:Reaction force

HIROSAKI UNIVERSITY

Kouta Iseki

Hirosaki University, Japan
with N.Harada, K.Yamada, H.Asada(Hirosaki)

JGRG24 in IPMU Nov. 10-14, 2014

Abstract: This poster gives an explicit expression for the reaction force of the Gravitational waves in the Lagrange's solution.

1 Introduction

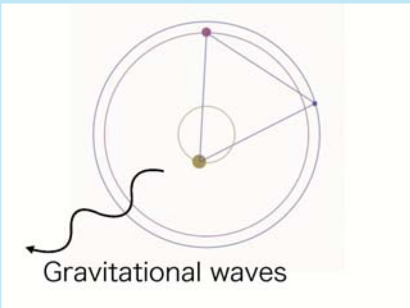


Figure 1: Lagrange's solution

・Gravitational waves from the Lagrange's solution have been studied in [1,2,3], but the radiation reaction on the solution is not fully discussed.
→ we examine the effect of the GW emission on the Lagrange's solution by adding the 2.5 post-Newtonian terms into EoM.
・As a Part I, As a result, this poster presents an explicit expression for the reaction force(Poster by Harada as a Part II will discuss an orbital evolution).
・In the following, we take the unit of G=c=1.

2 Radiation reaction by Gravitational waves

Radiation reaction potential in the Gravitational waves emission is expressed as [4]

$$\Phi = \frac{1}{5}\frac{d^5 {I\!\!\!-}_{ij}}{dt^5}x^i x^j \tag{1}$$

The reaction force around the unit mass in the Gravitational waves emission is expressed as

$$a_i = -\Phi_{,i} = -\frac{2}{5}\frac{d^5 {I\!\!\!-}_{ij}}{dt^5}x^j \tag{2}$$

Here,

$${I\!\!\!-}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} I^l{}_l \tag{3}$$

is the reduced quadrupole moment, and

$$I_{ij} = \sum_{A=1}^{N} m_A x_{Ai} x_{Aj} = \int \rho x_i x_j d^3x \tag{4}$$

is quantity called the quadrupole moment. In other words, a (2) is obtained if a (4) can calculate.

3 Two-body system

At first, we consider a two-body system in circular motion.

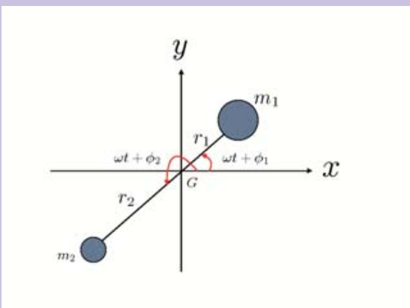


Figure 2: Two-body system in circular motion with angular velocity ω in xy plane.

We take the origin in the center of mass. m_I is mass of the heavenly bodies. r_I is distance from the centers of gravity. ϕ_I is the initial phase. Where, $\theta_I \equiv \omega t + \phi_I$

$$(x_1, y_1) = (r_1\cos\theta_1,\ r_1\sin\theta_1) \tag{5}$$
$$(x_2, y_2) = (r_2\cos\theta_2,\ r_2\sin\theta_2) \tag{6}$$

Substituting these into Eq.(4), Eq.(2) is rewritten as

$$\begin{pmatrix} a_{xI} \\ a_{yI} \end{pmatrix} = -\frac{32}{5}\omega^5 \begin{pmatrix} 0 & B_I \\ -B_I & 0 \end{pmatrix}\begin{pmatrix} r_I\cos\theta_I \\ r_I\sin\theta_I \end{pmatrix} \tag{7}$$

Where, I=1,2

$$B_1 = -(m_1 r_1^2 + m_2 r_2^2),\ B_2 = -B_1 \tag{8}$$

Eq.(7) implies that reaction force is always along to the tangential direction. Reaction force of each body is opposite to each other with the same magnitude.These lead to the inspiral phase of the binary.

4 Equilateral triangular configuration

Next, we consider the Lagrange's solution.

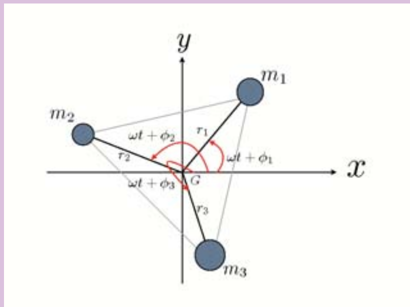


Figure 3: The Lagrange's solution with angular velocity ω in xy plane.

We take the origin in the center of gravity. m_I is mass of the heavenly bodies. r_I is distance from the centers of gravity. ϕ_I is the initial phase. Where, $\theta_I \equiv \omega t + \phi_I$

$$(x_1, y_1) = (r_1\cos\theta_1,\ r_1\sin\theta_1) \tag{9}$$
$$(x_2, y_2) = (r_2\cos\theta_2,\ r_2\sin\theta_2) \tag{10}$$
$$(x_3, y_3) = (r_3\cos\theta_3,\ r_3\sin\theta_3) \tag{11}$$

(2) can write in the following form that we substitute these for (4).

$$\begin{pmatrix} a_{xI} \\ a_{yI} \end{pmatrix} = -\frac{32}{5}\omega^5 \begin{pmatrix} A_I & B_I \\ -B_I & A_I \end{pmatrix}\begin{pmatrix} r_I\cos\theta_I \\ r_I\sin\theta_I \end{pmatrix} \tag{12}$$

Note: Diagonal components!

Where I=1,2,3

$$A_I = -\sum_{J=1}^{3} m_J r_J^2 \sin 2\theta_{IJ},\ B_I = -\sum_{J=1}^{3} m_J r_J^2 \cos 2\theta_{IJ} \tag{13}$$

$$\theta_{IJ} \equiv \theta_J - \theta_I = \phi_J - \phi_I \tag{14}$$

Eq.(12) implies that reaction force is not always along to the tangential direction. Reaction force of each body is not opposite to each other with the same magnitude. These may not lead to the inspiral phase of the binary.
→In the poster PART II, we discuss the orbital evolution.

5 Conclusion

・We studied the reaction force by gravitational waves.
・We obtained the expression of the reaction force to the Lagrange's solution.
・In the poster PARTII, we discuss the orbital evolution.

6 References

[1]H.Asada,PRD 80, 064021 (2009)
[2]Y.Torigoe, K.Hattori, and H.Asada, PRD 102, 251101 (2009)
[3]N.Seto and T.Muto, PRD 81, 103004 (2010)
[4]C.W.Misner,K.S.Thorne and J.A.Wheeler,「Gravitation」(Free-man, New York, 1973)

“Gravitational radiation reaction to the Lagrange’s solution of the three-body problem II: Orbital evolution”

Naoya Harada (Hirosaki)

[JGRG24(2014)P02]

Gravitational radiation reaction to the Lagrange's solution of the three-body problem II : Orbital evolution

NAOYA HARADA

Hirosaki University, Japan with K. Iseki, K. Yamada, and H. Asada.

JGRG24 in IPMU November 10 -14, 2014

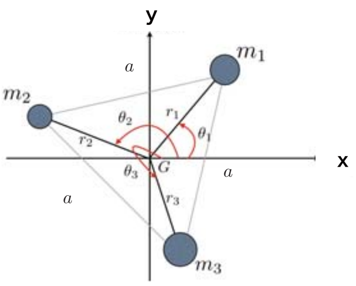
HIROSAKI UNIVERSITY

Abstract : We discuss orbital evolution of Lagrange's solution by taking account of gravitational radiation.
This poster gives the expression of the orbital evolution, and also we expressed the rate of change of the orbital period.

1. Introduction

- Gravitational waves from the Lagrange's solution have been studied in [1,2,3], but the radiation reaction on the solution is not fully discussed.

We examine the effect of GW emission on the Lagrange's solution by adding the 2.5 post-Newtonian terms into EoM.



Lagrange's solution with angular velocity ω in xy plane.

2. Equations of Motion

For radial part

$$\ddot{r}_I - r_I\omega^2 = -\frac{GM}{a^3}r_I - \frac{32}{5}\frac{G}{c^5}r_I\omega^5 A_I$$

Newtonian term (first term on the right); GW radiation reaction (second term)

For tangential part

$$2\dot{r}_I\omega + r_I\dot{\omega} = \frac{32}{5}\frac{G}{c^5}r_I\omega^5 B_I$$

We assume

$$r_I = r_{IN}e^{C\varepsilon t} \qquad \omega = \omega_N e^{D\varepsilon t}$$
$$a = ae^{C\varepsilon t}$$

*N:Newtonian value

Where,

$$A_I = -\sum_{j=1}^{3} m_J r_J^2 \sin 2\theta_{IJ} \qquad B_I = -\sum_{j=1}^{3} m_J r_J^2 \cos 2\theta_{IJ}$$

3. Condition to mass ratio I

Condition (1) is

$$A_I = 0 \qquad (1)$$

This is satisfied only in the following three cases:

(a) $m_1 = m_2 = m_3$

(b) $m_J = m_K = 0$

(c) $m_I = m_J$ and $m_K = 0$

4.Condition to mass ratio II

As a solution of equation of motion we can have

$$r_I = r_{IN}e^{\frac{64}{5}\omega_N B_I \varepsilon t}$$

$$\omega = \omega_N e^{-\frac{96}{5}\omega_N B_I \varepsilon t}$$

Thus, condition (2)

$$B_1 = B_2 = B_3 \qquad (2)$$

This is satisfied in the following three cases:

$$m_1 = m_2 = m_3$$

5.Preliminary

From Conditions (1) & (2), mass ratio is only

$$m_1 = m_2 = m_3$$

In this case, gravitational waves are not radiated [4].

6.Summary

- **In equilateral triangle, all the mass are the same.**
- **Is assumption appropriate, whether or not?**
- **As future work, we are going to consider post-Newtonian triangle.**

References

[1] H. Asada, PRD 80, 064021 (2009)
[2] Y. Torigoe, K. Hattori, and H. Asada, PRD 102, 251101 (2009)
[3] N.seto and T.Muto, PRD 81, 103004 (2010)
[4] Bernard Schutz,「A First Course in General Relativity, Second Edition」(Cambridge University Press, 2009)

“Probability distribution function for inclinations of merging compact binaries detected by gravitational wave interferometers”

Naoki Seto (Kyoto)

[JGRG24(2014)P03]

960

Probability distribution function for inclinations of merging compact binaries detected by gravitational wave interferometers

Naoki Seto (Kyoto)

arXiv:1406.4238 (event rate)
arXiv:1410.5136 (PDF of inclinations)

2014.11 JGRG24

Coherent analysis with detector network (HLVK+…)

- **detection rate**
  - basic measure
  - observational strategy
    - duty cycle
    - importance of LIGO-India
- **PDF of inclinations**
  - multi-messenger astronomy
    - SGRB?

detection rate: binary inspiral

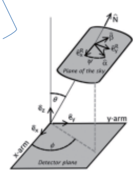
- detection rate
  (Merger rate [$Mpc^{-3}yr^{-1}$]) x (effective volume)
  geometry
  ⇒ relative event rate
- effective volume

$$SNR^2 \propto \frac{f(\mathbf{n}, \psi, I)}{r^2}$$

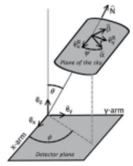
direction
orientation
I=cos[i]
important 4D

$$volume \propto r_{max}^3 \propto f(\mathbf{n}, \psi, I)^{3/2}$$

solid estimation : Monte Carlo etc



re-analysis



detector sensitivity (spin 2)

$$c_{i+}(\boldsymbol{n}, \psi) = a_i(\boldsymbol{n}) \cos 2\psi + b_i(\boldsymbol{n}) \sin 2\psi,$$
$$c_{i\times}(\boldsymbol{n}, \psi) = -a_i(\boldsymbol{n}) \sin 2\psi + b_i(\boldsymbol{n}) \cos 2\psi$$

GW amplitude (excellent for NS-NS)

$$d_+(I) = \frac{I^2+1}{2}, \quad d_\times(I) = I$$

SNR

$$SNR^2 \propto \sum_{i=1}^{m} \left[(c_{i+} d_+)^2 + (c_{i\times} d_\times)^2 \right] \equiv f(\boldsymbol{n}, I, \psi)$$

$$f(\boldsymbol{n}, \psi, I) = \sigma(\boldsymbol{n}) \left[(d_+^2 + d_\times^2) + \epsilon(\boldsymbol{n})(d_+^2 - d_\times^2) \cos 4\psi' \right]$$
$$\psi' = \psi + \delta(\boldsymbol{n})$$

offset (irrelevant for our analysis below)

Cutler & Flanagan 1994

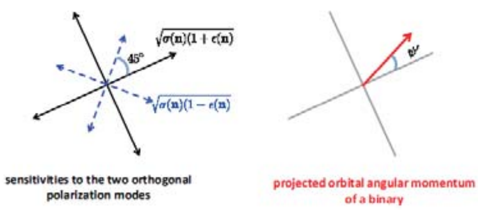


Figure 1. The geometric interpretation of Eq.(5) for incoming GW from a sky direction n. (Left panel) In the plane normal to n, the network has two orthogonal polarization bases at specific orientations, and measure these two modes with sensitivities proportional to $\sqrt{\sigma(n)(1+\epsilon(n))}$ and $\sqrt{\sigma(n)(1-\epsilon(n))}$. Here the parameter $\sigma(n)$ represents the total sensitivity to the two modes and $\epsilon(n)$ shows the asymmetry between them. (Right panel) The orbital angular momentum of the binary is projected to the normal plane. Its orientation is characterised by the angle ψ' measured from the better sensitivity mode in the left panel. The original amplitudes (1) are given for the polarization modes symmetric to this projected vector.

$$f(\boldsymbol{n}, \psi, I) = \sigma(\boldsymbol{n}) \left[(d_+^2 + d_\times^2) + \epsilon(\boldsymbol{n})(d_+^2 - d_\times^2) \cos 4\psi' \right]$$

Cutler & Flanagan 1994

$$\sigma(\boldsymbol{n}) \equiv \sum_{i=1}^{m} \left[a_i^2 + b_i^2 \right],$$

total sensitivity

$$\epsilon(\boldsymbol{n}) = \frac{\sqrt{\left[\sum_{i=1}^{m}(a_i^2 - b_i^2)\right]^2 + 4\left(\sum_{i=1}^{m} a_i b_i\right)^2}}{\sigma(\boldsymbol{n})}$$

anisotropy to 2 orthogonal modes

Cauchy-Schwarz inequality

$$0 \le \epsilon(\boldsymbol{n}) \le 1$$

$$|a||b| \ge |a \cdot b|$$

(a1,b1), (a2,b2),(a3,b3)… → (a1,a2,a3,…), (b1,b2,b3,…)

one interferometer (or aligned) $\epsilon = 1$
randomly placed $\epsilon = 0$

relative event rate

$$f(\boldsymbol{n},\psi,I)=\sigma(\boldsymbol{n})\left[(d_+^2+d_\times^2)+\epsilon(\boldsymbol{n})(d_+^2-d_\times^2)\cos 4\psi'\right]$$

$$d_+(I)=\frac{I^2+1}{2},\quad d_\times(I)=I$$

relative volume for $d\boldsymbol{n}d\psi dI$ $\quad f(\boldsymbol{n},\psi,I)^{3/2}d\boldsymbol{n}d\psi dI$

orientation integral

effective volume for a direction ***n***

$$\sigma(\boldsymbol{n})^{3/2}g(\epsilon(\boldsymbol{n}))d\boldsymbol{n}$$

$$g(\epsilon)\equiv\frac{1}{2^{5/2}\pi}\int_0^\pi d\psi\int_{-1}^1 dI\left[(d_+^2+d_\times^2)+\epsilon(d_+^2-d_\times^2)\cos 4\psi\right]^{3/2}$$

We can complete 2D integrals, but

profile of the function $g(\epsilon)$

$$0\le\epsilon(\boldsymbol{n})\le 1$$

$$\sigma(\boldsymbol{n})^{3/2}g(\epsilon(\boldsymbol{n}))d\boldsymbol{n}$$

$$g(0)=0.290451,\quad g(1)=0.293401=1.010125\times g(0)$$

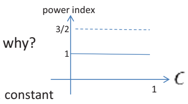
monotonic function

Taylor expansion (error less than 10^{-4})

$$g_{exp}(\epsilon)=0.290451(1+0.00978\epsilon^2+0.00026\epsilon^4+O(\epsilon^6))$$

$$g(\epsilon)\equiv\frac{1}{2^{5/2}\pi}\int_0^\pi d\psi\int_{-1}^1 dI\left[(d_+^2+d_\times^2)+\epsilon(d_+^2-d_\times^2)\cos 4\psi\right]^{3/2}$$

why?



3/2→1 constant
dg/dε=0@ε=0

approximation with $g(\epsilon)$=const

- guaranteed accuracy with error <1.0126%
  - integral of positive definite functions
  - identical to Schutz 2011 (taking ψ average for f)
    - validity: not clarified so far (in spite of quantitative arguments)
- we can effectively neglect orientation dependence of binaries
  - easy to evaluate
  - only consider face-on binaries

similarly for PDF of inclination

$$f(\boldsymbol{n},\psi,I)=\sigma(\boldsymbol{n})\left[(d_+^2+d_\times^2)+\epsilon(\boldsymbol{n})(d_+^2-d_\times^2)\cos 4\psi'\right]$$

$$d_+(I)=\frac{I^2+1}{2},\quad d_\times(I)=I$$

relative volume for $d\boldsymbol{n}d\psi dI$ $\quad f(\boldsymbol{n},\psi,I)^{3/2}d\boldsymbol{n}d\psi dI$

integrate dn dψ

1st dψ integral

$$\alpha(\boldsymbol{n},I)\equiv\frac{2}{\pi}\int_0^{\pi/2}f(\boldsymbol{n},I,\psi)^{3/2}d\psi.$$
$$=\sigma(\boldsymbol{n})^{3/2}D_0(I)^{3/2}\gamma\left[\epsilon(\boldsymbol{n})R(I)\right],$$

$$D_0(I)\equiv(d_+^2+d_\times^2)=\frac{I^4+6I^2+1}{4}$$

$$D_1(I)\equiv(d_+^2-d_\times^2)=\frac{(I^2-1)^2}{4}$$

$$R(I)\equiv\left(\frac{D_1}{D_0}\right)=\frac{(I^2-1)^2}{I^4+6I^2+1}$$

and

$$\gamma(x)\equiv\frac{2}{\pi}\int_0^{\pi/2}(1+x\cos 4\psi)^{3/2}d\psi.$$
$$=1+\frac{3}{16}x^2+\frac{9}{1024}x^4+\frac{35}{16384}x^6+\frac{3465}{4194304}x^8+\frac{27027}{67108864}x^{10}+\frac{969969}{4294967296}x^{12}+O(x^{14}).$$

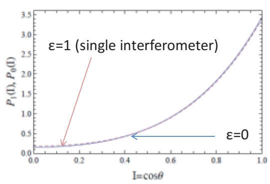
2nd dn and normalize

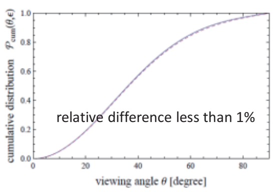
$$P_{net}(I)=\frac{\int_{4\pi}d\boldsymbol{n}\,\alpha(\boldsymbol{n},I)}{\int_0^1 dI\int_{4\pi}d\boldsymbol{n}\,\alpha(\boldsymbol{n},I)}.$$

For a network with ε=0

PDF of inclination becomes very simple (identical to Schutz 2011)

$$P_0(I)\equiv P(I,0)=\frac{D_0(I)^{3/2}}{N_0}\qquad N_0=0.82155$$





general network: basically bounded by ε=0 and 1

“Hilbert-Huang Transform in Search for Gravitational waves”

Hirotaka Takahashi (Nagaoka U. of Tech.)

[JGRG24(2014)P04]

Hilbert-Huang Transform in Search for Gravitational waves

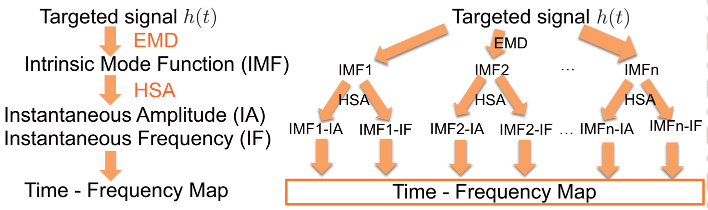
Hirotaka Takahashi (Nagaoka University of Technology, Japan)

Collaborate with

Satoshi Ueki, Yukitsugu Sasaki, Yoshihisa Kon (Nagaoka Univ. of Technology),
Ken-ichi Oohara, Masato Kaneyama, Takashi Wakamatsu (Niigata Univ.),
Jordan B. Camp (NASA GSFC)

Hilbert-Huang Transform (HHT)

- The HHT consists of two components [1];
  - ✓ Empirical Mode Decomposition (EMD)
  - ✓ Hilbert Spectral Analysis (HSA)



Hilbert Spectral Analysis (HSA)

- The Hilbert spectral analysis can be applied to investigate characteristics in non-stationary time series data.
- If $h(t)$ is the real part of a analytic complex function $F(z)$ on the real axis $z = t$ and $\lim_{|z|\to\infty} |z^k F(z)| < \infty$ for any positive value k, then the imaginary part $v(t)$ is given by Hilbert transform (HT) :

$$v(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{h(\tau)}{t-\tau} d\tau$$, where P indicates the Cauchy Principal value.

$$F(z) = h(t) + iv(t) = a_{\mathrm{HT}}(t) e^{i\theta(t)}$$ $a_{\mathrm{HT}}(t) = \sqrt{h(t)^2 + v(t)^2}$: Instantaneous Amplitude (IA)

$$\theta(t) = \tan^{-1}\left\{\frac{v(t)}{h(t)}\right\}$$: phase $f_{\mathrm{HT}}(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt}$: Instantaneous Frequency (IF)

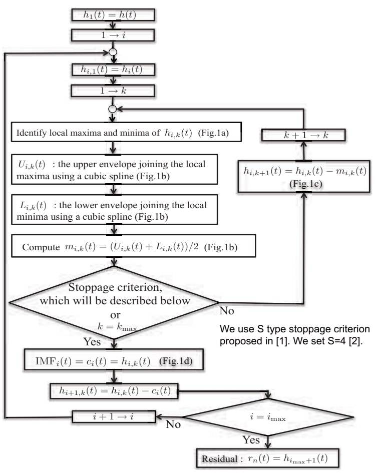
The signal $h(t)$ must satisfy the following conditions :

✓ (# of extrema) – (# of zero crossing) = 0 or ±1.

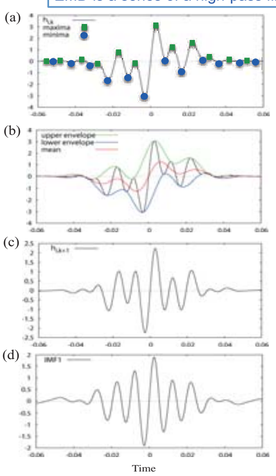
✓ The mean value of the envelope defined using the local maxima and the envelope defined using the local minima is zero.

Empirical Mode Decomposition (EMD)

Outline of EMD shifting algorithm :



EMD is a series of a high-pass filter



For further noise reduction, we apply ensemble EMD (EEMD) [1].

Wavelet Transform (WT)

- Continuous WT (CWT) : Time-Frequency analysis
- Discrete WT (DWT): Image processing, Image data compression etc

Continuous WT :

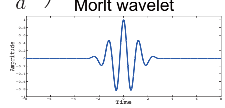
$$W_h(a,b) = \int_{-\infty}^{+\infty} h(t)\psi^*_{a,b}(t)dt, \quad \psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)$$

$\psi_{a,b}(t)$: mother wavelet (wavelet function)

a : scale factor

b : sift factor

Morlt wavelet



Simulation

- Signal1 : $s(t) = a_{\mathrm{sg}} \exp\left[-(t/\tau)^2\right] \sin\phi(t)$ $\mathrm{SNR} = \sqrt{\sum_i s_i^2}/\sigma$

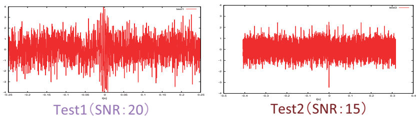
Frequency depends on time : $f(t) = \frac{1}{2\pi}\frac{d\phi}{dt} = \left[301.172 + 48\left(\frac{t}{0.01\mathrm{sec}}\right)\right]\mathrm{Hz}$

- Signal2 : Supernova waveform [3]

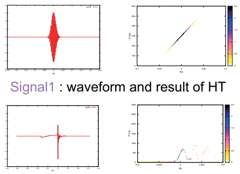
$h(t) = s(t) + n(t)$

Test1 : signal1 + white Gaussian noise (SNR=20)

Test2 : sginal2 + Adv. LIGO noise (SNR=15)



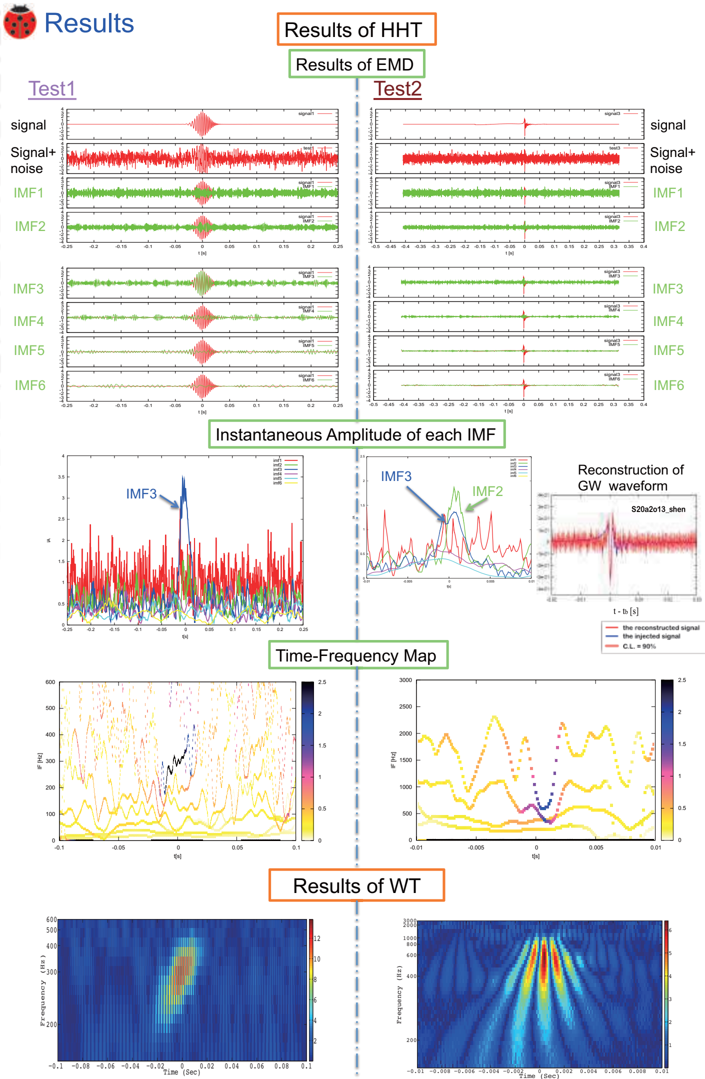
Test1 (SNR : 20) Test2 (SNR : 15)



Signal1 : waveform and result of HT

Signal2 : waveform and result of HT

Results



Summary

- We investigated the possibility of the application of a Hilbert-Huang transform (HHT) to the search for gravitational waves.
- We compared the Time-Frequency map obtained by HHT with the Time-Frequency map obtained by CWT.
- We also investigated the reconstruction of waveform with the HHT.
- More details of the results of systematic simulations will be discussed elsewhere.

Reference :

[1] N. E. Huang and Z. Wu, Rev. Geophys. **46**, RG2006 (2008).

[2] H.Takahashi *et. al.*, Advances in Adaptive Data Analysis, **Vol.5** No.2, 1350010 (2013).

[3] H. Dimmelmeier *et. al.*, Phys. Rev. D, **78**, 064056, (2008).

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“Dynamics of thick discs from a Schwarzschild-FRW Metric”

Guillaume Lambard (IBS - CUP)

[JGRG24(2014)P05]

(Thick discs) Dynamics embedded in a Schwarzschild-FRW Metric

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JGRG 2014 - The 24th Workshop on General Relativity and Gravitation in Japan

Guideline to a S-FRW metric

- FRW metric, homogeneous and isotropic, with local curvature in spherical co-moving coordinates (t, r, θ, φ) with k = 0 (*flat* FRW metric)

$$ds^2 = -dt^2 + a^2(t)[(1 - A/r)^{-1}dr^2 + r^2 d\Omega^2] ,$$

with a(t) the scale factor, A is an integration constant and $d\Omega^2$ is the metric on a unit 2-sphere.

- FRW metric can be expressed in spherical local coordinates (T, R, θ, φ) using the following transformations

$$R = R(r,t) = ra(t) , \quad \text{With,} \quad H = \dot{a}/a, \quad \text{(Hubble's parameter)}$$

(1) $$dR = r\dot{a}dt + adr \qquad adr = dR - HRdt$$

It comes

(2) $$ds^2 = -\left[\frac{\alpha}{\xi}\right]dT^2 + \alpha^{-1}dR^2 + R^2 dR , \qquad \alpha = 1 - \frac{Aa}{R} - H^2R^2 , \quad \xi = 1 - \frac{Aa}{R}$$

With,

$$dT = dt + \frac{HRdR}{\alpha}$$

Which, in the limit for weak fields (Aa << R), can be given by

(3) $$ds^2 = -\alpha\zeta dT^2 + \alpha^{-1}dR^2 + R^2 dR , \qquad \zeta = 1 + \frac{Aa}{R}$$

- The integration constant A is interpreted as A = 2M, with M the mass of the spherically symmetric object of radius r_s. Also, the Equation (3) is close to a Schwarzschild-deSitter metric, with H = constant = Λ/3 (Λ, the cosmological constant), and $a(t) = e^{(Ht)}$ as a scale factor. In the local Universe, the scale factor is assumed to be $a(t = t_0) = 1$ at the present cosmological time t_0.

Conserved quantities

- Trajectories of test particles with or without mass (a 'particle' and a 'photon' respectively) are investigated

- The present S-FRW metric with $a(t) = a(t = t_0) = 1$ is time independent and spherically symmetric. Also, conserved momentum component are associated to trajectories.

- Time independence of the metric means for the energy

(4) $$\text{particle}: \tilde{E} = -p_0/m , \qquad \text{photon}: E = -p_0$$

- Independence of the metric of the angle φ about the axis implies that the angular momentum p_φ is constant

(5) $$\text{particle}: \tilde{L} = p_\varphi/m , \qquad \text{photon}: L = p_\varphi$$

- Because of spherical symmetry, motion is confined to a single plane chosen to be the equatorial plane here (θ = constant = π/2 for the orbit). Then $p_\theta \propto d\theta/d\lambda = 0$, with λ any parameter on the trajectory. The non-vanishing components of momentum are

(6)
$$\text{particle}: p^0 = g^{00}p_0 = m(\alpha\zeta)^{-1}\tilde{E} , \qquad \text{photon}: p^0 = (\alpha\zeta)^{-1}E ,$$
$$p^r = m\, dR/d\tau , \qquad p^r = dR/d\lambda ,$$
$$p^\varphi = g^{\varphi\varphi}p_\varphi = m\,\tilde{L}/R^2 \qquad p^\varphi = d\varphi/d\lambda = L/R^2$$

- The scalar product $\vec{p}.\vec{p} = -m^2$ allows to give the following equations for orbits

(7)
$$\text{particle}: (dR/d\tau)^2 = \tilde{E}^2 - \tilde{V}^2(R) , \quad \tilde{V}^2(R) = (\alpha\zeta)(1 + \tilde{L}^2/R^2)$$
$$\text{photon}: (dR/d\lambda)^2 = E^2 - V^2(R) , \quad V^2(R) = (\alpha\zeta)(L^2/R^2)$$

(Effective potentials)

Context

- A homogeneous and isotropic Universe is well defined by a general FRW metric in spherical co-moving coordinates (t, r, θ, φ) which takes the form

$$ds^2 = -dt^2 + a^2(t)[e^{2\Sigma(r)}dr^2 + r^2 d\Omega^2] ,$$

With Σ(r) a function of the radial coordinate r to determine, and a(t) is the scale factor.

- The form of the metric implies the isotropy about a position, and the homogeneity is verified if the Ricci scalar of curvature of the three-dimensional metric, R^i_i (i = r, θ, φ), is independent of position at a fixed time. This last statement implies that the trace G of the three-dimensional Einstein tensor is a constant called κ.

$$G = G_{ij}g^{ij} = -\frac{1}{r^2}\left[r(1 - e^{-2\Sigma(r)})\right]' = \kappa$$

An integration gives

$$g_{rr} = e^{2\Sigma(r)} = \left(1 + \frac{1}{3}\kappa r^2 - \frac{A}{r}\right)^{-1}$$

Where A is a constant of integration which is commonly assumed to be zero to respect the local flatness of the metric at r = 0, $g_{rr}(r=0)=1$.

- Here, a local non-null curvature (A≠0) is approached by studying the shape of the metric at the exterior of an astrophysical object of mass M embedded in an expanding Universe. The resulting exterior dynamic is also described by computing the trajectory of a test particle. Following the latest cosmological data, the Universe in expansion is assumed to be flat, k = κ/3 = 0.

Types of orbits

- From the derivative of (7), it follows

(8)
$$\text{particle}: 0 = \frac{d}{dr}\left[(\alpha\zeta)(1 + \tilde{L}^2/R^2)\right]$$
$$\text{photon}: 0 = \frac{d}{dr}\left[(\alpha\zeta)(L^2/R^2)\right]$$

Which lead to non-trivial expressions for the trajectories radius

- It follows for the angular momentum of a particle

(9) $$\tilde{L}^2 = \frac{-R^2(8M^2 - 2MH^2R^3 - 2H^2R^4)}{16M^2 + 2MH^2R^3 - 2R^2}$$

- Considering the case of a stable circular orbit of a test particle, it comes $\tilde{E}^2 = \tilde{V}^2$

- In order to reach the angular velocity dφ/dt, we have

(10)
$$d\varphi/d\tau = U^\varphi = p^\varphi/m = g^{\varphi\varphi}p_\varphi/m = g^{\varphi\varphi}\tilde{L} = \tilde{L}/R^2$$
$$dt/d\tau = U^0 = p^0/m = g^{00}p_0/m = g^{00}(-\tilde{E}) = \tilde{E}/(\alpha\zeta)$$

Giving

(11) $$\frac{d\varphi}{dt} = \frac{d\varphi}{d\tau}\frac{d\tau}{dt} = \frac{1}{2R^2}\left[8M^2 - 2MH^2R^3 - 2H^2R^4\right]^{1/2}$$

This drives to a circular velocity R(dφ/dt). As one can see from (11), it exists an intrinsically bound to the angular velocity following this S-FRW metric, in sense that the radius R is restricted to the limit $R < (2M/H)^{1/2} \sim 3.6\times10^{20}$ m ~11.6 kpc, if one consider a central galactic mass $M = 10^{10}\, M_\odot = 10^{13}$ m, and the hubble constant $H = H_0 \sim 7.7\times10^{-27}$ m.

In Progress…

- Gravitational deflection of light following this S-FRW metric in the weak field approximation (R<<2M) will be stated.

- Following the image method ("displace, cut, fill and reflect") from Gonzalez and Letelier (2003), the dynamic of a thick disc embedded in a S-FRW metric will be stated.

- The final goal of the development being to compare the computed circular velocities to the data from the DiskMass Survey (Martinsson et al., 2013) to check the validity of the Keplerian model, and if an improvement in the understanding of the distribution of luminous and dark matter in spiral galaxies is available.

References

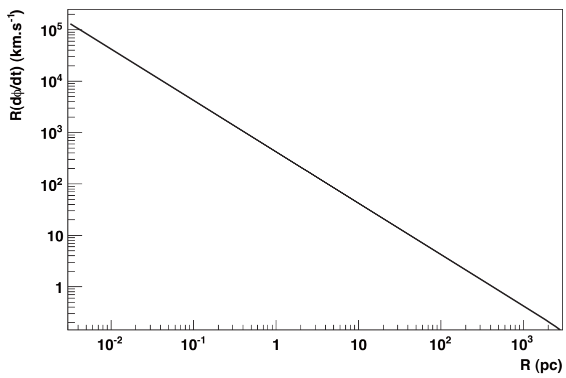
1. B. F. Schutz, *A First Course in General Relativity, Second Edition*, Cambridge University Press, 2009.
2. R. Adam et al., *Planck Collaboration, Planck intermediate results. XXX. The angular power spectrum of polarized dust emission at intermediate and high Galactic latitudes*, arXiv:1409.5738.
3. G. A. González, P. S. Letelier, *Exact General Relativistic Thick Disks*, arXiv:gr-qc/0311078

Acknowlegments

I would like to thank Ekaterina Gracheva for our fruitful discussions, and her useful comments and questions. Also, I would like to thank the Institute for Basic Science (IBS) and the Center for Underground Physics (CUP), to allow me to join the JGRG 2014 at the Kavli-IPMU, Tokyo, Japan.

Effective potential and circular velocity of a test particle

- Circular velocity R(dφ/dt) of a test particle in the gravitational field of a central point mass $M = 10^{10}\, M_\odot = 10^{13}$ m, embedded in a Universe in expansion following the S-FRW metric presented here, as a function of the distance R to the central potential.



- As one can see in the Figure above, the circular velocity is decreasing with the distance R to the center of the gravitational potential, here thought to be for a trajectory of a test particle (star) in a galactic plane. Sub-luminal velocities are reached close to the horizon (R->2M).

“Gravitational Faraday Effect for Cylindrical Gravitational Solitons”

Shinya Tomizawa (Tokyo U. of Tech.)

[JGRG24(2014)P06]

Gravitational Faraday Effect for Cylindrical Gravitational Solitons

Shinya Tomizawa (Tokyo U. of Tech)
& Takashi Mishima (Nihon U.)

- **Phys.Rev. D 90 (2014) 044036**
- **To be appeared**

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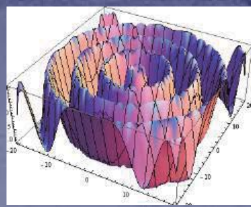
Introduction



- Cylinderical gravitational waves are the simplest form of gravitational radiation
- A diagonal metric form of a cylindrically symmetric spacetime makes the vacuum Einstein equation to an extremely simple structure of a linear wave equation in a at background.
  - Einstein-Rosen wave [1937] can be interpreted as superposition of cylindrical gravitational waves with a + mode only.
- However, the non-diagonal component of a metric drastically changes the structure of the Einstein equation since it generally yields a × mode together with non-linearity.
  - Piran et al . [1985] numerically studied non-linear interaction of cylindrical gravitational waves of both polarization modes and showed that the + mode converts to the × mode, whose phenomenon was named gravitational Faraday effect after the Faraday effect in electrodynamics.
  - Tomimatsu [1989] studied the gravitational Faraday rotation for cylindrical gravitational solitons by using the inverse scattering technique.
- Moreover, the interaction of gravitational soliton waves with a cosmic string was also studied
  - Interaction of GW pulse with a cosmic string (Economou-Tsoubelis 1987,Xanthopoulos 1986,1986)

Inverse scattering method

- Belinsky & Zakharov ('1979) showed that the vacuum Einstein equation with two commuting Killing vectors is completely integrable and admits a Lax pair of a linear equations.
- This BZ's method generated many physically interesting solutions such as cosmological, cylindrically symmetric, colliding plane waves, stationary axisymmetric solutions, … .
- The BZ's method can be simply extended to higher dimensional Einstein theories , but generally does not generate any regular solutions.
- Pomeransky ('2006) improved the original ISM formulated by BZ so that it can generate regular solutions.
- Actually, all known five-dimensional black hole (vacuum) solutions were found or re-derived by the Pomeransky's procedure not by BZ's procedure (Koikawa '05, Pomerasky '06, S.T-Morisawa-Yasui '06, S.T-Nozawa '06, Pomerasky-Sen'kov '06, Elvang-Figuras '07, Izumi '07…).

Our work

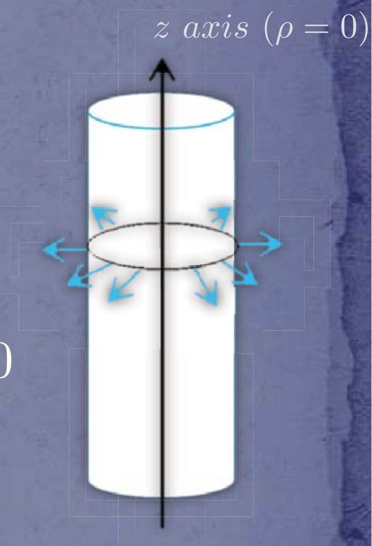
- In this work, using *the Pomeransky's improved Inverse Scattering Method for a cylindrical spacetime,* we construct a new gravitational two-soliton solution which describes non-linear GWs.

- In terms of the two-soliton solution, we analyze non-linear effects of GWs:
  - ❖ Time shift phenomenon :
    Wave packets can propagate at slower speed than light velocity
  - ❖ Interaction between solitons:
    Wave packets can collide or split as they collapse
  - ❖ Gravitational Faraday effect:
    Outgoing "+ mode" waves can convert to "× mode" waves when they interact with ingoing "× mode" waves

Einstein-Rosen wave

- Metric describing cylindrically symmetric spacetime

$$ds^2 = e^{2\psi} dz^2 + \rho^2 e^{-2\psi} d\phi^2 + e^{2(\gamma-\psi)}(-dt^2 + d\rho^2)$$



- The functions ψ、γ depend on ρ and t only
- Einstein eq. is reduced to a linear wave equation:
    - Ψ is determined by the linear wave equation

$$\Delta\psi \equiv \left(\frac{\partial^2}{\partial t^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{\partial^2}{\partial\rho^2}\right)\psi = 0$$

    - γ is determined by the harmonic function ψ

$$\gamma_{,\rho} = \rho(\psi_{,\rho}^2 + \psi_{,t}^2), \quad \gamma_{,t} = 2\rho\psi_{,t}\psi_{,\rho}$$

General cylindrically symmetric spacetime

The metric describing the most general cylindrically symmetric spacetime is written in Kampaneets-Jordan-Ehlers form ('58, '60)

$$ds^2 = e^{2\psi}(dz + \omega d\phi)^2 + \rho^2 e^{-2\psi} d\phi^2 + e^{2(\gamma-\psi)}(-dt^2 + d\rho^2)$$

The functions ψ、γ, and ω depend on ρ and t only

- Einstein eq. is reduced to non-linear equations:
    - ψ and ω are determined by 'non-linear' wave equations coupled with each other:

$$\Delta\psi = \frac{e^{4\psi}}{2\rho^2}(\omega_{,t}^2 - \omega_{,\rho}^2) \qquad \bar{\Delta}\omega = 4(\omega_{,\rho}\psi_{,\rho} - \omega_{,t}\psi_{,t})$$

    - γ is determined by the ψ and ω

$$\gamma_{,\rho} = \rho(\psi_{,\rho}^2 + \psi_{,t}^2) + \frac{e^{4\psi}}{4\rho}(\omega_{,t}^2 + \omega_{,\rho}^2), \quad \gamma_{,t} = 2\rho\psi_{,t}\psi_{,\rho} + \frac{e^{4\psi}}{4\rho}\omega_{,t}\omega_{,\rho}$$

Our new two-soliton solution:

- We choose Minkowski spacetime as a seed, and construct a 'two-soliton' solution with complex conjugate poles by using the Pomeransky's procedure.
- Tomimatsu ('89) constructed two-soliton from Minkowski, using the BZ procedure
- The metric in terms of Kampaneets-Jordan-Ehlers form:

$$ds^2 = e^{2\psi}(dz + \omega d\phi)^2 + \rho^2 e^{-2\psi} d\phi^2 + e^{2(\gamma-\psi)}(-dt^2 + dz^2)$$

where the metric functions are given by

$$e^{2\psi} = |w|^4 \left(1 - \frac{\mathcal{A}}{\mathcal{B}}\right),$$
$$\omega = \frac{(|w|^2-1)^2}{\rho} \frac{\mathcal{C}\mathcal{B}}{\mathcal{D}(\mathcal{B}-\mathcal{A})},$$
$$e^{2\gamma} = -\mathcal{C}\frac{\mathcal{F}(\mathcal{B}-\mathcal{A})}{\mathcal{B}},$$

$$\mathcal{A} = 2\Re\left[\frac{(|w|^2-1)^4(\bar{w}^2-1)^4}{\bar{w}^2(w^2-1)}(X^2+a^2Y^2)\right] - 2\frac{(|w|^2-1)^3|w^2-1|^4}{|w|^2}(|X|^2+|a|^2|Y|^2),$$
$$\mathcal{B} = \frac{1}{|w^2-1|^2}|X^2+a^2Y^2|^2 - \frac{1}{(|w|^2-1)^2}(|X|^2+|a|^2|Y|^2)^2,$$
$$\mathcal{C} = 2\Re\left[\frac{\bar{a}(\bar{w}^2-1)^2}{\bar{w}(w^2-1)}(X^2+a^2Y^2)\right] - 2\Re\left[\frac{\bar{a}(w^2-1)^2}{w(|w|^2-1)}\right](|X|^2+|a|^2|Y|^2),$$
$$\mathcal{D} = \frac{1}{|w^2-1|^2}|X^2+a^2Y^2|^2 - \frac{1}{(|w|^2-1)^2}(|X|^2+|a|^2|Y|^2)^2,$$
$$\mathcal{F} = \frac{1}{(w-\bar{w})^2|w|^4|w^2-1|^6(|w|^2-1)^6}[|X^2+a^2Y^2|^2 - (|X|^2+|a|^2|Y^2|)^2],$$

$$X = (w^2-1)^2(|w|^2-1)^2,$$
$$Y = \frac{|w|^2 w}{\rho^2},$$

- The solution has a complex parameter '$a=a_r+a_i$ i' only

The special choice of the parameter, a=0 corresponds to Minkowski spacetime
Hence, this does not have the limit to Einstein-Rosen wave

Following Piran et al. ('85) & Tomimatsu ('89), we define several useful quantities for analysis.

- Amplitudes for ingoing waves with + & × modes

$$\mathcal{A}_+ = 2\psi_{,v} \qquad \mathcal{A}_\times = \frac{2e^{2\psi}\omega_{,v}}{\rho}$$

- Amplitudes for outgoing waves with + & × modes

$$B_+ = 2\psi_{,u} \qquad B_\times = \frac{2e^{2\psi}\omega_{,u}}{\rho}$$

- Amplitudes:

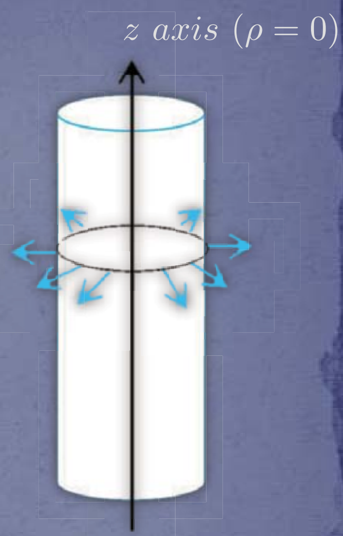
$$\mathcal{A} = (\mathcal{A}_+^2 + \mathcal{A}_\times^2)^{1/2} \quad \text{for ingoing waves}$$

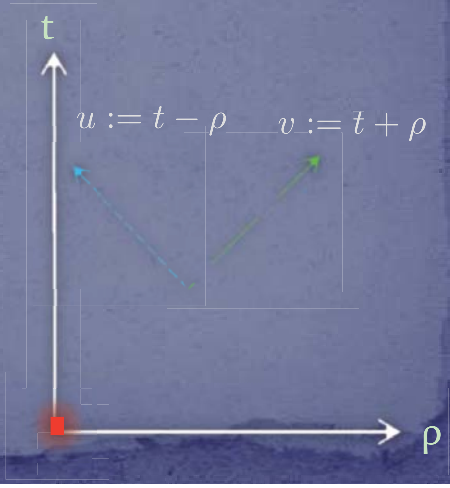
$$B = (B_+^2 + B_\times^2)^{1/2} \quad \text{for outgoing waves}$$

- Polarization angles:

$$\tan 2\theta_A = \frac{\mathcal{A}_\times}{\mathcal{A}_+} \quad \text{for ingoing waves}$$

$$\tan 2\theta_B = \frac{B_\times}{B_+} \quad \text{for outgoing waves}$$





Analysis of cylindrically symmetric gravitational waves

Metric form describing general cylindrically symmetric spacetime
(Kampaneets-Jordan-Ehlers form):

$$ds^2 = e^{2\psi}(dz + \omega d\phi)^2 + \rho^2 e^{-2\psi} d\phi^2 + e^{2(\gamma-\psi)}(-dt^2 + d\rho^2)$$

Einstein eq can be written in terms of (A_+,$A_\times$,B_+,$B_\times$) only.

- ψ and ω are determined by

$$A_{+,u} = \frac{A_+ - B_+}{2\rho} + A_\times B_\times \qquad A_{\times,u} = \frac{A_\times + B_\times}{2\rho} - A_+ B_\times$$

$$B_{+,v} = \frac{A_+ - B_+}{2\rho} + A_\times B_\times \qquad B_{\times,v} = -\frac{A_\times + B_\times}{2\rho} - A_\times B_+$$

Non-linear terms

- γ is determined by

$$\gamma_{,\rho} = \frac{\rho}{8}(A^2 + B^2) \qquad \gamma_{,t} = \frac{\rho}{8}(A^2 - B^2)$$

On Axis

Tomimatsu sol ('89)

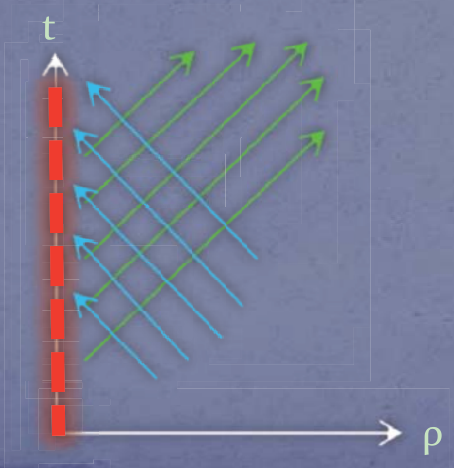
The singular source on the axis continues to absorb and emit gravitational waves with + mode only constantly

- Amplitudes:

$$A \to \infty, \quad B \to \infty$$

- Polarizations:

$$\tan\theta_A = -\tan\theta_B = 0$$



S.T. & Mishima

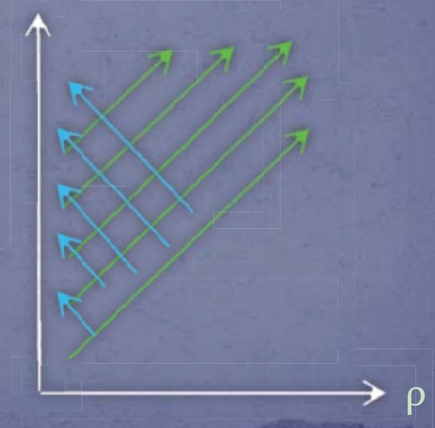
The polarization angles on the axis have time-depending behavior

- Amplitudes:

$$A \simeq B \simeq \frac{|a_r t^2 - 2a_i q t - a_r q^2|\sqrt{16(t^2+q^2)^2(a_i q - a_r t)^2 + \{(a_r t - a_i q)^2 - 4(t^2+q^2)^2\}^2}}{2(t^2+q^2)^2[(a_r t - a_i q)^2 + 4(q^2+t^2)^2]}.$$

- Polarizations:

$$\tan 2\theta_A \simeq \tan 2\theta_B \simeq -\frac{(2t^2+2q^2-a_r t + a_i q)(2t^2+2q^2+a_r t - a_i q)}{4(t^2+q^2)(a_i q - a_r t)}.$$



Timelike infinity

Tomimatsu sol ('89)

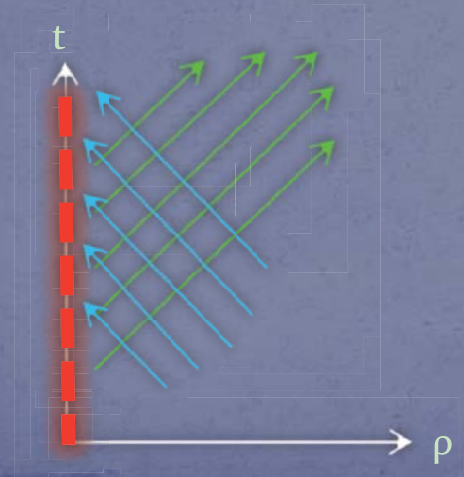
- The spacetime is not asymptotically Minkowsky spacetime because of emission from singular source
- The + mode dominates the × mode

- Amplitudes:

$$A \simeq B \simeq \frac{1}{\rho}$$

- Polarizations:

$$\tan\theta_A = -\tan\theta_B \simeq 0$$



S.T. & Mishima

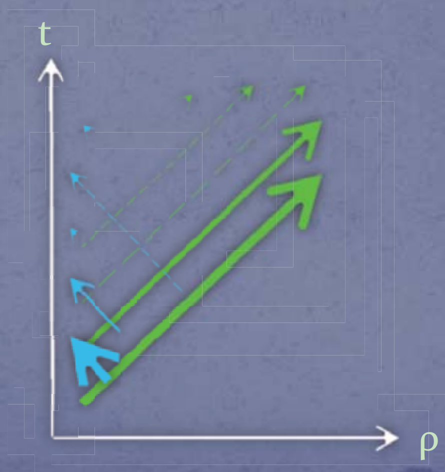
- The spacetime asymptotically behaves as Minkowski spacetime. Hence, both the ingoing and outgoing waves decay and finally vanish.
- The × mode dominates the +mode

- Amplitudes:

$$A \simeq B \simeq \frac{a_r}{2t^2} + \mathcal{O}(t^{-3}).$$

- Polarizations:

$$\tan\theta_A \simeq -\tan\theta_B \simeq 1$$



Null infinity

Tomimatsu sol ('89)

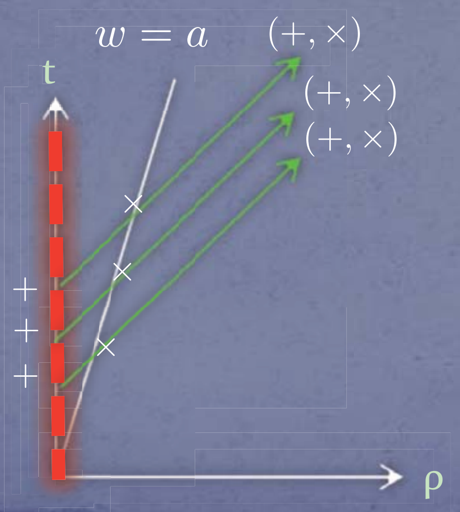
- The spacetime is asymptotically Minkowski spacetime. The ratio of × mode to + mode becomes constant

- Amplitudes:

$$A \simeq \frac{1}{v^{\frac{3}{2}}}, \quad B \simeq \frac{1}{v^{\frac{1}{2}}}$$

- Polarizations:

$$\tan\theta_A = -\tan\theta_B \simeq \frac{1}{a}$$



S.T. & Mishima

- The spacetime asymptotically behaves as Minkowski spacetime. Hence, both the ingoing and outgoing waves decay and finally vanish.
- The × mode dominates the +mode

- Amplitudes:

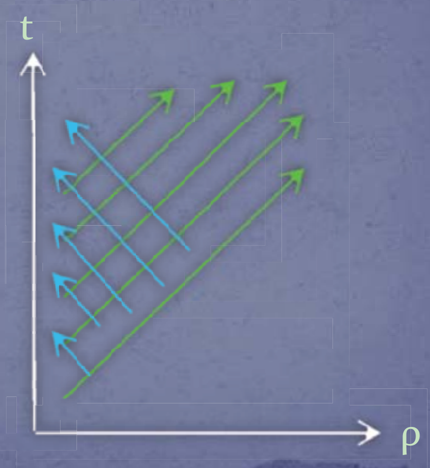
$$A \simeq \frac{1}{v^{\frac{3}{2}}}, \quad B \simeq \frac{1}{v^{\frac{1}{2}}}$$

- Polarizations:

⇓

More complex behaviors

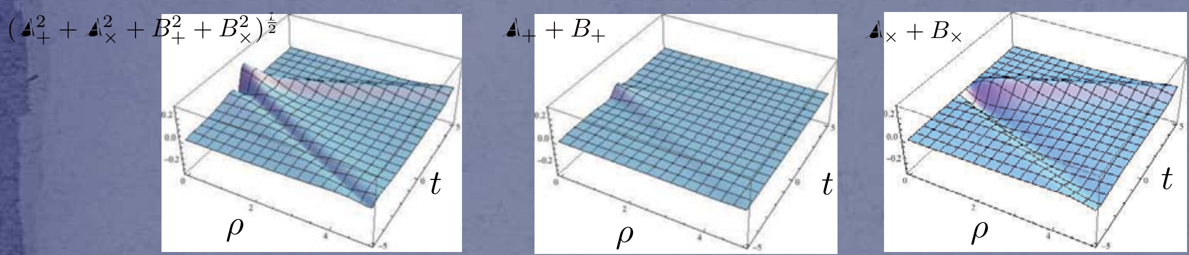
→ See next slide



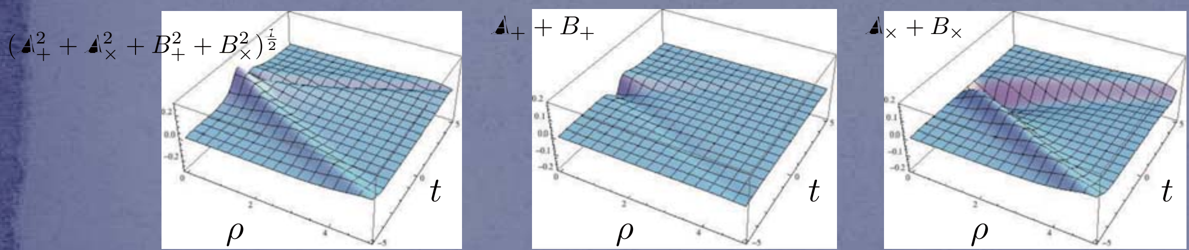
Amplitudes

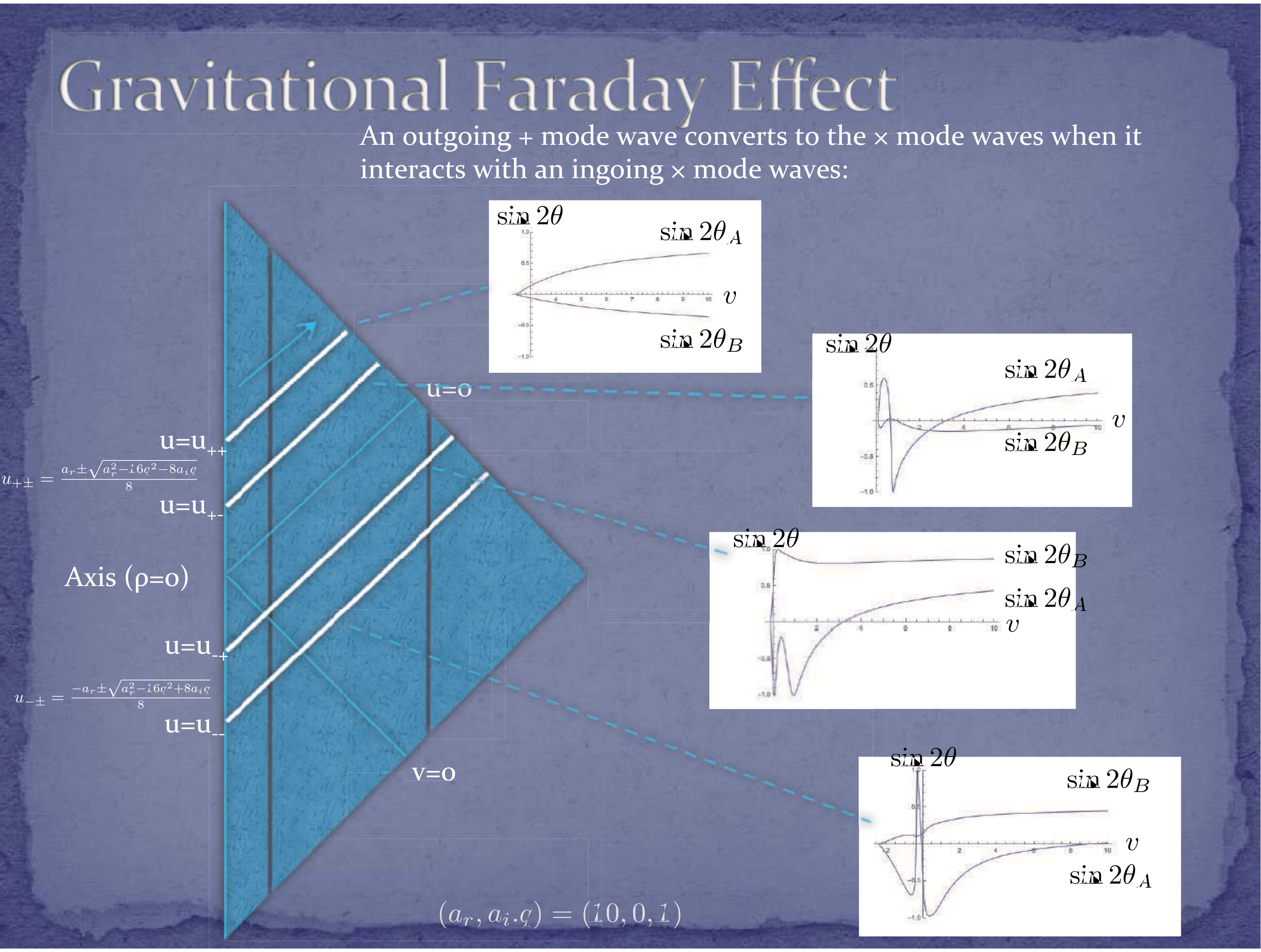
We define new parameters (k, θ) by $a = a_r + a_i i = ke^{i\theta}$

$(k,\theta) = \left(\frac{1}{3}, \frac{5\pi}{24}\right)$



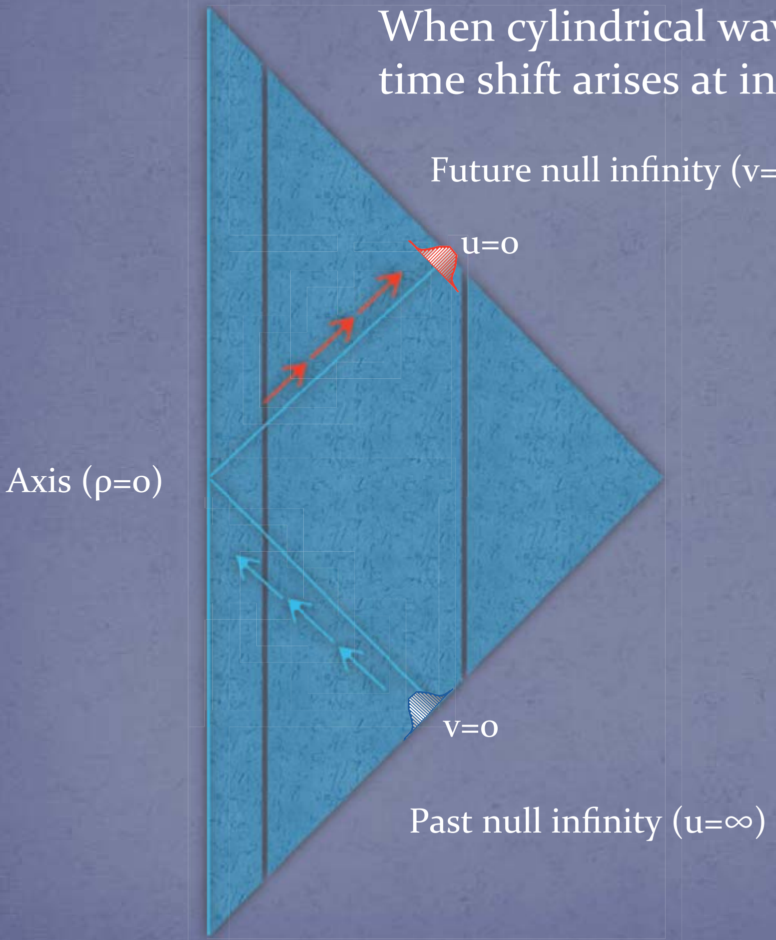
$(k,\theta) = \left(\frac{1}{3}, \frac{17\pi}{24}\right)$



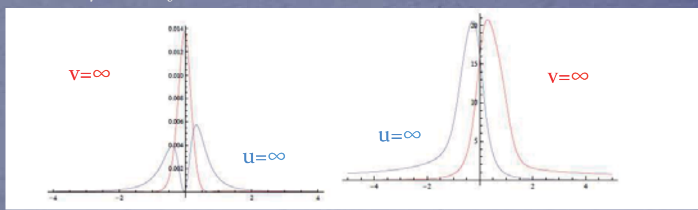

Gravitational Faraday Effect
An outgoing + mode wave converts to the × mode waves when it interacts with an ingoing × mode waves:
u=u++
$u_{+\pm} = \frac{a_r \pm \sqrt{a_r^2 - 16q^2 - 8a_i q}}{8}$
u=u+-
Axis (ρ=o)
u=u-+
$u_{-\pm} = \frac{-a_r \pm \sqrt{a_r^2 - 16q^2 + 8a_i q}}{8}$
u=u--
u=o
v=o
$(a_r, a_i.q) = (10, 0, 1)$
sin 2θ
sin 2θ_A
v
sin 2θ_B
sin 2θ
sin 2θ_A
v
sin 2θ_B
sin 2θ
sin 2θ_B
sin 2θ_A
v
sin 2θ
sin 2θ_B
v
sin 2θ_A

Time shift phenomenon

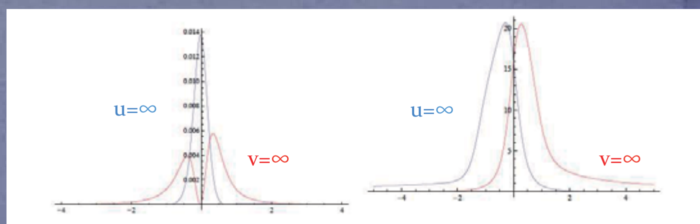
When cylindrical wave packets are reflected on the axis, time shift arises at infinity (due to the non-linear effect ?)



$a = a_r + a_i i = k e^{i\theta}$



$(k, \theta) = \left(\frac{1}{3}, \frac{5\pi}{24}\right)$ $(k, \theta) = \left(1000, \frac{5\pi}{24}\right)$

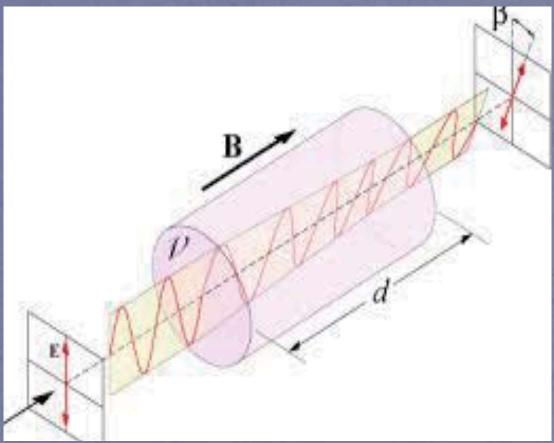


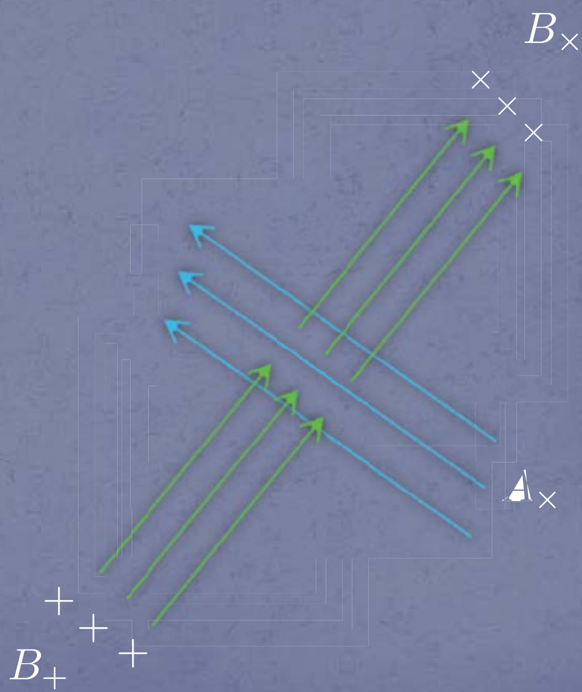
$(k, \theta) = \left(\frac{1}{3}, \frac{17\pi}{24}\right)$ $(k, \theta) = \left(1000, \frac{17\pi}{24}\right)$

Summary & Discussion

- In this work, applying the Pomeransky's procedure for the inverse scattering method to a cylindrically symmetric spacetime, we have obtained the gravitational two-soliton as an exact solution to vacuum Einstein equations with cylindrical symmetry.
- Our solution describes non-linear soliton-like cylindrical waves: an incident wave incoming from infinity collapses and then expands to infinity.
- The solution does not have any linear waves such that the ER wave.
- We have studied some non-linear effects:
  - <u>Gravitational Faraday effect</u>: An outgoing wave with a pure + mode can partially or completely convert to a × mode wave due to an ingoing wave with a × mode.
  - <u>Time Shift Phenomenon</u>: Wave packets can propagate at slower speed than light velocity.

Faraday effect





Future Works

- 2-solitonic solution with complex poles
- Levi-Civita family as a seed
- Gravitational plane waves & colliding gravitational waves
- Cosmological gravitational waves
- Higher dimension
- Kaluza-Klein theory
- · · ·

“Relativistic evolution of hierarchical triple systems”

Mao Iwasa (Kyoto)

[JGRG24(2014)P07]

(The presenter declined to upload the poster.)

“Negative time delay of light by a gravitational concave lens”

Koji Izumi (Hirosaki)

[JGRG24(2014)P08]

Negative time delay of light by a gravitational lens

Hirosaki University, **Koki Nakajima, Koji Izumi, Hideki Asada**

Ⅰ.Abstract

We re-examine the time delay of light in a gravitational concave lens as well as a gravitational convex one. The frequency shift due to the time delay is also investigated. We show that the sign of the time delay in the lens models is the same as that of the deflection angle of light. The size of the time delay decreases with increase in the parameter n. We also discuss possible parameter ranges that are relevant to pulsar timing measurements in our Galaxy.

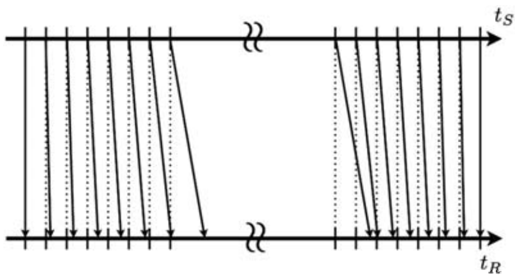


FIG.1. Frequency shift due to the gravitational time delay.

Ⅱ.Modified spacetime model

We consider the light propagation through a four-dimensional spacetime, though the whole spacetime may be higher dimensional. The four-dimensional space- time metric is

$$ds^2 = -\left(1 - \frac{\varepsilon_1}{r^n}\right)c^2dt^2 + \left(1 + \frac{\varepsilon_2}{r^n}\right)dr^2 + r^2(d\Theta^2 + \sin^2\Theta d\phi^2) + O(\varepsilon_1^2, \varepsilon_2^2, \varepsilon_1\varepsilon_2),$$

where r is the circumference radius and ε1 and ε2 are small bookkeeping parameters in iterative calculations.

The deflection angle of light becomes at the linear order

$$\alpha = \frac{\varepsilon}{b^n}\int_0^{\frac{\pi}{2}} \cos^n\Psi d\Psi + O(\varepsilon^2),$$

where the integral is positive definite, b denotes the impact parameter of the light ray, we denote ε ≡ nε1+ε2, and we define Ψ by r0/r = cos Ψ for the closest approach r0.

Ⅲ.Time delay and frequency shift

A.Time delay of a light signal

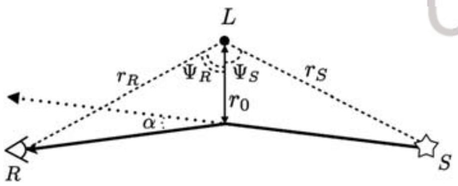


FIG.2. Schematic figure for a configuration of the source (emitter) of a signal of light S, the receiver of the signal R, and the lens L.

Subtracting the time in the flat spacetime from it provides the time delay at the linear order as

$$\delta t = \frac{1}{r_0^{n-1}}\int_{\Psi_S}^{\Psi_R}\left(\frac{\varepsilon_1(1-\cos^n\Psi)}{\sin^2\Psi} + \tilde{\varepsilon}\cos^{n-2}\Psi\right)d\Psi,$$

where ΨR and ΨS correspond to the direction from the lens to the receiver and that to the source of light, respectively.

For n=2p, the time delay is obtained as

$$\delta t_{2p} = \pi\frac{(2p-3)!!}{(2p-2)!!}\frac{(2p-1)\varepsilon_1 + \tilde{\varepsilon}}{r_0^{2p-1}},$$

and n=2p+1, it becomes

$$\delta t_{2p+1} = 2\frac{(2p-2)!!}{(2p-1)!!}\frac{2p\varepsilon_1 + \tilde{\varepsilon}}{r_0^{2p}},$$

where p is a positive integer.

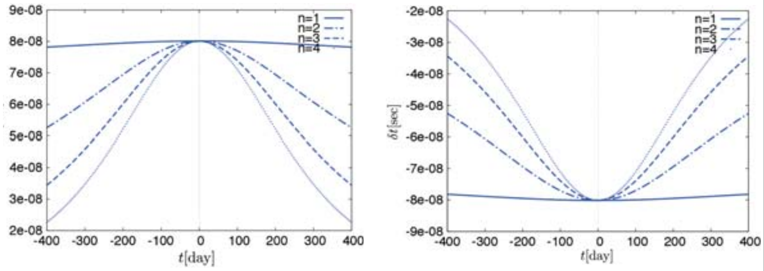


FIG.3. Time delay curves. The solid for dot-dashed, dashed, and dotted curves correspond to n = 1, 2, 3, and 4, respectively. The horizontal axis denotes the time t in days and the vertical axis means the time delay δt in seconds. Here, we assume r_{min} is 40 AU and v = 200 km=s. The lens is assumed to be a ten solar mass black hole for n = 1(ε/r_{min} ~ 10^{-8}), and the parameters for the other n are chosen such that the peak height of the time delay curve can remain the same as each other. left:ε < 0, right:ε < 0.

B.Frequency shift

The Frequency shift is y due to the time delay is defined as

$$y \equiv \frac{\nu(t) - \nu_0}{\nu_0} = -\frac{d(\delta t)}{dt},$$

For n=2p, the frequency shift is obtained as

$$y_{2p} = \frac{\pi}{c}\frac{(2p-1)!!}{(2p-2)!!}\frac{\varepsilon}{r_0^{2p+1}}v^2t,$$

and n=2p+1, it becomes

$$y_{2p+1} = \frac{2}{c}\frac{(2p)!!}{(2p-1)!!}\frac{\varepsilon}{r_0^{2p+2}}v^2t.$$

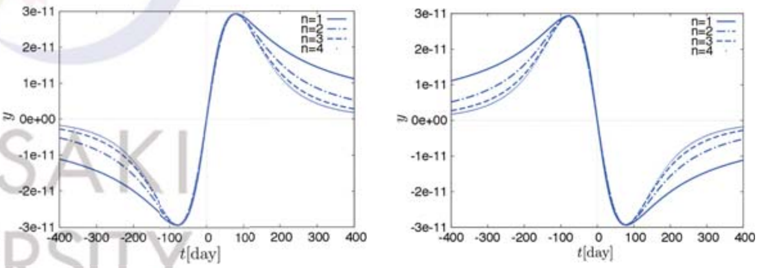


FIG.4. Frequency shift curves corresponding to Fig. 3. Here, the parameter values for n ≠ 1 are rearranged such that the peak heights of the time delay curve can remain the same as each other.

C.Possible parameter ranges in pulsar timing method

The number density of the lens objects ΩL would be constrained by no event detection as

$$\Omega_L < 10^3\ \mathrm{pc}^{-3}\left(\frac{40\mathrm{AU}}{r_0}\right)^2\left(\frac{1\ \mathrm{kpc}}{D_S}\right)\left(\frac{10\ \mathrm{year}}{T_{pt}}\right).$$

Although it seems very weak, this constraint might be interesting, because n > 1 models are massless at the spatial infinity and thus it is unlikely that these exoticobjects are constrained by other observations regarding stellar motions, galactic rotation, and so on.

Conclusion

We examined the arrival time delay of light and the frequency shift in the lens model with an inverse power law. The time delay by a gravitational convex lens (i.e., positive deflection angle of light) would be positive, even if the lens model had negative convergence like Ellis wormholes. On the other hand, time delay by a gravitational concave lens might become negative, even if the convergence were positive.

We find that negative time delay might appear not only in the strong gravitational field but also in the weak field.

Reference

Koki Nakajima, Koji Izumi, Hideki Asadam Phys. Rev. D 90, 084026 (2014)

Negative time delay of light by a gravitational lens

“Microlensing by an ultra-compact dark matter halo”

Chisaki Hagiwara (Hirosaki)

[JGRG24(2014)P09]

Microlensing by an ultra-compact dark matter halo

Chisaki Hagiwara

Hirosaki University, Japan
with H. Asada and K. Izumi (Hirosaki)

JGRG26 in Tokyo Nov. 9 - 11, 2014

Abstract: Inthis poster, we use a generalized NFW profile to study miclolensing by an ultra-compact dark matter halo in a collaboration with Izumi and Asada.

1 Motivation

Dark matter is one of the compotent that consists of the Univers. It is important to explain problems such as:

- The formation of large-scale structure
- The rotation cutve of galaxies.

The existence of dark matter was indirectly cinfirmed, but its nature has not been known. In 2009, ultra-compact minihalos (UCMHs) as nonbaryonic massive compact halo objects (MACHOs) are suggested by Ricotti and Gould [1]. Then, we concentrate on small-scale dark matter halos . If these structures are detected,

- the origin of structure in the Universe could be understood
- inflation models could be constrained.

Microlensign by halos with intermediate-mass ($10M_\odot \lesssim M \lesssim 10^6 M_\odot$) have been studied (e.g. [4], [5]). Thus, we study microlensing caused an ultra-compact (earth mass M $\lesssim 10^{-6} M_\odot$) dark matter halo with the density described by generalized NFW (gNFW) profile [2]

$$\rho_{gNFW} = \frac{\rho_s}{(\frac{r}{r_s})^\gamma (\frac{r}{r_s}+1)^{3-\gamma}} \qquad r_s : \text{scale radius}, \ \rho_s : \text{density inside } r_s$$

2 System of gravitational lensing

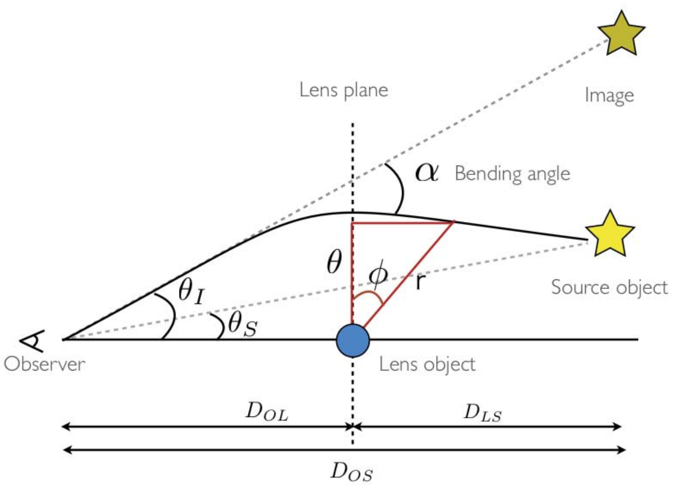


Figure 1: Diagram showing the position of the source object, its image, and the lens object.

Considering that a lens object is static spherical symmetry, surface density of the lens object projected on the lens plane is[3]

$$\Sigma(\theta) \equiv 2\int_0^{\pi/2} \rho(\theta,\phi)d\phi. \tag{1}$$

Bending angle is written as

$$\alpha(\theta_I) = \frac{4GD_{OL}}{c^2}\int_0^{\theta_I} \Sigma(\theta)\frac{\theta_I-\theta}{|\theta_I-\theta|^2}d\theta, \tag{2}$$

where G and c is the gravitational constant and light speed, respectively. The lens equation is[3] is

$$\theta_S = \theta_I - \frac{D_{LS}}{D_{OS}}\alpha(\theta_I). \tag{3}$$

The total magnification is[3]

$$A_{tot} = \Sigma\frac{1}{\frac{\theta_S}{\theta_I}\frac{\partial\theta_S}{\partial\theta_I}} \tag{4}$$

Considering the motion of the source object which performs linear motion of constan speed to the lens plane with the origin at the lens object, the angular position of the source object at the time t is

$$\theta_S(t) = \sqrt{t^2+\theta_{S0}^2}, \tag{5}$$

where θ_{S0} is the nearest distance between the source object and the lens object (the distance of closest approach).

3 Result

The surface density derived from gNFW profile is [6]

$$\Sigma(\theta) = 2\rho_s r_s\left(\frac{\theta}{r_s}\right)^{1-\gamma}\int_0^{\pi/2}\{\left(\cos\phi+\frac{\theta}{r_s}\right)^{\gamma-2} - \frac{\theta}{r_s}\left(\cos\phi+\frac{\theta}{r_s}\right)^{\gamma-3}\}d\phi. \tag{6}$$

The projected surface density (6) can be analytically calculated for each γ,

(a) $\gamma = 0$

$$\Sigma(\theta<|r_s|) = \frac{\rho_s r_s}{(1-\theta^2/r_s^2)^2}\{(1-\theta^2/r_s^2) - \frac{6\theta^2/r_s^2}{\sqrt{1-\theta^2/r_s^2}}\tanh^{-1}(\sqrt{\frac{1-\theta/r_s}{1+\theta/r_s}})\} \tag{7}$$

$$\Sigma(\theta>|r_s|) = \frac{\rho_s r_s}{(1-\theta^2/r_s^2)^2}\{(1+2\theta^2/r_s^2) - \frac{6\theta^2/r_s^2}{\sqrt{\theta^2/r_s^2-1}}\tan^{-1}(\sqrt{\frac{\theta/r_s-1}{\theta/r_s+1}})\} \tag{8}$$

(b) $\gamma = 1$

$$\Sigma(\theta<|r_s|) = \frac{2\rho_s r_s}{1-\theta^2/r_s^2}\{\frac{2}{\sqrt{1-\theta^2/r_s^2}}\tanh^{-1}(\sqrt{\frac{1-\theta/r_s}{1+\theta/r_s}})-1\} \tag{9}$$

$$\Sigma(\theta>|r_s|) = 2\frac{2\rho_s r_s}{\theta^2/r_s^2-1}\{-\frac{1}{\sqrt{\theta^2/r_s^2-1}}\tan^{-1}(\sqrt{\frac{\theta/r_s-1}{\theta/r_s+1}})+1\} \tag{10}$$

(c) $\gamma = 2$

$$\Sigma(\theta<|r_s|) = 2\rho_s r_s\{\frac{\pi}{2\theta/r_s} - \frac{2}{\sqrt{1-\theta^2/r_s^2}}\tanh^{-1}(\sqrt{\frac{1-\theta/r_s}{1+\theta/r_s}})\} \tag{11}$$

$$\Sigma(\theta>|r_s|) = 2\rho_s r_s\{\frac{\pi}{2\theta/r_s} - \frac{2}{\sqrt{\theta^2/r_s^2-1}}\tan^{-1}(\sqrt{\frac{\theta/r_s-1}{\theta/r_s+1}})\}. \tag{12}$$

The light curve is calculated numerically

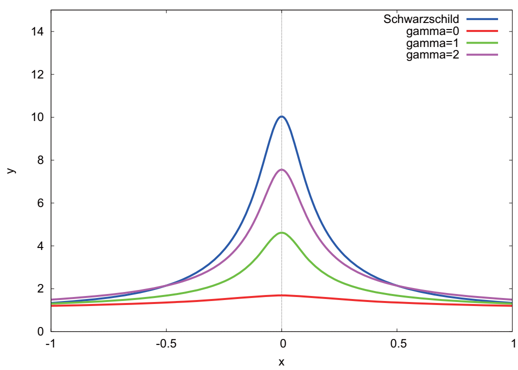


Figure 2: Magnification light curve. In this case, $D_{OL} = r_s/\sqrt{3}[Gpc], D_{OS} = 50[Gpc], M \simeq 10^{-4}[M_\odot]$. Horizontal axis is standardized by r_s and crossing time in units of r_s is 0.7day.

4 Conclusion

We compared light curves with the cases of (a), (b), (c), and Schwarzschild lens.

- It is distinguishable because the shape of the light curves are different from the Schwarzschild lens.
- It may be tested the density profile of ultra-compact dark halos by observation in the near future.

References

[1] M. Ricotti and A. Gould, 2009, ApJ 707 979

[2] Zhao, Hongsheng, 1996, Mon.Not.Roy.Astron.Soc.278:488-496

[3] P. Schneider, J. Ehlers, E. E. Falco, Gravitational Lenses

[4] A. L. Erickcek and N. M. Law, 2011, ApJ 729 49

[5] F. Li, A. L. Erickcek, and N. M. Law, 2012, Phys. Rev. D 86, 043519

[6] C. O. Wright and T. G. Brainerd, 2000, ApJ 534 34

“Blow-up behavior of Chern-Simons scalar field on the Kerr background”

Kohkichi Konno (National Inst. of Tech., Tomakomai)

[JGRG24(2014)P10]

JGRG24 (Kavli-IPMU, Nov 10-14, 2014)

Blow-up behavior of Chern-Simons scalar field on the Kerr background

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Introduction

Chern-Simons (CS) modified gravity is inspired by various theories.

- **Superstring theory**
  (see e.g. Smith et al. (2008))

- **Loop quantum gravity**
  (see e.g. Mercuri & Taveras (2009))

- **Effective field theory for inflation**
  (see S. Weinberg (2008))

CS modified gravity

Action (see e.g. Jackiw & Pi (2003)):

$$I = \int d^4x \sqrt{-g} \left[-\kappa R + \frac{\kappa\alpha}{4} \vartheta(x)\, {}^*R^{\alpha}{}_{\beta}{}^{\mu\nu} R^{\beta}{}_{\alpha\mu\nu} - \frac{1}{2} g^{\mu\nu} \left(\partial_\mu \vartheta\right)\left(\partial_\nu \vartheta\right) + \mathfrak{L}_m \right]$$

Field equations:

$$\begin{cases} G^{\mu\nu} + \alpha\, C^{\mu\nu} = -\dfrac{1}{2\kappa}\left(T_{\mathrm{m}}{}^{\mu\nu} + T_{\vartheta}^{\mu\nu}\right) \\ g^{\mu\nu} \nabla_\mu \nabla_\nu \vartheta = -\dfrac{\kappa\alpha}{4}\, {}^*R^{\tau}{}_{\sigma\alpha\beta} R^{\sigma}{}_{\tau}{}^{\alpha\beta} \end{cases}$$

C-tensor: $C^{\mu\nu} = -\frac{1}{2}\left[\left(\nabla_\sigma \vartheta\right) \varepsilon^{\sigma\mu\alpha\beta} \nabla_\alpha R^{\nu}{}_{\beta} + \left(\nabla_\tau \nabla_\sigma \vartheta\right) {}^*R^{\tau\mu\sigma\nu} + \left(\mu \leftrightarrow \nu\right)\right]$

Properties of CS gravity

- For spherically symmetric spacetimes, the CS corrections vanish.
- The static and asymptotically flat black hole spacetime is unique to be Schwarzschild spacetime. (see Shiromizu & Tanabe (2013))
- The rotating black hole solution has not yet been explored thoroughly. It should have different form from the Kerr solution.

Previous & Present works

Slowly rotating black hole solutions have been investigated by several authors.

- N. Yunes & F. Pretorius, PRD **79**, 084043 (2009)
- K. Konno, T. Matsuyama & S. Tanda, Prog. Theor. Phys. **122**, 561 (2009)
- K. Yagi, N. Yunes & T. Tanaka, PRD **86**, 044037 (2012)

Rapidly rotating black holes have not yet been investigated.

⇨ **We investigate the CS scalar field around a rapidly rotating black hole.**

K. Konno & R. Takahashi, PRD **90**, 064011 (2014)

Bootstrapping scheme

Let us assume weak CS coupling α and vacuum for ordinary matter $T_{\mathrm{m}}^{\mu\nu}=0$.

Assume GR solution $g^{(0)}{}_{\mu\nu}\sim O\left(\alpha^{0}\right)$

$$g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\vartheta^{(1)}=-\frac{\kappa\alpha}{4}\,{}^{*}R^{(0)\tau}{}_{\sigma\alpha\beta}R^{(0)\sigma}{}_{\tau}{}^{\alpha\beta}\rightarrow$$ 1st order solution $\vartheta^{(1)}\sim O\left(\alpha^{1}\right)$

$$G^{\mu\nu}\left(g^{(2)}{}_{\mu\nu}\right)=-\alpha C^{(1)\mu\nu}-\frac{1}{2\kappa}T^{(2)}{}_{\vartheta}{}^{\mu\nu}\rightarrow$$ 2nd order solution $g^{(2)}{}_{\mu\nu}\sim O\left(\alpha^{2}\right)$

Higher order of α

CS Scalar field solution

We assumed the Kerr spacetime as the background and solved the field equation for the CS scalar field.

The solution takes the form

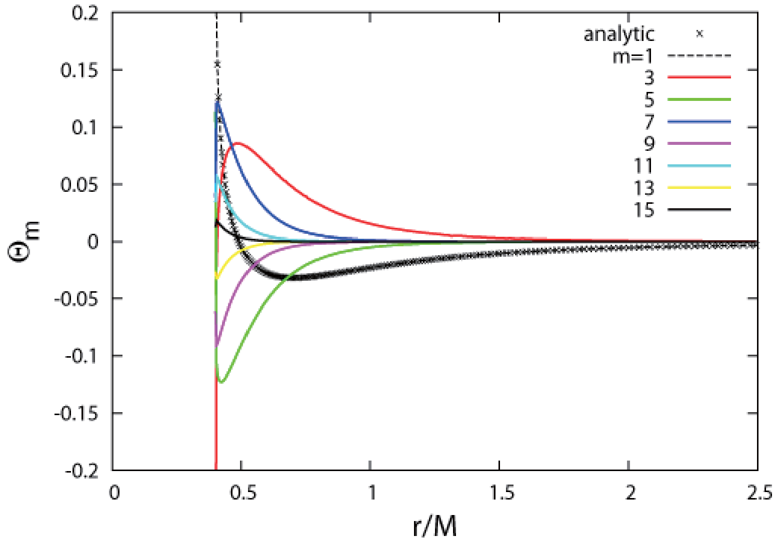
$$\vartheta^{(1)}(r,\cos\theta) = \mathrm{const} + \alpha\sum_{n=0}^{\infty}\Theta_{2n+1}(r)P_{2n+1}(\cos\theta)$$

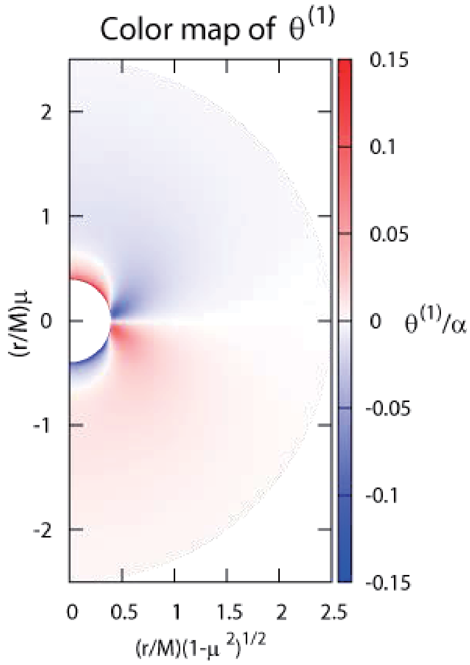
$$\Theta_1 = \frac{3\kappa\tilde{a}}{2}\left[\frac{1}{\beta^2}\left\{\frac{1+2\beta+2\beta^2}{(1+\beta)^2} - \frac{(\tilde{r}+1-2\beta^2)(\tilde{r}-1)}{(1-\beta^2)(\tilde{r}^2+1-\beta^2)} - \frac{2(\tilde{r}^2+1)}{(\tilde{r}^2+1-\beta^2)^2}\right\}\right.$$
$$\left. +(1-\beta^2)^{-\frac{3}{2}}\left(\pi - 2\arctan\frac{\tilde{r}}{\sqrt{1-\beta^2}}\right)(\tilde{r}-1) + (1-\beta^2)^{-2}\left(\log\frac{(\tilde{r}-1+\beta)^2}{\tilde{r}^2+1-\beta^2}\right)(\tilde{r}-1)\right]$$

where $\beta := \sqrt{1-\tilde{a}^2}, \quad \tilde{r} := \frac{r}{M}, \quad \tilde{a} := \frac{a}{M}$

The higher order terms $\Theta_{2n+1}\,(n \geq 1)$ were obtained numerically.

Results





Θ_m vs r/M when $a/M = 0.8$.
The inner and outer horizons are given by $r_- = 0.4$ and $r_+ = 1.6$, respectively.

Color map on the meridian place when $a/M = 0.8$.

Summary

We investigated the solution of the CS scalar field around a rapidly rotating black hole in CS modified gravity

- We obtained the solution analytically and numerically with the boundary condition that the scalar field be regular and vanish at infinity.

- We found the signature that the scalar field diverges at the inner horizon on the Kerr background.

“Recursive structure in the definitions of gauge-invariant variables for any order perturbations”

Kouji Nakamura (NAOJ)

[JGRG24(2014)P11]

Recursive structure
in
the definitions of gauge-invariant variables for any order perturbations

Kouji Nakamura (NAOJ)

Based on :

K.N. PTP **110** (2003), 723. (arXiv:gr-qc/0303039).
K.N. PTEP **2013** (2013), 043E02. (arXiv:1105.4007 [gr-qc]).
K.N. IJMPD **21** (2012), 1242004. (arXiv:1203.6448 [gr-qc]).
K.N. CQG **31** (2014), 064008. (arXiv:1403.1004 [gr-qc]).

I. Introduction

- The higher order perturbation theory in general relativity has very wide physical motivation.

  – Cosmological perturbation theory
    - Expansion law of inhomogeneous universe (ΛCDM v.s. inhomogeneous cosmology)
    - Non-Gaussianity in CMB.

  – Black hole perturbations
    - Radiation reaction effects due to the gravitational wave emission.

  – Binary coalessence through the post-Minkowski expansion
    - Target of GW detectors in 2^{nd} generation.

  – Perturbation of a star (Neutron star)
    - Rotation – pulsation coupling (Kojima 1997)

There are many physical situations to which higher order perturbation theory should be applied.

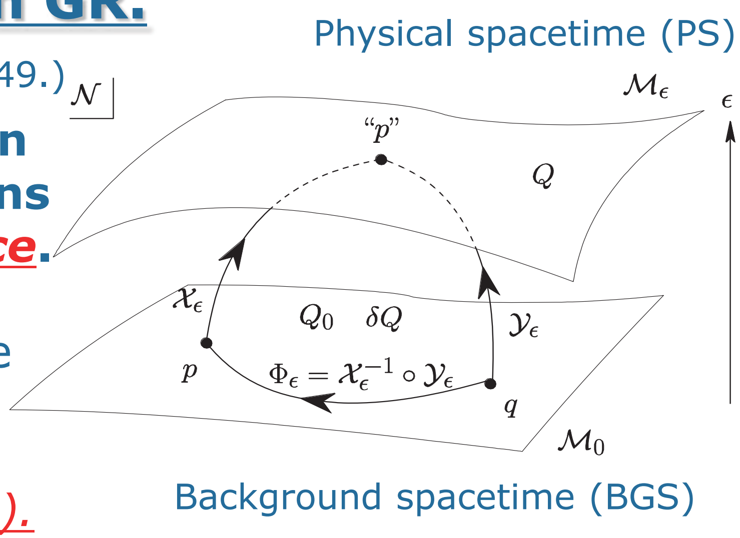
However, general relativistic perturbation theory requires very delicate treatments of “gauges”.

It is worthwhile to formulate the higher-order gauge-invariant perturbation theory from general point of view.

- According to this motivation, from 2003, we have been formulating a general-relativistic higher-order perturbation theory in a gauge-invariant manner.
  - **General formulation :**
    - Framework of higher-order gauge-invariant perturbations :
      - K.N. PTP**110** (2003), 723; *ibid.* **113** (2005), 413.
    - Construction of gauge-invariant variables for the linear-order metric perturbation :
      - K.N. CQG**28** (2011), 122001; PTEP **2013** (2013), 043E03; IJMPD**21** (2012), 1242004.
    - **The nth-order extension of the definitions of gauge-invariant variables :**
      - K.N. CQG 31 (2014), 135013. (**but this is still incomplete.**)
  - **Application to cosmological perturbation theory** :
    - Einstein equations : K.N. PRD**74** (2006), 101301R; PTP**117** (2007), 17.
    - Equations of motion for matter fields : K.N. PRD**80** (2009), 124021.
    - Consistency of the 2nd order Einstein equations : K.N. PTP**121** (2009), 1321.
    - Summary of current status of this formulation : K.N. Adv. in Astron. **2010** (2010), 576273.
    - Comparison with a different formulation : A.J. Christopherson, et al., CQG**28** (2011), 225024.

The second kind gauge in GR.

(Stewart and Walker, PRSL A341 (1974), 49.)



"*Gauge degree of freedom*" in general relativistic perturbations arises due to *general covariance*.

- In any perturbation theories, we always treat two spacetimes :
  - *Physical Spacetime (PS);*
  - *Background Spacetime (BGS).*

- In perturbation theories, we always write equations like

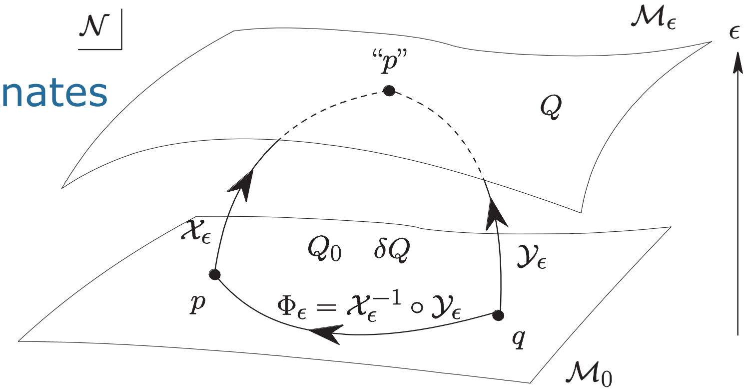
$$Q(\text{“}p\text{”}) = Q_0(p) + \delta Q(p)$$

Through this equation, we always identify the points on these two spacetimes and this identification is called "gauge choice" in perturbation theory.

The gauge choice is not unique by virtue of general covariance.

General covariance :

- "There is no preferred coordinates in nature" (intuitively).



Gauge transformation :

- The change of the point identification map.

- **Different gauge choice :** $\mathcal{X}_\epsilon$, $\mathcal{Y}_\epsilon$
- **Representation of physical variable :**

$$_{\mathcal{X}}Q := \mathcal{X}_\epsilon^* Q, \quad _{\mathcal{Y}}Q := \mathcal{Y}_\epsilon^* Q,$$

- **Gauge transformation :** $\mathcal{X} \to \mathcal{Y}$

$$\Phi_\epsilon := \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon, \quad _{\mathcal{Y}}Q = \Phi_\epsilon^* {}_{\mathcal{X}}Q$$

The is the basic understanding of gauge transformation.

In this poster, ...

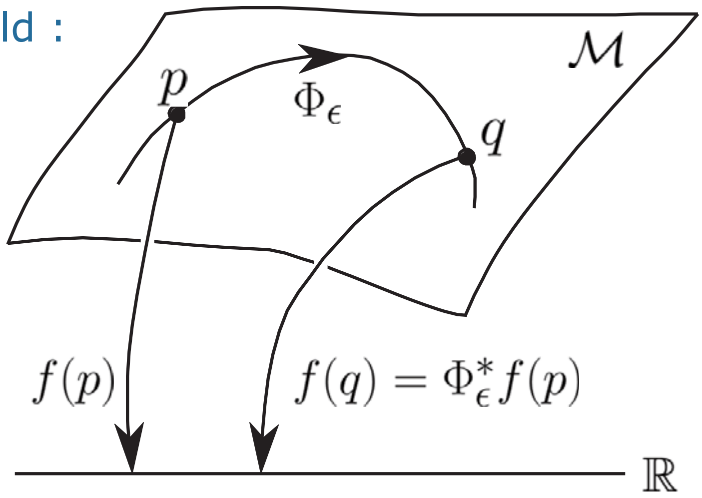
I point out the recursion structure in the definition of gauge-invariant variables for any order perturbations.

I also discuss the correspondence between the gauge issues in our framework and in an exact non-linear perturbation theory.

II. Order-by-order gauge-transformation rules

Taylor expansion of tensors on a manifold :

- The Taylor expansion of tensors is an approximated form of tensors at q (in $\mathcal{M}$) in terms of the variables at p (in $\mathcal{M}$).
- One parameter family of diffeomorphisms : Φ_ϵ
  - $\Phi_\epsilon : \mathcal{M} \to \mathcal{M}$
  - $\Phi_\epsilon(p) : q \neq p, \quad \Phi_{\epsilon=0}(p) = p.$



- Taylor expansion of a function $f(p)$

$$f(q) = (\Phi_\epsilon^* f)(p) = f(p) + \left(\frac{\partial}{\partial\epsilon}(\Phi_\epsilon^* f)\right)\Big|_p \epsilon + \frac{1}{2}\left(\frac{\partial^2}{\partial\epsilon^2}(\Phi_\epsilon^* f)\right)\Big|_p \epsilon^2 + O(\epsilon^3)$$

- **Taylor expansion of a function $f(p)$ is regarded as that of the diffeomorphism Φ_ϵ , and general arguments lead**

$$f(q) = (\Phi_\epsilon^* f)(p) = f(p) + \left(\pounds_{\xi_{(1)}} f\right)\Big|_p \epsilon + \frac{1}{2}\left(\pounds_{\xi_{(2)}} + \pounds^2_{\xi_{(1)}}\right) f\Big|_p \epsilon^2 + O(\epsilon^3)$$

nth-order representation of Taylor expansion

(Sonego and Bruni, CMP, **193** (1998), 209.)

- **Representation of general diffeomorphism :**

$$\Phi_\epsilon^* Q = \left(\phi_{\frac{\epsilon^n}{(n)!}\xi_{(n)}} \circ \phi_{\frac{\epsilon^{n-1}}{(n-1)!}\xi_{(n-1)}} \circ \cdots \circ \phi_{\frac{\epsilon^2}{(2)!}\xi_{(2)}} \circ \phi_{\frac{\epsilon}{(1)!}\xi_{(1)}} \right)^* Q + O(\epsilon^{n+1})$$

$$= \sum_{l=0}^{n} \epsilon^l \sum_{\{j_i\} \in J_l} C_l(\{j_i\}) \pounds_{\xi_{(1)}}^{j_1} \pounds_{\xi_{(2)}}^{j_2} \cdots \pounds_{\xi_{(l)}}^{j_l} Q + O(\epsilon^{n+1}),$$

where $C_l(\{j_i\}) := \prod_{i=1}^{l} \frac{1}{(i!)^{j_i} j_i!}, \quad J_l := \left\{ (j_1, ..., j_l) \in \mathbb{N}^l \middle| \sum_{i=1}^{l} i j_i = l \right\}.$

$\phi_{\frac{\epsilon^l}{(l!)}\xi_{(l)}}$: the exponential map generated by $\frac{\epsilon^l}{(l!)}\xi_{(l)}^a$.

- **Problem 1 :**
  General diffeomorphism should form a group.
  How to prove it from the above representation?

Key point : $\Phi_\sigma \circ \Phi_\lambda \neq \Phi_{\sigma+\lambda}, \quad \Phi_\lambda^{-1} \neq \Phi_{-\lambda}.$

Gauge transformation rules for nth-order perturbations (Sonego and Bruni, CMP, **193** (1998), 209.)

- **Representation of general diffeomorphism :**

$$\Phi_\epsilon^* Q = \left(\phi_{\frac{\epsilon^n}{(n)!}\xi_{(n)}} \circ \phi_{\frac{\epsilon^{n-1}}{(n-1)!}\xi_{(n-1)}} \circ \cdots \circ \phi_{\frac{\epsilon^2}{(2)!}\xi_{(2)}} \circ \phi_{\frac{\epsilon}{(1)!}\xi_{(1)}}\right)^* Q + O(\epsilon^{n+1})$$

$$= \sum_{l=0}^{n} \epsilon^l \sum_{\{j_l\}\in J_l} C_l(\{j_i\}) \pounds_{\xi_{(1)}}^{j_1} \pounds_{\xi_{(2)}}^{j_2} \cdots \pounds_{\xi_{(l)}}^{j_l} Q + O(\epsilon^{n+1})$$

where $J_l := \left\{ (j_1, ..., j_l) \in \mathbb{N}^l \,\middle|\, \sum_{i=1} i j_i = l \right\}$, $\phi_{\frac{\epsilon^l}{(l!)}\xi_{(l)}}$: the exponential map generated by $\frac{\epsilon^l}{(l!)}\xi_{(l)}^a$.

- **Expansion of the variable :** $Q = \sum_{l=0}^{n} \frac{\epsilon^l}{l!} Q_{(l)} + O(\epsilon^{n+1})$

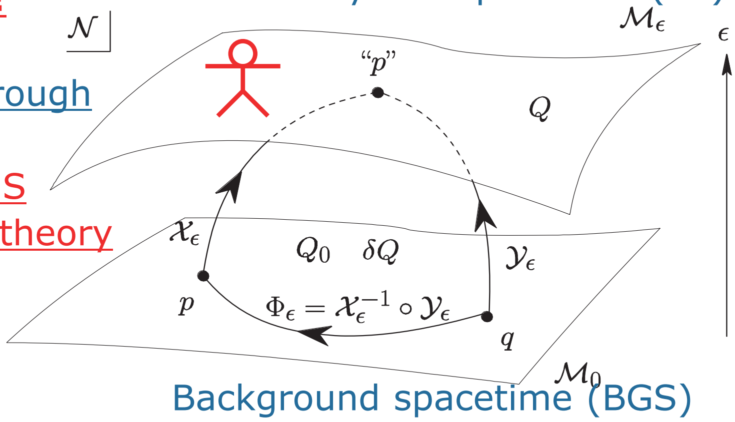
- **Order by order gauge transformation rules :**

$$ {}_{\mathcal{Y}}Q_{(k)} - {}_{\mathcal{X}}Q_{(k)} = \sum_{l=1}^{k} \frac{k!}{(k-l)!} \sum_{\{j_i\}\in J_l} C_l(\{j_i\}) \pounds_{\xi_{(1)}}^{j_1} \pounds_{\xi_{(2)}}^{j_2} \cdots \pounds_{\xi_{(l)}}^{j_l} {}_{\mathcal{X}}Q_{(k-l)} $$

To develop nth-order gauge-invariant perturbation theory, we have to construct gauge-invariant variables for each order perturbation through this gauge-transformation rule.

Gauge-invariant variables

- **Order-by-order gauge-invariance :**
  - We say that the k-th order perturbation $Q_{(k)}$ of the variable Q is gauge invariant iff $_{\mathcal{Y}}Q_{(k)} = {}_{\mathcal{X}}Q_{(k)}$ for any gauge-choice $\mathcal{X}$ and $\mathcal{Y}$.
- **Direct observables in experiments and observations should be gauge-invariant!!**
  - Any experiment or observation is carried out on PS (not on BGS) through the physical processes on PS and should have nothing to do with BGS nor gauge choices in perturbation theory



- In this sence, gauge-transformation rules

$$_{\mathcal{Y}}Q_{(k)} - {}_{\mathcal{X}}Q_{(k)} = \sum_{l=1}^{k} \frac{k!}{(k-l)!} \sum_{\{j_i\}\in J_l} C_l(\{j_i\})\, \pounds_{\xi_{(1)}}^{j_1} \pounds_{\xi_{(2)}}^{j_2} \cdots \pounds_{\xi_{(l)}}^{j_l}\, {}_{\mathcal{X}}Q_{(k-l)}$$

imply that perturbations $Q_{(k)}$ include unphysical degree of freedom. ---> gauge degree of freedom.

III. Construction of gauge-invariant variables

- metric perturbation : metric on PS : $\bar{g}_{ab}$, metric on BGS : g_{ab}

  metric expansion : $\bar{g}_{ab} = \sum_{l=0}^{n} \frac{\epsilon^l}{l!} {}^{(l)}g_{ab} + O(\epsilon^{n+1}), \quad {}^{(0)}g_{ab} := g_{ab}.$

  Our general framework of the higher-order gauge invariant perturbation theory is based on a single assumption.

- **linear order (decomposition conjecture) :**

Suppose that the linear order perturbation h_{ab} is decomposed as

$$h_{ab} = \mathcal{H}_{ab} + \pounds_X g_{ab}$$

so that the variable $\mathcal{H}_{ab}$ and X^a are the gauge invariant and the gauge variant parts of h_{ab} , respectively.

These variables are transformed as

$$ {}_{\mathcal{Y}}\mathcal{H}_{ab} - {}_{\mathcal{X}}\mathcal{H}_{ab} = 0 \qquad {}_{\mathcal{Y}}X^a - {}_{\mathcal{X}}X^a = \xi_1^a$$

under the gauge transformation $\Phi_\epsilon = \mathcal{X}_\epsilon^{-1} \circ \mathcal{Y}_\epsilon$.

This conjecture is almost proved but is still a conjecture due to the "zero-mode problem" !! (Problem 2)

K.N. CQG**28** (2011), 122001; PTEP 2013 (2013), 043E02; IJMPD**21** (2012), 1242004.

Example: Cosmological perturbations (1)

- Background metric

$$g_{ab} = a^2(\eta)\left[-(d\eta)_a(d\eta)_b + \gamma_{ij}(dx^i)_a(dx^i)_b\right]$$

γ_{ij}: metric on maximally symmetric 3-space

- metric perturbation

$$\bar{g}_{ab} = g_{ab} + \epsilon h_{ab} + \frac{1}{2}\epsilon^2 l_{ab} + O(\epsilon^3)$$

- decomposition of linear perturbation

$$h_{ab} = h_{\eta\eta}(d\eta)_a(d\eta)_b + 2h_{\eta i}(d\eta)_{(a}(dx^i)_{b)} + h_{ij}(dx^i)_a(dx^j)_b$$

$$\begin{aligned}
h_{\eta i} &= D_i h_{(VL)} + h_{(V)i}, \quad D^i h_{(V)i} = 0, \\
h_{ij} &= a^2 h_{(L)}\gamma_{ij} + a^2 h_{(T)ij}, \quad h_{(T)}{}^i_i := \gamma^{ij} h_{(T)ij} = 0, \\
h_{(T)ij} &= \left(D_i D_j - \frac{1}{3}\gamma_{ij}\Delta\right) h_{(TL)} + 2D_{(i}h_{(TV)j)} + h_{(TT)ij}, \\
D^i h_{(TV)i} &= 0, \quad D^i h_{(TT)ij} = 0, \quad D_k\gamma_{ij} = 0, \quad \Delta := D^i D_i.
\end{aligned}$$

Uniqueness of this decomposition

<--- Existence of Green functions Δ^{-1}, $(\Delta + 2K)^{-1}$, $(\Delta + 3K)^{-1}$

K : curvature constant associated with the metric γ_{ij}

Example: Cosmological perturbations (2)

Gauge variant and invariant variables of linear order metric perturbation:

- gauge variant variables : $X_a := X_\eta (d\eta)_a + X_i (dx^i)_a$

$$X_\eta := h_{(VL)} - \frac{1}{2} a^2 \partial_\eta h_{(TL)},$$

$$X_i := a^2 \left(h_{(TV)i} + \frac{1}{2} D_i h_{(TL)} \right),$$

where ${}_{\mathcal{Y}}X_a - {}_{\mathcal{X}}X_a = \xi_{(1)a}$.

- gauge invariant variables :

$$\mathcal{H}_{\eta\eta} := -2a^2\Phi = h_{\eta\eta} - 2\left(\partial_\eta - \mathcal{H}\right) X_\eta,$$

$$\mathcal{H}_{i\eta} := a^2\nu_i = h_{i\eta} - D_i X_\eta - \left(\partial_\eta - 2\mathcal{H}\right) X_i,$$

$$\mathcal{H}_{ij} := -2a^2\Psi + a^2\chi_{ij} = h_{ij} - 2D_{(i}X_{j)} + 2\mathcal{H}\gamma_{ij}X_\eta,$$

$$D^i\nu_i = 0, \quad \gamma^{ij}\chi_{ij} = 0 = D^i\chi_{ij},$$ (J. Bardeen (1980))

where ${}_{\mathcal{Y}}\mathcal{H}_{ab} - {}_{\mathcal{X}}\mathcal{H}_{ab} = 0.$

Once we accept the decomposition conjecture, we can construct higher-order gauge-invariant variables.

$$ {}^{(1)}g_{ab} = {}^{(1)}\mathcal{H}_{ab} + \pounds_{{}^{(1)}X} g_{ab} $$

<--- Conjecture

$$ {}^{(2)}g_{ab} =: {}^{(2)}\mathcal{H}_{ab} + 2\pounds_{{}^{(1)}X}\, {}^{(1)}g_{ab} + \left(\pounds_{{}^{(2)}X} - \pounds^2_{{}^{(1)}X}\right) g_{ab} $$

$$ Q_1 =: \mathcal{Q}_1 + \pounds_{{}^{(1)}X} Q_0 $$

$$ Q_2 =: \mathcal{Q}_2 + 2\pounds_{{}^{(1)}X} Q_1 + \left\{\pounds_{{}^{(2)}X} - \pounds^2_{{}^{(1)}X}\right\} Q_0 $$

Results

- As a corollary of these decomposition formulae, any order-by-order perturbative equation is automatically given in gauge-invariant form.
  (Gauge-variant parts are unphysical.)
- The decomposition of the metric perturbation into gauge-invariant and gauge-variant parts is not unique.
  (This corresponds to the fact that there are infinitely many gauge fixing procedure. Christopherson, et al., arXiv:1101.3525 [astro-ph.CO])
- Gauge-variant parts of metric perturbations also play an important role in the systematic construction of gauge-invariant variables for any perturbations.
  (In this sense, gauge-variant parts are also necessary.)

IV. nth-order extension of the definitions of gauge-invariant variables

Gauge transformation rule :

$$
\begin{aligned}
{}^{(n)}_{\mathcal{Y}}g_{ab} - {}^{(n)}_{\mathcal{X}}g_{ab} &= \sum_{l=1}^{n} \frac{n!}{(n-l)!} \sum_{\{j_i\}\in J_l} C_l\left(\{j_i\}\right) \pounds^{j_1}_{\xi_{(1)}} \pounds^{j_2}_{\xi_{(2)}} \cdots \pounds^{j_l}_{\xi_{(l)}} {}^{(n-l)}_{\mathcal{X}}g_{ab} \\
&=: F\left[\xi^a_{(1)}, \cdots, \xi^a_{(n-1)}, \xi^a_{(n)}; {}^{(0)}_{\mathcal{X}}g_{ab}, {}^{(1)}_{\mathcal{X}}g_{ab}, \cdots, {}^{(n-1)}_{\mathcal{X}}g_{ab}\right]
\end{aligned}
$$

Inspecting this gauge-transformation rule, we define the variable ${}^{(n)}\hat{H}_{ab}$ by

$$
{}^{(n)}\hat{H}_{ab} := {}^{(n)}g_{ab} + F\left[-{}^{(1)}X^a, \cdots, -{}^{(n-1)}X^a, 0; {}^{(0)}g_{ab}, {}^{(1)}g_{ab}, \cdots, {}^{(n-1)}g_{ab}\right]
$$

We have to prove the following statement :

There exists a vector field $\sigma^a_{(n)}$ such that the gauge-transformation rule for the variable ${}^{(n)}\hat{H}_{ab}$ is given by ${}^{(n)}_{\mathcal{Y}}\hat{H}_{ab} - {}^{(n)}_{\mathcal{X}}\hat{H}_{ab} = \pounds_{\sigma_{(n)}} {}^{(0)}g_{ab}$

When the proof of the above statement is accomplished, we may apply the decomposition conjecture and we can decompose the variable ${}^{(n)}\hat{H}_{ab}$ into its gauge-invariant and gauge-variant parts as

$$
{}^{(n)}\hat{H}_{ab} =: {}^{(n)}\mathcal{H}_{ab} + \pounds_{{}^{(n)}X} {}^{(0)}g_{ab}, \quad {}^{(n)}_{\mathcal{Y}}\mathcal{H}_{ab} - {}^{(n)}_{\mathcal{X}}\mathcal{H}_{ab} = 0, \quad {}^{(n)}_{\mathcal{Y}}X^a - {}^{(n)}_{\mathcal{X}}X^a = \sigma^a_{(n)}.
$$

This implies that we have decomposed ${}^{(n)}g_{ab}$ as

$$
{}^{(n)}g_{ab} := {}^{(n)}\mathcal{H}_{ab} + \pounds_{{}^{(n)}X}g_{ab} - F\left[-{}^{(1)}X^a, \cdots, -{}^{(n-1)}X^a, 0; {}^{(0)}g_{ab}, {}^{(1)}g_{ab}, \cdots, {}^{(n-1)}g_{ab}\right]
$$

(blue) : gauge-invariant part, (yellow) : gauge-variant part.

I have confirmed this to 4th-order perturbations.

We have to prove the following statement :
There exists a vector field $\sigma^a_{(n)}$ such that the gauge-transformation rule for the variable ${}^{(n)}\hat{H}_{ab}$ is given by ${}^{(n)}_{\mathcal{Y}}\hat{H}_{ab} - {}^{(n)}_{\mathcal{X}}\hat{H}_{ab} = \pounds_{\sigma_{(n)}} {}^{(0)}g_{ab}$

We have derived explicit expressions for $\sigma^a_{(n)}$ to 4th order:

- 1st order : $\sigma^a_{(1)} = \xi^a_{(1)},$

- 2nd order : $\sigma^a_{(2)} := \xi^a_{(2)} + \hat{\sigma}^a_{(2)} := \xi^a_{(2)} + \left[\xi_{(1)}, {}^{(1)}X\right]^a,$

- 3rd order : $\sigma^a_{(3)} := \xi^a_{(3)} + \hat{\sigma}^a_{(3)},$

$$\hat{\sigma}^a_{(3)} := 3\left[\xi_{(1)}, \xi_{(2)}\right]^a + 3\left[\xi_{(1)}, {}^{(2)}X\right]^a + 2\left[\xi_{(1)}, \left[\xi_{(1)}, {}^{(1)}X\right]\right]^a + \left[{}^{(1)}X, \left[\xi_{(1)}, {}^{(1)}X\right]\right]^a,$$

- 4th order : $\sigma^a_{(4)} := \xi^a_{(4)} + \hat{\sigma}^a_{(4)},$

$$\begin{aligned}\hat{\sigma}^a_{(4)} \quad := \quad & 4\left[\xi_{(1)}, \xi_{(3)}\right]^a + 6\left[\xi_{(1)}, \left[\xi_{(1)}, \xi_{(2)}\right]\right]^a + 4\left[\xi_{(1)}, {}^{(3)}X\right]^a \\ & +3\left[\xi_{(2)}, {}^{(2)}X\right]^a + 6\left[\xi_{(1)}, \left[\xi_{(1)}, {}^{(2)}X\right]\right]^a + 3\left[\xi_{(2)}, \left[\xi_{(1)}, {}^{(1)}X\right]\right]^a \\ & +3\left[{}^{(2)}X, \left[\xi_{(1)}, {}^{(1)}X\right]\right]^a + 3\left[\xi_{(1)}, \left[\xi_{(1)}, \left[\xi_{(1)}, {}^{(1)}X\right]\right]\right]^a \\ & +3\left[\xi_{(1)}, \left[{}^{(1)}X, \left[\xi_{(1)}, {}^{(1)}X\right]\right]\right]^a + \left[{}^{(1)}X, \left[{}^{(1)}X, \left[\xi_{(1)}, {}^{(1)}X\right]\right]\right]^a,\end{aligned}$$

These are evidences of the fact that I did check to the 4th order.

Through the confirmation to the 4th order, I also find the following identities for gauge-transformation rules of gauge-variant variables in metric perturbations.

- 1st order :

$$\sum_{\{j_i\}\in J_1} \mathcal{C}_1(\{j_i\}) \left(\pounds^{j_1}_{-\overset{(1)}{y}X} - \pounds^{j_1}_{-\overset{(1)}{x}X} + \pounds^{j_1}_{\xi_{(1)}} \right) = 0,$$

- 2nd order :

$$\sum_{\{j_i\}\in J_2} \mathcal{C}_2(\{j_i\}) \left(\pounds^{j_1}_{\xi_{(1)}} \pounds^{j_2}_{\xi_{(2)}} + \pounds^{j_1}_{-\overset{(1)}{y}X} \pounds^{j_2}_{-\overset{(2)}{y}X} - \pounds^{j_1}_{-\overset{(1)}{x}X} \pounds^{j_2}_{-\overset{(2)}{x}X} \right)$$
$$+ \sum_{\{j_i\}\in J_1} \mathcal{C}_1(\{j_i\}) \pounds^{j_1}_{-\overset{(1)}{y}X} \sum_{\{k_m\}\in J_1} \mathcal{C}_1(\{k_m\}) \pounds^{k_1}_{\xi_{(1)}} = 0,$$

- 3rd order :

$$\sum_{\{j_i\}\in J_3} \mathcal{C}_3(\{j_i\}) \left(\pounds^{j_1}_{\xi_{(1)}} \pounds^{j_2}_{\xi_{(2)}} \pounds^{j_3}_{\xi_{(3)}} + \pounds^{j_1}_{-\overset{(1)}{y}X} \pounds^{j_2}_{-\overset{(2)}{y}X} \pounds^{j_3}_{-\overset{(3)}{y}X} - \pounds^{j_1}_{-\overset{(1)}{x}X} \pounds^{j_2}_{-\overset{(2)}{x}X} \pounds^{j_3}_{-\overset{(3)}{x}X} \right)$$
$$+ \sum_{\{j_i\}\in J_1} \mathcal{C}_1(\{j_i\}) \pounds^{j_1}_{-\overset{(1)}{y}X} \sum_{\{k_i\}\in J_2} \mathcal{C}_2(\{k_m\}) \pounds^{k_1}_{\xi_{(1)}} \pounds^{k_2}_{\xi_{(2)}}$$
$$+ \sum_{\{j_i\}\in J_2} \mathcal{C}_2(\{j_i\}) \pounds^{j_1}_{-\overset{(1)}{y}X} \pounds^{j_2}_{-\overset{(2)}{y}X} \sum_{\{k_m\}\in J_1} \mathcal{C}_1(\{k_m\}) \pounds^{k_1}_{\xi_{(1)}} = 0,$$

- 4th order :

$$\sum_{\{j_l\}\in J_4} \mathcal{C}_4(\{j_i\}) \left(\pounds^{j_1}_{\xi_{(1)}} \cdots \pounds^{j_4}_{\xi_{(3)}} + \pounds^{j_1}_{-\overset{(1)}{y}X} \cdots \pounds^{j_4}_{-\overset{(3)}{y}X} - \pounds^{j_1}_{-\overset{(1)}{x}X} \cdots \pounds^{j_4}_{-\overset{(4)}{x}X} \right)$$
$$+ \sum_{n=1}^{3} \sum_{\{j_l\}\in J_n} \mathcal{C}_3(\{j_i\}) \pounds^{j_1}_{-\overset{(1)}{y}X} \cdots \pounds^{j_3}_{-\overset{(3)}{y}X} \sum_{\{k_m\}\in J_{4-n}} \mathcal{C}_3(\{k_m\}) \pounds^{k_1}_{\xi_{(1)}} \cdots \pounds^{k_3}_{\xi_{(3)}} = 0.$$

Example: 3rd-order perturbation (1) :

$$
\begin{aligned}
&{}^{(3)}_{\mathcal{Y}}\hat{H}_{ab} - {}^{(3)}_{\mathcal{X}}\hat{H}_{ab} \\
&= \frac{3!}{2!}\sum_{\{j_i\}\in J_1} \mathcal{C}_1(\{j_i\}) \left(\pounds^{j_1}_{-{}^{(1)}_{\mathcal{Y}}X} - \pounds^{j_1}_{-{}^{(1)}_{\mathcal{X}}X} + \pounds^{j_1}_{\xi_{(1)}} \right) {}^{(2)}_{\mathcal{X}}g_{ab} \\
&\quad +3!\Bigg[\sum_{\{j_i\}\in J_2} \mathcal{C}_2(\{j_i\}) \left(\pounds^{j_1}_{-{}^{(1)}_{\mathcal{Y}}X} \pounds^{j_2}_{-{}^{(2)}_{\mathcal{Y}}X} - \pounds^{j_1}_{-{}^{(1)}_{\mathcal{X}}X} \pounds^{j_2}_{-{}^{(2)}_{\mathcal{X}}X} - \pounds^{j_1}_{\xi_{(1)}} \pounds^{j_2}_{\xi_{(2)}} \right) \\
&\qquad + \sum_{\{j_i\}\in J_1} \mathcal{C}_1(\{j_i\}) \pounds^{j_1}_{-{}^{(1)}_{\mathcal{Y}}X} \sum_{\{k_m\}\in J_1} \mathcal{C}_1(\{k_m\}) \pounds^{k_1}_{\xi_{(1)}} \Bigg] {}^{(1)}_{\mathcal{X}}g_{ab} \\
&\quad +3!\Bigg[\sum_{\{j_i\}\in J_3 \backslash {}_3J_0^+} \mathcal{C}_2(\{j_i\}) \left(\pounds^{j_1}_{\xi_{(1)}} \pounds^{j_2}_{\xi_{(2)}} + \pounds^{j_1}_{-{}^{(1)}_{\mathcal{Y}}X} \pounds^{j_2}_{-{}^{(2)}_{\mathcal{Y}}X} - \pounds^{j_1}_{-{}^{(1)}_{\mathcal{X}}X} \pounds^{j_2}_{-{}^{(2)}_{\mathcal{X}}X} \right) \\
&\qquad + \sum_{\{j_i\}\in J_1} \mathcal{C}_1(\{j_i\}) \pounds^{j_1}_{-{}^{(1)}_{\mathcal{Y}}X} \sum_{\{k_m\}\in J_2} \mathcal{C}_2(\{k_m\}) \pounds^{k_1}_{\xi_{(1)}} \pounds^{k_2}_{\xi_{(2)}} \\
&\qquad + \sum_{\{j_i\}\in J_2} \mathcal{C}_2(\{j_i\}) \pounds^{j_1}_{-{}^{(1)}_{\mathcal{Y}}X} \pounds^{j_2}_{-{}^{(2)}_{\mathcal{Y}}X} \sum_{\{k_m\}\in J_1} \mathcal{C}_1(\{k_m\}) \pounds^{k_1}_{\xi_{(1)}} \Bigg] g_{ab} \\
&\quad + \pounds_{\xi_{(3)}} g_{ab} \\
&= \pounds_{\hat{\sigma}_{(3)}} g_{ab} + \pounds_{\xi_{(3)}} g_{ab}.
\end{aligned}
$$

These terms vanish due to the 1st and 2nd order identities.

Tough calculations yields … .

Here, ${}_{(3)}J_0^+ := \{j_3 = 1. j_i = 0, i = 1, 2, 4, 5, \ldots\}$.

Example: 3rd-order perturbation (2) :

Then, we may apply the decomposition conjecture which implies that the variable ${}^{(3)}\hat{H}_{ab}$ into its gauge-invariant and gauge-variant parts as

$$
{}^{(3)}\hat{H}_{ab} =: {}^{(3)}\mathcal{H}_{ab} + \pounds_{{}^{(3)}X} g_{ab}, \quad {}^{(3)}_{\mathcal{Y}}\mathcal{H}_{ab} - {}^{(3)}_{\mathcal{X}}\mathcal{H}_{ab} = 0, \quad {}^{(3)}_{\mathcal{Y}}X^a - {}^{(3)}_{\mathcal{X}}X^a = \hat{\sigma}^a_{(3)} + \xi^a_{(3)}.
$$

Through the last equation, the following 4th-order identity is derived:

$$
\sum_{\{j_l\}\in J_4} \mathcal{C}_4(\{j_i\}) \left(\pounds^{j_1}_{\xi_{(1)}} \cdots \pounds^{j_4}_{\xi_{(3)}} + \pounds^{j_1}_{-{}^{(1)}_{\mathcal{Y}}X} \cdots \pounds^{j_4}_{-{}^{(3)}_{\mathcal{Y}}X} - \pounds^{j_1}_{-{}^{(1)}_{\mathcal{X}}X} \cdots \pounds^{j_4}_{-{}^{(4)}_{\mathcal{X}}X} \right)
$$

$$
+ \sum_{n=1}^{3} \sum_{\{j_l\}\in J_n} \mathcal{C}_3(\{j_i\}) \pounds^{j_1}_{-{}^{(1)}_{\mathcal{Y}}X} \cdots \pounds^{j_3}_{-{}^{(3)}_{\mathcal{Y}}X} \sum_{\{k_m\}\in J_{4-n}} \mathcal{C}_3(\{k_m\}) \pounds^{k_1}_{\xi_{(1)}} \cdots \pounds^{k_3}_{\xi_{(3)}} = 0.
$$

This is the recursive structure in the definition of gauge-invariant variables for the metric perturbations.

V. Recursive structure in the definitions of gauge-invariant variables for n^th^-order perturbations

Through the construction of gauge-invariant variables for ${}^{(i)}g_{ab}$ ($i = 1, ..., n-1$) , we can define the vector fields ${}^{(i)}X^a$ ($i = 1, ..., n-1$) , whose gauge-transformations are given by ${}^{(i)}_{\mathcal{Y}}X^a - {}^{(i)}_{\mathcal{X}}X^a = \sigma^a_{(i)} = \xi^a_{(i)} + \hat{\sigma}^a_{(i)}$.

Furthermore, we obtain the n-1 identities which are expressed as

$$\sum_{p=1}^{i} \sum_{\{j_l\}\in J_p} \mathcal{C}_i(\{j_l\}) \pounds^{j_1}_{-{}^{(1)}_{\mathcal{Y}}X} \cdots \pounds^{j_i}_{-{}^{(i)}_{\mathcal{Y}}X} \sum_{\{k_m\}\in J_{i-p}} \mathcal{C}_i(\{k_m\}) \pounds^{k_1}_{\xi_{(1)}} \cdots \pounds^{k_i}_{\xi_{(i)}}$$
$$= \sum_{\{j_l\}\in J_i} \mathcal{C}_i(\{j_l\}) \pounds^{j_1}_{-{}^{(1)}_{\mathcal{X}}X} \cdots \pounds^{j_i}_{-{}^{(i)}_{\mathcal{X}}X}.$$

To define the gauge-invariant variables for ${}^{(n)}g_{ab}$, we consider the variable

$$\begin{aligned} {}^{(n)}\hat{H}_{ab} \;:=\; & {}^{(n)}g_{ab} + \sum_{l=1}^{n-1} \frac{n!}{(n-l)!} \sum_{\{j_i\}\in J_l} \mathcal{C}_l(\{j_i\}) \pounds^{j_1}_{-{}^{(1)}X} \cdots \pounds^{j_l}_{-{}^{(l)}X} {}^{(n-l)}g_{ab} \\ & + n! \sum_{\{j_i\}\in J_n \backslash {}_{n}J_0^+} \mathcal{C}_{n-1}(\{j_i\}) \pounds^{j_1}_{-{}^{(1)}X} \cdots \pounds^{j_{n-1}}_{-{}^{(n-1)}X} g_{ab}, \end{aligned}$$

where ${}_{(n)}J_0^+ := \{j_n = 1, j_i = 0, i \in \mathbb{N}\backslash\{n\}\}$.

Through the above identities, the gauge-transformation rule for the variable ${}^{(n)}\hat{H}_{ab}$ is given by

$$\begin{aligned} {}^{(n)}_{\mathcal{Y}}\hat{H}_{ab} - {}^{(n)}_{\mathcal{X}}\hat{H}_{ab} &= \pounds_{\xi_{(n)}} g_{ab} \\ &\quad + n! \left[\sum_{\{j_l\}\in J_n \backslash {}_nJ_0^+} \mathcal{C}_{n-1}(\{j_l\}) \left(\pounds^{j_1}_{\xi_{(1)}} \cdots \pounds^{j_{n-1}}_{\xi_{(n-1)}} + \pounds^{j_1}_{-{}^{(1)}_{\mathcal{Y}}X} \cdots \pounds^{j_{n-1}}_{-{}^{(n-1)}_{\mathcal{Y}}X} - \pounds^{j_1}_{-{}^{(1)}_{\mathcal{X}}X} \cdots \pounds^{j_{n-1}}_{-{}^{(n-1)}_{\mathcal{X}}X} \right) \right. \\ &\qquad \left. + \sum_{i=1}^{n-1} \sum_{\{j_l\}\in J_i} \mathcal{C}_{n-1}(\{j_l\}) \pounds^{j_1}_{-{}^{(1)}_{\mathcal{Y}}X} \cdots \pounds^{j_{n-1}}_{-{}^{(3)}\ {}_{\mathcal{Y}}X} \sum_{\{k_m\}\in J_{n-i}} \mathcal{C}_{n-1}(\{k_m\}) \pounds^{k_1}_{\xi_{(1)}} \cdots \pounds^{k_{n-1}}_{\xi_{(n-1)}} \right] g_{ab}. \end{aligned}$$

From the analyses to the 4th order, the following conjecture (algebraic conjecture) is reasonable:

There exists a vector field $\hat{\sigma}^a_{(n)}$ such that

$$\begin{aligned} & n! \left[\sum_{\{j_l\}\in J_n \backslash {}_nJ_0^+} \mathcal{C}_{n-1}(\{j_l\}) \left(\pounds^{j_1}_{\xi_{(1)}} \cdots \pounds^{j_{n-1}}_{\xi_{(n-1)}} + \pounds^{j_1}_{-{}^{(1)}_{\mathcal{Y}}X} \cdots \pounds^{j_{n-1}}_{-{}^{(n-1)}_{\mathcal{Y}}X} - \pounds^{j_1}_{-{}^{(1)}_{\mathcal{X}}X} \cdots \pounds^{j_{n-1}}_{-{}^{(n-1)}_{\mathcal{X}}X} \right) \right. \\ &\qquad \left. + \sum_{i=1}^{n-1} \sum_{\{j_l\}\in J_i} \mathcal{C}_{n-1}(\{j_l\}) \pounds^{j_1}_{-{}^{(1)}_{\mathcal{Y}}X} \cdots \pounds^{j_{n-1}}_{-{}^{(n-1)}_{\mathcal{Y}}X} \sum_{\{k_m\}\in J_{n-i}} \mathcal{C}_{n-1}(\{k_m\}) \pounds^{k_1}_{\xi_{(1)}} \cdots \pounds^{k_{n-1}}_{\xi_{(n-1)}} \right] \\ = {} & \pounds_{\hat{\sigma}_{(n)}}. \end{aligned}$$

To prove this conjecture, tough algebraic calculations are necessary, but we expect that there is no difficulty to prove this conjecture except for this tough calculations.

Actually, we have confirmed this conjecture to 4th order.

The above algebraic conjecture is true, the gauge-transformation rule for the variable ${}^{(n)}\hat{H}_{ab}$ is given by

$$ {}^{(n)}_{\ \mathcal{Y}}\hat{H}_{ab} - {}^{(n)}_{\ \mathcal{X}}\hat{H}_{ab} = \pounds_{\sigma_{(n)}} g_{ab} $$

Then, we may apply the decomposition conjecture to the variable ${}^{(n)}\hat{H}_{ab}$ and we can decompose it as

$$ {}^{(n)}\hat{H}_{ab} =: {}^{(n)}\mathcal{H}_{ab} + \pounds_{{}^{(n)}X} g_{ab}, \quad {}^{(n)}_{\ \mathcal{Y}}\mathcal{H}_{ab} - {}^{(n)}_{\ \mathcal{X}}\mathcal{H}_{ab} = 0, \quad {}^{(n)}_{\ \mathcal{Y}}X^{a} - {}^{(n)}_{\ \mathcal{X}}X^{a} = \sigma^{a}_{(n)} := \xi^{a}_{(n)} + \hat{\sigma}^{a}_{(n)}. $$

This implies that the original metric perturbation ${}^{(n)}g_{ab}$ is decomposed as

$$ {}^{(n)}g_{ab} \ := \ {}^{(n)}\mathcal{H}_{ab} - \sum_{l=1}^{n} \frac{n!}{(n-l)!} \sum_{\{j_i\}\in J_l} \mathcal{C}_l(\{j_i\}) \pounds^{j_1}_{-{}^{(1)}X} \cdots \pounds^{j_l}_{-{}^{(l)}X} {}^{(n-l)}g_{ab}, $$

Furthermore, the above algebraic conjecture and the gauge-variant variables ${}^{(n)}X^{a}$, we obtain the following identity

$$ \sum_{p=1}^{n} \sum_{\{j_l\}\in J_p} \mathcal{C}_n(\{j_l\}) \pounds^{j_1}_{-{}^{(1)}_{\mathcal{Y}}X} \cdots \pounds^{j_i}_{-{}^{(n)}_{\mathcal{Y}}X} \sum_{\{k_m\}\in J_{n-p}} \mathcal{C}_n(\{k_m\}) \pounds^{k_1}_{\xi_{(1)}} \cdots \pounds^{k_n}_{\xi_{(n)}} $$
$$ = \sum_{\{j_l\}\in J_n} \mathcal{C}_n(\{j_l\}) \pounds^{j_1}_{-{}^{(1)}_{\mathcal{X}}X} \cdots \pounds^{j_n}_{-{}^{(n)}_{\mathcal{X}}X}. $$

This identity is the i=n version of the previous set of identities and is used when we construct the gauge-invariant variables for more higher-order metric perturbations.

VI. Summary and Discussion

- **Summary**

We pointed out the recursive structure in the definition of gauge-invariant variables for higher-order general-relativistic perturbations.

We used the "decomposition conjecture" and the "algebraic conjecture" in our construction of gauge-invariant variables.

The "algebraic conjecture" is just algebraic one but tough algebraic calculations are necessary to show this.

On the other hand, "decomposition conjecture" is still a conjecture due to the "zero mode problem" [See K.N. PTEP **2013** (2013), 043E02; IJMPD **21** (2012), 1242004.]. In other words, the zero mode problem is an essential problem in our scenario of the higher-order gauge-invariant perturbation theory.

- **Discussion**

The full metric $\mathcal{X}_\epsilon^* \bar{g}_{ab}$, which is pulled back to the background spacetime, is given by

$$\begin{aligned}
\mathcal{X}_\epsilon^* \bar{g}_{ab} &= \sum_{k=0}^{n} \frac{\epsilon^k}{k!} {}^{(k)}g_{ab} \\
&= g_{ab} && \text{(background)} \\
&\quad + \sum_{k=1}^{n} \frac{\epsilon^k}{k!} {}^{(k)}\mathcal{H}_{ab} && \text{(gauge-invariant)} \\
&\quad - \sum_{k=1}^{n} \frac{\epsilon^k}{k!} \sum_{l=1}^{k} \frac{k!}{(k-l)!} \sum_{\{j_i\} \in J_l} \mathcal{C}_l(\{j_i\}) \pounds_{-{}^{(1)}X}^{j_1} \cdots \pounds_{-{}^{(l)}X}^{j_l} {}^{(k-l)}g_{ab} && \text{(gauge-variant)} \\
&\quad + o(\epsilon^n),
\end{aligned}$$

If the limit $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\epsilon^k}{k!} {}^{(k)}\mathcal{H}_{ab}$ converges, this corresponds to the gauge-invariant variable in an exact non-linear perturbation theory and the gauge issue in an exact non-linear perturbation theory will be justified in this way.

“Spherical Domain Wall Shell Collapse in a Dust Universe”

Chulmoon Yoo (Nagoya)

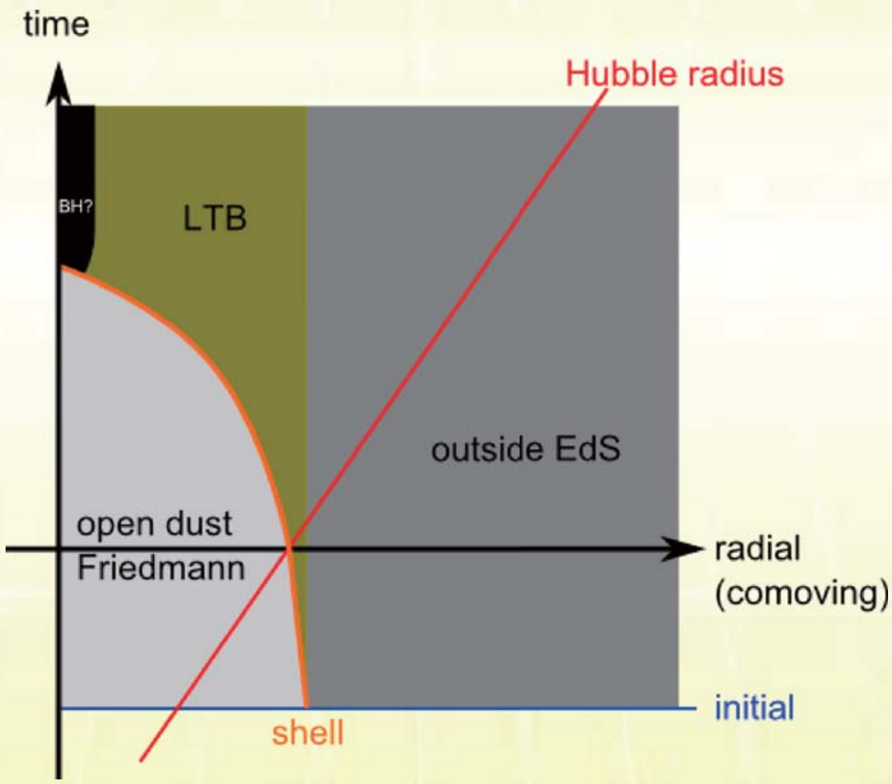
[JGRG24(2014)P12]

Spherical Domain Wall Shell Collapse in a Dust Universe

Chul-Moon Yoo
Graduate School of Science, Nagoya University
with Norihiro Tanahashi (DAMTP)

△Introduction

◎Domain wall shell dynamics in dust universe



◎Possible scenario

1. Bubble nucleation during inflation
   → lower density region + pure tension shell
2. The bubble enters the horizon after the inflation
3. Shrinks due to the tension
   → induced inhomogeneity? BH?

△Shell in a Dust-dominated Universe

◎Shell interior: FLRW universe

- **Metric**

$$ds_-^2 = -dt_-^2 + a^2(t_-)\left[d\chi^2 + f^2(\chi)d\Omega^2\right]$$

where $f(\chi) = \begin{cases} \sin\chi & \text{for } K = 1 \quad \text{(closed)} \\ \chi & \text{for } K = 0 \quad \text{(flat)} \\ \sinh\chi & \text{for } K = -1 \quad \text{(open)} \end{cases}$

- **Friedmann equations**

$$8\pi\rho_- = 3\left(\frac{\partial_t a}{a}\right)^2 + \frac{3K}{a^2}$$

$$0 = -2\left(\frac{\partial_t^2 a}{a}\right) - \left(\frac{\partial_t a}{a}\right)^2 - \frac{K}{a^2}$$

- **Energy-momentum tensor**

$$T_-^{\mu\nu} = \rho_- u_-^\mu u_-^\nu$$

◎Shell exterior: LTB model

- **Metric**

$$ds_+^2 = -dt_+^2 + \frac{(\partial_r R)^2}{1-k(r)r^2}dr^2 + R^2(t_+, r)d\Omega^2$$

- **Einstein equation**

$$(\partial_t R)^2 = -kr^2 + \frac{2M(r)}{R}$$

- **Energy density**

$$8\pi\rho_+ = \frac{2\partial_r M}{R^2\partial_r R} = \frac{r^2 m + \frac{1}{3}r^3\partial_r m(r)}{R^2\partial_r R}$$

where $m(r) = 6M(r)/r^3$

- **Solution(3 arbitrary functions: $k(r), m(r), t_\mathrm{B}(r)$)**

$$R(t_+, r) = rm^{1/3}(t_+ - t_\mathrm{B}(r))^{2/3}S(x)$$

where $x = km^{-2/3}(t_+ - t_\mathrm{B})^{2/3}$

$$S(x) = \frac{1-\cos\sqrt{\eta}}{6^{1/3}(\sqrt{\eta}-\sin\sqrt{\eta})^{2/3}} \quad \textbf{with } x = \frac{(\sqrt{\eta}-\sin\sqrt{\eta})^{2/3}}{6^{2/3}}$$

◎Shell

- Shell trajectory

$$(t_-,\chi) = (t^s_-(\tau),\ \chi^s(\tau))$$
$$(t_+,r) = (t^s_+(\tau),\ r^s(\tau))$$

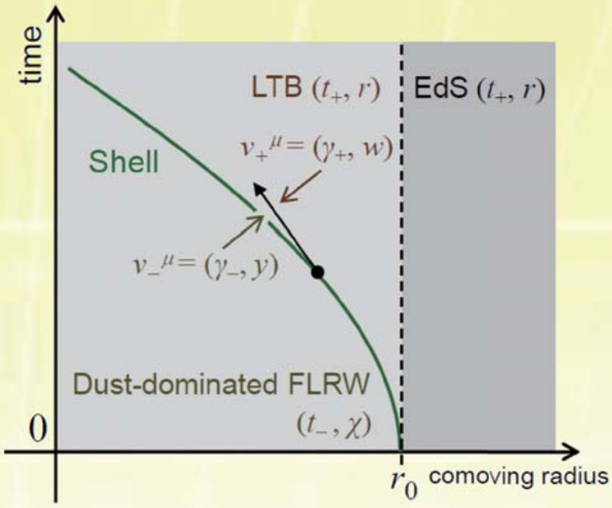
- Tangent vector

$$(v^t_-, v^\chi_-) = (\dot{t}^s_-, y)$$
$$(v^t_+, v^\chi_+) = (\dot{t}^s_+, w)$$

where $y := \dot{\chi}^s, w := \dot{r}^s$

- Normalization ($v^\mu v_\mu = -1$)

$$\dot{t}^s_- = \sqrt{1 + a^2 y^2} =: \gamma_-$$
$$\dot{t}^s_+ = \sqrt{1 + \frac{(w\partial_r R)^2}{1-kr^2}} =: \gamma_+$$



- Shell energy momentum tensor(pure tension)

$$S_{\mu\nu} = -\sigma h_{\mu\nu}$$

△Evolution Equations along the Shell Trajectory

◎Dynamical variables

- Variables for shell trajectory(6 variables)

$$t^s_\pm(\tau), \chi^s(\tau), r^s(\tau), y(\tau), w(\tau)$$

- Variables for LTB(3 variables)

$$m(r^s(\tau)), k(r^s(\tau)), t_B(r^s(\tau))$$

↓ **junction conditions + continuous four velocity of dust**
(Appendix B)

◎ODEs (Appendix C)

△Initial Conditions

- **Assumption: LTB region is initially infinitesimal**
- **Independent initial values**

  R_0 : Initial shell radius
  σ :shell tension
  δ_H :Deviation of the Hubble between FLRW and EdS

$$\delta_H := (H_{-0} - H_{\mathrm{EdS0}})/H_{\mathrm{EdS0}}$$

- **Others are fixed by these values(Appendix D)**

△Results

◎Useful unit

- **Hubble parameter H_{hc} at horizon crossing (shell radius=1/Hubble)**

◎Settings

- $\delta_H = 10^{-8}$, $R_0 H_{\mathrm{EdS0}} = 2, 3, 4$
- $\sigma/H_{\mathrm{hc}} = 1.72 \times 10^{-5}$ - 3.01×10^{-2}

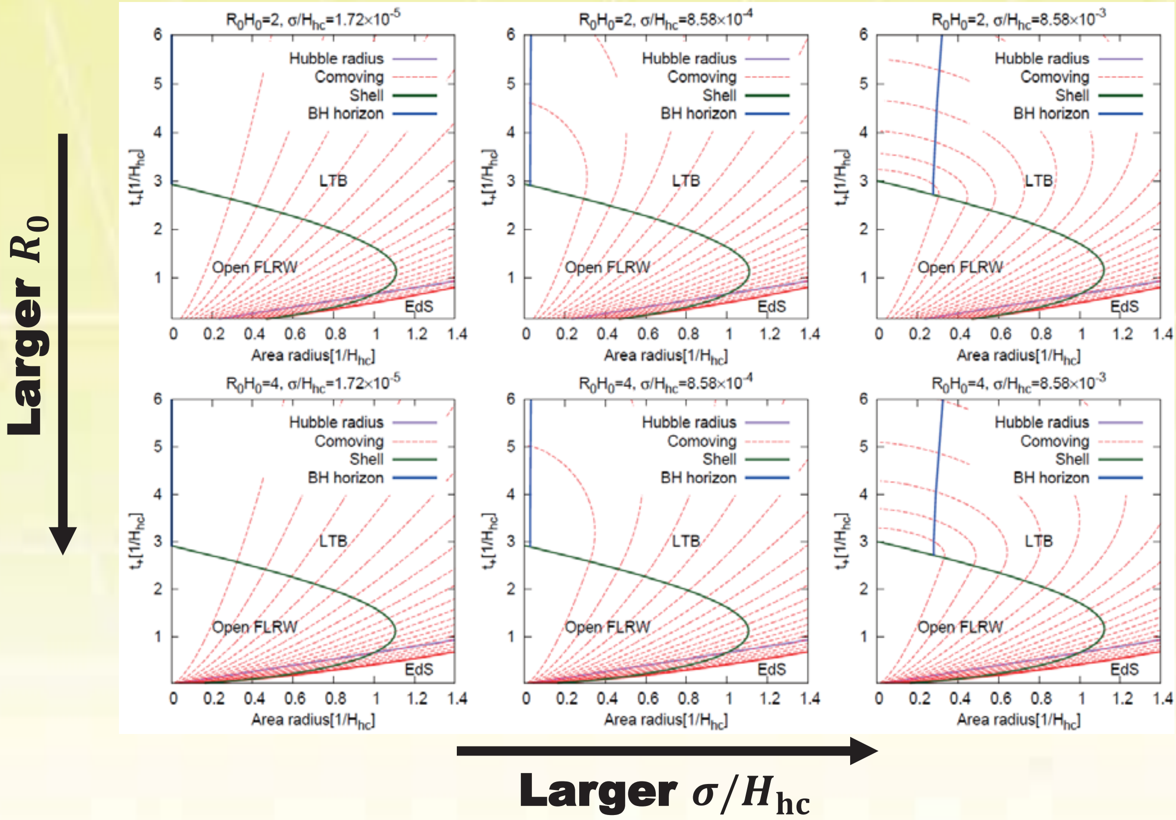
◎Summary of results

- **No essential dependence on $R_0 H_0$**
- **BH forms in the center in every case**
- $\rho \propto R^{-3/2}$ **near the center**
- **BH Mass increases with time due to dust accretion**
- $M_{\mathrm{BH}} H_{\mathrm{hc}} \simeq 17\sigma/H_{\mathrm{hc}}$ **at the moment of the formation**

$$M_{\mathrm{BH}} \sim 4.5 \times 10^{12} \left(\frac{\sigma}{\hbar^2 \mathrm{GeV}^3}\right)\left(\frac{H_{\mathrm{hc}}}{70\mathrm{km/s{\cdot}Mpc}}\right)^{-2} M_\odot$$

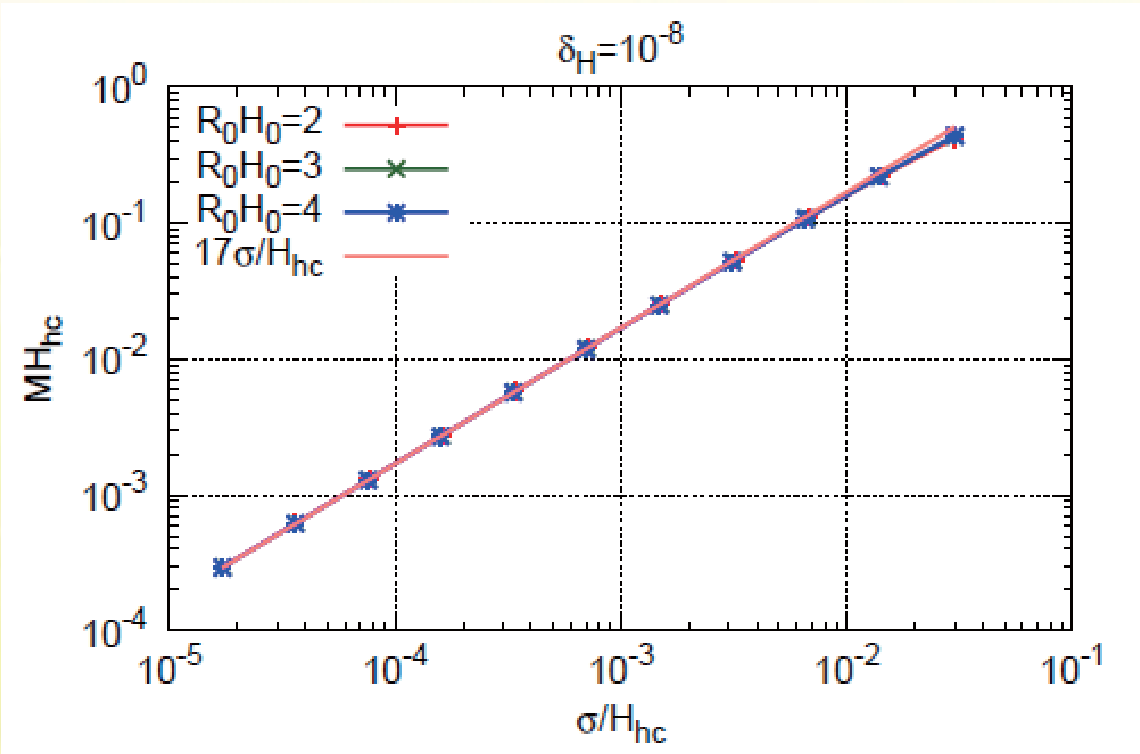
*Results do not change for $\delta_\rho = 0$ initial conditions

◎Spacetime figures



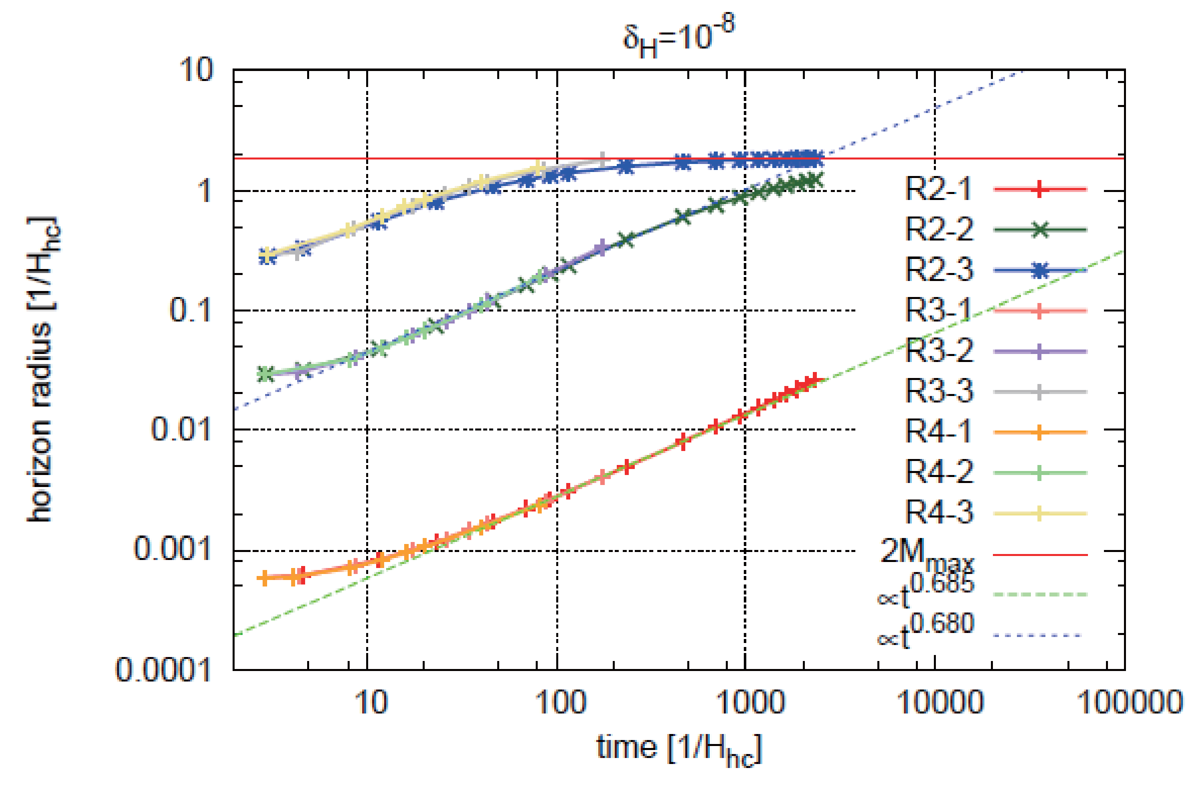
◎BH mass

- Mass at the moment of the formation

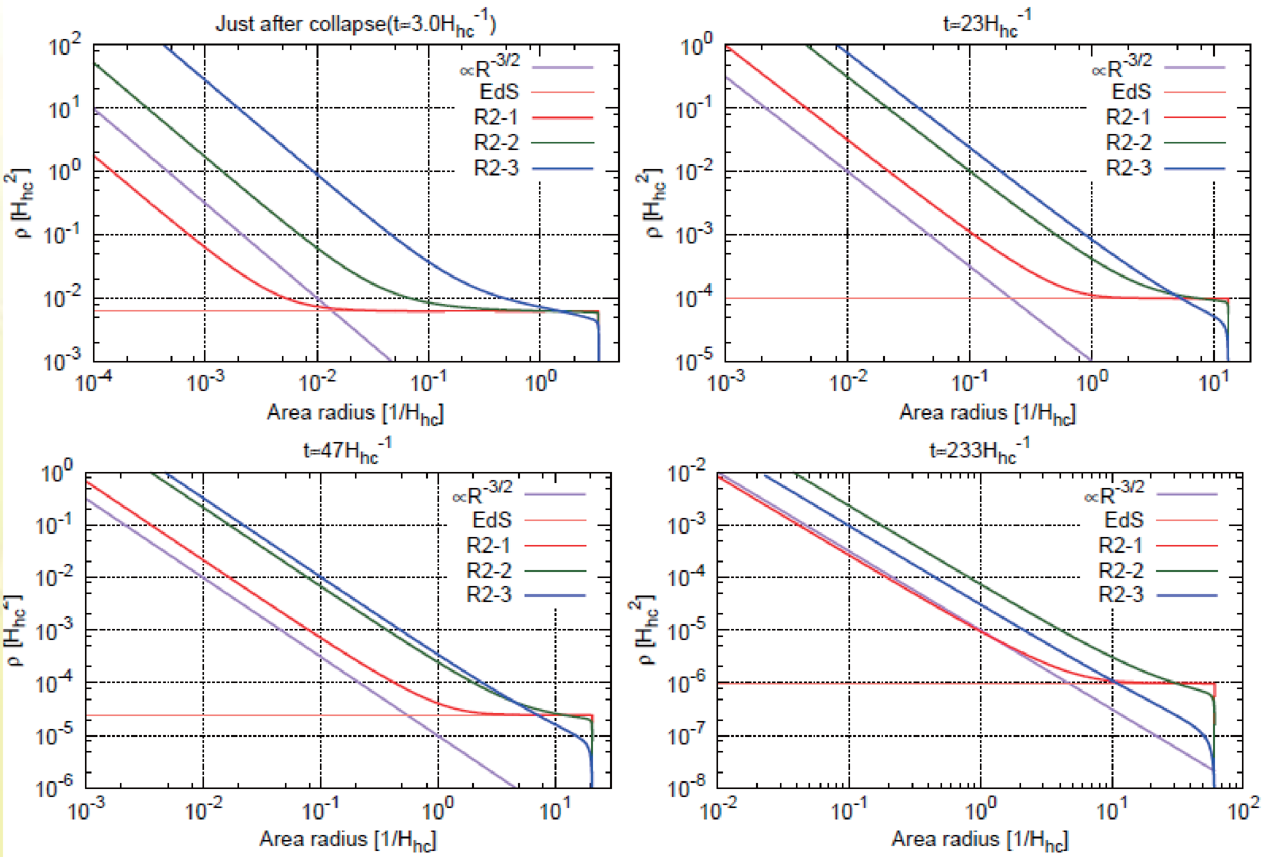


$$M_{\rm BH}H_{\rm hc} \simeq 17\sigma/H_{\rm hc}$$

- Time evolution ($M_{\max} = \frac{4}{3}\pi R_0^3 \rho_{\mathrm{EdS0}} = \frac{1}{2} H_{\mathrm{EdS0}}^2 R_0^3$)



◎Induced inhomogeneity



$\rho \propto R^{-3/2}$ **near the center**

△Appendix A: Junction Condition

◎Notation

- **Brackets**

$$[A]^{\pm} := A_+ - A_-$$

$$\{A\}^{\pm} := A_+ + A_- =: 2\bar{A}$$

- **Physical quantities**

s_μ: **surface normal unit vector**

χ : **normal coordinate**

$$\left(\frac{\partial}{\partial \chi}\right)^\mu := s^\mu$$

$h_{\mu\nu}$: **induced metric**

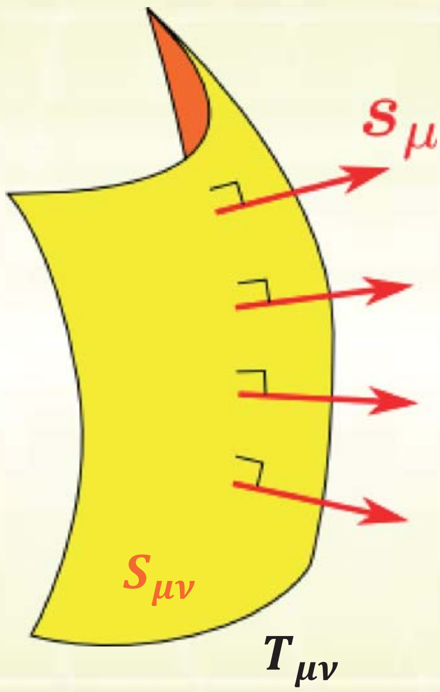
$$h_{\mu\nu} := g_{\mu\nu} - s_\mu s_\nu$$

$K_{\mu\nu}$: **extrinsic curvature**

$$K_{\mu\nu} := \frac{1}{2}\mathcal{L}_s h_{\mu\nu} = \frac{1}{2}\partial_\chi h_{\mu\nu}$$

$T^{\text{total}}_{\mu\nu}$: **energy momentum tensor**

$$T^{\text{total}}_{\mu\nu} = T_{\mu\nu} + S_{\mu\nu}\delta(\chi - \chi_s)$$



◎Israel's junction conditions

- **1st junction condition**

$$[h_{\mu\nu}]^{\pm} := 0$$

- **2nd junction condition**

$$[K_{\mu\nu}]^{\pm} = 8\pi\left(-S_{\mu\nu} + \frac{1}{2}h_{\mu\nu}S\right)$$

- Shell EoM

$$S_{\mu\nu}\bar{K}^{\mu\nu} = [T_{\mu\nu}s^\mu s^\nu]^\pm$$

- Energy momentum conservation

$$D_\mu S^\mu_\nu = -[T_{\mu\alpha}s^\mu h^\alpha_\nu]^\pm$$

◎Spherically symmetric case

- Metric

$$\mathrm{d}s^2 = -\mathrm{e}^{2\alpha(t,\chi)}\mathrm{d}t^2 + \mathrm{e}^{2\beta(t,\chi)}\mathrm{d}\chi^2 + R^2(t,\chi)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)$$

- Shell trajectory

$$t = t^{\mathrm{s}}(\tau), \quad \chi = \chi^{\mathrm{s}}(\tau)$$

- Tangent vector

$$v^\mu = (\dot{t}^{\mathrm{s}},\ \dot{\chi}^{\mathrm{s}},\ 0,\ 0)$$

- Energy momentum tensor

$$T^\pm_{\mu\nu} = (\rho_\pm + p_\pm)\, u^\pm_\mu u^\pm_\nu + p_\pm g^\pm_{\mu\nu}$$
$$S^{\mu\nu} = (\sigma + \varpi)v^\mu v^\nu + \varpi h^{\mu\nu}$$

- Equations

(θ,θ) **of 2nd junction conditions**

$$[s^\mu \partial_\mu \ln R]^\pm = -4\pi\sigma$$

(τ,τ) **of 2nd junction conditions**

$$\left[s_\mu \frac{Dv^\mu}{\mathrm{d}\tau}\right]^\pm = 4\pi(\sigma + 2\varpi)$$

Shell EoM

$$\left\{s_\mu \frac{Dv^\mu}{\mathrm{d}\tau}\right\}^\pm = -\frac{2}{\sigma}\left[(\rho + p)(u^\mu s_\mu)^2 + p\right]^\pm + \frac{2\varpi}{\sigma}\{s^\mu \partial_\mu \ln R\}^\pm$$

Shell energy momentum conservation

$$D_\mu\left(v^\mu(\sigma + \varpi)\right) - v^\mu D_\mu \varpi = [(p + \rho)u_\mu v^\mu u_\nu s^\nu]^\pm$$

△Appendix B: Equations

- **Gauge condition on r** **[dynamical-1]**

$$\partial_r R = \sqrt{1-kr^2} \quad \Rightarrow \quad C\dot{m} + D\dot{k} + E\dot{t}_{\mathrm{B}} = Fw$$

where

$$\begin{aligned} C &= \frac{1}{3} r m^{-2/3} (t-t_{\mathrm{B}})^{2/3} (S - 2xS'), \\ D &= r m^{-1/3} (t-t_{\mathrm{B}})^{4/3} S', \\ E &= -\frac{2}{3} r m^{1/3} (t-t_{\mathrm{B}})^{-1/3} (S + xS'), \\ F &= \sqrt{1-kr^2} - m^{1/3} (t-t_{\mathrm{B}})^{2/3} S. \end{aligned}$$

- **1st junction condition**

$$af = R$$ **[constraint-1]**

↓ **differentiate w.r.t τ** **[constraint-2]**

$$\gamma_- \partial_t (\ln a) + y \partial_\chi (\ln f) = \gamma_+ \partial_t (\ln R) + w \partial_r (\ln R)$$

- **2nd junction condition** (θ, θ) **[constraint-3]**

$$w \partial_t (\ln R) + \gamma_+ \partial_r (\ln R) - y \partial_t a - \frac{\gamma_-}{a} \partial_\chi (\ln f) = -4\pi\sigma$$

- **2nd junction condition** (τ, τ) **[dynamical-2]**

$$\frac{\dot{w}}{\gamma_+} - \frac{a\dot{y}}{\gamma_-} + \frac{G\dot{m}}{\partial_r R} + \frac{H\dot{k}}{\partial_r R} + \frac{I\dot{t}_{\mathrm{B}}}{\partial_r R} + \frac{wJ}{\partial_r R} - 2\partial_t a y = -4\pi\sigma$$

where

$$\begin{aligned} G &= \partial_t C = \frac{1}{6} r m^{-2/3} (t-t_{\mathrm{B}})^{-1/3} \frac{1}{S^2}, \\ H &= \partial_t D = \frac{2}{3} r m^{-1/3} (t-t_{\mathrm{B}})^{1/3} (2S' + xS''), \\ I &= \partial_t E = \frac{1}{6} r m^{1/3} (t-t_{\mathrm{B}})^{-4/3} \frac{1}{S^2}, \\ J &= \frac{2}{3} m^{1/3} (t-t_{\mathrm{B}})^{-1/3} (S + xS'). \end{aligned}$$

- **Shell equation of motion** **[dynamical-3]**

$$\frac{\dot{w}}{\gamma_+} + \frac{a\dot{y}}{\gamma_-} + \frac{G\dot{m}}{\partial_r R} + \frac{H\dot{k}}{\partial_r R} + \frac{I\dot{t}_{\mathrm{B}}}{\partial_r R} + \frac{wJ}{\partial_r R} + 2\partial_t a y = -\frac{2}{\sigma} (\rho_+ w^2 - \rho_- a^2 y^2)$$
$$-2 \left[w \partial_t (\ln R) + \gamma_+ \partial_r (\ln R) + y \partial_t a + \frac{\gamma_-}{a} \partial_\chi (\ln f) \right]$$

- **Shell energy conservation** **[dynamical-4]**

$$\rho_+\gamma_+ w - \rho_-\gamma_- ay = 0 \Leftrightarrow \boxed{\frac{\gamma_+ r^2(3mw + r\dot{m})}{24\pi R^2 \partial_r R} - \rho_-\gamma_- ay = 0}$$

- **Definitions**

$$\boxed{\begin{aligned} \dot{t^s}_\pm &= \gamma_\pm, \\ \dot{\chi^s} &= y, \\ \dot{r^s} &= w. \end{aligned}}$$

[dynamical-5,6,7,8]

3 constraints, 8 dynamical eqs

- **Consistency**

It can be shown that 3 constraints are automatically kept satisfied if they are initially imposed

- **From the 3 constraint equations**

$$M = \frac{1}{6}mr^3 = \frac{4\pi}{3}a^3 f^3 \rho_- + 4\pi\sigma a^2 f^2 \left[\partial_\chi f \gamma_- + af\left(-2\pi\sigma + \partial_t a y\right)\right]$$

(LTB mass) **(FLRW mass)** **(Shell energy)**

- **Continuous four velocity of dust**

$$\boxed{[u^\mu s_\mu]^\pm = 0 \quad \Leftrightarrow \quad w = ay}$$

[constraint-4]

↓ **differentiate w.r.t** τ

$$\boxed{\dot{w} = \partial_t a \gamma_- y + a\dot{y}}$$

[dynamical-9]

In other words,
- **no friction**
- **no interaction between the shell and the dust fluid**
- **individual realization of energy momentum conservation**

- **From [dynamical-4] and [constraint-4],**

$$\rho_+ = \rho_-$$

△Appendix C: ODEs

- **Use $a(t_-)$ as the independent variable**

$$\frac{\mathrm{d}\chi}{\mathrm{d}a} = \hat{y} := \frac{y}{\partial_t a \gamma_-},$$

$$\frac{\mathrm{d}\hat{y}}{\mathrm{d}a} = -\frac{2\partial_\chi f}{a^2(\partial_t a)^2 f \gamma_-^2} + \frac{6\pi\sigma}{a(\partial_t a)^2\gamma_-^3} - \frac{4\hat{y}}{a\gamma_-^2} - \frac{\partial_t^2 a \hat{y}}{(\partial_t a)^2} - a(\partial_t a)^2 \hat{y}^3,$$

$$\frac{\mathrm{d}r}{\mathrm{d}a} = \frac{w}{\partial_t a \gamma_-},$$

$$\frac{\mathrm{d}m}{\mathrm{d}a} = -\frac{3a\hat{y}m}{r} + \frac{24\pi a^3 f^2 \hat{y}\partial_r R \rho_-}{r^3},$$

- **How to calculate other quantities**

$$w = ay$$

$$\frac{t_-}{\alpha} = \sqrt{\frac{a}{\alpha}\left(1+\frac{a}{\alpha}\right)} - \operatorname{arcsinh}\sqrt{\frac{a}{\alpha}}$$ **where** $\alpha = \frac{8}{3}\pi\rho_- a^3 = \text{constant}$

$$\mathrm{d}t_-/\mathrm{d}t_+ = 1$$

$$k = \left(\frac{2M}{R} - (\partial_t R)^2\right)/r^2$$

$$t_+ - t_\mathrm{B} = \begin{cases} -\frac{\sqrt{R(mr-3kR)}}{\sqrt{3}kr} + \frac{m\arctan\left(\frac{\sqrt{3kR}}{\sqrt{mr-3kR}}\right)}{3k^{3/2}} & \text{for } \partial_t R > 0, \\ \frac{m\pi}{6k^{3/2}} + \frac{\sqrt{R(mr-3kR)}}{\sqrt{3}kr} - \frac{m\arctan\left(\frac{\sqrt{3kR}}{\sqrt{mr-3kR}}\right)}{3k^{3/2}} & \text{for } \partial_t R < 0. \end{cases}$$

△Appendix D: Boundary Conditions

◎Outer boundary(LTB | EdS)

- **No singular surface, comoving boundary, 1st junction**

$$\left(\frac{\partial_t R(t_+, r_0)}{R(t_+, r_0)}\right)^2 =: H_+^2(t_+, r_0) = H_\mathrm{EdS}(t_+)^2 = \frac{8\pi}{3}\rho_\mathrm{EdS}(t_+)$$

- **Mass compensation**

$$M(r_0) = \frac{4\pi}{3} R(t_+, r_0)^3 \rho_\mathrm{EdS}(t_+)$$

- **From the above two equations and Einstein eq.**

$$k(r_0) = 0$$

◎Initial hyper-surface

- **Assume that LTB region is infinitesimal**
- **[constraint-1]**

$$a(t_{-0})f(\chi_0) = R(t_{+0}, r_0) =: R_0$$

- **Continuous energy density on the shell**

$$\rho_{-0} = \rho_{+0} =: \rho_0$$

- **Continuous Hubble on the outer boundary**

$$H_{+0} = H_{\rm EdS}(t_{+0}) =: H_0$$

- **Deviations**

$$\delta_\rho := \frac{\rho_0 - \rho_{\rm EdS0}}{\rho_{\rm EdS0}}$$

$$\delta_H := \frac{H_{-0} - H_0}{H_0}$$

- **How to determine other initial values**

Gauge fix → $r_0 = R_0/b$ **: results do not depend on** b

Time shift→ $t_{\rm B}(r_0) = 0$

LTB eq. → $H_0^2 = \left.\frac{(\partial_t R)^2}{R^2}\right|_{t=t_{+0}} = \left.\frac{2M}{R^3}\right|_{t=t_{+0}} = \frac{1}{3b^3}m_0 \quad \Leftrightarrow \quad m_0 = 3b^3 H_0^2$

FLRW eq. → $\frac{1}{a_0^2} = H_{-0}^2 - \frac{8\pi}{3}\rho_0 \quad \Leftrightarrow \quad a_0 = \frac{1}{H_0\sqrt{\delta_H^2 + 2\delta_H - \delta_\rho}}$

1st junc. → $\sinh\chi_0 = \frac{R_0}{a_0} = R_0 H_0 \sqrt{\delta_H^2 + 2\delta_H - \delta_\rho}$

[constraint-2,3] →

$$y_0 = -\frac{H_0\delta_H}{4\pi\sigma a_0}$$

$$\delta_\rho = 2\delta_H - \frac{16\pi^2\sigma^2}{H_0^2} - \frac{2}{R_0 H_0^2}\sqrt{16\pi^2\sigma^2 + \delta_H^2 H_0^2}$$

- **Independent initial values in this presentation**

$$R_0 H_0,\ \sigma/H_0,\ \delta_H$$

- **We can also choose δ_ρ as a independent initial value instead of δ_H**

“Third Order Power Spectrum Using Uniform Approximation”

Allan L. Alinea (Osaka)

[JGRG24(2014)P13]

What is this all about?

Power spectrum is one of the most important physical quantities in inflationary cosmology. It is a measure of the variance in the distribution of (in this case,) the primordial cosmological perturbation ζ. This work is about the calculation of the power spectrum P, of ζ using the method called **uniform approximation** (UA). We calculate P up to third order with respect to the Hubble and sound flow functions evaluated at the turning point (see the discussion of uniform approximation). As demonstrated by (Martin, Ringeval, and Vennin, 2013), in the process of calculation, one encounters terms involving ln(η/η*), where η is the conformal time and (*) means evaluation at the turning point. Luckily enough, these terms cancel when η → 0 (the limiting process needed to calculate P) up to second-order with respect to the Hubble and sound flow functions. We demonstrate that such cancellation does not occur for the third-order part of the power spectrum. Some of the log terms survive rendering the resulting expression problematic.

Cosmic inflation

Our current understanding tells us that the universe started with a "Big Bang". Although successful in accounting many of the observed characteristics of the universe, the standard Big Bang cosmology suffers from two outstanding problems namely, horizon problem and flatness problem. **Cosmological inflation** is a rapid exponential expansion of the universe (or parts of the universe). The period of inflation is inserted right before the usual "slow" expansion of the universe as described by the Standard Big Bang cosmology. It solves the two mentioned problems. The merger of the Standard Big Bangcosmology and inflation forms a powerful hybrid theory : Big Bang + Inflation.

Prior to inflation, there were no large-scale structures (galaxies and clusters of galaxies). All that we had were quantum fluctuations. During inflation, these fluctuations were stretched due to the rapid expansion until they "went out" of the Hubble sphere. After inflation,the fluctuations "went back" to the Hubble sphere to form the structures that we nowadays observe as galaxies andclusters of galaxies.

Uniform Approximation

The **Mukhanov-Sasaki** equation (see pie #2) is a second-order linear homogeneous differential equation involving the primordial cosmological perturbation ζ and the conformal time η. One may "solve" this equation by dividing the domain into three regions and matching the three resulting solutions at the boundaries of these regions. **Uniform approximation** (UA) is a method of solving a differential equation using a (single) global interpolating solution (see pie #1) instead of three for the case of the Mukhanov-Sasaki equation. In applying this method to this differential equation, one performs Liouville transformation and defines the newly introduced independent variable in such a way that to the lowest order approximation, the resulting differential equation has one turning point characteristic of the original differential equation (Mukhanov-Sasaki-equation). At the turning point, the nature of the solution of a given differential changes; eg., from oscillatory to exponentially decaying.

uniform approximation ● uniform approximation ● uniform approximation ● uniform approximation ● uniform approximation ● uni

The solutions are Airy functions. The quantity ψ serves as a correction.

$$U(\xi) \approx c_1 \mathrm{Ai}(\xi) + c_2 \mathrm{Bi}(\xi)$$

To lowest order, ψ is ignored and what is left is an Airy differential equation.

$$\psi \approx 0$$

$$\frac{d^2U}{d\xi^2} = \left(\xi + \psi\right) U$$

$$\xi = (d\xi/dx)^{-2} g$$

Starting with the original differential equation, perform Liouville transformation.

$$\frac{d^2U}{d\xi^2} = \left[\left(\frac{d\xi}{dx}\right)^{-2} g(x) + \psi\right] U$$

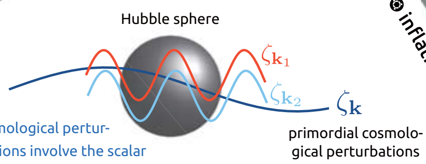
$$U \equiv u\sqrt{d\xi/dx}$$

$$\frac{d^2u}{dx^2} = \left[g(x) + q(x)\right] u$$

pie # 1

inflation and Mukhanov-Sasaki equation ● inflation and Mukhanov-Sasaki equation ● inflation and Mukhanov-Sasaki equa

The primordial cosmological perturbations are seeds of structures that we nowadays, observe as galaxies and clusters of galaxies.



The cosmological perturbations involve the scalar field called inflaton.

The equation of motion for the primordial cosmological perturbations is called the

Mukhanov-Sasaki Equation

$$\frac{d^2\mu_k}{d\eta^2} = \left[\left(\frac{\nu^2}{\eta^2} - k^2c_s^2\right) - \frac{1}{4\eta^2}\right]\mu_k$$

(the first bracketed term is labelled g, the second q)

Here, the quantity μ is related to ζ by the definition

$$\mu_k \equiv \frac{2a^2\epsilon}{c_s^2}\zeta_k$$

Apply uniform approx'n (UA)

UA

Power Spectrum

$$P = \lim_{\eta\to 0} \frac{k^3}{4\pi^2}\frac{c_s^2|\mu_k|^2}{2a^2\epsilon}$$

pie # 2

Power Spectrum

$$P = \lim_{\eta\to 0} \mathcal{N}_*\eta^3$$

$$\times \exp\left[\int \frac{d\eta}{\eta} f(\epsilon_i, \delta_i) - \frac{\Delta\nu}{\nu_*} + 2\Psi\right]$$

P (power spectrum), η (conformal time), $\mathcal{N}_*$ (constant), ϵ_i(Hubble flow f'n), δ_i(sound flow f'n), ν (index f'n), Ψ (integral of √g), g ≡ $\nu^2/\eta^2 - k^2c_s^2$, k (wavenumber), c_s (speed of sound)

Third-Order Power Spectrum Using Uniform Approximation

Allan L. Alinea
Osaka University

Takahiro Kubota
Yukari Nakanishi
Wade Naylor

power spectrum ● power spectrum ● power spectrum ● power spectrum ● power spectrum ● power spectrum ● power spectrum ● power

Unfortunately, there is an imperfect cancellation of log terms resulting in log divergences in the third-order power spectrum.

$$P = P^{(0)} + P^{(1)} + P^{(2)} + P^{(3)} + \cdots$$

pie # 3

Working Equation for the Power Spectrum

$$P = \lim_{\eta\to 0}\left[-\frac{k^3}{8\pi^2}\frac{\eta c_s^2}{a^2\nu\epsilon}\exp\left(\int_{\eta_*}^{\eta} d\tau\,\sqrt{g}\right)\right]$$

The quantity g is likewise, expanded about the turning point before integration.

Expand about the turning point where $g = g_* = 0$.

$g = 0$

$$c_s^2 = c_{s*}^2 + \left.\frac{dc_s^2}{dN}\right|_* \tilde{N} + \cdots$$

$$\epsilon = \epsilon_* + \left.\frac{d\epsilon}{dN}\right|_* \tilde{N} + \cdots$$

$$\nu = \nu_* + \left.\frac{d\nu}{dN}\right|_* \tilde{N} + \cdots$$

$$a = a_* \exp\left(\int d\eta\, \tilde{N}\right)$$

Here, N is the number of e-folds and $\tilde{N} = N - N_*$

Calculating the Power Spectrum through Uniform Approximation

To lowest order in UA, one arrives at the Airy differential equation after transforming the Mukhanov-Sasaki equation. The Airy function solutions are then used in the expression for the power spectrum (see pie# 2) subject to some conditions to fix some constants. The resulting working equation for P involves the speed of sound, index function, scale factor, slow-roll parameter, and the integral of √g (see pie #3). Starting from this working equation, the mentioned quantities are expanded about the turning point. The resulting equations after the expansion mainly involves the Hubble and sound flow functions evaluated at the turning point. With these equations at hand, the quantity √g in the working equation is integrated with respect to the conformal time. The exponential of this integral together with the other factors in the equation for P are then combined and the limit is taken as η → 0. What results is an expression for the power spectrum mainly involving the Hubble and sound flow functions evaluated at the turning point. Mathematically, one has $P = P(0) + P(1) + P(2) + P(3) + \ldots$, where the quantity n in $P(n)$ means the order of the expression with respect to the mentioned functions.

Logarithmic Divergences

As demonstrated in (Martin, Ringeval, and Vennin, 2013), in the process of calculating the power spectrum (P), one encounters terms involving ln(η/η*), where η is the conformal time and (*) means evaluation at the turning point. Luckily enough, these terms cancel when η → 0 (the limiting process needed to calculate P) up to second-order in the Hubble and sound flow functions. We demonstrate that such cancellation does not occur for the third-order part of the power spectrum. This result poses a challenge for the calculation of P beyond the second order.

Credits

S. Habib, etal, "The Inflationary Perturbation Spectrum," Phys.Rev.Lett. 89 (2002) 281301

J. Martin, C. Ringeval and V. Vennin, "K-inflationary Power Spectra at Second Order," JCAP 1306, 021 (2013)

F. W. J. Olver, "Asymptotics and Special Functions", AK Peters, (1997)

Poster template : based on http://blog.felixbreuer.net/2010/10/24/poster.html

Eyecandies : Libreoffice and Inkscape (on Ubuntu 14.04)

“Some insights into the cosmological four point function”

Nobuhiko Misumi (Osaka)

[JGRG24(2014)P14]

Some insights into cosmological four point correlation function

JGRG24@Kavli IPMU
2014/11/10-14

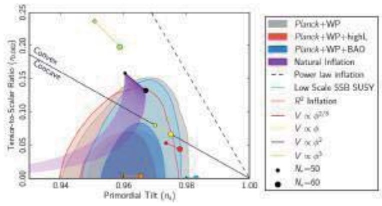
Nobuhiko Misumi (Osaka U.)
Collaboration with Takahiro Kubota (Osaka U.)

Abstract

We study cosmological four point correlation function with small speed of sound.
And we explore whether there exists the useful relation like consistency relation focusing on counter-collinear limit and double soft limit.

1.Introduction

Single-field inflation looks good (in 2pt. function)



→ Many models survive.

More informations are needed.

ex) 3pt function

$$\langle \zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\rangle = (\text{amplitude}) \times (2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\underbrace{F(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)}_{\text{shape of triangle}}$$

amplitude ∝ f_{NL}

Consistency relation

Creminelli, Norena and Simonovic 2012
Hinterbichler, Hui and Khoury 2012,2013

For single-field models, 3pt. function in the squeezed limit is given by

$$\lim_{q\to 0}\frac{\langle \zeta_{\vec{q}}\zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\rangle'}{P(q)} = -\left(3 + k\frac{d}{dk}\right)P(k)$$

(′ : extraction of $(2\pi)^3\delta^{(3)}(\vec{q}+\vec{k}_1+\vec{k}_2)$)

It's generalization is

$$\lim_{q\to 0}\frac{\langle \zeta_{\vec{q}}\zeta_{\vec{k}_1}\cdots\zeta_{\vec{k}_N}\rangle'}{P(q)} = -\left(3(N-1) + \sum_{a=1}^{N}\vec{k}_a\cdot\vec{\nabla}_{k_a}\right)P(k)$$

Current Observation

arXiv:1303.5084

Is non-Gaussianity dead ?

$$f_{NL}^{local} = 2.7 \pm 5.8(1\sigma) \qquad f_{NL}^{equil.} = -45 \pm 75(1\sigma)$$

2.Models

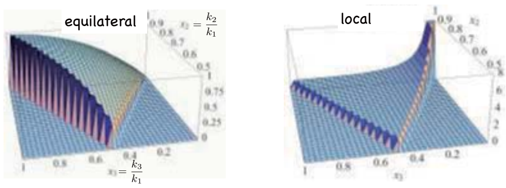
General single field inflation with non minimal coupling

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[f(\varphi)R + 2P(\varphi,X)\right]$$

$$P(\varphi,X) = K(\varphi)X + L(\varphi)X^2 + \cdots, \quad X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$$

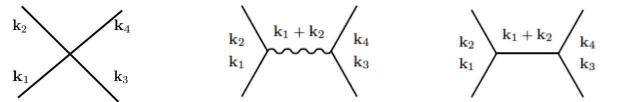
Features

1) Small speed of sound c_s (propagation speed of perturbation)
2) Shape of this model is equilateral.



3.Consistency relation with small c_s

Chen, Hu, Huang, Shiu and Wang 2009



Contact Interaction → vanishes is squeezed limit (leading order)

Graviton Exchange, Scalar Exchange → 4pt. func. is $1/c_s^4$

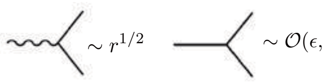
Squeezed limit is proportional to $1/c_s^2$

→ 4pt. function with $c_s \neq 1$ cannot have a squeezed limit.

4.Counter-collinear limit

Seery, Sloth & Vernizzi 2009

In slow-roll inflation, there exists similar relation to consistency relation for $k_1 \sim k_2$, $k_3 \sim k_4$. $T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) = 4\tau_{NL}P(k_{12})P(k_1)P(k_3)$



$\sim r^{1/2}$ $\quad \sim \mathcal{O}(\epsilon,\eta)$ → Graviton exchange is dominant.

Leading term in $S^{(3)}_{\mathrm{ssg}}$

$$S^{(3)}_{\mathrm{ssg}} = -\int d^4x\, a\bar{f}\left(\frac{H}{\dot{\theta}} - 1 + \frac{\ddot{\theta}}{\dot{\theta}^2}\right)\gamma^{ij}\partial_i\zeta\partial_j\zeta$$

4pt. function

$$\langle \zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\zeta_{k_4}\rangle = (2\pi)^3\delta^{(3)}\left(\sum_a \mathbf{k}_a\right)\bar{f}\left(\frac{H}{\dot{\theta}} - 1 + \frac{\ddot{\theta}}{\dot{\theta}^2}\right)^2\frac{16a^8H^6}{\prod_a(2c_s^3k_a^3)z^8}F(k_{12},k_1,k_2,k_3)$$

$$z^2 = 6e^{2\theta}\left(\frac{H}{\dot{\theta}} - 1\right)^2 + \frac{2a\Sigma}{\dot{\theta}^2}$$

Taking the limit,

1) non-canonical kinetic term with f=1
2) both canonical and non-canonical with f≠1

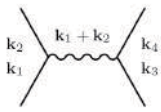
→ does not satisfy above relation

5.Double soft limit

Joyce, khoury & Simonovic 2014
Mirbabayi & Zaldarriaga 2014

$$\lim_{\vec{q}_1,\vec{q}_2\to 0}\nabla^j_{q_1}\nabla^i_{q_2}\left(\frac{\langle \zeta_{\vec{q}_1}\zeta_{\vec{q}_2}\zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\rangle'}{P(q_1)P(q_2)}\right) = P(k)\left(1 - \frac{1}{c_s^2}\right)\left\{\left(\frac{3}{4}\frac{\delta^{ij}}{k^2} + \frac{5}{2}\frac{k^ik^j}{k^4}\right) + \left(4\frac{\delta^{ij}}{k^2} - \frac{5}{2}\frac{k^ik^j}{k^4}\right)\frac{\langle \zeta_{\vec{q}_1}\zeta_{\vec{q}_2}\zeta_{-\vec{q}}\rangle'}{P(q_1)P(q_2)}\right\}$$

In 4pt. function, dominant contribution is graviton exchange diagram.



We have to include graviton exchange diagram and rederive above relation.

Conclusions

- 4pt. correlation function has rich information, although they cannot be observed in immediate future...
- In double soft limit, there may be useful relation, but we need further study.

“Numerical analysis of quantum cosmology”

Hiroshi Suenobu (Nagoya)

[JGRG24(2014)P15]

Numerical Analysis of Quantum Cosmology

JGRG24@IPMU (2014)

Nagoya University QG lab, Hiroshi Suenobu, Yasusada Nambu

Quantum Cosmology

- Investigating the initial state of the universe from quantum theory.
- It can be provided by the wave function of the universe.
- The boundary condition of the wave function plays a crucial roll in quantum cosmology.

Our research overview

- Use a no-inflaton-solution of the Wheeler-De Witt equation as the boundary condition.
- Solve the wave function of the universe numerically
- Define and Calculate a probability for a classical universe.
- Discuss about what type of the boundary condition can lead to sufficiently long duration of inflation.

1. Introduction

We consider General Relativity + massive scalar field in homogeneous and isotropic universe.

Canonical quantization leads the Wheeler-De Witt equation (WDW eq).

$$\frac{1}{2}\left[\frac{\hbar^2}{a^2}\frac{\partial}{\partial a}\left(a\frac{\partial}{\partial a}\right)-\frac{\hbar^2}{a^3}\frac{\partial^2}{\partial\phi^2}-a+a^3\left(\frac{\Lambda}{3}+m^2\phi^2\right)\right]\Psi(a,\phi)=0$$

Λ : cosmological constant

Ψ is the wave function of the universe.

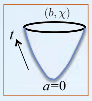
Problems in quantum cosmology

- The equation is hard to solve analytically.
- The wave function is often approximately evaluated by saddle point value of action using a complex classical solution.

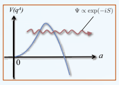
$$\Psi(a,\phi)\approx\exp\left(-I_{ext}/\hbar\right)$$

- What boundary condition of the wave function of the universe can predict our universe ? , such as how much inflate ?

No-boundary proposal by Hartle and Hawking



Tunneling proposal by Vilenkin



2. Our approach

- Boundary condition of the wave function (BC)

General solution of the WDW eq without inflaton

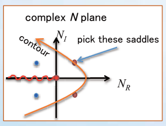
$$\Psi(a)=\alpha Ai(z)Ai(z_0)+\beta Bi(z)Bi(z_0)+\gamma Ai(z)Bi(z_0)+\delta Ai(z_0)Bi(z)$$

We can use it as the boundary condition at $a=0$. $\Psi(0,\phi),\partial_a\Psi(0,\phi)$

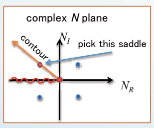
It is characterized by parameters $(\alpha,\beta,\gamma,\delta)$.

boundary condition	parameters $(\alpha,\beta,\gamma,\delta)$	Asymptotic form
(a) no-boundary proposal	(0,0,1,1)	Ψ~exp(1/V)cos$S$
(b) tunneling proposal	(1,0, 0,i)	Ψ ~exp(-1/V)e^{-iS}
(c) tunneling like real	(1,0,0,0)	Ψ~exp(-1/V)cos$S$

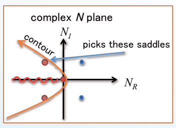
They correspond to integration contours in path integral.



(a) no-boundary proposal



(b) tunneling proposal



(c) tunneling like real

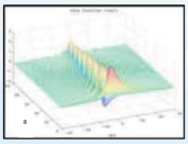
[J.J.Halliwell et.al. PRD39 (1989)]

- Solve the WDW eq

$$\frac{1}{2}\left[\frac{\hbar^2}{a^2}\frac{\partial}{\partial a}\left(a\frac{\partial}{\partial a}\right)-\frac{\hbar^2}{a^3}\frac{\partial^2}{\partial\phi^2}-a+a^3\left(\frac{\Lambda}{3}+m^2\phi^2\right)\right]\Psi(a,\phi)=0$$

Discretize and use RK4 algorism.

Obtain numerical wave function.

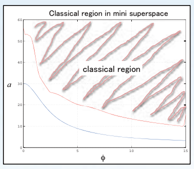


Wave function obtained by no-boundary BC

- Classicality

If we consider complex contours in path integral, the action becomes complex. $I\rightarrow I_R - iS$

Classicality condition : $\frac{|\nabla I_R|^2}{|\nabla S|^2}\ll 1$



In this region, the wave function becomes

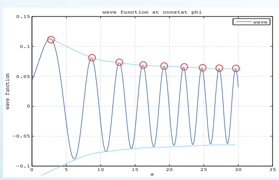
$$\Psi\approx A(q^A)\exp[iS(q^A)],\ A(q^A)=\exp(-I_R(q^A)),\ q^A=(a,\phi)$$

wave function oscillates

Extract pre-factor I_R and phase S from the numerical wave function

Amplitude $\rightarrow I_R$

Intervals of peaks $\rightarrow S$



- Definition of probability

Conserved current in mini superspace

$$J_A=\frac{i\hbar}{2}(\Psi^*\nabla_A\Psi-\Psi\nabla_A\Psi^*)\ \Rightarrow\ \nabla\cdot J=0\qquad,\nabla_A=\left(\frac{\partial}{\partial a},\frac{\partial}{\partial\phi}\right)$$

Probability measure

$$\mathcal{P}(q^A)\equiv J\cdot n=|A(q^A)|^2\nabla_n S(q^A)$$

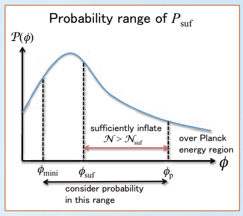
It can predict sets of initial data (b,χ,p_b,p_χ) for the classical equation.

How long inflation ? , What is a number of e-folding $\mathcal{N}$?

Probability of sufficient inflation P_{suf}

$$P_{suf}\equiv\frac{\int_{\phi_{suf}}^{\phi_p}d\phi\mathcal{P}(\phi)}{\int_{\phi_{min}}^{\phi_p}d\phi\mathcal{P}(\phi)}$$

In the region $\phi<\phi_{min}$, the universe can not reach current age of universe or there is no classical region in mini superspace.



Expectation value of e-folding number

$$\langle\mathcal{N}\rangle=\frac{\int_{\phi_m}^{\phi_p}d\phi\mathcal{N}(\phi)\mathcal{P}(\phi)}{\int_{\phi_m}^{\phi_p}d\phi\mathcal{P}(\phi)}$$

Probability of boundary condition $P(t|S)$

$$P(t|S)=\frac{P(S|t)}{\int_0^1 dt'P(S|t')}\ ,\ P(S|t)=P_{suf}(t)$$

t : parameter of boundary condition

$P(S|t)$: probability of sufficient inflation with BC=t.

$P(t|S)$ based on the Bayes' theorem

$$P(A_i|B)=\frac{P(A_i)P(B|A_i)}{\sum_{k=1}^{n}P(A_k)P(B|A_k)}$$

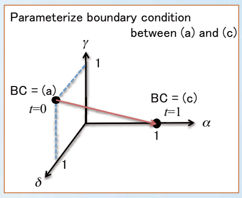


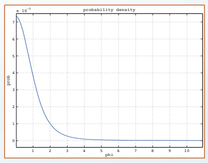
Fig1. Parameterize boundary condition between (a) and (c).

3. Our results

(1) Boundary condition : (a),(b),(c)

with Λ=0.3, m=0.1, ϕ_{min}=0.55, ϕ_p=10, $\mathcal{N}_{suf}$=60.

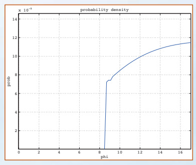
(a)no-boundary proposal



favors small ϕ ⇒ small $\mathcal{N}$

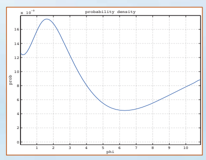
$\langle\mathcal{N}\rangle$ = 29.4

(b) tunneling proposal



doesn't classicalize at small ϕ

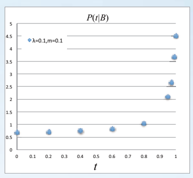
(c) tunneling like real

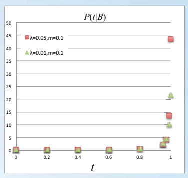


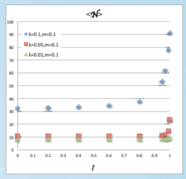
favors large ϕ ⇒ large $\mathcal{N}$

$\langle\mathcal{N}\rangle$ = 85.3

(2) Probability of boundary condition $P(t|S)$: (a) $t=0$ to (c) $t=1$ (Fig1)







4. Summary

- We could obtain the method to predict the classical universe from the numerical wave function of the universe.
- The no-boundary proposal is not favored so much to lead sufficiently long duration of inflation.
- We will explore furthermore about various combinations of boundary conditions and dependence on Λ and m.

“Adiabatic regularization of power spectrum

for non-minimal k-inflation”

Yukari Nakanishi (Osaka)

[JGRG24(2014)P16]

Adiabatic regularization of power spectrum for non-minimal k-inflation

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大阪大学
OSAKA UNIVERSITY

Primordial Perturbations

The power spectrum of the cosmic microwave background **is an observable** arising from cosmological primordial perturbations.

We can compare observables and inflation theories by using the power spectrum.

Definition and properties

The power spectrum of the scalar perturbation $|\mathcal{R}_k(\eta)|^2$ is defined by a Fourier transformation of the two point function.

$$\langle|\mathcal{R}(x)|^2\rangle = \int_0^\infty \frac{dk}{2\pi^2} k^2 |\mathcal{R}_k(\eta)|^2$$

The scalar perturbation obeys Mukhanov-Sasaki equation.

$$v_k'' + \left(c_s^2 k^2 - \frac{z''}{z}\right) v_k = 0,\ v_k \equiv z\mathcal{R}_k$$

where $f' = \frac{df}{d\eta}$, $z^2 = \frac{2a^2\epsilon_1}{c_s^2}$ and the "**sound speed**" is defined by $c_s^2 \equiv \frac{P_{,X}}{P_{,X}+2XP_{,XX}}$.

Our aim is checking the regularization of the power spectrum.

Adiabatic Regularization

Adiabatic regularization[1] is one of regularization schemes of QFT in curved spacetime. In this regularization, the physical amplitude is schematically given by

$$\langle|\mathcal{R}(x)|^2\rangle_{\text{phys}} \equiv \int_0^\infty \frac{dk}{2\pi^2} k^2 \left[|\mathcal{R}_k(\eta)|^2_{\text{bare}} - |\mathcal{R}_k(\eta)|^2_{\text{sub}}\right].$$

The bare power spectrum is derived from an inflation model. However, **we regard the bare spectrum minus the subtraction term as the observable power spectrum**.

How to make the subtraction term

1. introduce a fictitious parameter T in the metric.
$$g_{\mu\nu}(x) \to g_{\mu\nu}(x/T)$$
2. Require the adiabatic condition and do a WKB-like expansion.

Adiabatic condition

In lowest adiabatic order, v_k should have the form
$v_k \propto \omega_k(\eta)^{-\frac{1}{2}} \exp\left(-i\int^\eta \omega_k(\eta')d\eta'\right)$

3. Rearrange terms so that the power of $1/T$ are in ascending order.
4. Isolate divergent terms as the adiabatic subtraction term.
$$|\mathcal{R}_k(\eta)|^2_{\text{sub}} \equiv |\mathcal{R}_k(\eta)|^{2(0)} + |\mathcal{R}_k(\eta)|^{2(2)}$$

In slow-roll inflation, the "sound speed" is equal to one ($= c$).

The subtraction term for slow-roll inflation model becomes small because the coefficient of the second-order adiabatic term is exponentially suppressed.[2]

Subtraction Terms for K-inflation

In k-inflation model, which is motivated by string theory, the Lagrangian has non-canonical kinetic terms and **the "sound speed" is not constant.**[3]

$$\mathcal{L} = P(\phi, X),\ X \equiv \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,\ c_s^2 \neq \text{constant.}$$

By using the MS equation, we derived the subtraction term.

$$|\mathcal{R}_k(\eta)|^2_{\text{sub}} = \frac{1}{2z^2c_sk}\left\{1 + \frac{1}{2c_s^2k^2}\frac{z''}{z} + \frac{1}{c_s^2k^2}\left(\frac{1}{4}\frac{c_s''}{c_s} - \frac{3}{8}\frac{{c_s'}^2}{c_s^2}\right)\right\}$$

In conclusion, the subtraction term for k-inflation models depend on the "sound speed". Therefore **the time dependence of it is not obvious** unlike one of the slow-roll model.

Subtraction Terms for Non-minimal K-inflation

In non-minimal coupling model, the Einstein equation is more complicated to solve. Then we use the conformal transformation and make it simple.

$$S = \frac{1}{2}\int d^4x[f(\phi)R + 2P(\phi, X)] \quad \xrightarrow{\ \widehat{g}_{\mu\nu} = f(\phi)g_{\mu\nu}\ } \quad S = \frac{1}{2}\int d^4x[\widehat{R} + \widehat{P}(\phi, \widehat{X})]$$

where $\widehat{X} \equiv \frac{1}{2}\widehat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ and $\widehat{P}(\phi, \widehat{X}) \equiv f(\phi)^{-2}P(\phi, X) - 3\widehat{g}^{\mu\nu}(\partial_\mu \ln\sqrt{f})(\partial_\nu \ln\sqrt{f})$.

Comoving gauge

$$\delta\phi = 0$$
$$g_{00} = a(\eta)^2,\ g_{ij} = -a(\eta)^2 e^{2\mathcal{R}}\delta_{ij}$$

In this gauge, it is known that the scalar perturbation and its correlation functions are frame invariant.[4, 5]

$$\mathcal{R} = \widehat{\mathcal{R}},\ \langle|\mathcal{R}|^2\rangle_{\text{bare}} = \langle|\widehat{\mathcal{R}}|^2\rangle_{\text{bare}}$$

in the Jordan frame

$$v_k'' + \left(c_{s,\text{eff}}^2 k^2 - \frac{z_{\text{eff}}''}{z_{\text{eff}}}\right) v_k = 0,\ v_k \equiv z_{\text{eff}}\mathcal{R}_k$$

z_{eff} and $c_{s,\text{eff}}$ have been estimated directly by ADM formalism in [6].

in the Einstein frame

$$\widehat{v}_k'' + \left(\widehat{c}_s^2 k^2 - \frac{\widehat{z}''}{\widehat{z}}\right) \widehat{v}_k = 0,\ \widehat{v}_k \equiv \widehat{z}\widehat{\mathcal{R}}_k$$

$\widehat{z}$ and $\widehat{c}_s$ can be estimated by conformal transformation.

Because $\eta = \hat{\eta}$, the scalar perturbations in Jordan/Einstein frame obey the equations which have the same form. We showed that $z_{\text{eff}} = \widehat{z}$ and $c_{s,\text{eff}} = \widehat{c}_s$ as long as we take the same normalization manner, so we conclude that

$$|\mathcal{R}_k(\eta)|^2_{\text{sub}} = |\widehat{\mathcal{R}}_k(\eta)|^2_{\text{sub}}$$

Therefore, the physical power spectrum can be derived from both frames and we do not need to do the complicated calculation in Jordan frame to derive the adiabatic subtraction term for non-minimal k-inflation.

Future Work

We need next to constrain the model parameters.

However, in the non-minimal case, the result is obtained by arguments with some additional assumptions and conditions.

- We take the comoving gauge $\delta\phi = 0$.
- We set the non-diagonal components of the energy-momentum tensor to zero. (This is related to neglecting the anisotropic inertia.)

If we choose another gauge and throw away the second assumption, **the non-diagonal components which are gauge invariant and frame invariant appear**. Then the argument becomes unobvious because we cannot combine the Einstein equations into one MS equation without other equations or relations.

References

[1] L. Parker, [arXiv:hep-th/0702216].
[2] Y. Urakawa and A. A. Starobinsky, Proc. 19th Workshop in General Relativity and Gravitation in Japan (JGRG19), Tokyo, Japan, pp. 367-371.
[3] C. Armendariz-Picon, T. Damour and V. F. Mukhanov, Phys. Lett. B **458** (1999) 209 [arXiv:hep-th/9904075].
[4] T. Chiba and M. Yamaguchi, JCAP 10 (2008) 021 [arXiv:0807.4965 [astro-ph]].
[5] T. Kubota, N. Misumi, W. Naylor and N. Okuda, JCAP 02 (2012) 034 [arXiv:1112.5233 [gr-qc]].
[6] T. Qiu and K.-C. Yang, Phys. Rev. D **83**, 084022 (2011).

“Open inflation and scalar suppression on large scales”

Jonathan White (RESCEU)

[JGRG24(2014)P17]

Open inflation and scalar suppression on large scales

Jonathan White, RESCEU
with Ying-li Zhang (NAOC) and Misao Sasaki (YITP)
Based on Phys.Rev. D 90, 083517

JGRG24

Introduction

An epoch of **inflation** is in good **agreement with observations**

But - **what is the exact nature of inflation?**
Could **anomalies** represent important **clues?**

- e.g. **5-10% power deficit** in CMB temperature anisotropies on **large scales** ($l \lesssim 40$) with statistical significance 2.5—3σ [1]

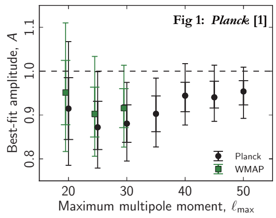
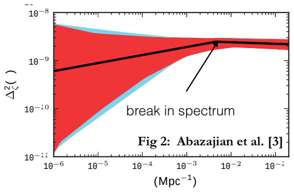


Fig 1: Amplitude of suppression, A, of low-l C_l^{TT}s relative to the best-fit *Planck* model. A is determined by fitting the two-parameter model shown below to the low-l *Planck* data, restricted to various ranges $2 \le l \le l_{max}$ [1].

$$C_l(A, n) = A C_l^{fid}\left(\frac{l}{l_0}\right)^n \qquad l_0 = \frac{(2 + l_{max})}{2}$$



Model	$\Delta \log Z_{Broad}$	$\Delta \log Z_{Informative}$	$2\Delta \log \mathcal{L}_{max}$
1 Knot	1.6	3.1	6.2

Fig 2: Form of "broken" primordial spectrum analysed in [3] and found to be preferred over the standard power-law form in light of BICEP2 results.

- Primordial **scalar power spectrum with suppression on large scales is favoured by *Planck*** even under the assumption $r=0$ [1].
- Non-zero **tensor modes,** as suggested by BICEP2 [2], would **contribute to C_l^{TT} on large scales** $l \lesssim 100$
- If the signal contains a contribution from r the **scalar contribution must be even more suppressed** on large scales

⇒ **Evidence for modified primordial scalar spectrum increased after BICEP2** [3]

Such a spectrum arises in Open Inflation

Open inflation

- String theory predicts a **landscape of vacua**
- Our **universe** may have **emerged after false-vacuum decay**
- There are two key features:
  1. **Universe after tunnelling is open** [4]
  2. **Steepening of potential** near barrier

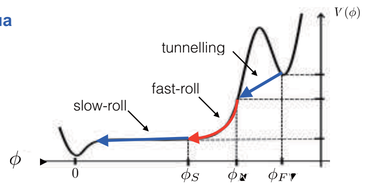


Fig 3: Form of potential associated with Open Inflation models, where standard inflation is preceded by tunnelling from a false vacuum.

After tunnelling our **open universe is described by the equations**

$$H^2 = \frac{1}{3}\left(\frac{\dot\phi^2}{2} + V\right) + \frac{1}{a^2} \qquad \ddot\phi + 3H\dot\phi + V_\phi = 0$$

with the **Initial conditions:**

$$a = 0 = \dot\phi,\ \phi = \phi_N,\ \dot a = 1$$

- Given these initial conditions we expect **three stages:**
  1. **Curvature domination:** $H = \frac{1}{a},\ a = t,\ \dot\phi = -\frac{V_\phi t}{4}$
     Large Hubble friction ⇒ field slowly rolling
  2. **Fast-roll phase:** After the transition to potential domination we are still in the vicinity of the tunnelling barrier where the potential is steep
  3. **Slow-roll inflation** If slow-roll phase is short enough, e.g. N ~ 60, expect to see **signatures of spatial curvature and steep potential** [5]

- **Scalar and tensor power spectra take non-standard form** in open inflation. Use **fitting functions** based on analytic results of Yamamoto et al. and Garriga et al. [6]

$$\mathcal{P}_\mathcal{R} = \left(\frac{H^2}{2\pi\dot\phi}\right)^2_{t=t_{\mathcal{R},p}} \frac{\cosh(\pi p) + \cos(\delta_p)}{\sinh(\pi p)}\frac{p^2}{c_1^2 + p^2}, \qquad \mathcal{P}_T = 4\left(\frac{H}{2\pi}\right)^2_{t=t_{T,p}}\frac{\cosh(\pi p) - 1}{\sinh(\pi p)}\frac{p^2}{c_2^2 + p^2}$$

- **Two sources of suppression** of scalar power on large scales:
  1. **Fast-roll:**
     - Inverse dependence on $\dot\phi^2$
     - Modified horizon crossing condition for $\mathcal{R}_c^p$

$$p^2 + 4 = a^2H^2\left(1 + \frac{\ddot\phi}{\dot\phi H}\right)^2 - \left(1 + 2\frac{\ddot\phi}{\dot\phi H}\right)$$

⇒ **large wavelength modes freeze later**
⇒ their **amplitudes** are **suppressed**

  2. **p-dependent suppression factor reflecting memory of tunnelling:**

$$\delta_p \begin{cases} p \gg 1 \to \text{irrelevant} \\ p \ll 1:\ \delta_p - \pi \propto p \to \text{take } \delta_p = \pi \end{cases}$$

$$\Rightarrow \quad \frac{\cosh(\pi p) - 1}{\sinh(\pi p)}\frac{p^2}{c_{1,2}^2 + p^2} \quad \begin{cases} p \gg 1 \to 1 \\ p \ll 1 \to \pi p^3/(2c_{1,2}^2) \end{cases}$$

Toy Models

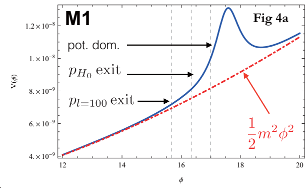
- Consider two **toy models** from Linde *et al.* [7]

Model 1: $V(\phi) = \frac{1}{2}m^2\phi^2\left(1 + \frac{\alpha^2}{\beta^2 + (\phi - \nu)^2}\right)$

$\phi_N = 17.14$

Model 2: $V(\phi) = \frac{m^2}{2}\left(\phi^2 - B^2\frac{\sinh[A(\phi - \nu)]}{\cosh^2[A(\phi - \nu)]}\right)$

$\phi_N = 16.55$



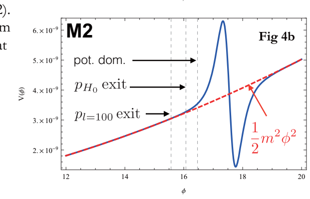


Fig 4: Tunnelling potentials of Model 1 (M1) and Model 2 (M2). The fiducial $m^2\phi^2$ potentials are plotted for comparison. From right to left, vertical lines correspond to the location of the field at

- Potential—Curvature equality
- Horizon exit of current Hubble scale p_{H_0} assuming $\Omega_K = 0.01$
- Horizon exit of scale associated with l=100, $p_{l=100}$

- **M2 "sharper" - expect suppression to affect a smaller range of scales**

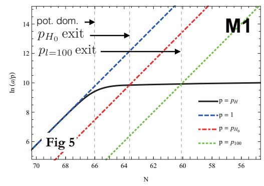


Fig 5: **Hubble evolution for M1** (black curve). Also plot the curvature scale (blue), current Hubble scale (red) (assuming $\Omega_K = 0.01$) and the scale associated with l=100 (green). Qualitatively similar for M2.

- See transition from curvature domination to slow-roll
- **Whether or not the suppressed scales leaving the horizon during the fast-rolling transitionary phase correspond to the largest observable scales in the CMB is determined by the separation between the curvature scale and the current Hubble scale:**

$$\ln\left(\frac{p_{H_0}}{p_K}\right) = \ln\left(\frac{1}{\sqrt{\Omega_K}}\right) \gtrsim \ln(10)$$

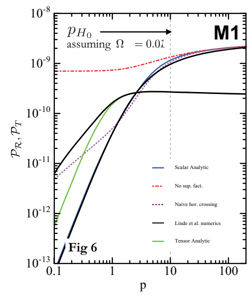
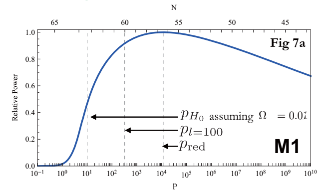
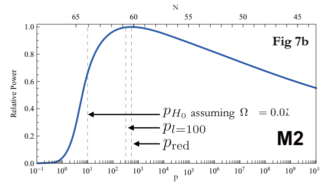


Fig 6: **Scalar and tensor power spectra for M1.** For the scalar spectrum we have four curves: Numerical results from [7] (black (upper)), full fitting formula with c_1 = 4 (blue), fitting formula without p-dependent suppression factor (red) and fitting formula using the naive horizon crossing condition. For the tensor spectrum we plot: Numerical results from [7] (black (lower)) and full fitting formula with c_2 = 1. Qualitatively similar for M2.

- **the fast-rolling of the inflaton, the p-dependent suppression factor and the modified horizon crossing condition are all important in determining the suppression on large scales**

Fig 7: **Relative power $\mathcal{P}_\mathcal{R}(p)/\mathcal{P}_\mathcal{R}(p_{red})$** where p_{red} is the scale at which the spectrum transitions from being blue- to red-tilted.





Model 1:

- Curvature—potential equality at N = 66
  ⇒ **~ 10 e-foldings of fast-roll**
- Even **for unobservable curvature,** i.e. when $\Omega_K \lesssim 10^{-4} \to p_{H_0} \gtrsim 10^2$, **get $\mathcal{O}(10\%)$ suppression for $p \lesssim p_{l=100}$**

Model 2:

- ~ 6 e-foldings of fast-roll
  ⇒ **suppression on smaller range of scales**
- **Can still satisfy constraints on Ω_K and get $\mathcal{O}(10\%)$ suppression for $p \lesssim p_{l=100}$**

Conclusions

- *Planck* and WMAP hint at a **deficit in primordial scalar power on large scales**
- This **tension is worsened if the BICEP2 signal is primordial**
- **Open Inflation** models offer a **viable explanation** for the deficit
- The **source of suppression** in Open Inflation is **two-fold:**
  1. **Fast-rolling** of the inflaton after tunnelling
  2. Additional **effects due to the tunnelling**
- Have studied **two toy models** that are **qualitatively viable**, but a more **quantitative analysis is required**

References:

1. *Planck* collaboration, A&A 571, A15 & A22 (2014).
2. BICEP2 collaboration, Phys. Rev. Lett. 112, 241101 (2014).
3. K. N. Abazajian *et al.*, JCAP 08 (2014) 053.
4. S. R. Coleman and F. De Luccia, Phys.Rev. D21, 3305 (1980).
5. B. Freivogel *et al.*, JHEP 0603, 039 (2006). R. Bousso *et al.*, arXiv:1309.4060 (2013). B. Freivogel *et al.*, arXiv:1404.2274 (2014). R. Bousso *et al.*, arXiv:1404.2278 (2014).
6. K. Yamamoto *et al.*, Phys.Rev. D54, 5031 (1996). J. Garriga *et al.*, Nucl.Phys. B551, 317 (1999).
7. A. D. Linde *et al.*, Phys.Rev. D59, 123522 (1999).

Orion Nebula image credit: Subaru Gallery http://subarutelescope.org/

"Effects of thermal fluctuations at the end of thermal inflation"

Yuhei Miyamoto (RESCEU)

[JGRG24(2014)P18]

The 24th Workshop on General Relativity and Gravitation (JGRG24)
@Kavli IPMU, November 10 (Mon) - 14 (Fri), 2014

Poster No.
A18

Effects of thermal fluctuations at the end of thermal inflation

Yuhei Miyamoto (RESCEU, The University of Tokyo)
Collaboration with
Takashi Hiramatsu (YITP) and Jun'ichi Yokoyama (RESCEU)

Introduction to thermal inflation

Motivation:
Solving the "gravitino problem" and "cosmological moduli problem" by diluting them using a short, secondary inflation (Lyth and Stewart, 1995)

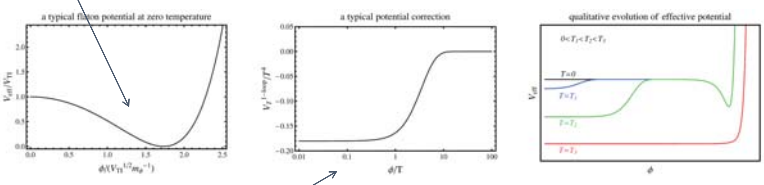
Mechanism:
Thermal inflation is driven by the potential energy of a scalar field, named flaton, with almost flat potential

$$V[\phi] = V_{\rm TI} - \frac{1}{2} m_\phi^2 \phi^2 + \cdots$$

Observational effects:
Accelerating period after the primordial inflation dilute the primordial GWs but create GWs from the collisions of bubbles

Scenario of thermal inflation

The flaton is fixed at the origin of the potential due to the thermal potential correction before the thermal inflation.
Then the thermal inflation begins when other energy density decays to be as small as the flaton potential energy.



Thermal effects: effective potential and thermal fluctuations

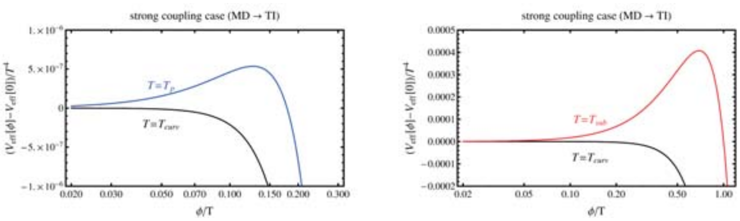
Interactions with fields in a thermal bath leads to
i) Corrections to effective potential
ii) Noise term coming from the imaginary part of the effective action (and additional friction)

EoM of the flaton

$$\ddot{\phi} - \nabla^2\phi + \eta\dot{\phi} + V'_{\rm eff}[\phi] = \xi$$

Effective potential only:
→ thermal inflation ends with (strong) first-order phase transition (at T=Tp)
→ production of GWs by bubble percolation

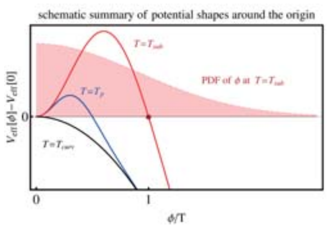
With thermal noise:
→ noise terms kick the flaton
→ flaton may escape from the dip before bubble nucleation (at T~Tsub > Tp)
→ no GWs



Results of numerical calculation

At T=Tsub>Tp, flaton has already overflowed.

We obtain $\sqrt{\langle\phi^2\rangle} \approx 0.9T$ or larger values.



The width of the PDF of the flaton becomes broader than the potential wall before critical bubble nucleation starts.

Conclusion

Thermal inflation ends with a weakly first-order phase transition.
We expect practically no GWs created at the end of thermal inflation.

The end of thermal inflation:
× critical bubble production → generation of GWs
○ gradual phase transition → no GWs

“Leptogenesis during axion inflation”

Hajime Fukuda (Kavli IPMU)

[JGRG24(2014)P19]

Gravi-Leptogenesis during Axion Inflation

Tomohiro Fujita, Hajime Fukuda, Ryo Namba, Yuichiro Tada (Kavli IPMU)
Naoyuki Takeda (ICRR)

Abstract

- Left-right asymmetric gravitational wave generates B-L through the anomaly.
- Axion inflation realizes such CP violation easily.
- We study axion-gauge interaction and its effect.

Conclusion

- We show axion-gauge interaction generates plausible amount of B-L. Our numerical simulations support this result.
- Elementary process of B-L production has not been clear yet. Hence the remnant of gravitational wave is unknown.

Introduction

- Baryon Asymmetry and Leptogenesis

$n_B/s \sim 10^{-10}$ ⇨ Too large!

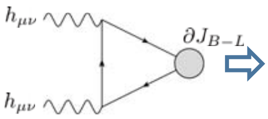
$$\partial_\mu J_B^\mu = \frac{N}{32\pi^2} W^a_{\mu\nu}\tilde{W}^a_{\mu\nu}$$

$$\partial_\mu J_L^\mu = \frac{N}{32\pi^2} W^a_{\mu\nu}\tilde{W}^a_{\mu\nu}$$

+ Sphaleron effect

We generate B-L instead of B. (Leptogenesis)

- How to Generate B-L?



⇨ $$\nabla_\mu J^\mu_{B-L} = \frac{N}{194\pi^2} R_{\mu\nu ab}\tilde{R}^{\mu\nu ab}$$

$$\Delta Q_{B-L} = \frac{N}{194\pi^2}\int d^4x\sqrt{-g}\langle R\tilde{R}\rangle$$

This term is a total derivative and thought to be zero in vacuum [2].

Any CP violating effects could generate B-L!

Axion inflation $\begin{cases}\phi(x) \neq 0 \\ \dot{\phi}(x) \neq 0\end{cases}$ ⇨ automatically violate CP

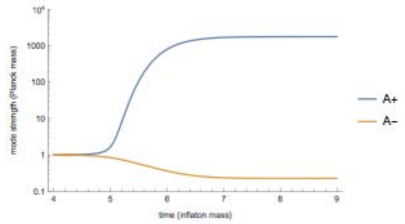
Our Setup

$$\Delta\mathcal{L} = \frac{\alpha}{4\Lambda}\phi F\tilde{F} \Rightarrow A'' - \nabla^2 A - \frac{\alpha}{f}\phi'\nabla\times A = 0$$

This violates CP and generates B-L through A.

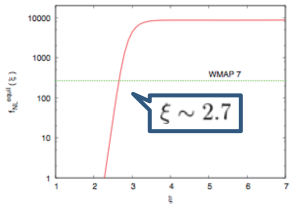
Feasible feature: exponential grow in super-horizon mode in terms of $\xi \equiv \alpha\dot{\phi}/2\Lambda H$

$$A_+(\tau, k) \simeq \frac{1}{\sqrt{2k}}\left(\frac{k}{2\xi aH}\right)^{1/4} e^{\pi\xi - 2\sqrt{2\xi k/(aH)}}$$



- The Constraint on ξ

1. A energy VS inflaton energy
2. backreaction of A
3. non-gaussianity ← Most stringent!



Previous Work and Problem

$$\Delta\mathcal{L} \sim \frac{\phi(x)}{\Lambda}R\tilde{R} \Rightarrow \Box h_\pm(z,t) \propto \pm\sqrt{\epsilon}\partial_t\partial_z h_\pm$$

This directly generate asymmetric h in perturbative calculation.

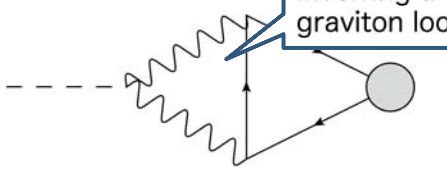
$$n_{B-L}/s \sim 6.3\times 10^{-4}(H/M_{\rm Pl})^{3/2}(\mu/M_{\rm Pl})^4$$

⇩

$$n_B/s \sim 10^{-10} \Longleftrightarrow \mu \to M_{\rm Pl}$$

- The Diagram and the Divergence

Involving a graviton loop



$\mu \to M_{PL}$ ⇨ Including vacuum divergence

References

[1] Stephon H. S. Alexander, Michael E. Peskin, and M. M. Sheikh-Jabbari, Phys. Rev. Lett. 96, 081301 (2006)
[2] Tohru Eguchi and Andrew J Hanson, General Relativity and Gravitation, 11, 5, (1979)
[3] Neil Barnaby, Ryo Namba and Marco Peloso, JCAP04(2011)009

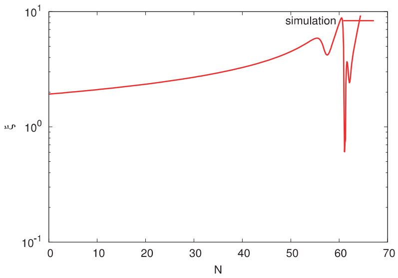
Result

With ε, η→0, we obtain

$n_B/s \sim 10^{-10} \Longleftrightarrow \xi \sim 6.09$ Is it too large? – No!

Non-gaussianity ⇩ CMB scale & $\xi \equiv \alpha\dot{\phi}/2\Lambda H$ ↗ ↘ @ inflation end

First we simply simulate ξ:



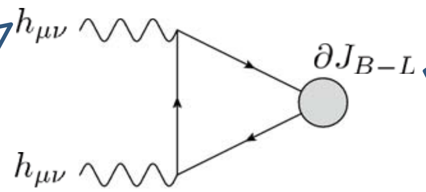
ξ seems large enough!

However, we should keep in mind that in the estimation ξ〜6 we do not take ξ time dependence into account. Also, preheating effect might be important.

- Remaining Work

How to create B-L charged particles?

Is this L-R asymmetry survive after B-L creation?



This blob does not mean particles.

“MSCO: Part 1 Formulation”

Tomohito Suzuki (Hirosaki)

[JGRG24(2014)P20]

MSCO: Part 1 Formulation

HIROSAKI UNIVERSITY

Tomohito Suzuki
Hirosaki University, Japan
with T. Ono, N. Fushimi, K. Yamada, and H. Asada (Hirosaki)

JGRG24 in Kavli IPMU Nov. 10 - 14, 2014

Abstract

We study a marginally stable circular orbit (MSCO) such as the innermost stable circular orbit (ISCO) of a timelike geodesic in any spherically symmetric and static spacetime. We present the equations describing the location of the MSCO [1]. It turns out that the metric components in this equations are separable from the constants of motion along geodescis. In addition, metric component g_{rr} (r is a radial coordinate) does not affect any MSCO radius. This suggests that, as a gravity test, any measurement of the ISCO may be put into the same category as gravitational redshift experiments, even in the strong field region.

1. Motivation

ISCOs [2] are useful for testing

- the strong gravity.
- the no-hair theorem for black holes.

They are

- important in gravitational waves astronomy [3].
- associated with the inner edge of an accretion disk around a black hole [4].

2. Timelike geodesic in spherically symmetric and static spacetimes

A general form ($G = c = 1$)

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

The Lagrangian in the equatorial plane $\theta = \pi/2$

$$\mathscr{L} = -A(r)\dot{t}^2 + B(r)\dot{r}^2 + C(r)\dot{\phi}^2, \quad \dot{} \equiv \frac{d}{d\tau} \tag{2}$$

Two constants of motion

$$\begin{aligned} E &\equiv \frac{1}{2}\frac{\partial \mathscr{L}}{\partial \dot{t}} & L &\equiv \frac{1}{2}\frac{\partial \mathscr{L}}{\partial \dot{\phi}} \\ &= -A(r)\dot{t} & &= C(r)\dot{\phi} \end{aligned} \tag{3}$$

E : the specific energy, L : the specific angular momentum

Orbit equation

$$\begin{aligned} \dot{r}^2 &= \frac{1}{B(r)}\left(\frac{E^2}{A(r)} - \frac{L^2}{C(r)} - 1\right) \\ &\equiv -V(r) \end{aligned} \tag{4}$$

$V(r)$ is not the same as the so-called effective potential.
By Eq. (5), we obtain the radial acceleration of the test body as

$$\ddot{r} = -\frac{1}{2}\frac{dV(r)}{dr} \tag{5}$$

3. Conditions for MSCO

Conditions for circular orbit

Momentarily circular condition (**Condition 1**)

$$\dot{r} = 0 \tag{6}$$

Permanently circular condition (**Condition 2**)

$$\ddot{r} = 0 \tag{7}$$

Linear stability of orbit

$$r = r_C + \delta r \tag{8}$$

r_C : the radius of a circular orbit ($\dot{r}_C = \ddot{r}_C = 0$), δr : perturbation

The equation of motion for perturbation

$$\frac{d^2}{d\tau^2}(\delta r) = -\frac{1}{2}\frac{d^2V(r_c)}{dr_c^2}\delta r \tag{9}$$

The condition for stable (or unstable)

$$\frac{d^2V(r_C)}{dr_C^2} > 0 \ (or \ \frac{d^2V(r_C)}{dr_C^2} < 0) \tag{10}$$

Marginally stable is a transition point between stable and unstable. (**Condition 3**)

Conditions of MSCO

$$V(r) = 0 \ \ and \ \ \frac{dV(r)}{dr} = 0 \ \ and \ \ \frac{d^2V(r)}{dr^2} = 0 \tag{11}$$

4. MSCO equation

The matrix

$$\begin{pmatrix} \frac{1}{A(r)} & -\frac{1}{C(r)} & -1 \\ \frac{d}{dr}\left(\frac{1}{A(r)}\right) & -\frac{d}{dr}\left(\frac{1}{C(r)}\right) & 0 \\ \frac{d^2}{dr^2}\left(\frac{1}{A(r)}\right) & -\frac{d^2}{dr^2}\left(\frac{1}{C(r)}\right) & 0 \end{pmatrix} \begin{pmatrix} E^2 \\ L^2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{12}$$

- $B(r)$ makes no contribution to the MSCO. Moreover, any circular orbit is not affected by $B(r)$.
- The geometrical part including $A(r)$ and $C(r)$ is separated from the particle motion parameters as E and L .

The determinant of this matrix vanishes : a *necessary* condition of MSCO.

MSCO equation

$$\frac{d}{dr}\left(\frac{1}{A(r)}\right)\frac{d^2}{dr^2}\left(\frac{1}{C(r)}\right) - \frac{d}{dr}\left(\frac{1}{C(r)}\right)\frac{d^2}{dr^2}\left(\frac{1}{A(r)}\right) = 0 \tag{13}$$

Eq. (13) can recover Eq. (41) in Ref. [2].

E^2 and L^2 on MSCO is

$$E^2 = -\frac{1}{\Delta}\frac{d}{dr}\left(\frac{1}{C(r)}\right), \quad L^2 = -\frac{1}{\Delta}\frac{d}{dr}\left(\frac{1}{A(r)}\right), \quad \Delta \equiv \begin{vmatrix} \frac{1}{A(r)} & -\frac{1}{C(r)} \\ \frac{d}{dr}\left(\frac{1}{A(r)}\right) & -\frac{d}{dr}\left(\frac{1}{C(r)}\right) \end{vmatrix} \tag{14}$$

The radius of the MSCO must satisfy not only the root of MSCO equation but also $0 < E^2 < \infty$ and $0 < L^2 < \infty$.

5. Conclusion

We studied a MSCO of a timelike geodesic in any spherically symmetric and static spacetime.

- The metric components are separable from the constants of motion along geodescis.
- g_{rr} does not affect MSCOs.
- Any ISCO measurement may be put into the same category as gravitational redshift experiments among gravity tests.

Applications to exact solutions to Einstein's equation are discussed in **Ono's poster**.

References

[1] T. Ono, T. Suzuki, N. Fushimi, K. Yamada, and H. Asada, arXiv:1410.6265 [gr-qc] (2014).

[2] L. Rezzolla, and A. Zhidenko, Phys. Rev. D **90**, 084009 (2014).

[3] L. Blanchet, Living Rev. Relativ. **9**, 4 (2006);
S. Isoyama, L. Barack, S. R. Dolan, A. Le Tiec, H. Nakano, A. G. Shah, T. Tanaka, and N. Warburton, Phys. Rev. Lett. **113**, 161101 (2014).

[4] M. A. Abramowicz, M. Jaroszynski, S. Kato, J. P. Lasota, A. Rozanska, and A. Sadowski, Astron. Astrophys. **521**, A15 (2010).

“MSCO: Part 2 Applications to exact solutions”

Toshiaki Ono (Hirosaki)

[JGRG24(2014)P21]

MSCO: Part 2 Applications to exact solutions

Toshiaki Ono

Hirosaki University, Japan
with T.Suzuki, N. Fushimi, K. Yamada, and H. Asada (Hirosaki)

JGRG24 in Tokyo Nov. 10 - 14, 2014

Abstract: We study a marginally stable circular orbit (MSCO) such as the innermost stable circular orbit (ISCO) of a timelike geodesic in any spherically symmetric and static spacetime. We discuss several examples: Schwarzschild, Kottler (Schwazschild-de Sitter), Reissner-Nordström, and Janis-Newman-Winicour (JNW) spacetimes.[1]

1 Introduction

We follow the Suzuki's poster. A general form of the line element for spherically symmetric and static spacetime that may have a deficit angle:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)(d\theta^2 + \sin^2\theta d\phi^2). \tag{1}$$

A **MSCO equation**:

$$\frac{d}{dr}\left(\frac{1}{A(r)}\right)\frac{d^2}{dr^2}\left(\frac{1}{C(r)}\right) - \frac{d}{dr}\left(\frac{1}{C(r)}\right)\frac{d^2}{dr^2}\left(\frac{1}{A(r)}\right) = 0. \tag{2}$$

In addition, E^2(E: energy) and L^2(L: angular momentum)

$$E^2 = -\frac{1}{\Delta}\frac{d}{dr}\left(\frac{1}{C(r)}\right), L^2 = -\frac{1}{\Delta}\frac{d}{dr}\left(\frac{1}{A(r)}\right), \tag{3}$$

where we define a determinant as

$$\Delta \equiv \begin{vmatrix} \frac{1}{A(r)} & -\frac{1}{C(r)} \\ \frac{d}{dr}\left(\frac{1}{A(r)}\right) & -\frac{d}{dr}\left(\frac{1}{C(r)}\right) \end{vmatrix}. \tag{4}$$

Effective potential:

$$V_{eff}(r) \equiv -\frac{1}{2}\left\{E^2\left(\frac{1}{A(r)B(r)} - 1\right) - \frac{L^2}{B(r)C(r)} - \frac{1}{B(r)} + 1\right\}. \tag{5}$$

In the following sections, we apply MSCO equation to some of exact solutions of the Einstein's equation. And also, we need to check whether L^2 is positive finite. We study whether the real roots are physical. Throughout this poster, we use the unit of G = c = 1.

2 Schwarzschild spacetime

The Schwarzschild spacetime:

$$ds^2 = -\left(1 - \frac{r_g}{r}\right)dt^2 + \left(1 - \frac{r_g}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{6}$$

where $r_g \equiv 2M$. From Eq.(2),

$$r_{MSCO} = 3r_g, \tag{7}$$

3 Kottler (Schwarzschild-de Sitter) spacetime

The Kottler spacetime[2]:

$$ds^2 = -\left(1 - \frac{r_g}{r} - \frac{\Lambda}{3}r^2\right)dt^2 + \frac{dr^2}{1 - \frac{r_g}{r} - \frac{\Lambda}{3}r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{8}$$

where Λ: the cosmological constant. For this spacetime, Eq.(2) becomes

$$8\lambda x^4 - 15\lambda x^3 - x + 3 = 0, \tag{9}$$

where $x \equiv rr_g^{-1}$ and $\lambda \equiv 3^{-1}\Lambda r_g^2$. We use the Sturm's theorem[3] in order to study the number of physical roots of this quartic equation.

- $0 < \lambda < 16/16875$: two MSCOs, where one is corresponding to the ISCO and the other is the OSCO.
- $16/16875 < \lambda$: no MSCO. Every circular orbit becomes unstable.
- $\lambda < 0$ (anti-de Sitter case) : single MSCO.

4 Reissner-Nordström spacetime

The Reisnner-Nordström spacetime[4]:

$$ds^2 = -\left(1 - \frac{r_g}{r} + \frac{e^2}{r^2}\right)dt^2 + \left(1 - \frac{r_g}{r} + \frac{e^2}{r^2}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{10}$$

For this spacetime, Eq.(2) becomes

$$x^3 - 3x^2 + 9q^2x - 8q^4 = 0, \tag{11}$$

where $x \equiv rr_g^{-1}$ and $q \equiv er_g^{-1}$. We use the Sturm's theorem for Eq.(11).

- $0 < e^2 < (5/16)r_g^2$: single MSCO.
- $(5/16)r_g^2 < e^2$: no MSCO. Every circular orbit becomes stable.

5 Janis-Newman-Winicour(JNW) spacetime

Finally, JNW spacetime[5]:

$$ds^2 = -\left(1 - \frac{r_g}{\gamma r}\right)^{\gamma}dt^2 + \left(1 - \frac{r_g}{\gamma r}\right)^{-\gamma}dr^2 + \left(1 - \frac{r_g}{\gamma r}\right)^{1-\gamma}r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{12}$$

In this spacetime, Eq.(2) becomes

$$2\gamma^2 r^2 - 2(1 + 3\gamma)\gamma r_g r + (1 + \gamma)(1 + 2\gamma)r_g^2 = 0. \tag{13}$$

Hence, the MSCO radius:

$$r_{MSCO} = \frac{(1 + 3\gamma) \pm \sqrt{-1 + 5\gamma^2}}{2}\frac{r_g}{\gamma}. \tag{14}$$

Therefore, there are three cases.

- $0 < \gamma < 1/\sqrt{5}$: no MSCO. Every circular orbit becomes stable.
- $1/\sqrt{5} < \gamma < 1/2$: two MSCOs. where one is corresponding to the ISCO and the other is the OSCO.
- $1/2 < \gamma < 1$: single MSCO.

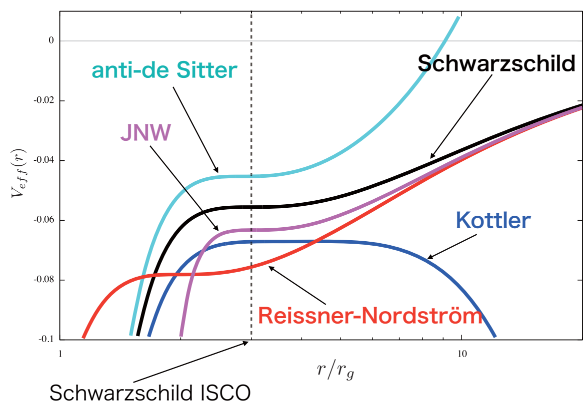


Fig 1: $V_{eff}(r)$: effective potential. Black line: Scwarzschild, blue line: Kottler, ISCO($\lambda = 1/1125$), cyan line: ($\lambda = -1/1125$), red line: Reissner-Nordström($q^2 = 1/4$), purple line: JNW($\gamma = 0.55$)

6 Conclusion

We examined roots of the MSCO equation to the Schwarzschild, Kottler, Reissner-Nordström, and Janis-Newman-Winicour spacetimes.

- If $0 < \lambda < 16/16875$ in Kottler spacetime, two MSCOs appear, where one is corresponding to the ISCO and the other is the OSCO.
- If $\lambda < 0$ (anti-de Sitter case), single MSCO appears.
- If $0 < e^2 < (5/16)r_g^2$ in Reissner-Nordström spacetime, single MSCO appear.
- If $1/\sqrt{5} < \gamma < 1/2$ in JNW spacetime, two MSCOs appear, where one is corresponding to the ISCO and the other is the OSCO.
- If $1/2 < \gamma < 1$ in JNW spacetime, single MSCO appear.

References

[1] T. Ono,T. Suzuki,N. Fushimi,K. Yamada and H. Asada, arXiv:1410.6265 [gr-qc] (2014).

[2] F. Kottler, Ann. Phys. (Leipzig), 361, 401, (1918).

[3] B. L. van der Waerden, Algebra I (Springer, 1966)

[4] H. Reissner, Ann. Phys. (Berlin), 50, 106 (1916): G. Nordström, Proc. K. Ned. Akad. Weten- sch., 20, 1238 (1918).

[5] A. I. Janis, E. T. Newman, and J. Winicour, Phys. Rev. Lett. 20, 878 (1968).

“Gravitational wave extraction from binary simulations”

Hiroyuki Nakano (Kyoto)

[JGRG24(2014)P22]

Gravitational wave extraction from binary simulations

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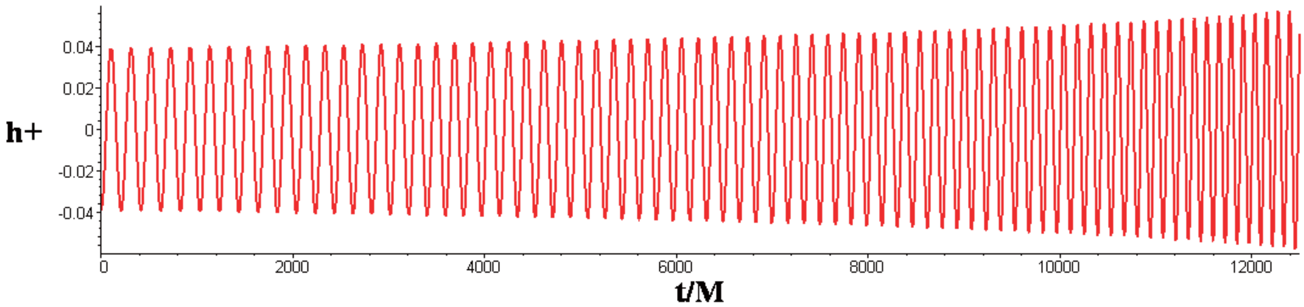
The 24th Workshop on General Relativity and Gravitation (JGRG24)
Nov. 10-14, 2014, Kavli IPMU, the University of Tokyo

Gravitational waves from merging binary black holes



A post-Newtonian waveform for m1/m2=10

Binary black holes: mass and spin $\{m_1, m_2, \mathbf{S}_1, \mathbf{S}_2\}$

Gravitational wave frequency:

$$f_{GW} \sim 13\text{[redacted]z} \frac{M\Omega_{\rm orb}}{0.02} \left(\frac{M}{100 M_\odot} \right)^{-1} .$$

M: Total mass of the binary

$\Omega_{\rm orb}$: Orbital frequency

From $M\Omega_{\rm orb} \sim 0.02$, we have 20-30 gravitational-wave cycles before merger.

Intro.: Wave extraction in numerical relativity

BEST Waveforms extracted at future null infinity

BETTER Waveforms extracted very far from the source but finite radius

Intro: Wave extraction in numerical relativity (cont'd)

BEST Directly computed using the method of Cauchy-characteristic extraction

—– Winicour, LRR 15, 2 (2012).

BETTER Computed at very large, or extrapolating several finite-radius measurements using the Regge-Wheeler-Zerilli formalism ($\psi_{\ell m}^{(even/odd)}$) or the Newman-Penrose formalism ($\psi_4^{\ell m}$)

Intro.: Perturbative extraction

An extrapolation formula for the Weyl scalar ψ_4:

$$r\,\psi_4^{\ell m}\Big|_{r=\infty} = r\,\psi_4^{\ell m}(t,r) - \frac{(\ell-1)(\ell+2)}{2\,r}\int dt\,[r\,\psi_4^{\ell m}(t,r)] + O(r^{-2})\,,$$

in the spin-weighted spherical harmonics $({}_{-2}Y_{\ell m})$ expansion.

r: an approximate areal radius

$\psi_4^{\ell m}(t,r)$: $(\ell,\, m)$ mode of ψ_4 at finite radius r

—— Lousto, Nakano, Zlochower, Campanelli, PRD 82, 104057 (2010).

Gravitational waveforms $h_{+/\times}$ are related to ψ_4 as

$$\psi_4 = \ddot{h}_+ - i\,\ddot{h}_\times\,.$$

This is true only at $r \to \infty$.

Intro.: Perturbative extraction (cont'd)

Why does this simple formula work?

For example, this formula has been used in

—— Babiuc *et al.*, PRD 84, 044057 (2011).

in the comparison with a characteristic evolution code
to obtain the gravitational waveform at null infinity, and

—— Kyutoku, Shibata and Taniguchi, PRD 90, 064006 (2014).

for numerical relativity simulations of neutron star binaries.

Basic idea

In the Regge-Wheeler-Zerilli (RWZ) formalism,

$$h_+ - i\,h_\times = \sum \frac{\sqrt{(\ell-1)\ell(\ell+1)(\ell+2)}}{2r} \left(\Psi_{\ell m}^{(\mathrm{even})} - i\,\Psi_{\ell m}^{(\mathrm{odd})}\right) {}_{-2}Y_{\ell m}\,,$$

in $r \to \infty$.

$\Psi_{\ell m}^{(\mathrm{even})}$: Even parity wave function

$\Psi_{\ell m}^{(\mathrm{odd})}$: Odd parity wave function

which satisfy

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^{*2}} - V_\ell^{(\mathrm{even/odd})}(r)\right] \psi_{\ell m}^{(\mathrm{even/odd})}(t,r) = S_{\ell m}^{(\mathrm{even/odd})}(t,r)\,.$$

$V_\ell^{(\mathrm{even/odd})}$: Potential, $S_{\ell m}^{(\mathrm{even/odd})}$: Source

---

The NR waveforms are usually obtained from the NR ψ_4 data, and

$$\psi_4 = \ddot{h}_+ - i\,\ddot{h}_\times\,.$$

Basic idea (cont'd)

In the analysis of the asymptotic behavior of the RWZ functions, we have

$$\Psi^{(\mathrm{even/odd})}_{\ell m}(t,r) = H_{\ell m}(t-r^*) + \frac{\ell(\ell+1)}{2\,r}\int dt\, H_{\ell m}(t-r^*) + O(r^{-2})\,,$$

$H_{\ell m}$: Wave observed at infinity

$r^* = r + 2M\ln[r/(2M) - 1]$,

for general ℓ modes.

- Error due to finite extraction radii arises from the integral term and higher orders in $1/r$.

Inverting the above relation, the wave function at $r \to \infty$ becomes

$$\Psi^{(\mathrm{even/odd})}_{\ell m}\Big|_{r=\infty} = \Psi^{(\mathrm{even/odd})}_{\ell m}(t,r) - \frac{\ell(\ell+1)}{2\,r}\int dt\,\Psi^{(\mathrm{even/odd})}_{\ell m}(t,r) + O(r^{-2})\,.$$

- This expression is applied to waveforms in the black hole perturbation approach.

Basic idea (cont'd)

Similarly, we discuss the mode function $\psi_4^{\ell m}$ of the Weyl scalar.

If the NR Weyl scalar satisfies the Teukolsky equation
in the Schwarzschild spacetime, $\psi_4^{\ell m}$ is expanded
with respect to $1/r$ as

$$r\,\psi_4^{\ell m}(t,r) = \quad \ddot{\tilde{H}}_{\ell m}(t-r^*) + \frac{(\ell-1)(\ell+2)}{2\,r}\,\dot{\tilde{H}}_{\ell m}(t-r^*) + O(r^{-2})\,,$$

where dot denotes the time derivative.

- The difference between this $\tilde{H}_{\ell m}$ and $H_{\ell m}$ of the RWZ function is only a numerical factor.

Inverting the above relation, we have

$$r\,\psi_4^{\ell m}\Big|_{r=\infty} = \quad r\,\psi_4^{\ell m}(t,r) - \frac{(\ell-1)(\ell+2)}{2\,r}\int dt\,[r\,\psi_4^{\ell m}(t,r)] + O(r^{-2})\,.$$

- This is used for extrapolating waveforms in the numerical relativity.

Basic idea (cont'd)

Phase and amplitude collections by the perturbative formula:

We assume

$$H_{\ell m}(t-r^*) = A_{\ell m}\exp(-i\omega_{\ell m}(t-r^*))\,.$$

Then, the RWZ functions $\Psi_{\ell m}^{(\mathrm{even/odd})}$ at a finite extraction radius are written as

$$\begin{aligned}\Psi_{\ell m}^{(\mathrm{even/odd})} =& A_{\ell m}\left[1+\frac{i\ell(\ell+1)}{2\omega_{\ell m}r}\right]\exp(-i\omega_{\ell m}(t-r^*))+O(r^{-2}) \\ =& A_{\ell m}\sqrt{1+\left(\frac{\ell(\ell+1)}{2\omega_{\ell m}r}\right)^2}\exp(-i\omega_{\ell m}(t-r^*))\exp(\delta\phi_{\ell m})+O(r^{-2}) \\ =& (A_{\ell m}+\delta A_{\ell m})\exp(-i\omega_{\ell m}(t-r^*))\exp(\delta\phi_{\ell m})+O(r^{-2})\,.\end{aligned}$$

Basic idea (cont'd)

- Amplitude correction:

$$\frac{\delta A_{\ell m}}{A_{\ell m}} = \frac{1}{2}\left(\frac{\ell(\ell+1)}{2\omega_{\ell m} r}\right)^2 + O(r^{-4})\,.$$

The amplitude collection will be $O(r^{-2})$ which we have ignored here.

- Phase correction:

$$\begin{aligned}\sin\delta\phi_{\ell m} &= \left(\frac{\ell(\ell+1)}{2\omega_{\ell m} r}\right) \Big/ \sqrt{1+\left(\frac{\ell(\ell+1)}{2\omega_{\ell m} r}\right)^2} \\ &= \frac{\ell(\ell+1)}{2\omega_{\ell m} r} + O(r^{-2})\,.\end{aligned}$$

The phase correction from the perturbative formula has $O(r^{-1})$, and is dominant.

Radiated energy

Teukolsky function ${}_{-2}\Psi = (r - ia\cos\theta)^4\psi_4$ (a: Kerr parameter) is written in the frequency domain as (**we need higher order!**)

$$
{}_{-2}\Psi_{\ell m\omega}(r) = \Bigg[\left(r^3 + i\left(ma + \frac{1}{2}\frac{\lambda}{\omega}\right) r^2 + \left(\frac{1}{2}\, i\left(-3\, ia + im^2 a + 2\, Mm\right) a \right.\right. \\
\left.\left. + \frac{1}{2}\, \frac{i\,(i\lambda ma + 3\, ima + 3\, M)}{\omega} - \frac{1}{8}\, \frac{\lambda\,(2+\lambda)}{\omega^2}\right) r + O(r^0) \Bigg] H_\omega \,,
$$

H_ω/r: Second time derivative of the waveform at infinity

The separation constant λ of the Teukolsky equation is obtained for $a\,\omega \ll 1$ as

$$
\lambda = (\ell+2)\,(\ell-1) - \frac{2\, m\left(\ell^2+\ell+4\right)}{\ell\,(\ell+1)}\, a\,\omega + (\mathcal{H}(\ell+1) - \mathcal{H}(\ell))\, a^2\omega^2 + O((a\omega)^3)\,;
$$

$$
\mathcal{H}(\ell) = 2\, \frac{(\ell-m)\,(\ell+m)\,(\ell-2)^2\,(\ell+2)^2}{(2\,\ell-1)\,\ell^3\,(2\,\ell+1)}\,.
$$

—— Mano, Suzuki and Takasugi, PTP 95, 1079 (1996).

Radiated energy (cont'd)

We discuss the $O(r^{-2})$ correction in the radiated energy.

The energy flux from the asymptotic expression is obtained as

$$\dot{E}_{\ell m\omega}(r) = \left(1 + \frac{6a\omega(a\omega - m) - \lambda}{2\omega^2 r^2} + O(r^{-3})\right) \dot{E}^{\infty}_{\ell m\omega}\,,$$

by the square of the time integration of ${}_{-2}\Psi_{\ell m\omega}$.

$\dot{E}^{\infty}_{\ell m\omega}$: Evaluated from the waveform at infinity, H_ω/r

- This is the same as

  —— Burko and Hughes, PRD 82, 104029 (2010).

  via the Sasaki-Nakamura equation.

  —— Sasaki and Nakamura, PTP 67, 1788 (1982).

More analysis of the perturbative formula

An extrapolation formula (with the Kinnersley tetrad):

$$r\,\psi_4^{\ell m}\Big|_{r=\infty} = r\,\psi_4^{\ell m}(t,r) - \frac{(\ell-1)(\ell+2)}{2\,r}\int dt\,[r\,\psi_4^{\ell m}(t,r)] + O(r^{-2})\,.$$

Difference between the Kinnersley (Kin) and a NR (num) tetrads

$$r\psi_4^{\rm Kin} = \frac{1}{2}\,[r\psi_4^{\rm num}] - \frac{M[r\psi_4^{\rm num}]}{r} + \frac{i\,a\cos(\theta)\,[r\psi_4^{\rm num}]}{r} + O(r^{-2})\,,$$

—— Campanelli, Kelly and Lousto, PRD 73, 064005 (2006).

The Teukolsky function ${}_{-2}\Psi$ for $\psi_4^{\rm num}$ is

$$\begin{aligned}{}_{-2}\Psi =& (r - ia\cos\theta)^4\left(\frac{1}{2}\,\psi_4^{\rm num} - \frac{M\psi_4^{\rm num}}{r} + \frac{i\,a\cos(\theta)\,\psi_4^{\rm num}}{r}\right)\\ =& \frac{r^4}{2}\,\psi_4^{\rm num} - M\,r^3\psi_4^{\rm num} - i\,a\,r^3\,\cos(\theta)\,\psi_4^{\rm num} + O(r^2)\\ =& \frac{r^4}{2}\left(1 - \frac{2M}{r} - \frac{2i\,a\,\cos(\theta)}{r}\right)\psi_4^{\rm num} + O(r^2)\,.\end{aligned}$$

More analysis of the perturbative formula (cont'd)

On the other hand, the asymptotic form of ${}_{-2}\Psi$

$$
\begin{aligned}
{}_{-2}\Psi_{\ell m} =& \dddot{H}_{\ell m}(t-r^*)r^3 + \left[\frac{(\ell-1)(\ell+2)}{2}\dot{\ddot{H}}_{\ell m}(t-r^*) - \frac{4\,i\,ma}{\ell\,(\ell+1)}\dddot{H}_{\ell m}(t-r^*)\right]r^2 \\
&+ O(r,\,(a\omega)^2)\,,
\end{aligned}
$$

for $a\,\omega \ll 1$.

Finally, $r\,\psi_4^{\ell m}$ at infinity is extrapolated from $r\psi_{4\ell m}^{\rm num}(t,r)$ as

$$
\begin{aligned}
r\,\psi_4^{\ell m}\Big|_{r=\infty} =& \left(1-\frac{2M}{r}\right)\left(r\psi_{4\ell m}^{\rm num}(t,r) - \frac{(\ell-1)(\ell+2)}{2\,r}\int dt[r\psi_{4\ell m}^{\rm num}(t,r)]\right) \\
&- \frac{2\,i\,a}{r}\sum_{\ell'\neq\ell,\,m'=m}[r\psi_{4\ell' m'}^{\rm num}(t,r)]C_{\ell m}^{\ell' m'} + O(1/r^2,\,(a\omega)^2)\,,
\end{aligned}
$$

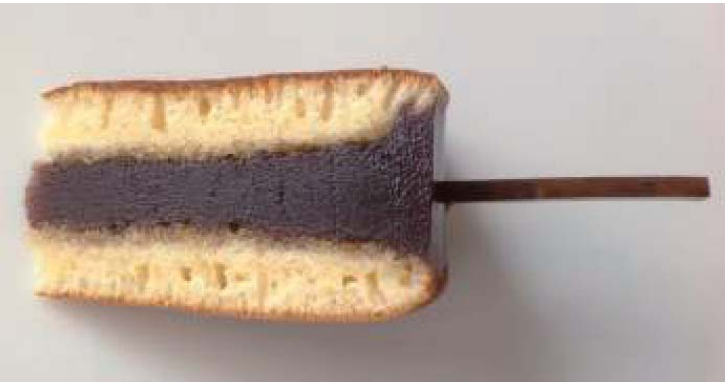
$$
C_{\ell m}^{\ell+1\,m} = \frac{1}{\ell+1}\sqrt{\frac{(\ell-1)\,(\ell+3)\,(\ell-m+1)\,(\ell+m+1)}{(2\,\ell+1)\,(2\,\ell+3)}}\,.
$$

—— See also, Berti and Klein, PRD 90, 064012 (2014).

Discussion

- What are M and a in the formula?
- The formula will give a good result for the $\ell = m = 2$ mode.
- How about the other modes?

Better extraction of GWs?



[Kyokado Toshiyasu, Shigure-gasa]

“On the creation of a baby universe”

Takahiro Tanaka (Kyoto)

[JGRG24(2014)P23]

B04 Tunneling with baby universe creation

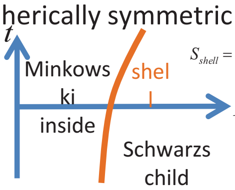
Takahiro Tanaka (Kyoto u)
with Stefano Ansoldi (Udina u)

arXiv:1410.6202

The description of quantum tunnelling in the presence of gravity shows subtleties in some cases. Here we discuss wormhole production in the context of the spherically symmetric thin-shell approximation. By presenting a fully consistent treatment based on canonical quantization, we solve a controversy present in literature.

1) Thin wall dynamics

Spherically symmetric shell motion



$S_{shell} = -\sigma \int \hat{R}^2 d\tau$

$ds^2 = -f dt^2 + \frac{1}{f} dR^2 + R^2 d\Omega^2$

Effective Lagrangian is derived by using this t coordinate.

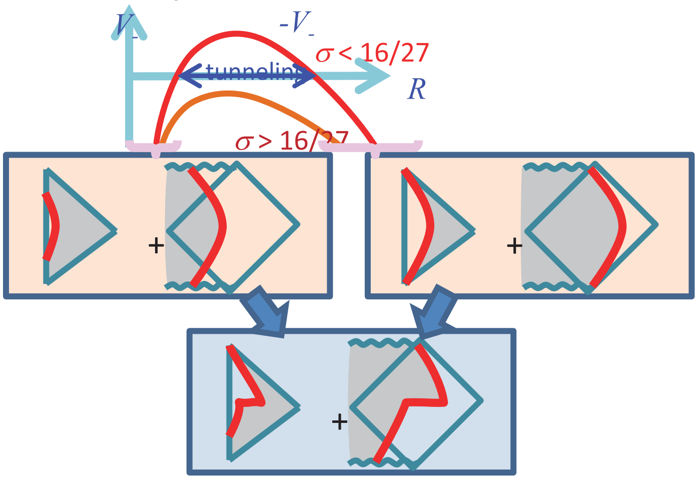
$L = P_{eff}\dot{R} - H_{eff}$

$P_{eff} = R\left[\log \frac{s\sqrt{f+\hat{R}_{,\tau}^2} + \hat{R}_{,\tau}}{\sqrt{f}}\right]_{-\varepsilon}^{+\varepsilon}$ $\quad H_{eff} = -R\left[s\sqrt{f+\hat{R}_{,\tau}^2}\right]_{-\varepsilon}^{+\varepsilon} - \sigma R^2$

Junction condition gives H_{eff} ⇒ $\hat{R}_{,\tau}^2 + V(\hat{R}) = 0$

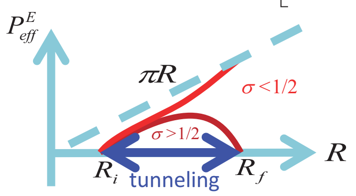
Setting $\begin{cases} f_+ = 1 - \frac{2}{R} \\ f_- = 1 \end{cases}$ $\quad s_\pm = sign\left[2R \mp \sigma^2 R^4\right]$ ⇒ $s_- = 1$, $s_+ = sign(\sigma^2/2 - R^3)$

$V(R) = -\frac{1}{4\sigma R^4}\left(\sigma^2 R^3 - 2\sigma R^2 + 2\right)\left(\sigma^2 R^3 + 2\sigma R^2 + 2\right)$



2) Strange Tunneling

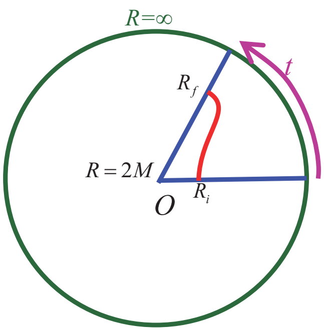
$S^E = \int_{R_i}^{R_f} P_{eff}^E dR \qquad P_{eff}^E = -iR\left[\log \frac{s\sqrt{f-\hat{R}_{,\tau}^2} + i\hat{R}_{,\tau}}{\sqrt{f}}\right]_{-\varepsilon}^{+\varepsilon}$



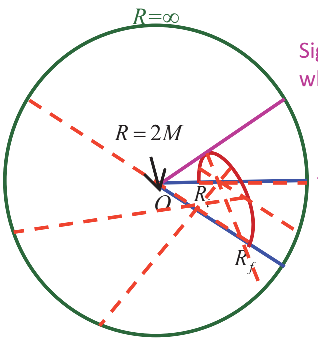
$s_- = 1$, $s_+ = sign(\sigma^2/2 - R^3)$

At the turning point $R_{,\tau}$=0.
Usually s_+=1 and so $P_{eff}^E = 0$ at the turning point.
However, when s_+ flips the sign + → −, $P_{eff}^E = \pi R$ at $R=R_f$.
Something is wrong?

Usual case



Case with flipping sign



$s\sqrt{f-\hat{R}_{,\tau}^2}$

Sign flip occurs when $f - \hat{R}_{,\tau}^2 = 0$ ⇒ $\frac{dt}{d\tau} = 0$

t =constant foliation breaks down ⇓ Alternative foliation

3) Foliation independent approach

$$ds^2 = N^2 dt^2 + L^2(dr + \beta\, dt)^2 + R^2 d\Omega^2$$

$S_{shell}^E = \sigma \int R^2 \sqrt{\hat{N}^2 + \hat{L}^2(\hat{r}_{,t} + \beta)^2}\, dt$

$S_{grav}^E = \frac{-1}{16\pi}\int d^4x \sqrt{g}\mathcal{R} + \text{(boundary terms)}$

$s_- = 1$, $s_+ = sign(\sigma^2/2 - R^3)$

Hamilton formalism

$S^E = \int dt\, p\hat{r}_{,t} + \int dt dr\left[\pi_R R_{,t} + \pi_L L_{,t} - NH_t - \beta H_r\right]$

$H_t = \left(\frac{RR'}{L}\right)' + \cdots + \sqrt{\sigma^2 \hat{R}^4 - \frac{p^2}{\hat{L}^2}}\,\delta(r - \hat{r})$

$H_r = -L\pi_L' + R'\pi_R - p\delta(r - \hat{r})$

If we use the time coordinate in the static chart, the bulk term completely vanishes, but here we do not do so.
Without fixing the gauge, we only use the constraint equations: $H_t = 0 \quad H_r = 0$

In the bulk constraint equations can be solved as

$\pi_L^{\pm} = R\sqrt{1 - \frac{2M}{R} - X^2} \qquad \pi_R = \frac{\pi_L'}{X} \qquad \pi_L = R\sqrt{1 - X^2} \qquad X \equiv \frac{R'}{L}$

Integration of the constraints across the shell gives junction conditions.

$\left[\pi_L\right]_{-\varepsilon}^{+\varepsilon} = -\frac{p}{\hat{L}} \qquad \left[R'\right]_{-\varepsilon}^{+\varepsilon} = -\frac{1}{\hat{R}}\sqrt{\sigma^2 \hat{R}^4 - p^2}$

We introduce a potential $\Phi(R, R', L)$ such that satisfies

$\pi_R = \frac{\partial\Phi}{\partial R} - \frac{\partial}{\partial r}\left(\frac{\partial\Phi}{\partial R'}\right) \qquad \pi_L = \frac{\partial\Phi}{\partial L}$

$S^E = \int dt\, p\hat{r}_{,t} + \int dt \int dr \frac{\partial\Phi}{\partial t} + \int dt\left[\frac{\partial\Phi}{\partial R'}R_{,t}\right]_{-\varepsilon}^{+\varepsilon}$

$= \int dr\, \Phi\Big|_{t_i}^{t_f} + \int dt\, \hat{r}_{,t}\left[\Phi\right]_{-\varepsilon}^{+\varepsilon} \qquad = \left[\frac{\partial\Phi}{\partial R'}\left(\hat{R}_{,t} - R'\hat{r}_{,t}\right)\right]_{-\varepsilon}^{+\varepsilon}$

$= \int dt\, \hat{r}_{,t}\left(p - \left[\frac{\partial\Phi}{\partial R'}R' - \Phi\right]_{-\varepsilon}^{+\varepsilon}\right) + \int dr\, \Phi\Big|_{t_i}^{t_f} + \int dt\, \hat{R}_{,t}\left[\frac{\partial\Phi}{\partial R'}\right]_{-\varepsilon}^{+\varepsilon}$

$$\frac{\partial\Phi}{\partial R'} = -iR\log\left(\frac{X + i\sqrt{f - X^2}}{\sqrt{f}}\right)$$

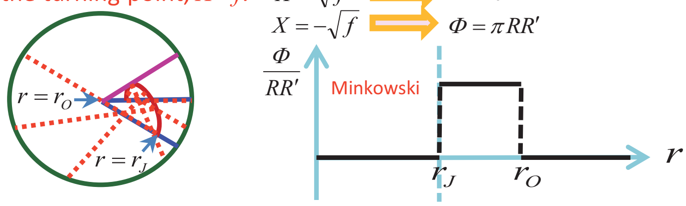
Using the junction conditions, we have

$$\left[\frac{\partial\Phi}{\partial R'}\right]_{-\varepsilon}^{+\varepsilon} = -i\hat{R}\left[\log\left(\frac{s\sqrt{f - \hat{R}_{,\tau}^2} + i\hat{R}_{,\tau}}{\sqrt{f}}\right)\right]_{-\varepsilon}^{+\varepsilon} = P_{eff}^E$$

Difference from the gauge fixed approach.

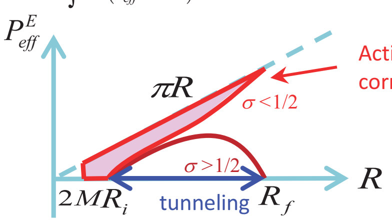
$\int dr\, \Phi\Big|_{t_i}^{t_f} \qquad \Phi = -iRR'\log\left(\frac{X + i\sqrt{f - X^2}}{\sqrt{f}}\right) + RL\sqrt{f - X^2}$

At the turning point, $X^2 = f$. $\quad X = \sqrt{f}$ ⇒ $\Phi = 0$; $\quad X = -\sqrt{f}$ ⇒ $\Phi = \pi RR'$



$S^E = \int dR P_{eff}^E + \int dr\, \Phi\Big|_{t_i}^{t_f}$

$\approx \int dR\left(P_{eff}^E - \pi R\right)$ In this combination integrand vanishes at $R=R_f$

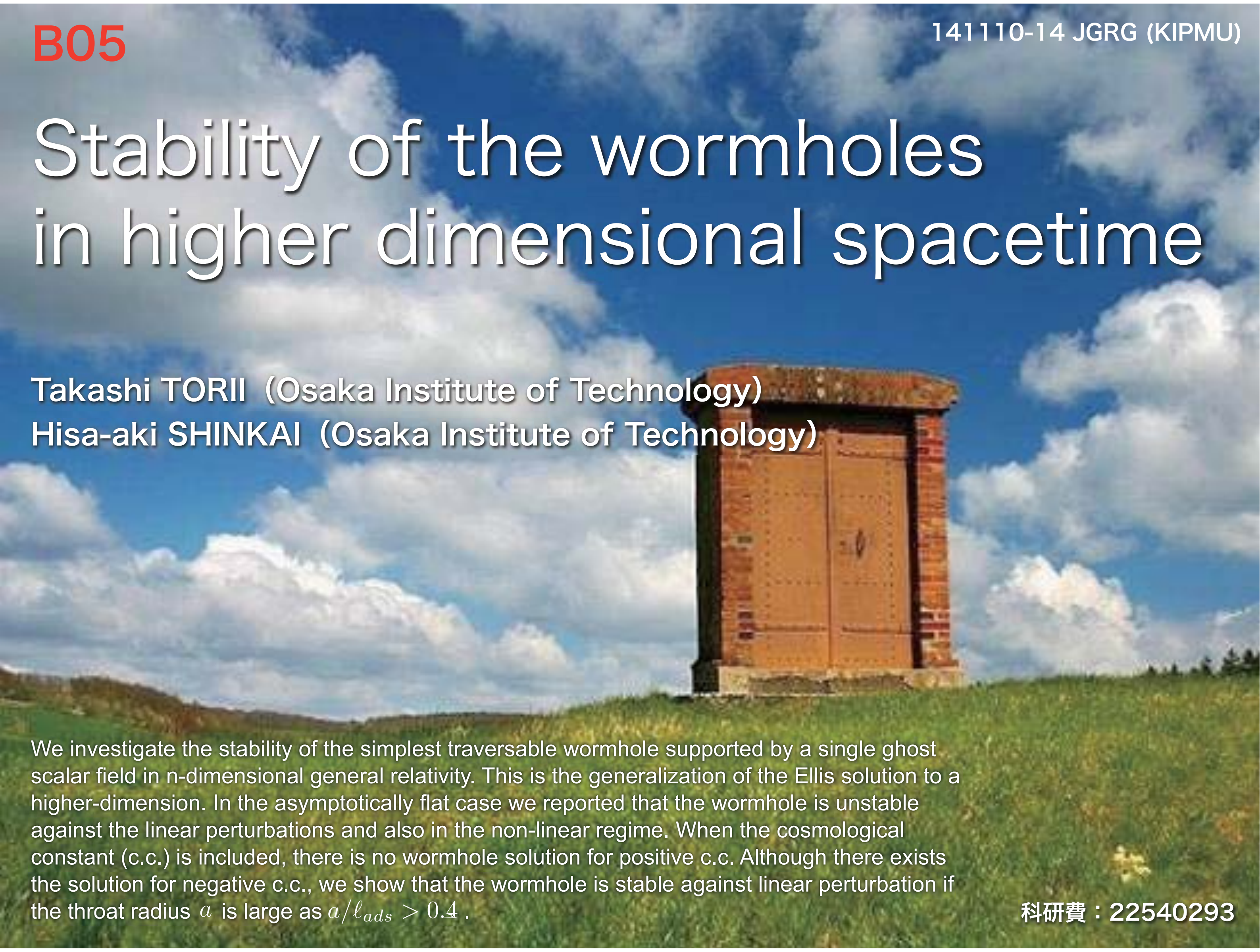


Action to be evaluated is corresponding to this area.

“Stability of the wormholes in higher dimensional spacetime”

Takashi Torii (Osaka Inst. of Tech.)

[JGRG24(2014)P24]



B05

141110-14 JGRG (KIPMU)

Stability of the wormholes in higher dimensional spacetime

Takashi TORII（Osaka Institute of Technology）
Hisa-aki SHINKAI（Osaka Institute of Technology）

We investigate the stability of the simplest traversable wormhole supported by a single ghost scalar field in n-dimensional general relativity. This is the generalization of the Ellis solution to a higher-dimension. In the asymptotically flat case we reported that the wormhole is unstable against the linear perturbations and also in the non-linear regime. When the cosmological constant (c.c.) is included, there is no wormhole solution for positive c.c. Although there exists the solution for negative c.c., we show that the wormhole is stable against linear perturbation if the throat radius a is large as $a/\ell_{ads} > 0.4$.

科研費：22540293

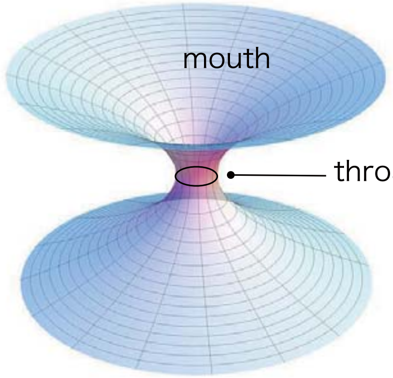
2

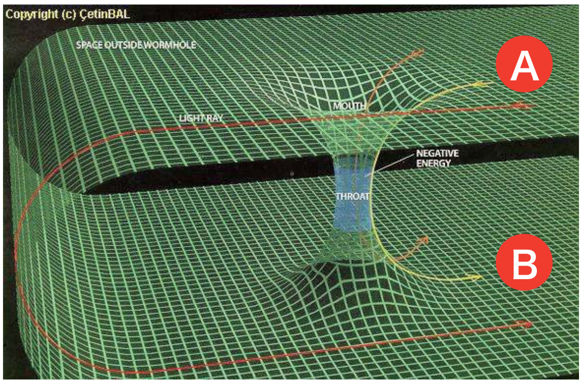
What is wormhole?

There are some definitions of a wormhole.

- Visser, 1995 :
  If a Minkowski spacetime contains a compact region Ω, and if the topology of Ω is of the form $\Omega \sim R\times\Sigma$, where Σ is a three-manifold of the nontrivial topology, whose boundary has topology of the form $\partial\Sigma \sim S^2$, and if, furthermore, the hypersurfaces Σ are all spacelike, then the region Ω contains a quasipermanent intra-universe wormhole.

- We employ the "naive definition".
  - Two asymptotic regions are connected.
  - The spacetime has a throat structure.
  - Two asymptotic regions can be the same.
    (The throat is a handle)





3

"desirable" wormhole

Let us list up the conditions of "desirable" traversable wormhole for passing through.
(Ellis 1973, Morris-Thorn 1988)

The "desirable" wormhole for passing through is

- There is no horizon for coming back.
- The tidal force should be small enough.
- It takes finite and short proper time to passing through.
- It is constructed of physically reasonable matters.
  (But the energy conditions are violated (Visser 1994))
  - perfect fluid with negative energy density
  - the ghost field
  - the tachyonic field (Das & Kar, 2005)
  - → generalized gravity
- It should be stable for perturbations at least.
- It should be possible for human being to construct it.

◎ Ultimately, we want to construct the desirable wormhole !

★ First, we should find wormhole solutions.

4

stable wormhole ?

- Bronnikov, et al (Grav. Coamol. 19 (2013) 269, arXiv:1312.6929)
  - ★In 4-dim. GR. perfect fluid and source free electro-magnetic field.
  - ★The pressure of the fluid is zero for the static solution. However, if we perturb it, the pressure appears ! → stable wormhole

    This is the key!!
  - ★However, the matter field must satisfy a certain EOS.

    ?

    Does the matter behaves like this?

- Kanti, Kleihaus and Kunz, (PRL107 (2011) 271101)
  - ★In dilatonic Einstein-Gauss-Bonnet theory.
  - ★No exotic matter and linearly stable !
  - ★However, they fix the throat radius.

    !

    The stability analysis is insufficient.

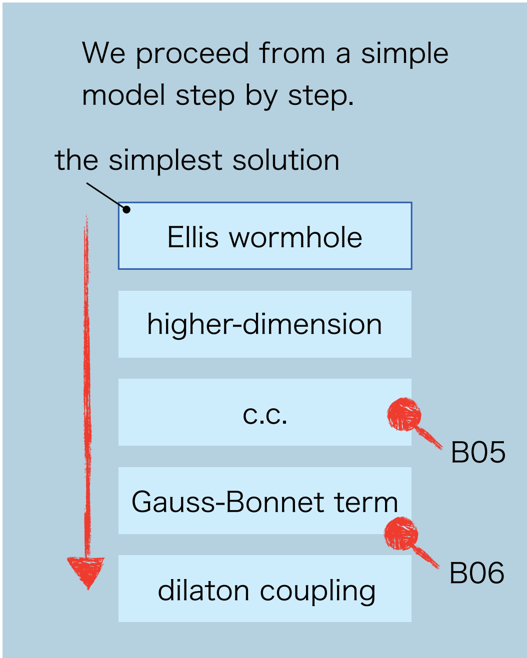
5

why Λ?

several reasons why we consider the cosmological constant

We include the cosmological constant.

- stabilize by the negative c.c.?
  - In black hole physics, a Yang-Mills hair and a scalar hair can be stabilized by adding the negative c.c.
- If wormhole is stabilized …
  - construction of a time machine and "*dokodemo* door" may be possible theoretically.
  - adS/CFT : What effects appear on the boundary theory in the wormhole bulk.
  - Bizon & Rostworowski showed that the pure adS is unstable in some sense. (PRL107 (2011) 031102)



6

model & equations

- general relativity, n-dimensions

massless scalar field

$$S = \int d^n x \sqrt{-g} \left[\frac{1}{2\kappa_n^2} (R - 2\Lambda) - \frac{1}{2} \epsilon (\nabla \phi)^2 - V(\phi) \right]$$

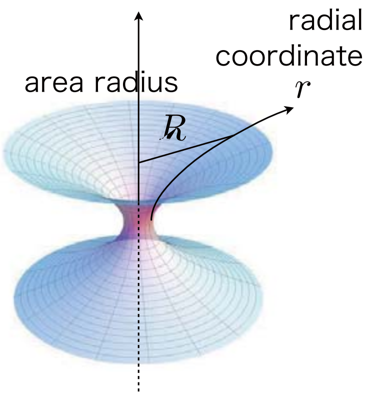
$\epsilon = -1$ (ghost)

- static spacetime

R is the area radius.

$$ds_n^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + R(r)^2 h_{ij} dx^i dx^j$$

the line element of the (n-2)-dimensional sub-manifold. It is assumed to be a constant curvature space with curvature k.



- $\Lambda = 0$
    - 4-dim : Ellis wormhole (1973)
    - n-dim : T.T & Shinkai (2013)

7

model & equations

- Einstein equations and the Klein-Gordon equation

$$(t,t)\quad -\frac{n-2}{2} f^2\left[\frac{2R''}{R}+\frac{f'R'}{fR}+\frac{(n-3)R'^2}{R^2}\right]+\frac{(n-2)(n-3)kf}{2R^2}-\Lambda f=\frac{\kappa_n^2}{2}\epsilon f^2\phi'^2,$$

$$(r,r)\quad \frac{n-2}{2}\frac{R'}{R}\left[\frac{f'}{f}+\frac{(n-3)R'}{R}\right]-\frac{(n-2)(n-3)k}{2fR^2}+\frac{\Lambda}{f}=\frac{\kappa_n^2}{2}\epsilon\phi'^2,$$

$$(i,j)\quad \frac{f''}{2}+(n-3)f\left(\frac{R''}{R}+\frac{f'R'}{fR}+\frac{n-4}{2}\frac{R'^2}{R^2}\right)-\frac{(n-3)(n-4)k}{2R^2}=\frac{\kappa_n^2}{2}\epsilon f\phi'^2.$$

$$(KG)\quad \frac{1}{R^{n-2}}\left(R^{n-2}f\phi'\right)'=0.$$

The Klein-Gordon equation can be integrated, and the scalar field is obtained by integrating the metric functions. → $\phi'=\frac{C}{fR^2},$ ①

integration constant

The Einstein equations are reduced to two equations. →

$$\frac{(n-2)R'}{R}\left[\frac{f'}{f}+\frac{(n-3)R'}{R}\right]-\frac{(n-2)(n-3)k}{fR^2}+\frac{2\Lambda}{f}=-\frac{\kappa_n^2C^2}{f^2R^{2(n-2)}} \quad ②$$

$$\frac{(n-2)R''}{R}=\frac{\kappa_n^2C^2}{f^2R^{2(n-2)}} \quad ③$$

8

boundary conditions

- regularity condition (+ symmetry) at the throat $r = 0$

throat radius → $R = a$

$R' = 0$

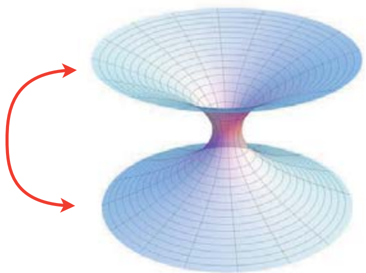
$f = f_0$

$f' = 0$

We also assume the mirror symmetry at the throat. We can extend the solution to non-symmetric one.

shift symmetry → $\phi = 0$

mirror sym.



- Asymptotically AdS

9

existence of solutions

- At the throat, Einstein equation ② becomes

② → $\kappa_n^2 C^2 = f_0\Big[(n-2)(n-3)ka^{2(n-3)} - 2\Lambda a^{2(n-2)}\Big]$ $\therefore\ \Lambda < \frac{(n-2)(n-3)}{2a^2}k.$ ④

- For the positive c.c., k is positive and the cosmological horizon should appear.

④ → $k=1$ and $f=0$ at $r=r_C$

① ③ → $\phi' \to \infty,\ \ R'' \to \infty$ at $r=r_C$ The spacetime becomes singular!

There is no regular wormhole solution for positive cosmological constant.

- For the negative c.c.,

there is no constraint for $k=1,\ 0$.

$k=-1$ ④ → $a > \sqrt{\frac{(n-2)(n-3)}{2|\Lambda|}}.$

Throat radius has the lower limit.

	$\Lambda=0$	$\Lambda>0$	$\Lambda<0$
$k=1$	exist	×	exist
$k=0$	×	×	exist
$k=-1$	×	×	exist

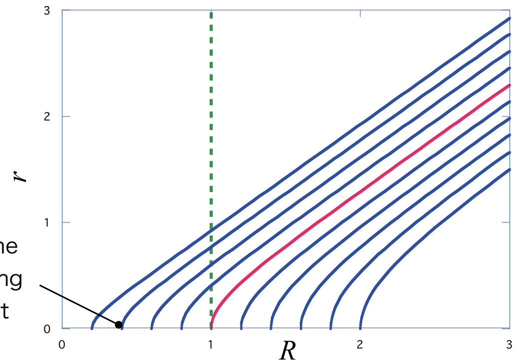
10

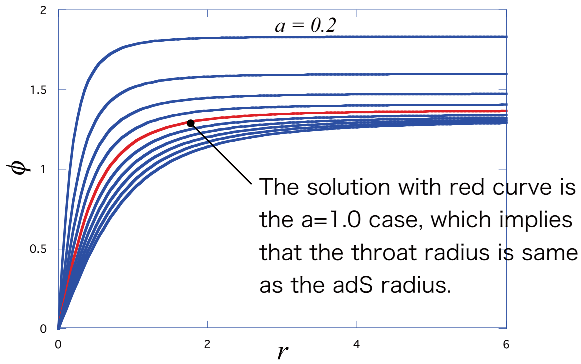
configurations

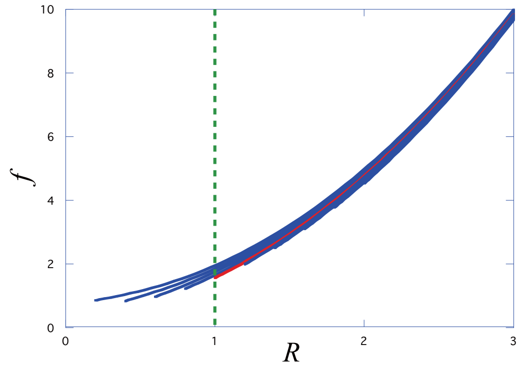
examples of the solution

- configurations $(n = 4,\ k = 1,\ \ell_{ads} = 1,\ a = 0.2 - 2.0)$

The expansions of the out-going and in-going directions are zero at the throat. It can be regarded as a double trapping horizon.







★ We find that they have qualitatively same configurations independently of their size.

11

linear analysis

In the rest of this section, we examine the linear stability of the higher-dimensional Ellis wormhole.

- metric ansatz

$$ds_n^2 = -f(t,r)e^{-2\delta(t,r)}dt^2 + f(t,r)^{-1}dr^2 + R(t,r)^2 h_{ij}dx^i dx^j$$

We consider only the spherically symmetric perturbations.

- These functions are expanded.

The variables with 0 are the static solutions.

$$f = f_0(r) + f_1(r)e^{i\omega t}, \quad R = R_0(r) + R_1(r)e^{i\omega t},$$
$$\delta = \delta_0(r) + \delta_1(r)e^{i\omega t}, \quad \phi = \phi_0(r) + \phi_1(r)e^{i\omega t}.$$

ω is a frequency.

The variables with 1 are the perturbations.

- By taking linear combination, we can find the single master equation.

$$\psi_1 = R_0^{\frac{n-2}{2}}\left(\phi_1 - \frac{\phi_0'}{R_0'}R_1\right),$$

gauge invariant under spherical symmetry

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perturbation equation

- By taking linear combination, we can find the single master equation.

$$\psi_1 = R_0^{\frac{n-2}{2}}\left(\phi_1 - \frac{\phi_0'}{R_0'}R_1\right),$$

gauge invariant under spherical symmetry

$$-\frac{d^2\psi_1}{dr_*^2} + V(r)\psi_1 = \omega^2\psi_2$$

$$V(r) = \frac{2C^2 R_0^{-2n+4}}{(n-2)f_0 R_0'^2}\left[(n-3)k - \frac{2\Lambda R_0^2}{n-2}\right] - \Lambda f_0 + \frac{(n-2)f_0}{4R_0^2}\left[2(n-3)k - (n-2)f_0 R_0'^2\right].$$

diverges at the throat !

➡ The potential is positive definite. ∴ stable ?

- 0-mode solution $\bar{\psi}_1$ ⬅ The mode which changes the throat radius.

The 0-mode diverges at the throat.

This divergence is canceled by the divergence of the potential function.

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regularization

- regularize the perturbation equation by the 0-mode

$$\mathcal{D}_\pm = \pm\frac{d}{dr} - \frac{1}{\bar{\psi}_1}\frac{\bar{\psi}_1}{dr_*}$$

the perturbation equation

$$\mathcal{D}_-\mathcal{D}_+\psi_1 = \omega^2\phi_1 .$$

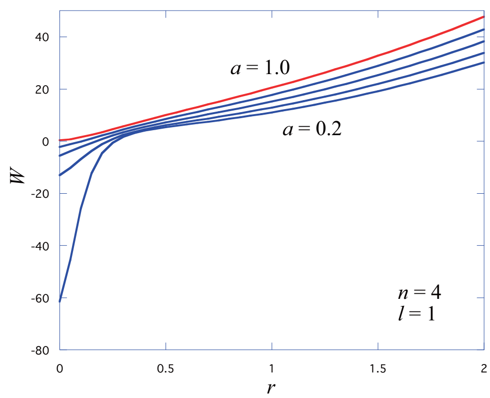
- Operating $D+$ on the eqaution and defining $\Psi_1 = \mathcal{D}_+\psi_1$, ...

- We find the regularized equation.

$$-\frac{d^2\Psi_1}{dr_*^2} + W(r)\Psi_1 = \omega^2\Psi_1$$

$$W(r) = 2f_0^2\left(\frac{1}{\bar{\psi}_1}\frac{d\bar{\psi}_1}{dr_*}\right)^2 - V(r)$$

$n = 4,\ \ell_{ads} = 1.0$



For $n = 4$ and $l_{ads} = 1$, the potential W is positive definite for $a > 1$. Hence these wormholes are stable !!

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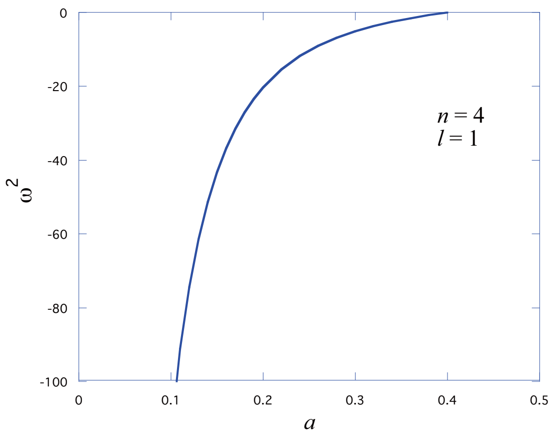
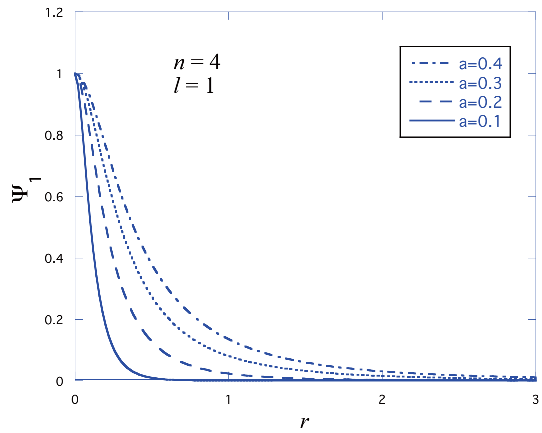
stable or unstable?

- Solving this equation numerically, we can find a negative mode for $a < 0.4$.



eigenvalue of negative mode



eigenfunction of the negative mode

★ For n=4 and l_{ads}=1,

$a > 0.4$	stable
$a < 0.4$	unstable

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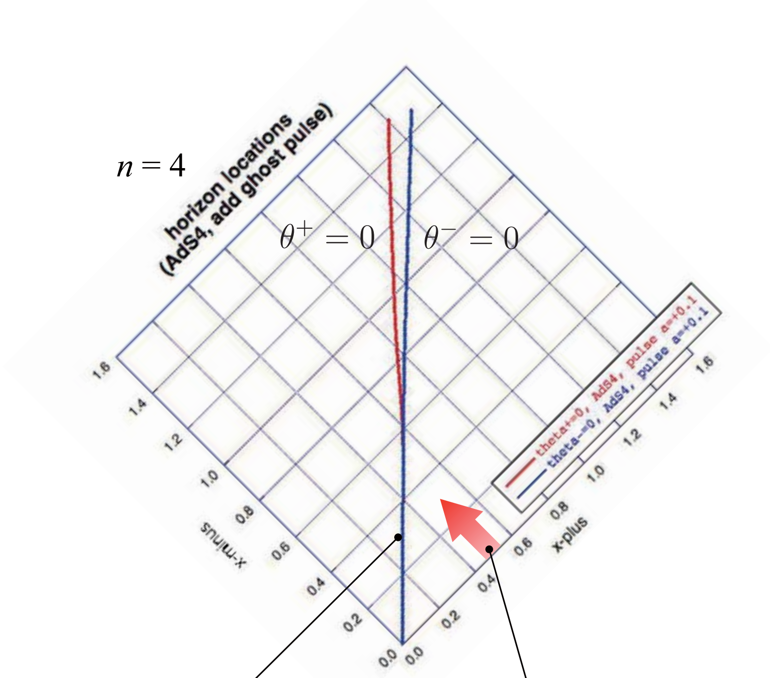
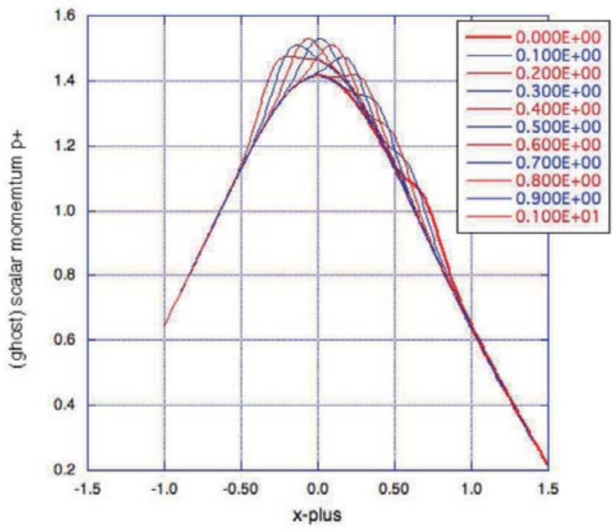
dynamical evolution

- we add the pulse to the momentum of the ghost field, and investigate the evolution of the wormhole.

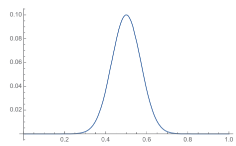




The throat can be considered as a double trapping horizon.

pulse

$$p_+ = p_{+\mathrm{sol}} + a \exp\{-100(x^+ - 0.5)^2\}$$



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Summary

- We derived the Ellis wormhole solution in higher dimensions including the c.c..
  - For positive Lambda no solution exists.
  - For negative Lambda there can be the solution with not only k=1 but k=0, -1.
- We investigated their linear stability, and found the large ($a/\ell_{ads} > 0.4$) wormhole is linearly stable.
- We performed the dynamical simulation to investigate the evolution of the wormhole.

Anyway, we want to construct stable wormhole solution because we want a *dokodemo* door and a time machine !!

“Wormhole evolutions in n-dimensional Gauss-Bonnet gravity”

Hisaaki Shinkai (Osaka Inst. of Tech.)

[JGRG24(2014)P25]

poster B06

Wormhole Evolutions in n-dim Gauss-Bonnet gravity

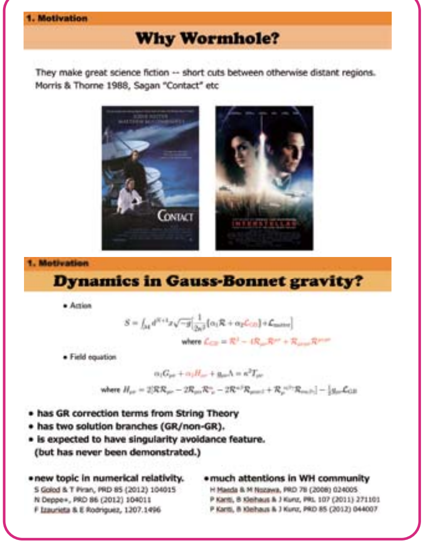
Hisaaki Shinkai & Takashi Torii (Osaka Inst. Technology, Japan)
真貝寿明 & 鳥居隆 (大阪工業大学)

Previous Stories

(a) "Fate of Morris-Thorne (Ellis) wormhole" was numerically investigated in 2002. [HS & Hayward, PRD66, 044005].
The fate is either black-hole collapse or inflationary expansion, depending on the excessed energy.

(b) The n-dimensional GR Ellis wormhole solutions are obtained. Perturbation study suggests instability. [TT & HS, PRD88 (2013), 064023]. Numerical evolutions in 4-6 dim confirm its instability. [HS & TT, in preparation]

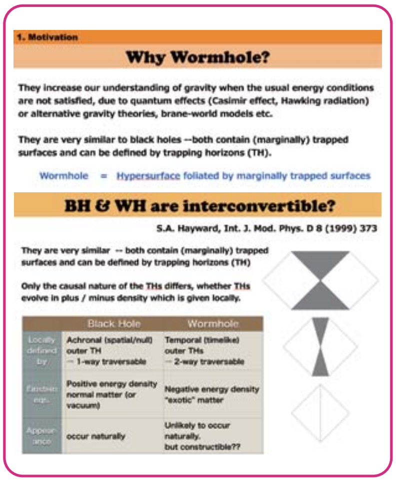
(c) The wormholes in anti-de Sitter spacetime are analyzed. Perturbation study suggests instability if throat is smaller than a half of AdS horizon. Numerical evolutions support this prediction. [TT & HS, in Poster B05.]



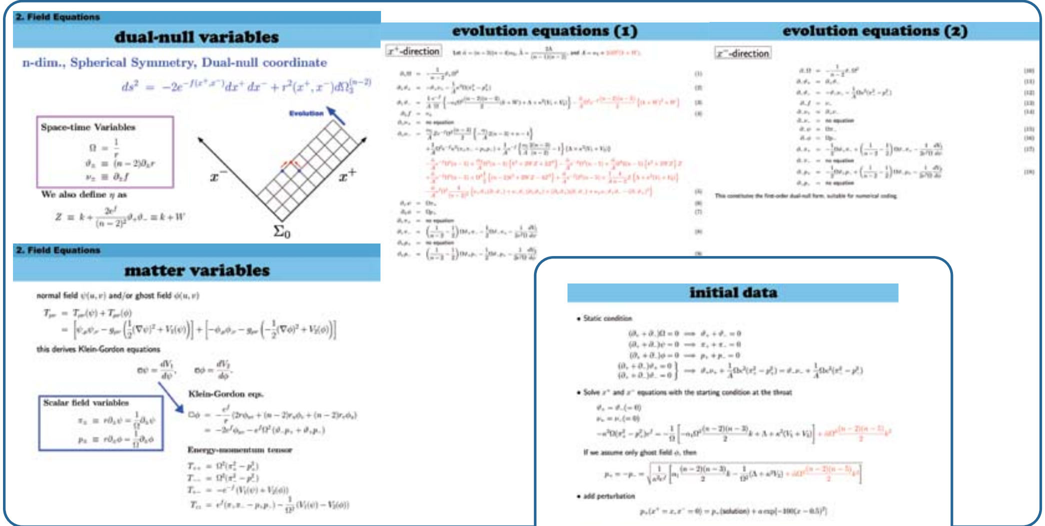
Outline & Summary

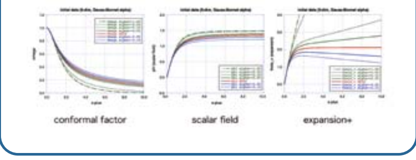
The dynamics of the simplest wormhole solutions in n-dimensional Gauss-Bonnet gravity are investigated numerically.
The solutions catch an unstable mode, and the throat begins inflate if GB coupling term is positive, while it turns into a black-hole if the coupling is negative. This horizon bifurcation can be seen easily in higher-dimensional spacetime. There exists the optimized positive coupling constant which maximizes the throat expansion.

Motivations



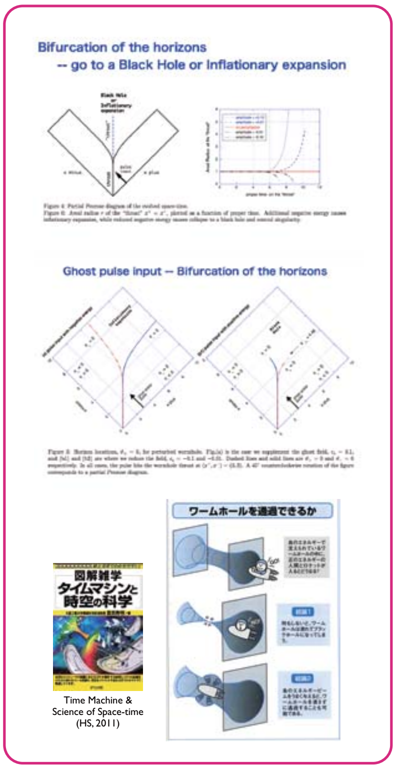
Field Eqs.



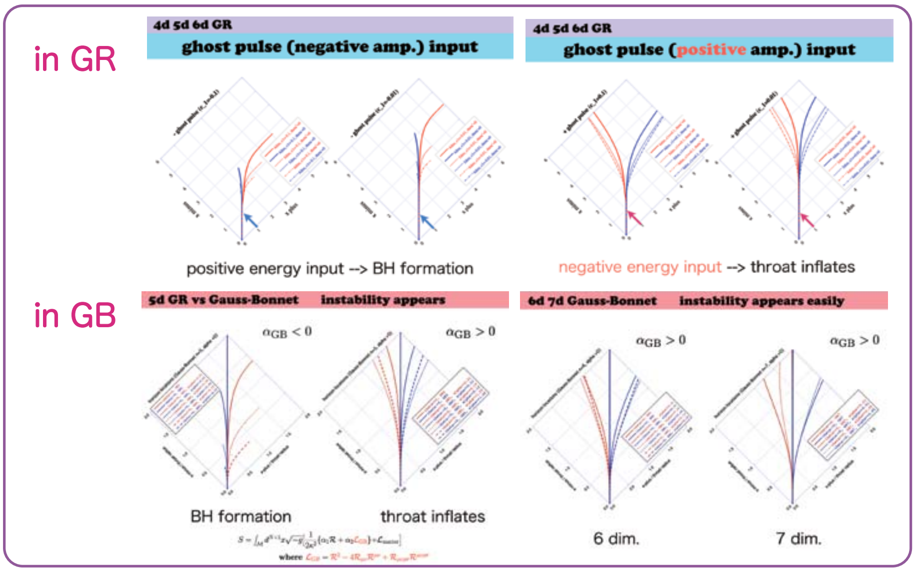


Results in 4-dim. GR

PRD66 (2002) 044005



WH evolution in 5/6/7-dim.



“Wormhole on DGP brane”

Yoshimune Tomikawa (Nagoya)

[JGRG24(2014)P26]

JGRG24 Nov. 10-14 (2014)

Wormhole on DGP brane

Yoshimune Tomikawa
Department of Mathematics, Nagoya University

based on
Y.Tomikawa, T.Shiromizu, K.Izumi, arXiv:1409.6816, to appear in PRD

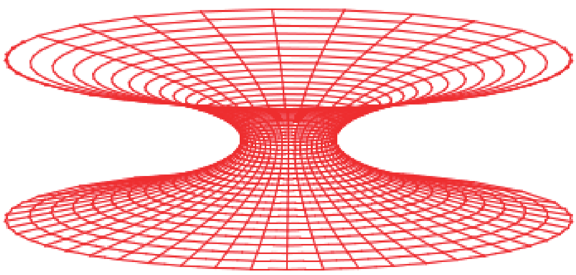
1. Introduction

Wormhole and Exotic matter

M.S.Morris, K.S.Thorne (1988), D.Hochberg, M.Visser (1997)

Exotic matters are required to construct wormhole

(at least for static)



The presence of throat violates null energy condition

Braneworld and wormhole

-On brane, in general, gravity is modified from Einstein's one.

-Without introducing of exotic matters, we may be able to construct the wormhole in the braneworld.

-A candidate has been constructed recently in DGP(Dvali, Gabadadze, Porrati) model. K.Izumi, T.Shiromizu (2014)

⇒ We examine the detail of spacetime structure on brane focusing on wormhole aspect

2. Setup

DGP braneworld – single vacuum brane-

action

$$S = 2M^3 \int_{\text{bulk}} d^5x \sqrt{-g} R + 2M^3 r_c \int_{\text{brane}} d^4x \sqrt{-q}\, {}^{(4)}R(q)$$

G.R.Dvali, G.Gabadadze, M.Porrati (2000)

$M \cdots$ five dimensional Planck scale

$r_c \cdots$ a constant having lenth scale

$g_{\mu\nu} \cdots$ bulk metric

$q_{\mu\nu} \cdots$ induced metric on brane

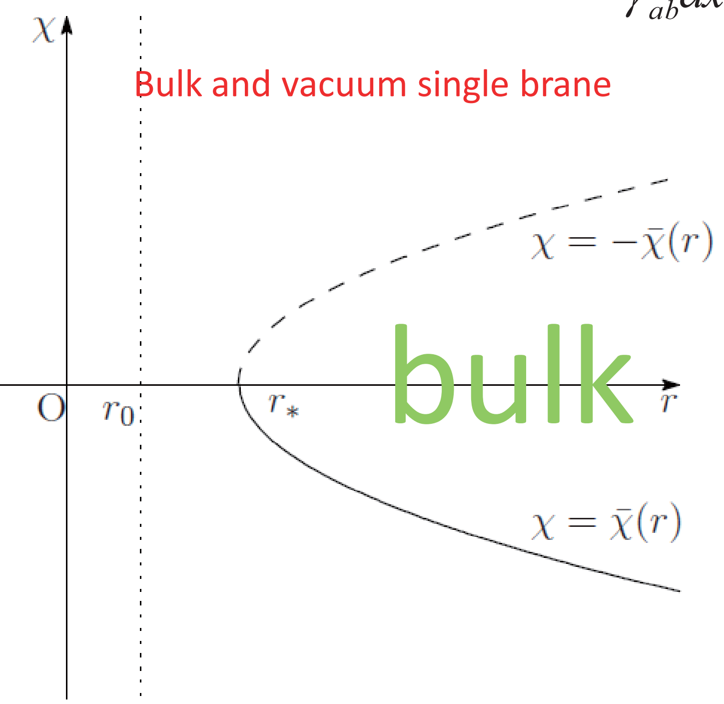
New configuration

K.Izumi, T.Shiromizu (2014)

Bulk spacetime: five dimensional Kaluza-Klein bubble

$$g_{\text{bulk}} = f(r)d\chi^2 + f^{-1}(r)dr^2 + r^2\gamma_{ab}dx^a dx^b, \quad f(r) = 1-(r_0/r)^2$$

$$\gamma_{ab}dx^a dx^b = -d\tau^2 + \cosh^2\tau d\Omega_2^2 \ \ (3\text{-dim. de Sitter})$$



-Brane location $\chi = \bar{\chi}(r)$ is determined by junction condition

-Single vacuum brane solution is realized for $r_0 > r_c$ (regular brane)

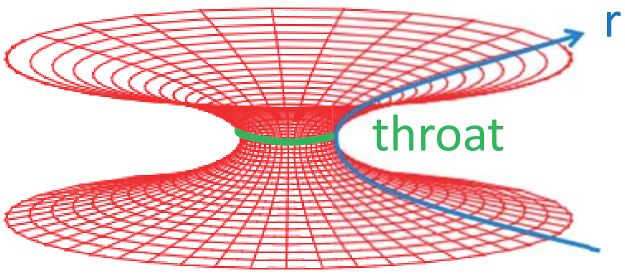
Induced metric on brane

K.Izumi, T.Shiromizu (2014)

$$ds^2 = \alpha^{-2} dr^2 + r^2 \gamma_{ab} dx^a dx^b$$

$$\alpha^2 = \frac{-(r^2 - 2r_c^2) + \sqrt{r^4 - 4r_0^2 r_c^2}}{2r_c^2}$$

α^2 : real and positivity $\Leftrightarrow$ $r \geq r_* := \sqrt{r_0^2 + r_c^2}$



3. Wormhole on DGP brane

Induced metric on brane

$$ds^2 = \alpha^{-2} dr^2 + r^2 \gamma_{ab} dx^a dx^b$$

$$\gamma_{ab} dx^a dx^b = -d\tau^2 + \cosh^2 \tau d\Omega_2^2 \quad \text{(3-dim. de Sitter)}$$

$$\alpha^2 = \frac{-(r^2 - 2r_c^2) + \sqrt{r^4 - 4r_0^2 r_c^2}}{2r_c^2}$$

$$r \geq r_* := \sqrt{r_0^2 + r_c^2}$$

We examine the detail of this spacetime.

Location of throat

Maeda, Harada, Carr's definition (2009)

Throat is defined as the minimal surface in the trapped region or at the bifurcating trapping horizon.

(i) In (τ, r) coordinate

minimal surface at $r = r_* = \sqrt{r_0^2 + r_c^2}$

(ii) In (T, R) coordinate

$$T = rh(r)\sinh\tau, R = rh(r)\cosh\tau$$

$$\Rightarrow ds^2 = h^{-2}(-dT^2 + dR^2 + R^2 d\Omega_2^2)$$

$$\log(h(r)) = \int \frac{1-\alpha}{\alpha r} dr$$

$$\lim_{r\to\infty} h(r) = 1$$

(aymptotically flat)

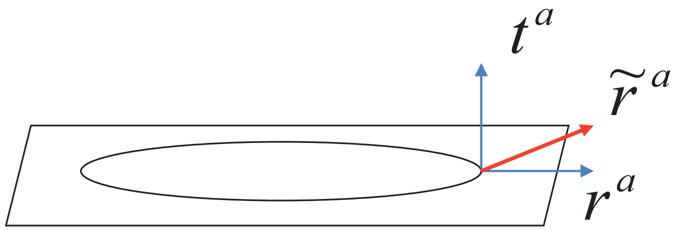
Location of minimal surface on T=const.

$$r_{\min}^2(\tau) = r_c^2(1-\tanh^4\tau) + r_0^2(1-\tanh^4\tau)^{-1}$$

$$\Rightarrow r_{\min}^2(\tau) - r_*^2 \geq r_0^2 \frac{\tanh^8\tau}{1-\tanh^4\tau} \geq 0$$

Minimal surface depends on slice

$$k = h^{ab}\nabla_a r_b = \frac{1}{\beta}\tilde{k} \mp \frac{\sqrt{1-\beta^2}}{\beta} h^{ab} K_{ab}$$



$$\tilde{r}_a = \beta r_a \pm \sqrt{1-\beta^2}\, t_a, \;\; r_a t^a = 0$$

$$K_{ab} = (\delta_a^c + t_a t^c)\nabla_c t_b$$

$$h_{ab} = g_{ab} + t_a t_b - r_a r_b$$

Location of minimal surface depends on slice

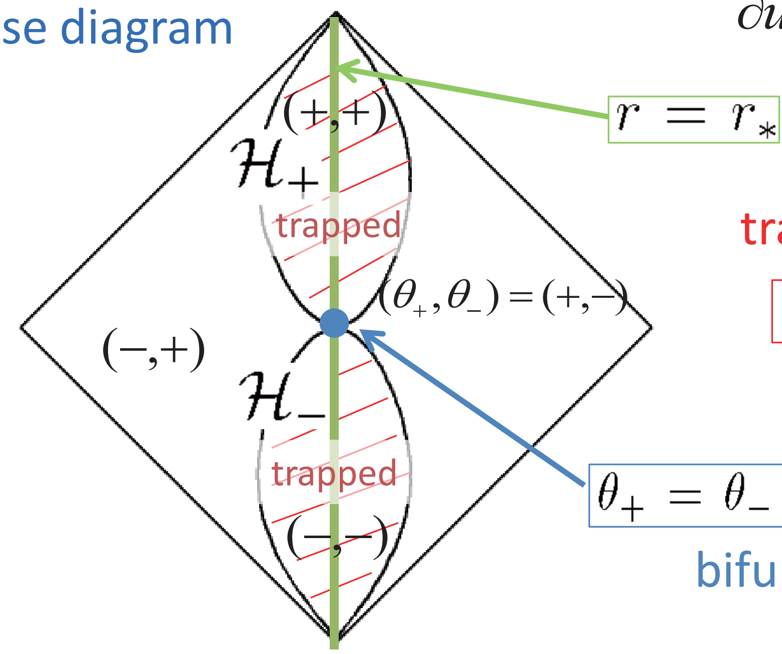
Spacetime structure on brane

$$ds^2 = -r^2 du_+ du_- + r^2 \cosh^2 \tau d\Omega_2^2 \qquad du_\pm = d\tau \pm dr/(r\alpha)$$

null expansion rate for outgoing/ingoing

$$\theta_\pm = \frac{\partial \ln(r\cosh\tau)}{\partial u_\pm} = \frac{1}{2}(\tanh\tau \pm \alpha)$$

Penrose diagram

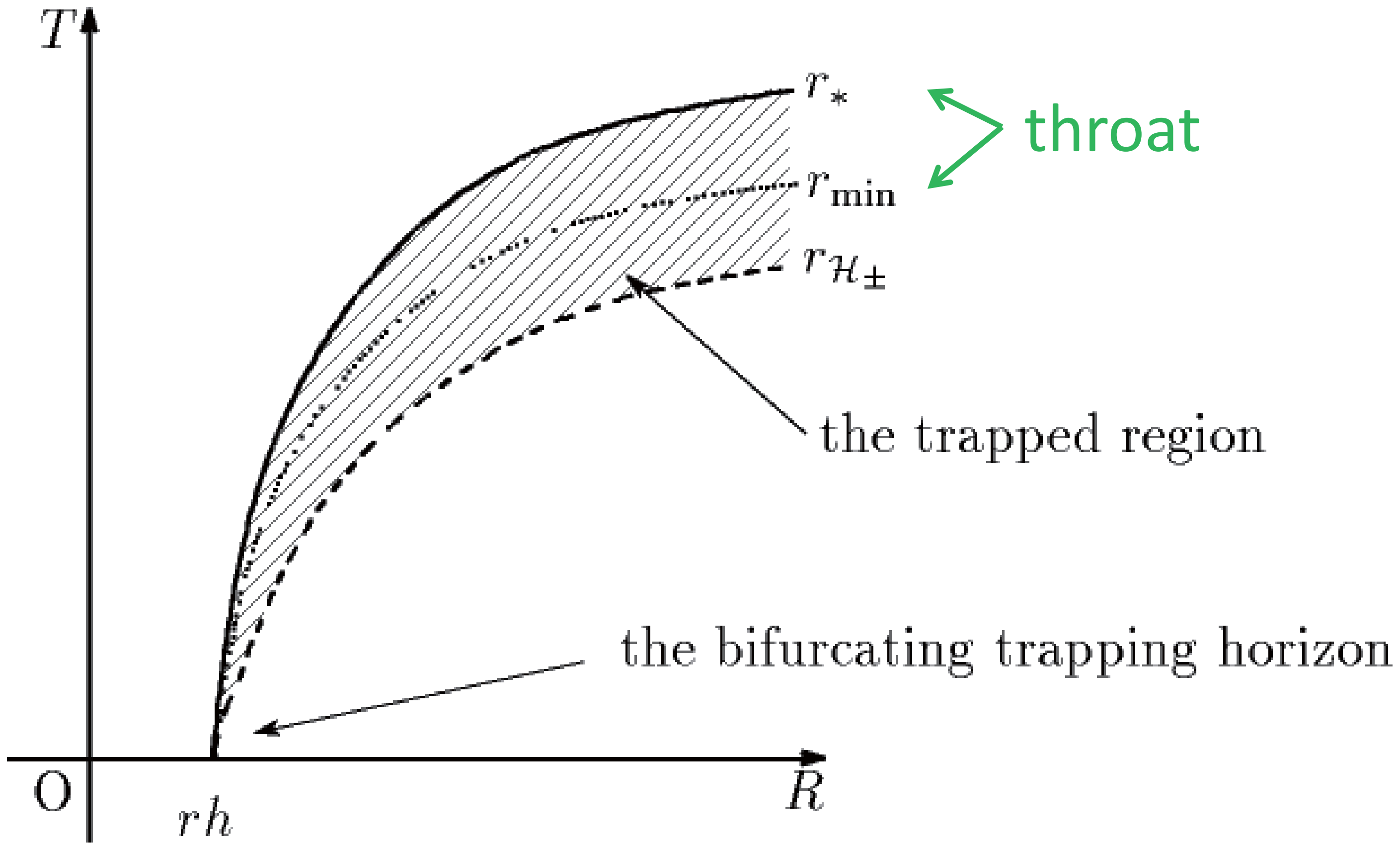


trapped region

$\theta_+\theta_- > 0$

bifurcating trapping horizon

In (T,R) plane


T
R
O
rh
r_*
$r_{\min}$
$r_{\mathcal{H}_\pm}$
throat
the trapped region
the bifurcating trapping horizon

No exotic, but effectively…

effectively energy-momentum tensor

$$ {}^{(4)}G_{\mu\nu} = T^{(\mathrm{eff})}_{\mu\nu} $$

$$ T^{(\mathrm{eff})}_{\hat{\mu}\hat{\nu}} = \mathrm{diag}\left[\rho^{(\mathrm{eff})}, p_r^{(\mathrm{eff})}, p^{(\mathrm{eff})}, p^{(\mathrm{eff})}\right] $$

$$ \begin{cases} \rho^{(\mathrm{eff})} = -p^{(\mathrm{eff})} = \dfrac{1}{r^2}(1-\alpha^2 - 2r\alpha\alpha') < 0 \\ p_r^{(\mathrm{eff})} = -\dfrac{3}{r^2}(1-\alpha^2) < 0 \\ \rho^{(\mathrm{eff})} + p_r^{(\mathrm{eff})} = -\dfrac{2}{r^2}(1-\alpha^2 + r\alpha\alpha') < 0 \end{cases} $$

all energy conditions are not satisfied

Traversability -Acceleration and tidal force-

c : speed of light

g_{Earth}: acceleration on the Earth

-acceleration

$$|a^i| \sim c^2 / r_0 < g_{Earth}$$

$$\Rightarrow \quad r_0 > c^2 / g_{Earth} \sim 10^{18}\mathrm{cm} \sim 1\mathrm{pc}$$

(depend on traveler)

-tidal force

$$|R_{0i0j}| \sim r_0^{-2}$$

$|\xi|$: size of traveler

$$\Rightarrow \text{tidal acceleration } |\Delta a^i| \sim c^2 |R_{0i0j}| \times |\xi|$$

$$\Rightarrow \quad r_0 > \sqrt{(|\xi| c^2) / g_{Earth}} \sim 10^{10}\mathrm{cm}$$

4. Summary

Summary

-we confirmed that wormhole spacetime is realized on DGP brane. No exotic matters!

-traversable wormhole is too large, say $10^{10}\mathrm{cm}$.

-a mechanism to keep the size compact?

“Charged multi-black strings

in a five-dimensional Kaluza-Klein universe”

Hideki Ishihara (Osaka City)

[JGRG24(2014)P27]

Charged multi-black strings in a Kaluza-Klein universe

Hideki Ishihara, Masashi Kimura*, and Ken Matsuno

Department of Physics, Osaka City University

* DAMTP, University of Cambridge

2014/11/22

Solutions

We consider the 5-dimensional Einstein-Maxwell equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2\left(F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right),$$

$$F^{\mu\nu}{}_{;\nu} = 0 .$$

Exact solutions :

$$ds^2 = -H^{-2}dt^2 + H\left[V\left(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right) + V^{-1}dw^2\right],$$

$$H = 1 + \sum_i \frac{M_i}{|\boldsymbol{x} - \boldsymbol{x}_i|},$$: harmonics on the 4-dim. Ricci flat space

$$V = \frac{t}{t_0}.$$

$$A_\mu dx^\mu = \pm\frac{\sqrt{3}}{2}H^{-1}dt,$$

Physical properties of Solutions

Single solution :

$$ds^2 = -H^{-2}dt^2 + H\left[V\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right) + V^{-1}dw^2\right],$$

$$H = 1 + \frac{M}{r}, \qquad V = \frac{t}{t_0}.$$

$$A_\mu dx^\mu = \pm\frac{\sqrt{3}}{2}H^{-1}dt,$$

We investigate

- geometrical properties of the metric,
- motion of a test particle.

Geometrical Properties

Asymptotic structure

At a large distance $r \to \infty$ $\quad H = 1 + \frac{M}{r} \to 1$

$$ds^2 = -H^{-2}dt^2 + H\left[\frac{t}{t_0}\left(dr^2 + r^2 d\Omega_{\mathrm{S}^2}^2\right) + \frac{t_0}{t}dw^2\right],$$

$$\to -dt^2 + \frac{t}{t_0}\left(dr^2 + r^2 d\Omega_{\mathrm{S}^2}^2\right) + \frac{t_0}{t}dw^2,$$

5-dim. Kasner universe with the parameter (½, ½, ½, -½)

Expanding 3-dimensions and a contracting extra dimension

Effectively 4-dimensional Friedmann universe

$$ds_4^2 = -dt^2 + a(t)^2\left(dr^2 + r^2 d\Omega_{S^2}^2\right)$$

$$a(t) = \sqrt{V(t)} = \sqrt{t/t_0}$$

Singularities

The Kretschmann scalar :

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{18(r+M)^5\left[(r+M)^5 - t_0 M^2 tr\right] + (t_0 M tr)^2\left(144r^2 + 48Mr + 31M^2\right)}{4t^4r^4(r+M)^6},$$

curvature singularities exist at $t = 0,\ r = 0,$ and $r = -M.$

$t = \infty,\ r = 0$ with $rt = \text{finite}.$ seems to be non singular.

Ingoing null geodesics

$$\theta = \text{const.}, \phi = \text{const.}, \psi = \text{const.},$$

$$-H^{-2}dt^2 + H\frac{t}{t_0}dr^2 = 0,$$

Then

$$\left(\frac{dt}{dr}\right)^2 = \frac{t}{t_0}\left(\frac{M}{r}+1\right)^3.$$

An approximate solution is

$$tr = a + ur^{1/2} + \left(\frac{u^2}{4a} - \frac{3M^2}{t_0}\right)r + \cdots$$

$$a = \frac{M^3}{t_0},$$

New coordinates

$$r = \rho^2,$$
$$t = \frac{M^3}{t_0\rho^2} + \frac{u}{\rho} + \frac{t_0}{4M^3}u^2 - \frac{3M^2}{t_0},$$

Metric in the coordinates is

$$ds^2 = \frac{L(u,\rho)^2}{t_0 M^3 \rho^4 H(\rho)^2}\left(-\frac{t_0}{M^3}\rho^2 du^2 + 4du d\rho\right) + 4\frac{t_0^2\rho^6 H(\rho)^3 K(u,\rho) - L(u,\rho)^2}{t_0^2\rho^6 H(\rho)^2}d\rho^2$$
$$+\rho^2 H(\rho)\left[K(u,\rho)d\Omega_{S^2}^2 + K(u,\rho)^{-1}dw^2\right],$$

$$K(u,\rho) = \frac{M^3}{t_0^2} + \frac{u\rho}{t_0} + \left(\frac{u^2}{4M^3} - \frac{3M^2}{t_0^2}\right)\rho^2, \quad H(\rho) = 1 + \frac{M}{\rho^2}, \quad L(u,\rho) = M^3 + \frac{t_0}{2}u\rho.$$

In the limit $\rho \to 0$ with $u =$ finite we have

$$ds^2 \to \frac{4M}{t_0}du d\rho + \frac{M^4}{t_0^2}d\Omega_{S^2}^2 + \frac{t_0^2}{M^2}dw^2, \quad A_\mu dx^\mu \to \pm\frac{\sqrt{3}}{2}\frac{t_0}{M}u d\rho,$$

regular !

Global structure

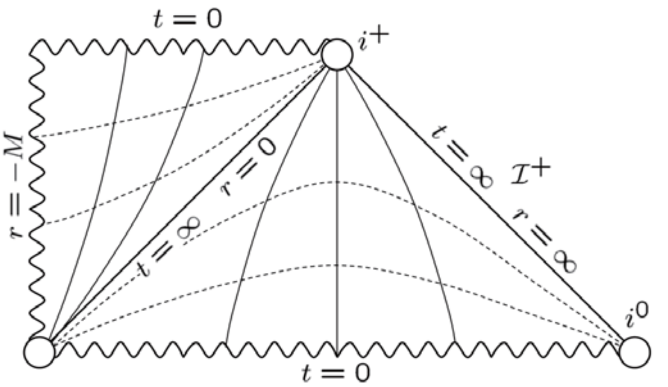
The metric

$$ds^2 = \frac{L(u,\rho)^2}{t_0 M^3 \rho^4 H(\rho)^2}\left(-\frac{t_0}{M^3}\rho^2 du^2 + 4du d\rho\right) + 4\frac{t_0^2\rho^6 H(\rho)^3 K(u,\rho) - L(u,\rho)^2}{t_0^2 \rho^6 H(\rho)^2} d\rho^2$$
$$+\rho^2 H(\rho)\left[K(u,\rho) d\Omega_{S^2}^2 + K(u,\rho)^{-1} dw^2\right],$$

(

provides the analytic extension.

The Penrose diagram :



The metric describes charged black string in a KK universe.

Staticity near horizon

we consider the limit $r \to 0_+$, $t \to \infty$ keeping $rt \to$ finite

$$ds^2 = -\frac{r^2}{M^2}dt^2 + \frac{tM}{t_0 r}dr^2 + \frac{trM}{t_0}d\Omega^2_{\mathrm{S}^2} + \frac{t_0 M}{tr}dw^2,$$

Introducing coordinates and constants,

$$R^2 = \frac{Mtr}{t_0}, \quad R_h^2 = \frac{M^4}{t_0^2}, \quad dT = \frac{t_0}{tM^2}dt + \frac{2R_h}{R\left(R^2 - R_h^2\right)}dR, \quad W = Mw,$$

the metric has the form of

$$ds^2 = -R^2\left(R^2 - R_h^2\right)dT^2 + \frac{4R^2}{R^2 - R_h^2}dR^2 + R^2 d\Omega^2_{\mathrm{S}^2} + \frac{dW^2}{R^2},$$

We see that

the spacetime has non-degenerate horizon, and
the size of horizon is constant during evolution of the universe.

Motion of a Test Particle

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Test particles

Lagrangian of a test particle around the black string is

$$\mathcal{L} = \frac{1}{2}\left[-H^{-2}\dot{t}^2 + HV\dot{r}^2 + HVr^2\dot{\theta}^2 + HVr^2\sin^2\theta\dot{\phi}^2 + HV^{-1}\dot{w}^2\right].$$

Conserved quantities are (The dot denotes derivative w.r.t. the proper time.)

$$L = HVr^2\sin^2\theta\dot{\phi}, \quad p_w = HV^{-1}\dot{w}.$$

Concentrate on a particle with $\theta = \pi/2, \ p_w = 0$, we have

$$H^{-1}V\dot{r}^2 + U_{\text{eff}} = E^2, \quad E = H^{-2}\dot{t}.$$

$$U_{\text{eff}} = H^{-2}\left(1 + \frac{L^2}{HVr^2}\right) = \frac{r\left[L^2 t_0 + rt(M+r)\right]}{t(M+r)^3}.$$

U_{eff} is time dependent, and E is not conserved.

There is no circular orbit !

Physical length

At a large distance, the 3-dimensional metric becoms

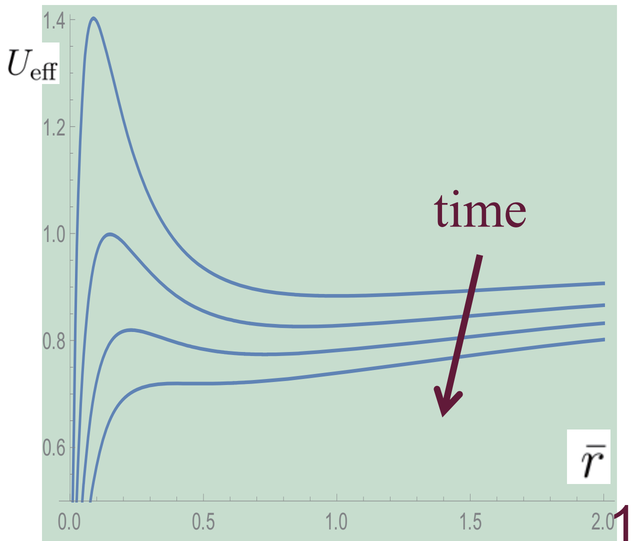
$$ds^2 = -dt^2 + V(t)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right)$$
$$V = \frac{t}{t_0}.$$

The physical length is given by

$$\bar{r} = \sqrt{V(t)}\; r = \sqrt{t/t_0}\; r\,.$$

Effective potential as the function of physical radius becomes

$$U_{\rm eff}(\bar{r}, t) = \frac{\bar{r}\left[L^2 + \bar{r}(\sqrt{t}M + \bar{r})\right]}{(\sqrt{t}M + \bar{r})^3}.$$



Quasi circular orbits

At a late stage, the cosmological expansion

$$\frac{\dot{a}(t)}{a(t)} = \frac{1}{2t}$$ becomes small.

Then, the scale factor $a(t) = \sqrt{V(t)}$ can be considered as a constant during an orbital motion of a particle at the late stage.

At the late stage, for a particle motion of small duration from a time $t = t_1$, we set $V = V(t_1)$.

Under this assumption we can find quasi circular orbits by

$$U_{\text{eff}} - E = 0, \;\; U'_{\text{eff}} = 0.$$

The radius of quasi circular orbit is

$$\bar{r}_c = \frac{L^2}{M\sqrt{t/t_0}} \quad .$$

Kepler's 3rd. law

Then, the period of the particle is

$$T^2 = 4\pi^2 \left(\frac{dt}{d\phi}\right)^2 = 2\pi^2 \frac{t^2(M+r_c)^3(M+2r_c)}{tMr_c t_0}$$

$$\xrightarrow{\text{large radius}} 4\pi^2 \frac{(t/t_0) r_c^3}{M} = 4\pi^2 \frac{\bar{r}_c^3}{\sqrt{t/t_0}M}.$$

We can define effective mass by Kepler's 3rd law in the form,

$$(GM)_{\text{eff}} := 4\pi^2 \frac{\bar{r}_c^3}{T^2} = \sqrt{t/t_0}GM.$$

Quasi ISCO

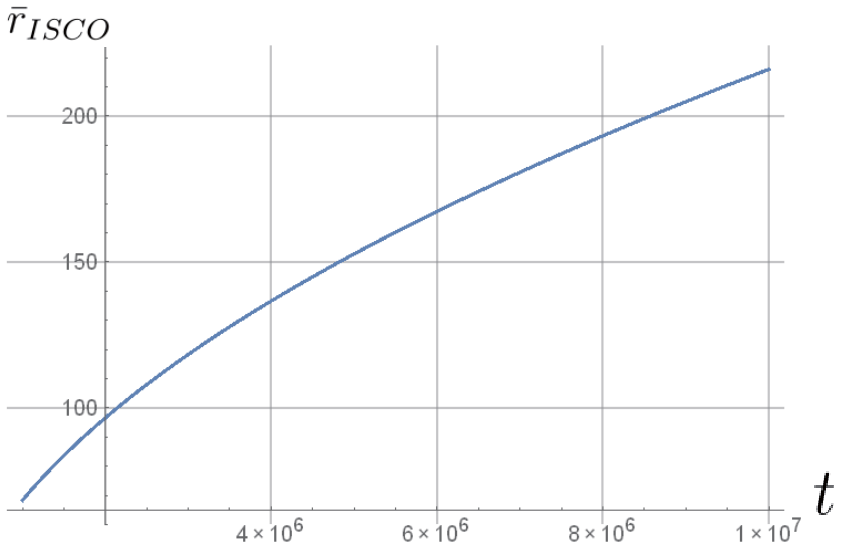
Quasi Innermost Stable Circular Orbits are determined by

$$U_{\rm eff} - E = 0,\ \ U'_{\rm eff} = 0,\ \ U''_{\rm eff} = 0$$

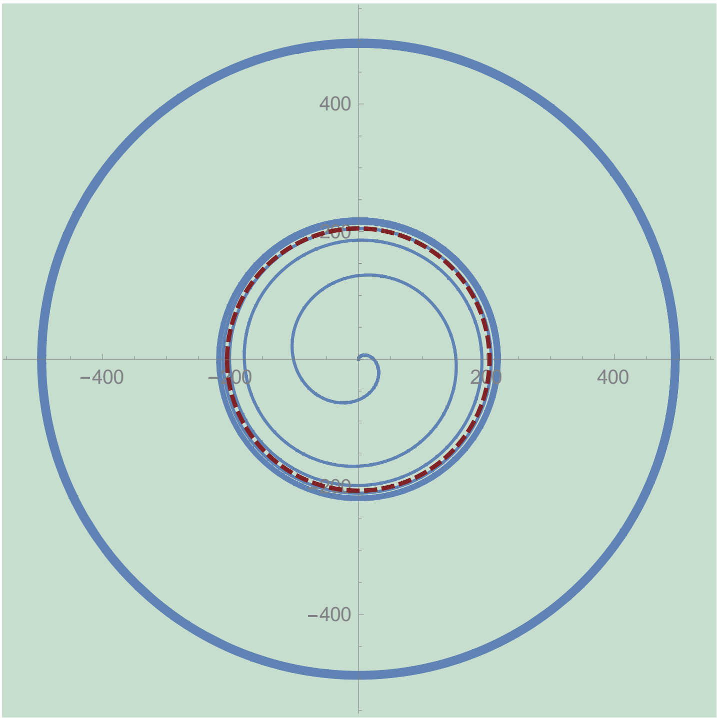
$$r_{ISCO} = \frac{1}{2}(1+\sqrt{3})M\,.$$

The physical radius is time dependent in the form

$$\bar{r}_{ISCO} = \sqrt{t/t_0}\,r_{ISCO} = \frac{1}{2}(1+\sqrt{3})\sqrt{t/t_0}\,M = 1.366 M_{\rm eff}\,.$$



Numerical plot of orbits



$$M = \frac{1}{20}$$

$$\frac{t}{t_0} = 10,000,000 - 11,000,000$$

$$\bar{r}_{\text{init}} = 500$$

$$E_{\text{init}} = 0.89, \quad L = 351.6$$

 $\bar{r}_{\text{ISCO}} = 216$

Conclusion

Geometry of the spacetime

- We study exact solutions which describe charged black strings in a dynamical Kaluza-Klein universe.
- The metric is analytic at the horizon.
- The spacetime admits no timelike Killing vector, but the size of horizon is constant.

Motion of a test particle

- There exist quasi circular orbits which are shrinking gradually.
- Kepler's 3rd law almost holds.
- Quasi ISCO can be defind. It increases as time.

“Polarization of photons around black holes in non-minimally coupled Einstein-Maxwell theory”

Daisuke Nitta (Nagoya)

[JGRG24(2014)P28]

Polarization of photons around black holes in non-minimally coupled Einstein-Maxwell theory

JGRG24 @IPMU

Daisuke Nitta (Nagoya University)

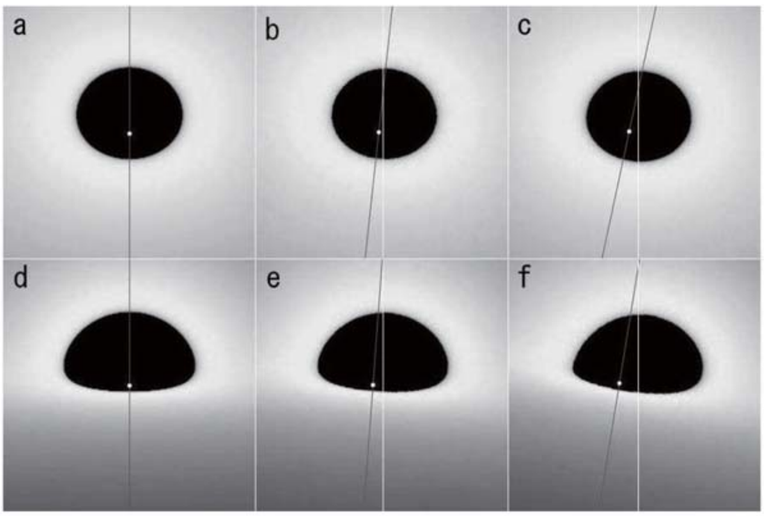
MOTIVATION

Black hole observation as a test of GR.

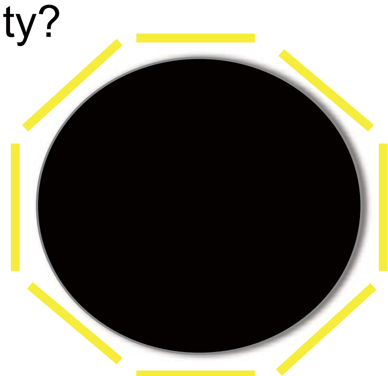
- Sg A* is the most promising target for the direct observation of black holes
- An observable wave length region is sub-mm radio wave.

→ polarizations are observed simultaneously

Black hole shadow (Rohta Takahashi '04)



Does polarization have new information about a theory of gravity?



NON-MINIMALLY COUPLED EINSTEIN-MAXWELL THEORY

(e.g. Drummond-Hathrell effective action, Horndeski vector-tensor theory)

Lagrangian

$$\mathcal{L} = \frac{m_{pl}^2}{2} R + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \boxed{\frac{\alpha}{4} R_{\rho\mu\sigma\nu} F^{\rho\mu} F^{\sigma\nu}}$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu$$

Field equations

$$\begin{cases} A^{\mu;\nu}{}_{;\nu} = \alpha R^{\mu\nu\rho\sigma} A_{\sigma;\nu;\rho}. \\ A^{\mu}{}_{;\mu} = 0 \end{cases}$$

(Lorentz gauge)

with R>>F^2, and we will use a vacuum solution of the Einstein equation

Today the parameter α is constrained by the solar system as α<1.1×10^20 cm^2 (Prasanna&Subhendra 2003)

WKB APPROXIMATION

To derive the equation of motion for a photon, we assume that the Radius of curvature is much greater than photon's wave length. Then the solution of the field equation is given by

$$A^\mu = a^\mu e^{i\Theta/\epsilon}, \quad \partial_\mu \Theta = p_\mu \ , \qquad \varepsilon \ll 1$$

· Minimal coupling (α=0)

Maxwell equations in WKB approximation up to second-order give well known relations

1st order: $p^\mu p_\mu = 0 \Rightarrow p^\nu \nabla_\nu p^\mu = 0,$ Geodesic equiation

2nd order: $a^{\mu;\nu} p_\nu + \frac{1}{2} a^\mu p^\nu{}_{;\nu} = 0$

$$\Downarrow$$

$$\begin{cases} \nabla_\mu (a^2 p^\mu) = 0 & \text{Photon number conservation} \\ p^\nu \nabla_\nu f^\mu = 0, & \text{Parallel transport of polarization vector} \end{cases}$$

$(a^2 \equiv a^\mu a_\mu,\ f^\mu \equiv a^\mu / a)$

WKB APPROXIMATION

・Non-minimal coupring

We obtain

1st order $a^{\mu} p^{\nu} p_{\nu} = \alpha R^{\mu\nu\rho\sigma} p_{\nu} p_{\rho} a_{\sigma},$

⇒ ・Violation of the Equivalence principle

・Birefringence (Drummond&Hathrell 1979)

2nd order $a^{\mu;\nu} p_{\nu} + \frac{1}{2} a^{\mu} p^{\nu}{}_{;\nu} = \frac{1}{2} \alpha R^{\mu\nu\rho\sigma} (p_{\rho;\nu} a_{\sigma} + p_{\rho} a_{\sigma;\nu}),$

⇒ Generation of polarization

We assume α satisfies αR << 1

GENERATION OF POLARIZATION

Introduce null tetrad

Polarization vector

$$e_+{}^{\mu} \equiv p^{\mu}, \quad e_-{}^{\mu} \equiv l^{\mu}, \quad e_A{}^{\mu}, \quad (A, B = 1, 2),$$

$$\eta_{ab} \equiv e_a{}^{\mu} e_{b\mu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{matrix} + \\ - \\ 1 \\ 2 \end{matrix}$$

Stokes parameters

$$\langle a^{\alpha} a^{\beta} \rangle = \langle a^A a^B \rangle e_A{}^{\alpha} e_B{}^{\beta},$$

$$\langle a^A a^B \rangle = I\delta^{AB} + V\omega^{AB} + Q\chi^{AB} + U\psi^{AB},$$

where

$$\delta^{AB} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \omega^{AB} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\chi^{AB} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \psi^{AB} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

GENERATION OF POLARIZATION

$$p^\nu \nabla_\nu a^\mu + \frac{1}{2}\theta a^\mu = \frac{\alpha}{2} R^{\mu\nu\rho\sigma}(p_{\rho;\nu} a_\sigma + a_B p_\rho e^B_{\sigma;\nu} + a_{B,\nu} p_\rho e^B{}_\sigma),$$

where θ denotes the expansion

Introducing Ricci rotation coefficients

$$\gamma_{abc} = \gamma_{[ab]c} \equiv e_a{}^\mu e_{b\mu;\nu} e_c{}^\nu,$$

$$\gamma_{ab} \equiv \gamma_{a+b} = \gamma_{(ab)}, \quad \gamma_a \equiv \gamma^b{}_{ab}$$

Ricci rotation coefficients are determined by the following equations

$$\frac{d\gamma_{abc}}{d\lambda} = -\gamma_{abd}\gamma^d_c - R_{abc+},$$

(of course the equations for γ_{A+B} are equivalent to
the Raychaudhuri equations)

GENERATION OF POLARIZATION

we obtain

$$\frac{da^A}{d\lambda} + \frac{\theta}{2} a^A = \frac{\alpha}{2} D^A{}_B a^B,$$

$$D^A{}_B \equiv R^A{}_{aBb} \gamma^{ab} + R^A{}_{a+b\gamma B}{}^{ba} + \frac{1}{2} R^A{}_{aB+} \gamma^a,$$

Above equations can be solved approximately
(note that we assume αR<< 1).

$$\langle a_A a_B \rangle(\lambda) = I(\lambda) \left[\delta_{AB} + \alpha \int d\lambda D_{(AB)} \right]$$

where, ***I*** denotes intensity given by the homogeneous solution of the above equations.

$$I(\lambda) = a^2(0) e^{-\int d\lambda \theta},$$

POLARIZATION IN SCHWARZSCHILD SPACE-TIME

The Schwarzschild metric is given by (G=1)

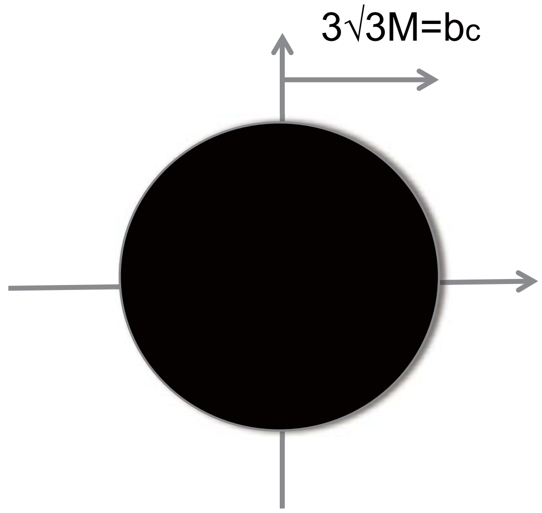
$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

At the first order of α, the photons orbits are regarded as null geodesics.

$$\left(\frac{dr}{d\lambda}\right)^2 = 1 - \frac{b^2}{r^2}(1 - \frac{2M}{r}), \qquad \frac{d\phi}{d\lambda} = \frac{b}{r^2},$$

b: impact parameter

$b_c=3\sqrt{3}M$: critical impact parameter



A black hole shadow in a celestial coordinate

POLARIZATION IN SCHWARZSCHILD SPACE-TIME

Riemann tensor is given by using bivectors as

$$R_{acbd} = \frac{M}{r^3}\left[\eta_{ab}\eta_{cd} - \eta_{ad}\eta_{cb} + 3(U_{ac}U_{bd} - V_{ac}V_{bd})\right],$$

$$U_{ab} = {e_a}^0{e_b}^1 - {e_b}^0{e_a}^1, \quad V_{ab} = r^2\sin\theta({e_a}^2{e_b}^3 - {e_b}^2{e_a}^3),$$

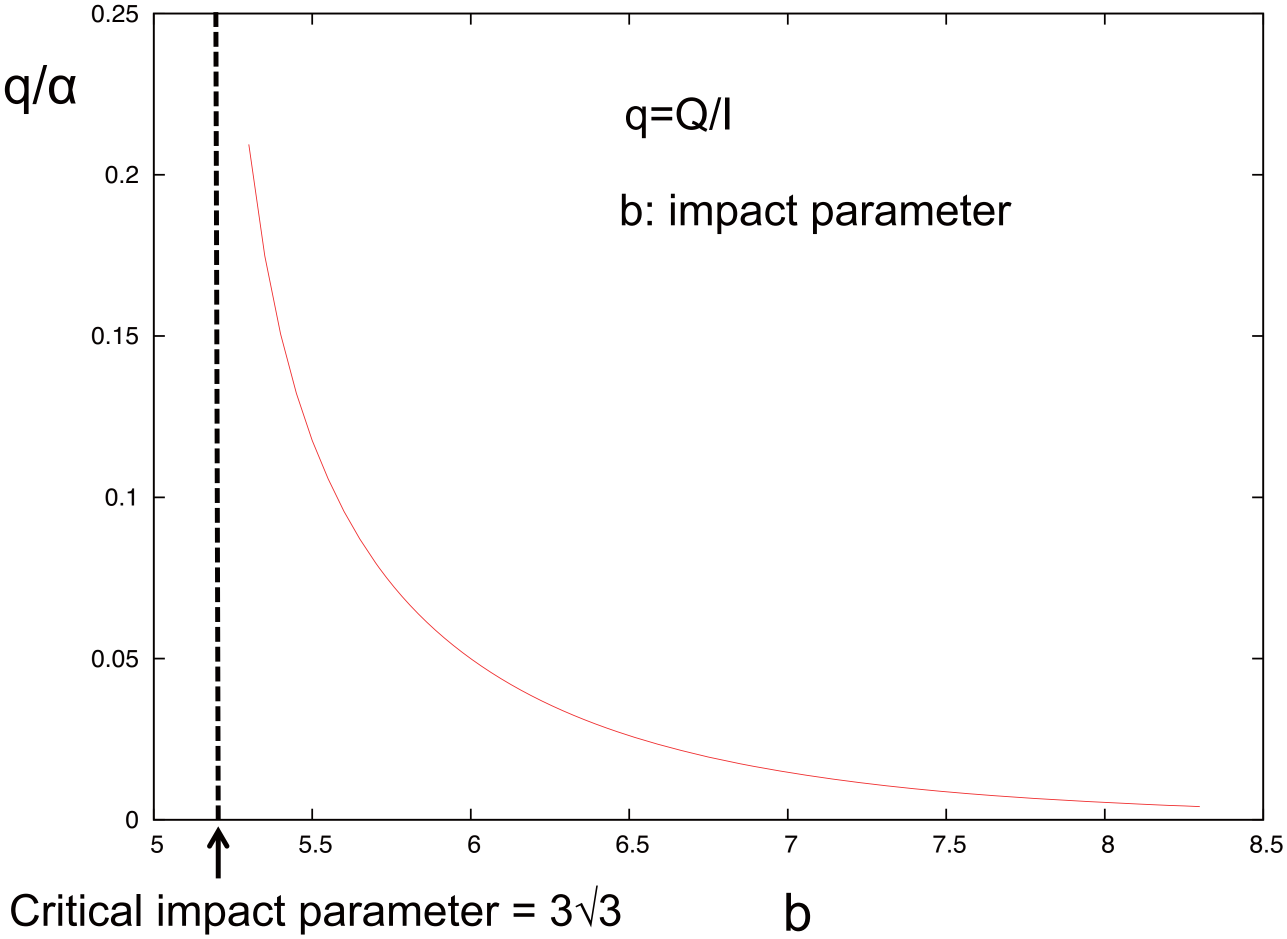
We obtain

shear ↓ expansion ↓

$$D_{AB} = -\frac{M}{r^3}\left[(1-3\frac{e_{13}^2}{r^2})\sigma + (1-3\frac{bl_3}{r^2})\theta + 3\frac{be_{13}}{r^2}\gamma_{1-} - 3\frac{b^2}{r^2}(\gamma_- + \gamma_{--})\right]\chi_{AB},$$

Then we compute the photon polarization.

RESULT


q/α
0.25
0.2
0.15
0.1
0.05
0
5
5.5
6
6.5
7
7.5
8
8.5
q=Q/I
b: impact parameter
Critical impact parameter = 3√3
b

SUMMARY

- We obtain the geodesic equation and for non-minimally coupled photons.
- We compute the polarization around Schwartzshild black hole.
- We would like to emphasis that these polarized photons has no wave length dependence.
- the polarization around Sgr A* may be contaminated by synchrotron radiation. According to Bower et al. 1999, this degree of polarization is ~0.01. This corresponds to α~10^20 cm^2, however, these are distinguishable from wave length dependence.

“Multi-black holes on Kerr-Taub-bolt space in five-dimensional Einstein-Maxwell theory”

Ken Matsuno (Osaka City)

[JGRG24(2014)P29]

Multi-black holes on Kerr-Taub-bolt space in 5D Einstein-Maxwell theory

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Introduction

- String theory, brane world models:

Studies of higher-dim. black holes, rings, strings, branes, ...

- Construction of more and more general solutions (charges, horizon topologies, asymptotic structures, ...)
- Physical properties of solutions (geodesics, stabilities, thermodynamics, uniqueness, ...)
- Applying to our spacetime (mini-black holes, classical tests, black hole shadows, ...)

We construct extremal charged static multi-black holes on Kerr-Taub-bolt space in 5D Einstein-Maxwell theory

- Five-dimensional Einstein-Maxwell theory

$$R_{\mu\nu} = 2\left(T_{\mu\nu} - \frac{T}{3}g_{\mu\nu}\right), \quad \nabla_\mu F^{\mu\nu} = 0$$

- Five-dimensional static exact solution

$$\begin{cases} ds^2 = -H(x^i)^{-2}dt^2 + H(x^i)ds^2_{\text{Ricci flat}} \\ \boldsymbol{A} = \pm\frac{\sqrt{3}}{2}H(x^i)^{-1}dt \\ \Delta_{\text{Ricci flat}}H(x^i) = 0 \quad \text{: Laplace's equation} \end{cases}$$

✓ harmonic function $H(x^i)$ with point sources

⇒ extremal charged multi-black holes

$ds_{\text{Ricci flat}}{}^2$ = (4D Kerr-Taub-bolt space) : new multi-BHs

5D multi-black holes on Kerr-Taub-bolt space

$$\begin{cases} ds^2 = -H(r,\theta)^{-2}dt^2 + H(r,\theta)ds_4^2 \\ \boldsymbol{A} = \pm\dfrac{\sqrt{3}}{2}H(r,\theta)^{-1}dt \end{cases}$$

- Four-dimensional Kerr-Taub-bolt space

$$\begin{aligned} ds_4^2 =& \Xi(r,\theta)\left[\frac{dr^2}{\Delta(r)} + d\theta^2\right] + \frac{\sin^2\theta}{\Xi(r,\theta)}\left[2\alpha\nu d\psi - (r^2-\nu^2-\alpha^2)d\phi\right]^2 \\ &+ \frac{\Delta(r)}{\Xi(r,\theta)}\left[2\nu d\psi + (2\nu\cos\theta + \alpha\sin^2\theta)d\phi\right]^2 \end{aligned}$$

$$\Delta(r) = r^2 - 2\mu r + \nu^2 - \alpha^2, \quad \Xi(r,\theta) = r^2 - (\nu - \alpha\cos\theta)^2 > 0$$

$$\begin{cases} \Delta(r_b) = 0,\ r_b = \mu + \sqrt{\mu^2 - \nu^2 + \alpha^2} > 0,\ \mu > \nu > 0,\ \alpha > 0 \\ -\infty < t < \infty,\ r_b < r < \infty,\ 0 \le \theta \le \pi,\ m_i > 0 \end{cases}$$

Asymptotic behaviors

- Far region $r \to \infty$:

$$ds^2 \to -dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) + 2\alpha\nu \sin^2\theta \cos\theta d\phi^2$$
$$+ 4\nu^2 \left(d\psi + \cos\theta d\phi\right)^2$$

Effectively four-dimensional spacetime

- Mass M, Charge Q :

Extremal charged solution

- Near horizon $r=r_b$, $\theta=0$: $(t, r, \theta, \phi, \Psi) \Rightarrow (v, \rho, \theta_N, \phi_N, \Psi_N)$

$(0 \leqq \theta_N \leqq \pi/2)$

$$ds^2 \simeq 4\sqrt{2(r_b-\mu)\left[r_b^2-(\nu-\alpha)^2\right]/m_1}\,dv\,d\rho$$
$$+\frac{2m_1\left[r_b^2-(\nu-\alpha)^2\right]}{r_b-\mu}\left(d\theta_N^2+\sin^2\theta_N d\phi_N^2+\cos^2\theta_N d\psi_N^2\right)$$

➢ $\rho=0$ ($r=r_b$, $\theta=0$) : smooth, round Killing horizon

✓ Regularity conditions

$\theta_N=0$: identify $\phi_N \sim 2\pi$ along Ψ_N =const.

$\theta_N=\pi/2$: identify $\Psi_N \sim 2\pi$ along ϕ_N =const.

Equivalently,

identify $\phi \sim 2\pi$ along $\psi+\phi=$ const.

identify $\psi \sim 2\pi\frac{r_b^2-\nu^2-\alpha^2}{2\nu(r_b-\mu)}$ along $\frac{-2\alpha\nu}{r_b^2-\nu^2-\alpha^2}\psi+\phi=$ const.

- Near horizon $r=r_b$, $\theta=\pi$: $(t, r, \theta, \phi, \Psi) \Rightarrow (v', \rho', \theta_S, \phi_S, \Psi_S)$

$(0 \le \theta_S \le \pi/2)$

$$ds^2 \simeq 4\sqrt{2(r_b-\mu)\left[r_b^2-(\nu+\alpha)^2\right]}/m_2 dv' d\rho' + \frac{2m_2\left[r_b^2-(\nu+\alpha)^2\right]}{r_b-\mu}\left(d\theta_S^2+\sin^2\theta_S d\phi_S^2+\cos^2\theta_S d\psi_S^2\right)$$

➢ $\rho'=0$ ($r=r_b$, $\theta=\pi$) : smooth, round Killing horizon

✓ Regularity conditions

$\theta_S=0$: identify $\phi_S \sim 2\pi$ along Ψ_S =const.

$\theta_S=\pi/2$: identify $\Psi_S \sim 2\pi$ along ϕ_S =const.

Equivalently,

identify $\phi \sim 2\pi$ along $\psi - \phi =$ const.

identify $\psi \sim 2\pi \dfrac{r_b^2-\nu^2-\alpha^2}{2\nu(r_b-\mu)}$ along $\dfrac{-2\alpha\nu}{r_b^2-\nu^2-\alpha^2}\psi + \phi =$ const.

Regularity of 4D Kerr-Taub-bolt space

$$ds_4^2 = \Xi(r,\theta)\left[\frac{dr^2}{\Delta(r)} + d\theta^2\right] + \frac{\sin^2\theta}{\Xi(r,\theta)}\left[2\alpha\nu d\psi - (r^2-\nu^2-\alpha^2)d\phi\right]^2$$

$$+ \frac{\Delta(r)}{\Xi(r,\theta)}\left[2\nu d\psi + (2\nu\cos\theta + \alpha\sin^2\theta)d\phi\right]^2$$

$$\Delta(r) = r^2 - 2\mu r + \nu^2 - \alpha^2, \quad \Xi(r,\theta) = r^2 - (\nu - \alpha\cos\theta)^2 > 0$$

$$\Delta(r_b) = 0, \quad r_b = \mu + \sqrt{\mu^2 - \nu^2 + \alpha^2} > 0, \quad \mu > \nu > 0, \quad \alpha > 0$$

Killing vector field	fixed point
$\frac{\partial}{\partial\psi} - \frac{\partial}{\partial\phi}$	r=finite, θ=0
$\frac{\partial}{\partial\psi} + \frac{\partial}{\partial\phi}$	r=finite, $\theta=\pi$
$\frac{\partial}{\partial\psi} + \frac{2\alpha\nu}{r_b^2-\nu^2-\alpha^2}\frac{\partial}{\partial\phi}$	$r=r_b$
$\frac{\partial}{\partial\psi}, \quad \frac{\partial}{\partial\phi}$	$r=r_b$, $\theta=0,\pi$

- Near r=finite, θ=0 :

$$ds_4^2 \simeq \left[r^2 - (\nu-\alpha)^2\right] \left[\frac{dr^2}{\Delta(r)} + d\theta^2 + \theta^2 \left\{\frac{-2\alpha\nu}{r^2-(\nu-\alpha)^2} d(\psi+\phi) + d\phi\right\}^2\right]$$
$$+ \frac{4\nu^2\Delta(r)}{r^2-(\nu-\alpha)^2}\left[d(\psi+\phi)\right]^2$$

identify $\varphi \sim 2\pi$ along $\psi + \phi =$ const.

- Near r=finite, $\theta = \pi$:

$$ds_4^2 \simeq \left[r^2 - (\nu+\alpha)^2\right] \left[\frac{dr^2}{\Delta(r)} + d\theta^2 + (\theta-\pi)^2 \left\{\frac{-2\alpha\nu}{r^2-(\nu+\alpha)^2} d(\psi-\phi) + d\phi\right\}^2\right]$$
$$+ \frac{4\nu^2\Delta(r)}{r^2-(\nu+\alpha)^2}\left[d(\psi-\phi)\right]^2$$

- Near r=r_b : $R = \sqrt{\dfrac{2(r - r_b)}{r_b - \mu}}, \quad \chi = \dfrac{-2\alpha\nu}{r_b^2 - \nu^2 - \alpha^2}\psi + \phi$

$$ds_4^2 \simeq \Xi(r_b, \theta)\left[dR^2 + R^2\left\{\frac{r_b - \mu}{\Xi(r_b,\theta)}(2\nu\cos\theta + \alpha\sin^2\theta)d\chi + \frac{2\nu(r_b - \mu)}{r_b^2 - \nu^2 - \alpha^2}d\psi\right\}^2\right]$$
$$+\Xi(r_b,\theta)\left[d\theta^2 + \frac{(r_b^2 - \nu^2 - \alpha^2)^2}{\Xi(r_b,\theta)^2}\sin^2\theta d\chi^2\right]$$

pear-shaped bolt (R=0)

- Near arbitrary point on bolt R=0 (θ, χ=const.) :

$$ds_4^2 \to \Xi(r_b, \theta)\left[dR^2 + R^2 d\left(\frac{2\nu(r_b - \mu)}{r_b^2 - \nu^2 - \alpha^2}\psi\right)^2\right]$$

identify $\psi \sim 2\pi \dfrac{r_b^2 - \nu^2 - \alpha^2}{2\nu(r_b - \mu)}$ along $\chi = \dfrac{-2\alpha\nu}{r_b^2 - \nu^2 - \alpha^2}\psi + \phi = \text{const.}$

- Near θ=0 on bolt r=r_b :

$$ds_4^2 \simeq \left[r_b^2 - (\nu-\alpha)^2\right] \left[dR^2 + R^2 d\left(\frac{2\nu(r_b-\mu)}{r_b^2-(\nu-\alpha)^2}\chi + \frac{2\nu(r_b-\mu)}{r_b^2-\nu^2-\alpha^2}\psi \right)^2 \right.$$
$$\left. + d\theta^2 + \theta^2 d\left(\frac{r_b^2-\nu^2-\alpha^2}{r_b^2-(\nu-\alpha)^2}\chi \right)^2 \right]$$

$$\begin{cases} \text{identify } \dfrac{2\nu(r_b-\mu)}{r_b^2-(\nu-\alpha)^2}\chi + \dfrac{2\nu(r_b-\mu)}{r_b^2-\nu^2-\alpha^2}\psi \sim 2\pi \text{ along } \dfrac{r_b^2-\nu^2-\alpha^2}{r_b^2-(\nu-\alpha)^2}\chi = \text{const.} \\ \text{identify } \dfrac{r_b^2-\nu^2-\alpha^2}{r_b^2-(\nu-\alpha)^2}\chi \sim 2\pi \text{ along } \dfrac{2\nu(r_b-\mu)}{r_b^2-(\nu-\alpha)^2}\chi + \dfrac{2\nu(r_b-\mu)}{r_b^2-\nu^2-\alpha^2}\psi = \text{const.} \end{cases}$$

Equivalently,

$$\begin{cases} \text{identify } \psi \sim 2\pi \dfrac{r_b^2-\nu^2-\alpha^2}{2\nu(r_b-\mu)} \text{ along } \dfrac{-2\alpha\nu}{r_b^2-\nu^2-\alpha^2}\psi + \phi = \text{const.} \\ \text{identify } \phi \sim 2\pi \text{ along } \psi + \phi = \text{const.} \end{cases}$$

- Near $\theta = \pi$ on bolt $r = r_b$:

$$ds_4^2 \simeq \left[r_b^2 - (\nu+\alpha)^2\right] \left[dR^2 + R^2 d\left(-\frac{2\nu(r_b-\mu)}{r_b^2-(\nu+\alpha)^2}\chi + \frac{2\nu(r_b-\mu)}{r_b^2-\nu^2-\alpha^2}\psi \right)^2 \right.$$

$$\left. + d\theta^2 + (\theta-\pi)^2 d\left(\frac{r_b^2-\nu^2-\alpha^2}{r_b^2-(\nu+\alpha)^2}\chi \right)^2 \right]$$

Equivalently,

$$\text{identify } \psi \sim 2\pi \frac{r_b^2-\nu^2-\alpha^2}{2\nu(r_b-\mu)} \text{ along } \frac{-2\alpha\nu}{r_b^2-\nu^2-\alpha^2}\psi + \phi = \text{const.}$$

$$\text{identify } \phi \sim 2\pi \text{ along } \psi - \phi = \text{const.}$$

Required identifications in Kerr-Taub-bolt space

$$ds_4^2 = \Xi(r,\theta)\left[\frac{dr^2}{\Delta(r)} + d\theta^2\right] + \frac{\sin^2\theta}{\Xi(r,\theta)}\left[2\alpha\nu d\psi - (r^2-\nu^2-\alpha^2)d\phi\right]^2$$
$$+ \frac{\Delta(r)}{\Xi(r,\theta)}\left[2\nu d\psi + (2\nu\cos\theta + \alpha\sin^2\theta)d\phi\right]^2$$

$$\Delta(r) = r^2 - 2\mu r + \nu^2 - \alpha^2, \quad \Xi(r,\theta) = r^2 - (\nu - \alpha\cos\theta)^2 > 0$$

$$\Delta(r_b) = 0, \quad r_b = \mu + \sqrt{\mu^2 - \nu^2 + \alpha^2} > 0, \quad \mu > \nu > 0, \quad \alpha > 0$$

✓ Regularity conditions :

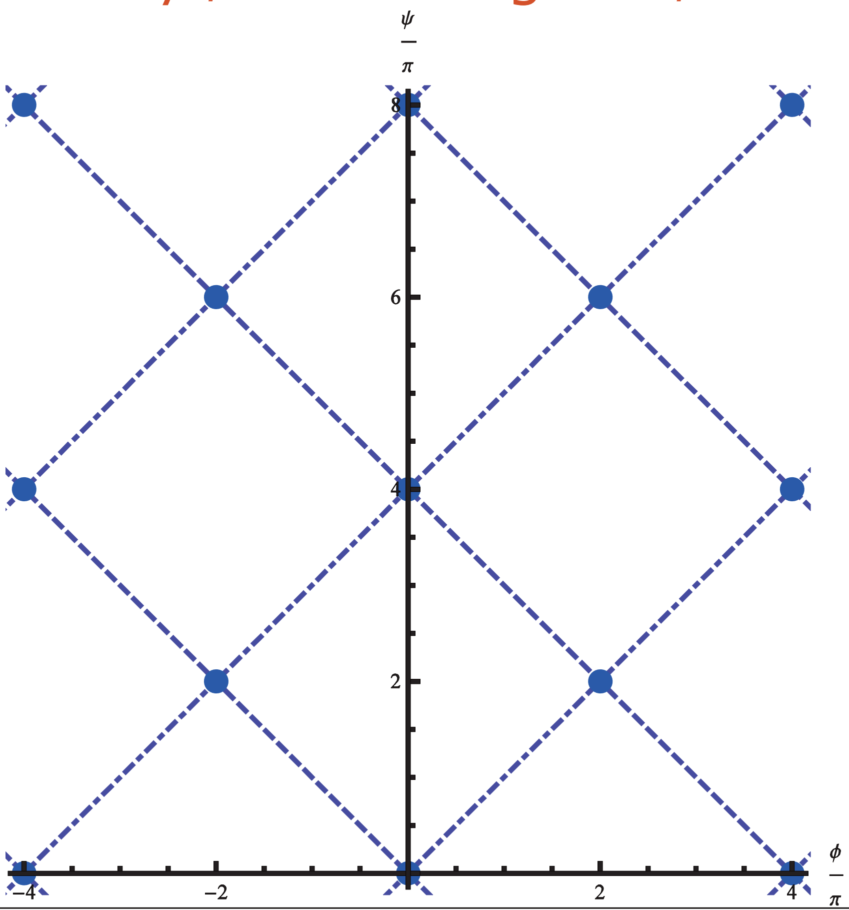
A. identify $\phi \sim 2\pi$ along $\psi + \phi = \text{const.}$

B. identify $\phi \sim 2\pi$ along $\psi - \phi = \text{const.}$

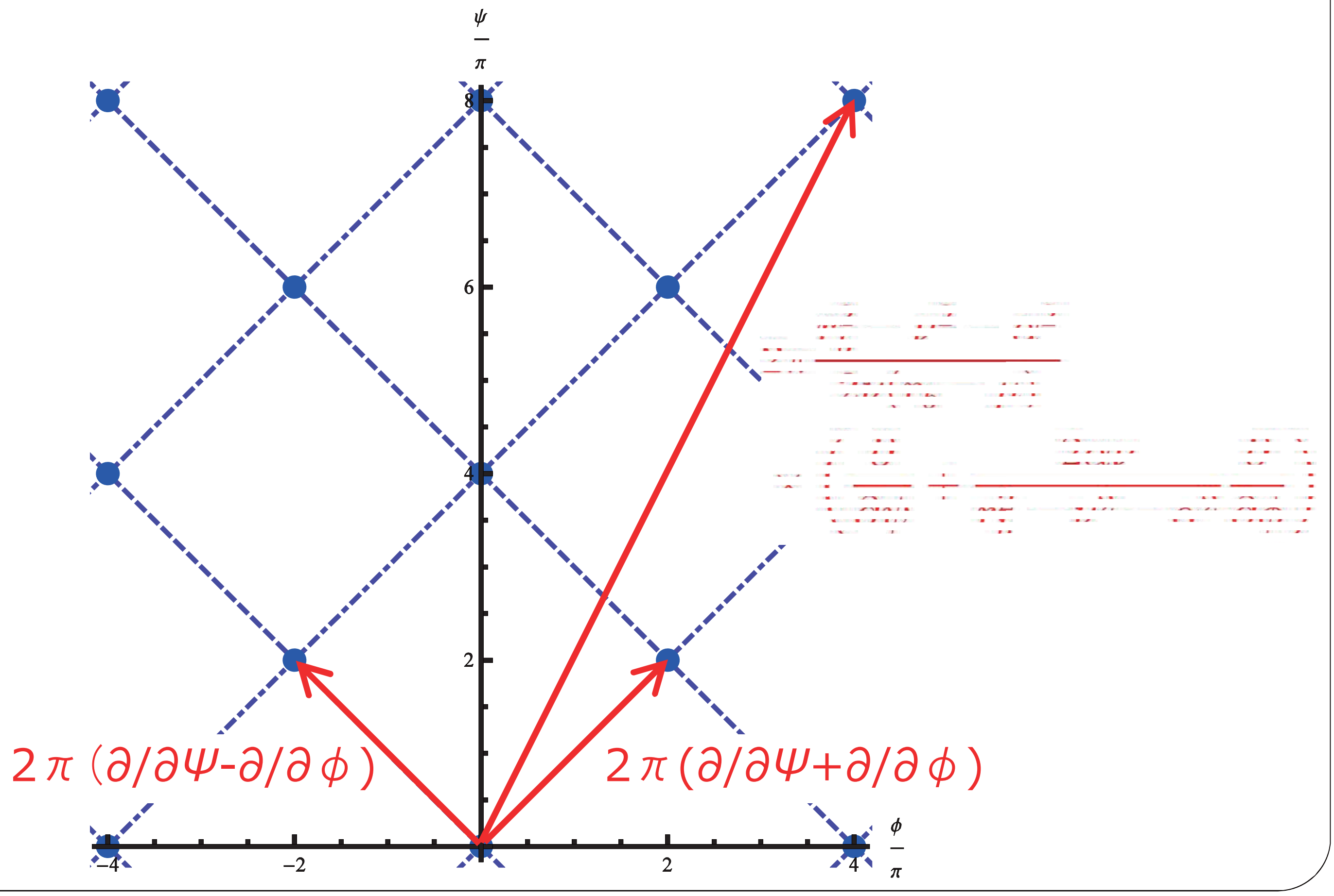
C. identify $\psi \sim 2\pi \dfrac{r_b^2 - \nu^2 - \alpha^2}{2\nu(r_b - \mu)}$ along $\dfrac{-2\alpha\nu}{r_b^2 - \nu^2 - \alpha^2}\psi + \phi = \text{const.}$

A. identify $\phi \sim 2\pi$ along $\Psi + \phi =$ const.

B. identify $\phi \sim 2\pi$ along $\Psi - \phi =$ const.



C. identify $\psi \sim 2\pi \dfrac{r_b^2 - \nu^2 - \alpha^2}{2\nu(r_b - \mu)}$ along $\dfrac{-2\alpha\nu}{r_b^2 - \nu^2 - \alpha^2}\psi + \phi = \text{const.}$



$$\text{C. identify } \psi \sim 2\pi \frac{r_b^2 - \nu^2 - \alpha^2}{2\nu(r_b - \mu)} \text{ along } \frac{-2\alpha\nu}{r_b^2 - \nu^2 - \alpha^2}\psi + \phi = \text{const.}$$

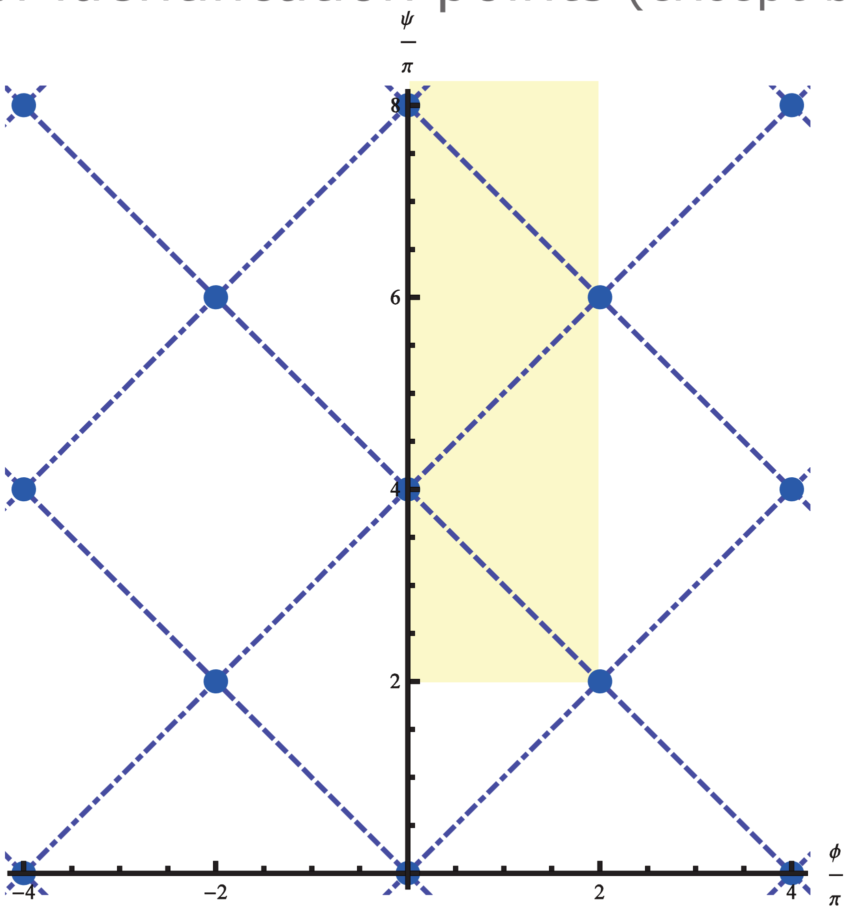
C. identify $(\phi,\ \psi)$:

$$(0,\ 0) \sim 2\pi \left(\frac{\alpha}{r_b - \mu}, \frac{r_b^2 - \nu^2 - \alpha^2}{2\nu(r_b - \mu)} \right) := 2\pi \left(\frac{q_+ - q_-}{p}, \frac{q_+ + q_-}{p} \right)$$

$$\begin{cases} \dfrac{\mu}{\nu} = \dfrac{p(q_+ + q_-) - 2\sqrt{q_+ q_- [p^2 - (q_+ - q_-)^2]}}{(q_+ - q_-)^2} < 1 \\[2ex] \dfrac{\alpha}{\nu} = \dfrac{2p}{q_+ - q_-}\sqrt{\dfrac{q_+ q_-}{p^2 - (q_+ - q_-)^2}} - \dfrac{q_+ + q_-}{q_+ - q_-} > 0 \end{cases}$$

$$\Rightarrow \quad 0 < q_- < q_+, \quad q_+ - q_- < p < q_+ + q_-$$

Region of identification points (except boundaries)



No regular S^3 topology Kerr-Taub-bolt space satisfying conditions A∼C simultaneously

Kerr-Taub-bolt space topology: Lens space $L(p;q)$

- Add identification points satisfying condition C

except on
- identification points satisfying conditions A, B
- orbits of $\partial/\partial\Psi \pm \partial/\partial\phi$

C. identify (ϕ, ψ) : $(0, 0) \sim 2\pi($[illegible], [illegible]$)$

relatively prime natural numbers p, q_+

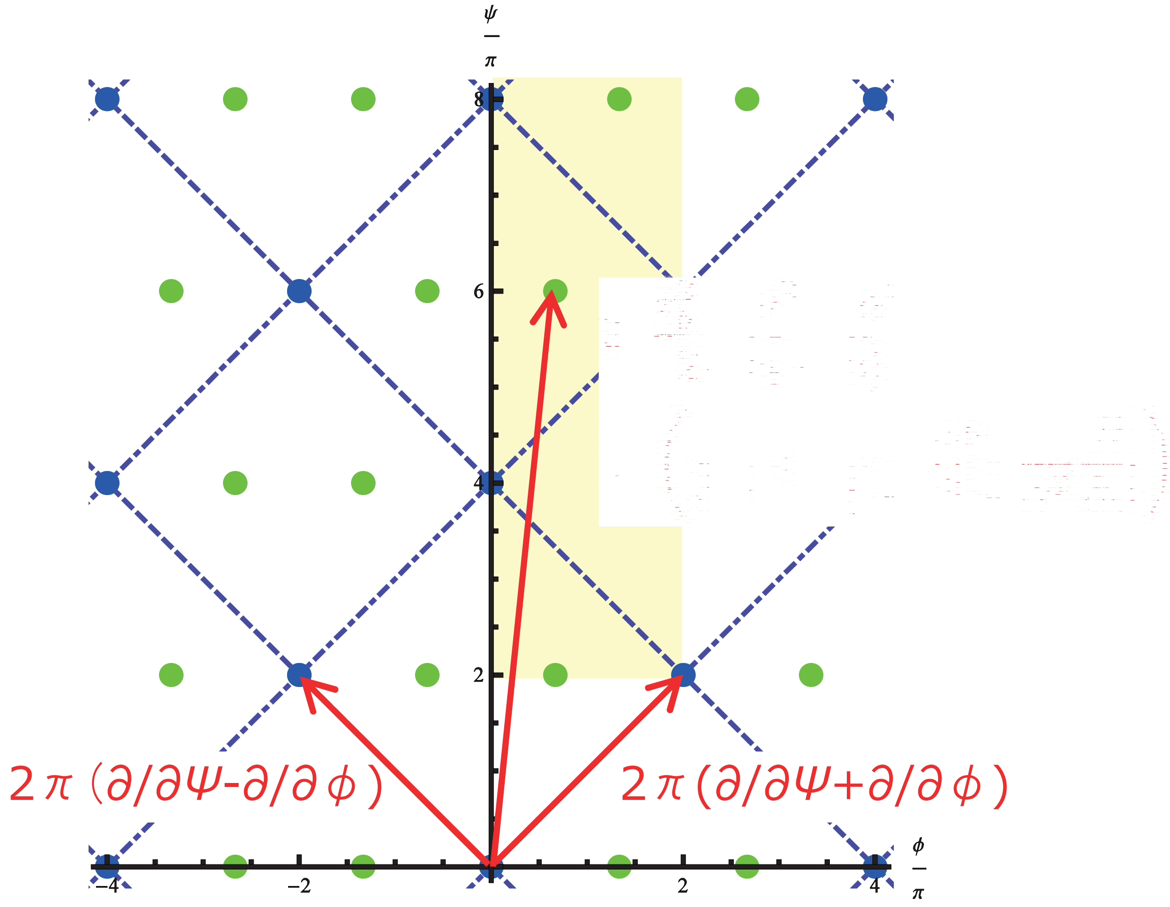
$(0 < q_- < q_+,\ q_+ - q_- < p < q_+ + q_-)$

➢ Adding identification points on orbits of $\partial/\partial\Psi$, $\partial/\partial\phi$ (fixed points $r=r_b$, $\theta=0,\pi$)

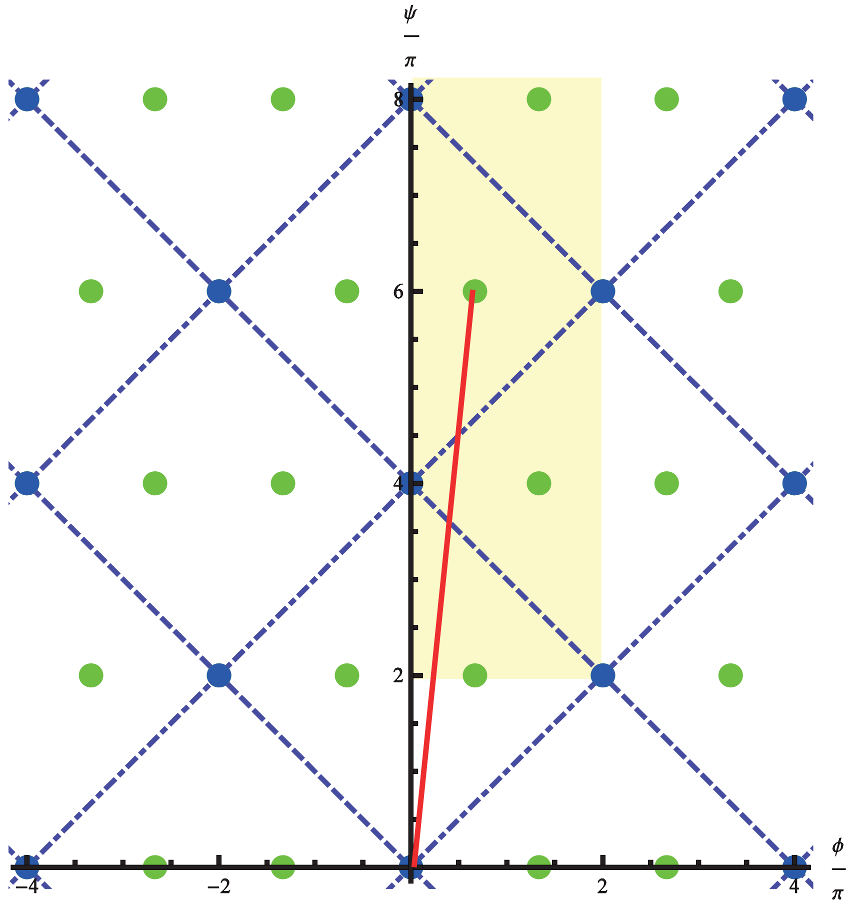
➢ Regular Kerr-Taub-bolt space except $\theta=0,\pi$ on bolt $r=r_b$

5D regular multi-BHs by putting BHs on poles of bolt

Example: Lens space $L(3;2)$ topology

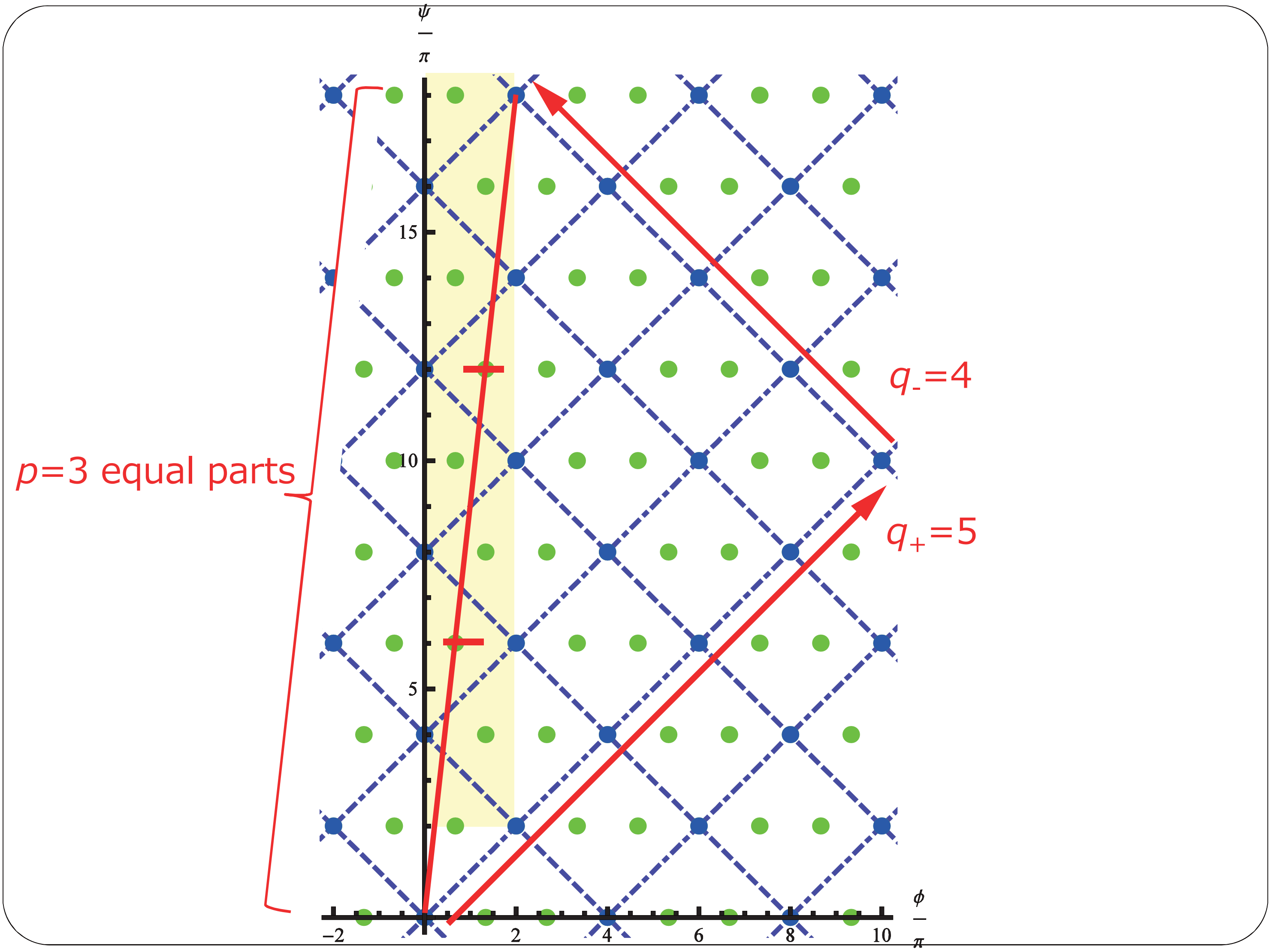


$L(3;2)$: C. identify $(\phi,\ \psi):\ (0,\ 0) \sim 2\pi\left(\frac{q_+ - q_-}{p},\ \frac{q_+ + q_-}{p}\right)$



$$2\pi\left(\frac{1}{3},\ 3\right)$$

$$\begin{cases} p = 3 \\ q_+ = 5 \\ q_- = 4 \end{cases}$$



ψ/π
15
10
5
−2
2
4
6
8
10
ϕ/π
p=3 equal parts
q-=4
q+=5

Topology of 4D Kerr-Taub-bolt space

- Lens space $L(p;q)$ (coprime natural numbers $p,q\neq 1$)

$$\text{division number } p, \quad q = \left| \frac{nq_+ - kp}{nq_- - lp} \right|$$

relatively prime natural numbers $p, q_\pm$, arbitrary integers n, k, l

- $p=3$, $q_+=5$, $q_-=4$:

$$q = \left| \frac{5n - 3k}{4n - 3l} \right| \xrightarrow{n=l=1} \left| \frac{5 - 3k}{1} \right| \xrightarrow{k=1} \left| \frac{2}{1} \right| = 2$$

- $p=3$, $q_+=7$, $q_-=5$:

[illegible]

- Topology: everywhere $L(3;2)$

Geometry of 4D Kerr-Taub-bolt space

- Topology $L(3;2)$: identification points satisfying condition C

[illegible]

✓ Various geometries are possible:

[illegible]

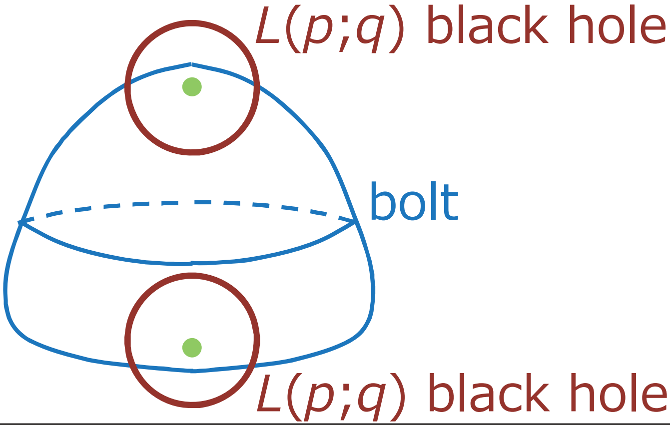
[illegible]

➢ $ds_4^2 = ds_4^2(v, p, q)$: Quantized 4D Kerr-Taub-bolt space

Summary

We construct multi-BHs on Kerr-Taub-bolt space in five-dimensional Einstein-Maxwell theory

- ✓ Far region: Effectively 4D spacetime
- ✓ Near horizon: 5D smooth black hole spacetime
- ✓ Topology: Everywhere lens space $L(p;q)$ (coprime natural numbers $p,q\neq 1$)



“Time Variability of an orbiting Hot Spot around a Black Hole”

Masaaki Takahashi (Aichi U. of Education)

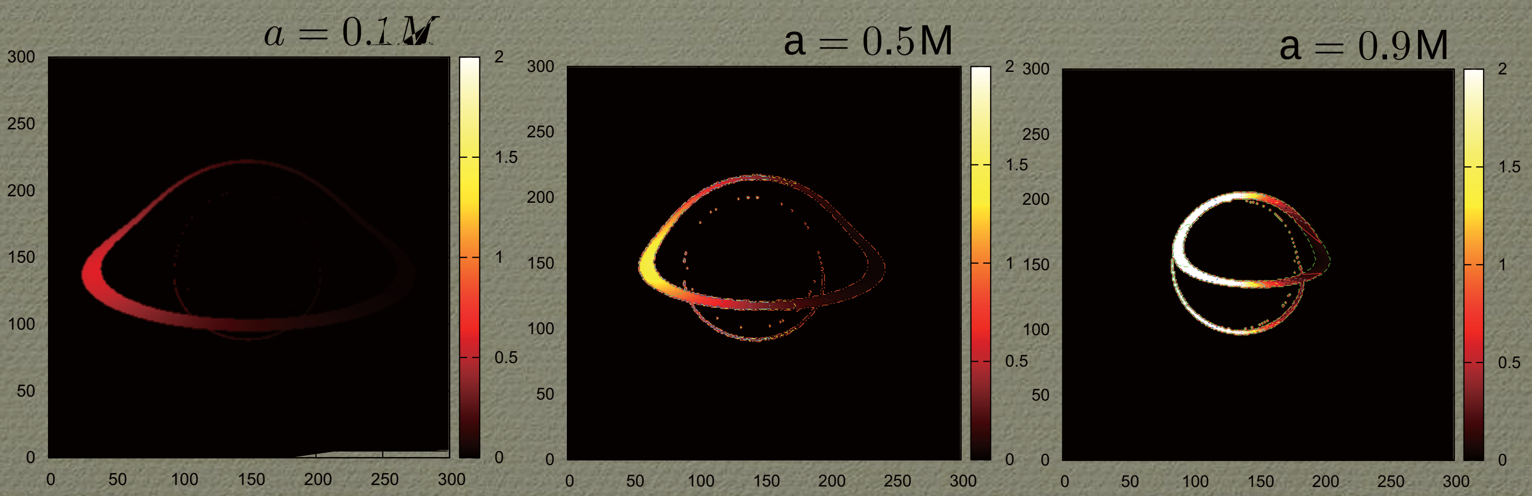
[JGRG24(2014)P30]

B11

JGRG2014@IPMU

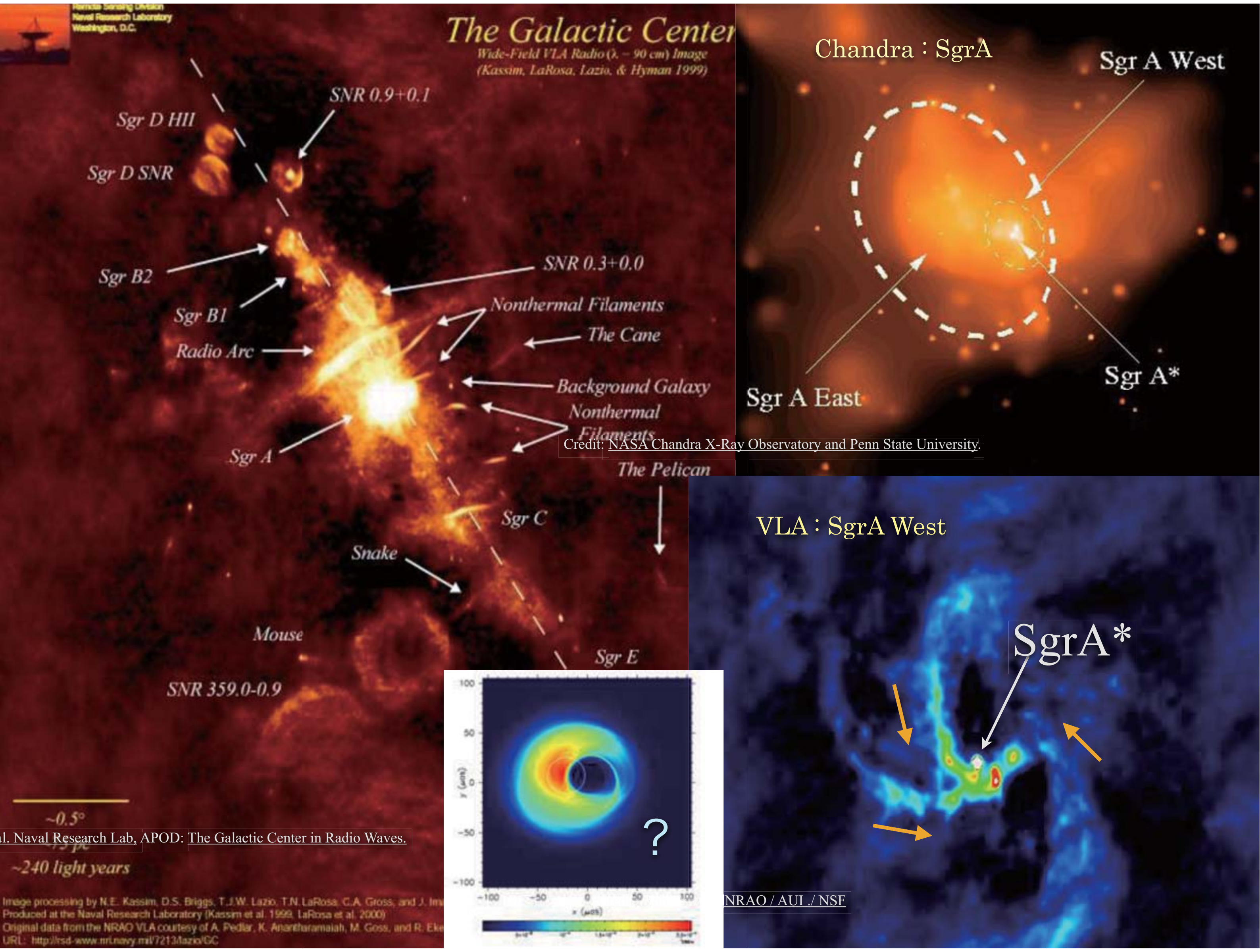
Time Variability of a orbiting Hot Spot around a Black Hole

Masaaki Takahashi (Aichi U. of Education)



Equatorial ring around Kerr BH $\theta = 0.4\pi$

To show the evidence of the super-massive black hole in our Galactic center (Sgr A* BH) by observations, I discuss time-variability of plasma surrounding the black hole. Here, I consider the emission from a hot spot orbiting around the black hole. For the Sgr A* BH, the accreting plasma onto the black hole is optically-thin, so we can observe the multiple images (emissions) from a hot spot. The rays from the hot spot are influenced by the general/special relativistic effects (i.e., the gravitational lens effect, gravitational redshift effect, Doppler beaming effect). By comparing the flux of the first and second images with the time-lag of two images, we can get some information of the black hole space-time. Thus, we can expect that more careful observations by sub-mm VLBI and/or X-ray can probe the existence of the black hole.


The Galactic Center
Wide-Field VLA Radio (λ ~ 90 cm) Image
(Kassim, LaRosa, Lazio, & Hyman 1999)
Sgr D HII
Sgr D SNR
SNR 0.9+0.1
Sgr B2
Sgr B1
SNR 0.3+0.0
Nonthermal Filaments
The Cane
Radio Arc
Background Galaxy
Nonthermal Filaments
Sgr A
The Pelican
Sgr C
Snake
Mouse
Sgr E
SNR 359.0-0.9
~0.5°
~240 light years
l. Naval Research Lab, APOD: The Galactic Center in Radio Waves.
Image processing by N.E. Kassim, D.S. Briggs, T.J.W. Lazio, T.N. LaRosa, C.A. Gross, and J. Im
Produced at the Naval Research Laboratory (Kassim et al. 1999, LaRosa et al. 2000)
Original data from the NRAO VLA courtesy of A. Pedlar, K. Anantharamaiah, M. Goss, and R. Eke
URL: http://rsd-www.nrl.navy.mil/7213/lazio/GC
Chandra : SgrA
Sgr A West
Sgr A East
Sgr A*
Credit: NASA Chandra X-Ray Observatory and Penn State University.
VLA : SgrA West
SgrA*
NRAO / AUI / NSF
?

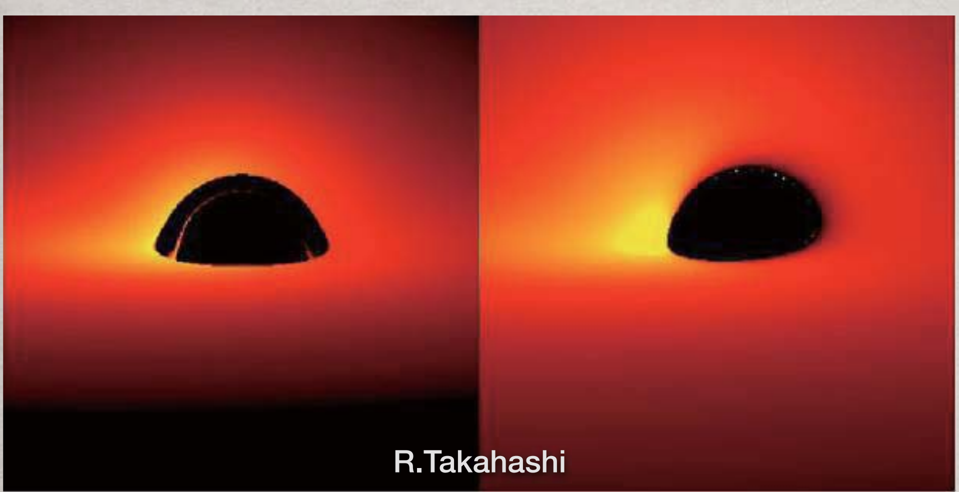
How can we see the Central Region of Galaxy ?

<Theoretical Model>

R.Takahashi & M.Takahashi

Black Hole Shadow with Disk

"Mirage of the space"



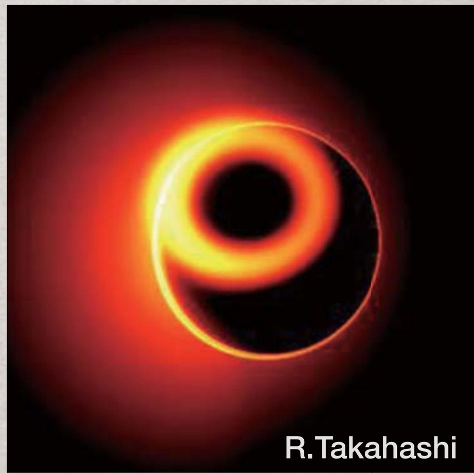
non-rotating BH   rotating BH

For a light bending effect by the BH, it seems that the other side of the thin disk rises.

Hot plasma by MHD Shock

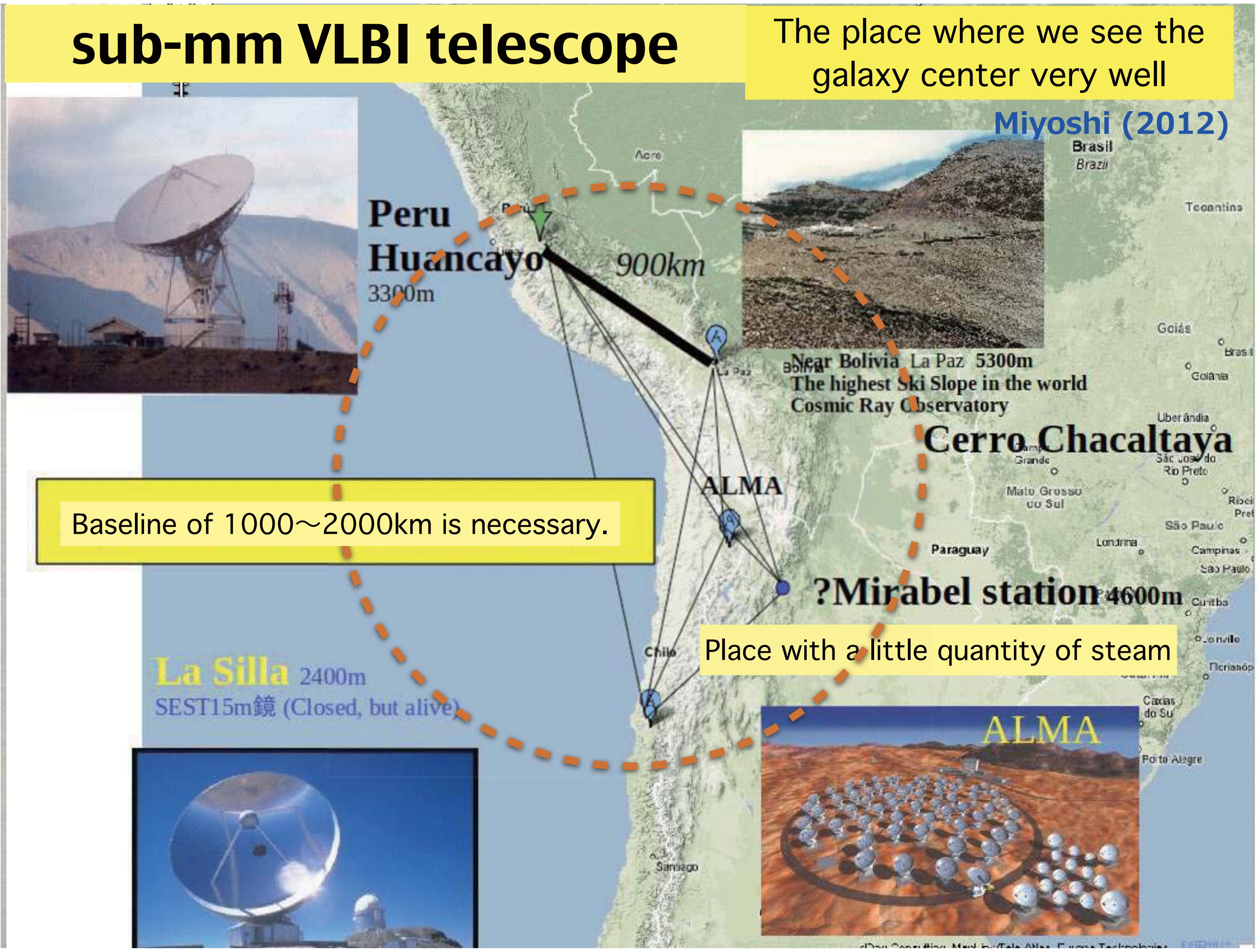
"Black Hole Aurora"

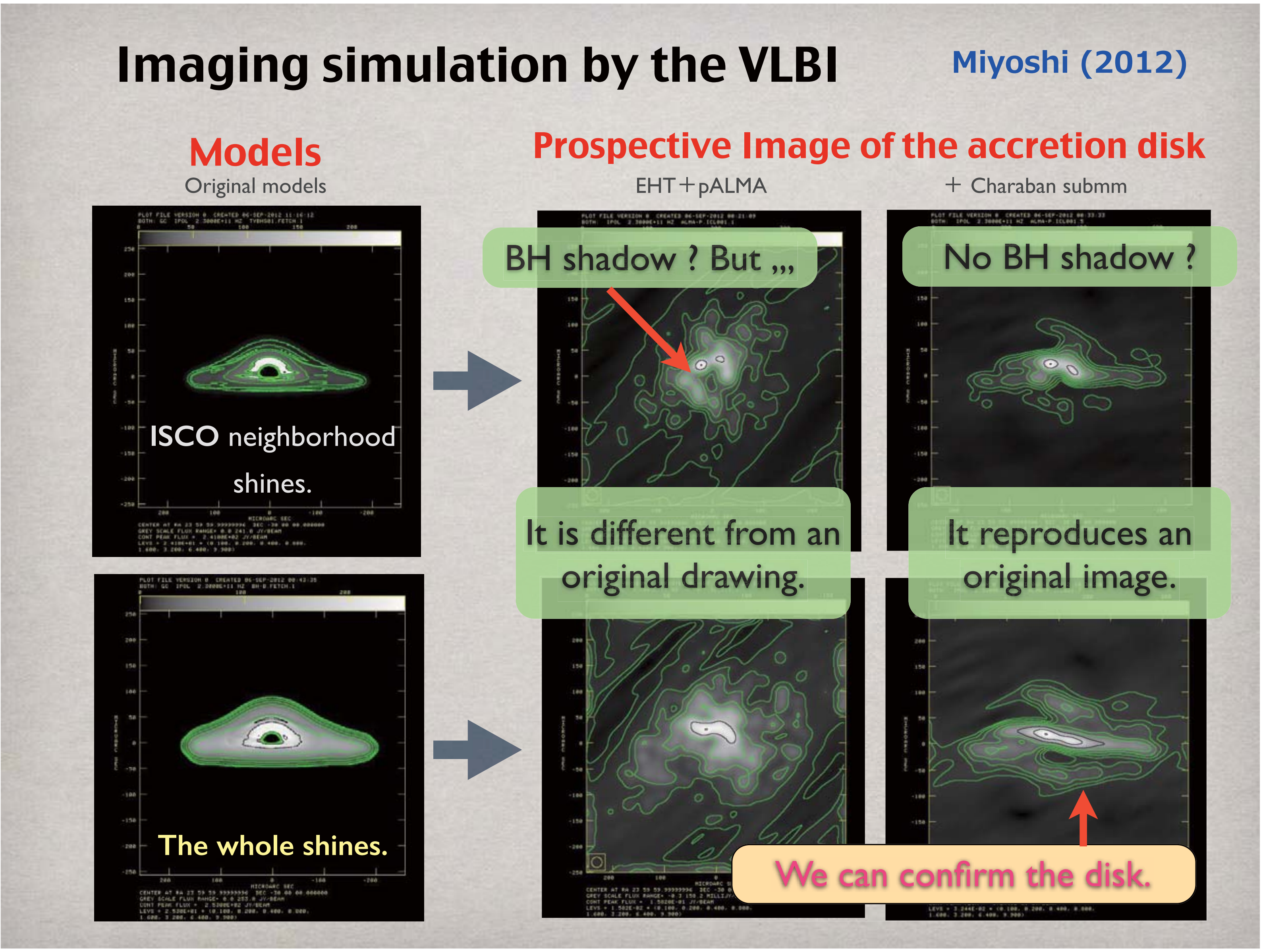
+



in a BH Magnetosphere

Fast-Magnetosonic shock can occur near the event horizon


sub-mm VLBI telescope
The place where we see the galaxy center very well
Miyoshi (2012)
Peru
Huancayo
3300m
900km
Near Bolivia La Paz 5300m
The highest Ski Slope in the world
Cosmic Ray Observatory
Cerro Chacaltaya
ALMA
Baseline of 1000～2000km is necessary.
?Mirabel station 4600m
Place with a little quantity of steam
La Silla 2400m
SEST15m鏡 (Closed, but alive)
ALMA


Imaging simulation by the VLBI
Miyoshi (2012)
Models
Original models
Prospective Image of the accretion disk
EHT＋pALMA
＋ Charaban submm
BH shadow ? But ,,,
No BH shadow ?
ISCO neighborhood shines.
It is different from an original drawing.
It reproduces an original image.
The whole shines.
We can confirm the disk.

Black Hole exploration

1。Orbit Determination of the S2 star

2。More inside ? …. Direct image of the BH Shadow

3。Time variability of accreting gases !

Prior to the observation of the BH shadow,,,

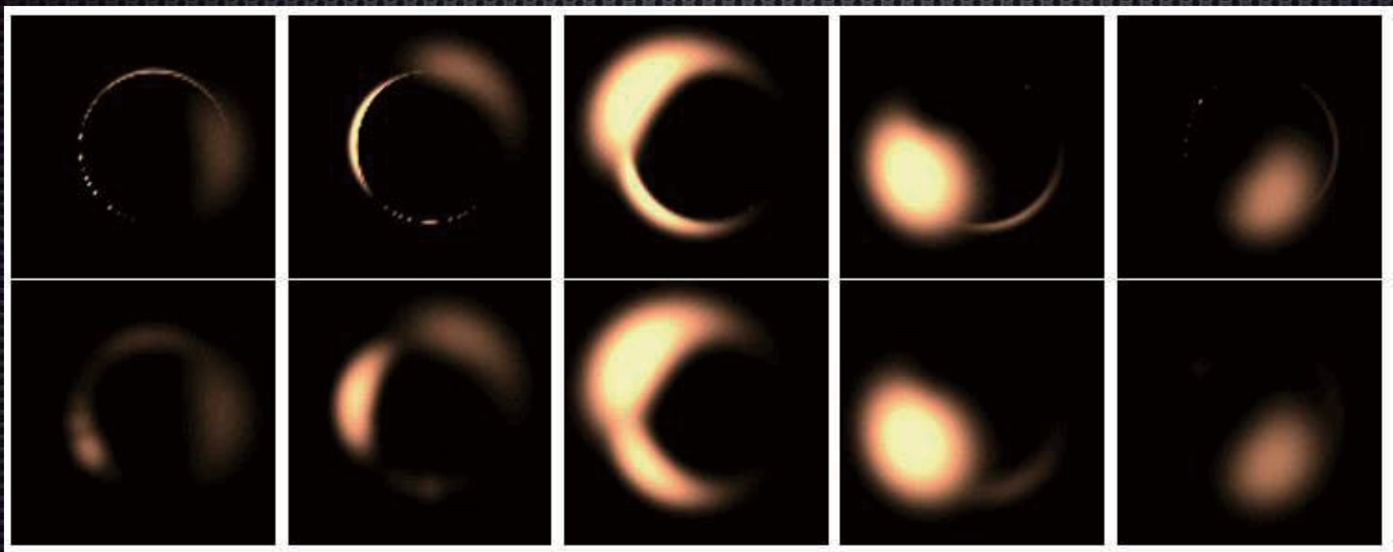
Time variability of Hot spot

The information from a black hole space-time is obtained.

Bright spot ! orbting a rotating Black Hole

BH evidence ← size + time variability

BH shadow images (theoretical)



Bottom panels include the effect of the light scattering by the electron between the star.

https://www.cfa.harvard.edu:~/loeb/im.pdf

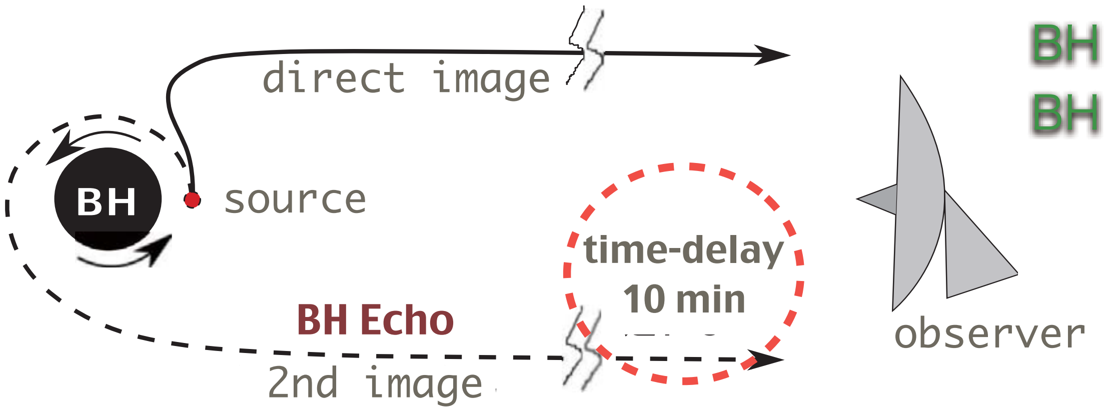
Time variability of a Hot spot

Hot spot :

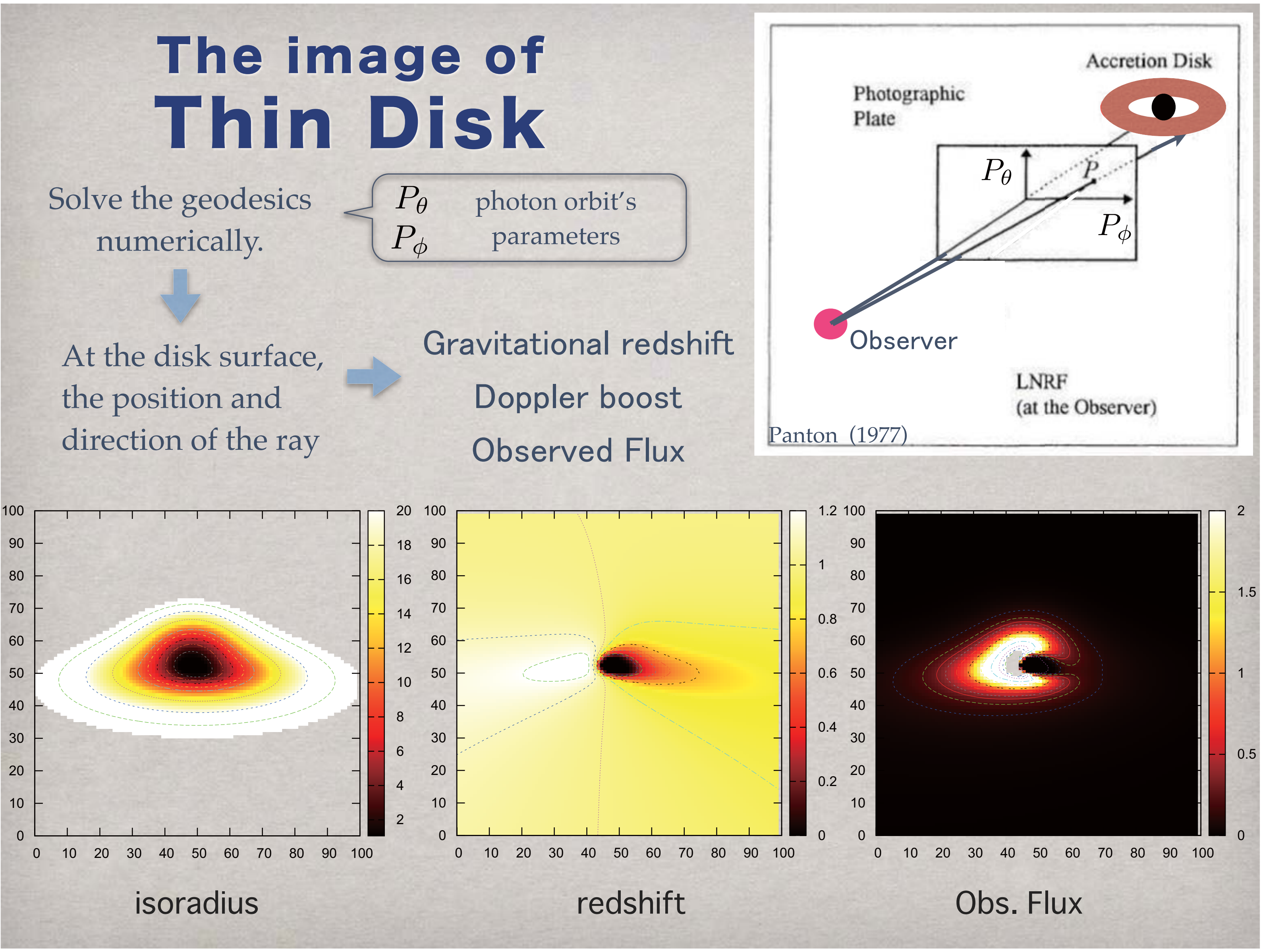
Flares on the disk surface,
MHD shocks in the BH magnetosphere, etc

Direct(1st) + BH Echo (2nd) + ...

(Fukumura+2008)



BH mass
BH spin


The image of
Thin Disk
Solve the geodesics numerically.
P_θ P_ϕ photon orbit's parameters
At the disk surface, the position and direction of the ray
Gravitational redshift
Doppler boost
Observed Flux
Accretion Disk
Photographic Plate
P_θ
P
P_ϕ
Observer
LNRF
(at the Observer)
Panton (1977)
isoradius
redshift
Obs. Flux

$$F_{\rm obs} = g^4 \; F_{\rm emi}$$

The Observed Flux $F_{\rm obs}$ (for Line profiles)

From the relativistic invariance of I_ν/ν^3 $(= I_\mu/E^3)$, i.e., $I_\nu^{\rm obs}/\nu_{\rm obs}^3 = I_\nu^{\rm em}/\nu_{\rm em}^3$, the observed flux distribution F_ν is given by [11]

$$dF_\nu^{\rm obs}(E_{\rm obs}) = I_\nu^{\rm obs}(E_{\rm obs})\, d\Theta = (E_{\rm obs}/E_{\rm em})^3 I_\nu^{\rm em}(E_{\rm em})\, d\Theta \,, \tag{1.87}$$

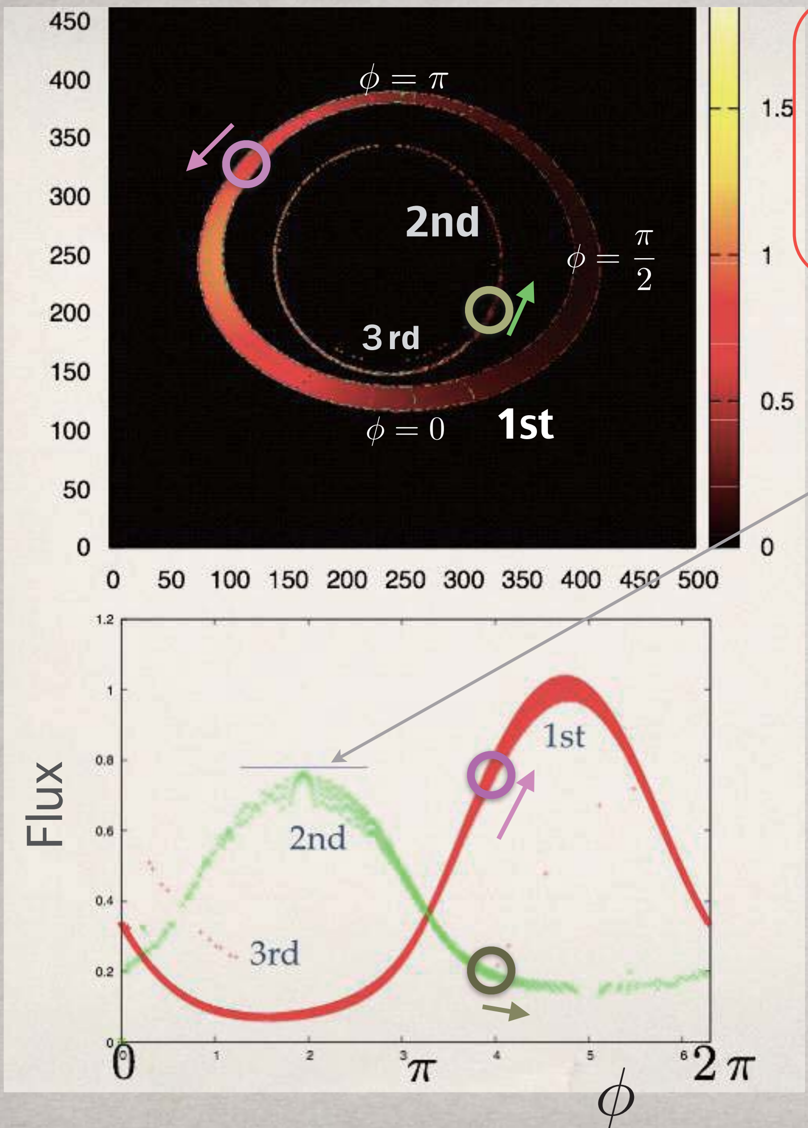
where $d\Theta$ is the solid angle subtended by the disk in the observer's sky. Here, we define the redshift factor (GR version of Doppler factor) as

$$g \equiv \frac{\nu_{\rm obs}}{\nu_{\rm em}} = \frac{E_{\rm obs}}{E_{\rm em}} = \frac{(p_\alpha u^\alpha)_{\rm obs}}{(p_\alpha u^\alpha)_{\rm em}} = \frac{(p_t u^t)_{\rm obs}}{[p_t u^t(1+\Omega p_\phi/p_t)]_{\rm em}} = \tilde{\alpha}_{\rm Z}^{-1/2}[1+\Omega(p_\phi/p_t)]_{\rm em}^{-1} \,, \tag{1.88}$$

where $\Omega \equiv u^\phi/u^t$ is the angular velocity of the emitting particle and

$$u_{\rm em}^t = (g_{tt} + 2g_{t\phi}\Omega + g_{\phi\phi}\Omega^2)^{-1/2} \equiv \tilde{\alpha}_{\rm Z}^{-1/2} \tag{1.89}$$

is the gravitational redshift factor for the rotating emitting matter. Note that $u_{\rm obs}^t = 1$ and $p_t^{\rm em} = p_t^{\rm obs}$. The value of constant $p_\phi/p_t (\equiv -\lambda)$ is determined by the angle between the rotational direction of the emitter and the direction of the photon trajectory at the disk surface (Luminet 1979) [12].



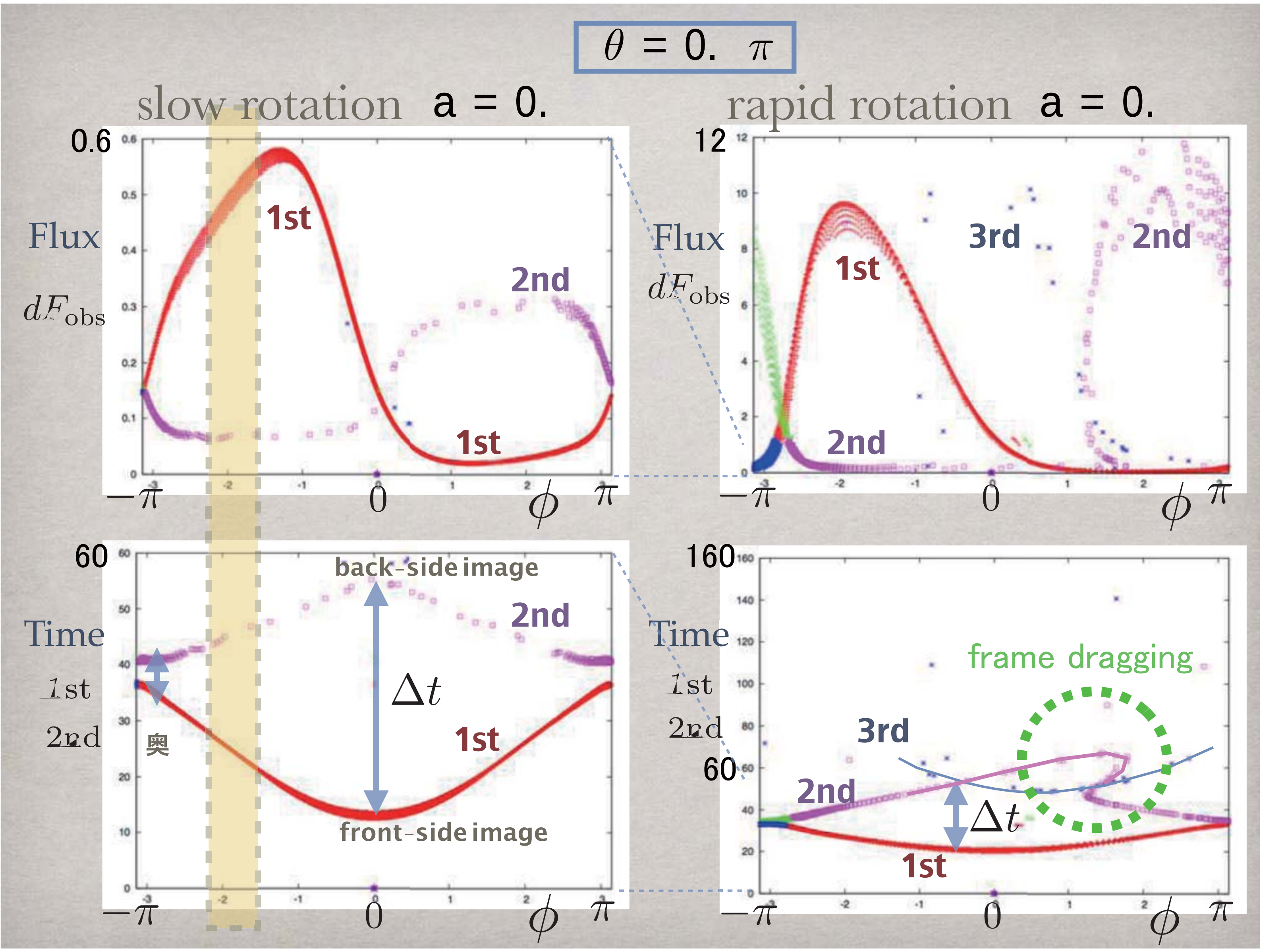
Page-Thorne thin Disk model (1974)

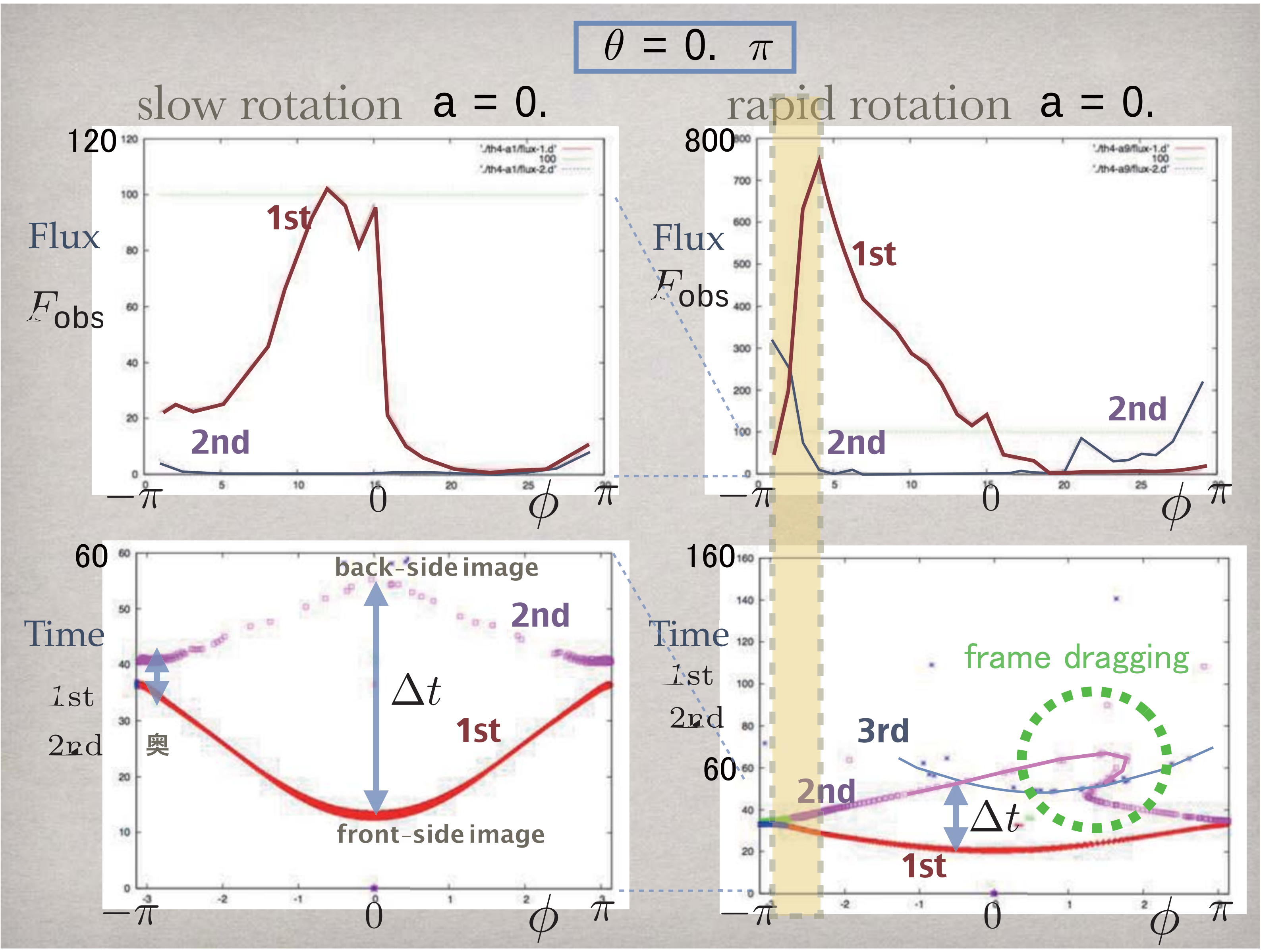
Time-averaged Hot Spot

= ring-like shape

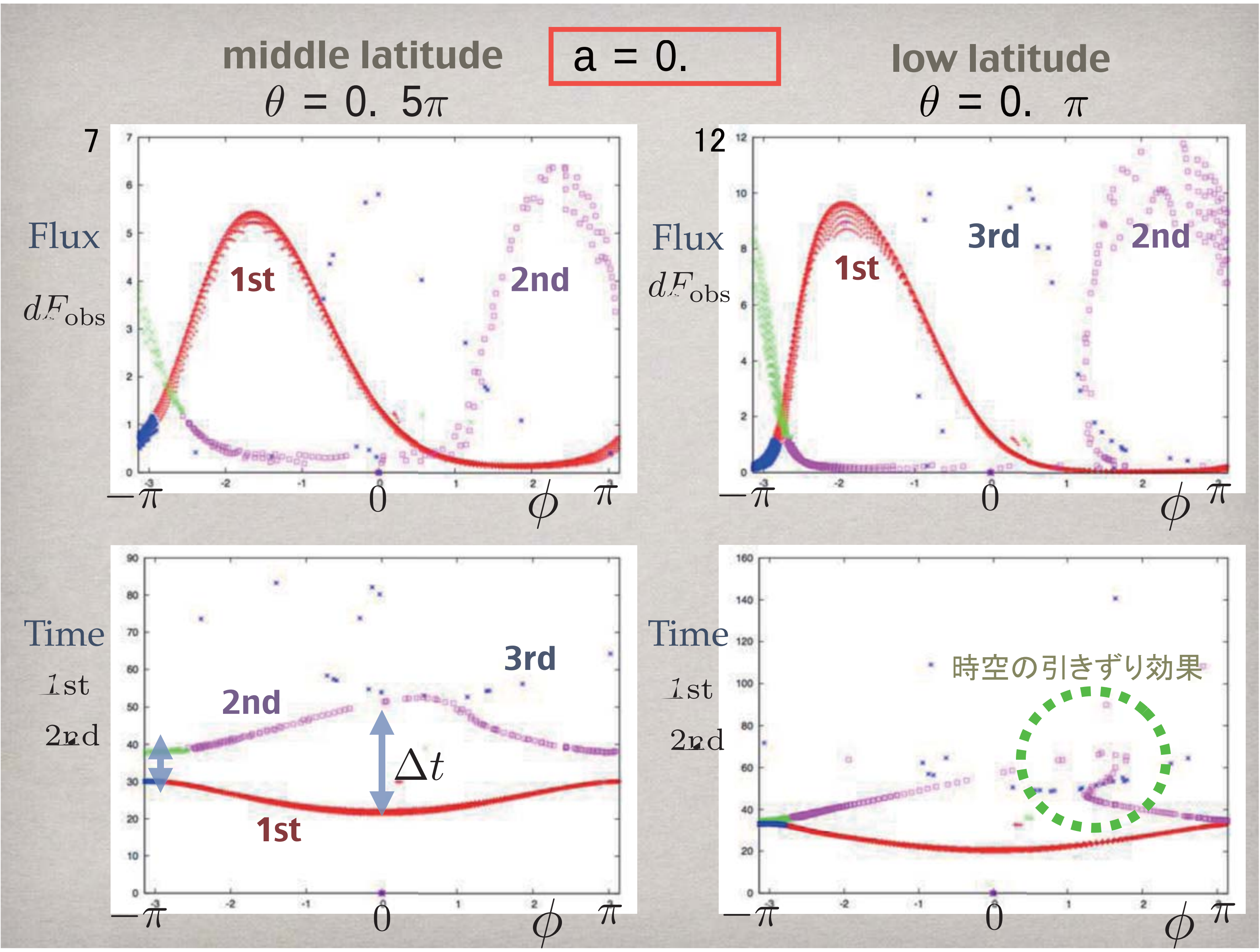
$a = 0.5\,m \qquad \theta = 0.25\,\pi$

- The brightness of the second light can be at the same level as primary light.
- The image of the second is small, so it has a small total flax.
- We may estimate the space-time and / or disk parameters from a change at the time of the flux ratio.


θ = 0. π
slow rotation a = 0.
rapid rotation a = 0.
Flux
dF_obs
1st
2nd
3rd
Time
1st
2nd
back-side image
front-side image
Δt
frame dragging
φ
−π
0
π



θ = 0. π
slow rotation a = 0.
rapid rotation a = 0.
Flux
F obs
1st
2nd
Time
1st
2nd
奥
back-side image
front-side image
Δt
3rd
frame dragging
−π
0
φ
π


middle latitude
θ = 0. 5π
a = 0.
low latitude
θ = 0. π
Flux
dF_obs
1st
2nd
3rd
Time
1st
2nd
Δt
−π
0
φ
π
時空の引きずり効果

Information from BH space-time

- We have discussed the images and fluxes from a orbiting hot spot around a black hole.
- We may see two (or more) images of the hot spot. These two images have different brightness, flux and red shift.
- The time-lag of two images also give us the very important information about the scale of the horizon (i.e., mass and spin).
- Thus, by comparing two images, we can get some information of the black hole space-time, in addition to a state of the hot spot.

“Hawking-Page phase transition in AdS 3 and extremal CFTs”

Yasunari Kurita (Kanagawa Inst. of Tech.)

[JGRG24(2014)P31]

JGRG24@Kavli IPMU, Nov. 10-14 2014

Hawking-Page phase transition in AdS_3 and extremal CFTs

Yasunari KURITA（Kanagawa Inst. Tech.）

Collaborator： Masaru Siino（Tokyo Inst. Tech.）

Ref. : Hawking and Page, Comm.Math.Phys. 152 (1984) 220
Banados, Teitelboim and Zanelli, PRL 69 (1992) 1849
Witten, arXiv:0706.3359,

For multiple-BTZ: YK and Masaru Siino, PRD89, 024018 (2014)

Contents

（１）３-dim. pure AdS Gravity and extremal CFTs (ECFTs)
(Witten ’07）

（２）BTZ Black holes entropy as number of primary fields
(Witten ’07)

（３）Emergence of Hawking-Page transition from ECFT partition functions

(1) 3-dim. pure AdS Gravity and extremal CFTs

Witten, arXiv:0706.3359

Witten's conjecture

3-dim. pure gravity with negative Λ corresponds to extremal CFT (ECFT)

- ECFT has central charge c=24k, and its lowest dimension of primary operators is precisely k+1.

k: natural number

- A k=1 ECFT is known as **FLM** model.

having Monster symmetry

Frenkel, Lepowsky, Meurman('88)

- It is not yet known whether the k>1 ECFTs exist, but this is fascinating conjecture!

Partition functions of genus one ECFTs

The partition functions for each k:

Index is k

$$Z_1(q) = |J(q)|^2 = \left|\frac{41E_4(\tau)^3 + 31E_6(\tau)^2}{72\eta(\tau)^{24}}\right|^2$$

$$Z_2(q) = |J(q)^2 - 393767|^2$$

$$Z_3(q) = |J(q)^3 - 590651J(q) - 64481279|^2$$

$$Z_4(q) = |J(q)^4 - 787535J(q)^2 - 85975039J(q) + 74069025266|^2$$

（For arbitrary each k, partition function is calculable. ）

where $J(q) = 1728j(\tau) - 744$

Klein's modular invariant

nome: $q = e^{2\pi i\tau}$

moduli parameter of boundary torus

(2) BTZ black hole entropy as number of primary op.

Witten, arXiv:0706.3359

Expansion of partition functions

- Note the coefficients!

$$Z_1(q) = |q^{-1} + \underline{196884}q + \mathcal{O}(q^2)|^2$$

$$Z_2(q) = |q^{-2} + 1 + \underline{42987520}q + \mathcal{O}(q^2)|^2$$

$$Z_4(q) = |q^{-4} + q^{-2} + q^{-1} + 2 + \underline{81026609428}q + \mathcal{O}(q^2)|^2$$

- Take log!

k=1	$\ln 196884 \approx 12.19$		$4\pi\sqrt{1} \approx 12.57$
k=2	$\ln 42987520 \approx 17.58$	⟺	$4\pi\sqrt{2} \approx 17.77$
k=4	$\ln 81026609428 \approx 25.12$		$4\pi\sqrt{4} \approx 25.13$

For large k, good approximation!

(2) BTZ black hole entropy as number of primary op.

Witten, arXiv:0706.3359

Entropy of BTZ black holes

$$S = \pi \left(\frac{\ell}{2G}\right)^{1/2} \left(\sqrt{M\ell - J} + \sqrt{M\ell + J}\right) = 4\pi\sqrt{k}\left(\sqrt{L_0} + \sqrt{\bar{L}_0}\right)$$

$$M\ell = L_0 + \bar{L}_0, \quad J = L_0 - \bar{L}_0, \quad c = \frac{3\ell}{2G} = 24k$$

- For $L_0 = 1$, Log of coefficients are nealy equals to entropy (for each holomorphic sector and anti-holomorphic sector)
- For k=1, FLM interprets 196883 as the number of primary operators.
- Witten interprets that, including the case of k>1, the coefficients are the number of primary operators creating BTZ black holes.
- Witten have also shown that it agrees with the Bekenstein-Hawking entropy in the limit: $k \to \infty, \quad L_0 \to \infty, \quad L_0/k$ fixed

3-dim. version of Hawking Page ('84)

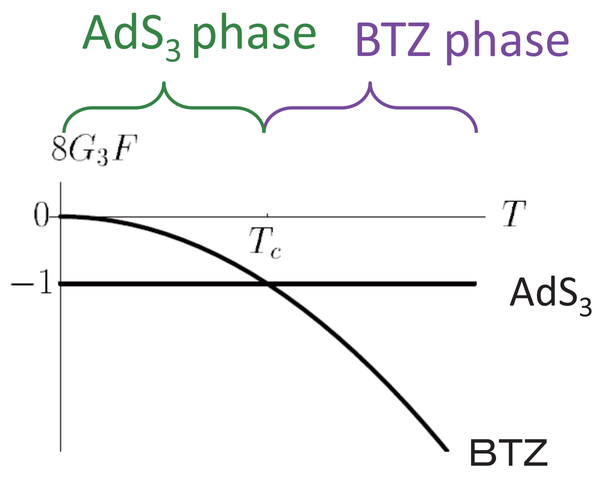
Hawking-Page transition (semi-classical)

- Free energy based on Euclidean classical action

$$Z \approx e^{-I_E[\hat{g}]} \quad \Rightarrow \quad F = -\frac{1}{\beta} \ln Z$$

- Critical temperature:

$$T_c = \frac{\sqrt{1+\Omega_E^2 \ell^2}}{2\pi\ell}$$



(3) Emergence of Hawking-Page in ECFT.

Behavior of ECFT partition functions : low temperature limit

- Leading behavior :

$$J(q) \to q^{-1} \text{ as } T \to 0 \quad \Rightarrow \quad Z_k \to |q|^{-2k} = \exp\left(\frac{2k}{T\ell}\right) = e^{-I_{AdS_3}}$$

AdS$_3$ dominant !

$$q = e^{2\pi i\tau} = \exp\left[\frac{\Omega_E}{T}i - \frac{1}{T\ell}\right]$$

- Thermodynamical relation :

$$F = -T\ln Z_k = -1 \quad \Rightarrow \quad S = -\frac{\partial F}{\partial T} = 0, \quad J_E = \frac{\partial F}{\partial \Omega_E} = 0$$

When $8G_3 = 1,\ k = 2\ell$

(3) Emergence of Hawking-Page in ECFT.

Behavior of ECFT partition functions : high temperature limit

- Leading term (after the modular transformation) :

$$Z_k(\tilde{q}) \to |\tilde{q}|^{-2k} = \exp\left(\frac{8\pi^2 k\ell T}{\Omega_E^2 \ell^2 + 1}\right) = e^{-I_{BTZ}}$$

BTZ dominant !

$$\tilde{q} = e^{-\frac{2\pi i}{\tau}} = \exp\left[-\frac{4\pi^2 T\ell^2 \Omega_E}{\Omega_E^2 \ell^2 + 1} i - \frac{4\pi^2 T\ell}{\Omega_E^2 \ell^2 + 1}\right]$$

- Thermodynamical relation

$$F = -T \ln Z_k = -\frac{4\pi^2 l^2 T^2}{\Omega_E^2 l^2 + 1}$$

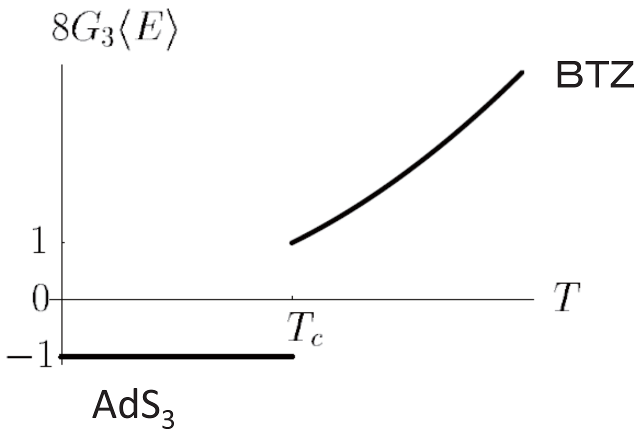
$$\Rightarrow \quad -\frac{\partial F}{\partial T} = 4\pi r_+ = S, \quad \frac{\partial F}{\partial \Omega_E} = 2\frac{|r_-| r_+}{l} = J_E$$

(3) Emergence of Hawking-Page in ECFT.

Internal Energy (semi-classical)

- The semi-classical result

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln Z = \sqrt{M^2 + J_E^2/\ell^2}$$



We take the internal energy as order parameter for the Hawking-Page transition

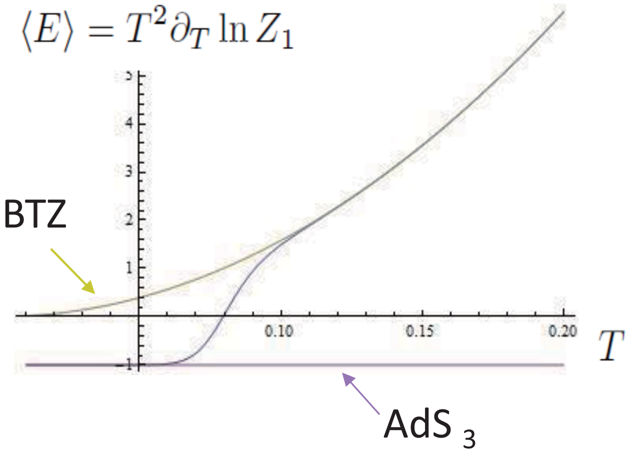
(3) Emergence of Hawking-Page in ECFT.

Internal Energy obtained from ECFT partition functions

(we set J=0, for simplicity)

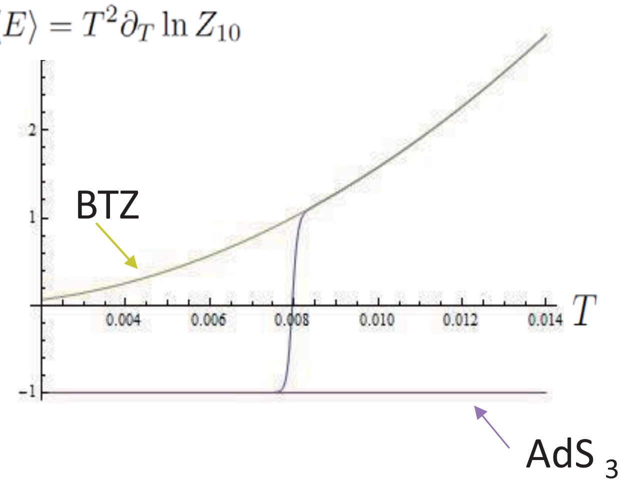
- k=1 case

$$\langle E \rangle = T^2 \partial_T \ln Z_1$$



- k=10 case

$$\langle E \rangle = T^2 \partial_T \ln Z_{10}$$

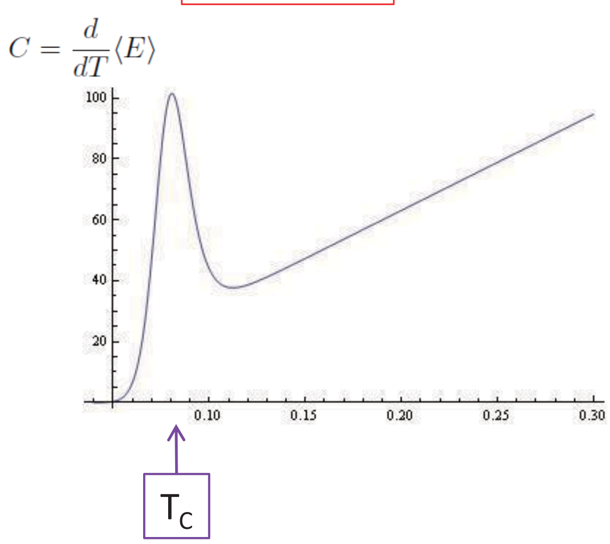


It agrees with mass of AdS_3 at low T and with mass of BTZ at high T. The transition becomes sharper with increasing k (corresponding to thermodynamic limit).

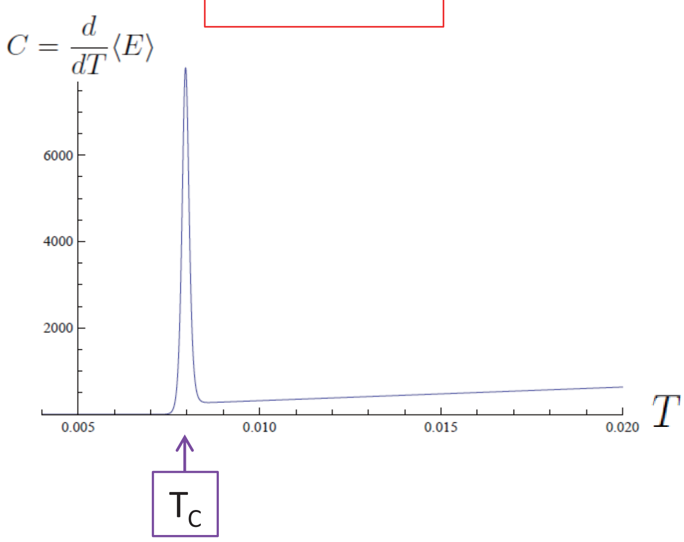
(3) Emergence of Hawking-Page in ECFT.

Specific heat from ECFT partition functions

k=1 case



k=10 case



The specific heat will diverge at T_C in thermodynamic limit($k \to \infty$).
⇒The internal energy will jump discontinuously at the critical temperature.
⇒ ECFT implies that the HP transition is a first order transition.

Summary

- In AdS_3 pure gravity/ECFT correspondence, the number of primary op. in ECFT corresponds to number of microscopic black holes states. (review)
- Hawking–Page transition emerges from the ECFT partition functions.
- ECFTs imply that the Hawking–Page transition is a first order phase transition.
- ECFTs show thermodynamical properties of AdS_3 gravity especially of BTZ black holes.

Appendix

- Partition funcion of k=10 ECFT:

$$\begin{aligned} Z_{10} \quad = \quad & \Big| J^{10} - 1968839J^{8} - 214937599J^{7} + 1348071256190J^{6} \\ & +253704014739574J^{5} - 361538450036076764J^{4} \\ & -824143081027930253330J^{3} + 301233730723154384160085J^{2} \\ & +621970556517352063759223 6J - 2643904925535517177481002 92\Big|^{2} \end{aligned}$$

“Bimetric gravity and the AdS/CFT correspondence”

Kouichi Nomura (Kyoto)

[JGRG24(2014)P32]

Bimetric gravity and the AdS/CFT correspondence

arXiv:1407.1160 [hep-th]

Nomura Kouichi (Kyoto University)

Abstract

We study bimetric gravity through the context of the AdS/CFT correspondence, especially, in the first order hdrodynamic limit.
In the case of general relativity, we have the N = 4 supersymmetric Yang-Mills plasma as the boundary field, and the transport coefficients are computed via the AdS/CFT correspondence.
Then, we put bimetric gravity on the bulk side, where the interaction generates a massive graviton. We see that this massive mode leads to the extra divergences which are absent in the case of general relativity.
Our first investigation is how to cancel these divergences. After that, we find the emergence of two-component fluid and calculate their pressure and sheer viscosity.

1. the AdS/CFT correspondnce

The correspondence between
(d+1)-dimensional gravity theory ↔ d-dimensional matter field theory

We can investigate complicated (quantum) matter field theory,
thorough the rather simple (classical) gravity theory

A lot of examples are known, but the most classic one is
five-dimensional general relativity ↔ four-dimensional Yang-Mills theory

Especially, in the first order hydrodynamic limit (derivative exoansion),
we can easily calculate the transport coefficients such as sheer viscosity.

We consider an extension of this method to the case of bimetric gravity

2. The case of general relativity (a short review)

We begin with the action

$$S = S_{EH} + S_{GH} + S_{ct}$$

$$S_{EH} = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}(R-2\Lambda)$$ ········ Einsyein-Hilbert term

$$S_{GH} = \frac{2}{16\pi G_5}\int_{AdS-bdy} d^4x\sqrt{-\gamma}K$$ ········ Gibbons-Hawking term

$$S_{ct} = \frac{1}{16\pi G_5}\int_{AdS-bdy} d^4x\sqrt{-\gamma}\left(\frac{6}{L}+\frac{L}{2}\mathcal{R}+\cdots\right)$$ ········ counter term

γ: induced metric on the AdS-boundary

K :extrinsic curvature

$\mathcal{R}$: spatial curvature, $\Lambda = -6/L^2$

The background metric is set to be

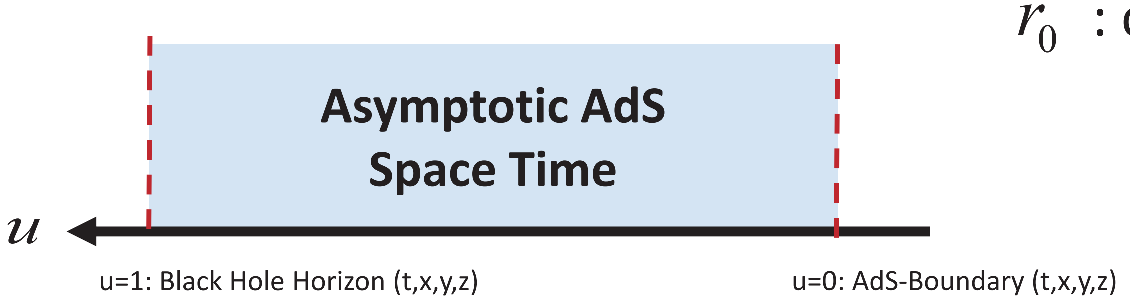
$$g_{\mu\nu}dx^{\mu}dx^{\nu} = \left(\frac{r_0}{L}\right)^2 \frac{1}{u^2}\left(-hdt^2 + dx^2 + dy^2 + dz^2\right) + \frac{L^2}{hu^2}du^2$$

(Schwarzschild AdS Black-Hole)

$$h = 1 - u^4 \quad (0 < u < 1)$$

$$\begin{cases} u = 0 & \text{:AdS-boundary} \\ u = 1 & \text{:Black Hole Horizon} \end{cases}$$

r_0 : constant



We take a perturbation $\delta g^{\mu}{}_{\nu} = \bar{g}^{\mu\lambda}\delta g_{\lambda\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi & 0 & 0 \\ 0 & \phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $\phi = \phi(t, u)$

Fourier transform (t → ω)

and solve the EOM $\left(\frac{h}{u^3}\phi'_\omega\right)' + \left(\frac{L^2}{r_0}\omega\right)^2 \frac{1}{u^3 h}\phi_\omega = 0$ with the ingoing wave condition at the horizon

$$\phi_\omega(u) \cong \phi_\omega^{(0)} + \frac{i\omega L^2}{4r_0}\phi_\omega^{(0)} u^4$$ + higher order terms of u and ω

$\phi^{(0)}$:field value at the AdS-boundary

This solution is substituted back into the action,

and we obtain the on-shell action

$$S = \frac{V_4}{16\pi G_5}\frac{r_0^4}{L^5} + \frac{V_3}{16\pi G_5}\frac{r_0^4}{L^5}\int \frac{d\omega}{2\pi}\left\{ -\frac{1}{2}\phi_{-\omega}^{(0)}\phi_\omega^{(0)} + \frac{1}{2}\phi_{-\omega}^{(0)}\left(i\frac{L^2}{r_0}\omega\right)\phi_\omega^{(0)} \right\}$$

Through the AdS/CFT prescription, we obtain

the (perturbed) enegy-momentum tensor for the boundary field theory

$$< \delta T^{xy}_{\omega} >= \frac{\delta S}{\delta \phi^{(0)}_{-\omega}} = -\frac{1}{16\pi G_5}\frac{r_0^4}{L^5}\phi^{(0)}_{\omega} + i\frac{1}{16\pi G_5}\left(\frac{r_0}{L}\right)^3 \omega\phi^{(0)}_{\omega}$$

On the other hand, if the boundary space-time is slightly distorted from the flat space-time $\eta_{\mu\nu} \to g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}$.

the linear response of the energy momentum tensor can be written as

$$\delta T^x_y = -P\delta g^x_y + i\omega\eta\delta g^x_y.$$

Therefore, we conclude that

pressure: $P = \frac{1}{16\pi G_5}\frac{r_0^4}{L^5}$

Sheer viscosity: $\eta = \frac{1}{16\pi G_5}\left(\frac{r_0}{L}\right)^3$

and the ratio to the entropy density is

$$\eta/s = 1/4\pi$$

($s = \frac{1}{4G_5}\left(\frac{r_0}{L}\right)^3$ from the background)

3.The case of dRGT massive gravity (16πG=1,L=1)

We add the mass (interaction) term

$$S = S_{EH} + S_{GH} + S_{ct} + S_{int}$$

$$S_{int} = m^2 \int d^5x \sqrt{-g}\, e\left(\sqrt{g^{-1}\bar{g}}\right)$$

$$e(A) = \sum_{n=0}^{5} \beta_n \epsilon_{\mu_1 \cdots \mu_n \lambda_{n+1} \cdots \lambda_5} \epsilon^{\nu_1 \cdots \nu_n \lambda_{n+1} \cdots \lambda_5} A^{\mu_1}_{\nu_1} \cdots A^{\mu_n}_{\nu_n}$$

$\bar{g}$: fixed background metric (Shwartzschild AdS Black-Hole)

Parameters β are chosen to reduce to the Fiertz-Pauli mass term in the linear level

$$S_{int} = -\frac{1}{4} m^2 \int d^5x \sqrt{-\bar{g}} \left(\mathrm{Tr}(\delta g)^2 - \mathrm{Tr}^2(\delta g)\right)$$

We take a perturbation

$$\delta g^{\mu}{}_{\nu} = \bar{g}^{\mu\lambda}\delta g_{\lambda\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi & 0 & 0 \\ 0 & \phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \phi = \phi(t,u)$$

Fourier transform (t → ω)

and solve the EOM $\left(\frac{h}{u^3}\phi'_\omega\right)' - m^2\frac{1}{u^5}\phi_\omega + \frac{\omega^2}{u^3 h}\phi_\omega = 0.$

$$\phi_\omega(u) = A_\omega\left\{u^{2-2\alpha} + \frac{1}{4}(1-\alpha)u^{6-2\alpha} + O[u^{10-2\alpha}]\right\} + B_\omega\left\{u^{2+2\alpha} + \frac{1}{4}(1+\alpha)u^{6+2\alpha} + O[u^{10+2\alpha}]\right\}$$

A_ω, B_ω :ω dependent coefficient, $\alpha = \sqrt{1+m^2/4}$

This solution is substituted to the action, and we encounter divergences

$$S = V_4 + \int \frac{d\omega}{2\pi}\left\{(1+\alpha)A_{-\omega}B_\omega + (1-\alpha)B_{-\omega}A_\omega + A_{-\omega}A_\omega\left((1-\alpha)u^{-4\alpha} + \frac{1}{2}(\alpha^2-\alpha-1)u^{4-4\alpha} + O[u^{8-4\alpha}]\right)\right\}\Bigg|_{u=0}$$

In order to remove the divergence,
we introduce a new counter term

$$S_{mct} \propto \int_{Ads-bdy} d^4x\sqrt{-\gamma}\, e(\gamma^{-1}\bar{\gamma})$$

which reduces to the Fiertz-Pauli form in the linear level

$$S_{mct} = -\frac{1}{2}(1-\alpha)\int_{Ads-bdy} d^4x\sqrt{-\bar{\gamma}}\Big(\mathrm{Tr}(\delta\gamma)^2 - \mathrm{Tr}^2(\delta\gamma)\Big)$$

Then, the leading divergence is removed

$$S + S_{mct} = V_4 + \int \frac{d\omega}{2\pi}\Big\{2\alpha A_{-\omega}B_\omega + A_{-\omega}A_\omega\Big(-\frac{1}{2}u^{4-4\alpha} + O[u^{8-2\alpha}]\Big)\Big\}\Big|_{u=0}$$

but other divergent terms remain.
We eliminate them by the BF-bound like condition $0 < \alpha < 1$ $(-4 < m^2 < 0)$

and obtain the finite on-shell action

$$S + S_{mct} = V_4 + \int \frac{d\omega}{2\pi}(2\alpha A_{-\omega}B_\omega)$$

We fix the remaining constant A_ω, B_ω by the condition that
the solution of the EOM should coincide with that of general relativity in the massless limit.

Then, the action is
$$S + S_{mct} = V_4 + \int \frac{d\omega}{2\pi}\left(\frac{i\alpha\omega}{2}\right)\phi^{(0)}_{-\omega}\phi^{(0)}_{\omega}$$

and the energy momentum tensor for the boundary field is
$$< \delta T^{xy}_\omega >= \frac{\delta S}{\delta\phi^{(0)}_{-\omega}} = i\omega\alpha\phi^{(0)}_\omega$$

Comparing to the linear response formula,
$$\delta T^x_y = -P\delta g^x_y + i\omega\eta\delta g^x_y.$$
we find that the pressure is zero $P = 0$.

However, the pressure can be calculated from the background metric
$$P = \frac{1}{\beta}\partial_{V_3}\ln Z = \frac{1}{16\pi G_5}\frac{r_0^4}{L^5} \qquad Z = e^{-S_E}$$
S_E :Euclidean on-shell action

which contradicts with our result. It seems to be unphysical .

4. The case of bimetric gravity

We give dynamics to the fixed metric

$$S = S_{EH}[g] + S_{GH}[\gamma] + S_{ct}[\gamma]$$ for metric g (induced metric ϒ)

$$+ S_{EH}[f] + S_{GH}[\rho] + S_{ct}[\rho]$$ for metric f (induced metric ρ)

$$+ S_{int}[g, f] + S_{int,ct}[\gamma, \rho],$$

$S_{int}[g,f]$: interaction term; $S_{int,ct}[\gamma,\rho]$: new counter term

$$S_{EH}[g] + S_{GH}[\gamma] + S_{ct}[\gamma]$$
$$= M_g^2 \int d^5x \sqrt{-g}(R[g] - 2\Lambda) + 2M_g^2 \int_{AdS-bdy} d^4x \sqrt{-\gamma} K[\gamma] + M_g^2 \int_{AdS-bdy} d^4x \sqrt{-\gamma}\left(\frac{6}{L} + \cdots\right)$$
$$S_{EH}[f] + S_{GH}[\rho] + S_{ct}[\rho]$$
$$= M_f^2 \int d^5x \sqrt{-f}(R[f] - 2\Lambda) + 2M_f^2 \int_{AdS-bdy} d^4x \sqrt{-\rho} K[\rho] + M_f^2 \int_{AdS-bdy} d^4x \sqrt{-\rho}\left(\frac{6}{L} + \cdots\right)$$

$$M_g^2 = \frac{1}{16\pi G_g}, M_f^2 = \frac{1}{16\pi G_f}$$

(M_g, M_f does not have mass dimension)

The interaction term is given by

$$S_{int}[g,f] = 2m^2 M_{eff}^2 \int d^5x \sqrt{-g}\, e\left(\sqrt{g^{-1}f}\right) \qquad M_{eff}^2 = \left(\frac{1}{M_g^2} + \frac{1}{M_f^2}\right)^{-1}$$

and the newly introduced counter term is

$$S_{int,ct}[\gamma,\rho] \propto \frac{M_{eff}^2}{L} \int_{AdS-bdy} d^4x \sqrt{-\gamma}\, e\left(\sqrt{\gamma^{-1}\rho}\right)$$

Under a perturbation $g = \bar{g} + \delta g, \quad f = \bar{g} + \delta f$ ($\bar{g}$: background)

we have

$$S_{int}[g,f] = -\frac{1}{4} m^2 M_{eff}^2 \int d^5x \sqrt{-\bar{g}} \left(\mathrm{Tr}(\delta g - \delta f)^2 - \mathrm{Tr}^2(\delta g - \delta f)\right)$$

$$S_{int,ct}[\gamma,\rho] = -\frac{1}{2}(1-\alpha)\frac{M_{eff}^2}{L} \int_{Ads-bdy} d^4x \sqrt{-\bar{\gamma}} \left(\mathrm{Tr}(\delta\gamma - \delta\rho)^2 - \mathrm{Tr}^2(\delta\gamma - \delta\rho)\right)$$

We take a perturbation on the common background (Schwartzschild AdS BH)

$$\delta g^{\mu}{}_{\nu} = \bar{g}^{\mu\lambda}\delta g_{\lambda\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \phi & 0 & 0 \\ 0 & \phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \phi = \phi(t,u)$$

$$\delta f^{\mu}{}_{\nu} = \bar{g}^{\mu\lambda}\delta f_{\lambda\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \psi & 0 & 0 \\ 0 & \psi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \psi = \psi(t,u)$$

where the background $\bar{g}$ is Schwartzschild AdS Black Hole.

We solve the EOM and substitute the solution to the action,

and obtain the on-shell action

$$
\begin{aligned}
S = &\frac{r_0^4}{L^5} V_4 (M_g^2 + M_f^2) \\
&+ \frac{r_0^4}{L^5}\left(\frac{1}{M_g^2 + M_f^2}\right) V_3 \int \frac{d\omega}{2\pi} \Big\{ -\frac{1}{2} M_g^4 \phi_{-\omega}^{(0)} \phi_{\omega}^{(0)} + i \frac{L^2\omega}{2r_0} M_g^2 (M_g^2 + \alpha M_f^2) \phi_{-\omega}^{(0)} \phi_{\omega}^{(0)} \\
&\qquad - \frac{1}{2} M_f^4 \psi_{-\omega}^{(0)} \psi_{\omega}^{(0)} + i \frac{L^2\omega}{2r_0} M_f^2 (M_f^2 + \alpha M_g^2) \psi_{-\omega}^{(0)} \psi_{\omega}^{(0)} \\
&\qquad - \frac{1}{2} M_g^2 M_f^2 \left(\phi_{-\omega}^{(0)} \psi_{\omega}^{(0)} + \psi_{-\omega}^{(0)} \phi_{\omega}^{(0)}\right) \\
&\qquad + i \frac{L^2\omega}{2r_0} M_g^2 M_f^2 (1-\alpha) \left(\phi_{-\omega}^{(0)} \psi_{\omega}^{(0)} + \psi_{-\omega}^{(0)} \phi_{\omega}^{(0)}\right) \Big\}.
\end{aligned}
$$

$\alpha = \sqrt{1 + (mL)^2/4}$. $0 < \alpha < 1$

The coupling between ϕ and ψ suggests emergence of two-component fluid.

To interpret this result, we assume that there are two AdS-boundaries at u=0, which correspond to metric g and f respectively.
Focusing on the boundary for g
the field sourced by ϕ has the energy momentum tensor

$$\left\langle \delta T^{xy}_{(\phi)} \right\rangle = \frac{\delta S}{\delta \phi^{(0)}_{-\omega}}\bigg|_{\psi=0} = -\left(\frac{r_0^4}{L^5}\right)\frac{M_g^4}{M_g^2+M_f^2}\phi^{(0)}_{\omega} + i\omega\left(\frac{r_0^3}{L^3}\right)\frac{M_g^2(M_g^2+\alpha M_f^2)}{M_g^2+M_f^2}\phi^{(0)}_{\omega}$$

sourced by ϕ

we are focusing on the boundary not for f

and the field sourced by ψ has the energy momentum tensor

$$\left\langle \delta T^{xy}_{(\psi)} \right\rangle = \frac{\delta S}{\delta \psi^{(0)}_{-\omega}}\bigg|_{\psi=0} = -\left(\frac{r_0^4}{L^5}\right)\frac{M_g^2 M_f^2}{M_g^2+M_f^2}\phi^{(0)}_{\omega} + i\omega\left(\frac{r_0^3}{L^3}\right)\frac{M_g^2 M_f^2(1-\alpha)}{M_g^2+M_f^2}\phi^{(0)}_{\omega}$$

We compare these results to the linear response formula

$$\delta T^x_y = -P\delta g^x_y + i\omega\eta\delta g^x_y.$$

and read off the pressure and the sheer viscosity

$$P[g]_\phi = \left(\frac{r_0^4}{L^5}\right)\frac{M_g^4}{M_g^2 + M_f^2} \qquad \eta[g]_\phi = \left(\frac{r_0^3}{L^3}\right)\frac{M_g^2(M_g^2 + \alpha M_f^2)}{M_g^2 + M_f^2}$$

for the ϕ sourced fluid

and

$$P[g]_\psi = \left(\frac{r_0^4}{L^5}\right)\frac{M_g^2 M_f^2}{M_g^2 + M_f^2} \qquad \eta[g]_\psi = \left(\frac{r_0^3}{L^3}\right)\frac{M_g^2 M_f^2(1-\alpha)}{M_g^2 + M_f^2}$$

for the ψ sourced fluid

The total pressure coincides with that calculated from the background

$$P[g]_\phi + P[g]_\psi = \frac{r_0^4}{L^5} M_g^2 = \frac{r_0^4}{16\pi G_g L^5}$$

The entropy density of the boundary for the metric g is $s[g] = 4\pi M_g^2 (r_0/L)^3$ and the ratio is

$$\frac{\eta[g]_\phi}{s[g]} = \left(\frac{1}{4\pi}\right) \frac{M_g^2 + \alpha M_f^2}{M_g^2 + M_f^2}, \qquad \frac{\eta[g]_\psi}{s[g]} = \left(\frac{1}{4\pi}\right) \frac{M_f^2 (1-\alpha)}{M_g^2 + M_f^2}$$

If $M_g = M_f$, we obtain

$$\frac{\eta[g]_\phi}{s[g]} = \left(\frac{1}{4\pi}\right) \frac{1+\alpha}{2}, \qquad \frac{\eta[g]_\psi}{s[g]} = \left(\frac{1}{4\pi}\right) \frac{1-\alpha}{2}$$

"Anti-evaporation in bigravity"

Taishi Katsuragawa (Nagoya)

[JGRG24(2014)P33]

Anti-evaporation in Bigravity

Taishi Katsuragawa (QG-lab. Nagoya Univ.)
In collaboration with S.Nojiri and S.D.Odintsov

Based on the works
arXiv:1312.1550
1411.1610

1. Introduction

It is well known that horizon radius of the black hole usually **degreases** by the Hawking radiation.

→ **Black hole evaporation** [Hawking (1974)]

However, the black hole radius can **increases** by the quantum correction for the **Nariai space-time**.

→ **Black hole anti-evaporation** [Bousso and Hawking (1997)]

The anti-evaporation can occur in **F(R) gravity without quantum correction**. [Nojiri and Odintsov (2013,2014)].

It might be general phenomena in modified gravity.

In this work, we study if the **anti-evaporation could occur on the classical level in bigravity**.

2. Natiai space-time and Quantum correction

Schwarzschild-de Sitter space-time

$$ds^2=-V(r)dt^2+V(r)^{-1}dr^2+r^2d\Omega^2$$
$$V(r)=1-\frac{2\mu}{r}-\frac{\Lambda}{3}r^2$$

→ $\mu\to\frac{1}{3}\Lambda^{-1/2}$

Nariai space-time

$$ds^2=\frac{1}{\Lambda\cos^2 t}(-dt^2+dx^2)+\frac{1}{\Lambda}d\Omega^2$$

(S^1: $(-dt^2+dx^2)$, S^2: $\frac{1}{\Lambda}d\Omega^2$)

Topology of $S^1\times S^2$

When we consider the Hawking radiation from BH, this quantum correction leads to **the trace anomaly** of energy-momentum tensor.

Classical $\langle T^\mu_{\ \mu}\rangle=0$ → **Quantum** $\langle T^\mu_{\ \mu}\rangle\neq 0$

The effective action corresponding to the trace anomaly can be written by a covariant form.
For instance, in the case of massless scalar field,

$$S_{\rm eff}=-\frac{1}{48\pi G}\int d^2x\sqrt{-g}\left[\frac{1}{2}R\frac{1}{\Box}R-6(\nabla\phi)^2\frac{1}{\Box}R-\omega\phi R\right]$$

ϕ is dilaton due to dimensional reduction, ω is redundancy parameter.

3. Anti-evaporation in GR and beyond

The effective action leads to the **modification for the EOM**.
In GR, specific perturbations around the Nariai space-time shrink from its initial values, and **the size of black hole horizon increases** at least initially

→ **Black hole anti-evaporation**

On the other hand, anti-evaporation may occur **without quantum corrections in F(R) gravity**. [Nojiri and Odintsov(2013,2014)]

Modification of EOM may be important (?)

It might be interesting to study if **the anti-evaporation may occur on the classical level in other modified gravity.**

In Bigravity, the interaction terms between two metric may affect to the time-evolution of the perturbation, and its behavior is not so trivial.

Therefore, it is worth studying **if the anti-evaporation occurs even on the classical level.**

4. Bigravity

Bigravity describes interacting **massive spin-2 field** and **gravitational field.**

- **Two dynamical metrics** $g_{\mu\nu}$ and $f_{\mu\nu}$
- Background independence (general coordinate trans. inv.)

$$S=M_g^2\int d^4x\sqrt{-g}R(g)+M_f^2\int d^4x\sqrt{-f}R(f)$$
$$-2m_0^2M_{\rm eff}^2\int d^4x\sqrt{-g}\sum_{n=0}^{4}\beta_n e_n\left(\sqrt{g^{-1}f}\right)$$

[Hassan and Rosen (2011)]

Planck mass scales $M_g, M_f,\ \frac{1}{M_{\rm eff}^2}=\frac{1}{M_g^2}+\frac{1}{M_f^2}$ $\quad\left(\sqrt{g^{-1}f}\right)^\mu_{\ \rho}\left(\sqrt{g^{-1}f}\right)^\rho_{\ \nu}=g^{\mu\rho}f_{\rho\nu}$

Free parameters: β_n Mass of massive spin-2 field (massive graviton): m_0

$$e_0(\mathbf{X})=1,\quad e_1(\mathbf{X})=[\mathbf{X}],\quad e_2(\mathbf{X})=\frac{1}{2}\left([\mathbf{X}]^2-[\mathbf{X}^2]\right),\quad e_3(\mathbf{X})=\frac{1}{6}\left([\mathbf{X}]^3-3[\mathbf{X}][\mathbf{X}^2]+2[\mathbf{X}^3]\right)$$
$$e_4(\mathbf{X})=\frac{1}{24}\left([\mathbf{X}]^4-6[\mathbf{X}]^2[\mathbf{X}^2]+3[\mathbf{X}^2]^2+8[\mathbf{X}][\mathbf{X}^3]-6[\mathbf{X}^4]\right)=\det(\mathbf{X}),\quad [\mathbf{X}]=X^\mu_{\ \mu}$$

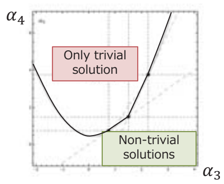
5. Nariai solution in Bigravity

In order to obtain background Nariai space-time, we impose some condition for the metrics and parameters.

$f_{\mu\nu}=C^2 f_{\mu\nu}$ (C is a constant) — **$f_{\mu\nu}$ and $g_{\mu\nu}$ are determined by Einstein's eq.**

$$\beta_0=6-4\alpha_3+\alpha_4,\quad \beta_1=-3+3\alpha_3-\alpha_4$$
$$\beta_2=1-2\alpha_3+\alpha_4,\quad \beta_3=\alpha_3-\alpha_4,\quad \beta_4=\alpha_4$$

Required so that
(i) theory reproduce FP-theory
(ii)theory has asymptotically flat solution



[TK (2013)]

$M_g=M_f$ **To make the calculation easy.**

For instance,
the model with $(\alpha_3,\alpha_4)=(1,-1),(-1,1),(-1,-1)$ have asymptotically de Sitter solution.

We can obtain Nariai solutions for some parameter.

6. Perturbations and Stability

We introduce the two metric ansatz with topology of $S^1\times S^2$ in **conformal gauge**.

$$g_{\mu\nu}dx^\mu dx^\nu=e^{2\rho_1(t,x)}\left(-dt^2+dx^2\right)+e^{-2\varphi_1(t,x)}d\Omega^2$$
$$f_{\mu\nu}dx^\mu dx^\nu=e^{2\rho_2(t,x)}\left(-dt^2+dx^2\right)+e^{-2\varphi_2(t,x)}d\Omega^2$$

Nariai bg.: $e^{2\rho_1(t,x)}=\frac{1}{\Lambda\cos^2 t},\quad e^{-2\varphi_1(t,x)}=\frac{1}{\Lambda},\quad e^{2\rho_2(t,x)}=\frac{C^2}{\Lambda\cos^2 t},\quad e^{-2\varphi_2(t,x)}=\frac{C^2}{\Lambda}$

Then, we make perturbations around the Nariai solutions.

$$\rho_1\equiv\bar\rho_1+\delta\rho_1(t,x),\quad \varphi_1\equiv\bar\varphi_1+\delta\varphi_1(t,x),$$
$$\rho_2\equiv\bar\rho_2+\delta\rho_2(t,x),\quad \varphi_2\equiv\bar\varphi_2+\delta\varphi_2(t,x).$$

$\bar\rho$, $\bar\varphi$ are the Nariai background, $\delta\rho$, $\delta\varphi$ are the perturbations.
And **these two sets of perturbations are independent.**

When we substitute above perturbations into EOMs, we obtain the equations for perturbations.
And we study **the time evolution of the location of BH horizon**.

7. Horizon trace

Eqs. For $\delta g_{\mu\nu}$

(t,t) $\quad 0=\delta\varphi_1''-\tan t\,\delta\dot\varphi_1+\frac{1}{\cos^2 t}\delta\varphi_1+\frac{C_1}{2\Lambda\cos^2 t}(\delta\zeta-2\delta\xi)$

(t,x) $\quad 0=\delta\dot\varphi_1'-\tan t\,\delta\varphi_1'$

(x,x) $\quad 0=\delta\ddot\varphi_1-\tan t\,\delta\dot\varphi_1-\frac{1}{\cos^2 t}\delta\varphi_1-\frac{C_1}{2\Lambda\cos^2 t}(\delta\zeta-2\delta\xi)$

$(\theta,\theta)(\phi,\phi)$ $\quad 0=2\delta\rho_1+\cos^2 t\,(-\delta\ddot\rho_1+\delta\rho_1''+\delta\ddot\varphi_1-\delta\varphi_1'')-\frac{C_1}{\Lambda}(2\delta\zeta-\delta\xi)$

Eqs. For $\delta f_{\mu\nu}$

$0=\delta\varphi_2''-\tan t\,\delta\dot\varphi_2+\frac{1}{\cos^2 t}\delta\varphi_2-\frac{C_1}{2C^2\Lambda\cos^2 t}(\delta\zeta-2\delta\xi)$

$0=\delta\dot\varphi_2'-\tan t\,\delta\varphi_2'$

$0=\delta\ddot\varphi_2-\tan t\,\delta\dot\varphi_2-\frac{1}{\cos^2 t}\delta\varphi_2+\frac{C_1}{2C^2\Lambda\cos^2 t}(\delta\zeta-2\delta\xi)$

$0=2\delta\rho_2+\cos^2 t\,(-\delta\ddot\rho_2+\delta\rho_2''+\delta\ddot\varphi_2-\delta\varphi_2'')+\frac{C_1}{C^2\Lambda}(2\delta\zeta-\delta\xi)$

Contributions from interaction terms

We specify the **perturbation along S^1.coordinate.** $\quad\delta\varphi(t,x)=\epsilon\sigma(t)\cos x$

Substituting the perturbation into (t,x) component, we obtain

$$\sigma(t)=\frac{\sigma_0}{\cos t},\quad \sigma_0=\sigma(t=0)\quad \dot\sigma(t=0)=0$$

The size of the BH horizon is $\quad r_h(t)^{-2}=e^{2\phi}=\Lambda[1+2\epsilon\delta(t)],\quad \delta(t)=\sigma\left(1+\frac{\dot\sigma^2}{\sigma^2}\right)^{-1/2}=\sigma_0$

The size of horizon remains that of the initial perturbation.
Therefore **the anti-evaporation does not occur on classical level in bigravity.**

8. Summary and Discussion

- ☐ We found that the **anti-evaporation does not take place on the classical level in the bigravity.**
- ☐ Time evolution is defined by the (t,x) component of the equations. On the other hand, **off-diagonal components are not modified** and they take the same form as those in GR.
- ☐ Moreover, we obtain $\delta\zeta-2\delta\xi=0$. Then, contributions from the interaction terms vanish in (t,t), (t,x), and (x,x) components, and **these equations take the same forms as those in GR**.
- ☐ **The time-evolution of $\delta\varphi$ is exactly same as that in GR.**
- ☐ In order to realize the anti-evaporation, we need
  - To find the **spherical BH solution with off-diagonal components.**
  - To introduce **the quantum corrections** in $g_{\mu\nu}$ and/or $f_{\mu\nu}$ sector.
  - To **modify** the bigravity to **F(R) bigravity** in $g_{\mu\nu}$ and/or $f_{\mu\nu}$ sector.

“Black holes in non-projectable HoravaLifshitz gravity”

Yosuke Misonoh (Waseda)

[JGRG24(2014)P34]

Black holes in non-projectable Horava-Lifshitz gravity

Yosuke Misonoh (Waseda University)
: in preparation

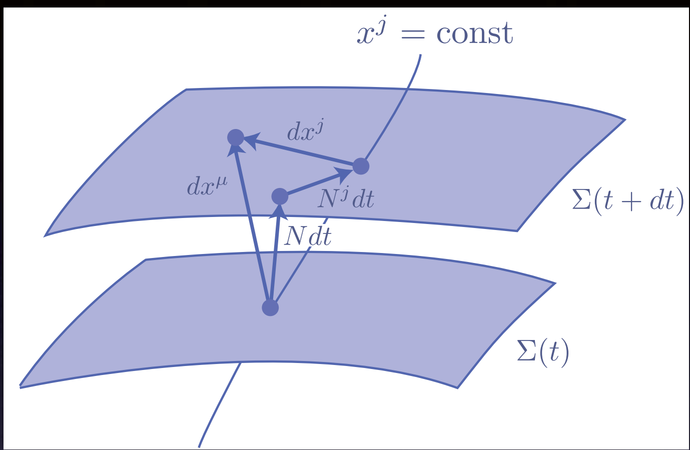
JGRG24@Kavli-IPMU 10-14/11/2014

abstract and contents

We investigate the black holes solution in Lorentz violating spacetime in the context of Horava-Lifshitz(HL) gravity considering the higher order spacial curvature correction as a counter term of quantum renormalization. It is already known that HL gravity in low energy (IR) limit is equivalent to Einstein-aether theory and the black hole solutions are already known in the context of this theory. However if higher order spacial curvature corrections are considered, the analysis becomes difficult, that is caused by lack of null coordinate.

In our analysis we rewrite the theory in the Stueckelberg field called khronon, which restore the choice of time direction. And then, the static and spherically symmetric solution with higher order spacial curvature corrections is discussed by comparing the solution without such a correction.

HL action in ADM formalism



$$ds^2 = -N^2dt^2 + \gamma_{ij}(dy^i + N^i dt)(dy^j + N^j dt)$$

N : lapse function

N_i : shift vector

γ_{ij} : induced metric on Σ

$u_\mu = (N,0,0,0)$: unit normal to Σ

$P^\mu_{\ i} = \dfrac{\partial x^\mu}{\partial y^i}$: projection to Σ

✓Gauss relation ($R^{(3)}_{ijkl}$: Riemann tensor on Σ, $K_{ij} = \frac{1}{2N}(\partial_t\gamma_{ij} - D_iN_j - D_jN_i)$)

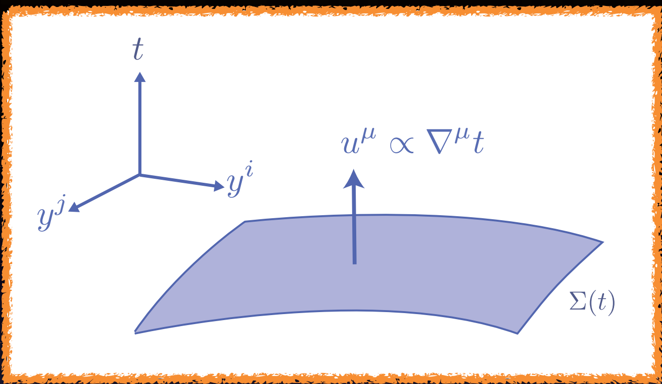
$$R^{(3)}_{ijkl} = P^\mu_{\ i}P^\nu_{\ j}P^\rho_{\ k}P^\sigma_{\ l}R_{\mu\nu\rho\sigma} + K_{il}K_{jk} - K_{ik}K_{jl}$$

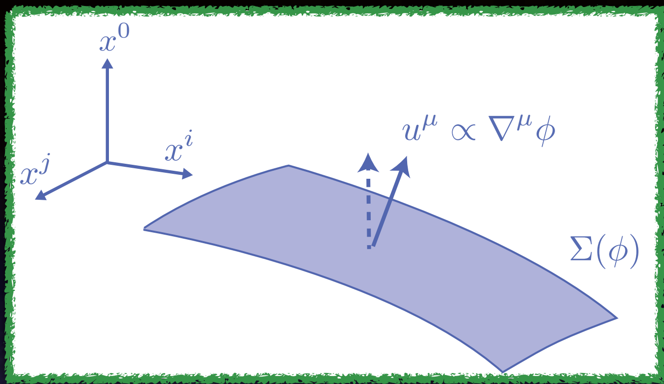
✓Possible terms in HL action (P.Horava, 2009, D.Blas, O.Pujolas and S.Sibiryakov, 2010)

time derivative : $K_{ij}K^{ij}$, K^2 (2nd order)

spacial derivative : combination of $(R^{(3)}_{ij}$, $a_i := \partial_i N/N$, $D_i)$ (up to 6th order)

ADM vs khronon formalism





ADM formalism

coordinate : $t,\ y^i (i = 1, 2, 3)$

variables : $N\ , N^i\ , \gamma_{ij}$

foliation : a priori fixed

null coordinate : No

khronon formalism

coordinate : $x^\mu\ \ (\mu = 0, 1, 2, 3)$

variables : $g_{\mu\nu}\ , \phi$

foliation : can be changed by setting ϕ

null coordinate : Yes

Null coordinate is prohibited in HL gravity

Stueckelberg formalism in HL gravity in IR

$$S_{\rm kh} = \frac{1}{16\pi G_{\rm kh}} \int d^4x\sqrt{-g}\left[R - c_1\left(\nabla_\alpha u^\beta\right)\left(\nabla^\alpha u_\beta\right) - c_2\left(\nabla\cdot u\right)^2 - c_3\left(\nabla_\alpha u^\beta\right)\left(\nabla_\beta u^\alpha\right) + c_4 a^2\right],$$

$u_\mu := \nabla_\mu\phi/\sqrt{-(\nabla_\alpha\phi)(\nabla^\alpha\phi)}$: twistless timilike unit normal to Σ

$a^\mu := u^\alpha(\nabla_\alpha u^\mu)$, $\quad c_1, c_2, c_3, c_4$: arbitrary coupling constants

khronon theory (restricted Einstein-aether) (T.Jacobson, 2010)

ADM formalism ↓

$$S_{\rm HL} = \frac{1}{16\pi G_{\rm HL}} \int dt d^3y\sqrt{\gamma}N\left[K^{ij}K_{ij} - \lambda K^2 + g_1 R^{(3)} + \alpha a^2\right],$$

$G_{\rm HL} := G_{\rm kh}/(1-c_{13})$, $\lambda := (1+c_2)/(1-c_{13})$, $g_1 := 1/(1-c_{13})$,

$\alpha := c_{14}/(1-c_{13})$, $c_{ij} := c_i + c_j$

IR limit of HL gravity

khronon theory is equivalent to HL gravity in IR region

The theory we consider is :

$$S_{\rm khHL} = \frac{1}{16\pi G_{\rm HL}} \int d^4x \sqrt{-g}\, [\mathcal{K}^{ij}\mathcal{K}_{ij} - \lambda \mathcal{K}^2 + g_1 \mathcal{R} + \alpha a^2 + \mathcal{V}_{\rm higher}[\mathcal{R}_{\mu\nu}, a_\mu, \mathcal{D}_\mu]] \ ,$$

$u_\mu := \nabla_\mu \phi / \sqrt{-(\nabla_\alpha \phi)(\nabla^\alpha \phi)}$: twistless timilike unit normal to Σ

$\mathcal{V}_{\rm higher}$: higer spacial derivative terms (up to 6th order)

$\mathcal{R}_{\mu\nu}[g_{\mu\nu}, \phi],\ \mathcal{K}_{\mu\nu}[g_{\mu\nu}, \phi]$: 3D Ricci tensor and extrinsic curvature associated with $g_{\mu\nu},\ \phi$

full HL gravity with khronon

khronon formalism

$$S_{\rm HL} = \frac{1}{16\pi G_{\rm HL}} \int dt d^3x \sqrt{\gamma} N \left[K^{ij}K_{ij} - \lambda K^2 + g_1 R^{(3)} + \alpha a^2 + \mathcal{V}_{\rm higher}[R^{(3)}_{ij}, a_i, D_i]\right] \ ,$$

$G_{\rm HL} := G_{\rm kh}/(1-c_{13})\ ,\ \lambda := (1+c_2)/(1-c_{13})\ ,\ g_1 := 1/(1-c_{13})\ ,$

$\alpha := c_{14}/(1-c_{13})\ ,\ c_{ij} := c_i + c_j$

$\mathcal{V}_{\rm higher}$: higer spacial derivative terms (up to 6th order)

full HL gravity

action

In khronon formalism, 3-quantites can be written by ($P^{\mu}_{\ \nu} := \delta^{\mu}_{\ \nu} + u^{\mu}u_{\nu}$)

$$\mathcal{R}_{\mu\nu} = P^{\beta}_{\ \mu}P^{\delta}_{\ \nu}P^{\gamma}_{\ \alpha}R^{\alpha}_{\ \beta\gamma\delta} + \mathcal{K}_{\mu\alpha}\mathcal{K}^{\alpha}_{\ \nu} - \mathcal{K}\mathcal{K}_{\mu\nu}$$

$$a_{\mu} := u^{\alpha}(\nabla_{\alpha}u_{\mu})\,,\ \mathcal{D}_{\mu}A_{\nu\rho\ldots} := P_{\mu}^{\ \alpha}P_{\nu}^{\ \beta}\cdots(\nabla_{\alpha}A_{\beta\ldots})$$

where, $\mathcal{K}_{\mu\nu} := P^{\alpha}_{\ \mu}P^{\beta}_{\ \nu}(\nabla_{\alpha}u_{\beta})$, and $A_{\beta\ldots}$ is a tensor on Σ. (C.Germani et.al. 2009, etc…)

for simplicity, we consider $\mathcal{R}^n$ terms as higher order spacial derivative.

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu} - \lambda\mathcal{K}^2 + g_1\mathcal{R} + \alpha a^2 + \mathcal{V}_{\rm higher}[\mathcal{R}_{\mu\nu}, a_{\mu}, \mathcal{D}_{\mu}]\right]\,,$$

$$\mathcal{V}_{\rm higher}[\mathcal{R}_{\mu\nu}, a_{\mu}, \mathcal{D}_{\mu}] = g_2\mathcal{R}^2 + g_5\mathcal{R}^3$$

$$\mathcal{R} = R - (\nabla_{\alpha}u_{\beta})(\nabla^{\beta}u^{\alpha}) + (\nabla\cdot u)^2 + 2\nabla_{\alpha}[a^{\alpha} - u^{\alpha}(\nabla\cdot u)]$$

$$\lambda := (1+c_2)/(1-c_{13})\,,\ g_1 := 1/(1-c_{13})\,,\ \alpha := c_{14}/(1-c_{13})$$

theory	variables	aether	higher spacial derivative	null coordinate	e.o.m
Einstein-aether	$g_{\mu\nu}\,,\,u^\mu$	$u^2 = -1$	N/A	possible	$E_{\mu\nu} - (\text{Æ}_\alpha u^\alpha) u_\mu u_\nu = 0$ $(g_{\mu\alpha} + u_\mu u_\alpha)\text{Æ}^\alpha = 0$
khronon theory (IR-HL gravity)	$g_{\mu\nu}\,,\,\phi$	$u_\mu = \frac{\nabla_\mu \phi}{\sqrt{-(\nabla_\alpha\phi)(\nabla^\alpha\phi)}}$	N/A	possible	$E_{\mu\nu} + u_\mu u_\nu (\text{Æ}_\alpha u^\alpha) + 2\text{Æ}_{(\mu} u_{\nu)} = 0\,,$ $\nabla_\mu \left[\frac{(g^{\mu\nu} + u^\mu u^\nu)\text{Æ}_\nu}{\sqrt{-g^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi)}}\right] = 0$
full HL gravity	$N\,,N^i\,,\gamma_{ij}$	$u_\mu = N\delta^t{}_\mu$	$R^{(3)}_{ij}\,,\,a_i\,,\,D_i$	impossible	$\frac{\delta\tilde{S}}{\delta N^i} = 0\,,\ \frac{\delta\tilde{S}}{\delta N} = 0\,,\ \frac{\delta\tilde{S}}{\delta\gamma^{ij}} = 0\,,$
full HL gravity with khronon	$g_{\mu\nu}\,,\,\phi$	$u_\mu = \frac{\nabla_\mu \phi}{\sqrt{-(\nabla_\alpha\phi)(\nabla^\alpha\phi)}}$	$\mathcal{R}_{\mu\nu}\,,\,a_\mu\,,\,\mathcal{D}_\mu$	possible	$E_{\mu\nu} + \mathcal{V}_{\mu\nu} + u_\mu u_\nu (\text{Æ}_\alpha + \mathcal{V}_\alpha)u^\alpha + 2[\text{Æ}_{(\mu} + \mathcal{V}_{(\mu}]u_{\nu)} = 0\,,$ $\nabla_\mu \left[\frac{(g^{\mu\nu} + u^\mu u^\nu)(\text{Æ}_\nu + \mathcal{V}_\nu)}{\sqrt{-g^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi)}}\right] = 0$

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R - M^{\mu\nu}{}_{\alpha\beta}\left(\nabla_\mu u^\alpha\right)\left(\nabla_\nu u^\beta\right)\right]$$

$$\delta S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[E_{\mu\nu}(\delta g^{\mu\nu}) + 2\text{Æ}_\mu(\delta u^\mu)\right]$$

$$\tilde{S} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R - M^{\mu\nu}{}_{\alpha\beta}\left(\nabla_\mu u^\alpha\right)\left(\nabla_\nu u^\beta\right) + \mathcal{V}[\partial_i^6]\right]$$

$$\delta\tilde{S} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[(E_{\mu\nu} + \mathcal{V}_{\mu\nu})(\delta g^{\mu\nu}) + 2(\text{Æ}_\mu + \mathcal{V}_\mu)(\delta u^\mu)\right]$$

spherically symmetric spacetime

✓ansatz : spherically symmetric spacetime in Eddington-Finkelstein coordinate

$$ds^2 = -T(r)dv^2 + 2B(r)dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$u^\mu = \left(a(r)\,, \frac{a(r)^2T(r)-1}{2a(r)B(r)}, 0, 0\right)$$

(1) u^r is determined by norm fixed condition $u^2 = -1$.

(2) In spherical symmetric spacetime, these two equations are equivalent.

$$E_{\mu\nu} + \mathcal{V}_{\mu\nu} + u_\mu u_\nu (Æ_\alpha + \mathcal{V}_\alpha)u^\alpha = 0\,,$$

$$(\delta_\mu{}^\alpha + u_\mu u^\alpha)(Æ_\alpha + \mathcal{V}_\alpha) = 0\,,$$

⟷

$$E_{\mu\nu} + \mathcal{V}_{\mu\nu} + u_\mu u_\nu (Æ_\alpha + \mathcal{V}_\alpha)u^\alpha + 2[Æ_{(\mu} + \mathcal{V}_{(\mu}]u_{\nu)} = 0\,,$$

$$\nabla_\mu \left[\frac{(g^{\mu\nu} + u^\mu u^\nu)(Æ_\nu + \mathcal{V}_\nu)}{\sqrt{-g^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi)}}\right] = 0$$

perturbative solution around asymptotic flat region

explicit form of asymptotic solution up to 4th order

higher order curvature correction g2

$$T(r) = 1 + \frac{t_1}{r} + \left(\frac{t_1^3 c_{14}}{48}\right)\frac{1}{r^3} + \left[\frac{c_{14}\{(19+4c_{14})t_1^4 - 144t_1^2 a_2 + 192 a_2^2 + 384 g_2 t_1^2\}}{192(2-c_{14})}\right]\frac{1}{r^4} + \cdots$$

$$B(r) = 1 + \left(\frac{c_{14}t_1^2}{16}\right)\frac{1}{r^2} - \left(\frac{c_{14}t_1^3}{12}\right)\frac{1}{r^3} + \left[\frac{c_{14}\{3(c_{14}^2 + 14c_{14} + 4)t_1^4 - 576t_1^2 a_2 + 768 a_2^2 + 512(2c_{14}-1)g_2 t_1^2\}}{512(c_{14}-2)}\right]\frac{1}{r^4} + \cdots$$

$$a(r) = 1 - \frac{t_1}{2r} + \frac{a_2}{r^2} - \left[\frac{(c_{14}-6)t_1^3 + 96 t_1 a_2}{96}\right]\frac{1}{r^3}$$
$$+ \frac{[5c_2\{5c_{14}(2c_{14}-1)+24\} + 18c_{14}(c_{14}-2)]t_1^4 + 48[\{c_{14}(c_{14}-2) + 10c_2(c_{14}-5)\}a_2 + 40 g_2 c_{14} c_2]t_1^2 - 1920 c_2(c_{14}-1)g_2 a_2^2}{1920 c_2 (2-c_{14})}\frac{1}{r^4} + \cdots$$

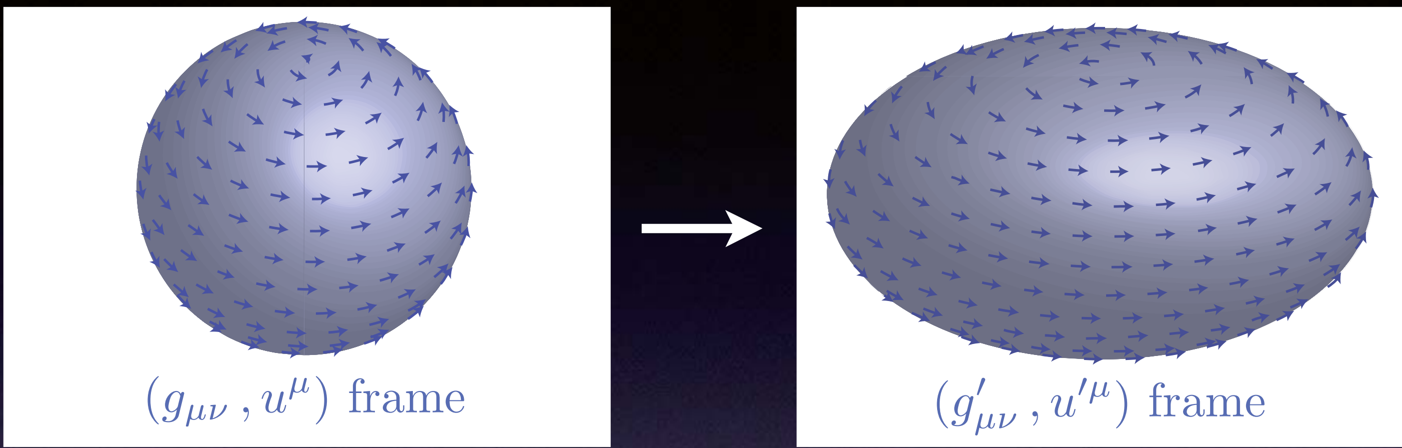
then, it is confirmed that…

(1) every function depends on only t1 and a2 (at least up to 8th order).

(2) g2 first appears in 4th order.

(3) g5 first appears in 8th order.

"time rescaling"



stretching out coordinate along u^{μ}

$$g'_{\mu\nu} = g_{\mu\nu} + (1-\sigma)u_{\mu}u_{\nu}\,, u'^{\mu} = \frac{1}{\sqrt{\sigma}}u^{\mu}$$

their inverse are given by

$$g'^{\mu\nu} = g^{\mu\nu} + \left(1 - \frac{1}{\sigma}\right)u^{\mu}u^{\nu}\,, u'_{\mu} = \sqrt{\sigma}u_{\mu}$$

under this transformation, form of the action is invariant.

However, the value of coupling constants $\lambda\,, g_1$ and α are changed.

horizon for gravitons in Einstein-aether or IR limit of HL

(B.Z.Foster,2005)

$$S_{æ} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R - c_1\left(\nabla_\alpha u^\beta\right)\left(\nabla^\alpha u_\beta\right) - c_2\left(\nabla\cdot u\right)^2 - c_3\left(\nabla_\alpha u^\beta\right)\left(\nabla_\beta u^\alpha\right) + c_4 a^2\right],$$

$$(s_2)^2 = \frac{1}{1-c_{13}}\,,(s_1)^2 = \frac{(c_{13}-1)(c_1-c_3-1)-1}{2c_{14}(c_{13}-1)}\,,(s_0)^2 = \frac{c_{123}(2-c_{14})}{c_{14}(1-c_{13})(2+c_{13}+3c_2)}$$

$u^2 = -1$ and s_i gives the sound speed of spin-i graviton.

following transformation does not change the action except coupling constants

$$g'_{\mu\nu} = g_{\mu\nu} + (1-\sigma)u_\mu u_\nu\,, u'^\mu = \frac{1}{\sqrt{\sigma}}u^\mu$$

that is,

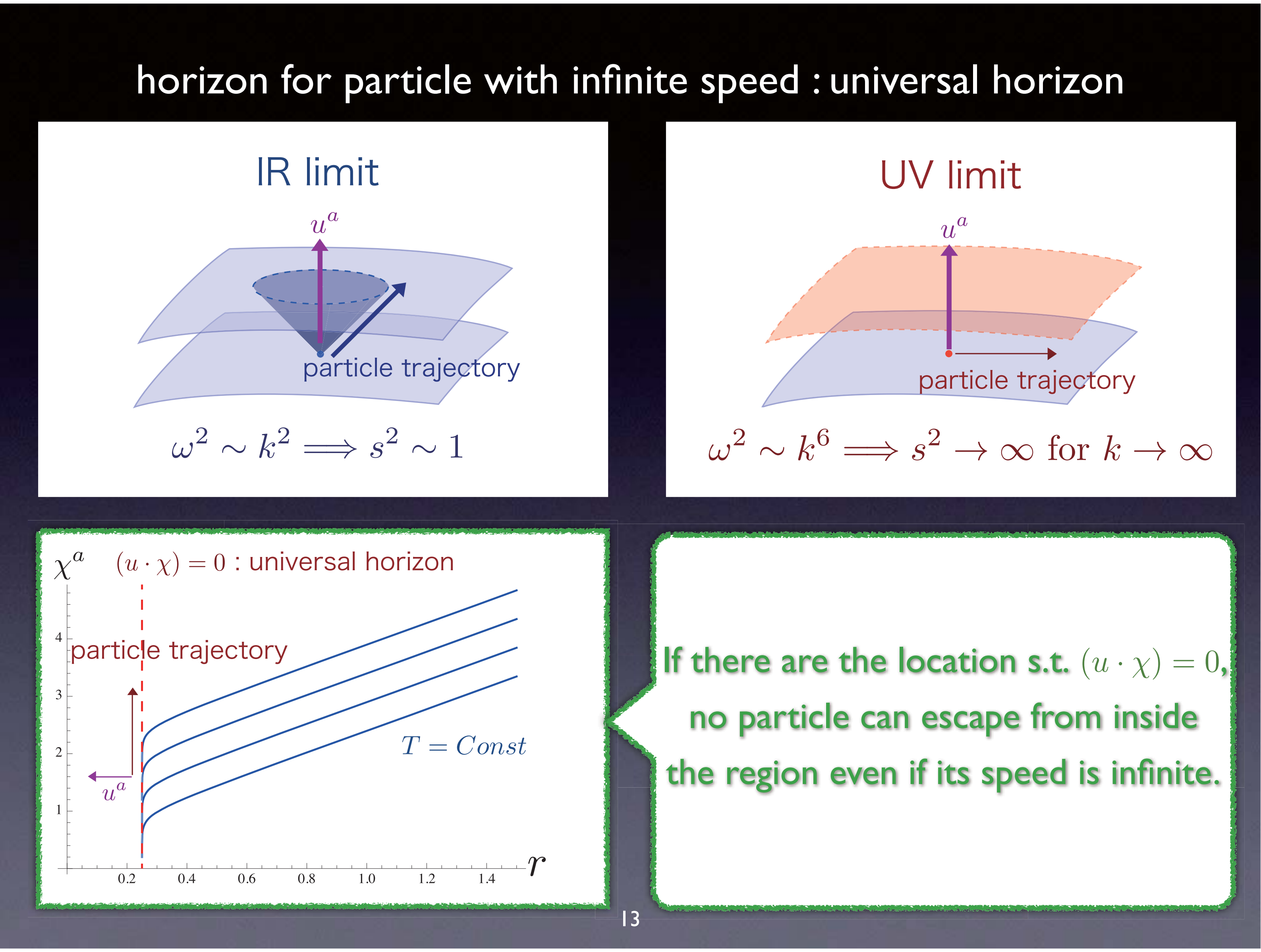
$$c'_{14} = c_{14}\,,\ c'_{123} = \sigma c_{123}\,,\ c'_{13} - 1 = \sigma(c_{13}-1)\,,\ c'_1 - c'_3 - 1 = \sigma^{-1}(c'_1 - c'_3 - 1)$$

which gives,

$$(s'_i)^2 = (s_i)^2/\sigma$$

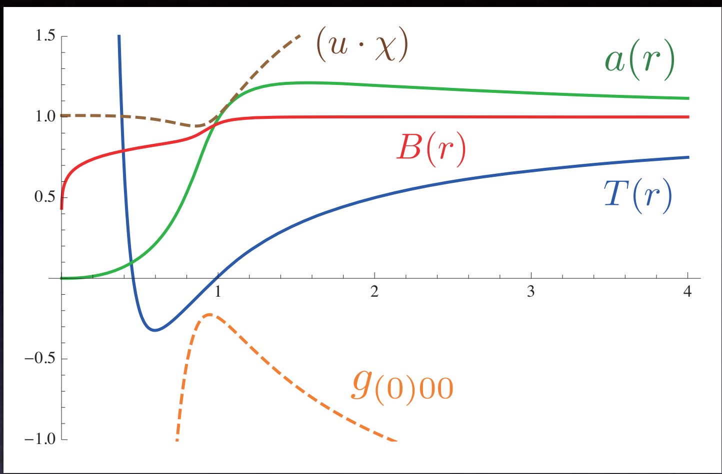
thus, the location of horizon for spin-i graviton given by

$$g^{(i)00} := g_{00} + [1-(s_i)^2]u_0u_0 = 0\,.$$


horizon for particle with infinite speed : universal horizon
IR limit
u^a
particle trajectory
$\omega^2 \sim k^2 \Longrightarrow s^2 \sim 1$
UV limit
u^a
particle trajectory
$\omega^2 \sim k^6 \Longrightarrow s^2 \to \infty$ for $k \to \infty$
χ^a
$(u \cdot \chi) = 0$: universal horizon
particle trajectory
u^a
$T = Const$
r
If there are the location s.t. $(u \cdot \chi) = 0$,
no particle can escape from inside
the region even if its speed is infinite.
13

solution (g2=g5=0)

✓spherical solution without spin-0 horizon (C. Elling and T. Jacobson 2006)



$$\lambda = 279/250\,,\, \alpha = 51/1000\,,\, g_1 = 1\,,$$

$$g_2 = g_5 = 0\,,$$

$$t_1 = -1\,,\, a_2 = -1/10\,.$$

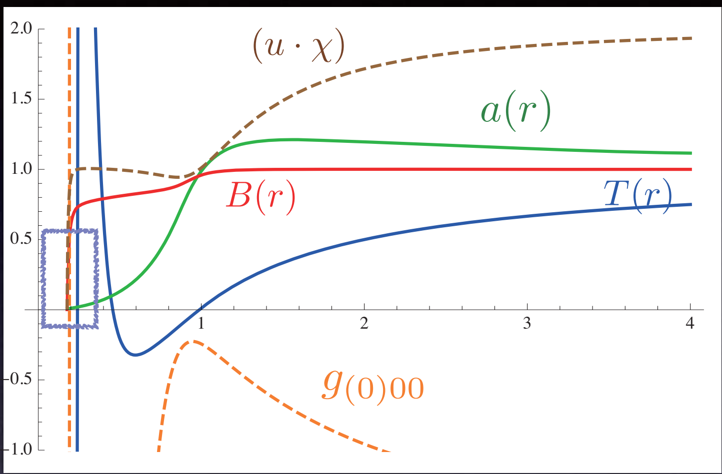
$$(s_2)^2 = 1\,,\, (s_2)^2 = 56521/29937\,,$$

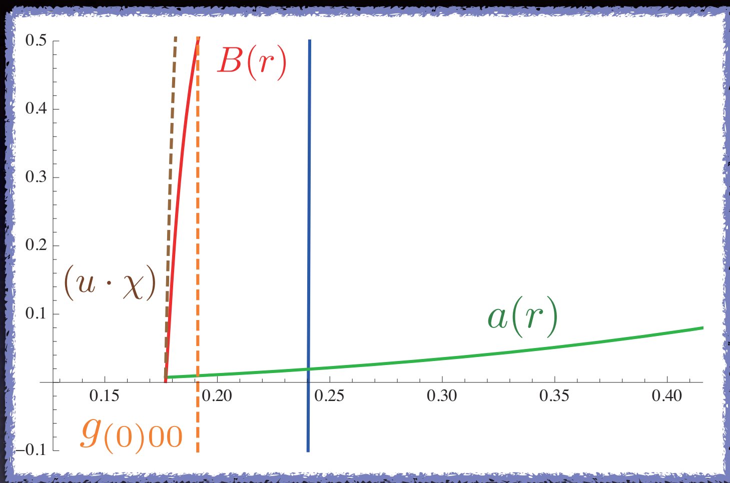
there is double spin-2 horizon, however no spin-0 horizon…

essentially naked singularity

solution with higher order spacial curvature

✓spherical solution with spin-2, 0 horizon ($g_2 = 5.0 \times 10^{-8}$, $g_5 = 0$)





✓horizon for low-energy graviton :

there are triple spin-2 horizon and single spin-0 horizon

✓horizon for high-energy graviton :

universal horizon seems to appear, however it is irregular.

summary and future work

- We consider the HL gravity in khronon formalism considering higher order spacial curvature corrections.
- The effect of such a correction to the static and spherically symmetric black hole is studied.
  - outside the horizon, there is little effect, on the other hand, near or inside the horizon, the spacetime structure drastically changed.
  - we find the black holes solution in IR region, however, it has irregular horizon for high-energy particle.
- Can we impose the regularity on the horizon for high-energy particle?
- How about the effect from other types of correction such as $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$, ...

“Relativistic Sagnac effect by CS gravity”

Daiki Kikuchi (Hirosaki)

[JGRG24(2014)P35]

HIROSAKI UNIVERSITY

Relativistic Sagnac effect by CS gravity

PRD in press

Daiki Kikuchi

Hirosaki University, Japan
with N. Omoto, K. Yamada, and H. Asada (Hirosaki)

JGRG24 in Kavli IPMU Nov. 10 - 14, 2014

Abstract: We discuss relativistic Sagnac effect in Chern-Simons (CS) modified gravity [1]. In particular, we examine possible altitudinal, latitudinal, and directional dependence comparing the CS effects with the general relativistic Lense-Thirring (LT) effects.

1 Motivation

The Chern-Simons (CS) correction is one of the most interesting modified gravity models.

- The CS modification motivated by both string theory and quantum gravity.
- A possible constraint by neutron interferometers has recently been studied [2, 3].

⇒ We improve the previous results regarding two points[1].

- a point-like spinning object [4] → an extended one [5]
- neutron interferometers → Optical (Sagnac) one

2 Relativistic Sagnac effect

○The time shift $c\Delta t$ is given by the relativistic version of Sagnac effect

$$c\Delta t = -2\oint_C \frac{g_{0i}}{g_{00}}dx^i = -2\int_S \left(\vec{\nabla}\times\vec{h}\right)\cdot\vec{N}_I dS + O(h^2). \quad (1)$$

$$\left(g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \;\Rightarrow\; \vec{h} \equiv h_{0i} = (h_{01}, h_{02}, h_{03}) \right)$$

C : a clockwise closed path of a light beam, S : the area of the Sagnac interferometer
$\vec{N}_I$: unit normal vector

3 Time shift and Chern-Simons(CS)modified gravity

○The action of CS gravity theory [5]

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{2\kappa^2}\boldsymbol{R} + \frac{\ell}{12}\theta\boldsymbol{R}\tilde{\boldsymbol{R}} - \frac{1}{2}(\partial\theta)^2 - V(\theta) + \mathcal{L}_{mat}\right], \quad (2)$$

$$\kappa^2 = \frac{8\pi G}{c^4}, \quad \boldsymbol{R}\tilde{\boldsymbol{R}} \equiv R^{\alpha}{}_{\beta}{}^{\gamma\delta}R^{\beta}{}_{\alpha\gamma\delta} = \frac{1}{2}\varepsilon^{\gamma\delta\mu\nu}R^{\alpha}{}_{\beta\mu\nu}R^{\beta}{}_{\alpha\gamma\delta}.$$

ℓ : the parameter of the theory , θ : the scalar field

○The metric as the weak-field solution by Earth (**extended source**)

$$\vec{h}_{LT} = \frac{4GM_E r_E^2}{5c^3r^2}\left(\vec{n}\times\vec{\omega}\right). \quad (3)$$

$$\vec{h}_{CS} = \frac{12GM_E}{m_{CS}c^3r_E}\left[C_1(r)\vec{\omega} + C_2(r)\vec{n}\times\vec{\omega} + C_3(r)\vec{n}\times(\vec{n}\times\vec{\omega})\right], \; m_{CS} \equiv -\frac{3}{\ell\kappa^2\dot{\theta}}. \quad (4)$$

$\vec{\omega}$: angular velocity vector, $\vec{n}$: unit vertical vector

○The time shift

$$(c\Delta t)_{LT} = \frac{8GM_E{r_E}^2S}{5c^3r^3}\vec{N}_I\cdot\left[2\vec{\omega} - 3\vec{\rho}\right]. \quad (5)$$

$$(c\Delta t)_{CS} = \frac{24GM_ES}{c^3r_E}\vec{N}_I\cdot\left[D_1(r)\vec{\omega} - D_2(r)\vec{\lambda} - D_3(r)\vec{\rho}\right]. \quad (6)$$

where $\vec{\rho} \equiv \vec{n}\times(\vec{\omega}\times\vec{n})$, $\vec{\lambda} \equiv \vec{\omega}\times\vec{n}$. $r \geq r_E$,

$$C_1(r) = \frac{2{r_E}^3}{15r^3} + \frac{2r_E}{r}j_2(m_{CS}r_E)y_1(m_{CS}r), \qquad D_1(r) = \frac{2r_E}{r}j_2(m_{CS}r_E)y_1(m_{CS}r),$$

$$C_2(r) = m_{CS}r_E j_2(m_{CS}r_E)y_1(m_{CS}r), \qquad D_2(r) = m_{CS}r_E j_2(m_{CS}r_E)y_1(m_{CS}r),$$

$$C_3(r) = \frac{{r_E}^3}{5r^3} + m_{CS}r_E j_2(m_{CS}r_E)y_2(m_{CS}r), \quad D_3(r) = m_{CS}r_E j_2(m_{CS}r_E)y_2(m_{CS}r). \quad (7)$$

$j_n(z)$, $y_n(z)$: spherical Bessel function of the first and second kind, respectively

⇒ *(5) and (6) depend on interferometers'*

latitude $(\vec{\rho}, \vec{\lambda})$, *direction* $(\vec{N}_I)$, *and* *altitude* (r).

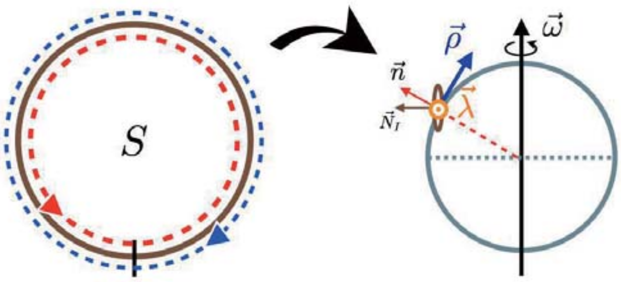


Figure 1: Sagnac interferometer on Earth and related vectors.

4 Dependence of time shift

○The angular part of $(c\Delta t)_{LT}$ ○The angular part of $(c\Delta t)_{CS}$

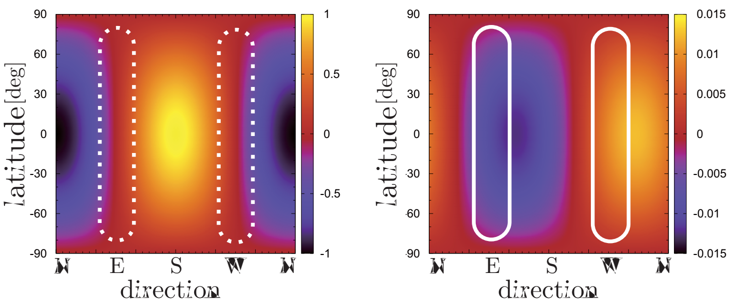


Figure 2: Contour maps for the latitudinal and directional dependence of time shift.

○ $(c\Delta t)_{CS}/(c\Delta t)_{LT}$

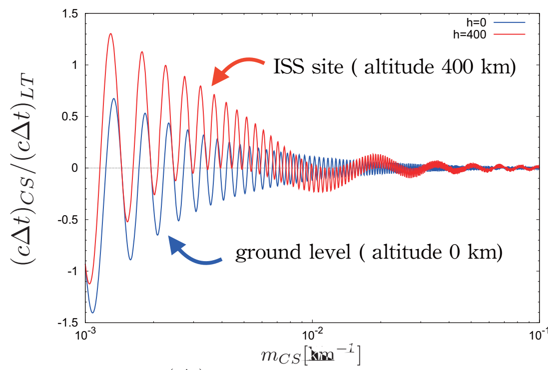


Figure 3: The ratio of $\frac{(c\Delta t)_{CS}}{(c\Delta t)_{LT}}$ at the equatorial case and the northbound direction.

5 Order of magnitude estimation

○The strain of the time shift

$$\frac{(c\Delta t)_{LT}}{\sqrt{S}} \sim 10^{-21}\left(\frac{\sqrt{S}}{10\text{m}}\right) \quad (8)$$

$$\frac{(c\Delta t)_{CS}}{\sqrt{S}} \sim 10^{-22}\left(\frac{\sqrt{S}}{10\text{m}}\right)\left(\frac{0.01\text{km}^{-1}}{m_{CS}}\right) \quad (9)$$

★ *GINGER* experiment will measure LT effect with 1 % accuracy by reducing various sources of noises [6] .

6 Conclusion

We investigated relativistic Sagnac effects in CS modified gravity.

⟨**The latitudinal and directional dependence** ⟩

LT effects on the eastbound interferometer cancel out.
⇒ The eastbound Sagnac interferometer might be preferred for testing CS separately.

⟨**The altitudinal dependence**⟩

The altitudinal effect makes a more complicated form of oscillating behavior in terms of m_{CS} at the ISS site compared with the ground level.
⇒ Space experiments might place tighter constraints on m_{CS}.

References

[1] D. Kikuchi et al., Phys. Rev. D **90**, 064036 (2014).

[2] H. Okawara, K. Yamada, and H. Asada, Phys. Rev. Lett. **109**, 231101 (2012).

[3] H. Okawara, K. Yamada, and H. Asada, Phys. Rev. D **87**, 084038 (2013).

[4] S. Alexander and N. Yunes, Phys. Rev. Lett. **99**, 241101 (2007).

[5] T. L. Smith et al., Phys. Rev. D **77**, 024015 (2008).

[6] A. Tartaglia, J. Phys. Conf. Ser. **453**, 012019 (2013).

“Disformal transformation of cosmological perturbations”

Masato Minamitsuji (IST, Lisbon)

[JGRG24(2014)P36]

Disformal transformation of cosmological perturbations

Masato Minamitsuji (IST, University of Lisbon)

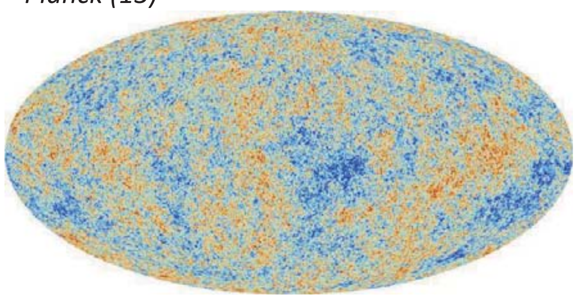
Physics Letters B 737 (2014) 139-150

Inflation and Modifications of GR

Inflation

- Explaining flatness and homogeneity of the Universe
- Generating successful seed of Large Scale Structures
- Supported by observations.

Planck (13)



➢ Models with nonminimal coupling

$$L = \sqrt{-g}[f(\phi)R - \omega(\phi)(\partial\phi)^2 - V(\phi)]$$

□ Ubiquitous in high energy physics

String theory
Copeland, Easther and Wands (97)

Renormalization
Bezrukov and Shaposhnikov (08)

Higher curvature theory
Starobinsky (81)

□ Comfortably consistent with observation data Kallosh and Linde (13,14)

□ Frame independence of observables Makino and Sasaki (86), Chiba and Yamaguchi (08), Gong, et.al (11)

Comoving curvature perturbation $R_c \; (\leftrightarrow \frac{\delta T}{T})$

$$R_c := R - \frac{H}{d\phi/dt}\delta\phi \quad \Rightarrow \widetilde{R_c} = R_c$$

...may be evaluated in the convenient Einstein frame

The Horndeski Scalar-Tensor Theory

- Most general scalar-tensor theory with 2nd order EOMs.

$\Rightarrow$ *Ghost-free*

Horndeski (74), Deffayet, Esposito-Farese & Vikman (09), Kobayashi, Yamaguchi & Yokoyama (11)

$$\mathcal{L}_g[g,\phi] = \sum_{i=2}^{5} \mathcal{L}_i[g,\phi] \qquad X := -\frac{1}{2} g^{\mu\nu}\phi_\mu \phi_\nu \quad \phi_\mu = \nabla_\mu \phi$$

$$\begin{aligned}
\mathcal{L}_2 &= P(X,\phi), \quad \mathcal{L}_3 = -G(X,\phi)\Box\phi, \\
\mathcal{L}_4 &= G_4(X,\phi)R + G_{4,X}\left((\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right), \\
\mathcal{L}_5 &= G_5(X,\phi)G_{\mu\nu}\phi^{\mu\nu} \\
&\quad - \frac{1}{6}G_{5,X}\left[(\Box\phi)^3 - 3(\Box\phi)\phi_{\mu\nu}\phi^{\mu\nu} + 2\phi_{\mu\alpha}\phi^{\alpha\nu}\phi^{\mu}_{\nu}\right]
\end{aligned}$$

Realistic models of Inflation, Dark Energy and Modified Gravity belong to the Horndeski scalar-tensor theory.

Framing the Horndeski theory

- Conformal transformation $\bar{g}_{\mu\nu} = \alpha(\phi)g_{\mu\nu}$

  can frame the scalar-tensor theory with nonminimal coupling $f(\phi)R$

$\Longrightarrow$ The Horndeski theory is framed within the *disformal* transformation

- Disformal transformation Bekenstein (93)

  The transformation including up to the 1st order derivative of ϕ

$$\bar{g}_{\mu\nu} = \alpha(\phi)g_{\mu\nu} + \beta(\phi)\phi_\mu\phi_\nu$$

$$\Longrightarrow \quad \bar{\mathcal{L}}_g[\bar{g},\phi] = \sum_{i=2}^{5} \bar{\mathcal{L}}_i[\bar{g},\phi]$$

Bettoni & Liberati (13)

The theory written in terms of $\bar{g}_{\mu\nu}$ belongs to another class of the Horndeski theory.

$\Longrightarrow$ Here we will restrict to the class of the Horndeski theory.

C.f. Framing the scalar-tensor theory beyond Horndeski. Zumalacárregui & Garcia-Bellido (13)
Gleyzes, Langlois, Piazza & Vernizzi (14)

$$\bar{g}_{\mu\nu} = \alpha(X,\phi)g_{\mu\nu} + \beta(X,\phi)\phi_\mu\phi_\nu$$

Disformal Transformation

Bekenstein (93) Bettoni & Liberati (13)

- Keeping causality for the conformal part $\alpha > 0$
- Disformal transformation modifies the causal structure of spacetime

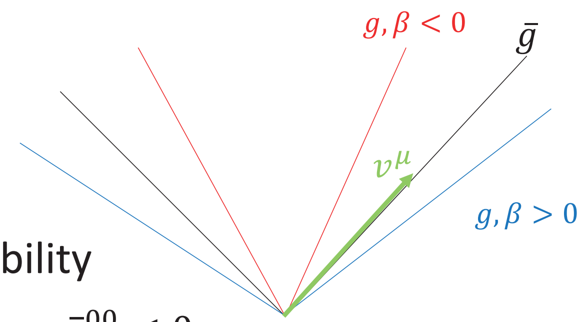
  v^μ : a null vector field in the barred frame $\bar{g}_{\mu\nu} v^\mu v^\nu = 0$

$\beta > 0 \Rightarrow g_{\mu\nu} v^\mu v^\nu < 0$

Timelike in the original frame

$\beta < 0 \Rightarrow g_{\mu\nu} v^\mu v^\nu > 0$

Spacelike in the original frame



- Lorentz signature, causal behavior and invertibility

$\bar{g}_{00} < 0$ $\quad d\bar{s}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu < 0$ $\quad -\bar{g} > 0$ $\quad \bar{g}^{00} < 0$

$$\Rightarrow \quad \alpha g_{00} + \beta \phi_0^2 < 0 \qquad \beta < 0 \qquad \alpha - 2\beta X > 0$$

Disformal Coupling to the Matter Sector

$$S = S_g + S_m \quad S_g := \int d^4x\sqrt{-g}\mathcal{L}_g[g,\phi]$$

Gravity sector (Horndeski theory)

$$S_m := \int d^4x\sqrt{-\bar{g}}\mathcal{L}_m[\bar{g},\Psi]$$

Matter sector (disformally coupled)

- Gravity and Matter frames related by the disformal relation.

$$\bar{g}_{\mu\nu} = \alpha(\phi)g_{\mu\nu} + \beta(\phi)\phi_\mu\phi_\nu$$

Matter frame ($\bar{g}_{\mu\nu}$) Gravity frame ($g_{\mu\nu}$)

- The energy-momentum is not conserved in gravity frame but in matter frame.

$$-\nabla_\mu E^{\mu\nu} = \nabla_\mu T^{\mu\nu}_{(m)} \qquad \bar{\nabla}_\mu \bar{T}^{\mu\nu}_{(m)} = 0$$

- We will investigate the relation of curvature perturbations associated with the scalar and matter fields between frames and their evolution.

Disformal Inflation

Kaloper (04)

- Deceleration in the gravity frame : $\alpha = 1 \quad \beta = -\frac{1}{m^4}$

$$ds^2 = -dt^2 + t^{\frac{2}{3}}\delta_{ij}dx^i dx^j \quad \phi = \phi_0 + m_p \ln t$$

$\Rightarrow$ Inflation in the matter frame

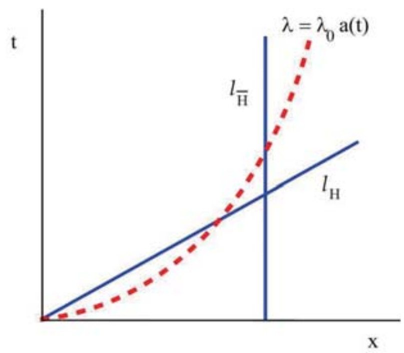
$t \simeq 0 \qquad d\bar{s}^2 \simeq -d\tau^2 + e^{H\tau}\delta_{ij}dx^i dx^j \qquad H := \frac{m^2}{3m_p}$

- $\delta\phi$ cannot be the responsible source for curvature perturbations.

$\Downarrow$

- Density perturbations should be sourced by matter propagating on the matter frame

$$L_m = \sqrt{-\bar{g}}\left(-\bar{g}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - m_\chi^2\chi^2\right) \qquad m_\chi \ll H$$



Taken from hep-ph/0312002

$\Rightarrow$ In general, both scalar field and matter contribute to density perturbations.

Need of a general formulation of cosmological perturbations

Perturbations in the Gravity Sector

- Curvature perturbations from the inflaton fluctuations $\phi + \delta\phi$
- Perturbed FLRW universe in the gravity frame

$$\begin{aligned} ds^2 &= -(1+2A(t,x^i))dt^2 + 2a(t)\partial_i B(t,x^i)dtdx^i \\ &+ a(t)^2\Big[(1-2\psi(t,x^i))\delta_{ij} + 2\partial_i\partial_j E(t,x^i)\Big]dx^i dx^j \end{aligned}$$

- Perturbed FLRW universe in the matter frame

$$d\hat{t} = \sqrt{\alpha - \beta\dot{\phi}^2}dt \quad \hat{x}^i = x^i; \qquad a \longrightarrow \hat{a} \quad A \longrightarrow \hat{A} \quad B \longrightarrow \hat{B} \quad \psi \rightarrow \hat{\psi} \quad E \rightarrow \hat{E}$$

Proper time in matter frame

Relating frames

$$\hat{a} = \sqrt{\alpha}a \qquad \hat{A} = \frac{\alpha A + \frac{\alpha'\delta\phi}{2} - \frac{1}{2}\dot{\phi}^2\beta'\delta\phi - \beta\dot{\phi}\dot{\delta\phi}}{\alpha - \beta\dot{\phi}^2} \qquad \hat{B} = \frac{B + \frac{\beta\dot{\phi}}{a\alpha}\delta\phi}{\sqrt{1-\frac{\beta}{\alpha}\dot{\phi}^2}}$$

$$\hat{\psi} = \psi - \frac{\alpha'\delta\phi}{2\alpha}, \qquad \hat{E} = E$$

- Gauge-invariant metric perturbations

$$\Phi = A - \frac{d}{dt}\Big[a^2(\dot{E} - \frac{B}{a})\Big] \qquad \Psi = \psi + a^2\frac{\dot{a}}{a}(\dot{E} - \frac{B}{a}) \qquad \mathcal{R}_c^{(\phi)} = \psi + \frac{1}{\dot{\phi}}\frac{\dot{a}}{a}\delta\phi.$$

$$\hat{\Phi} = \hat{A} - \frac{d}{d\hat{t}}\Big[\hat{a}^2(\hat{E}_{,\hat{t}} - \frac{\hat{B}}{\hat{a}})\Big] \qquad \hat{\Psi} = \hat{\psi} + \hat{a}^2\frac{\hat{a}_{,\hat{t}}}{\hat{a}}(\hat{E}_{,\hat{t}} - \frac{\hat{B}}{\hat{a}}) \qquad \hat{\mathcal{R}}_c^{(\phi)} = \hat{\psi} + \frac{1}{\phi_{,\hat{t}}}\frac{\hat{a}_{,\hat{t}}}{\hat{a}}\delta\phi.$$

Relating frames

$$\hat{\Psi} = \Psi - \frac{1}{\alpha - \beta\dot{\phi}^2}\Big(\beta\dot{\phi}^2\frac{\dot{a}}{a} + \frac{\dot{\alpha}}{2}\Big)\frac{\delta_g\phi}{\dot{\phi}}$$

$$\hat{\Phi} = \frac{1}{\alpha - \beta\dot{\phi}^2}\Big\{\alpha\Phi + \frac{\dot{\alpha}(\alpha - 2\beta\dot{\phi}^2) + \alpha(\dot{\phi}^2\dot{\beta} + 2\beta\dot{\phi}\ddot{\phi})}{2(\alpha - \beta\dot{\phi}^2)}\frac{\delta_g\phi}{\dot{\phi}}\Big\}$$

$$\hat{R}_C^{(\phi)} = R_C^{(\phi)} \qquad \delta_g Y = \delta Y - a^2\dot{Y}\Big(\dot{E} - \frac{B}{a}\Big)$$

Comoving curvature perturbation $R_C^{(\phi)}$ ($\longleftrightarrow \frac{\delta T}{T}$) is disformally invariant as well as conformally invariant.

⇒ may be evaluated in any disformally related frames.

Perturbations in the Matter Sector

- Disformally coupled matter could be the dominant source of density perturbations via the curvaton mechanism   Kaloper (04)
- The non-interacting fluids $T^{\mu\nu}_{(m)} = \sum_a T^{(a)\mu\nu}, \quad \hat{T}^{\mu\nu}_{(m)} = \sum_a \hat{T}^{(a)\mu\nu}$

$$T^{(a)0}{}_0 = -\rho^{(a)} - \delta\rho^{(a)}, \quad T^{(a)i}{}_0 = -\frac{\rho^{(a)} + p^{(a)}}{a}\partial^i v^{(a)}$$
$$T^{(a)i}{}_j = (p^{(a)} + \delta p^{(a)})\delta^i{}_j + p^{(a)}\left[\partial^i\partial_j - \frac{1}{3}\delta^i{}_j\Delta\right]\Pi^{(a)}$$

"Hatted" components for the matter frame counterparts.

- Background $\hat{\rho}^{(a)} = f\rho^{(a)}, \quad \hat{p}^{(a)} = \frac{\alpha}{\alpha - \beta\dot{\phi}^2} f p^{(a)} \quad f := \frac{\sqrt{\alpha - \beta\dot{\phi}^2}}{\alpha^{\frac{5}{2}}}$
- Perturbations are also related, as $\hat{\Pi}^{(a)} = \Pi^{(a)}$

Disformal transformation keeps the structure of the energy-momentum tensors.

➢Curvature perturbation in the uniform energy density hypersurface

Gravity frame $$-\zeta^{(a)} := \psi + \frac{\dot{a}}{a}\frac{\delta\rho^{(a)}}{\dot{\rho}^{(a)}} \qquad \zeta = \sum_a \frac{\dot{\rho}^{(a)}}{\dot{\rho}}\zeta^{(a)}$$

Matter frame $$-\hat{\zeta}^{(a)} := \hat{\psi} + \frac{\hat{a}_{,\hat{t}}}{\hat{a}}\frac{\delta\hat{\rho}^{(a)}}{\hat{\rho}^{(a)}_{,\hat{t}}} \qquad \hat{\zeta} = \sum_a \frac{\hat{\rho}^{(a)}_{,\hat{t}}}{\hat{\rho}_{,\hat{t}}}\hat{\zeta}^{(a)}$$

$$\Rightarrow \quad -(\hat{\zeta}^{(a)} - \zeta^{(a)}) = \frac{\frac{\dot{\rho}^{(a)}}{\rho^{(a)}}\frac{\dot{\alpha}}{2\alpha} - \frac{\dot{a}}{a}\frac{\dot{f}}{f}}{\frac{\dot{f}}{f} + \frac{\dot{\rho}^{(a)}}{\rho^{(a)}}}\frac{1}{\dot{\rho}^{(a)}}\delta_\phi\rho^{(a)} + \beta\frac{\frac{\dot{a}}{a} + \frac{\dot{\alpha}}{2\alpha}}{\frac{\dot{f}}{f} + \frac{\dot{\rho}^{(a)}}{\rho^{(a)}}}\frac{\Sigma^{(\phi)}}{\alpha - \beta\dot{\phi}^2}$$

Curvature perturbations in two frames are *not* equivalent
by the isocurvature perturbations associated with the scalar field.

$$\Sigma^{(\phi)} := A\dot{\phi}^2 - \dot{\phi}\dot{\delta\phi} + \ddot{\phi}\delta\phi \Leftrightarrow \Gamma^{(\phi)} := \delta p^{(\phi)} - \frac{\dot{p}^{(\phi)}}{\dot{\rho}^{(\phi)}}\delta\rho^{(\phi)}$$

$$\delta_\phi Y := \delta Y - \frac{\dot{Y}}{\dot{\phi}}\delta\phi$$

Not suppressed on superhorizon scales
in the presence of disformal coupling

Evolution of Curvature Perturbations

- In the matter frame

Wands, Malik, Lyth and Liddle (00)

$$\hat{\nabla}_\mu \hat{T}^{(a)\mu}{}_\nu = 0 \quad \Rightarrow \quad \hat{\zeta}^{(a)}_{,\hat{t}} = -\frac{1}{\hat{\rho}^{(a)} + \hat{p}^{(a)}} \frac{\hat{a}_{,\hat{t}}}{\hat{a}} \hat{\Gamma}^{(a)}$$

$$\hat{\Gamma}^{(a)} := \delta\hat{p}^{(a)} - \frac{\hat{p}^{(a)}_{,\hat{t}}}{\hat{\rho}^{(a)}_{,\hat{t}}} \delta\hat{\rho}^{(a)}$$

Entropy perturbation

- In the gravity frame

$$\nabla_\mu T^{(a)\mu}{}_\nu = -Q^{(a)}\phi_\nu \qquad Q^{(a)} := \nabla_\rho\left(\frac{\beta}{\alpha} T^{(a)\rho\sigma}\phi_\sigma\right) - \frac{1}{2\alpha} T^{(a)\rho\sigma}\left(\alpha' g_{\rho\sigma} + \beta'\phi_\rho\phi_\sigma\right)$$

$$\Rightarrow \quad \dot{\zeta}^{(a)} = C_1^{(a)} \frac{\delta_{\rho^{(a)}}\phi}{\dot{\phi}} + C_2^{(a)} \frac{d}{dt}\left(\frac{\delta_{\rho^{(a)}}\phi}{\dot{\phi}}\right) + C_3^{(a)} \Sigma^{(\phi)} + C_4^{(a)} \Gamma^{(a)}$$

Sourcing by the coupling to the scalar field

$$\Gamma^{(a)} := \delta p^{(a)} - \frac{\dot{p}^{(a)}}{\dot{\rho}^{(a)}} \delta\rho^{(a)}$$

Entropy perturbation

$C_i^{(a)}$: Background dependent coefficients

- The adiabaticity conditions in both frames are not equivalent:

$$\hat{\Gamma}^{(a)} = 0 \not\Leftrightarrow \Gamma^{(a)} = 0$$

Conservation in the matter frame does not lead to that in gravity frame

$$\hat{\zeta}^{(a)}_{,\hat{t}} = 0 \Rightarrow \dot{\zeta}^{(a)} \approx \left[C_1^{(a)} - \frac{p^{(a)}}{\rho^{(a)}}\dot{\rho}^{(a)}\Big(\frac{\hat{\rho}^{(a)}}{\hat{p}^{(a)}}\frac{\hat{p}^{(a)}_{,\hat{t}}}{\hat{\rho}^{(a)}_{,\hat{t}}} - \frac{\rho^{(a)}}{p^{(a)}}\frac{\dot{p}^{(a)}}{\dot{\rho}^{(a)}}\Big)C_4^{(a)}\right]\frac{\delta_{\rho^{(a)}}\phi}{\dot{\phi}}$$
$$+ C_2^{(a)}\frac{d}{dt}\Big(\frac{\delta_{\rho^{(a)}}\phi}{\dot{\phi}}\Big)$$
$$+ \left[C_3^{(a)} + \frac{\beta p^{(a)}}{\alpha - \beta\dot{\phi}^2}\Big(1 + \frac{\hat{p}^{(a)}_{,t}}{\hat{\rho}^{(a)}_{,t}}\frac{\hat{\rho}^{(a)}}{\hat{p}^{(a)}}\Big)C_4^{(a)}\right]\Sigma^{(\phi)},$$

- Curvature perturbation should be finally evaluated in the matter frame where CMB photons propagate along the null geodesics.

$$-\hat{\rho}^{(a)}_{,t}\hat{\zeta}^{(a)} = -f\dot{\rho}^{(a)}\zeta^{(a)} + \dot{f}\rho^{(a)}\mathcal{R}_c^{(\phi)} + f\frac{\dot{\alpha}}{2\alpha}\delta_\phi\rho^{(a)}$$
$$+ \beta f\rho^{(a)}\Big(\frac{\dot{a}}{a} + \frac{\dot{\alpha}}{2\alpha}\Big)\frac{\Sigma^{(\phi)}}{\alpha - \beta\dot{\phi}^2}.$$

Summary

- Disformal transformation can frame the Horndeski theory:

$$\bar{g}_{\mu\nu} = \alpha(\phi) g_{\mu\nu} + \beta(\phi)\partial_\mu\phi\partial_\nu\phi$$

$\Leftarrow$ Conformal transformation $\bar{g}_{\mu\nu} = \alpha(\phi) g_{\mu\nu}$
for the ordinary nonminimally coupling $f(\phi)R$

- Comoving curvature perturbation $R_C^{(\phi)}$ is disformally invariant.

- Curvature perturbations associated with matter are not equivalent, but straightforwardly related between frames.

- Vector and tensor perturbations, and tensor-to-scalar ratio are manifestly disformally invariant.

“New model of massive spin-2 on curved spacetime”

Yuichi Ohara (Nagoya)

[JGRG24(2014)P37]

New model of massive spin-2 on curved space-time

Yuichi Ohara (QG lab. Nagoya univ.)

This work is in collaboration with Satoshi Akagi and Shin'ichi Nojiri

Phys.Rev.D90 (2014) 043006, arXiv : 1410. 5553

Fierz-Pauli theory

$$\mathcal{L}_{\rm FP} = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2)$$

The system has 5 degrees of freedom thanks to the Fierz-Pauli mass term.

Interactinon of massive spin-2 theories

Ghost-free interaction for Fierz-Pauli theory

Hinterbichler, JHEP 10 (2013) 102

$$\mathcal{L}_{d,n} \sim \eta^{\mu_1\nu_1\cdots\mu_n\nu_n}\partial_{\mu_1}\partial_{\nu_1}h_{\mu_2\nu_2}\cdots\partial_{\mu_{d-1}}\partial_{\nu_{d-1}}h_{\mu_d\nu_d}h_{\mu_{d+1}\nu_{d+1}}$$

d : the number of derivatives, n: the number of the field

$\eta^{\mu_1\nu_1\cdots\mu_n\nu_n}$ is products of Minkowski metrics anti-symmetrized over v

In 4 dimensions, there exist 3 interaction terms

$$\mathcal{L}_{0,3} \sim \eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}$$

$$\mathcal{L}_{0,4} \sim \eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4}$$

$$\mathcal{L}_{2,3} \sim \eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\partial_{\mu_1}\partial_{\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4}$$

From the ghost-free interaction terms, we construct a new model of massive spin-2 particles.

New model of massive spin-2 in flat space-time

$$\mathcal{L} = \mathcal{L}_{FP} - \frac{\mu}{3!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3} - \frac{\lambda}{4!}\eta^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4}$$

μ , λ : coupling constants.

Application of the new model

1. SUSY breaking

One of the SUSY breaking model uses V.E.V of a scalar field theory with the potential because V.E.V of the scalar field does not break the isotropy.

The V.E.V of the trace part of the rank 2 tensor can break SUSY keeping the isotropy.

2. BH physics, Cosmology

Before considering the application……

We have to consider whether a ghost appears or not when the model is coupled with gravity.

Ghost-free model in curved space-time

$$S = \int d^4x\sqrt{-g}\left\{\mathcal{L}_{FP} + \frac{\xi}{4}Rh_{\mu\nu}h^{\mu\nu} + \frac{1-2\xi}{8}Rh^2 - \frac{\mu}{3!}g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3} - \frac{\lambda}{4!}g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4}\right\}$$

Here the background metric satisfies $R_{\mu\nu} = \frac{1}{4}g_{\mu\nu}R$

- h is not the perturbation of g but a independent tensor field of g
- Non-minimal couplings and the restriction of the space-time are required for the ghost-free property.

Counting d.o.f in the lagrangian formalism

e.o.m

$$E_{\mu\nu} = -g_{(\mu\nu)}{}^{\mu_1\nu_1\mu_2\nu_2}\nabla_{\mu_1}\nabla_{\nu_1}h_{\mu_2\nu_2} + (\text{terms without } \nabla\nabla h)$$

The second time derivatives are not defined for h_{00} and h_{0i}.

Primary constraint

$$\begin{aligned} E^0{}_\nu &= g^{00}E_{0\nu} + g^{0i}E_{i\nu} \\ &= -g_{\nu\sigma}g^{(0\sigma)\mu_1\nu_1\mu_2\nu_2}\nabla_{\mu_1}\nabla_{\nu_1}h_{\mu_2\nu_2} + (\text{terms without } \nabla\nabla h) \equiv \phi^{(1)}_\nu \approx 0 \end{aligned}$$

Secondary constraint

$$\partial_0\phi^{(1)}_\nu = \partial_0 E^0{}_\nu \approx \nabla^\mu E_{\mu\nu} \equiv \phi^{(2)}_\nu \approx 0$$

$\phi^{(2)}_\nu$ do not contain any time derivative of h_{00}

$$\begin{aligned} \partial_0\phi^{(2)0} &= \partial_0\nabla^\mu E_{\mu\nu} \\ &\approx \nabla^\nu\nabla^\mu E_{\mu\nu} + \frac{m^2}{2}g^{\mu\nu}E_{\mu\nu} - \mu h^{\mu\nu}E_{\mu\nu} + \frac{1-\xi}{4}Rg^{\mu\nu}E_{\mu\nu} + \cdots \equiv \phi^{(3)} \approx 0 \end{aligned}$$

$\phi^{(3)}$does not contain any time derivative of h_{00}

$$\partial_0\phi^{(3)} \approx (\text{terms without } \nabla_0\nabla_0 h) \equiv \phi^{(4)} \approx 0$$

($\partial_0\phi^{(2)}_i$ and $\partial_0\phi^{(4)}$give $\ddot{h}_{0i}$ and $\ddot{h}_{00}$ respectively.)

10 constraints for $h_{\mu\nu}$ and $\dot{h}_{\mu\nu}$ and the system has 5 d.o.f

Derivative interaction

$$g^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4}\nabla_{\mu_1}\nabla_{\nu_1}h_{\mu_2\nu_2}\cdot h_{\mu_3\nu_3}h_{\mu_4\nu_4}$$

Is this interaction ghost-free on the Einstein manifold?

$$\phi^{(2)\mu} = \nabla_\nu E^{\mu\nu} \supset \left\{-C^{\mu\alpha0\beta}g^{00} + C^{\alpha0\beta0}g^{\mu0} + C^{\mu00\alpha}g^{\beta0}\right\}h_{\alpha\beta}\nabla_0 h_{00}$$

C : Weyl tensor

$\phi^{(2)}$ should not contain any time derivative of h_{00} for the ghost-free property. Then, can we eliminate $\dot{h}_{00}$ using the non-minimal coupling terms?

$$c_1 C^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\beta}h + c_2 C^{\mu\alpha\nu\beta}h_{\mu\nu}h_\alpha{}^\lambda h_{\lambda\beta}$$

$$\phi^{(2)\nu} = \nabla_\mu E^{\mu\nu} \supset \left\{(2c_1+c_2)C^{\mu\alpha0\beta}g^{00} + (2c_1+c_2)\,C^{0\alpha0\beta}g^{\mu0}\right\}h_{\alpha\beta}\nabla_0 h_{00} + (\text{terms not including } \nabla_0 h_{00})$$

The derivative interaction induces a ghost.

New non-minimal coupling terms

Can we have non-minimal coupling terms with the Weyl tensor?

We eliminate $\dot{h}_{00}$ in $\phi^{(2)}$ by tuning c_1 and c_2

$$C^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\beta}h - 2C^{\mu\alpha\nu\beta}h_{\mu\nu}h_{\alpha\lambda}h^\lambda{}_\beta$$

We confirmed this term is really ghost-free on the Einstein manifold by repeating the above procedure.

We also obtain the following new interaction terms

$$\left\{\begin{array}{l} C^{\mu_1\mu_2\nu_1\nu_2}h_{\mu_1\nu_1}h_{\mu_2\nu_2} \\ \delta^{\mu_1}{}_{\rho_1}{}^{\mu_2}{}_{\rho_2}{}^{\mu_3}{}_{\rho_3}{}^{\mu_4}{}_{\rho_4}\delta^{\nu_1}{}_{\sigma_1}{}^{\nu_2}{}_{\sigma_2}{}^{\nu_3}{}_{\sigma_3}{}^{\nu_4}{}_{\sigma_4}C^{\rho_1\rho_2\sigma_1\sigma_2}g^{\rho_3\sigma_3\rho_4\sigma_4}h_{\mu_1\nu_1}h_{\mu_2\nu_2}h_{\mu_3\nu_3}h_{\mu_4\nu_4} \end{array}\right.$$

“Multi-scalar Extention of Horndeski Theory”

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[JGRG24(2014)P38]

Multi-Scalar Extension of Horndeski Theory

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1.Introduction

- Horndeki Theory is the most general single-scalar tensor theory with second order field equations.

$$\mathcal{L} = G_2(X,\phi) - G_3(X,\phi)\Box\phi + G_4(X,\phi)R + \frac{\partial G_4}{\partial X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right]$$
$$+G_5(X,\phi)G^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{1}{6}\frac{\partial G_5}{\partial X}\left[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right]$$

where $X := -\partial_\mu\phi\partial^\mu\phi/2$

- That gives us theoretical framework to describe all single-scalar tensor theory in a unified manner.

- Recently, Horndeski theory has been extensively studied in the context of inflation.

2.Introduction

- Aim is to give more general theoretical framework, and to apply them to multi-field inflationary scenario.

- Straightforward generalization of Horndeski theory into multi-scalar field (covariant multi-Galileon)

$$\begin{aligned}\mathcal{L} \;=\; & G_2(X^{IJ},\phi^K) - G_{3L}(X^{IJ},\phi^K)\Box\phi^L + G_4(X^{IJ},\phi^K)R \\ & + G_{4,\langle IJ\rangle}\left(\Box\phi^I\Box\phi^J - \nabla_\mu\nabla_\nu\phi^I\nabla^\mu\nabla^\nu\phi^J\right) + G_{5L}(X^{IJ},\phi^K)G^{\mu\nu}\nabla_\mu\nabla_\nu\phi^L \\ & - \frac{1}{6}G_{5I,\langle JK\rangle}\left[\Box\phi^I\Box\phi^J\Box\phi^K - 3\Box\phi^{(I}\nabla_\mu\nabla_\nu\phi^J\nabla^\mu\nabla^\nu\phi^{K)} + 2\nabla_\mu\nabla_\nu\phi^I\nabla^\nu\nabla^\lambda\phi^J\nabla_\lambda\nabla^\mu\phi^K\right]\end{aligned}$$

where $X^{IJ} := -\frac{1}{2}\partial_\mu\phi^I\partial^\mu\phi^J$

- The theory is ***NOT*** the most general theory !

[Tsutomu Kobayashi, Norihiro Tanahashi and Masahide Yamaguchi,PRD(2013)]

3.Introduction

- What is the most general multi-scalar tensor theory with 2^{nd} order field equations ?

- We constructed the most general **two**-scalar + gravity theory with 2^{nd} order field equations in four dimension.

[SO, Tanahashi, Kobayashi and Yamaguchi]

4.Assumption

- Two-scalar field extension of Horndeski Theory

(i) The system has covariant Lagrangian

$$\mathcal{L} = \mathcal{L}(g, \partial g, \ldots, \partial^p g, \phi_I, \partial \phi_I, \ldots, \partial^{q_I} \phi_I)$$

$$I = 1, 2 \qquad p, q_I \geq 2$$

(ii) The field equations are 2nd order

$$\frac{\delta \mathcal{L}}{\delta g_{ab}} \equiv E^{ab}(g, \partial g, \partial^2 g, \phi_I, \partial \phi_I, \partial^2 \phi_I)$$

$$\frac{\delta \mathcal{L}}{\delta \phi_I} \equiv E_I(g, \partial g, \partial^2 g, \phi_J, \partial \phi_J, \partial^2 \phi_J)$$

5.Constraints from covariance

- Covariance

Coordinate transformation $x^a \to x^a + \xi^a$

$$\delta \int \mathcal{L} d^4x = \int \left(\nabla^a E_{ab} - \sum_{I=1}^{2} \frac{1}{2} E_I \nabla_b \phi_I \right) \xi^b d^4x \qquad = 0$$

$$E^{ab} \equiv \frac{\delta \mathcal{L}}{\delta g_{ab}} \qquad E_I \equiv \frac{\delta \mathcal{L}}{\delta \phi_I}$$

- Constraints from covariance

$$\nabla_b E^{ab} = \sum_{I=1}^{2} \frac{1}{2} E_I \nabla^a \phi_I$$

6.Constraints from covariance

$$\nabla_b E^{ab} = \sum_{I=1}^{2} \frac{1}{2} E_I \nabla^a \phi_I$$

3rd order — 2nd order

- Right-hand side is 2nd order
- Left-hand side is 3rd order in general

- Left-hand side is also 2nd order

Outline of Construction

- Starting from field equations

$$E_{ab} = E_{ab}(g, \partial g, \partial^2 g, \phi_I, \partial\phi_I, \partial^2\phi_I)$$

$$E_I = E_I(g, \partial g, \partial^2 g, \phi_J, \partial\phi_J, \partial^2\phi_J)$$

- No 3rd derivative conditions

$$\frac{\partial \nabla_b E^{ab}}{\partial g_{cd,efg}} = 0 \qquad \frac{\partial \nabla_b E^{ab}}{\partial \phi_{I,cde}} = 0$$

- General covariance conditions

$$\nabla_b E^{ab} = \sum_{I=1}^{2} \frac{1}{2} E_I \nabla^a \phi_I$$

7.No 3rd derivative conditions

$$\frac{\partial \nabla_b E^{ab}}{\partial g_{cd,efg}} = 0 \qquad \frac{\partial \nabla_b E^{ab}}{\partial \phi_{I,cde}} = 0$$

- These conditions are equivalent to

$$\frac{\partial^2 E^{ab}}{\partial g_{cd,ef}\ \partial g_{hi,jk}} = 0$$
$$\frac{\partial^3 E^{ab}}{\partial g_{cd,ef}\ \partial \phi_{I,hi}\ \partial \phi_{J,jk}} = 0$$
$$\frac{\partial^4 E^{ab}}{\partial \phi_{I,cd}\ \partial \phi_{J,ef}\ \partial \phi_{K,hi}\ \partial \phi_{K,jk}} = 0$$

- We can determine the structure of field equations by integrating above equations

8.No 3rd derivative conditions

$$\frac{\partial}{\partial g_{c_1c_2,c_3c_4}}\frac{\partial}{\partial g_{c_5c_6,c_7c_8}}E^{ab}=0$$

$$\frac{d^2y}{dx^2}=0$$

$$y=ax+b$$

$$E^{ab}=\tilde{\xi}^{abc_1c_2c_3c_4}g_{c_1c_2,c_3c_4}+\tilde{\xi}^{ab}$$
$$=\xi^{abc_1c_2c_3c_4}R_{c_3c_1c_2c_4}+\xi^{ab}$$

independent of 2nd derivatives of metric

9.No 3rd derivative conditions

- By integration,

$$E^a_b = A\delta^a_b + \sum_{I,J=1}^{2} A_{IJ}\phi_I{}^a\phi_{Jb} + \sum_{I=1}^{2} B_I\delta^{ac}_{bd}\phi_{Ic}^{\ d} + \sum_{I,J,K=1}^{2} C_{IJK}\delta^{ace}_{bdf}\phi_{Ic}\phi_J{}^d\phi_{K}{}_e^f$$
$$+ \sum_{I,J,K,L,M=1}^{2} D_{IJKLM}\delta^{aceg}_{bdfh}\phi_{Ic}\phi_J{}^d\phi_{Ke}\phi_L{}^f\phi_{M}{}_g^h$$
$$+ \sum_{I,J,K,L,M=1}^{2} E_{IJKLM}\delta^{aceg}_{bdfh}\epsilon_{cepq}\phi_I{}^p\phi_J{}^q\phi_K{}^d\phi_L{}^f\phi_{M}{}_g^h$$
$$+ \sum_{I,J=1}^{2} G_{IJ}\delta^{ace}_{bdf}\phi_{I}{}_c^d\phi_{J}{}_e^f + \sum_{I,J,K,L=1}^{2} H_{IJKL}\delta^{aceg}_{bdfh}\phi_{Ic}\phi_J{}^d\phi_{K}{}_e^f\phi_{L}{}_g^h + I\delta^{ace}_{bdf}R^{df}_{ce}$$
$$+ \sum_{I,J=1}^{2} J_{IJ}\delta^{aceg}_{bdfh}\phi_{Ic}\phi_J{}^d R^{fh}_{eg} + \sum_{I=1}^{2} K_I\delta^{aceg}_{bdfh}\phi_{I}{}_c^d R^{fh}_{eg} + \sum_{I,J,K=1}^{2} L_{IJK}\delta^{aceg}_{bdfh}\phi_{I}{}_c^d\phi_{J}{}_f^e\phi_{K}{}_g^h$$

- 38 arbitrary function of ϕ_I and X_{IJ} where $X_{IJ} = \partial^a\phi_I\partial_a\phi_J$

10.General covariance conditions

$$\nabla_b E^{ab} = \sum_{I=1}^{2} \frac{1}{2} E_I \nabla^a \phi_I$$

- Divergence of field equation is proportional to gradient of scalar field.
- These conditions determine the relations between functions and reduce the number of arbitrary functions

e.g.

$$L_{IJK} = -\frac{4}{9}\left(\frac{\partial K_I}{\partial X_{JK}} + \frac{\partial K_J}{\partial X_{IK}} + \frac{\partial K_K}{\partial X_{IJ}}\right)$$

$$G_{IJ} = 2J_{IJ} - \left(\frac{\partial K_I}{\partial \phi_J} + \frac{\partial K_J}{\partial \phi_I}\right) - \sum_{K,L=1}^{2} H_{KLIJ}\phi_{KL}$$

11.Most general field equation with 2nd order

$$E^a_b = A\delta^a_b + \sum_{I,J=1}^{2} A_{IJ}\phi_I{}^a\phi_{Jb} + G\delta^{ace}_{bdf}R^{df}_{ce} - 4\sum_{I,J=1}^{2}\frac{\partial G}{\partial X_{IJ}}\delta^{ace}_{bdf}\phi_{I}{}^{d}_{c}\phi_{J}{}^{f}_{e}$$
$$+\sum_{I=1}^{2}\left(-4\frac{\partial G}{\partial\phi_I} - \sum_{J,K=1}^{2}\left(C_{JKI} - 4\frac{\partial J_{IJ}}{\partial\phi_K} + 4\frac{\partial J_{JK}}{\partial\phi_I}\right)\phi_{JK} - 2\sum_{J,K,L,M=1}^{2} D_{JKLMI}X_{JK}X_{LM}\right)\delta^{ac}_{bd}\phi_{I}{}^{d}_{c}$$
$$+\sum_{I,J,K=1}^{2} C_{IJK}\delta^{ace}_{bdf}\phi_{Ic}\phi_J{}^d\phi_{K}{}^{f}_{e} + \sum_{I,J,K,L,M=1}^{2} D_{IJKLM}\delta^{aceg}_{bdfh}\phi_{Ic}\phi_J{}^d\phi_{Ke}\phi_L{}^f\phi_{M}{}^{h}_{g}$$
$$+\sum_{I,J=1}^{2} J_{IJ}\delta^{aceg}_{bdfh}\phi_{Ic}\phi_J{}^d R^{fh}_{eg} - 4\sum_{I,J,K,L=1}^{2}\frac{\partial J_{IK}}{\partial X_{JL}}\delta^{aceg}_{bdfh}\phi_{Ic}\phi_J{}^d\phi_{K}{}^{f}_{e}\phi_{L}{}^{h}_{g}$$
$$+\sum_{I=1}^{2} K_I\delta^{aceg}_{bdfh}\phi_{I}{}^{d}_{c}R^{fh}_{eg} - \frac{4}{3}\sum_{I,J,K=1}^{2}\frac{\partial K_I}{\partial X_{JK}}\delta^{aceg}_{bdfh}\phi_{I}{}^{d}_{c}\phi_{J}{}^{e}_{f}\phi_{K}{}^{h}_{g}$$

- 12 arbitrary functions (6 arbitrary function in Gen. Cov. Multi-Galileon)

Summary

- We studied the multi-scalar generalization of Horndeski theory.
- We derived most general EoM for **two**-scalar + gravity theory.
- Finding Lagrangian is in progress