

**PROCEEDINGS OF THE EIGHTH WORKSHOP
ON
GENERAL RELATIVITY AND GRAVITATION**

**Graduate School of Science and Technology,
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edited by

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PREFACE

The Eighth Workshop on General Relativity and Gravitation was held at the Graduate School of Science and Technology, Niigata University on October 19–22, 1998. The purpose was to review the latest developments in general relativity and gravitation and related fields and to discuss future prospects. There were 110 participants and 59 talks which include 10 invited talks. We wish to thank all the participants for their contributions to the workshop. Our efforts to organize the workshop were highly rewarded by their enthusiasm.

The number of researchers in this field is increasing in Japan. We are very happy to announce that during the workshop all of the participants agreed to establish the Japanese Society on General Relativity and Gravitation (JGRG) and to hold this series of workshops continually in the forthcoming years.

We would like to thank Ms. Yokota, the secretary at the Department of Physics, Kyoto University, for her devoted transaction of various tedious official works. We are also grateful to the graduate students of the astrophysics group in Niigata University for their cooperation in the management of the workshop. The workshop would not have been successful without their help. This workshop was financially based on the Grant-in-Aid for Creative Basic Research (09NP0801) from the Japanese Ministry of Education, Science, Sports and Culture.

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Resolution of Spacetime Singularities

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Abstract

We review the possibility of the resolution of spacetime singularities by higher-dimensional “oxidation”. We apply this approach to the Hawking-Turok instanton recently proposed to describe the creation of an open universe from nothing.

1 Introduction

The singularity theorems[1] in classical general relativity indicate that the occurrence of spacetime singularities are inevitable if the expansion of a causal geodesic congruence turns negative and the matter satisfies some energy condition and the spacetime satisfies some global condition. The notion of singularity here is defined as the geodesically incompleteness. Therefore, the singularity theorems say that the evolution of a test particle must be terminated within a finite proper time: dead end.

The usual interpretation of the theorems is that the singularity merely indicates the breakdown of the “classical” general relativity; presumably new quantum physics intervenes near the singularity and thus in full quantum perspective the singularity will hopefully be resolved by analogy with the hydrogen atom, and several efforts toward quantum resolution are attempted[2, 3].

Although it may be true, we point out here that there can intervene new “classical” higher-dimensional physics in such a way that singularities are resolved[5, 6]. That can occur well below the Planck scale and thus new quantum physics needs not be taken into account.

For example, let us consider any non-singular solution in a higher-dimensional theory which has a spacelike isometry with a fixed point. Then reduction along the isometry induces a lower-dimensional solution with a singularity. Examples are the dimensional reduction of spherical symmetric spacetime to two-dimensional system and the Kaluza-Klein monopole[4].

After reviewing the procedure of the higher resolution of black-hole/cosmological singularities, we apply it to the Hawking-Turok instanton recently proposed to describe the creation of an open universe from nothing. Finally we give some comments and speculation.

2 Higher-dimensional resolution of black-hole singularities

We show that dilatonic black hole solutions in four dimensions can be regularly embedded in a black p -brane solution[5]. The starting point is the four-dimensional action for gravity with dilaton coupled to 2-form field strength

$$S_4 = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2(\nabla\phi)^2 - e^{-2a\phi} F_2^2), \quad (1)$$

where a is the dilaton coupling constant which we can assume to be positive without loss of generality. It can be shown that the above action with $a = \sqrt{p/(p+2)}$ can be derived from 4 + p -dimensional action for gravity with 2-form field strength but without dilaton[5]

$$S_{4+p} = \int d^{4+p}x \sqrt{-g} (R - F_2^2) \quad (2)$$

if we assume the following form of the metric

$$ds^2 = e^{2a\phi(x)} dy \cdot dy + e^{2\beta\phi(x)} g_{\mu\nu} dx^\mu dx^\nu, \quad (3)$$

$$F_2 = \frac{1}{2} F_{\mu\nu}(x) dx^\mu \wedge dx^\nu \quad (4)$$

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with $\alpha = -2/\sqrt{p(p+2)}$, $\beta = \sqrt{p/(p+2)}$. Here y denotes the coordinates of the additional p -dimensional direction, $g_{\mu\nu}$ is the four-dimensional metric.

Now as an example, let us consider the extremally charged dilatonic black hole[7] as a solution of the action Eq.(1):

$$ds^2 = -\left(1 - \frac{\mu}{r}\right)^{2/(1+a^2)} dt^2 + \left(1 - \frac{\mu}{r}\right)^{-2/(1+a^2)} dr^2 + \left(1 - \frac{\mu}{r}\right)^{2a^2/(1+a^2)} d\Omega^2, \quad (5)$$

$$e^{a\phi} = \left(1 - \frac{\mu}{r}\right)^{-a^2/(1+a^2)}, \quad (6)$$

$$F_2 = Q\epsilon_2, \quad (7)$$

where the volume 2-form of the unit 2-sphere, and $\mu = \sqrt{1+a^2}|Q| = (1+a^2)M$ with M being the ADM mass. This solution has a curvature singularity at $r = \mu$ with null for $a \leq 1$ and timelike for $a > 1$. Now its $4 + p$ -dimensional counterpart is given by

$$ds^2 = \left(1 - \frac{\mu}{r}\right)^{2/(p+1)} (-dt^2 + dy \cdot dy) + \left(1 - \frac{\mu}{r}\right)^{-2} dr^2 + r^2 d\Omega^2, \quad (8)$$

which is a black p -brane aligned with the y -axis[8]. The singularity at $r = \mu$ is now merely a coordinate singularity, but now there exists in general a new singularity at $r = 0$. However, it can be shown that if p is odd, the interior region of the horizon ($r = \mu$) is isometric to the exterior region, and thus the solution is completely non-singular[5].

3 Higher-dimensional resolution of cosmological singularities

We show that FRW universe model can be embedded in a non-singular fashion[6]. Following Larsen and Wilczek[6], let us start the discussion by reference to the five-dimensional interior Schwarzschild metric. It can be written as

$$ds^2 = e^{-4\sigma} dy^2 - e^{4\sigma} d\tau^2 + \tau^2 d\Omega_3^2, \quad (9)$$

$$e^{-4\sigma} = \frac{\mu}{\tau^2} - 1, \quad (10)$$

where $\mu > 0$ is a mass parameter. The range of τ is $0 < \tau^2 < \mu$. Compactification along y induces an effective four-dimensional dilaton $e^{-2\phi_4} = e^{-2\sigma}$ with the string frame metric $g_{S\mu\nu}$ in which the dy^2 term is omitted in Eq.(9). In terms of the Einstein frame metric $g_{E\mu\nu} = e^{-2\phi_4} g_{S\mu\nu}$, the four-dimensional cosmology becomes

$$ds_E^2 = -e^{2\sigma} d\tau^2 + e^{-2\sigma} \tau^2 d\Omega_3^2. \quad (11)$$

We shall work in the Einstein frame line element because it is invariant under the standard duality transformations in the string theory context and because a test particle follow geodesic in this frame[6]. Since in the Einstein frame the effective four-dimensional action is that of a massless scalar, minimally coupled to gravity, the four-dimensional FRW cosmology is the closed model with matter of $\rho = p$ and singularities at both the big-bang and the big-crunch. Thus, the five dimensional vacuum black hole solution with a coordinate singularity (the horizon) and a physical singularity (the origin) becomes four dimensional cosmology with a massless scalar field and physical singularities at both initial and final times. As a side remark, we note that the compactification along y in the interior implies the emergence of closed timelike curves in the exterior, however, they are restricted to the epoch after the end of time!

Larsen-Wilczek further generalized the model and embedded the cosmology in a solution that originates from string theory[6]. The starting point is the classical solution associated with the black p -brane[8]. In the string frame it is written as

$$ds_S^2 = e^{2\ell} (e^{-4\sigma} dy^2 + dx_1^2 + \dots + dx_p^2) + e^{-2\ell} (dx_{p+1}^2 + \dots + dx_s^2) + e^{-2\ell} (-e^{4\sigma} d\tau^2 + \tau^2 d\Omega_3^2), \quad (12)$$

$$e^{-2\phi_{10}} = e^{-2\ell(p-3)}, \quad (13)$$

$$F_{p+2} = \partial_\mu e^{4\ell} dy \wedge dx_1 \wedge \dots \wedge dx_p \wedge dx^\mu, \quad (14)$$

where F_{p+2} is the $(p+2)$ -form Ramond-Ramond field strength and ξ and σ are defined as

$$e^{-4\xi} = \frac{q}{\tau^2} + 1, \quad (15)$$

$$e^{-4\sigma} = \frac{\mu}{\tau^2} - 1. \quad (16)$$

Compactification to four-dimensions induces an effective dilaton $e^{-2\phi_4} = e^{2\xi-2\sigma}$ independent of p . Remarkably, the four dimensional Einstein metric is again Eq.(11). Thus the black p -brane generates the same four-dimensional cosmology as the five-dimensional Schwarzschild solution does. Further dimensional oxidation to eleven dimensions can be performed rather straightforwardly along the line of [9].

A recurrent theme of recent works on duality is that the low energy field theory develops singularities when heavy states of the full theory become light[10]. In the five-dimensional Schwarzschild solution, the Kaluza-Klein modes from the compactified dimension become light near the big-bang singularity. In the black p -brane solutions, the apparent singularity of the low energy field theory (at $\tau = 0$) arises because open string excitations, which are ignored in the effective field theory, become light close to the p -branes. That singularity is plausibly resolved in an intrinsically string-theoretic way that has no conventional spacetime interpretation[6].

4 Application to Hawking-Turok singular instanton

Hawking and Turok(HT) have recently proposed an instanton for the creation of an open universe from nothing[11]. This leads to an interesting possibility of realization of open inflation with more realistic form of the inflaton potential. However, the HT-instanton contains a genuine singularity: both the curvature and the scalar field diverge there. Although the action is finite, further justification for allowing such a singular instanton is necessary. In fact, as Vilenkin has shown[12], instantons with the same singularity structure as that of HT-instanton lead to unacceptable physical consequences: the decay of flat space.

We show that the singular instanton of Hawking-Turok type, in which the singularity occurs due to the divergence of a massless scalar field, can be generated by Euclideanized regular p -brane solutions in string theory upon compactification to four dimensions[13].

We perform the required Euclideanization in five dimensions. The Euclideanized p -brane solution in ten or eleven dimensions can be immediately obtained by following the same procedure. We introduce the time coordinate t by

$$dt = e^{2\sigma} d\tau, \quad (17)$$

so that $t^2 = \mu - \tau^2$. Then Eq.(9) becomes

$$ds^2 = -dt^2 + \tau^2 d\Omega_3^2 + e^{-4\sigma} dy^2, \quad (18)$$

$$= -dt^2 + (\mu - t^2)d\Omega_3^2 + \frac{t^2}{\mu - t^2} dy^2. \quad (19)$$

Thus, if we complexify t and y such that

$$T = it, \quad Y = iy \quad (20)$$

with T and Y being real, then Eq.(19) is Euclideanized as

$$ds^2 = dT^2 + (\mu + T^2)d\Omega_3^2 + \frac{T^2}{\mu + T^2} dY^2, \quad (21)$$

which is the same as the Garriga's regular five-dimensional instanton solution (if $0 \leq Y \leq 2\pi\mu^{1/2}$) and solves the Euclidean field equations for pure gravity in five dimensions[14]. We note that Garriga's instanton can also be obtained from the exterior Schwarzschild metric. Changing into the Einstein frame after compactifying Y yields the following metric and massless scalar field:

$$ds^2 = \frac{T}{\sqrt{\mu + T^2}} (dT^2 + (\mu + T^2)d\Omega_3^2) \quad (22)$$

$$\equiv d\bar{T}^2 + b(\bar{T})^2 d\Omega_3^2, \quad (23)$$

$$\phi \propto -\ln\left(\frac{T^2}{\mu + T^2}\right). \quad (24)$$

The internal radius of the fifth dimension can be zero, but it is just a coordinate singularity. However, that zero-point induces a singularity of the Einstein frame metric in four dimensions. In that way, the singularity in the four-dimensional world is induced. Near the singularity $T \simeq 0$, $\bar{T} - \bar{T}_f \simeq 2\mu^{-1/4}T^{3/2}/3$ with $\bar{T}_f$ being a constant. Thus the scale factor $b(\bar{T})$ and the scalar field behave as

$$b(\bar{T}) \simeq \mu^{1/4}T^{1/2} \propto |\bar{T} - \bar{T}_f|^{1/3}, \quad (25)$$

$$\phi \propto -\ln T + \text{const.} \propto -\ln |\bar{T} - \bar{T}_f| + \text{const.} \quad (26)$$

This is the same singularity structure as that of the Hawking-Turok(or Vilenkin) instanton[11, 12, 14].

5 Conclusion

We have shown that certain singularities can be resolved by simply passing to a higher-dimensional theory of gravity for which spacetime is only effectively four-dimensional below some compactification scale.

We have applied the above idea to the Hawking-Turok type instanton and have shown that the singularity of the instanton of Hawking-Turok type can be resolved from higher-dimensional viewpoint within the context of M-(or string) theory. The same singularity structure emerges when a regular instanton in higher dimensions is compactified to four dimensions. The fundamental length scale in M-theory[15] is much larger than the four dimensional Planck length[16], and therefore the decay of flat space is strongly suppressed.

We make a comment on the relation to the singularity theorem; which condition in the theorems is violated? Matter fields in higher dimensions do not violate energy conditions. One of the key ingredient in the conditions is the existence of *closed* trapped surface. We considered a dimensional oxidation of four dimensional gravity, and therefore four dimensional trapped surface is now equipped with the brane structure, black brane, which is now not closed.

We have seen that in higher-dimensional view point, four-dimensional solutions are highly degenerate; the same four-dimensional cosmology is generated either by the five dimensional Schwarzschild solution or by ten-dimensional black p -brane solutions or eleven-dimensional ones. It is natural to ask whichever choice is possible? Is there any selection rule to restrict possible dimensional reduction? It is also interesting to explore the idea of a possible quantum resolution of singularities along the line of ref.[2]. The idea is that a spacetime can be said nonsingular if a gauge field is defined in nonsingular way. We hope to address these problems in the future.

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The expectation value of the massive scalar fields $\langle\phi^2\rangle$ in Reissner-Nordstrom spacetime

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Abstract

The expectation value of the massive scalar fields $\langle\phi^2\rangle_{reg}$ near the horizon of Reissner-Nordstrom black hole is evaluated. It is assumed that the field is a thermal state at temperature T with respect to the timelike Killing vector which always exists in a static spacetime. We correct the analytic approximation of Anderson *et al.* [4] by adding the contribution from zero mode. Our new results show that $\langle\phi^2\rangle_{reg}$ does not diverge on the horizon when the field has the same temperature as the surface gravity of the black hole.

1 Introduction

Since Hawking's discovery of black hole radiation, a great deal of effort has been made to understand quantum effects on gravitational fields. Early calculations of black hole radiance proceeded by studying the modes of the quantized field in the fixed background spacetime. This approach allows one to obtain information about particle production. However, possible quantum effect is not only particle creation, but also vacuum polarization and its back reaction on the metric. To obtain useful information about such quantum effects it is important to evaluate the quantities $\langle\phi^2\rangle_{reg}$ and $\langle T_{\mu\nu}\rangle_{reg}$. $\langle\phi^2\rangle_{reg}$ is a useful quantity because it gives one qualitative information about $\langle T_{\mu\nu}\rangle_{reg}$ with much less effort.

To date most calculations of these quantities have centered on massless scalar fields in Schwarzschild spacetime. For massive scalar fields the conventional method to evaluate $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}\rangle$ is DeWitt-Schwinger approximation, which is made without resorting to mode sums.

Anderson *et al.* [1, 2, 3, 4] have computed $\langle\phi^2\rangle_{reg}$ and $\langle T_{\mu\nu}\rangle_{reg}$ numerically in Schwarzschild and Reissner-Nordstrom spacetimes. Their derivation is based solely on quantum field theory. They have calculated analytically also [4] for the scalar field with arbitrary coupling in general static spherically symmetric spacetime. For massive scalar field their analytical approximation shows that the logarithmic divergence occurs on the horizon independent of the field temperature, although their numerical calculation does not show. these two results are not consistent.

In this article, we evaluate the regularized $\langle\phi^2\rangle_{reg}$ as a preparation for calculations of $\langle T_{\mu\nu}\rangle_{reg}$, for massive scalar fields with finite temperature near the horizon of Reissner-Nordstrom black hole, correcting Anderson's analytic approximation by adding the contribution from zero mode. In section 2, we review the analytic approximation algorithm of $\langle\phi^2\rangle_{reg}$ [2],[3],[4]. In section 3, we investigate the contribution from $n=0$ mode to logarithmic divergence of $\langle\phi^2\rangle_{reg}$. We suggest that this logarithmic divergence is an artifact of the approximation which is caused by the infrared cutoff made by not including the $n=0$ contribution.

2 Analytical calculation of $\langle\phi^2\rangle$ from WKB approximation

In this section the expression for $\langle\phi^2\rangle$ is derived for massive scalar fields in asymptotically flat, static, spherically symmetric spacetimes, using WKB approximation for the modes. The derivation in this section is based on [2], [4].

An Euclidean space approach is used and it is assumed that the field is a finite-temperature state at temperature T with respect to the timelike Killing vector which always exists in a static spacetime. The

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metric for a general static spherically symmetric spacetime when continued analytically into Euclidean space is

$$ds^2 = f(r)d\tau^2 + h(r)dr^2 + r^2 d\Omega^2, \quad (1)$$

here $\tau = it$ is the Euclidean time, and f, h are arbitrary functions of r which, if the space is asymptotically flat, become constant in the limit $r \rightarrow \infty$.

The field ϕ is a scalar field with mass m and coupling ξ to the scalar curvature R . $\langle \phi^2 \rangle$ can be computed from the Euclidean Green's function by noting the identities

$$\langle \phi^2 \rangle_{unreg} = \frac{1}{2} G^{(1)}(x, x) = iG_F(x, x) = G_E(x, x), \quad (2)$$

where $G^{(1)}$ is the Hadamard Green's function, G_F is the Feynman Green's function, and G_E is the Euclidean Green's function for the scalar field.

$G_E(x, x)$ is divergent and must be regularized. In this paper point splitting method is used as the renormalization scheme. This technique requires that we have an expression for $G_E(x, x')$ when the points x and x' are separated. In the rest of this section such an expression is derived. For a scalar field $G_E(x, x')$ obeys the equation:

$$[\Box_x - m^2 - \xi R(x)]G_E(x, x') = -\frac{\delta^4(x, x')}{g^{1/2}(x)}, \quad (3)$$

with $\Box_x$ evaluated using the Euclidean metric (1). With the metric (1) one finds that when the points have the same values of r, θ , and ϕ , and different values of τ , such that $\tau - \tau' = \epsilon$ for Eq.(3), $G_E(x, x')$ is given by

$$\begin{aligned} \langle \phi^2 \rangle_{unreg} &= G_E(x, x') \\ &= \frac{T}{4\pi} \sum_{l=0}^{\infty} (2l+1) p_{0l}(r) q_{0l}(r) + \frac{T}{2\pi} \sum_{n=1}^{\infty} \cos[n2\pi T(\tau - \tau')] \sum_{l=0}^{\infty} (2l+1) p_{nl}(r) q_{nl}(r). \end{aligned} \quad (4)$$

The radial modes $p_{nl}(r)$ and $q_{nl}(r)$ obey the mode equation:

$$\frac{d^2 S_{nl}}{dr^2} + \left[\frac{2}{r} + \frac{f'}{f} \right] \frac{dS_{nl}}{dr} - \left[\frac{(2\pi nT)^2}{f^2} + \frac{l(l+1)}{r^2 f} + \frac{m^2}{f} \right] S_{nl} = 0. \quad (5)$$

$p_{nl}(r)$ is chosen to be the solution which is well behaved at the event horizon and divergent at $r = \infty$, and $q_{nl}(r)$ is chosen to be the solution which is well behaved at $r = \infty$ and diverge at the event horizon. Normalization for the modes is achieved by the Wronskian condition:

$$p_{nl} \frac{dq_{nl}}{dr} - q_{nl} \frac{dp_{nl}}{dr} = -\frac{1}{r^2} \left[\frac{h}{f} \right]^{1/2}. \quad (6)$$

WKB approximation for the modes is obtained as follow. First, we make the change of variables

$$\begin{cases} p_{nl}(r) = (2r^2 W)^{-1/2} \exp \left[\int^r W \sqrt{\frac{h}{f}} dr \right] \\ q_{nl}(r) = (2r^2 W)^{-1/2} \exp \left[- \int^r W \sqrt{\frac{h}{f}} dr \right]. \end{cases} \quad (7)$$

Substitution these into the mode equation(5) show that W obeys the equation

$$W^2 = \Omega^2(r) + V_1(r) + V_2(r) + \frac{1}{2} \left[\frac{f}{hW} \frac{d^2 W}{dr^2} + \left[\frac{1}{h} \frac{df}{dr} - \frac{f}{h^2} \frac{dh}{dr} \right] \frac{1}{2W} \frac{dW}{dr} - \frac{3}{2} \frac{f}{h} \left[\frac{1}{W} \frac{dW}{dr} \right]^2 \right], \quad (8)$$

where

$$\Omega^2(r) = (2\pi nT)^2 + m^2 f + (l+1)^2 \frac{f}{r^2}, \quad (9)$$

$$V_1(r) = \frac{1}{2rh} \frac{df}{dr} - \frac{f}{2rh^2} \frac{dh}{dr} - \frac{f}{4r^2}, \quad (10)$$

$$V_2(r) = \xi R f. \quad (11)$$

The WKB approximation for the modes is obtained by solving Eq.(8) iteratively for W . Thus to zeroth-order $W^{(0)} = \Omega$ while to second-order

$$W^{(2)} = \Omega + \frac{1}{2}\Omega^{-1}(V_1 + V_2) + \frac{1}{4\Omega^3}V_1 + \frac{1}{4} \left[\frac{f}{h\Omega^2} \frac{d^2\Omega}{dr^2} + \left[\frac{1}{h} \frac{df}{dr} - \frac{f}{h^2} \frac{dh}{dr} \right] \frac{1}{2\Omega^2} \frac{d\Omega}{dr} - \frac{3}{2} \frac{f}{h} \frac{1}{\Omega^3} \left[\frac{d\Omega}{dr} \right]^2 \right]. \quad (12)$$

The derivation of $\langle \phi^2 \rangle_{unreg}$ proceeds as follows: First one substitutes the second-order WKB expansion for the modes into Eq.(4). Using Plana sum formula to compute the sums over l , which is

$$\sum_{j=k}^{\infty} g(j) = \frac{1}{2}g(k) + \int_k^{\infty} g(\tau) d\tau + i \int_0^{\infty} \frac{dt}{e^{2\pi t} - 1} [g(k+it) - g(k-it)], \quad (13)$$

we can calculate the unregularized expression for $\langle \phi^2 \rangle_{unren}$. For regularization of $\langle \phi^2 \rangle$, point splitting method is used. In point splitting one uses the DeWitt-Schwinger expansion [5] to obtain the approximation for $G_E(x, x')$ when the points x and x' are separated by a distance of $(2\sigma)^{1/2}$ along the shortest geodesic connecting them. This approximate expression or counterterm is then subtracted from the unrenormalized exact expression for $G_E(x, x')$ and the limit $x' \rightarrow x$ is taken. The regularization counterterms for $\langle \phi^2 \rangle_{unren}$ is

$$\begin{aligned} \langle \phi^2 \rangle_{DS} &= G_{DS}(x, x') \\ &= \frac{1}{8\pi^2\sigma} + \frac{1}{8\pi^2} \left[m^2 + \left(\xi - \frac{1}{6} \right) R \right] \times \left[\gamma + \frac{1}{2} \ln \left(\frac{m^2 |\sigma|}{2} \right) \right] \\ &\quad - \frac{m^2}{16\pi^2} + \frac{1}{96\pi^2} R_{\alpha\beta} \frac{\sigma^\alpha \sigma^\beta}{\sigma}. \end{aligned} \quad (14)$$

Taking point separation as timelike, $\epsilon = t - t'$, $r = r'$, $\theta = \theta'$, $\phi = \phi'$, we have $\sigma = \frac{1}{2}f\epsilon^2$. Substituting it into eq.(14), we obtain

$$\begin{aligned} \langle \phi^2 \rangle_{DS} &= G_{DS}(x, x') \\ &= \frac{1}{4\pi^2 f \epsilon^2} + \frac{1}{8\pi^2} m^2 \left[\gamma + \frac{1}{2} \ln \left(\frac{m^2 f \epsilon^2}{4} \right) \right] \\ &\quad - \frac{m^2}{16\pi^2} + \frac{f'^2}{192\pi^2 f} - \frac{f''}{96\pi^2} - \frac{f'}{48\pi^2 r}. \end{aligned} \quad (15)$$

The regularized expression for $\langle \phi^2 \rangle_{reg}$ is

$$\begin{aligned} \langle \phi^2 \rangle_{reg} &= \langle \phi^2 \rangle_{unreg} - \langle \phi^2 \rangle_{DS} \\ &= \frac{1}{12f} \left[T^2 - \left(\frac{\kappa}{2\pi} \right)^2 \right] - \frac{1}{8\pi^2} m^2 \ln \left(\frac{m^2 f}{4} \right) \\ &\quad - \frac{f'^2}{96\pi^2 f^2 h} - \frac{f' h'}{192\pi^2 f h^2} + \frac{f''}{96\pi^2 f h} + \frac{f'}{48\pi^2 f h^2} + \frac{f''}{48\pi^2 r f h}. \end{aligned} \quad (16)$$

To compare their analytic approximations with their numerical results, we apply (16) to Reissner-Nordstrom spacetime. The metric and surface gravity are

$$f = \frac{1}{h} = 1 - \frac{2M}{r^2} + \frac{Q^2}{r^2}, \quad (17)$$

$$\kappa = \frac{r_+ - r_-}{2r_+^2}, \quad (18)$$

here, M is the mass and Q is the charge of the black hole, and r_+ and r_- are outer and inner horizon:

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2} \quad (19)$$

Substituting the metric (17) into the Eq.(16), we have the WKB approximation for $\langle \phi^2 \rangle$ in Reissner-Nordstrom spacetime:

$$\begin{aligned} \langle \phi^2 \rangle_{reg} &= \langle \phi^2 \rangle_{1 \leq n \leq \infty} - \langle \phi^2 \rangle_{DS} \\ &= \frac{1}{12f} \left[T^2 - \left(\frac{\kappa}{2\pi} \right)^2 \right] - \frac{1}{8\pi^2} m^2 \ln \left(\frac{m^2 f}{4} \right) + \mathcal{O}(m^{-2}). \end{aligned} \quad (20)$$

This analytic calculations show that the logarithmic divergence on the event horizon occurs independent of temperature. On the other hand, however, their numerical calculation does not show such a divergence when the field has the same temperature as the surface gravity of black hole [1], [2]. these two results are not consistent.

3 Contribution from $n=0$ mode to logarithmic divergence near the horizon

In this section we investigate the contribution from $n=0$ mode to logarithmic divergence of $\langle \phi^2 \rangle_{reg}$. It is noted that Anderson's analytic calculation [3], [4] does not include $n=0$ contribution, by imposing infrared cutoff. In this section we investigate the contribution from $n=0$ mode to logarithmic divergence. Since WKB approximation is broken for $n=0$ mode, we analyze asymptotic behavior of radial mode equation near horizon. For easy, we focus the metric on Reissner-Nordstrom. We change the variable

$$\begin{aligned} x &= \left[\frac{2}{\kappa} \left\{ \frac{l(l+1)}{r_+^2} + m^2 \right\} (r - r_+) \right] \\ &= \alpha s. \end{aligned} \quad (21)$$

We introduce the following dimensionless variables

$$\alpha = \left[\frac{2}{\tilde{\kappa}} \{ l(l+1) + \tilde{m}^2 \} \right]^{1/2}, \quad (22)$$

$$s = \sqrt{\frac{r - r_+}{r_+}}, \quad (23)$$

$$\tilde{\kappa} = \kappa r_+, \quad (24)$$

$$\tilde{m}^2 = m^2 r_+^2, \quad (25)$$

We assume that the field is super massive, $\tilde{m} \gg 1$. Then we can regard $\frac{1}{\tilde{\kappa}\alpha^2}$ as an order parameter, since $\frac{1}{\tilde{\kappa}\alpha^2} \ll 1$ holds for all value of l . We expand the radial mode equation (5) near horizon with respect to $\frac{1}{\tilde{\kappa}\alpha^2}$, we obtain

$$\frac{d^2 S_l}{dx^2} + \left[\frac{1}{x} + ax^2 \right] \frac{dS_l}{dx} - [1 - bx^2] S_l = 0, \quad (26)$$

where,

$$a = \frac{1}{\tilde{\kappa}\alpha^2}, \quad (27)$$

$$b = \frac{1}{2\tilde{\kappa}\alpha^2} - \frac{4\tilde{m}^2}{\tilde{\kappa}^2\alpha^4}. \quad (28)$$

It is found that 0th-order equation is modified Bessel equation, and we write 0th-order solutions of eq.(26) as

$$\begin{cases} p_{0l}^{(0)} = I_0, \\ q_{0l}^{(0)} = \frac{2}{r_+ - r_-} K_0, \end{cases} \quad (29)$$

here, I_0 and K_0 are modified Bessel functions. Next, we write 1st-order solutions as

$$\begin{cases} p_{0l}^{(1)} = C(x)I_0, \\ q_{0l}^{(1)} = \frac{2}{r_+ - r_-} D(x)K_0, \end{cases} \quad (30)$$

1st-order equations become

$$\begin{cases} C''I_0 + (2I_0' + \frac{1}{x}I_0)C' = -axI_0' - bx^2I_0 \\ D''K_0 + (2K_0' + \frac{1}{x}K_0)D' = -axK_0' - bx^2K_0. \end{cases} \quad (31)$$

Integrating (31), we obtain asymptotic solutions for $n=0$ mode near the horizon until 1st-order

$$\begin{aligned} p_{0l}(r) &= p_{0l}^{(0)}(r) + p_{0l}^{(1)}(r) \\ &= I_0(x) + \left(-\frac{a}{4} + \frac{b}{6}\right)x^2I_0 + \left(\frac{a}{2} - \frac{b}{3}\right)xI_1 - \frac{b}{6}x^3I_1, \end{aligned} \quad (32)$$

$$\begin{aligned} q_{0l}(r) &= q_{0l}^{(0)}(r) + q_{0l}^{(1)}(r) \\ &= \frac{2}{r_+ - r_-} \left[K_0 + \left(-\frac{a}{4} + \frac{b}{6}\right)x^2K_0 + \left(-\frac{a}{2} + \frac{b}{3}\right)xK_1 + \frac{b}{6}x^3K_1 \right]. \end{aligned} \quad (33)$$

the contribution from $n=0$ mode to $\langle \phi^2 \rangle$ is

$$\begin{aligned} \frac{T}{4\pi} \sum_{l=0}^{\infty} (2l+1)p_{0l}q_{0l} &= \frac{2}{r_+ - r_-} \sum_{l=0}^{\infty} (2l+1) \left[I_0K_0 + \left(-\frac{a}{2} + \frac{b}{3}\right)x^2I_0K_0 \right. \\ &\quad \left. + \left(-\frac{a}{2} + \frac{b}{3}\right)x(I_0K_1 - I_1K_0) + \frac{b}{6}x^3(I_0K_1 - I_1K_0) \right]. \end{aligned} \quad (34)$$

Using Plana sum formula (13), we computing l sum, and finally, find the corrected result of (16) by adding the contribution from zero mode:

$$\begin{aligned} \langle \phi^2 \rangle_{reg} &= \langle \phi^2 \rangle_{n=0} + \langle \phi^2 \rangle_{0 \leq n \leq \infty} - \langle \phi^2 \rangle_{DS} \\ &= \left(T - \frac{\kappa}{2\pi}\right) \left\{ -\frac{m^2}{16\pi\kappa} + \frac{m^2}{8\pi\kappa} \left[2\gamma + \ln \left\{ \frac{m^2(r-r_+)}{2\pi T} \right\} \right] - \frac{1}{12\pi r_+} \right\} \\ &\quad + \left\{ T^2 - \left(\frac{\kappa}{2\pi}\right)^2 \right\} \left[\frac{1}{24\kappa(r-r_+)} - \frac{r_-}{48r_+^3\kappa^2} \right] + \frac{m^2}{16\pi^2\kappa} (\kappa + 2\pi T) \ln \left(\frac{2\pi T}{\kappa} \right) \\ &\quad + \mathcal{O}(m^{-2}). \end{aligned} \quad (35)$$

Our new result shows that the logarithmic divergence does not occur only if the field has same temperature as the surface gravity of black hole, that is, $T = \frac{\kappa}{2\pi}$.

4 Conclusion

We evaluate the regularized $\langle \phi^2 \rangle_{reg}$, as a preparation for the calculation of $\langle T_{\mu\nu} \rangle_{reg}$, for massive scalar field with finite temperature near the horizon of Reissner-Nordstrom black hole. We correct the Anderson's analytic approximation by adding contribution from zero mode to logarithmic divergence on the horizon. Our new result shows that the regularized expression for $\langle \phi^2 \rangle_{reg}$ is not diverge on the horizon when the field has the same temperature as the surface gravity of the black hole.

The Asymptotic analysis near horizon shows that there are two types of divergence in black hole spacetimes; One is ultraviolet divergence which occurs even in flat spacetime. It is local and independent

of temperature. The other is peculiar in black hole spacetimes, which depends on the global structure of spacetime, or field states. The former can be regularized independent of the temperature, by using point splitting method, while the latter can be, only when the field has the same temperature as the surface gravity of the black hole, so called, Hartle-Hawking state. This regularizability suggests that, at least in the case of large mass the structure of divergence constructed from the field of Hartle-Hawking state is identified with DeWitt-Schwinger counterterm. It means that DeWitt-Schwinger counterterm, that is, the coefficients of DeWitt-Schwinger expansion involve not only ultraviolet divergence, but also non-local divergence.

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Some Consequences of Black-Hole Scaling Laws

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Abstract

A critical phenomenon takes place at the extremal limit of a Kerr-Newman black hole. Analogously to such phenomena in usual material physics, the scaling laws among various critical exponents hold also in the black-hole cases. Combined with the results from the analyses of black-hole thermodynamic fluctuations, we conclude that the effective spatial dimension of a black hole is 1. This implies the existence of an unusual spatial correlation. A procedure reproducing such a correlation on a hypersurface of constant universal time leads us to the expression of “renormalized” correlation length, the existence of some kind of non-locality in a black-hole spacetime and the notion of hybrid thermodynamics of a black hole.

1 Introduction

In the beginning, let us introduce black-hole thermodynamics briefly. It is well known that macroscopic properties of a black hole is described in terms of thermodynamic language. For example, the area of the horizon and the surface gravity correspond to entropy and temperature, respectively. It should be noted first that such quantities are actually defined not only on the outer horizon but also on the inner horizon.

An arbitrary Kerr-Newman black hole generally has two horizons whose radii are given as the solutions to $\Delta = 0$ (+ sign is for the outer horizon, and $-$ sign for the inner horizon):

$$\Delta \equiv r^2 - 2Mr + a^2 + Q^2, \quad a = J/M \geq 0,$$

$$\Delta = 0 \Rightarrow r_H = M \pm \sqrt{M^2 - a^2 - Q^2} \equiv r_{\pm}.$$

Here, M is the mass, J is the angular momentum, a is the specific angular momentum and Q is the electric charge of the hole.

There is a nice set of non-dimensional parameters, α and h , which specifies the thermodynamic state of a Kerr-Newman hole. α specifies

the charge to angular momentum ratio,

$$\alpha = \tan^{-1}[Q/a],$$

and h specifies the departure from the extremal state in which the two horizons coincide each other. The definition of h is given by

$$h \equiv \sqrt{a^2 + Q^2}/r_H = \begin{cases} h_+, & (0 \leq h_+ \leq 1) \\ h_-, & (1 \leq h_-) \end{cases} \begin{array}{l} \text{: for outer horizon} \\ \text{: for inner horizon,} \end{array}$$

and h_+ and h_- satisfy the duality conditions,

$$h_+ + h_- = \frac{2M}{\sqrt{a^2 + Q^2}}, \quad h_+ h_- = 1.$$

In the parameter space, an arbitrary Kerr-Newman black hole is represented by a pair of points, h_+ and h_- , on a line of constant α . At the extremal limit, these points merge on the circle of unit radius.

All other thermodynamic quantities than the mass M are expressed in terms

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of M and the above parameters. Here are such expressions for angular momentum, electric charge, entropy, temperature, angular velocity, electric potential and associated potentials:

$$J = \frac{2h \cos \alpha}{1 + h^2} M^2, \quad Q = \frac{2h \sin \alpha}{1 + h^2} M, \quad S = \frac{A_{\pm}}{4} = \frac{1 + h^2 \cos^2 \alpha}{2(1 + h^2)^2} M^2,$$

$$T = \frac{\pm 16\pi \sqrt{M^2 - a^2 - Q^2}}{A_{\pm}} = \frac{1 - h^4}{1 + h^2 \cos^2 \alpha} M^{-1} \equiv \beta^{-1},$$

$$\Omega = \frac{4\pi J}{MA_{\pm}} = \frac{h(1 + h^2) \cos \alpha}{2(1 + h^2 \cos^2 \alpha)} M^{-1}, \quad \Phi = \frac{4\pi Q r_H}{A_{\pm}} = \frac{h \sin \alpha}{1 + h^2 \cos^2 \alpha},$$

$$\mu = \beta \Omega = \frac{h \cos \alpha}{2(1 - h^2)}, \quad \phi = \beta \Phi = \frac{h \sin \alpha}{1 - h^4} M.$$

The double-signed expressions for entropy, temperature etc. are unified in the parameter expressions.

2 Critical Phenomenon

Although there have been some debates, it is now widely accepted that a critical point of a Kerr-Newman hole exists at its extremal limit [1]. In the extremal limit ($h \rightarrow 1$), the response coefficients such as

$$\bar{\chi}_M \equiv \left(\frac{\partial^2 S}{\partial M^2} \right)_{J,Q}, \quad \bar{\chi}_J \equiv \left(\frac{\partial^2 S}{\partial J^2} \right)_{M,Q}, \quad \bar{\chi}_Q \equiv \left(\frac{\partial^2 S}{\partial Q^2} \right)_{M,J}$$

exhibit divergences.

This phenomenon can be well understood if one regards the black hole as a coexistence system of the outer-horizon and inner-horizon phases. For a given set of M , J , and Q , these phases have different values for intensive quantities such as temperature, angular velocity and electrostatic potential:

$$\begin{cases} \text{outer horizon: } T_+ = \beta_+^{-1}, \quad \Omega_+ = \beta_+^{-1} \mu_+, \quad \Phi_+ = \beta_+^{-1} \phi_+ \\ \text{inner horizon: } T_- = \beta_-^{-1}, \quad \Omega_- = \beta_-^{-1} \mu_-, \quad \Phi_- = \beta_-^{-1} \phi_- \end{cases}$$

The differences in these quantities, however, vanish and the two phases merge together in the limit of extremality.

As the order parameters of this phenomenon, one may choose the differences of these quantities between the two phases, since the differences vanish in the critical limit. However, somewhat different definitions are required

from the general scheme of thermodynamics [1]. They are the differences in the conjugate variables:

$$\eta_M \equiv \beta_+ - \beta_-, \quad \eta_J \equiv \mu_+ - \mu_- = \beta_+ \Omega_+ - \beta_- \Omega_-, \quad \eta_Q \equiv \phi_+ - \phi_- = \beta_+ \Phi_+ - \beta_- \Phi_-.$$

Then, owing to the presence of inverse temperature β , these order parameters also diverge in the extremal limit (i.e. $T \rightarrow 0$). This is one of the peculiarities of black-hole critical phenomenon.

Associated with the asymptotic form ($r \rightarrow \infty$) of the correlation function,

$$G_{\alpha}(r) \sim \frac{\exp[-r/l_{\alpha}]}{r^{d-2+\zeta}}, \quad (\alpha = J, Q),$$

the correlation length l_{α} and the effective spatial dimension of the system d are introduced. Here, α denotes the different modes of energy exchange.

Various critical exponents are introduced to the theory in order to describe the ways of divergences of various quantities:

response coefficients

$$\bar{\chi}_1 \sim \begin{cases} \epsilon_M^{-\alpha_*} & (Q \neq 0, J = 0 \text{ or } J \neq 0, Q = 0) \\ \epsilon_J^{-\varphi_1} & (M \neq 0, Q \neq 0) \\ \epsilon_Q^{-\varphi_2} & (M \neq 0, J \neq 0), \end{cases}$$

$$\bar{\chi}_2 \sim \begin{cases} \epsilon_M^{-\gamma_1} & (J \neq 0, Q = 0) \\ \epsilon_J^{-\sigma_1} & (M \neq 0, Q \neq 0), \end{cases} \quad \bar{\chi}_3 \sim \begin{cases} \epsilon_M^{-\gamma_2} & (Q \neq 0, J = 0) \\ \epsilon_Q^{-\sigma_2} & (M \neq 0, J \neq 0), \end{cases}$$

order parameters

$$\eta_J \sim \begin{cases} \epsilon_M^{\beta_1^*} & (J \neq 0, Q = 0) \\ \epsilon_J^{\delta_1^{-1}} & (M \neq 0, Q \neq 0), \end{cases} \quad \eta_Q \sim \begin{cases} \epsilon_M^{\beta_2^*} & (Q \neq 0, J = 0) \\ \epsilon_Q^{\delta_2^{-1}} & (M \neq 0, J \neq 0), \end{cases}$$

correlation lengths

$$l_J \sim \begin{cases} \epsilon_M^{-\nu_1} & (J \neq 0, Q = 0) \\ \epsilon_J^{-\mu_1^*} & (M \neq 0, Q \neq 0), \end{cases} \quad l_Q \sim \begin{cases} \epsilon_M^{-\nu_2} & (Q \neq 0, J = 0) \\ \epsilon_Q^{-\mu_2^*} & (M \neq 0, J \neq 0). \end{cases}$$

Since the black-hole entropy is a generalized homogeneous function which satisfies the relations,

$$M = \left(2S + \frac{J^2}{8S} + \frac{Q^2}{2} + \frac{Q^4}{32S} \right)^{1/2} \quad (\text{Smarr's formula})$$

$$\Rightarrow S(\lambda^{1/2}M, \lambda J, \lambda^{1/2}Q) = \lambda S(M, J, Q),$$

scaling laws among the critical exponents are expected to hold analogously to the cases in usual material physics.

We divide these laws into two classes, for convenience. The scaling laws of “the 1st kind” are the equalities among the exponents associated with the response coefficients and order parameters only. They are

$$\begin{cases} \alpha^* + 2\beta^* + \gamma = 2, \\ \beta^*(\delta - 1) = \gamma, \\ \varphi(\beta^* + \gamma) = \alpha^*, \end{cases}$$

and it is easy to confirm that these equalities are actually satisfied by the exponents

$$\alpha^* = \frac{3}{2}, \quad \gamma_1 = \gamma_2 = \frac{3}{2}, \quad \varphi_1 = \varphi_2 = \frac{3}{2}, \\ \beta_1^* = \beta_2^* = -\frac{1}{2}, \quad \delta_1 = \delta_2 = -2, \quad \sigma_1 = \sigma_2 = \frac{3}{2},$$

which are directly calculated from black-hole thermodynamics.

On the other hand, the scaling laws of “the 2nd kind” contain the exponents associated with the correlation length:

$$\begin{cases} \nu(2 - \zeta) = \gamma, \\ \nu d = 2 - \alpha^*, \\ \mu^*(\beta^* + \gamma) = \nu. \end{cases}$$

If we further know the explicit expressions for the correlation length l_α , we can derive from this set of equations the effective spatial dimension d and Fisher's exponent ζ for a Kerr-Newman black hole.

In a recent work on the dynamical evolution of black-hole thermodynamic fluctuations on the outer horizon [2], the author found that the damping constant for every fluctuation mode is proportional to temperature. This means that the correlation length is proportional to inverse temperature of the outer horizon and diverge at the critical point:

$$l_\alpha \sim c \Gamma_\alpha^{-1} \propto T_+^{-1} \rightarrow \infty \quad (\text{as } h \rightarrow 0).$$

This result is consistent with that of Traschen who inferred the correlation length of a black hole from Green's function of a massless charged field around an extremal Kerr-Newman hole [3].

Anyway, the above result enables us to calculate the critical exponents associated with the correlation length as

$$\nu_1 = \nu_2 = \mu_1^* = \mu_2^* = \frac{1}{2},$$

and then the scaling laws of the 2nd kind give

$$\zeta = -1, \quad d = 1 :$$

i.e. the effective spatial dimension of a black hole is one. This implies the existence of an unusual spatial correlation of the form

$$G(r) \propto r^2 \exp[-r/l].$$

These conclusions have obtained already by Cai [4] based on Traschen's guess and our scaling laws [1]. He interpreted the black-hole critical phenomenon

as the merging of the usual black-hole description and the string description near the extremal limit. Therefore, he regarded $d = 1$ as a reflection of the string dimension.

3 Reproduction of Spatial Correlation

Our interpretation of the above results is, however, quite different from that of Cai. We pursue the way of reproducing the spatial correlation function on a hypersurface of constant universal time t , extending in the spacetime outside the event horizon. In the following discussion, we employ the ingoing time defined by

$$\bar{t} = t + \frac{1}{2\kappa_+} \ln(r - r_+) + \frac{1}{2\kappa_-} \ln(r - r_-),$$

instead of the universal time which is a natural time for distant observers. Here κ 's are the surface gravities of the outer and inner horizons. The ingoing time coincides at infinity with the universal. Both times tick at the same rate (i.e. $d\bar{t} = dt$) at an arbitrarily fixed r . Nevertheless, near the event horizon, the ingoing time describes infinite future which the universal time cannot. Therefore, it has definite advantages in describing an eternal black hole.

From the analyses of thermodynamic fluctuations [2], we already know the spectral intensity of the fluctuations:

$$(\xi_\alpha^2)_\omega = \int_{-\infty}^{\infty} G_\alpha(\bar{t}) e^{i\omega\bar{t}} d\bar{t} = \frac{2\Gamma_\alpha}{\omega^2 + \Gamma_\alpha^2} (\xi_\alpha^2(0))_f,$$

where $G_\alpha(\bar{t})$ is the time correlation function on the event horizon and ω is the redshifted frequency. On the event horizon, the time correlation function is therefore reproduced by Fourier-synthesizing this spectrum.

From a more general spacetime viewpoint, we introduce here the spacetime correlation functional which is Fourier-synthesized formally from a given spectrum as

$$G_\alpha[w]_S = \int_{-\infty}^{\infty} (\xi_\alpha^2)_\omega e^{-i\omega w} \frac{d\omega}{2\pi} = \frac{(\xi_\alpha^2(0))_f}{2\pi i} \int_{-\infty}^{\infty} \left\{ \frac{1}{\omega - i\Gamma_\alpha} - \frac{1}{\omega + i\Gamma_\alpha} \right\}_S e^{-i\omega w} d\omega.$$

This is a function of the functional forms of w which is the Fourier conjugate variable to ω . Generally, the integral cannot be performed unless the functional form of w and the hypersurface S on which the integral is evaluated are both specified. From this general expression, the spatial correlation function in particular is obtained by performing the integration on a hypersurface of constant t .

It is rather easy to conclude by inspection that the functional form of w should be

$$w(t, r) = \bar{t} - r.$$

Then, on a hypersurface of constant t , the spatial correlation function is written as

$$G_\alpha(r) \propto r^{(\Gamma_\alpha/2)(1/\kappa_+ + 1/\kappa_-)} e^{-\Gamma_\alpha r},$$

at large distances. In order for this expression to coincide with the asymptotic form that the scaling laws predict, the correlation length should be

$$l_\alpha = \Gamma_\alpha^{-1} = \frac{1}{4} \left(\frac{1}{\kappa_+} + \frac{1}{\kappa_-} \right) = \beta_+ + \beta_- = M.$$

This result is as expected in the sense that l_α is determined directly by the inverse temperature, but is unexpected in that it include both β_+ and β_- . It should be noted that the resulting correlation length has been renormalized in the sense that it doesn't contain divergence any longer.

The summary of the present considerations are as follows. (1) The result that $d = 1$ is merely a reflection of temporal dimension, since the time and spatial correlations are essentially the same thing. (2) The existence of unusual spatial correlation predicted by the scaling laws may resolve the information paradox proposed by Hawking [5]. However, further investigations are needed of this point. (3) The scaling laws predict the existence of such kind of “non-locality” as Lee [6] has proposed more than ten years ago in the context of quantum field theory. Indeed, the inner horizon, nevertheless it is causally disconnected from the outer spacetime, plays an essential role in determining the interactions of a black hole with its surroundings. (4) The above results suggest the importance of hybrid thermodynamics, which will be taken up as a next topic.

4 Hybrid Thermodynamics

As the scaling laws have shown, a black hole should be regarded as a hybrid system of the outer and inner horizons. The first laws of thermodynamics on both of the two horizons are represented by

$$dS_{\pm} = \beta_{\pm} dM - \mu_{\pm} dJ - \phi_{\pm} dQ,$$

where the double signs represent the thermodynamic quantities on these horizons. The following assertions in this section belong to a conjecture.

The addition of the above two first laws yields the first law

$$dS = \beta dM - \mu dJ - \phi dQ,$$

for added quantities such as

$$\begin{aligned} S &\equiv S_+ + S_- = \frac{1}{2} \left(M^2 - \frac{Q^2}{2} \right), & \beta &\equiv \beta_+ + \beta_- = M, \\ \mu &\equiv \mu_+ + \mu_- = 0, & \phi &\equiv \phi_+ + \phi_- = \frac{Q}{2}. \end{aligned}$$

This law seems to describe the thermal (or disordered) properties of the hybrid system, since the potential μ associated with hole’s rotation, which is

a completely ordered motion, becomes zero. There also appears a hybrid inverse temperature which have determined the correlation length in the previous section. The expressions of other quantities of our interest are

$$\begin{aligned} T &\equiv \left(\frac{\partial M}{\partial S} \right)_{J,Q} = \beta^{-1} = M^{-1}, \\ \Omega &\equiv \left(\frac{\partial M}{\partial J} \right)_{S,Q} = \beta^{-1} \mu = 0, \\ \Phi &\equiv \left(\frac{\partial M}{\partial Q} \right)_{S,J} = \beta^{-1} \phi = \frac{1}{2} \frac{Q}{M}. \end{aligned}$$

On the other hand, the subtraction of the first laws

$$\begin{aligned} d\bar{S} &= \bar{\beta} dM - \bar{\mu} dJ - \bar{\phi} dQ, \\ \bar{S} &\equiv S_+ - S_- = \frac{1}{2} M \sqrt{M^2 - (a^2 + Q^2)}, \\ \bar{\beta} &\equiv \beta_+ - \beta_- = \frac{M^2 - Q^2/2}{\sqrt{M^2 - (a^2 + Q^2)}}, \\ \bar{\mu} &\equiv \mu_+ - \mu_- = \frac{1}{2} \frac{a}{\sqrt{M^2 - (a^2 + Q^2)}}, \\ \bar{\phi} &\equiv \phi_+ - \phi_- = \frac{1}{4} \frac{Q \sqrt{a^2 + Q^2}}{\sqrt{M^2 - (a^2 + Q^2)}}, \end{aligned}$$

seems to describe the ordered aspects of the hole, since these subtracted quantities ($\bar{\beta}$, $\bar{\mu}$, $\bar{\phi}$) are the order parameters defined before.

However, the detailed meaning of these quantities and formalism are not so clear at present.

5 Evolution of Fluctuations

Finally in this section, the way of calculating the correlation length of a black hole is described. We consider the time evolution of thermodynamic fluctuations of extensive variables around their slowly varying quasi-equilibrium values:

$$\xi_\alpha(t) \equiv x_\alpha(t) - \bar{x}_\alpha(t) \quad (\bar{x}_\alpha(t) : \text{quasi-equil. values})$$

Physically, the fluctuations are considered to be driven by the vacuum fluctuations and the back reactions of particle emissions. These effects are represented by the deviations of intensive variables from their equilibrium values:

$$f_\alpha(t) \equiv -\{X_\alpha(t) - \bar{X}_\alpha(t)\},$$

where the equilibrium values are given by

$$\bar{X}_\alpha = \left(\frac{\partial S}{\partial \bar{x}_\alpha} \right)_{\{\bar{x}_\beta\}} = \bar{X}_\alpha(\bar{x}_1, \dots, \bar{x}_n) \quad (\beta \neq \alpha)$$

The static admittance defined by

$$\bar{\mathcal{A}}_\alpha \equiv - \left(\frac{\partial^2 S}{\partial \bar{x}_\alpha^2} \right)_{\{\bar{x}_\beta\}}^{-1} = - \left(\frac{\partial \bar{X}_\alpha}{\partial \bar{x}_\alpha} \right)_{\{\bar{x}_\beta\}}^{-1} \quad (\beta \neq \alpha)$$

specifies the response of the system in an equilibrium state to quasi-static change of variables. The virtual fluctuation is defined as the amount of change in an extensive variable caused in response to a driving force if the system were restricted to follow the equilibrium sequence:

$$\xi_\alpha^* = \bar{\mathcal{A}}_\alpha f_\alpha.$$

Here, we demand as a general principle that nature tries to minimize the differences between the actual and virtual fluctuations. This “principle of virtual fluctuations” yields, in the linear regime, a dynamical equation of

the Langevin type:

$$\frac{d\xi_\alpha(t)}{dt} = -\Gamma_\alpha [\xi_\alpha(t) - \xi_\alpha^*(t)] = -\Gamma_\alpha [\xi_\alpha(t) - \bar{\mathcal{A}}_\alpha f_\alpha(t)].$$

The general solution to this equation is written as

$$\xi_\alpha(t) - \xi_\alpha(0) e^{-\Gamma_\alpha t} = \Gamma_\alpha \bar{\mathcal{A}}_\alpha \int_0^t f_\alpha(s) e^{-\Gamma_\alpha(t-s)} ds,$$

and this implies that the dynamic response function of the system is given by

$$R_\alpha(t) = \begin{cases} \Gamma_\alpha \bar{\mathcal{A}}_\alpha e^{-\Gamma_\alpha t} & (t \geq 0) \\ 0 & (t < 0). \end{cases}$$

The Fourier component of the response function is called the complex admittance and is written as

$$\mathcal{A}_\alpha[\omega] = \int_{-\infty}^{\infty} R_\alpha(t) e^{i\omega t} dt = \frac{\Gamma_\alpha}{\Gamma_\alpha - i\omega} \bar{\mathcal{A}}_\alpha$$

in the present case.

The spectral intensity of the two-time correlation function

$$G_\alpha(t-t') \equiv \langle \xi_\alpha(t) \xi_\alpha(t') \rangle_f$$

is calculated from the above dynamical equation. The result is

$$(\xi_\alpha^2)_\omega = \int_{-\infty}^{\infty} G_\alpha(t) e^{i\omega t} dt = \frac{2\Gamma_\alpha}{\omega^2 + \Gamma_\alpha^2} \langle \xi_\alpha^2(0) \rangle_f.$$

Based on the same dynamical equation, we can also demonstrate that the fluctuation-dissipation theorem of the first and the second kinds hold in their classical limits:

FDT of the 1st kind

$$(\xi_\alpha^2)_\omega = \frac{2}{\omega} \frac{\langle \xi_\alpha^2(0) \rangle_f}{\bar{\mathcal{A}}_\alpha} \text{Im } \mathcal{A}_\alpha[\omega] \simeq \frac{2k_B}{\omega} \text{Im } \mathcal{A}_\alpha[\omega],$$

FDT of the 2nd kind

$$g_\alpha(t) \equiv \langle f_\alpha(t) f_\alpha(0) \rangle_f = 2D_\alpha \delta(t) \quad \Rightarrow \quad \Gamma_\alpha = \frac{2\langle \xi_\alpha^2(0) \rangle_f}{\bar{\mathcal{A}}_\alpha^2 (f_\alpha^2)_\omega} \simeq \frac{k_B}{D_\alpha \bar{\mathcal{A}}_\alpha}$$

In the latter derivation, it is assumed that the driving force is completely random and has white spectrum.

If we can express D_α in terms of mean square fluctuations $\langle f_\alpha^2 \rangle_f$, the damping constant Γ_α is obtained from the FDT of the 2nd kind. However, in the limit of white noise, the integration of the spectral intensity of f diverges, and the explicit expression for D cannot be obtained. In order to overcome this difficulty, we introduce here

a harmonic oscillator model, and thereby introduce a quantum correction of the form

$$(f_\alpha^2)_\omega = \frac{2D_\alpha x}{e^x - 1}, \quad x = (\hbar\omega - \hbar m\Omega - e\Phi)/k_B T_+$$

to the spectral intensity. In the classical limit of small x , this form really reduces to a white noise. Integrating this expression, we obtain

$$D_\alpha = \frac{3\hbar}{\pi k_B T_+} \langle f_\alpha^2(0) \rangle_f \simeq \frac{3\hbar}{\pi T_+} \bar{\mathcal{A}}_\alpha^{-1}.$$

Then, substituting this into the FDT of the 2nd kind, we obtain the final result,

$$\Gamma_\alpha = \frac{\pi k_B}{3\hbar} T_+, \quad l_\alpha \sim c \Gamma_\alpha^{-1} \propto T_+^{-1}$$

That is, the correlation length of every fluctuation mode is common and inversely proportional to the horizon temperature.

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Exact Trajectories of Two Body Motion in 2-dim Gravity with Charge and Cosmological Constant

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Abstract

We develop the canonical formalism for a system of N charged particles in lineal gravity and get the exact solutions for $N = 2$. The determining equation of the Hamiltonian is derived in a form of a transcendental equation, which leads to the exact Hamiltonian to infinite order of the gravitational coupling constant. The effect of the cosmological constant on the particles' motion is incorporated in a potential. In the equal mass case an explicit expression of the particle's trajectory for arbitrary charges is given as a function of the proper time.

1 Introduction

Solving the motion of a system of N particles interacting through their mutual gravitational forces has long been one of the important themes in physics [1]. Though the exact solution is known in $N = 2$ case in Newtonian theory, in a context of general theory of relativity the N particles' motion can not be solved exactly due to a dissipation of energy in a form of gravitational radiation. Then for the actual analysis of the binary pulsars we have to resort to an approximation method such as a post-Newtonian expansion.

On the other hand, in this decade, the lower dimensional versions (both in (1+1) and (2+1) dimensions) of general relativity have been extensively investigated in both classical and quantum issues, because non-existence of gravitational radiation may simplify the problem and lead to a deep understanding of the nature of gravitation [2, 3]. Actually for the problem of particles' motion in (2+1) dimensions one can generalize the static 2-body metric to that of two bodies moving with any speed [4]. In (1+1) dimensions we formulated the canonical formalism of N particles in lineal gravity and derived the exact Hamiltonian for $N = 2$ as a solution to a transcendental equation which is valid to infinite order in the gravitational coupling constant [5]. In the equal mass case the exact solutions to the equations of motion were given in a form of the proper separation of the two point masses as a function of their mutual proper time. The trajectories showed characteristic structures depending on the values of a cosmological constant Λ .

Introducing charge in the lineal gravity was first brought to people's attention because it may provide a key to a neutralization of the cosmological constant due to the effect of pair creation [6]. But the motion of particles itself was not fully understood even in the classical context. It is governed by a complex combination of four factors, namely, a gravitational attraction, an electrical force (attractive or repulsive), a cosmological constant (effectively attractive or repulsive) and a relativistic effect. Corresponding to the values of the magnitudes (and signs) of the energy and the parameters as a gravitational coupling constant, mass, charge, and a cosmological constant, the particles are expected to have different types of motion, bounded or unbounded or balanced.

In this paper we generalize our previous formalism [7, 8] to a system of charged particles in (1+1) dimensional gravity. This is an extended theory of Jackiw-Teitelboim lineal gravity [2], introducing

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the coupling to N charged point masses. Since the Einstein action is a topological invariant in (1+1) dimensions, we must incorporate a scalar (dilaton) field into the action. By a canonical reduction of the action, the Hamiltonian is defined as a spatial integral of the second derivative of the dilaton field, which is a function of the canonical variables of the particles (coordinates and momenta) and is determined from the constraint equations. This definition of the Hamiltonian is quite analogous to the ADM Hamiltonian in (3+1) dimensions, since both can be expressed as surface integrals at spatial infinity [9]. For a system of two particles a determining equation of the Hamiltonian is derived from the matching conditions of the solution to the constraint equations and is a kind of transcendental equation. The canonical equations of motion are derived from the Hamiltonian and for the equal mass case they are transformed to be expressed in terms of the proper time. And this latter form of equations can be solved exactly in terms of hyperbolic and/or trigonometric functions of the proper time.

2 Canonically reduced Hamiltonian of N -charged particles

Since the Einstein action is a topological invariant in (1+1) dimensions, we must incorporate a scalar (dilaton) field into our action. The action integral for the gravitational and the electric fields coupled with N charged point masses is

$$\begin{aligned}
I = & \int dx^2 \left[\frac{1}{2\kappa} \sqrt{-g} g^{\mu\nu} \left\{ \Psi R_{\mu\nu} + \frac{1}{2} \nabla_\mu \Psi \nabla_\nu \Psi + \frac{1}{2} g_{\mu\nu} \Lambda \right\} \right. \\
& - A_\mu \mathcal{F}^{\mu\nu}_{,\nu} + \frac{1}{4\sqrt{-g}} \mathcal{F}^{\mu\nu} \mathcal{F}^{\alpha\beta} g_{\mu\alpha} g_{\nu\beta} \\
& \left. + \sum_a \int d\tau_a \left\{ -m_a \left(-g_{\mu\nu}(x) \frac{dz_a^\mu}{d\tau_a} \frac{dz_a^\nu}{d\tau_a} \right)^{1/2} + e_a \frac{dz_a^\mu}{d\tau_a} A_\mu(x) \right\} \delta^2(x - z_a(\tau_a)) \right] \quad (1)
\end{aligned}$$

where Ψ is the dilaton field, A_μ and $\mathcal{F}^{\mu\nu}$ are the vector potential and the field strength, $g_{\mu\nu}$ and g are the metric and its determinant, R is the Ricci scalar, and e_a and τ_a are the charge and the proper time of a -th particle, respectively, with $\kappa = 8\pi G/c^4$. The symbol ∇_μ denotes the covariant derivative associated with $g_{\mu\nu}$.

On transforming the action (1) to the canonical form we use the constraint

$$\frac{\partial E}{\partial x} = \sum_a e_a \delta(x - z_a(t)) \quad (2)$$

derived from the variation of a Lagrange multiplier A_0 , where $E \equiv \mathcal{F}^{01}$. The solution to (2) is

$$E = \frac{1}{2} \sum_a e_a |x - z_a(t)|' \quad (3)$$

This means that in (1+1) dimensions there is no independent degree of freedom of the electric field. We use also the decomposition of the scalar curvature in terms of the extrinsic curvature K via

$$\sqrt{-g} R = -2\partial_0(\sqrt{\gamma} K) + 2\partial_1(\sqrt{\gamma} N^1 K - \gamma^{-1} \partial_1 N_0)$$

where $K = (2N_0\gamma)^{-1}(2\partial_1 N_1 - \gamma^{-1} N_1 \partial_1 \gamma - \partial_0 \gamma)$, with $\gamma = g_{11}$, $N_0 = (-g^{00})^{-1/2}$ and $N_1 = g_{10}$, and change the action (1) into the first order form

$$I = \int dx^2 \left\{ \sum_a p_a \dot{z}_a \delta(x - z_a(t)) + \pi \dot{\gamma} + \Pi \dot{\Psi} + N_0 R^0 + N_1 R^1 \right\} \quad (4)$$

where π and Π are conjugate momenta to γ and Ψ , respectively, and

$$R^0 = -\kappa \sqrt{\gamma} \gamma \pi^2 + 2\kappa \sqrt{\gamma} \pi \Pi + \frac{1}{4\kappa \sqrt{\gamma}} (\Psi')^2 - \frac{1}{\kappa} \left(\frac{\Psi'}{\sqrt{\gamma}} \right)' - \frac{1}{2} \sqrt{\gamma} (E^2 - \frac{\Lambda}{\kappa})$$

$$- \sum_a \sqrt{\frac{p_a^2}{\gamma} + m_a^2} \delta(x - z_a(t)) \quad (5)$$

$$R^1 = \frac{\gamma'}{\gamma} \pi - \frac{1}{\gamma} \Pi \Psi' + 2\pi' + \sum_a \frac{p_a}{\gamma} \delta(x - z_a(t)) \quad (6)$$

with the symbols $(\cdot)$ and $(\cdot)'$ denoting ∂_0 and ∂_1 , respectively.

Since N_0 and N_1 are Lagrange multipliers, the equations $R^0 = 0 = R^1$ are constraints. We may solve the constraints for $(\Psi'/\sqrt{\gamma})'$ and π' in terms of the dynamical and gauge (*i.e.* co-ordinate) degrees of freedom, since these are the only linear terms in these constraints. We identify these latter degrees of freedom by writing the generator arising from the variation of the action at the boundaries in terms of $(\Psi'/\sqrt{\gamma})'$ and π' , and then finding which quantities serve to fix the frame of the physical space-time coordinates in a manner similar to the $(3+1)$ -dimensional case.

Carrying out the same procedure as that of the case of $\Lambda = 0$ and $e_a = 0$, we find that we can consistently choose the coordinate conditions

$$\gamma = 1 \quad \text{and} \quad \Pi = 0 \quad . \quad (7)$$

Eliminating the constraints, the action reduces to

$$I = \int dx^2 \left\{ \sum_a p_a \dot{z}_a (x - z_a) - \mathcal{H} \right\} \quad . \quad (8)$$

Thus the reduced Hamiltonian for the system of particles is

$$H = \int dx \mathcal{H} = -\frac{1}{\kappa} \int dx \Delta \Psi \quad (9)$$

where Ψ is a function of z_a and p_a and is determined by solving the constraints which are under the coordinate conditions (7)

$$\Delta \Psi - \frac{1}{4}(\Psi')^2 + \kappa^2 \pi^2 + \frac{1}{2}(\kappa E^2 - \Lambda) + \kappa \sum_a \sqrt{p_a^2 + m_a^2} \delta(x - z_a) = 0 \quad (10)$$

$$2\pi' + \sum_a p_a \delta(x - z_a) = 0 \quad . \quad (11)$$

The expression of the Hamiltonian (9) is analogous to the reduced Hamiltonian in $(3+1)$ dimensional general relativity. In $(1+1)$ dimensions it is determined by the dilaton field at spatial infinity. The consistency of this canonical reduction is proved in quite an analogous way to the case of no cosmological constant and no charges : namely the canonical equations of motion derived from the reduced Hamiltonian (9) are identical with those derived from the starting action (1).

3 Solution to the constraint equations and the Hamiltonian for a system of two particles

The standard approach for investigating the dynamics of particles is to derive an explicit expression of the Hamiltonian, from which the equations of motion and the solution of trajectories are obtained. In this section we explain how to derive the Hamiltonian from the solution to the constraint equations (10) and (11) and get the explicit Hamiltonian for the case of two charged particles. It has been known that in $(1+1)$ dimensional gravity a constant part of the electric field plays a similar role as the cosmological constant. Since the electric field appears in a combination of $(E^2 - \Lambda/\kappa)$ in all equations we set

$$V(x) \equiv E^2 - C \quad \Lambda_e \equiv \Lambda_0 - \kappa C \quad (12)$$

with $C \equiv \frac{1}{4}(\sum_a e_a)^2$. Thus $V(x)$ vanishes in the outer regions ($|x| \rightarrow \infty$) and Λ_e is an effective cosmological constant. When Λ_e vanishes, we get the Hamiltonian which leads to the correct electrodynamics in (1+1) flat space-time in the limit of $\kappa \rightarrow 0$.

Now the equations (10) and (11) are expressed as

$$\Delta\Psi = \frac{1}{4}(\Psi')^2 - \kappa^2(\chi')^2 - \frac{1}{2}(\kappa V - \Lambda_e) - \kappa \sum_a \sqrt{p_a^2 + m_a^2} \delta(x - z_a), \quad (13)$$

$$\Delta\chi = -\frac{1}{2} \sum_a p_a \delta(x - z_a), \quad (14)$$

where we set $\chi' = \pi$.

Defining ϕ by

$$\Psi = -4\log|\phi| \quad (15)$$

the constraints (13) and (14) for a two-particle system become

$$\Delta\phi - \frac{1}{4} \left\{ \kappa^2(\chi')^2 + \frac{\kappa}{2}V - \frac{1}{2}\Lambda_e \right\} \phi = \frac{\kappa}{4} \left\{ \sqrt{p_1^2 + m_1^2} \phi(z_1) \delta(x - z_1) + \sqrt{p_2^2 + m_2^2} \phi(z_2) \delta(x - z_2) \right\} \quad (16)$$

$$\Delta\chi = -\frac{1}{2} \{ p_1 \delta(x - z_1) + p_2 \delta(x - z_2) \}. \quad (17)$$

The general solution to (17) is

$$\chi = -\frac{1}{4} \{ p_1 |x - z_1| + p_2 |x - z_2| \} - \epsilon X x + \epsilon C_\chi. \quad (18)$$

The factor ϵ ($\epsilon^2 = 1$) has been introduced in the constants X and C_χ so that the T-inversion (time reversal) properties of χ are explicitly manifest. By definition, ϵ changes sign under time reversal and so, therefore, does χ .

Consider first the case $z_2 < z_1$, for which we may divide space into three regions: $z_1 < x$ ((+) region), $z_2 < x < z_1$ ((0) region) and $x < z_2$ (-) region). In each region, V is constant:

$$V = \begin{cases} 0 & (+) \text{ region,} \\ -\frac{1}{2}e_1e_2 & (0) \text{ region,} \\ 0 & (-) \text{ region,} \end{cases} \quad (19)$$

The field ϕ obeys a harmonic oscillator equation, whose solutions are linear combinations of exponential functions of x in the $\pm$ regions and of either trigonometric or exponential functions of x in the 0 region depending on the size of Λ relative to the other integration constants. Matching these solutions at the boundaries $x = z_1$ and $x = z_2$ of each region allows a determination of the coefficients of these linear combinations in the $+$ and $-$ regions in terms of those in the 0 region.

The magnitudes of both ϕ and χ increase with increasing $|x|$ and so we must employ a boundary condition which guarantees that the surface terms obtained in passing from (1) to (4) vanish and the Hamiltonian remain finite. A perturbative analysis indicates that this condition must be $\Psi^2 - 4\kappa^2\chi'^2 + 2\Lambda x^2 = C_\pm x$ in the $\pm$ regions. Incorporating this into the matching conditions determines the coefficients of the linear combination in the 0 region in terms of the momenta and positions of the point masses. This procedure is sufficient to solve for N_0 , N_1 and all other field variables exactly.

The ϕ solutions and the metric components are

$$\begin{aligned} \phi_+ &= \left(\frac{K_1}{\mathcal{M}_1} \right)^{\frac{1}{2}} e^{-\frac{1}{4}(K_{01}z_1 - K_{02}z_2) + \frac{1}{2}K_+(x - z_1)} \\ \phi_0 &= \frac{1}{4K_0} e^{-\frac{1}{4}(K_{01}z_1 - K_{02}z_2)} \left\{ (K_1\mathcal{M}_1)^{1/2} e^{-\frac{1}{2}K_0(x - z_1)} + (K_2\mathcal{M}_2)^{1/2} e^{\frac{1}{2}K_0(x - z_2)} \right\} \\ \phi_- &= \left(\frac{K_2}{\mathcal{M}_2} \right)^{\frac{1}{2}} e^{-\frac{1}{4}(K_{01}z_1 - K_{02}z_2) - \frac{1}{2}K_-(x - z_2)} \end{aligned}$$

$$\begin{aligned}
N_{0(+)}(x) &= \frac{8}{J} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \frac{K_0 K_1}{\mathcal{M}_1} e^{K_+(x-z_1)} \\
N_{0(0)}(x) &= \frac{1}{2K_0 J} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \left[(K_1 \mathcal{M}_1)^{\frac{1}{2}} e^{-\frac{1}{2} K_0(x-z_1)} + (K_2 \mathcal{M}_2)^{\frac{1}{2}} e^{\frac{1}{2} K_0(x-z_2)} \right]^2 \\
N_{0(-)}(x) &= \frac{8}{J} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \frac{K_0 K_2}{\mathcal{M}_2} e^{-K_-(x-z_2)} \\
N_{1(+)} &= \epsilon \frac{Y_+}{K_+} \left\{ \frac{8}{J} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \frac{K_0 K_1}{\mathcal{M}_1} e^{K_+(x-z_1)} - 1 \right\} \\
N_{1(0)} &= \epsilon \left\{ \frac{Y_0}{2JK_0^2} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \left[K_2 \mathcal{M}_2 e^{K_0(x-z_2)} - K_1 \mathcal{M}_1 e^{-K_0(x-z_1)} \right. \right. \\
&\quad \left. \left. + 2K_0(K_1 K_2 \mathcal{M}_1 \mathcal{M}_2)^{1/2} e^{\frac{1}{2} K_0(z_1-z_2)} x \right] + D_0 \right\} \\
N_{1(-)} &= -\epsilon \frac{Y_-}{K_-} \left\{ \frac{8}{J} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \frac{K_0 K_2}{\mathcal{M}_2} e^{-K_-(x-z_2)} - 1 \right\}
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
K_{\pm} &= \sqrt{\kappa^2 \left(X \pm \frac{\epsilon}{4} (p_1 + p_2) \right)^2 - \frac{1}{2} \Lambda_{\epsilon}}, & K_0 &= \sqrt{\kappa^2 \left(X - \frac{\epsilon}{4} (p_1 - p_2) \right)^2 - \frac{\kappa}{2} e_1 e_2 - \frac{1}{2} \Lambda_{\epsilon}} \\
K_1 &= 2K_0 + 2K_- - \kappa \sqrt{p_2^2 + m_2^2}, & K_2 &= 2K_0 + 2K_+ - \kappa \sqrt{p_1^2 + m_1^2}, \\
K_{01} &= K_0 - K_+ + \frac{\kappa \epsilon}{2} p_1, & K_{02} &= K_0 - K_- - \frac{\kappa \epsilon}{2} p_2, \\
\mathcal{M}_1 &= \kappa \sqrt{p_1^2 + m_1^2} + 2K_0 - 2K_+, & \mathcal{M}_2 &= \kappa \sqrt{p_2^2 + m_2^2} + 2K_0 - 2K_-, \\
Y_{\pm} &= \kappa \left[X \pm \frac{\epsilon}{4} (p_1 + p_2) \right], & Y_0 &= \kappa \left[X - \frac{\epsilon}{4} (p_1 - p_2) \right], \\
J &= 2 \left\{ \left(\frac{Y_0}{K_0} + \frac{Y_+}{K_+} \right) K_1 + \left(\frac{Y_0}{K_0} + \frac{Y_-}{K_-} \right) K_2 \right\} \\
&\quad - 2 \left\{ \left(\frac{Y_0}{K_0} - \frac{Y_+}{K_+} \right) \frac{1}{\mathcal{M}_1} + \left(\frac{Y_0}{K_0} - \frac{Y_-}{K_-} \right) \frac{1}{\mathcal{M}_2} \right\} K_1 K_2 - \frac{Y_0}{K_0} K_1 K_2 (z_1 - z_2). \\
D_0 &= -\frac{1}{2} \left(\frac{Y_+}{K_+} - \frac{Y_-}{K_-} \right) + \frac{1}{2J} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \left\{ 2 \left[\left(\frac{Y_0}{K_0} + \frac{Y_+}{K_+} \right) K_1 - \left(\frac{Y_0}{K_0} + \frac{Y_-}{K_-} \right) K_2 \right] \right. \\
&\quad \left. - 2 \left[\left(\frac{Y_0}{K_0} - \frac{Y_+}{K_+} \right) \frac{1}{\mathcal{M}_1} - \left(\frac{Y_0}{K_0} - \frac{Y_-}{K_-} \right) \frac{1}{\mathcal{M}_2} \right] K_1 K_2 - \frac{Y_0}{K_0} K_1 K_2 (z_1 + z_2) \right\}.
\end{aligned} \tag{21}$$

Repeating the analysis for $z_1 < z_2$ yields a similar solution with $p_i \rightarrow -p_i$ and so the full solution is obtained by replacing p_i and $z_1 - z_2$ by $\tilde{p}_i = p_i \operatorname{sgn}(z_1 - z_2)$ and $|z_1 - z_2|$, respectively.

The matching conditions at (z_1, z_2) force the relation

$$K_1 K_2 = \mathcal{M}_1 \mathcal{M}_2 e^{K_0 |z_1 - z_2|} \tag{22}$$

or equivalently

$$\begin{aligned}
&\left(4K_0^2 + [\kappa \sqrt{p_1^2 + m_1^2} - 2K_+] [\kappa \sqrt{p_2^2 + m_2^2} - 2K_-] \right) \tanh \left(\frac{1}{2} K_0 |z_1 - z_2| \right) \\
&= -2K_0 \left([\kappa \sqrt{p_1^2 + m_1^2} - 2K_+] + [\kappa \sqrt{p_2^2 + m_2^2} - 2K_-] \right)
\end{aligned} \tag{23}$$

which determines X . From this the Hamiltonian

$$H = -\frac{1}{\kappa} \int dx \Delta \Psi = -\frac{1}{\kappa} [\Psi']_{-\infty}^{\infty} = \frac{2(K_+ + K_-)}{\kappa}. \tag{24}$$

may be explicitly determined as a function of the coordinates and momenta of the particles.

For the expression (24) to have a definite meaning as the Hamiltonian, $K_{\pm}$ should be real. This impose a restriction on X corresponding to a value of the cosmological constant Λ_e . But K_0 is not necessarily real. If $e_1 e_2$ takes a large positive value (a strong repulsive case), K_0 may be imaginary. In this case we need to change the above analysis, because in (0) region the solution to the ϕ equation (16) becomes

$$\phi_0(x) = A_s \sin \frac{1}{2} \tilde{K}_0 x + A_c \cos \frac{1}{2} \tilde{K}_0 x \quad (25)$$

where

$$\tilde{K}_0 = -iK_0 = \sqrt{\frac{\kappa}{2} e_1 e_2 + \frac{1}{2} \Lambda_e - \kappa^2 \left(X - \frac{\epsilon}{4} (\tilde{p}_1 - \tilde{p}_2) \right)^2}. \quad (26)$$

Under the same matching conditions and the boundary condition we get, instead of (23), a new determining equation of the Hamiltonian

$$\begin{aligned} & \left(4\tilde{K}_0^2 - [\kappa\sqrt{p_1^2 + m_1^2} - 2K_+] [\kappa\sqrt{p_2^2 + m_2^2} - 2K_-] \right) \tan \left(\frac{1}{2} \tilde{K}_0 |z_1 - z_2| \right) \\ & = 2\tilde{K}_0 \left([\kappa\sqrt{p_1^2 + m_1^2} - 2K_+] + [\kappa\sqrt{p_2^2 + m_2^2} - 2K_-] \right). \end{aligned} \quad (27)$$

Actually, this is just the equation derived from (23) by formally replacing K_0 with $i\tilde{K}_0$. The solution of ϕ for an imaginary K_0 is also identical with that obtained from (20) by the same replacement. So we have only to treat the equation (22) for all values of K_0 , which is a kind of transcendental equation for H .

4 Canonical equations of motion and the exact solutions of the trajectories for the equal mass case

The canonical equations for the particles are derived by differentiating the determining equation (22) with respect to the variables z_i and p_i :

$$\dot{p}_i = -\frac{\partial H}{\partial z_i} = (-1)^i \frac{2}{\kappa} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \frac{K_0 K_1 K_2}{J} \quad (28)$$

$$\dot{z}_i = \frac{\partial H}{\partial p_i} = (-1)^{i+1} \epsilon \frac{Y_{\pm}}{K_{\pm}} + \frac{8}{J} \left(\frac{Y_+}{K_+} + \frac{Y_-}{K_-} \right) \frac{K_0 K_1}{\mathcal{M}_i} \left\{ \frac{p_i}{\sqrt{p_i^2 + m_i^2}} + (-1)^i \epsilon \frac{Y_{\pm}}{K_{\pm}} \right\}$$

These canonical equations guarantee the conservation of the Hamiltonian and the total momentum $p_1 + p_2$.

For a system of two particles with equal mass in the center of inertia (C.I.) system $p_1 = -p_2 = p$, the canonical equations of motion are

$$\dot{p} = -\frac{\kappa \mathcal{J}_\Lambda^2 (\mathcal{J}_\Lambda^2 - B^2)}{16C} \text{sgn}(r), \quad (29)$$

$$\dot{r} = 2\epsilon \text{sgn}(r) + \frac{2\mathcal{J}_\Lambda^2}{C} \left(\frac{p}{\sqrt{p^2 + m^2}} - \epsilon \text{sgn}(r) \right). \quad (30)$$

where

$$\begin{aligned} \mathcal{J}_\Lambda^2 &= \left(\sqrt{H^2 + \frac{8\Lambda_e}{\kappa^2}} - 2\epsilon \tilde{p} \right)^2 - \frac{8e_1 e_2}{\kappa} - \frac{8\Lambda_e}{\kappa^2}, \quad B = H - \sqrt{p^2 + m^2}, \\ C &= \mathcal{J}_\Lambda^2 - (H - 2\epsilon \tilde{p})B - \frac{\kappa}{16} (H - 2\epsilon \tilde{p})(\mathcal{J}_\Lambda^2 - B^2) r. \end{aligned} \quad (31)$$

For given values of Λ_e and e_a , the equations (23) or (27) describes the surface in (r, p, H) space of all allowed phase-space trajectories. Since H is a constant of motion, we can draw a phase space trajectory in (r, p) space by setting $H = H_0$ in (23) or (27). The same trajectory should be obtained directly from the solution $r(t), p(t)$ to the canonical equations (29) and (30) by eliminating a time variable t . Numerical solutions to (29) and (30) surely affirm this, but in $r(t)$ and $p(t)$ there appear superficial singularities due to the zero points of C . It is a general manner to describe the particles' trajectories in some invariant parameter and the proper time τ_a of each particle is the best candidate as seen in the starting action (1). From the metric components (20) and the canonical equations (28), the proper time is

$$d\tau_a^2 = dt^2 \{ N_0(z_a)^2 - (N_1(z_a) + \dot{z}_a)^2 \} = dt^2 N_0(z_a)^2 \frac{m_a^2}{p_a^2 + m_a^2} \quad (a = 1, 2) . \quad (32)$$

For the equal mass case it is common for the particle 1 and 2 as

$$d\tau = d\tau_1 = d\tau_2 = \frac{m}{\sqrt{p^2 + m^2}} \frac{\mathcal{J}_\Lambda^2}{C} dt , \quad (33)$$

with which the canonical equations (29) and (30) are expressed in the form

$$\frac{dp}{d\tau} = - \frac{\kappa \sqrt{p^2 + m^2} (\mathcal{J}_\Lambda^2 - B^2)}{16m} \text{sgn}(r) , \quad (34)$$

$$\frac{dr}{d\tau} = \frac{2\epsilon}{m} \left\{ \frac{\sqrt{p^2 + m^2} C}{\mathcal{J}_\Lambda^2} - (\sqrt{p^2 + m^2} - \epsilon \tilde{p}) \right\} \text{sgn}(r) . \quad (35)$$

First Eq.(34) is solved for $p(\tau)$ and then $r(\tau)$ is obtained, either by directly solving (35) after substituting the solution for p or by solving (23)-(27) for r . In $r > 0$ region Eq.(34) leads to

$$\begin{aligned} \int_{p_0}^p \frac{dp}{\left\{ \sqrt{p^2 + m^2} - \epsilon \sqrt{1 + \frac{8\Lambda_e}{\kappa^2 H^2}} p - \frac{1}{H} \left(m^2 + \frac{2e_1 e_2}{\kappa} \right) \right\} \sqrt{p^2 + m^2}} &= - \frac{\kappa H}{4m} \int_{\tau_0}^{\tau} d\tau \\ &= - \frac{\kappa H}{4m} (\tau - \tau_0) . \end{aligned} \quad (36)$$

This expression implies a condition

$$1 + \frac{8\Lambda_e}{\kappa^2 H^2} \geq 0 . \quad (37)$$

Then for a negative Λ_e the motion is allowed for H satisfying

$$H \geq \sqrt{-\frac{8\Lambda_e}{\kappa^2}} . \quad (38)$$

The integration of the LHS of (36) is performed in three cases, separately, depending on the value of Λ_e . The solution $p(\tau)$ is

$$p(\tau) = \frac{\epsilon m}{2} \left(f(\tau) - \frac{1}{f(\tau)} \right) \quad (39)$$

with

$$f(\tau) = \begin{cases} \frac{\frac{H}{m} (1 + \sqrt{\gamma_H}) \left\{ 1 - \eta e^{\frac{\kappa \epsilon m}{4} \sqrt{\gamma_m} (\tau - \tau_0)} \right\}}{\gamma_e + \sqrt{\gamma_m} + (\sqrt{\gamma_m} - \gamma_e) \eta e^{\frac{\kappa \epsilon m}{4} \sqrt{\gamma_m} (\tau - \tau_0)}} & \gamma_m > 0 , \\ \frac{1 + \sqrt{\gamma_H}}{\frac{m}{H} \gamma_e + \frac{m - \sigma}{\frac{\kappa \epsilon m}{4} \sqrt{\gamma_m} (\tau - \tau_0)}} & \gamma_m = 0 , \\ \frac{\frac{H}{m} (1 + \sqrt{\gamma_H})}{\gamma_e + \sqrt{-\gamma_m} \frac{\sigma + \frac{m^2}{H^2} \sqrt{-\gamma_m} \tanh \left[\frac{\kappa \epsilon m}{4} \sqrt{-\gamma_m} (\tau - \tau_0) \right]}{\frac{m^2}{H^2} \sqrt{-\gamma_m} - \sigma \tanh \left[\frac{\kappa \epsilon m}{4} \sqrt{-\gamma_m} (\tau - \tau_0) \right]}} & \gamma_m < 0 , \end{cases} \quad (40)$$

where

$$\begin{aligned}\gamma_H &= 1 + \frac{8\Lambda_c}{\kappa^2 H^2}, & \gamma_e &= 1 + \frac{2\epsilon_1 \epsilon_2}{\kappa m^2}, \\ \gamma_m &= \gamma_e^2 + \frac{8\Lambda_c}{\kappa^2 m^2}, & \sigma &= (1 + \sqrt{\gamma_H})(\sqrt{p_0^2 + m^2} - \epsilon p_0) - \frac{m^2}{H} \gamma_e, \\ \eta &= \frac{\sigma - \frac{m^2}{H} \sqrt{\gamma_m}}{\sigma + \frac{m^2}{H} \sqrt{\gamma_m}},\end{aligned}\quad (41)$$

with p_0 being the initial momentum at $\tau = \tau_0$.

Similarly the solution in $\tau < 0$ region is

$$p(\tau) = -\frac{\epsilon m}{2} \left(\bar{f}(\tau) - \frac{1}{\bar{f}(\tau)} \right) \quad (42)$$

with

$$\bar{f}(\tau) = \begin{cases} \frac{\frac{H}{m}(1+\sqrt{\gamma_H}) \left\{ 1 - \eta e^{\frac{\epsilon \kappa m}{4} \sqrt{\gamma_m}(\tau - \tau_0)} \right\}}{\gamma_e + \sqrt{\gamma_m} + (\sqrt{\gamma_m} - \gamma_e) \eta e^{\frac{\epsilon \kappa m}{4} \sqrt{\gamma_m}(\tau - \tau_0)}} & \gamma_m > 0, \\ \frac{1 + \sqrt{\gamma_H}}{\frac{H}{m} \gamma_e + \frac{\sigma}{m - \sigma \frac{\epsilon \kappa H}{8}}(\tau - \tau_0)} & \gamma_m = 0, \\ \frac{\frac{H}{m}(1+\sqrt{\gamma_H})}{\gamma_e + \sqrt{-\gamma_m} \frac{\sigma + \frac{m^2}{H} \sqrt{-\gamma_m} \tanh\left[\frac{\epsilon \kappa m}{8} \sqrt{-\gamma_m}(\tau - \tau_0)\right]}{\frac{m^2}{H} \sqrt{-\gamma_m} - \sigma \tanh\left[\frac{\epsilon \kappa m}{8} \sqrt{-\gamma_m}(\tau - \tau_0)\right]}} & \gamma_m < 0, \end{cases} \quad (43)$$

where

$$\bar{\sigma} = (1 + \sqrt{\gamma_H})(\sqrt{p_0^2 + m^2} + \epsilon p_0) - \frac{m^2}{H} \gamma_e, \quad \bar{\eta} = \frac{\bar{\sigma} - \frac{m^2}{H} \sqrt{\gamma_m}}{\bar{\sigma} + \frac{m^2}{H} \sqrt{\gamma_m}}. \quad (44)$$

The solution for $r(\tau)$ is obtained by substituting the above $p(\tau)$ into (23) and (27).

The exact trajectories $(r(\tau), p(\tau))$ show many interesting features : a negative cosmological constant $\Lambda_c < 0$ acts effectively as an attractive force leading to bounded motions and for $0 < \Lambda_c < \Lambda_c = \frac{\kappa^2 m^4}{2(H^2 - 4m^2)}$ both bounded and unbounded motions are realized for a fixed value of H , and for $\Lambda_c > \Lambda_c$ only unbounded motion is possible. The details of the analysis of the trajectories will be reported in another occasion.

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Boson Stars with Large Self-interaction in (2+1) dimensions

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Abstract

We consider a solution for boson stars with large self interaction in (2+1) dimensions. For a nonrotating boson stars, an exact solution is found. And for a rotating boson stars, a numerically solution is obtained.

1 Introduction

Self-gravitating systems have been investigated in various situations. Boson stars ([1][2] for reviews) have a very simple constituent, a complex scalar field which is bound by gravitational attraction. Thus the boson star provides us with the simplest model of relativistic stars.

The solutions for relativistic boson stars are only numerically obtained in four dimensions. In (2 + 1) dimensions, static equilibrium configurations have been argued [3] in Einstein gravity with a negative cosmological constant.

We consider a solution for the two boson stars, a nonrotating boson star and a rotating boson star, in (2+1) dimensional gravity with a negative cosmological constant. We assume that the scalar field has a strong self-interaction. An infinitely large self-interaction term in the model leads to much simplifications as in the (3 + 1) dimensional case [4]. For a nonrotating boson star, we obtained the exact solutions [5]. For a rotating boson star, we cannot solve the field equations analytically. Thus, we obtained numerical solutions.

2 Nonrotating boson stars

2.1 Exact solutions

We consider a complex scalar field with mass m and a quartic self-coupling constant λ . The action for the scalar field coupled to gravity can be written down as

$$S = \int d^3x \sqrt{-g} \left[\frac{1}{16\pi G} (R + 2C) - |\nabla_\mu \varphi|^2 - m^2 |\varphi|^2 - \frac{\lambda}{2} |\varphi|^4 \right], \quad (1)$$

where R is the scalar curvature and the positive constant C stands for the (negative) cosmological constant. G is the Newton constant.

Varying the action (1) with respect to the scalar field and the metric yields equations of motion. The equation of motion for the scalar field is

$$\nabla^2 \varphi - m^2 \varphi - \lambda |\varphi|^2 \varphi = 0, \quad (2)$$

while the Einstein equation is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \left[2\text{Re} \left(\nabla_\mu \varphi^* \nabla_\nu \varphi \right) - \frac{1}{2} |\nabla \varphi|^2 g_{\mu\nu} \right] - m^2 |\varphi|^2 g_{\mu\nu} - \frac{\lambda}{2} |\varphi|^4 g_{\mu\nu} + C g_{\mu\nu}. \quad (3)$$

We assume the three-dimensional metric for a static circularly symmetric spacetime as

$$ds^2 = -\alpha^2(r) dt^2 + \beta^2(r) dr^2 + \frac{\gamma^2(r)}{\gamma_0^2} d\theta^2, \quad (4)$$

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where α , β , and γ are functions of the radial coordinate r only, while γ_0 is a constant.

We also assume the complex scalar as

$$\varphi = e^{-i\omega t} \varphi(r), \quad (5)$$

where $\varphi(r)$ is a function of the radial coordinate r only and ω is a constant.

For convenience, we rescale the variables as

$$\tilde{r} = mr, \quad \tilde{\varphi} = \sqrt{8\pi G} \varphi, \quad \Lambda = \frac{\lambda}{8\pi G m^2}, \quad \tilde{\omega} = \omega/m, \quad \text{and} \quad \tilde{C} = C/m^2. \quad (6)$$

Then, using these variables, the equation (2) and (3) are written as

$$\frac{\tilde{\omega}^2}{\alpha^2} \tilde{\varphi} + \frac{1}{\alpha\beta\gamma} \frac{d}{d\tilde{r}} \left(\frac{\alpha\gamma}{\beta} \frac{d\tilde{\varphi}}{d\tilde{r}} \right) - \tilde{\varphi} - \Lambda \tilde{\varphi}^3 = 0, \quad (7)$$

$$\frac{1}{\beta\gamma} \frac{d}{d\tilde{r}} \left(\frac{1}{\beta} \frac{d\gamma}{d\tilde{r}} \right) = -\frac{1}{\beta^2} \left(\frac{d\tilde{\varphi}}{d\tilde{r}} \right)^2 - \frac{\tilde{\omega}^2}{\alpha^2} \tilde{\varphi}^2 - \tilde{\varphi}^2 - \frac{\Lambda}{2} \tilde{\varphi}^4 + \tilde{C}, \quad (8)$$

$$\frac{1}{\alpha\beta^2\gamma} \frac{d\alpha}{d\tilde{r}} \frac{d\gamma}{d\tilde{r}} = +\frac{1}{\beta^2} \left(\frac{d\tilde{\varphi}}{d\tilde{r}} \right)^2 + \frac{\tilde{\omega}^2}{\alpha^2} \tilde{\varphi}^2 - \tilde{\varphi}^2 - \frac{\Lambda}{2} \tilde{\varphi}^4 + \tilde{C}, \quad (9)$$

$$\frac{1}{\alpha\beta} \frac{d}{d\tilde{r}} \left(\frac{1}{\beta} \frac{d\alpha}{d\tilde{r}} \right) = -\frac{1}{\beta^2} \left(\frac{d\tilde{\varphi}}{d\tilde{r}} \right)^2 + \frac{\tilde{\omega}^2}{\alpha^2} \tilde{\varphi}^2 - \tilde{\varphi}^2 - \frac{\Lambda}{2} \tilde{\varphi}^4 + \tilde{C}. \quad (10)$$

Next, we study the limit of large self-interaction. The study of boson stars, with finite value of the self-coupling, is very difficult. Thus, we simplify by the limit of large self-interaction

New variable are:

$$r_* = \tilde{r}/\sqrt{\Lambda}, \quad \varphi_* = \sqrt{\Lambda} \tilde{\varphi}, \quad \text{and} \quad C_* = \Lambda \tilde{C}. \quad (11)$$

For the limit of large self-coupling, $\Lambda \rightarrow \infty$, the field equations (7- 10) will be reduced to

$$\frac{\tilde{\omega}^2}{\alpha^2} \varphi_* - \varphi_* - \varphi_*^3 = 0, \quad (12)$$

$$\frac{1}{\beta\gamma} \left(\frac{\gamma'}{\beta} \right)' = -\frac{\tilde{\omega}^2}{\alpha^2} \varphi_*^2 - \varphi_*^2 - \frac{1}{2} \varphi_*^4 + C_*, \quad (13)$$

$$\frac{1}{\beta^2} \frac{\alpha'}{\alpha} \frac{\gamma'}{\gamma} = +\frac{\tilde{\omega}^2}{\alpha^2} \varphi_*^2 - \varphi_*^2 - \frac{1}{2} \varphi_*^4 + C_*, \quad (14)$$

$$\frac{1}{\alpha\beta} \left(\frac{\alpha'}{\beta} \right)' = +\frac{\tilde{\omega}^2}{\alpha^2} \varphi_*^2 - \varphi_*^2 - \frac{1}{2} \varphi_*^4 + C_*. \quad (15)$$

Now, let us solve the set of an algebraic equation and differential equations (12-15).

These equation can be solved. We get two solutions, outside and inside boson stars. The exterior solution is

$$ds^2 = -\frac{1}{C_*^2} (CR^2 - 8GM_o) dt^2 + \frac{1}{CR^2 - 8GM_o} dR^2 + R^2 d\theta^2, \quad (16)$$

where B is an integration constant and $R \equiv \sqrt{\Lambda/m^2} \gamma$. This solution is precisely the same as the well-known BTZ vacuum solution [6]. And interior solution is

$$ds^2 = -\frac{1 + \varphi_{*0}^2}{1 + \varphi_*^2} dt^2 + \frac{1 + \varphi_*^2}{1 + \varphi_{*0}^2} \left(\frac{C_* + \frac{1}{2}\varphi_{*0}^4}{C_* + \frac{1}{2}\varphi_*^4} \right)^2 dR^2 + R^2 d\theta^2, \quad (17)$$

where φ_{*0} is the value of φ_* at the center of the star.

2.2 Physical quantities

The outside solution (16) is precisely the same as the BTZ vacuum solution. Thus, we identify the BTZ mass M_o with the *boson star mass*.

We obtain the relation between the BTZ mass of the boson star and φ_{*0} by the solutions:

$$\frac{8GM_o}{C_*} = \frac{1}{(\frac{1}{2}\varphi_{*0}^4 + C_*)^2} \left[\frac{1}{2}\varphi_{*0}^4 + \varphi_{*0}^2 - (\varphi_{*0}^2 + 1) \ln(\varphi_{*0}^2 + 1) - C_* \right]. \quad (18)$$

The value of BTZ mass M_o is negative for small φ_{*0} . In particular, as φ_{*0} goes to zero, M_o approaches $-1/(8G)$. One can take

$$M_i = M_o + \frac{1}{8G}, \quad (19)$$

as a definition of the intrinsic mass M_i , we call here. When φ_{*0} vanishes, M_i becomes zero, and spacetime becomes a usual anti-de Sitter spacetime everywhere.

The total particle number N is given by

$$N = \int d^2x \sqrt{-g} |g^{tt}| i(\varphi \partial_t \varphi^* - \varphi^* \partial_t \varphi). \quad (20)$$

Therefore we can calculate the particle number of the boson star by making use of the above solution. It is given by

$$\begin{aligned} N &= 2\pi \int \frac{2m\tilde{\omega}}{8\pi G\Lambda} \varphi_*^2 \frac{1}{\alpha^2} \frac{\alpha\beta\gamma}{\gamma_0} d\tau \\ &= \frac{1}{4Gm} \frac{\sqrt{\varphi_{*0}^2 + 1} [\varphi_{*0}^2 - \ln(\varphi_{*0}^2 + 1)]}{\frac{1}{2}\varphi_{*0}^4 + C_*}. \end{aligned} \quad (21)$$

The intrinsic mass and particle number of the boson star as functions of φ_{*0} are plotted in FIG. 1.

The radius of the boson stars is given by

$$R^2 = \frac{\Lambda}{m^2} \frac{1}{(\frac{1}{2}\varphi_{*0}^4 + C_*)^2} \left\{ \left[\frac{1}{2}(\varphi_{*0}^2 + 1) + \frac{\frac{1}{2} + C_*}{\varphi_{*0}^2 + 1} \right] (\varphi_{*0}^2 - \varphi_*^2) + (\varphi_{*0}^2 + 1) \ln \frac{\varphi_{*0}^2 + 1}{\varphi_*^2 + 1} \right\}. \quad (22)$$

Now, we consider the binding energy $E_b = M_i - mN$. If the binding energy is positive, the boson star is not stable. To exist the stable boson star, the binding energy is negative. Thus, the value of C_* have the critical value.

To see the critical value, we plot the dependence of binding energy $E_b = M_i - mN$ on the particle number N for a fixed C_* , Fig 2. FIG. 2 is called *bifurcation diagrams* [7]. The cuspidal point corresponds to the boson star with the mass $M_{i, \max}$ and the particle number $N_{\max}$. This point represents an absolute stable solution [7]. For $C_* > C_{*, \text{crit}} \approx 2.5268$, however, the cuspidal point is not a global minimum of the binding energy. A global minimum of the binding energy is positive for all finite N . Thus, in this case, the boson star is not energetically stable.

Also, for $C_* < C_{*, \text{crit}}$, the binding energy is positive when N is small. This means that the small lumps cannot coalesce to make a large star.

To summarize, we have obtained an exact solution describing a nonrotating boson star with very large self-coupling constant in $(2+1)$ dimensions. There is a critical value for $C_* = (\lambda/(8\pi G m^4))C$, $C_{*, \text{crit}} \approx 2.5268$. For $C_* > C_{*, \text{crit}}$ the binding energy turns to be positive, the boson star configuration is not energetically favorable.

3 Rotating boson stars

3.1 Field equations

Similarly to the nonrotating case, we consider a complex scalar field with mass m and a quartic self-coupling constant λ . The action for the scalar field coupled to gravity can be written down as

$$S = \int d^3x \sqrt{-g} \left[\frac{1}{16\pi G} (R + 2C) - |\nabla_\mu \varphi|^2 - m^2 |\varphi|^2 - \frac{\lambda}{2} |\varphi|^4 \right], \quad (23)$$

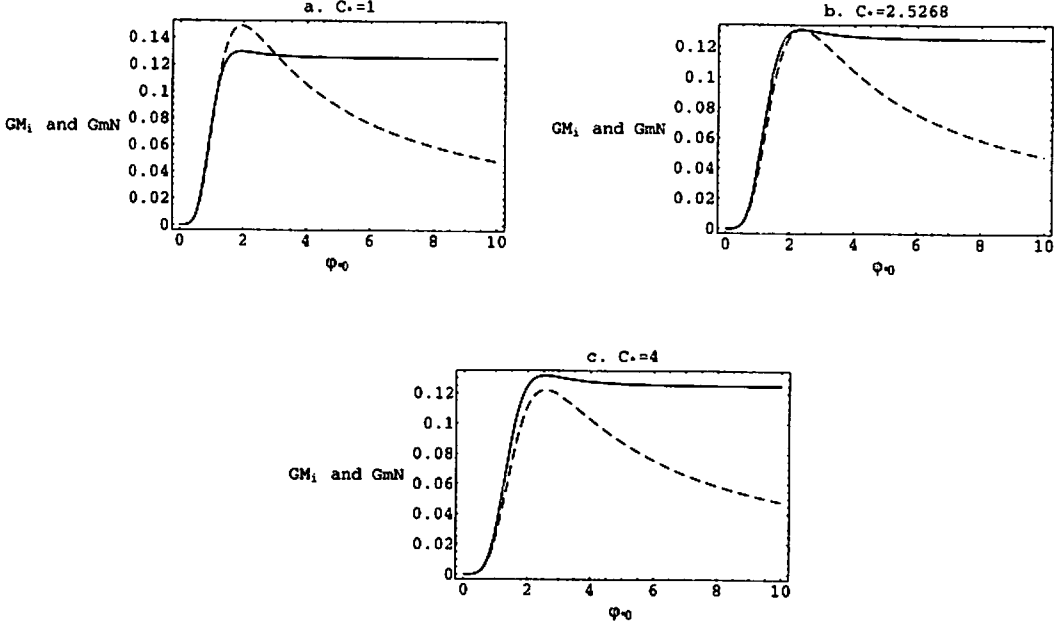


Figure 1: Boson star mass M_i in units of G^{-1} (solid line) and particle number N in units of $(Gm)^{-1}$ (broken line) as functions of φ_{*0} for different values of C_* : (a) $C_* = 1$. (b) $C_* = 2.5268$. (c) $C_* = 4$.

where R is the scalar curvature and the positive constant C stands for the (negative) cosmological constant. G is the Newton constant.

Varying the action (23) with respect to the scalar field and the metric yields equations of motion. The equation of motion for the scalar field is

$$\nabla^2 \varphi - m^2 \varphi - \lambda |\varphi|^2 \varphi = 0, \quad (24)$$

while the Einstein equation is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \left[2\text{Re} \left(\nabla_\mu \varphi^* \nabla_\nu \varphi - \frac{1}{2} |\nabla \varphi|^2 g_{\mu\nu} \right) - m^2 |\varphi|^2 g_{\mu\nu} - \frac{\lambda}{2} |\varphi|^4 g_{\mu\nu} \right] + C g_{\mu\nu}. \quad (25)$$

We assume the three-dimensional metric for a static circularly symmetric spacetime as

$$ds^2 = -e^{-2\delta(r)} \Delta(r) dt^2 + \frac{1}{\Delta(r)} dr^2 + r^2 (d\theta - \Omega(r) dt)^2, \quad (26)$$

where Δ , δ , and Ω are functions of the radial coordinate r only. We will use $\Delta(r) = C r^2 - 8GM(r)$, where M is a function of r .

We also assume that the complex scalar has the following dependence on the coordinates:

$$\varphi = e^{-i\omega t + i\ell\theta} \varphi(r), \quad (27)$$

where $\varphi(r)$ is a function of the radial coordinate r only and ω and ℓ is a constant.

For convenience, we rescale the variables as

$$\tilde{r} = mr, \quad \tilde{\varphi} = \sqrt{8\pi G} \varphi, \quad \Lambda = \frac{\lambda}{8\pi G m^2}, \quad \tilde{\Omega} = \frac{1}{m} \left(\Omega - \frac{\omega}{\ell} \right), \quad (28)$$

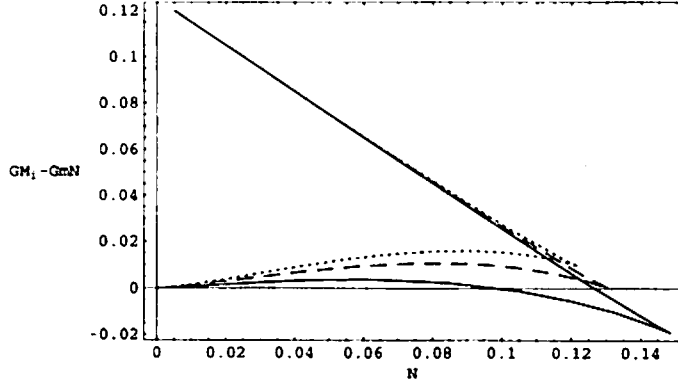


Figure 2: Binding energy E_b (in units of G^{-1}) as a function of N (in units of $(Gm)^{-1}$) for $C_* = 1$ (solid line), $C_{* \text{ crit}} (\approx 2.5268)$ (broken line), and 4 (dotted line).

We also denote $\Delta = Cr^2 - 8GM = (C/m^2)\tilde{r}^2 - 8GM$.

Using these variables, we can rewrite the field equations (24) and (25), as

$$-\frac{1}{2\tilde{r}} \frac{d(8GM)}{d\tilde{r}} + \frac{1}{4}\tilde{r}^2 e^{2\delta} \left(\frac{d\tilde{\Omega}}{d\tilde{r}} \right)^2 = - \left[\Delta \left(\frac{d\tilde{\varphi}}{d\tilde{r}} \right)^2 + \frac{\ell^2}{\tilde{r}^2} \tilde{\varphi}^2 + \frac{e^{2\delta}}{\Delta} \ell^2 \tilde{\Omega}^2 \tilde{\varphi}^2 + \tilde{\varphi}^2 + \frac{\Lambda}{2} \tilde{\varphi}^4 \right], \quad (29)$$

$$\frac{\Delta}{\tilde{r}} \frac{d\delta}{d\tilde{r}} = -2 \left[\Delta \left(\frac{d\tilde{\varphi}}{d\tilde{r}} \right)^2 + \frac{e^{2\delta}}{\Delta} \ell^2 \tilde{\Omega}^2 \tilde{\varphi}^2 \right], \quad (30)$$

$$-\frac{1}{2\tilde{r}^2} \frac{d}{d\tilde{r}} \left(\tilde{r}^3 e^{\delta} \frac{d\tilde{\Omega}}{d\tilde{r}} \right) = -2 \frac{e^{\delta}}{\Delta \tilde{r}} \ell^2 \tilde{\Omega} \tilde{\varphi}^2, \quad (31)$$

$$\frac{1}{\tilde{r} e^{-\delta}} \frac{d}{d\tilde{r}} \left(\tilde{r} e^{-\delta} \Delta \frac{d\tilde{\varphi}}{d\tilde{r}} \right) = \frac{\ell^2}{\tilde{r}^2} \tilde{\varphi} - \frac{e^{2\delta}}{\Delta} \ell^2 \tilde{\Omega}^2 \tilde{\varphi} + \tilde{\varphi} + \Lambda \tilde{\varphi}^3. \quad (32)$$

Next, we rescale the variables again, to study the limit of large self-interaction. New variables are:

$$r_* = \tilde{r}/\sqrt{\Lambda}, \quad \varphi_* = \sqrt{\Lambda} \tilde{\varphi}, \quad L^2 = \frac{\ell^2}{\Lambda}, \quad \Omega_* = \sqrt{\Lambda} \tilde{\Omega}. \quad (33)$$

Then $\Lambda = Cr^2 - 8GM = C_* r_*^2 - 8GM$, where $C_* = C\Lambda/m^2$. For the limit of large self-coupling, $\Lambda \rightarrow \infty$, these equations will be reduced to

$$\frac{1}{2r_*} \tilde{M}' = \frac{1}{4L^2} r_*^2 e^{2\delta} (\tilde{\Omega}')^2 + \frac{L^2}{r_*^2} \varphi_*^2 + \frac{e^{2\delta}}{\Delta} \tilde{\Omega}^2 \varphi_*^2 + \varphi_*^2 + \frac{1}{2} \varphi_*^4, \quad (34)$$

$$\frac{1}{r_*} \delta' = -2 \frac{e^{2\delta}}{\Delta^2} \tilde{\Omega}^2 \varphi_*^2, \quad (35)$$

$$\frac{1}{r_*} (r_*^3 e^{\delta} \tilde{\Omega}')' = 4 \frac{e^{\delta}}{\Delta} L^2 \tilde{\Omega} \varphi_*^2, \quad (36)$$

$$\varphi_*^3 = \varphi_* \left(\frac{e^{2\delta}}{\Delta} \tilde{\Omega}^2 - \frac{L^2}{r_*^2} - 1 \right), \quad (37)$$

where $\tilde{M} = 8GM$ and $\tilde{\Omega} = L\Omega_*$.

Unfortunately, these equations cannot be solved analytically. We will find the numerical solution.

3.2 Numerical solution

We can solve about φ_* by Eq. (37). A trivial solution is

$$\varphi_*^2 = 0 \quad (38)$$

This corresponds to the region of no bosonic matter field, i.e., the vacuum outside star. The interior solution for φ_* is

$$\varphi_*^2 = \frac{e^{2\delta}}{\Delta} \bar{\Omega}^2 - \frac{L^2}{r_*^2} - 1 \quad (39)$$

This describes the configuration of the boson field inside a boson star.

The term L^2/r_*^2 yields a remarkable consequence. For nonzero L^2 , any positive value for the solution becomes impossible in the vicinity of the origin. Thus, for a sufficiently small value for r_* , the solution for φ_*^2 must be $\varphi_*^2 = 0$. In other words, a rotating boson star has a vacuum “hole” in its central region!

Therefore, we consider that the central vacuum region is in the range $0 < r_* < r_{*,i}$. The value for $r_{*,i}$ gives the location of the inside boundary of rotating boson star.

For solving the full solution numerically, we take the following conditions as the boundary conditions:

$$\varphi_*^2(r_{*,i}) = 0, \quad \delta(r_{*,i}) = 0, \quad \bar{\Omega}(r_{*,i}) = \bar{\Omega}_i, \quad \bar{M}(r_{*,i}) = -1. \quad (40)$$

By these conditions, we will solve the differential equations numerically.

Fig 3 is φ_* as a function of the radial coordinate r_* . We see that the boson star has a vacuum hole in the center of the star.

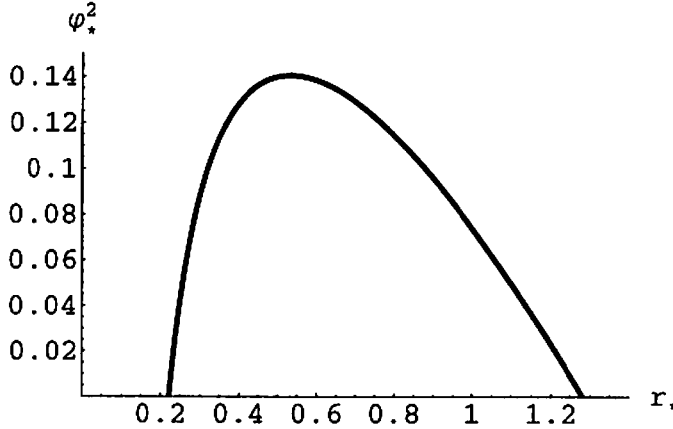


Figure 3: φ_* as a function of the star's radial r_* , where $C_* = 0.1$, $\bar{\Omega}_i = 1.1$, and $L = 0.1$.

Although we cannot calculate numerical values of φ_* , Δ , δ , and $\bar{\Omega}$ for all set of the conditions $\bar{\Omega}_i$, L and C_* . Because there may be a numerical singularity. Therefore, it is worth noting that the parameter region for the possible solutions is restricted.

The mass of the boson star is given,

$$8GM_{BTZ} = \bar{M}_o + \frac{r_{*,o}^4}{4L^2} e^{2\delta_o} (\bar{\Omega}'_o)^2. \quad (41)$$

where $\bar{M}_o = \bar{M}(r_{*,o})$, $\delta_o = \delta(r_{*,o})$, $\bar{\Omega}'_o = \bar{\Omega}'(r_{*,o})$ and $r_* = r_{*,o}$ is the point where φ_*^2 becomes zero again. And the particle number of the boson star is given,

$$N = \frac{1}{2Gm} \int dr_* \frac{r_* e^{\delta}}{\Delta} \bar{\Omega} \varphi_*^2. \quad (42)$$

By the obtained mass and particle number, The binding energy is given,

$$E = M_{BTZ} + \frac{1}{8G} - mN. \quad (43)$$

The binding energy must be negative for the stable boson star. Therefore, there is the region of the stable solution. Fig 4 is shown the region of the stable solution.

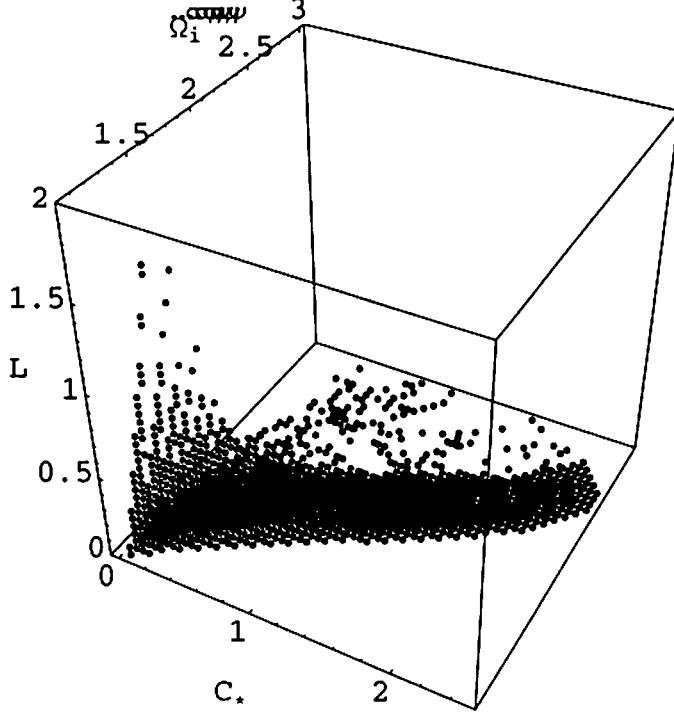


Figure 4: The region of the stable solution. These points are shown in the region of the possible solutions for the stable boson stars.

The the maximum value of the cosmological constant is $C_* = C_{crit} \approx 2.5268$ for $L \rightarrow 0$. This fact is trivial, because rotating bosonstar is approached nonrotating boson star when $L \rightarrow 0$.

To summarize, we have obtained a numerical solution of a rotating boson star in $(2+1)$ dimensions. We found that the rotating boson star has a vacuum hole in the center of the star. And there is the region of the stable solution. The maximum value for $C_* = (\lambda/(8\pi Gm^4))C$ is $C_{crit} \approx 2.5268$ when $L \rightarrow 0$.

4 Summary and prospect

Here, we studied the boson stars, nonrotating and rotating boson star, with very large self-coupling constant in $(2+1)$ dimensions. For a nonrotating boson stars, an exact solution is found. From the stability of boson star, we obtained the critical value of cosmological constant $C_* = (\lambda/(8\pi Gm^4))C$, $C_{crit} \approx 2.5268$. For a rotating boson stars, a numerically solution is obtained. We found that the rotating boson star has a vacuum hole in the center of the star. And the region of the solution for the stable boson star is shown.

The future study of boson stars is the following. The analysis of boson star with the actually finite value of self-coupling is of much interest and will be nessessary. More general cases including such as a

$|\varphi|^6$ coupling may exhibit more complicated results, but the analysis of them can be carried out in the same manner as in the present study.

And for the rotating boson star, the set of the input conditions, $\bar{\Omega}_i$, L , and C_* , is not the meaningful physical quantities. Thus these variables must be replaced with other physical variables. Moreover, the model of the spinning boson star in (3+1) dimensions is studied [8]. The similarity and the difference, between the our model and the model in other dimensions, must be studied.

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A Maximum Mass-to-Size Ratio in Scalar-Tensor Theories of Gravity

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Abstract

We derive a modified Buchdahl inequality for scalar-tensor theories of gravity. In general relativity, Buchdahl has shown that the maximum value of the mass-to-size ratio, $2M/R$, is $8/9$ for static and spherically symmetric stars under some physically reasonable assumptions. We formally apply Buchdahl's method to scalar-tensor theories and obtain theory-independent inequalities. After discussing the mass definition in scalar-tensor theories, these inequalities are related to a theory-dependent maximum mass-to-size ratio. We show that its value can exceed not only Buchdahl's limit, $8/9$, but also unity, which we call *the black hole limit*, in contrast to general relativity. Next, we numerically examine the validity of the assumptions made in deriving the inequalities and the applicability of our analytic results. We find that the assumptions are mostly satisfied and that the mass-to-size ratio exceeds both Buchdahl's limit and the black hole limit. However, we also find that this ratio never exceeds Buchdahl's limit when we impose the further condition, $\rho - 3p \geq 0$, on the density, ρ , and pressure, p , of the matter.

1 Introduction

The scalar-tensor theories of gravity (scalar-tensor theories) are natural alternatives to general relativity, and gravity is mediated not only by a tensor field but also by a scalar field in these theories [1],[2],[3],[4]. It has been found that scalar-tensor theories exhibit different aspects of gravity in strong gravitational fields in contrast to general relativity [4],[5],[6].

In general relativity, the mass-to-size ratio of a star has physical significance, especially for an isolated system. Buchdahl has obtained a maximum value of the mass-to-size ratio of a static and spherically symmetric star under the physically reasonable assumptions [7],[8]. Buchdahl has obtained an upper limit of the mass-to-size ratio as $2M/R \leq 8/9$. We shall refer to this as the *Buchdahl inequality*.

Motivated by Buchdahl's theorem, we shall derive a modified Buchdahl inequality to obtain the maximum mass-to-size ratio in scalar-tensor theories. We then numerically examine the validity of the assumptions made in deriving the inequality. The applicability of our analytic results is also examined. In section 2, we summarize the basic equations in the scalar-tensor theories. In section 3, we derive a modified Buchdahl inequality in the scalar-tensor theories, and the numerical results are compared with the analytic results in section 4. A brief summary is given in section 5.

2 Basic equations

We shall consider the simplest scalar-tensor theory [1],[4],[9]. In this theory, gravitational interactions are mediated by a tensor field, $g_{\mu\nu}$, and a scalar field, ϕ . The action of the theory is

$$S = \frac{1}{16\pi} \int \sqrt{-g} \left[\phi R - \frac{\omega(\phi)}{\phi} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \right] d^4x + S_{\text{matter}}[\Psi_m, g_{\mu\nu}], \quad (2.1)$$

where $\omega(\phi)$ is a dimensionless arbitrary function of ϕ , Ψ_m represents matter fields, and S_{matter} is the action of the matter fields. The scalar field, ϕ , plays the role of an effective gravitational constant as $G \sim 1/\phi$.

Now we perform the conformal transformation,

$$g_{\mu\nu} = A^2(\varphi) g_{*\mu\nu}, \quad (2.2)$$

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such that

$$G_* A^2(\varphi) = \frac{1}{\phi}, \quad (2.3)$$

where G_* is a bare gravitational constant, and we call $A(\varphi)$ a *coupling function*. Hereafter, the symbol, $*$, denotes quantities or derivatives associated with $g_{*\mu\nu}$. Then the action can be rewritten as

$$S = \frac{1}{16\pi G_*} \int \sqrt{-g_*} (R_* - 2g_*^{\mu\nu} \varphi_{*,\mu} \varphi_{*,\nu}) d^4x + S_{\text{matter}}[\Psi_m, A^2(\varphi) g_{*\mu\nu}], \quad (2.4)$$

where the scalar field, φ , is defined by

$$\alpha^2(\varphi) \equiv \left(\frac{d \ln A(\varphi)}{d\varphi} \right)^2 = \frac{1}{3 + 2\omega(\phi)}. \quad (2.5)$$

Varying the action by $g_{*\mu\nu}$ and φ yields, respectively,

$$G_{*\mu\nu} = 8\pi G_* T_{*\mu\nu} + 2 \left(\varphi_{*,\mu} \varphi_{*,\nu} - \frac{1}{2} g_{*\mu\nu} g_*^{\alpha\beta} \varphi_{*,\alpha} \varphi_{*,\beta} \right), \quad (2.6)$$

$$\square_* \varphi = -4\pi G_* \alpha(\varphi) T_*, \quad (2.7)$$

where $T_*^{\mu\nu}$ represents the energy-momentum tensor with respect to $g_{*\mu\nu}$ defined by

$$T_*^{\mu\nu} \equiv \frac{2}{\sqrt{-g_*}} \frac{\delta S_{\text{matter}}[\Psi_m, A^2(\varphi) g_{*\mu\nu}]}{\delta g_{*\mu\nu}} = A^6(\varphi) T^{\mu\nu}. \quad (2.8)$$

The conservation law for $T_*^{\mu\nu}$ is given by

$$\nabla_{*\nu} T_*^{\nu\mu} = \alpha(\varphi) T_* \nabla_{*\mu} \varphi. \quad (2.9)$$

The field equations (2.7) and (2.9) tell us that the *coupling strength*, $\alpha(\varphi)$, plays a role in mediating interactions between the scalar field, φ , and the matter.

3 Modified Buchdahl's theorem in scalar-tensor theories

In this section, we consider a static and spherically symmetric space-time with a perfect fluid. First, we derive a modified Buchdahl inequality. Then the inequality is reformulated to obtain the maximum value of the mass-to-size ratio in the scalar-tensor theories. Hereafter, we refer to $(g_{\mu\nu}, \phi)$ and $(g_{*\mu\nu}, \varphi)$, respectively, as the physical frame and the Einstein frame.

3.1 A modified Buchdahl inequality in scalar-tensor theories

In the Einstein frame, a line element of the static and spherically symmetric space-time is written as [8]

$$ds_*^2 = -f_*(r) dt^2 + h_*(r) dr^2 + r^2 d\Omega^2. \quad (3.1)$$

The stress-energy tensor for the perfect fluid in the Einstein frame is given by

$$T_*^{\mu\nu} = (\rho_* + p_*) u_*^\mu u_*^\nu + p_* g_*^{\mu\nu}, \quad u_{*\alpha} = -\sqrt{f_*(r)} (dt)_\alpha, \quad (3.2)$$

where u_*^α is the four velocity of the matter.

Now the field equations (2.6) and (2.7) are reduced to the following equations:

$$(r(1 - h_*^{-1}))' = 8\pi G_* \rho_* r^2 + \frac{r^2}{h_*} \varphi'^2, \quad (3.3)$$

$$-r^{-2} h_* (1 - h_*^{-1}) + r^{-1} f_*^{-1} f_*' = 8\pi G_* p_* h_* + \varphi'^2, \quad (3.4)$$

$$\left(\frac{f_*'}{2f_*} \right)' + \left(\frac{f_*'}{2f_*} \right)^2 + \frac{1}{r} \frac{f_*'}{2f_*} - \frac{h_*'}{2h_*} \frac{f_*'}{2f_*} - \frac{1}{r} \frac{h_*'}{2h_*} = 8\pi G_* p_* h_* - \varphi'^2, \quad (3.5)$$

$$r^{-2} (f_* h_*)^{-\frac{1}{2}} \left(\left(\frac{f_*}{h_*} \right)^{\frac{1}{2}} r^2 \varphi' \right)' = 4\pi G_* \alpha(\varphi) (\rho_* - 3p_*), \quad (3.6)$$

where a prime denotes differentiation with respect to r . As is often done in the cases of general relativity, we define a *mass function*, $m_*(r)$, in the Einstein frame as follows:

$$h_*(r) \equiv \left[1 - \frac{2m_*(r)}{r} \right]^{-1}. \quad (3.7)$$

Then (3.3) is rewritten as

$$m'_*(r) = 4\pi G_* \rho_{eff} r^2, \quad (3.8)$$

where

$$\rho_{eff} \equiv \rho_* + \frac{\dot{\varphi}^2}{8\pi G_* h_*}. \quad (3.9)$$

That is, ρ_{eff} plays the role of an effective density in the Einstein frame.

In order to derive a modified Buchdahl inequality, we assume

$$h_*(r) \geq 0, \quad f_*(r) \geq 0, \quad (3.10)$$

$$\rho_{eff}(r) \geq 0, \quad \rho'_{eff}(r) \leq 0. \quad (3.11)$$

These assumptions imply the following:

- No black hole exists in the Einstein frame.
- The effective density, ρ_{eff} , is a positive and monotonously decreasing function of the radius.

Moreover, we assume that an interior solution of the above field equations smoothly matches the corresponding exterior one. Note that these assumptions are concerned with the unphysical variables and that their validity should be examined. This will be done later.

Using the assumption (3.11), it is easy to verify the following inequality:

$$\left(\frac{m_*}{r^3} \right)' \leq 0. \quad (3.12)$$

Moreover, with (3.4), (3.5) and (3.8), we obtain

$$- \left(\frac{(\sqrt{f_*})'}{r\sqrt{h_*}} \right)' = \sqrt{f_* h_*} \left(- \left(\frac{m_*}{r^3} \right)' + 2 \frac{\dot{\varphi}^2}{r h_*} \right) \geq 0. \quad (3.13)$$

Now let the inequality (3.13) be reformulated in terms of variables of the exterior solution. The exterior solution is [10],

$$ds_*^2 = -e^{\gamma(\chi)} dt^2 + e^{-\gamma(\chi)} d\chi^2 + e^{\lambda(\chi)-\gamma(\chi)} d\Omega^2, \quad (3.14)$$

where

$$e^{\lambda(\chi)} = \chi^2 - a\chi, \quad e^{\gamma(\chi)} = \left(1 - \frac{a}{\chi} \right)^{\frac{b}{a}}, \quad \varphi(\chi) = \varphi_0 + \frac{c}{a} \ln \left(1 - \frac{a}{\chi} \right), \quad a^2 - b^2 = 4c^2. \quad (3.15)$$

Here a , b , c and φ_0 are constants of integration, and φ_0 is the asymptotic value of φ at infinity.

By matching the interior solution to the exterior solution, we obtain the following relations:

$$r_s = \chi_s \left(1 - \frac{a}{\chi_s} \right)^{\frac{a-b}{2a}}, \quad f_s(r_s) = \left(1 - \frac{a}{\chi_s} \right)^{\frac{b}{a}}, \quad h_*(r_s) = \left(1 - \frac{a}{\chi_s} \right) \left(1 - \frac{a+b}{2\chi_s} \right)^{-2}, \quad (3.16)$$

where the subscript, s , refers to χ evaluated at the surface, $r = r_s$. Note that

$$\chi_s > a \geq b. \quad (3.17)$$

Since $h_*(r)^{-1} = 1 - 2m_*(r)/r$, 3rd equation of (3.16) becomes

$$2m_*(r_s) = \left(b - \frac{(a+b)^2}{4\chi_s} \right) \left(1 - \frac{a}{\chi_s} \right)^{-\frac{(a+b)}{2a}}. \quad (3.18)$$

Accordingly, by virtue of the positivity of $m_*(r)$, we obtain the additional inequality

$$b - \frac{(a+b)^2}{4\chi_s} > 0. \quad (3.19)$$

With (3.13) and (3.16), we obtain the following relation for $r \leq r_s$:

$$\frac{(\sqrt{f_*})'}{r\sqrt{h_*}} \geq \frac{(\sqrt{f_*(r_s)})'}{r_s\sqrt{h_*(r_s)}} = \frac{b}{2r_s^3}. \quad (3.20)$$

Integrating (3.20) from the center, $r = 0$, to the surface, $r = r_s$, we obtain

$$\begin{aligned} 0 &\leq \sqrt{f_*(0)} \\ &\leq \sqrt{f_*(r_s)} - \frac{b}{2r_s^3} \int_0^{r_s} r \sqrt{h_*(r)} dr \\ &\leq \left(1 - \frac{a}{\chi_s} \right)^{\frac{3a}{2}} + \frac{b}{4m_*(r_s)} \left(\sqrt{1 - \frac{2m_*(r_s)}{r_s}} - 1 \right), \end{aligned} \quad (3.21)$$

where the inequality (3.12) has been used.

The inequalities obtained to this point can be simplified in terms of new parameters defined by

$$a_s \equiv \frac{a}{\chi_s}, \quad b_s \equiv \frac{b}{\chi_s}, \quad c_s \equiv \frac{c}{\chi_s}. \quad (3.22)$$

Substituting (3.18) into (3.21), we obtain the following inequality:

$$\begin{aligned} 0 &\leq \sqrt{f_*(0)} \\ &\leq \frac{1}{2}(1 - a_s)^{\frac{3a_s}{2}} \left(b_s - \frac{(a_s + b_s)^2}{4} \right)^{-1} \times \\ &\quad \times \left[2b_s - \frac{(a_s + b_s)^2}{2} - b_s \sqrt{1 - a_s} \left(1 - \sqrt{1 - \frac{4b_s - (a_s + b_s)^2}{4(1 - a_s)}} \right) \right]. \end{aligned} \quad (3.23)$$

The above inequality can be further simplified, and we finally obtain a modified Buchdahl inequality in the scalar-tensor theories as

$$F(a_s, b_s) \equiv 3b_s - \frac{1}{2}(a_s + b_s)(a_s + 2b_s) - b_s \sqrt{1 - a_s} \geq 0, \quad (3.24)$$

supplemented with (3.17) and (3.19).

The modified Buchdahl inequality can be solved to yield

$$\left. \begin{array}{ll} b_s \leq a_s \leq 2\sqrt{b_s} - b_s & \text{for } 0 \leq b_s \leq 4(3 - 2\sqrt{2}), \\ b_s \leq a_s \leq 2\sqrt{2b_s} - 2b_s & \text{for } 4(3 - 2\sqrt{2}) \leq b_s \leq \frac{8}{5}, \\ \text{Forbidden} & \text{for } b_s > \frac{8}{5}. \end{array} \right\} \quad (3.25)$$

Fig. 1 displays the allowed region, D , of (a_s, b_s) .

In addition, we can obtain, with 4th equation of (3.15), the following upper limit on $|c_s|$:

$$|c_s| \leq \frac{2\sqrt{3}}{9}. \quad (3.26)$$

The inequality (3.25) is significant. The new and important inequality (3.26) is characteristic of the scalar-tensor theories. Our result implies that, even in a strong gravitational field, the characteristic amplitude of the scalar field, $|c_s|$, is bounded.

It is important to note that we have not used any assumption regarding the coupling function, $A(\varphi)$, in deriving the inequalities. In particular, the inequalities (3.25) and (3.26) give theory-independent constraints on the parameters, b_s and c_s .

3.2 The mass-to-size ratio

Now we reformulate the inequalities derived in the previous section, which are in terms of variables in the Einstein frame, in order to obtain the mass-to-size ratio in the physical frame. To do this, the coupling function, $A(\varphi)$, should be specified. In this paper, we take the following simple form [5],[6]

$$A(\varphi) = e^{\frac{1}{2}\beta\varphi^2}, \quad \beta \equiv \text{const.}, \quad (3.27)$$

$$\alpha(\varphi) = \beta\varphi. \quad (3.28)$$

A natural definition of the radius of a spherical star is obtained by using its (physical) surface area. In the physical frame, the surface area, S , is given by

$$S = 4\pi A^2(\varphi_s) e^{\lambda(\chi_s) - \gamma(\chi_s)} = 4\pi \chi_s^2 (1 - a_s)^{1 - \frac{b_s}{a_s}} \exp \left[\beta \left(\frac{c_s}{a_s} \ln(1 - a_s) \right)^2 \right], \quad (3.29)$$

where we take the asymptotic value of the scalar field as $\varphi_0 = 0$. This surface area defines the physical radius, R , of the star is given by $R \equiv \sqrt{S/4\pi}$. When $\varphi_0 = 0$, the effective gravitational constant, G , is equal to G_* [5],[6],[10].

The definition of the mass in the Brans-Dicke theory is found in Ref.[11], and the mass in the scalar-tensor theories is defined in the same manner. By introducing the asymptotic Cartesian coordinates such that $\bar{r} = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$, the asymptotic form of the solution is easily found to be

$$\begin{aligned} G_* \phi &\sim 1 + \frac{0}{\bar{r}} \equiv 1 + \frac{2M_S}{\bar{r}}, & g_{00} &\sim -1 + \frac{b}{\bar{r}} \equiv -1 + \frac{2(M_T + M_S)}{\bar{r}}, \\ g_{ij} &\sim \left(1 + \frac{b}{\bar{r}} \right) \delta_{ij} \equiv \left(1 + \frac{2(M_T - M_S)}{\bar{r}} \right) \delta_{ij}, \end{aligned} \quad (3.30)$$

where the quantities, M_S and M_T , are called, respectively, the scalar mass and the tensor mass. At Newtonian order, their sum, $M \equiv M_T + M_S$, plays the role of the mass and is called the active gravitational mass. In our model, $M_S = 0$, and, accordingly, $M_T = M = b/2$. Hereafter, we call M the mass for simplicity. Now we are ready to calculate the mass-to-size ratio in the scalar-tensor theory as a function of a_s , b_s , χ_s and a specific parameter of our model, β . We obtain

$$H(a_s, b_s; \beta) \equiv \frac{b}{R} = b_s (1 - a_s)^{\frac{b_s - c_s}{1 - a_s}} \exp \left[-\frac{1}{2} \beta \left(\frac{c_s}{a_s} \ln(1 - a_s) \right)^2 \right]. \quad (3.31)$$

In Fig.2, in the allowed region of (a_s, b_s) , we display lines on which $H(a_s, b_s; \beta)$ is equal to 8/9 for various values of β . For a fixed value of β , the region above the line corresponds to the case that the mass-to-size ratio, $2M/R$, exceeds Buchdahl's limit, 8/9. Moreover, in some cases, it may be greater than unity. We refer to this case as the *black hole limit*. The maximum values of $H(a_s, b_s; \beta)$ for various values of β are shown in Fig.3. Indeed, the maximum mass-to-size ratio can sometimes become larger than the black hole limit.

4 Numerical results

Equations (3.3) ~ (3.6) are numerically solved to obtain an interior solution. Some details of the numerical methods are summarized in Ref.[5]. Since we take $\varphi_0 = 0$, G_* is equal to G [5],[6],[10]. Hereafter, we use units in which $G_* = G = 1$.

As for the matter, we assume the following polytropic equation of state [5]:

$$\rho = m_b n + \frac{K n_0 m_b}{\Gamma - 1} \left(\frac{n}{n_0} \right)^\Gamma, \quad p = K m_b n_0 \left(\frac{n}{n_0} \right)^\Gamma, \quad m_b = 1.66 \times 10^{-24} \text{g}, \quad n_0 = 0.1 \text{fm}^{-3}. \quad (4.1)$$

We take the parameters values, $\Gamma = 2.34$ and $K = 0.0195$ [5], which fit a realistic equation of state of high density nuclear matter, and probably also that of a neutron star quite well. Our numerical solutions are therefore parametrized by β and $n_c \equiv n(0)$. It has been shown numerically that significant effects of φ appear when $\beta \leq -4.35$ [5],[6], and we are mostly interested in cases of negative values of β .

First, we examine whether our assumption, $\rho'_{\text{eff}} \leq 0$, is satisfied. We find that the assumption, $\rho'_{\text{eff}}(r) \leq 0$, is mostly satisfied. Next, we examine an extreme example in which the assumption, $\rho'_{\text{eff}} \leq 0$, is violated. For example, when $\beta = -30$ and $n_c/n_0 = 10$, we find that the assumption, $\rho'_{\text{eff}}(r) \leq 0$, is violated. However, we are forbidden to take β smaller than -5 because of experimental constraints [5],[6],[12].

For each value of β , the mass-to-size ratio, $2M/R = H(a_s, b_s; \beta)$, can be numerically calculated as a function of n_c . By changing n_c , we search for a maximum value of $H(a_s, b_s; \beta)$ for each β . In the 5th column of Table 1, we summarize our results for the maximum mass-to-size ratio, where the parameters are chosen such that the assumption, $\rho'_{\text{eff}} \leq 0$, is satisfied. For $\beta < -12.07$, we find numerically that the assumption is always violated. The first interesting example is found in the case, $\beta = -12.07$, in which the maximal mass-to-size ratio is obtained as $H_{\text{MAX}} = 1.018$ when $n_c/n_0 = 11.2$. This is a case in which H_{MAX} exceeds the black hole limit, $H = 1$. Another interesting example is found in the case, $\beta = -11$, in which $H_{\text{MAX}} = 0.919$ when $n_c/n_0 = 11.3$. This is a case in which H_{MAX} exceeds Buchdahl's limit, $H = 8/9 \approx 0.889$. These examples have academic importance in the sense that our analytic results in the previous section are partially realized also in the numerical solutions.

To this point, the stability of our numerical solutions has not been taken into account. We have found that $\rho - 3p$ may be a good estimator of the stability, as described below. The baryonic mass of a star is defined as [5]

$$\tilde{m} = m_b \int_0^{r_s} 4\pi n A^3(\varphi) r^2 \left(1 - \frac{2m_s}{r} \right)^{-\frac{1}{2}} dr. \quad (4.2)$$

We have numerically examined how $\tilde{m}$ depends on n_c and find a significant correlation between the signature of $d\tilde{m}/dn_c$ and that of $\rho - 3p$. That is, the cases, $\rho - 3p < 0$ and $\rho - 3p > 0$, approximately correspond to the cases, $d\tilde{m}/dn_c < 0$ and $d\tilde{m}/dn_c > 0$, respectively. Accordingly, we shall interpret the signature of $\rho - 3p$ as a measure of the onset of the instability in our numerical calculations. When we impose the condition, $\rho - 3p \geq 0$, we cannot numerically find cases in which H_{MAX} exceeds Buchdahl's limit, as is seen in the 4th and 5th columns in Table 1. Note that the condition, $\beta < -5$, also excludes all the interesting cases in which H_{MAX} exceeds Buchdahl's limit.

5 Summary

We have derived a modified Buchdahl inequality in scalar-tensor theories of gravity. As a result, we have obtained two theory-independent inequalities, $b_s \leq 8/9$ and $|c_s| \leq 2\sqrt{3}/9$. The first inequality corresponds to the Buchdahl inequality in general relativity. The second inequality is characteristic of scalar-tensor theories. Consequently, even if the scalar field is locally amplified due to non-perturbative effects in a strong gravitational field, the characteristic amplitude of the scalar field, $|c_s|$, is bounded in this sense.

The modified Buchdahl inequality is then reformulated to obtain a theory-dependent mass-to-size ratio, $2M/R$, with an example of the coupling function, $A(\varphi)$, in a simple form. If we take $\varphi_0 = 0$, the mass in the physical frame is the same as that in general relativity, $M = b/2$. However, the physical radius, R , of the star can be smaller than the general relativistic one. As a result, the mass-to-size ratio can exceed not only Buchdahl's limit but also the black hole limit in contrast to general relativity.

Our analytic results have been numerically confirmed when we assume a polytropic equation of state for the matter. In particular, we have found numerical solutions in which the mass-to-size ratio exceeds

both Buchdahl's limit and the black hole limit. However, these theoretically interesting stars could not be found numerically under the condition, $\rho - 3p \geq 0$, which is interpreted as a numerical measure of the onset of the instability of a star. Moreover, under this condition, we find numerically that any quantitative deviation from general relativity due to the scalar field remains comparatively small in contrast to our analytic results, where possible significant effects of the scalar field are expected. However, some measurable effects in astronomical observations may exist.

For details, see Ref.[13].

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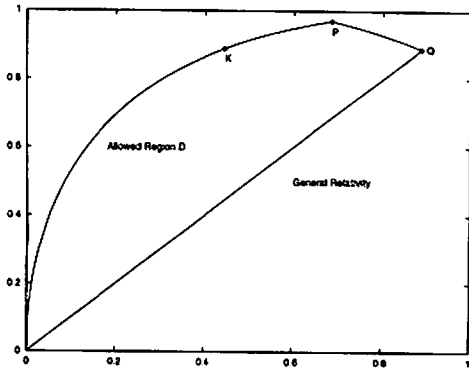


Figure 1: The allowed region, D , is depicted, where horizontal and vertical axes denote, respectively, b_s and a_s . The characteristic points, $P = (4(3 - 2\sqrt{2}), 4(3\sqrt{2} - 4))$, $Q = (8/9, 8/9)$ and $K = (4/9, 8/9)$, are shown. In general relativity, $a_s = b_s$. Buchdahl's limit is denoted by Q . On K , $|c_s|$ takes the maximum value, $|c_s|_{\text{MAX}} = 2\sqrt{3}/9$.

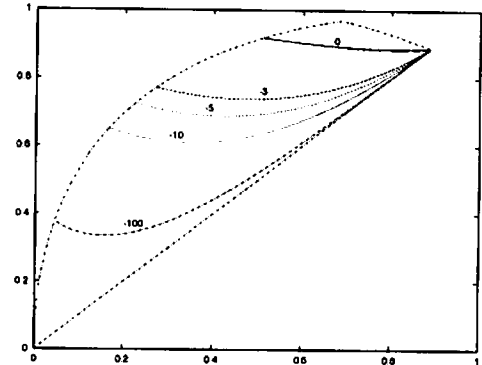


Figure 2: On each lines, $H(a_s, b_s; \beta) = 8/9$ for various values of β : 0, -3, -5, -10 and -100. Horizontal and vertical axes denote, respectively, b_s and a_s .

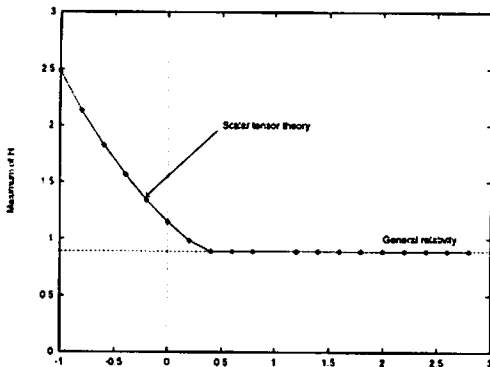


Figure 3: The Maximum mass-to-size ratio, H_{MAX} , is shown as a function of β . Horizontal and vertical axes denote, respectively, β , and H_{MAX} . A horizontal line, $H_{\text{MAX}} = 8/9$, denotes Buchdahl's limit in general relativity. When $\beta \gtrsim 0.4$, H_{MAX} exceeds Buchdahl's limit. When $\beta \gtrsim 0.2$, H_{MAX} exceeds unity, a black hole limit.

Table 1: We summarize our numerical results. In the 1st column, β is given. In the 2nd column, we give a range of n_c from our numerical studies. In the 3rd and 4th columns, we indicate, respectively, whether the assumption, $\rho'_{\text{eff}} \leq 0$, and the condition, $\rho - 3p \geq 0$, are satisfied. In the 5th column, the maximum mass-to-size ratio is shown for each β .

β	n_c/n_0	$\rho'_{\text{eff}} \leq 0$	$\rho - 3p \geq 0$	H_{MAX}
-12.07	0.1 ~ 10.3	○	○	0.679
	10.3 ~ 11.2	○	×	1.018
-11.0	0.1 ~ 10.3	○	○	0.666
	10.3 ~ 11.3	○	×	0.919
-10.0	0.1 ~ 10.3	○	○	0.651
	10.3 ~ 11.4	○	×	0.834
-6.0	0.1 ~ 10.3	○	○	0.556
	10.3 ~ 11.2	○	×	0.569
-5.0	0.1 ~ 10.3	○	○	0.517
	10.3 ~ 15.7	○	×	0.567
-4.0	0.1 ~ 10.3	○	○	0.514
	10.3 ~ 14.9	○	×	0.562
0.0	0.1 ~ 10.3	○	○	0.514
	10.3 ~ 14.9	○	×	0.562

The exterior solutions and their geometry in scalar-tensor theories of gravity

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Abstract

We study series of stationary solutions with asymptotic flatness properties in the Einstein-Maxwell-free scalar system because they are locally equivalent to the exterior solutions in some class of scalar-tensor theories of gravity. First, we reduce the field equations for the axisymmetric exterior solutions. We find that the reduced equations are partially the same as the Ernst equations. As simple examples, we derive new series of static, axisymmetric exterior solutions, which correspond to Voorhees's solutions. We then establish a non-trivial relation between the spherical exterior solutions and our new solutions. Second, since null geodesics have conformally invariant properties, we study the local geometry of the exterior solutions by using the optical scalar equations and find some anomalous behavior of the null geodesics.

1 Introduction

Recently, as natural alternatives to general relativity, scalar-tensor theories have been studied by many theoretical physicists. In these theories, gravity is mediated not only by a tensor field but also by a scalar field. Also, such theories have been of interest as effective theories of string theory at low energy scales[1].

Several theoretical predictions in scalar-tensor theories have been obtained (see e.g. Ref.[2]~[5]). It has been found that a wide class of the scalar-tensor theories can pass all the experimental tests in weak gravitational fields. However, it has also been found that the scalar-tensor theories display different aspects of the gravity in the strong gravitational fields in comparison with general relativity (see e.g. Ref.[5]). It has been shown numerically that nonperturbative effects in scalar-tensor theories increase the maximum mass of an isolated system such as a neutron star [2-5]. In these works, numerical methods play an important role, and, in interpreting the numerical results, a static, spherically symmetric vacuum solution (a spherical exterior solution) is matched to the numerical solution.

The spherical exterior solution is an analytic exact solution, and, as far as the authors know, only a few exact solutions are known. In particular, the axisymmetric exterior solution corresponding to the Kerr solution must be of significant interest. Motivated by this fact, we study a series of stationary solutions with asymptotic flatness properties in the Einstein-Maxwell-free scalar system because they are locally equivalent to the exterior solutions in some class of the scalar-tensor theories of gravity. The field equations are given by

$$R_{\mu\nu} = 8\pi T_{\mu\nu} + 2\varphi_{,\mu}\varphi_{,\nu} , \quad (1.1)$$

$$\square\varphi = 0 , \quad (1.2)$$

where $T_{\mu\nu}$ is an energy-momentum tensor of the Maxwell field.

In this paper we first reduce the field equations for the axisymmetric exterior solutions and show that the reduced equations are partially the same as the Ernst equations. As simple examples, we derive a new series of static, axisymmetric exterior solutions, which will be referred to as scalar-tensor-Weyl solutions. We then establish a non-trivial relation between the spherical exterior solutions and our new solutions. Second, we study the local geometry of the exterior solutions by using the optical scalar equations and find some anomalous behavior of the null geodesics.

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2 The Ernst equations in scalar-tensor theories

In stationary, axisymmetric spacetime, there exist two Killing vectors, $\xi = \partial_t$ and $\eta = \partial_\phi$. It is shown that the energy-momentum tensor, $T^{\alpha\beta}$, of the Maxwell field satisfies the following relations [6]:

$$\xi^\alpha T_\alpha^{[\beta} \xi^\gamma \eta^{\delta]} = \eta^\alpha T_\alpha^{[\beta} \xi^\gamma \eta^{\delta]} = 0. \quad (2.1)$$

Moreover, one finds that similar relations hold for $S_{\mu\nu}^{[\varphi]} \equiv 2\partial_\mu \varphi \partial_\nu \varphi$. Accordingly, with Theorem 7.1.1 in Ref.[7], the metric can be reduced to the following form:

$$ds^2 = -e^{2\psi}(dt - \omega d\phi)^2 + e^{-2\psi} [e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2], \quad (2.2)$$

where the metric functions, ψ, γ and ω , are functions of $x^1 \equiv \rho$ and $x^2 \equiv z$.

After long and complicated calculations, we find that the field equations are reduced to the following:

$$e^{2\psi} = \frac{1}{2} (\mathcal{E} + \bar{\mathcal{E}}) + \Phi \bar{\Phi}, \quad (2.3a)$$

$$e^{2\psi} \nabla^2 \mathcal{E} = (\nabla \mathcal{E}) \cdot [(\nabla \mathcal{E}) + 2\bar{\Phi} \nabla \Phi], \quad (2.3b)$$

$$e^{2\psi} \nabla^2 \bar{\Phi} = (\nabla \Phi) \cdot [(\nabla \mathcal{E}) + 2\bar{\Phi} \nabla \Phi], \quad (2.3c)$$

$$\begin{aligned} \gamma_{,1} = & \rho [(\psi_{,1})^2 - (\psi_{,2})^2] - \frac{1}{4\rho} [(\omega_{,1})^2 - (\omega_{,2})^2] e^{4\psi} \\ & - \rho (\Phi_{,1} \bar{\Phi}_{,1} - \Phi_{,2} \bar{\Phi}_{,2}) e^{-2\psi} + \rho [(\varphi_{,1})^2 - (\varphi_{,2})^2], \end{aligned} \quad (2.3d)$$

$$\gamma_{,2} = 2\rho \psi_{,1} \psi_{,2} - \frac{1}{2\rho} \omega_{,1} \omega_{,2} e^{4\psi} - \rho (\Phi_{,1} \bar{\Phi}_{,2} + \Phi_{,2} \bar{\Phi}_{,1}) e^{-2\psi} + 2\rho \varphi_{,1} \varphi_{,2}, \quad (2.3e)$$

$$\nabla^2 \varphi = 0, \quad (2.3f)$$

where, for any functions, f and h ,

$$\nabla^2 f \equiv \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{\partial^2 f}{\partial z^2}, \quad \nabla f \cdot \nabla h \equiv \frac{\partial f}{\partial \rho} \frac{\partial h}{\partial \rho} + \frac{\partial f}{\partial z} \frac{\partial h}{\partial z}, \quad f_{,i} \equiv \frac{\partial f}{\partial x^i}. \quad (2.4)$$

In Ref.[8], there appear explicit forms of the metric function, ω , and the Maxwell field in terms of the complex potentials, $\mathcal{E}$ and Φ . Note that the first, second and third equations are the same as those in general relativity and are referred to as *the Ernst equations* of the Einstein-Maxwell system. The remaining equations contain the scalar field contributions, and, as for the metric functions, effects of the scalar field appear only in γ .

3 Scalar-tensor-Weyl solutions.

In this section, we consider static, axisymmetric vacuum solutions. In this case, $\omega = \Phi = 0$, and the field equations (2.3a)~(2.3f) are reduced to the following:

$$e^{2\psi} = \mathcal{E}, \quad (3.1a)$$

$$\mathcal{E} \nabla^2 \mathcal{E} = (\nabla \mathcal{E}) \cdot (\nabla \mathcal{E}), \quad (3.1b)$$

$$\gamma_{,1} = \rho [(\psi_{,1})^2 - (\psi_{,2})^2] + \rho [(\varphi_{,1})^2 - (\varphi_{,2})^2], \quad (3.1c)$$

$$\gamma_{,2} = 2\rho \psi_{,1} \psi_{,2} + 2\rho \varphi_{,1} \varphi_{,2}, \quad (3.1d)$$

$$\nabla^2 \varphi = 0, \quad (3.1e)$$

where $\mathcal{E}$ is a real function. One finds that the Ernst equation (3.1b) is reduced to

$$\nabla^2 \psi = 0. \quad (3.2)$$

That is, φ and ψ are harmonic functions.

We introduce *prolate* coordinates, (x, y) , defined by

$$\rho = \sigma \sqrt{(x^2 - 1)(1 - y^2)}, \quad z = \sigma xy \quad (3.3)$$

where σ is a positive constant. The Ernst equation is then reduced to

$$\frac{\partial}{\partial x} \left[(x^2 - 1) \frac{\partial \psi}{\partial x} \right] + \frac{\partial}{\partial y} \left[(1 - y^2) \frac{\partial \psi}{\partial y} \right] = 0. \quad (3.4)$$

A similar equation holds for φ .

The simplest prolate solutions for ψ and for φ are given by

$$\psi = \frac{\delta}{2} \ln \left(\frac{x-1}{x+1} \right), \quad \varphi = \varphi_0 + \frac{d}{2} \ln \left(\frac{x-1}{x+1} \right), \quad (3.5)$$

where δ , φ_0 and d are integration constants. Then the corresponding solution for γ becomes

$$e^{2\gamma} = \left(\frac{x^2 - 1}{x^2 - y^2} \right)^{\Delta^2}, \quad \Delta^2 \equiv \delta^2 + d^2. \quad (3.6)$$

For completeness, we give an explicit form of the metric:

$$ds^2 = - \left(\frac{x-1}{x+1} \right)^{\delta} dt^2 + \sigma^2 \left(\frac{x-1}{x+1} \right)^{-\delta} \times \left[\left(\frac{x^2 - 1}{x^2 - y^2} \right)^{\Delta^2} (x^2 - y^2) \left(\frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right) + (x^2 - 1)(1 - y^2) d\phi^2 \right]. \quad (3.7)$$

Note that, though ψ and φ are independent of y , γ depends on both x and y . The metric contains two parameters, δ and Δ . Now we show that the metric is reduced to the previously known one when we take specific values of the parameters. When $\Delta = \delta$, i.e., $d = 0$, the metric is reduced to one of the Weyl series of solutions in general relativity (Voorhees's prolate solution [9]). Therefore, we refer to (3.7) as the scalar-tensor-Weyl solution. Voorhees's prolate solution contains the Schwarzschild solution as a specific case, $\delta = 1$, and the Schwarzschild coordinates are related to (x, y) as

$$x = \frac{r}{m} - 1, \quad y = \cos \theta, \quad \sigma = m, \quad (3.8)$$

where m is the usual mass parameter.

The case, $\Delta = 1$, is the most interesting one. We introduce the new parameters, a , b and c , and new coordinates (the Just coordinates) by

$$d = \frac{2c}{a}, \quad \sigma = \frac{a}{2}, \quad \delta = \frac{b}{a}; \quad x = \frac{2\chi}{a} - 1, \quad y = \cos \theta. \quad (3.9)$$

Then we find that the solution coincides with a spherical exterior solution with a vanishing electric charge:

$$ds^2 = - \left(1 - \frac{a}{\chi} \right)^{\frac{b}{a}} dt^2 + \left(1 - \frac{a}{\chi} \right)^{-\frac{b}{a}} d\chi^2 + \chi^2 \left(1 - \frac{a}{\chi} \right)^{\frac{a-b}{a}} (d\theta^2 + \sin^2 \theta d\phi^2). \quad (3.10)$$

Moreover, when $\delta = 1$, i.e., $a = b$, the solution is reduced to Schwarzschild's one.

The right hand sides of (3.1c) and (3.1d) are in the forms of the superposition of the contributions of ψ and φ . It is important to note that the superposition of the *effectively non-spherical* ψ -terms corresponding to Voorhees's prolate solution with $\delta \neq 1$ and the *spherical* configuration of φ reduces to the solution containing the spherical exterior solution. When the spherical scalar field configuration is added to the ψ -terms corresponding to Voorhees's prolate solution with $\delta = 1$, namely the Schwarzschild solution, we cannot have spherical solutions unless $a = b$. The relation among these solutions is schematically shown in Figure 1. The scalar-tensor-Weyl solution generically has a singularity, $x = 1$, whose topology is S^2 , and qualitative features of the singularity may be similar to those in Voorhees's prolate solution. When $\Delta = 1$, the singularity, $x = 1$, becomes a point, which is given by $\chi = a$ in the Just coordinates. Moreover, when $\Delta = \delta = 1$, the topology of $\chi = a$, corresponding to the event horizon, $r = 2m$, becomes S^2 .

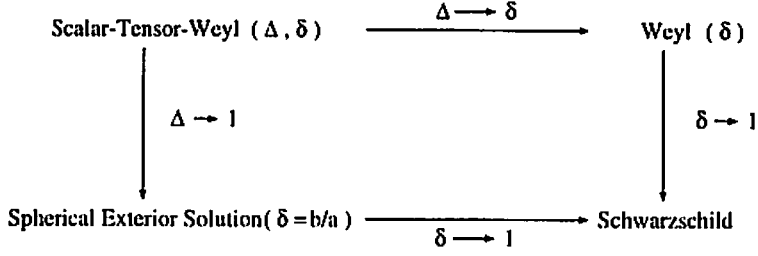


Figure 1: A schematic depiction of series of the scalar-tensor-Weyl solutions.

4 Local behavior of null geodesics

4.1 The spherical exterior solution

In this section, we examine local behavior of irrotational null geodesics in the spherical exterior solution with a vanishing electric charge. The metric is rewritten in the Schwarzschild coordinates as

$$\begin{aligned}
 ds^2 &= - \left(1 - \frac{a}{\chi(r)}\right)^{\frac{1}{a}} dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \\
 2m(r) &= \left(b - \frac{(a+b)^2}{4\chi(r)}\right) \left(1 - \frac{a}{\chi(r)}\right)^{-\frac{(a+b)}{2a}}, \quad r = \chi(r) \left(1 - \frac{a}{\chi(r)}\right)^{\frac{(a-b)}{2a}},
 \end{aligned} \tag{4.1}$$

where the mass function, $m(r)$, can be interpreted as a local energy [10]. One immediately finds that $m(r)$ becomes negative when $(a+b)^2/4b > \chi > a$. Therefore, one may expect that there will be a significant difference not only in the global causal structures but also in the local geometry, depending on whether or not a is equal to b . However, we will find that it is not the case and that local behavior of null geodesics depend significantly on whether a is greater than $2b$.

Due to the conformally invariant nature of null geodesics, we consider null geodesics in the spacetime with the metric, g . Let $k^\mu = dx^\mu/d\lambda$ be a null geodesic associated with g with the affine parameter, λ . The optical scalar equations become [11, 12, 14]

$$\begin{aligned}
 \frac{d\theta}{d\lambda} + \theta^2 + |\sigma|^2 &= -\frac{1}{2} R_{\mu\nu} k^\mu k^\nu \equiv \mathcal{R}, \\
 \frac{d\sigma}{d\lambda} + 2\theta\sigma &= -R_{\mu\alpha\nu\beta} k^\mu k^\nu \bar{l}^\alpha \bar{l}^\beta = -C_{\mu\alpha\nu\beta} k^\mu k^\nu \bar{l}^\alpha \bar{l}^\beta \equiv F,
 \end{aligned} \tag{4.2}$$

where $C_{\mu\alpha\nu\beta}$ is the Weyl curvature, and $\bar{l}^\mu$ is a null tetrad[11].

First, we examine circular orbits of null geodesics in the Just coordinates. Without loss of generality, we consider null geodesics in the $\theta = \pi/2$ plane. Then the geodesic equation is reduced to

$$\begin{aligned}
 \dot{\chi}^2 &= E^2 - \frac{L^2}{\chi^2} \left(1 - \frac{a}{\chi}\right)^{\frac{(2b-a)}{a}} \equiv E^2 - V(\chi), \\
 E &= \left(1 - \frac{a}{\chi}\right)^{\frac{1}{a}} \dot{t}, \quad L = \chi^2 \left(1 - \frac{a}{\chi}\right)^{\frac{(a-b)}{a}} \dot{\phi},
 \end{aligned} \tag{4.3}$$

where the dot denotes a differentiation with respect to the affine parameter, λ , and E and L are integration constants. The circular orbits are determined by the conditions that $\dot{\chi} = 0$ and $dV/d\chi = 0$. We find that a circular orbit is obtained as $\chi = \chi_C \equiv (a + 2b)/2$ for $a < 2b$, and that, surprisingly, no circular orbit exists for $a \geq 2b$ in contrast to the case in the Schwarzschild spacetime.

Dyer [12] has obtained general solutions to the optical scalar equations (4.2) in the general static, spherically symmetric spacetime. In static, spherically symmetric spacetime, one finds that the Weyl

driving term, F , and the shear, σ , can be regarded as real quantities without loss of generality [12]. By defining the new optical scalars, $C_{\pm}$, as

$$\frac{d}{d\lambda} \ln C_{\pm} = \theta \pm \sigma, \quad (4.4)$$

Dyer has obtained the following equations:

$$\frac{d^2 C_{\pm}}{d\lambda^2} = (\mathcal{R} \pm F) C_{\pm}. \quad (4.5)$$

We evaluate the Ricci and Weyl driving terms, $\mathcal{R}$ and F , as

$$\begin{aligned} \mathcal{R} &= -\frac{(a^2 - b^2)}{4\chi^4} \left(1 - \frac{a}{\chi}\right)^{-2} \left\{1 - \frac{h^2}{\chi^2} \left(1 - \frac{a}{\chi}\right)^{\frac{(2b-a)}{a}}\right\} \\ &= -\frac{(a^2 - b^2)}{4\chi^4} \left(1 - \frac{a}{\chi}\right)^{-2} (k^1)^2, \\ F &= \frac{h^2}{4\chi^6} \left(1 - \frac{a}{\chi}\right)^{\frac{(2b-3a)}{a}} \{6b\chi - (a+b)(a+2b)\}, \end{aligned} \quad (4.6)$$

where $h = L/E$ is an impact parameter, and the affine parameter, λ , is chosen such that $E = 1$. It is immediately found that the Ricci driving term, $\mathcal{R}$, vanishes at the perihelion where $k^1 = 0$. A particularly interesting case is a circular orbit on which $k^1 = 0$ identically. If F vanishes on a circular orbit, the image shape of an infinitesimal light ray congruence remains unchanged. The condition, $F = 0$, is reduced to $\chi = \chi_F \equiv (a+b)(a+2b)/(6b)$ for $a > 2b$. When $a < 2b$, F is strictly positive. Since a circular orbit exists only when $a < 2b$, the condition, $\mathcal{R} = F = 0$, is never satisfied on a circular orbit. However, the condition, $\mathcal{R} = F = 0$, can be satisfied at $\chi = \chi_F$ for a scattering orbit when the impact parameter, h , is chosen as

$$h = \frac{(a+b)(a+2b)}{6b} \left[\frac{(a-b)(a-2b)}{(a+b)(a+2b)} \right]^{\frac{(a-2b)}{3a}}, \quad \rightarrow \quad \mathcal{R}(\chi_F) = F(\chi_F) = 0. \quad (4.7)$$

That is, *the gravity effectively vanishes at χ_F* in this sense.

In summary, we have found that local behavior of null geodesics significantly depends on whether a is greater than $2b$. When $a < 2b$, the behavior is similar to that in the Schwarzschild spacetime. That is, circular orbit exists, and the Weyl driving term, F , is strictly positive. When $a > 2b$, on the contrary, there are no circular orbits, and the sign of F changes at χ_F .

General solutions to (4.5) with (4.6) can be obtained using Dyer's formula, however, we have only examined quantitative behaviors of the Ricci and Weyl driving terms, $\mathcal{R}$ and F . Further quantitative studies of the solutions and their astronomical applications, in particular the implications with regard to gravitational lens effects, will be discussed in a forthcoming paper.

4.2 The scalar-tensor-Weyl solution

It is difficult to study generic null geodesics in the scalar-tensor-Weyl solution. Therefore, we shall examine specific null geodesics on the $y = 0$ plane. When $y = 0$, the geodesic equation is reduced to

$$\begin{aligned} \dot{W}^2 &= E^2 - \frac{L^2}{x^2 - 1} \left(\frac{x-1}{x+1} \right)^{2\delta} \equiv E^2 - V(x), \\ E &= \frac{1}{\sigma} \left(\frac{x-1}{x+1} \right)^{\delta} \dot{t}, \quad L = (x^2 - 1) \left(\frac{x-1}{x+1} \right)^{-\delta} \dot{\phi}, \end{aligned} \quad (4.8)$$

where W is a function of x defined by

$$\frac{dx}{dW} = \left(\frac{x^2}{x^2 - 1} \right)^{\frac{\delta^2 - 1}{2}}, \quad (4.9)$$

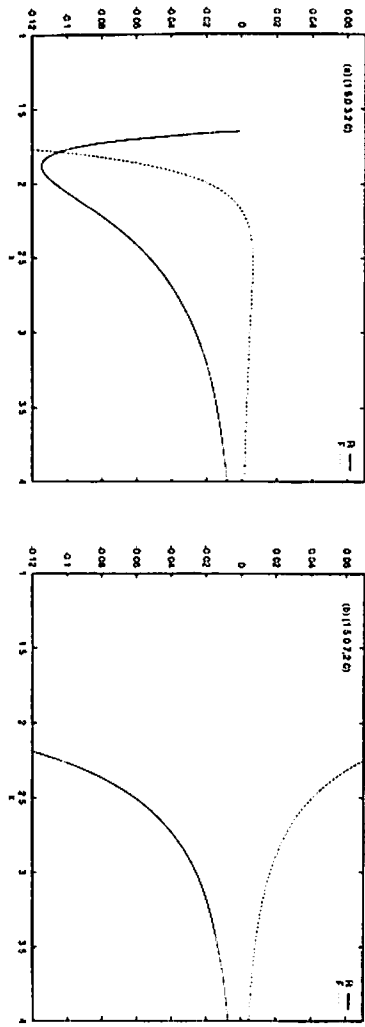


Figure 2: The Weyl and Ricci driving terms, F and R , are shown, where $(\Delta, \delta, h) = (1.5, 0.3, 2, 0)$ in (a) and $(1.5, 0.7, 2, 0)$ in (b). The vertical and horizontal axes denote, respectively, the driving terms and x .

the dot denotes a differentiation with respect to the affine parameter, λ , and E and L are integration constants. The circular orbits are determined by the conditions that $W = 0$ and $dV/dx = 0$. We find that these conditions are reduced to $x = x_c \equiv 2\delta$. Since x should be larger than unity, circular orbit exists when $\delta > 1/2$.

We evaluate the Ricci and Weyl driving terms, $\mathcal{R}$ and F , and find that [14]

$$\begin{aligned}
 \mathcal{R} &= -\frac{\Delta^2 - \delta^2}{\sigma^2} \frac{1}{x^2(x^2 - 1)} \left(\frac{x^2}{x^2 - 1} \right)^{\Delta^2} \left[1 - \frac{(h/\sigma)^2}{x^2 - 1} \left(\frac{x - 1}{x + 1} \right)^{2\delta} \right] \\
 &= -\frac{\Delta^2 - \delta^2}{(x^2 - 1)^2} (h^1)^2, \\
 F &= \frac{1}{\sigma^2} \frac{1}{x^3(x^2 - 1)} \left(\frac{x^2}{x^2 - 1} \right)^{\Delta^2} \times \\
 &\quad \times \left[\delta(\Delta^2 - 1) + \frac{(h/\sigma)^2}{x^2 - 1} \left(\frac{x - 1}{x + 1} \right)^{2\delta} \{ \delta(\Delta^2 - 1) - (\Delta^2 + 2\delta^2)x + 3\delta x^2 \} \right],
 \end{aligned} \tag{4.10}$$

where $h = L/E$ is an impact parameter. One finds that these are reduced to (4.6) when $\Delta^2 = 1$. We show $\mathcal{R}$ and F in Figure 2. We numerically find that the qualitative features of $\mathcal{R}$ and F are the same as those in the spherical case ($\Delta = 1$) and that the significant qualitative changes in $\mathcal{R}$ and F depend on whether $\delta < 1/2$.

5 Discussion

We have derived a new series of the axisymmetric exterior solutions, which are reduced to Voorhees's solutions in general relativity. We have found that the prolate solution, namely the scalar-tensor-Weyl solution, contains the spherical exterior solution as a special case in which we take some non-trivial choice of the parameters in the solution. We also tried to find exterior solutions corresponding to the Kerr solution in general relativity and have not yet succeeded in it. Note that the generic scalar-tensor-Weyl solution is not reduced to the Schwarzschild solution in the limit, $\varphi \rightarrow 0$. Therefore, we guess that the Kerr-like exterior solution must be reduced to the Tomimatsu-Sato-like solution in the limit, $\varphi \rightarrow 0$, where, by the term, “Kerr-like exterior solution”, we mean the exterior solution such that it coincides with the Kerr solution when $\varphi = 0$ and that it is reduced to the spherical exterior solution when the Kerr parameter (the angular momentum of the matter) vanishes. An example of the solution with the first property is obtained by using the metric function, ψ , corresponding to the Kerr solution. However, the resulting solution does not satisfy the second property. We think that generating methods of exact

solutions (e.g. the Newman-Janis algorithm [13]) may play an important role in obtaining further exterior solutions.

It is important to note that one must specify an explicit form of the coupling function, $A(\varphi)$, in order to obtain complete information, in particular the global geometry, of the exterior solutions in the scalar-tensor theories. We have not given any discussion on $A(\varphi)$, and, instead, we have examined local geometrical properties of the solutions by taking account of the conformally invariant nature of null geodesics. We have evaluated the Ricci and Weyl driving terms, $\mathcal{R}$ and F , in the scalar-tensor-Weyl solution and have found that the local behavior of null geodesics significantly depend on whether the parameter, δ , which corresponds to b/a in the spherical exterior solution, is greater than $1/2$. When $\delta > 1/2$, the behavior is similar to that in the Schwarzschild spacetime. That is, the circular orbit exists, and the Weyl driving term, F , is strictly positive. When $\delta < 1/2$, on the contrary, there are no circular orbits, and F changes its sign at χ_F . In the later case, one may expect to observe a gravitational lensing anomaly for images of distant sources appearing near χ_F . The ratio,

$$C(h) \equiv \lim_{\lambda \rightarrow \infty} \frac{C_+ - C_-}{C_+}, \quad (5.1)$$

is a measure of the image deformation. In the spherical exterior solution with $\delta > 1/2$, C may behave in a similar manner as in the Schwarzschild solution. However, in the spherical exterior solution with $\delta < 1/2$, C will show quite different behavior compared with that in the Schwarzschild solution when h is near χ_F . Further quantitative studies of the solutions, C_+ and C_- , and their astronomical applications, in particular the implications with regard to gravitational lens effects including their observational detectability, will be discussed in a forthcoming paper.

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Scale-factor duality and dimensional reduction

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Abstract

In the low-energy effective actions of superstring theory, there is a dual relation of cosmological solutions known as the scale-factor duality. It is also known that 4-dimensional cosmological solutions can be obtained by the dimensional reduction of higher, say 5-, dimensional black hole solutions. We study how the dual relation appears as we derive a 4-dimensional cosmological solution from a higher dimensional blackhole solution, and discuss the correspondence between the duality and the dimensional reduction map. We also discuss some generalization of this correspondence.

1 Introduction

As currently the leading candidate for the theoretical framework including quantum gravity, the superstring theory is studied intensively not only in the search of consistent unified theory but also from the cosmological interest. The ordinary approach in the study of string-motivated cosmology is based on the 4-dimensional low-energy effective action of the string theory, and the most widely studied model would be the pre-big-bang model, which assumes a pole-like inflationary phase preceding the Friedmann-like decelerating expansion phase.

A characteristic feature of the low-energy effective action of the string theory is that it involves a scalar field called dilaton. However, following the philosophy of the Hořava-Witten theory[4], the dilaton can be understood as a scalar field appearing through the dimensional reduction, and the distinction between the dilaton and the moduli is now quite unclear. In this sense, it is expected that the 4-dimensional effective string theory can be seen from higher dimensions in a clear manner. It is well known that the low-energy string effective action provides disconnected branches of the cosmological solutions, which is an essential point of the pre-big-bang scenario. These branches are related to each other by discrete transformations, i.e. the time reversal and the scale factor duality. In this paper we discuss how these branches, and these symmetries, are interpreted in relation with the dimensional reduction.

In section 2, we briefly review the scale factor duality and the pre-big-bang scenario. In section 3, we apply the dimensional reduction technique to a simple pre-big-bang cosmology¹, and show how the branches are derived from the higher dimensional solution. In sections 4, we discuss the correspondence between the scale factor duality and the non-uniqueness of the dimensional reduction in more general cases, where the 4-dimensional theories are Brans-Dicke and scalar-tensor theories. In section 5, we summarize our conclusion.

¹Similar discussions are found in [3] and [8].

2 Scale factor duality

Cosmological solutions from 4-dimensional low-energy string effective action provides distinct and disconnected branches. As a simple example, we consider the dilaton-gravity system

$$\mathcal{S} = \int d^4x \sqrt{-g} e^{-\phi} [R + (\nabla\phi)^2], \quad (1)$$

with a flat FRW metric

$$ds^2 = -dt^2 + a^2(dx^2 + dy^2 + dz^2). \quad (2)$$

The equations of motion

$$\dot{\phi}^2 - 6H\dot{\phi} + 6H^2 = 0, \quad (3)$$

$$2\ddot{\phi} + 4H\dot{\phi} - \dot{\phi}^2 - 4\dot{H} - 6H^2 = 0, \quad (4)$$

$$2\ddot{\phi} - \dot{\phi}^2 + 6H\dot{\phi} - 12H^2 - 6\dot{H} = 0, \quad (5)$$

give 4 branches of the solution for the scale factor a and the dilaton ϕ :

$$\begin{aligned} 1: & \quad a = t^{1/\sqrt{3}}, & \phi &= (1 + \sqrt{3}) \ln t & (t > 0) \\ 2: & \quad a = (-t)^{1/\sqrt{3}}, & \phi &= (1 + \sqrt{3}) \ln(-t) & (t < 0) \\ 3: & \quad a = t^{-1/\sqrt{3}}, & \phi &= (1 - \sqrt{3}) \ln t & (t > 0) \\ 4: & \quad a = (-t)^{-1/\sqrt{3}}, & \phi &= (1 - \sqrt{3}) \ln(-t) & (t < 0) \end{aligned} \quad (6)$$

The branch 1 is a decelerating expansion which, in the pre-big-bang scenario, leads to our universe, and the branches 2, 3, 4 are accelerating contraction, decelerating contraction, accelerating expansion, respectively. Branches 1 and 2, 3 and 4 transform to each other under the time-reversal transformation:

$$\text{TR} : t \leftrightarrow -t \quad (7)$$

and 1 and 3, 2 and 4 interchange under the transformation

$$\text{SFD} : \begin{cases} a & \leftrightarrow \frac{1}{a} \\ \phi & \leftrightarrow \phi - 6 \ln a, \end{cases} \quad (8)$$

which is called the scale factor duality.

In the pre-big-bang scenario, the super-inflationary branch 4 is assumed to precede our universe (branch 1). This scenario provides an explanation to the inflationary behavior of the early universe, and it describes the whole history of the universe from non-singular past to non-singular future. However, in the classical solution the two branches are separated by a curvature singularity, and the mechanism of the smooth transition is yet to be known (graceful-exit problem).

3 String cosmology seen from higher dimension

It is known that 4-dimensional cosmological solutions of the low-energy effective string theory are obtained by the dimensional reduction of 5-dimensional blackhole or Schwarzschild-de Sitter solution[3]. The same technique is used in the resolution of singularities of dilatonic blackhole[5], cosmological initial singularity[6], and the singularity of Hawking-Turok type instanton[7]. Before discussing the relation to the scale factor duality, we shortly review the standard formalism of dimensional reduction.

We assume a $(d + p)$ -dimensional metric in a block-diagonalized form:

$$ds_{d+p}^2 = e^{2v(x)} g_{\mu\nu}(x) dx^\mu dx^\nu + e^{2\phi(x)} \delta_{mn} dy^m dy^n \quad (9)$$

i.e.

$$G_{MN} = \begin{pmatrix} e^{2v(x)} g_{\mu\nu}(x) & 0 \\ 0 & e^{2\phi(x)} \delta_{mn} \end{pmatrix}, \quad (10)$$

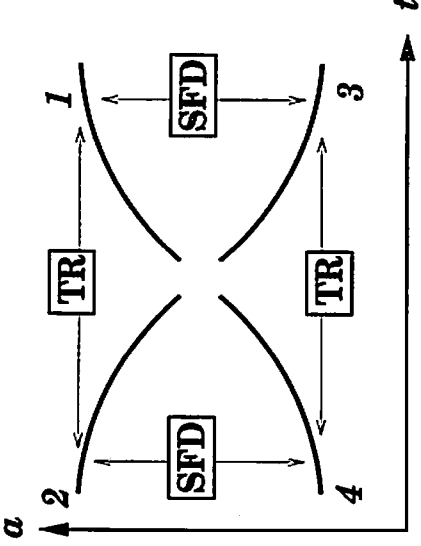


Figure 1: Four branches of cosmological solutions in four-dimensional simple dilaton-gravity system. TR is the time reversal transformation, and SFD stands for the scale factor duality transformation. In the pre-big-bang scenario the branch of our universe (1) emerged by a transition from the super-inflationary branch (4) by some unknown mechanism.

where x^μ denote $x^0, \dots, x^{d-1}$ and y^m denote $x^d, \dots, x^{d+p-1}$. An important postulate is that metric does not depend on the coordinate y^m . To keep the dynamical degrees of freedom we assume that $\psi(x)$ is not an independent field but dependent on the field ϕ , i.e. $\psi = \psi(\phi(x))$. Under this assumption the $(d+p)$ -dimensional curvature is written in terms of d -dimensional curvature and the scalar field ϕ as

$$R\sqrt{-G} = e^{(d-2)\psi+p\phi}\sqrt{-g} \left[{}^{(d)}R - p\{2\nabla^2\phi + (p+1)(\nabla\phi)^2\} - 2(d-2)(\nabla\phi)(\nabla\psi) - (d-1)(d-2)(\nabla\psi)^2 - 2(d-1)\nabla^2\psi \right]. \quad (11)$$

The $(d+p)$ -dimensional vacuum action is therefore considered as equivalent to a d -dimensional action with a scalar field:

$$S = \int d^{d+p}x \sqrt{-G} R = \int d^p x \quad (12)$$

$$\times \int d^d x \sqrt{-g} e^{(d-2)\psi+p\phi} [{}^{(d)}R + (p-1)p(\nabla\phi)^2 + (d-1)(d-2)(\nabla\psi)^2 + 2p(d-1)\nabla\psi\nabla\phi]. \quad (13)$$

Next we see 4-dimensional pre-big-bang solution can be derived from 5-dimensional vacuum solution. Setting $d=4$, $p=1$ and replacing $\psi(\phi) \rightarrow \beta\phi$, $\phi \rightarrow \alpha\phi$ in (13) the 5-dimensional action, apart from the surface term, will be

$$S = \int d^5 x \sqrt{-G} R = \text{const.} \times \int d^4 x \sqrt{-g} e^{(2\beta+\alpha)\phi} [{}^{(4)}R + 6\beta(\alpha+\beta)(\nabla\phi)^2]. \quad (14)$$

We consider ϕ as the dilaton. Comparing (1) and (14), we have two equations for α and β

$$\alpha + 2\beta = -1, \quad (15)$$

$$6\beta(\alpha + \beta) = 1, \quad (16)$$

which lead to two distinct solutions:

$$I: \quad (\alpha, \beta) = \left(\frac{1}{\sqrt{3}}, -\frac{1}{2} \left(1 + \frac{1}{\sqrt{3}} \right) \right), \quad (17)$$

$$II: \quad (\alpha, \beta) = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{2} \left(1 - \frac{1}{\sqrt{3}} \right) \right). \quad (18)$$

This means there exist two choices of dimensional reduction for a given pair of 5- and 4-dimensional actions. Appropriate 5 dimensional vacuum solutions which will become cosmological one in 4 dimension can be found by putting 5 dimensional metric in the form:

$$ds_5^2 = -d\tau^2 + b^2 \gamma_{ij} dx^i dx^j + c^2 dy^2, \quad (19)$$

where

$$\gamma_{ij} = \delta_{ij} \left\{ 1 + \frac{\mathcal{K}}{4} (x^2 + y^2 + z^2) \right\} \quad (20)$$

and $\mathcal{K}$ is assumed to be normalized as 1, 0, -1. The consequent 5 dimensional solutions are

$$\mathcal{K} = 0 : ds_5^2 = -d\tau^2 + \tau \gamma_{ij} dx^i dx^j + \frac{1}{\tau} dy^2, \quad (21)$$

$$\mathcal{K} = 1 : ds_5^2 = -d\tau^2 + |1 - \tau^2| \gamma_{ij} dx^i dx^j + \frac{\tau^2}{|1 - \tau^2|} dy^2, \quad (22)$$

$$\mathcal{K} = -1 : ds_5^2 = -d\tau^2 + (1 + \tau^2) \gamma_{ij} dx^i dx^j + \frac{\tau^2}{1 + \tau^2} dy^2. \quad (23)$$

The flat ($\mathcal{K} = 0$) and closed ($\mathcal{K} = 1$) solutions are less interesting in the view of the higher-dimensional singularity-free solutions, since they have physical singularities at $\tau = 0$ for $\mathcal{K} = 0$ and $\tau = \pm 1$ for $\mathcal{K} = 1$. On the other hand, $\mathcal{K} = -1$ solution merely has a polar coordinate singularity and the manifold itself is smooth. This is a 5 dimensional blackhole solution with an adequate rotation of axes. In the following, therefore, we consider only the open case.

If we assume the 4-dimensional metric in the ordinary open FRW form

$$ds_4^2 = a^2 (-d\eta^2 + \gamma_{ij} dx^i dx^j), \quad (24)$$

where η is the conformal time and a represents the scale factor, the dimensional reduction (9) associate the 5-dimensional metric with the 4-dimensional metric as

$$d\eta = \frac{d\tau}{\sqrt{1 + \tau^2}}, \quad (25)$$

$$a^2 = (1 + \tau^2)^{1+\beta/\alpha} \tau^{-2\beta/\alpha}, \quad (26)$$

$$\phi = \frac{1}{2\alpha} \ln \frac{\tau^2}{1 + \tau^2}. \quad (27)$$

Integration of (25) gives $\tau = \sinh \eta$. There are two possible 4-dimensional solutions corresponding to the choice of (α, β) pair. The resulting solutions from (17) and (18) are:

$$I \begin{cases} a = (\cosh \eta)^{\frac{1-\sqrt{3}}{2}} |\sinh \eta|^{\frac{1+\sqrt{3}}{2}}, \\ \phi = \sqrt{3} \ln \tanh \eta \end{cases} \quad (28)$$

$$II \begin{cases} a = (\cosh \eta)^{\frac{1+\sqrt{3}}{2}} |\sinh \eta|^{\frac{1-\sqrt{3}}{2}}, \\ \phi = -\sqrt{3} \ln \tanh \eta \end{cases} \quad (29)$$

Taking the flat limit ($|\eta| \rightarrow 0$), these solutions coincide with the flat pre-big-bang solutions (6). It is easily verified that 1, 2 are included in *I*, and 3, 4 in *II*. That is, the branches interchanging under time reversal (TR) transformation are joined as solutions *I* or *II*. On the other hand, *I* and *II* interchange under SFD transformation (fig.1). Therefore, the scale-factor duality of 4-dimensional solution can be attributed to the non-uniqueness, or the duality, of the dimensional reduction. The relation between the scale-factor duality and dimensional reduction is diagrammatically shown in the fig.2.

4 Extension to Brans-Dicke and scalar-tensor theories

We have argued that, in the pre-big-bang type low-energy string effective theory in 4-dimension, the scale-factor duality can be understood as a consequence of the dual choice of mapping from 5-dimensional

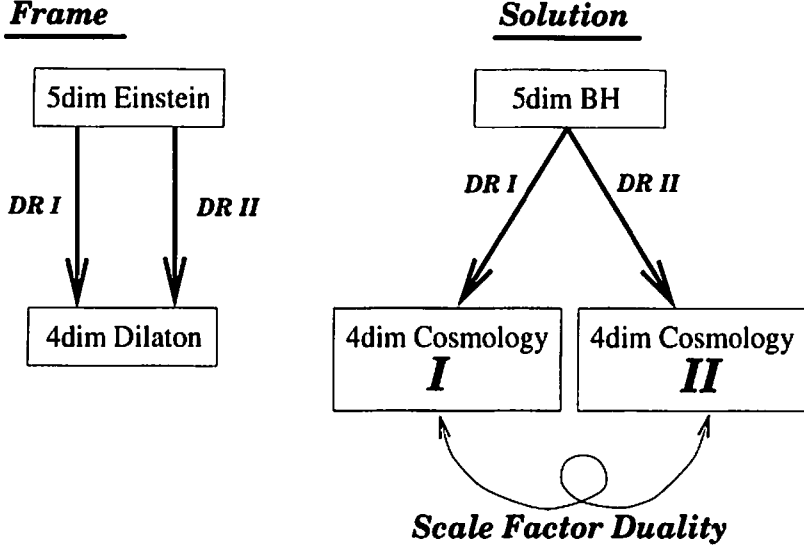


Figure 2: Relation between the dimensional reduction and the scale-factor duality. If we give a 5-dimensional action and 4-dimensional action, there are in general two possible mappings of dimensional reduction (DR1 and DR2). Under these mappings, the 5-dimensional solution is transformed into two different solutions in 4-dimension. These two solutions are related to each other by the scale-factor duality.

action down to 4-dimensional action. This means we can construct 4-dimensional dual solutions from 5-dimensional solution by the dimensional reduction. As a generalization of this formalism, we shall first discuss the case where the 4-dimensional action is that of the Brans-Dicke theory, and next that of the scalar-tensor theory.

Neglecting the matter content except the Brans-Dicke scalar field, the Lagrangian of the Brans-Dicke theory can be written as

$$S = \int d^4x \sqrt{-g} e^{-\phi} [(^{(4)}R - \omega(\nabla\phi)^2], \quad (30)$$

where ω is a constant. We compare this and the action (14), and find the relation between (α, β) and ω :

$$\alpha + 2\beta = -1, \quad (31)$$

$$6(\alpha + \beta)\beta = -\omega. \quad (32)$$

The cosmological solutions resulting from the dimensiona reduction are (25,26,27), which are in this case written as

$$a = \cosh \eta |\coth \eta|^{\beta/\alpha}, \quad (33)$$

$$\phi = \frac{1}{2\alpha} \ln \tanh \eta, \quad (34)$$

with

$$I : (\alpha, \beta) = \left(\sqrt{\frac{3+2\omega}{3}}, -\frac{1}{2} \left(1 + \sqrt{\frac{3+2\omega}{3}} \right) \right), \quad (35)$$

$$II : (\alpha, \beta) = \left(-\sqrt{\frac{3+2\omega}{3}}, -\frac{1}{2} \left(1 - \sqrt{\frac{3+2\omega}{3}} \right) \right). \quad (36)$$

In order that α and β be real, ω has to be larger than $-3/2$.

In the flat limit ($|\eta| \rightarrow 0$) the scale factor and the scalar field behave like

$$a \sim |\eta|^{-\beta/\alpha}, \quad (37)$$

$$\phi \sim \frac{1}{2\alpha} \ln |\eta|. \quad (38)$$

In this flat limit, the dual relation ($\dot{a} > 0$ for one and $\dot{a} < 0$ for the other) is realized when

$$-\frac{4}{3} < \omega < 0. \quad (39)$$

The dilaton-metric model ($\omega = -1$) well satisfies this condition. When $\omega \neq -4/3$, the scale factor (37) and the scalar field (38) for the two possible pair of (α, β) interchange under the generalized scale-factor duality[9]

$$\ln a \leftrightarrow \frac{1}{3\omega + 4} \{ (3\omega + 2) \ln a - (2\omega + 2)\phi \}, \quad (40)$$

$$\phi \leftrightarrow \frac{1}{3\omega + 4} \{ -6 \ln a - (3\omega + 2)\phi \}. \quad (41)$$

Next, we extend our formalism to derive a pair of d -dimensional solutions of scalar-tensor theories from $(d + p)$ -dimensional vacuum solution. We write the d -dimensional action in the form:

$$S = \int d^d x \sqrt{-g} e^{-\Phi} [{}^{(d)}R - \omega(\Phi)(\nabla\Phi)^2]. \quad (42)$$

We keep the function $\psi(\phi)$ unspecified in the action (13) in order to derive $\omega(\Phi)$ in d -dimension. Then the relation between Φ , ω and ψ , ϕ will be

$$\Phi = -(d - 2)\psi(\phi) + p\phi, \quad (43)$$

$$\omega(\Phi) = -\frac{(d - 1)(d - 2)f^2 + 2p(d - 1)f + p(p - 1)}{((d - 2)f + p)^2}, \quad (44)$$

where $f := d\psi/d\phi$.

The right hand side of (44) is a function of f , related to Φ implicitly through ϕ . For a given $\omega(\Phi)$, (44) is a quadratic algebraic equation of f , having two real solutions when

$$p(d + p - 2)\{(d - 2)\omega + d - 1\} > 0. \quad (45)$$

In our dimensional reducing point of view, this ‘quadratic’ness is the origin of the ‘dual’ity of the low-dimensional solutions. If the two real f ’s are found, we can map a higher dimensional solution to lower dimension in two ways, and the dual lower dimensional solutions are found.

We will concentrate on $d = 4$, $p = 1$ case and show a simple example here. We consider the case where ω is given by

$$\frac{1}{\sqrt{3 + 2\omega}} = \alpha_0 + \beta_0\phi. \quad (46)$$

Then, neglecting the integration constant,

$$\psi = \left(-\frac{1}{2} \pm \frac{\sqrt{3}}{2}\alpha_0\right)\phi \pm \frac{\sqrt{3}}{4}\beta_0\phi^2. \quad (47)$$

The solution for 4-dimensional scale factor and the scalar field Φ will be

$$a = a_0 \exp \left[\frac{1}{2}(1 \pm \sqrt{3}\alpha_0) \ln \tanh \eta \pm \frac{\sqrt{3}}{4}\beta_0(\ln \tanh \eta)^2 \right] \cosh \eta, \quad (48)$$

$$\Phi = \pm \sqrt{3}\alpha_0 \ln \tanh \eta \pm \frac{\sqrt{3}}{2}\beta_0(\ln \tanh \eta)^2. \quad (49)$$

$\beta_0 = 0$ case reduces to the Brans-Dicke theory. In the flat limit the scale factor behaves as

$$a \sim \exp(\pm \frac{\sqrt{3}}{4}\beta_0(\ln \eta)^2) = |\eta|^{\pm \frac{\sqrt{3}}{4}\beta_0 \ln \eta}, \quad (50)$$

indicating a dual relation of the two solutions for the scale factor.

5 Summary

We have discussed the dual cosmological solutions appearing in the low-energy string effective theories in terms of the dimensional reduction. It has been shown that this dual relation of the scale factor can be attributed to the non-uniqueness of the dimensional reduction. That is, if we fix the higher dimensional action and lower dimensional action, there are in general two mapping of dimensional reduction. As for the solution, one solution (our example is black hole solution) in higher dimension is transferred to two different solutions in lower dimension, resulting in general the different growth rates of the scale factor. We have also shown that this correspondence also holds if we generalize the low-dimensional theory to Brans-Dicke or scalar-tensor theories.

Acknowledgements

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Do naked singularities generically occur in generalized theories of gravity?

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Abstract

A new mechanism for causing naked singularities is found in an effective superstring theory. We investigate the gravitational collapse in a spherically symmetric Einstein-Maxwell-dilaton system in the presence of a pure cosmological constant “potential”, where the system has no static black hole solution. We show that once gravitational collapse occurs in the system, naked singularities necessarily appear in the sense that the field equations break down in the domain of outer communications. This suggests that in generalized theories of gravity, the non-minimally coupled fields generically cause naked singularities in the process of gravitational collapse if the system has no static or stationary black hole solution.

The singularity theorem[1] states that the occurrence of singularities is inevitable under some physical conditions in general relativity. There are two notable scenarios where singularities may appear in our universe. One is the initial singularity at the birth of the universe and the other is the final stage of gravitational collapse. In the latter case we believe that an event horizon is formed which encloses all occurring singularities as the collapse proceeds, following the cosmic censorship hypothesis (CCH) [2]. CCH is classified into two types, the weak cosmic censorship hypothesis (WCCH) and the strong one (SCCH). WCCH says that observers at an infinity should not see singularities while SCCH says that no observer should see them and the whole region of a space-time can be uniquely determined by initial regular data. Mathematically, SCCH is equivalent to the statement that no Cauchy horizon can be formed in a physical gravitational collapse.

Recently many elegant results [3] suggest that SCCH holds for charged and/or rotating black holes due to the destruction of the Cauchy horizon by the mass inflation phenomenon. The general proof of CCH is, however, far from complete and many counter examples have been found in the framework of general relativity [4]. A new approach has been considered in the context of generalized theories of gravity. Gibbons and Maeda [5] discovered the static black hole solutions in the Einstein-Maxwell-dilaton system, which comes from an effective superstring theory, and later Garfinkel, Horowitz and Strominger [6] showed that the inner (Cauchy) horizon in the Reissner-Nordström solution is replaced by a spacelike singularity. This suggests that the occurrence of the inner horizon is not generic and hence SCCH holds if we take the effect of string theory into account. On the other hand Horne and Horowitz [7] obtained the opposite result that extremal electrically charged non-static black hole solutions in the presence of a central charge have timelike singularities. It seems, however, not to be a counter example of SCCH in the sense that such solutions have no regular initial spacelike hypersurface because of the central singularities. Thus, the following question naturally arises. Does CCH really hold in generalized theories of gravity? There is not much evidence deciding the matter yet because the above results do not take any physical process of the gravitational collapse from initial regular data into account.

Although a large number of studies have been made on finding static/stationary black hole solution and investigating their thermodynamical properties and geometrical stability and so on in generalized theories of gravity, only few studies have so far been made on investigating gravitational collapse. We should not overlook that the gravitational collapse in generalized theories of gravity will be much different from that in general relativity. In general relativity, the black hole no-hair conjecture states that the matter fields are swallowed by a black hole and a space-time asymptotically approaches the electrovacuum Kerr-Newman solution after gravitational collapse. On the other hand, the above aspects are not necessarily

satisfied in generalized theories of gravity because the dilaton field couples to the curvature and/or other matter fields.

In this letter we investigate the gravitational collapse in the spherically symmetric Einstein-Maxwell-dilaton system analytically in the presence of a pure positive cosmological constant, which corresponds to $g_1 = 0$ in the Liouville-type potential [8]. This system is interesting because it has been proved that no static black hole solution exists for spherically symmetric space-times [9]. Such property seems to be general in the sense that there exists no asymptotically flat, asymptotically de Sitter, or asymptotically anti-de Sitter solution in the arbitrary exponential dilaton potential [8]. Furthermore, it is known that there is a critical mass below which no asymptotically flat black hole solution exists in the system which includes the Gauss-Bonnet term which was neglected in the former works [10, 11]. In these systems the following three interpretations are available: (i) the matter fields do not cause the gravitational collapse but escape to infinity without forming a black hole event horizon (BEH); (ii) the space-time does not approach a stationary space-time and the matter fields oscillate forever or, (iii) naked singularities are necessarily formed. If the case (i) is true for all initial data, we should find that the strong cosmic no-hair conjecture holds in the present system. By investigating numerically, however, we confirmed that this is not the case and some initial data lead to the gravitational collapse, namely, a trapped surface forms [12]. This result is consistent with the fact that the dilaton field satisfies the dominant energy condition in the Einstein frame. Then it is enough to consider only the cases (ii) and (iii). The case (ii) seems incompatible with our naive expectation that after gravitational collapse, the space-time settles down to an asymptotically stationary space-time with a black hole. If the case (iii) is true, we must say that CCH is violated even in generalized theories of gravity.

Using the double null coordinates, we will show that the field equations necessarily break down in the domain of outer communications or at the BEH, once the gravitational collapse has occurred. It is worth to note that “break down” does not imply a coordinate singularity because our coordinates are, probably, the most extended coordinates in spherically symmetric space-times. It is also supported by numerical calculations which show the formation of naked singularities in the system [12]. Thus, our result suggests that naked singularities generically occur in the gravitational collapse of spherically symmetric systems.

The low-energy effective action (Einstein frame) of string theory as a dilaton model is

$$S = \int d^4x \sqrt{-g} [-R + 2(\nabla\phi)^2 + e^{-2\phi} F^2 + 2\Lambda], \quad (1)$$

where R is the Ricci scalar, ϕ is a massless dilaton field, Λ is a positive constant, and $F_{\mu\nu}$ is the field strength of the Maxwell field. The double-null coordinates for spherically symmetric space-time are

$$ds^2 = -2e^{-\lambda}(U, V)dUdV + R(U, V)^2 d\Omega, \quad (2)$$

where ∂_U and ∂_V are future-ingoing and outgoing null geodesics, respectively. The Maxwell equation is automatically satisfied for a purely magnetic Maxwell field $F = Q \sin\theta d\theta \wedge d\phi$ ($F^2 = 2Q^2/R^4$), where Q is the magnetic charge. An electrically charged solution is obtained by a duality rotation from the magnetically charged one [5, 6]. Therefore we shall consider only a purely magnetic case in this letter. The dynamical field equations are

$$\lambda_{,UV} - \frac{2R_{,UV}}{R} = 2\phi_{,U}\phi_{,V} + e^{-\lambda} \left(\frac{Q^2 e^{-2\phi}}{R^4} - \Lambda \right), \quad (3)$$

$$R_{,UV} + \frac{R_{,U}R_{,V}}{R} = -\frac{e^{-\lambda}}{2R} \left(1 - \frac{Q^2 e^{-2\phi}}{R^2} - \Lambda R^2 \right), \quad (4)$$

$$2R^3(R_{,U}\phi_{,V} + R\phi_{,UV} + R_{,V}\phi_{,U}) = Q^2 e^{-2\phi-\lambda}, \quad (5)$$

where $A_{,a}$ is a partial derivative of A with respect to a . The constraint equations are

$$R_{,UU} + \lambda_{,U}R_{,U} = -(\phi_{,U})^2 R, \quad (6)$$

$$R_{,VV} + \lambda_{,V} R_{,V} = -(\phi_{,V})^2 R. \quad (7)$$

We consider the evolution of the field equations with initial regular data on a null characteristic hypersurface whose boundary is a closed future-trapped surface (see Fig. 1) because we are interested in the gravitational collapse. As shown in Theorem 1 of Ref. [13], the trapped surface causes the formation of a BEH ($U = U_B$ null hypersurface) if there are no singularities observed from $\mathcal{I}^+$. For this initial value problem, we will show the following theorem.

Theorem

Let us consider the dynamical evolution of the model (1) in a spherically symmetric space-time or equivalently for the equations (3)-(7) with an initial data on the characteristic null hypersurface N . Then, there is $U_1 (\leq U_B)$ such that the system of equations breaks down at $U = U_1$.

We shall show the above theorem by contradiction below. First, we will consider the asymptotic behavior of field functions near the cosmological event horizon (CEH), which is defined as a past Cauchy horizon $H^-(\mathcal{I}^+)$ when a BEH exists (see Ref. [14]). It is convenient to rescale the coordinate U such that U is an affine parameter of a null geodesic of the CEH, i.e., λ is constant along the CEH. Hereafter we use a character $u (= f(U))$ instead of U for parameterization to avoid confusion. Under such coordinates, Eq. (6) on the CEH is

$$-R_{,uu} = (\phi_{,u})^2 R. \quad (8)$$

If there are no singularities in $\overline{J^-(\mathcal{I}^+) \cap J^+(N)}$, the null geodesic generators of the BEH and the CEH are future complete. Then, by the non-decreasing area law for the CEH and by existence of an upper bound of the area [14], $\lim_{u \rightarrow \infty} R = C_1$ (hereafter, $C_i (i = 1, 2, \dots)$ means a positive constant), and hence future asymptotic behavior of $R_{,u}$ on the CEH is represented as follows [15],

$$R_{,u} \sim C_2 u^{-\alpha-1} \quad (\alpha > 0). \quad (9)$$

Then, by Eq. (8)

$$\phi_{,u} \sim u^{-\alpha/2-1}, \quad (10)$$

and hence $\lim_{u \rightarrow \infty} \phi(u) = \text{const.}$ We shall obtain the asymptotic value of $R_{,V}$ on the CEH by solving Eq. (4) as

$$R_{,V} = \left(\int_{u_i}^u \frac{KR}{R_i} du + R_{,V}|_i \right) \frac{R_i}{R}, \quad (11)$$

where $R_{,V}|_i, R_i$ are initial values of $R_{,V}, R$ on $u = u_i$, respectively and $K = -(1 - Q^2 e^{-2\phi}/R^2 - \Lambda R^2)/2Re^\lambda$. K must approach a positive constant K_∞ as $u \rightarrow \infty$ because $R, \phi \rightarrow \text{const.}$ and the expansion $\theta_+ \equiv R_{,V}/R$ of each outgoing null geodesic is positive (if it were negative, the area element $d\Omega$ would become 0 along the outgoing null geodesics). Then, the asymptotic value of $R_{,V}$ is

$$R_{,V} \sim K_\infty u > 0. \quad (12)$$

We shall also obtain the asymptotic value of $\phi_{,V}$ on the CEH by solving Eq. (5) with the following solution

$$\phi_{,V} = \left(\int_{u_i}^u \frac{HR}{R_i} du + \phi_{,V}|_i \right) \frac{R_i}{R}, \quad (13)$$

where $\phi_{,V}|_i$ is an initial value of $\phi_{,V}$ on $u = u_i$ and $H = (Q^2 e^{-2\phi-\lambda} - 2R^3 R_{,V} \phi_{,u})/2R^4$. H approaches a positive constant H_∞ as $u \rightarrow \infty$ because $R_{,V} \phi_{,u} \sim u^{-\alpha/2} \rightarrow 0$ by Eqs. (10) and (12). The asymptotic value of $\phi_{,V}$ is

$$\phi_{,V} \sim H_\infty u > 0. \quad (14)$$

Next, let us consider an infinitesimally small neighborhood $\mathcal{U}_C$ of the CEH which contains a timelike hypersurface T_C such that T_C is in the past of the CEH by $\epsilon (> 0)$, where ϵ is a fixed affine parameter

distance along the outgoing null geodesic intersecting the CEH [16]. We denote each point of the intersection of T_C and $u = \text{const.}$ hypersurface by $p(u)$. By Eq. (5), the solution of $h \equiv \phi_{,u}$ along each $u = \text{const.}$ is

$$h = \left(- \int_V^{V_C} \frac{LR}{R_C} dV + h_C \right) \frac{R_C}{R}, \quad (15)$$

where R_C and h_C are values of R and h at the CEH ($V = V_C$), respectively, and $L = (Q^2 e^{-2\phi-\lambda} - 2R^3 R_{,u} \phi_{,V})/2R^4$. Since $(R_{,u} \phi_{,V}/R)|_{V_C} \sim u^{-\alpha} \rightarrow 0$ for large values of u , $L \sim L_\infty > 0$ asymptotically. Differentiating h by V in Eq. (15),

$$h_{,V} = L + \frac{R_C R_{,V}}{R^2} \left(\int_V^{V_C} \frac{LR}{R_C} dV - h_C \right). \quad (16)$$

By the relation $\epsilon = dr \sim u dV$ for large u [17], $h_{,V} \sim L_\infty + O(\epsilon) + O(u^{-\alpha/2})$. Since ϵ is an arbitrary small value, asymptotically $h_{,V} > 0$ on the $u = \text{const.}$ null segment $[V]_{p(u), V_C} \subset \mathcal{U}_C$.

In the next step we investigate the behavior of $\phi, \phi_{,V}$ and R on BEH, just like the CEH case. We rescale V into v such that v is an affine parameter and $\lambda = \text{const.}$ on the BEH, while we leave U unchanged. Since the area of the BEH is non-decreasing and also has an upper bound as shown in [13, 14], $\lim_{v \rightarrow \infty} R = C_3$, and $R_{,v} \sim C_4 v^{-\beta-1}$ ($\beta > 0$). This means that by Eq. (7) $\phi_{,v} \sim v^{-\beta/2-1}$ as obtained in the CEH case. By replacing u by v in the argument of the CEH case and solving Eqs. (4) and (5), the asymptotic values of $\phi_{,U}$ and $R_{,U}$ become $R_{,U} \sim v$ and $\phi_{,U} \sim C_5 v$, respectively.

Hence the first and third terms in the l.h.s. of the dilaton field Eq. (5) are negligible asymptotically and

$$\lim_{v \rightarrow \infty} k_{,U} = \lim_{v \rightarrow \infty} \phi_{,vU} = C_6 > 0. \quad (17)$$

We define $k \equiv \phi_{,v}$. Consider an infinitesimally small neighborhood $\mathcal{U}_B$ of the BEH. There is a small ϵ such that the timelike hypersurface T_B which is in the past of the BEH by a fixed affine parameter distance ϵ of ingoing null geodesics intersecting the BEH is contained in $\mathcal{U}_B$. By the relation between the affine parameter and dU , i.e., $dU \propto \epsilon/v$, k on T_B is asymptotically

$$\begin{aligned} k|_{T_B} &\sim k|_{BEH} + k_{,U}|_{BEH} (-dU) \\ &\sim k|_{BEH} - C_6 \epsilon v^{-1} \sim (v^{-\beta/2} - C_6 \epsilon) v^{-1}. \end{aligned} \quad (18)$$

This indicates that $\phi_{,V}$ is negative on T_B for large values $v(> v_1)$. Now, consider each null segment $N_u(u \geq u_I) : [V_C - \epsilon/u_I, V_C]$ where $h_{,V} > 0$ on N_{u_I} (see Fig. 2). If one takes u_I large enough, N_u intersects $T_B(v > v_1)$ at $u = u_F$. Let us take a sequence of $N_{u_J}(J = 1, 2, \dots, L+1)$ (L is a natural number large enough), where $\delta u = (u_F - u_I)/L$ and $u_J = u_I + (J-1)\delta u$. Assume that $\phi_{,V} > 0$ on $N_J(\equiv N_{u_J})$, for a moment, then $h_{,V} > 0$ on N_J in the same way as $h_{,V} > 0$ on $[V]_{p(u), V_C}$ [18]. Hence we can show that $\phi_{,V}|_{N_{J+1}} \cong (\phi_{,V}|_{N_J} + h_{,V}|_{N_J} \delta u) > 0$. On the other hand, $\phi_{,V} > 0$ on $u = u_I$ by Eq. (14), hence $\phi_{,V} > 0$ for each u_J by induction. This is a contradiction because as we showed $\phi_{,V} < 0$ on $T_B(u = u_F)$. $\square$

It is worth to comment that the possibility of the formation of null singularities on BEH, continuing to i^+ is not excluded from our theorem. In this case the BEH is a singular null hypersurface even if WCCH holds (SCCH is, probably, violated).

Our theorem says that the case (ii) is not true, as we expected. When we consider the dyon solution or a rotating space-time, non-trivial 3-rank anti-symmetric tensor fields inevitably appear. However, we expect that such fields do not change our result drastically. This implies that if there are no static/stationary black hole solutions, naked singularities generically occur in the process of gravitational collapse in generalized theories of gravity.

Since the model under consideration is the low energy limit of string theory, we cannot exclude the possibility that higher curvature terms and higher order α' corrections, where α' is the inverse string tension, may prevent the space-time from causing naked singularities. However, the theory is useful near the Planck scale. Thus, we can at least say that *string theory predicts existence of space-time points with*

very high curvature in the domain of outer communications. This picture is quite different from that of general relativity because it is strongly believed that space-time points with high curvature do not appear generically in the outer region of a black hole.

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- [15] It is enough to consider the case of the power law function, $R_{,u} \sim u^{-\alpha-1}$ to show our theorem because it gives the mildest dumping.
- [16] The neighborhood of CEH can be covered by Gaussian null coordinates, $ds^2 = -2drd\eta + f(d\eta)^2 + R^2d\Omega$, where r is an affine parameter of outgoing null geodesics and $f = 0$ on the CEH. Then, $\epsilon = dr = \text{const.}$
- [17] It seems reasonable to assume that the surface gravity of CEH, $\kappa_c \equiv -f_{,r}/2$, is asymptotically constant. In this case, $e^{\kappa_c \eta} \sim u$ and hence $d\eta \sim du/u$. This indicates that $dr \sim u dV$.
- [18] One can show that $R_{,u} < 0$ on T_C for large u by using Eq. (4). Therefore, $R_{,u} < 0$ inside T_C because each expansion of ingoing null geodesics from T_C must decrease monotonically. Thus, we can easily get $h_{,V} > 0$ by considering each segment; $[V_C - \epsilon/u_I, V_C - \epsilon/u_J] \notin \mathcal{U}_C$, $[V_C - \epsilon/u_J, V_C] \in \mathcal{U}_C$. The detailed proof can be found in Ref. [12].

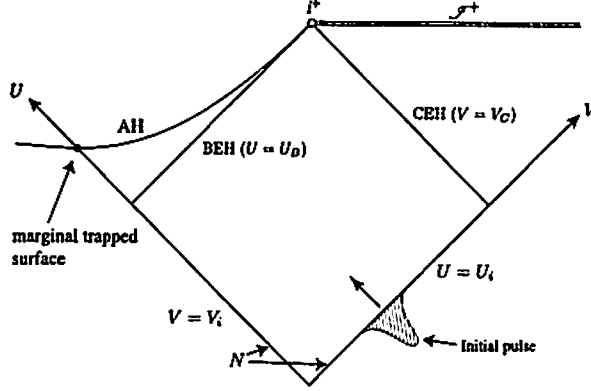


Figure 1: A Penrose diagram in asymptotically de Sitter space-time. N and AH are the characteristic hypersurface and the apparent horizon of the black hole, respectively.

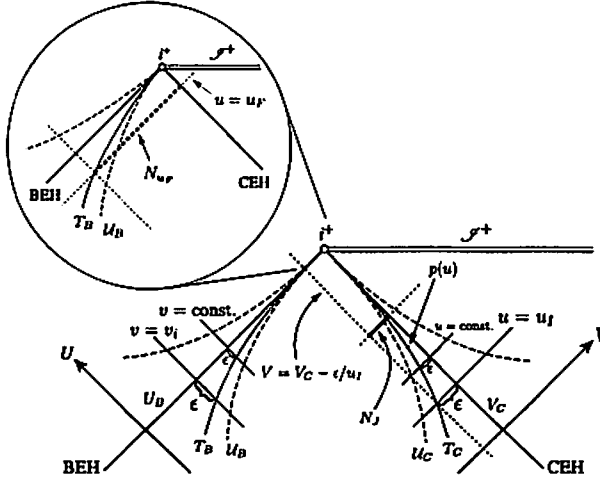


Figure 2: Timelike hypersurfaces T_B, T_C are displayed in the neighborhoods $\mathcal{U}_B, \mathcal{U}_C$ of BEH and CEH , respectively. A null segment N_J is displayed by a thick line. ϵ is a fixed affine parameter distance along outgoing and ingoing null geodesics.

Generalized Scalar-Tensor Cosmology

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Abstract

A generalized scalar-tensor(GST) theory is investigated in which the cosmological term Λ depends not only on the scalar field ϕ but also on its time derivative $\dot{\phi}^2$. Solution for the cosmological term is studied in addition to solutions for the scale-factor and the scalar field.

1 INTRODUCTION

Generalizations of Brans-Dicke theory^[1] such as those involving cosmological terms^[2] and other modifications^[3] have come under increasing scrutiny in recent years^[4], particularly in connection with the early Universe.

A variable cosmological term has also been studied in relation to the scalar field to solve the cosmological constant problem^[5,6].

We take here the generalized scalar-tensor theory which will make it possible for us to study the early Universe in relation to the cosmological constant problem.

2 BASIC EQUATIONS

We start with the following variational principle:

$$0 = \delta \int \left(\phi (R - 2\Lambda(\phi, \phi_i \phi^i)) + \left(\frac{16\pi}{c^4} \right) L_m - \omega(\phi) \frac{\phi_{,i} \phi^{,i}}{\phi} \right) \sqrt{-g} d\Omega, \quad (1)$$

where Latin index i runs over 0,1,2 and 3. The scalar field ϕ is assumed to be a function of time $x^0 = ct$ alone, $\phi \equiv \phi(ct)$. We emphasize Λ is assumed to be a function of $\phi_{,i} \phi^{,i}$ as well as ϕ , to give more freedom to the functional forms of Λ and scalar field as well. That is, Λ thus depends on the kinetic energy of Brans-Dicke type scalar field too. That is a generalization of previous work^[5] in which $\Lambda = \Lambda(\phi)$ was assumed. A similar idea has been considered by Moffat^[7] in connection with the age problem. The field equation for the metric field g_{ij} is

$$\begin{aligned} R_{ij} &= \frac{1}{2} g_{ij} R + g_{ij} \Lambda - 2 \frac{\partial \Lambda}{\partial b} \phi_{,i} \phi_{,j} \\ &= \frac{8\pi}{\phi c^4} T_{ij} + \frac{\omega}{\phi^2} \left(\phi_{,i} \phi_{,j} - \frac{1}{2} g_{ij} \phi_{,l} \phi^{,l} \right) + \frac{1}{\phi} (\phi_{,i;j} - g_{ij} \square \phi), \end{aligned} \quad (2)$$

where $b = \phi_{,i} \phi^{,i}$ and

$$T^{ij} = \frac{2}{\sqrt{-g}} \frac{\partial(\sqrt{-g} L_m)}{\partial g_{ij}}.$$

Eq.(2) with $\omega = \text{const.}$ reduces to that of the Brans-Dicke theory^[1] in case of $\Lambda = 0$ and to that of the modified Brans-Dicke theory^[5] in case of $\Lambda = \Lambda(\phi)$.

While the field equation for ϕ is obtained by varying ϕ and $\phi_{,i}$,

$$R - 2\Lambda - 2\phi \frac{\partial \Lambda}{\partial \phi} + 4 \frac{\partial}{\partial x^i} \left(\phi \frac{\partial \Lambda}{\partial b} \right) \phi^{,i} + 4\phi \frac{\partial \Lambda}{\partial b} \square \phi = \frac{\omega}{\phi^2} \phi_{,i} \phi^{,i} - \frac{2\omega}{\phi} \square \phi - \frac{\phi_{,i} \phi^{,i}}{\phi} \frac{\partial \omega}{\partial \phi}, \quad (3)$$

Again Eq.(3) with $\omega = \text{const.}$ gives back earlier results of the Brans-Dicke when $\Lambda = 0$ and of the modified Brans-Dicke when $\Lambda = \Lambda(\phi)$. Eq.(3) also ensures that the conservation law $T_{i;k}^k = 0$ holds as required.

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3 APPLICATION TO COSMOLOGY

We apply the above equations to the homogeneous and isotropic Universe. Then the metric is Friedmann-Robertson-Walker, and the energy-momentum tensor, that of a perfect fluid.

The field equations are derived from the time-time and space-space components of Eq.(2) as follows: (0,0) component of Eq.(2);

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} - \frac{\Lambda c^2}{3} + \frac{2}{3} \frac{\partial \Lambda c^2}{\partial \dot{\phi}^2} \dot{\phi}^2 = \frac{8\pi}{3\phi c^2} \epsilon + \frac{\omega}{6} \left(\frac{\dot{\phi}}{\phi}\right)^2 - \frac{\dot{a}}{a} \frac{\dot{\phi}}{\phi}, \quad (4)$$

(1,1) = (2,2) = (3,3) components of Eq.(2);

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} - \Lambda c^2 = -\frac{8\pi}{\phi c^2} p - \frac{\omega}{2} \left(\frac{\dot{\phi}}{\phi}\right)^2 - \frac{\ddot{\phi}}{\phi} - 2\frac{\dot{a}}{a} \frac{\dot{\phi}}{\phi}. \quad (5)$$

Eqs.(4) and (5) reduce to the appropriate expression in [8,9] and [5] when $\Lambda = 0$ and $\Lambda = \Lambda(\phi)$ respectively. From Eq.(4),

$$\begin{aligned} \frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} - \frac{\Lambda c^2}{3} - \frac{\phi}{3} \frac{\partial \Lambda c^2}{\partial \phi} + \frac{2}{3} \frac{\partial}{\partial t} \left(\phi \frac{\partial \Lambda c^2}{\partial \dot{\phi}^2} \right) \cdot \dot{\phi} + \frac{2}{3} \phi \frac{\partial \Lambda c^2}{\partial \dot{\phi}^2} \left(\ddot{\phi} + 3\frac{\dot{a}}{a} \frac{\dot{\phi}}{\phi} \right) \\ = \omega \frac{\dot{a}}{a} \frac{\dot{\phi}}{\phi} + \frac{\omega}{3} \frac{\ddot{\phi}}{\phi} + \frac{1}{6} \frac{\partial \omega}{\partial \phi} \frac{\dot{\phi}^2}{\phi} - \frac{\omega}{6} \left(\frac{\dot{\phi}}{\phi}\right)^2. \end{aligned} \quad (6)$$

The basic equations for the scale-factor $a(ct)$ and the scalar field $\phi(ct)$ are Eqs.(4), (5) and (6).

Since a and ϕ are independent of each other, the cross terms of a and ϕ in the right and left hand sides of Eq.(6) should be equated, that is,

$$\frac{\partial \Lambda c^2}{\partial \dot{\phi}^2} = \frac{\omega}{2\dot{\phi}^2}. \quad (7)$$

Then Eq.(6) is separable with a and ϕ as follows,

$$\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{1}{3} (f + \Lambda_\gamma c^2 + f' \phi) = \text{const.} (\equiv K). \quad (6)'$$

The scale factor $a(ct)$ is obtained by solving Eq.(6)' as follows:

$$K = 0 \quad ; \quad a = \sqrt{-kc^2 t^2 + \alpha t + \beta}, \quad (8)$$

$$K \neq 0 \quad ; \quad a = \left(C_1 e^{\sqrt{2K}t} + C_2 e^{-\sqrt{2K}t} + \frac{kc^2}{K} \right)^{1/2}. \quad (9)$$

The cosmological term is obtained by solving Eqs.(6)^{prime} and (7),

$$\Lambda c^2 = \frac{\omega}{2} \left(\frac{\dot{\phi}}{\phi}\right)^2 + \frac{f_0}{\phi} + 3K, \quad (10)$$

where f_0 is a constant.

The scalar field $\phi(ct)$ is obtained by solving Eqs.(4) and (5). Both Eq.(4) and Eq.(5), however, give an identical equation for ϕ under the relation, Eq.(7).

Here we take the equation of state as follows,

$$p = \gamma \epsilon. \quad (11)$$

Then we have, for example, $\gamma = -1, 1, 1/3$ and 0 for false-vacuum energy, stiff matter, radiation and pressureless dust respectively.

From the conservation law, $T^k_{;k} = 0$, the well known equation is derived :

$$\epsilon = \epsilon_\gamma a^{-3(1+\gamma)}, \quad (12)$$

where ϵ_γ is a constant of dimension $ML^{2+3\gamma}T^{-2}$. We adopt the usual conventions^[1] in which $[a] = L$, $[\Lambda] = L^{-2}$, $[\omega] = \text{dimensionless}$ and $[\phi] = ML^{-3}T^2$.

From Eqs.(4) and (5), the following differential equation for ϕ is derived:

$$K = 0; \quad \dot{\phi} + \frac{\alpha^2 + 4k\beta c^2}{2(-2kc^2t + \alpha)a^2} \phi = \frac{2a^2}{-2kc^2t + \alpha} \left(\frac{8\pi\epsilon_\gamma}{3c^2} a^{-3(1+\gamma)} + \frac{f_0}{3} \right), \quad (13)$$

$$\begin{aligned} K \neq 0; \quad \dot{\phi} - \frac{\sqrt{2K}}{2} \left(2\frac{x+y}{x-y} - \frac{x-y}{x+y + \frac{kc^2}{K}} \right) \phi \\ = \frac{16\pi\epsilon_\gamma}{3\sqrt{2K}c^2} \frac{\left(x+y + \frac{kc^2}{K} \right)^{-(3\gamma+1)/2}}{x-y} + \frac{2f_0}{3\sqrt{2K}} \frac{x+y + \frac{kc^2}{K}}{x-y}, \end{aligned} \quad (14)$$

where $x \equiv C_1 e^{\sqrt{2K}t}$, $y \equiv C_2 e^{-\sqrt{2K}t}$. Those two equations (13) and (14) are linear differential equations of first order. Therefore they can be easily solved for each case of $k = \pm 1$ and $\gamma = -1, 1, 1/3$ and 0 to result in giving the functional forms of the scalar field with time. We are not going to present them $\phi(ct)$ here for the reason of space and time, which will appear elsewhere.

4 COMMENTS AND DISCUSSION

In the present work, we have extended the generalized scalar-tensor theory by taking the cosmological term as a function not only of the scalar field but also of the gradient of the scalar field. Since we regard the scalar field as a function only of time, the cosmological term depends on $\dot{\phi}^2$ and ϕ as well.

After a conformal transformation, $g_{ij} = \hat{g}_{ij}/\phi$, and re-definition of the scalar-field,

$$\Phi \equiv \frac{d\phi}{\phi} \sqrt{\frac{2\omega(\phi) + 3}{2}},$$

the vacuum action in the Jordan-Brans-Dicke frame, Eq.(1) with $L_m = 0$, is transformed to the equivalent action in the Einstein-Hilbert frame^[4] in the simplest case of $\sqrt{(2\omega(\phi) + 3)/2} \sim \phi$ as:

$$s = \int \left(\hat{R} - \Phi_{,i} \hat{g}^{ij} \Phi_{,j} - \frac{2\Lambda(\Phi, \Phi \Phi_{,i} \hat{g}^{ij} \Phi_{,j})}{\Phi} \right) \sqrt{-\hat{g}} d\Omega.$$

This implies that the scalar field in the EH frame, Ω could be canceled by choosing a functional form of Λ such as $2\Lambda = \Lambda_0 \Phi + \Phi \Phi_{,i} \hat{g}^{ij} \Phi_{,j}$. This enables us to speculate that the vacuum action in the JBD frame could be transformed to the action of the gravitational field with or without a cosmological constant Λ_0 in the EH frame. In [7], the cosmological term is identified with $\Phi_{,i} \hat{g}^{ij} \Phi_{,j}$. The present theory is somewhat different in that $\Lambda(\Phi \Phi_{,i} \hat{g}^{ij} \Phi_{,j})$ contributes to cancel the $\Phi_{,i} \hat{g}^{ij} \Phi_{,j}$ -term in the action.

As is seen in Eq.(9), the cosmological term evolves differently with time for each period because ϕ is obtained for each γ . This result is one way to approach the cosmological "constant" problem^[5,6].

When we apply the present model to a numerical calculation of the nucleosynthesis in the early Universe of which the periods concerned are $\gamma = 1/3$ and $\gamma = 0$, we can take $K = 0$ and $\beta = 0$ for the Universe expanding in a power-law fashion from $a = 0$ at $t = 0$.

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Random Density Field and Multifractal Structure in the Universe

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Abstract

Recent observations indicate the structure of the universe is more complicated than the one previously imagined. It is required to seek a good statistical measure to characterize the pattern of the universe. As a such quantity, the multifractal $f(\alpha)$ spectrum might be useful. Indeed, it was shown that the CfA data can be quantified by the $f(\alpha)$ spectrum. A natural theoretical attempt is to explain this result from the fundamental theory. We will demonstrate that the inflationary motivated CDM model can fit the $f(\alpha)$ spectrum and explore its dynamical evolution.

1 Introduction

The most important postulate of the modern cosmology is the validity of the cosmological principle. Based on this principle, we can quantify the structure of the universe composed by galaxies, clusters of galaxies, and so on. The most popular measure to quantify the pattern is the two-point correlation function or the power spectrum. Needless to say, more sophisticated measures are necessary to characterize the complex pattern of the universe. The multi-point function might be useful. [1] However, intuitively appealing measures would be desirable. The genus statistics is one of such measures and widely used. [2] Here, we will discuss alternative, i.e., multifractal $f(\alpha)$ spectrum. [3, 4, 5, 6]

At a glance, the fractality of the distribution contradicts with the cosmological principle. Indeed, some people insist that the fractal principle can take the position for the cosmological principle. It may be difficult to reject this opinion from the observations. Nevertheless, we will keep the attitude to regard that the cosmological principle is valid. If so, how do we understand the fractality in the analysis of the data? The answer is simple; it is nothing but the mapping of the real distribution to the fractal distribution. It is an approximation in nature. Nevertheless, the $f(\alpha)$ spectrum as a measure of the cosmological pattern is important and should not be overlooked.

Our work is motivated by the analysis of the CfA data for which the multifractal spectrum $f(\alpha)$ is used. It is needed to explain the result from the theoretical point of view. The strategy of our analysis is simple. The random density field originated from the inflationary stage will be evolved by the gravitational force. As we are interested in the large scale, the quasi-nonlinear analysis is used for this purpose. To obtain the $f(\alpha)$ spectrum at each time from the random density field, we adopt the Poisson model. The resulting $f(\alpha)$ spectrum turns out to be in a good agreement with the CfA observation.

Organization of this paper is as follows. In sec.2, the concept of the multifractal is presented and the calculational procedures for the multifractal $f(\alpha)$ spectrum is explained. In sec.3, we adopt the spherical collapse model to calculate the dynamical evolution of the higher order moments. The validity of the approximation is discussed in detail. The Poisson model is used to connect the random density field to the distribution of the galaxies. Based on this formalism, the quasi-nonlinear evolution of the $f(\alpha)$ spectrum is calculated. The result shows good agreements with that of CfA data analysis. The final section is devoted to the summary.

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2 Multifractal as a statistical measure

The best way to introduce the concept of multifractal structure is to start with the partition function. Let V and r be the total volume of the sample and the size of the small cell, respectively. The number of the cells becomes M . Introducing the parameter q , we define the partition function as

$$Z(q, r) = \sum_{i=1}^M [p_i(r)]^q, \quad M = \frac{V}{r^d}, \quad (1)$$

where p_i denotes the probability to find the particles in the i -th cell. This quantity provides the biased view of the distribution, namely, the positive q strength the clump part and the negative q do opposite. Each biased distribution has the generalized dimension

$$D_q = \frac{1}{q-1} \frac{d \log Z(q, r)}{d \log r}. \quad (2)$$

It is easy to see that $D_q = d$ in case of the uniform distribution. It is a special case of the monofractal distribution. Of course, in general, D_q has different value for each q .

To reveal the geometrical picture of the multifractal distribution, let us recall that the scaling

$$p_i = r^{\alpha_i}, \quad (3)$$

is assumed in the usual definition of the multifractal, where α_i represents the local scaling exponent. For multifractal distribution, the set of the points with the exponent α has also the scaling with the exponent $f(\alpha)$:

$$n(\alpha) = r^{-f(\alpha)}. \quad (4)$$

Provided the above scaling properties, it is straightforward to obtain the relation

$$D_q = \frac{1}{q-1} [q\alpha_q - f(\alpha_q)], \quad (5)$$

which yields

$$\alpha_q = \frac{d}{dq} [(q-1)D_q] \quad (6)$$

and

$$f(\alpha_q) = \frac{d}{dq} [(q-1)D_q] - (q-1)D_q. \quad (7)$$

By eliminating the variable q from the above equations, $f(\alpha)$ spectrum can be derived.

In the cosmological context, the partition function can be connected with the count in cell as

$$Z(q, r) = \sum_i p_i^q = \left(\frac{L}{r}\right)^3 \sum_{N=0}^{\infty} \left(\frac{N}{N_0}\right)^q f(N), \quad (8)$$

where $f(N)$ is the probability to have N particles in the cell. Then the generalized dimensions are given by

$$D_q = \frac{1}{q-1} \left[-3 + \frac{d \log \frac{\langle N^q \rangle}{N_0^q}}{d \log r} \right]. \quad (9)$$

The remaining calculation for obtaining the $f(\alpha)$ spectrum can be performed easily.

3 Quasi-nonlinear evolution of $f(\alpha)$ spectrum

The non-linear evolution of the density fluctuations can be calculated by the Newtonian equations for the density field ρ . It is impossible to solve the above equations exactly. As we are considering the large

scale structure of the universe, the perturbative approach is viable. The perturbative expansion

$$\delta(\mathbf{x}, t) = \frac{\rho - \rho_0}{\rho_0} \quad (10)$$

$$= \sum_{n=1}^{\infty} \delta^{(n)}(\mathbf{x}, t) \quad (11)$$

can be solved iteratively. Assuming the random Gaussian nature of the density field, we can calculate the multi-point correlation functions in principle. For our purpose, the averaged quantities of those are necessary. For example,

$$\bar{\xi}_n = \frac{1}{V^n} \int d\mathbf{x}_1 d\mathbf{x}_2 \cdots d\mathbf{x}_n < \delta(\mathbf{x}_1, t) \delta(\mathbf{x}_2, t) \cdots \delta(\mathbf{x}_n, t) > . \quad (12)$$

How can we calculate the time evolution of these quantities? Let us use the spherical collapse model for this purpose. The reason is understood in the following way. [7] The equation describing the evolution of a spherically symmetric perturbation in an expanding universe is given by

$$\frac{d^2 R}{dt^2} = -\frac{GM}{R^2} = -\frac{4\pi G \rho}{3} R , \quad (13)$$

where ρ is the mass density of the over-dense homogeneous region. The equation describing the evolution of a background universe is given by

$$\frac{d^2 a}{dt^2} = -\frac{4\pi G \rho_0}{3} a , \quad (14)$$

where ρ_0 is the average mass density of the universe. The density perturbation

$$\delta = \frac{a^3}{R^3} - 1 \quad (15)$$

is governed by the following equation of motion

$$\ddot{\delta} + \mathcal{H} \dot{\delta} - \frac{4}{3} \frac{\dot{\delta}^2}{(1 + \delta)} = 4\pi G \rho_0 a^2 \delta (1 + \delta) , \quad (16)$$

where the dot denotes the derivative with respect to the conformal time τ defined as $a(t) d\tau = dt$.

On the other hand, the cosmological density perturbation is described by the fluid equations

$$\frac{\partial \delta}{\partial t} + \frac{1}{a} \nabla \cdot [(1 + \delta) \mathbf{v}] = 0 , \quad (17)$$

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{\dot{a}}{a} \mathbf{v} + \frac{1}{a} (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{a} \nabla \phi , \quad (18)$$

$$\nabla^2 \phi = 4\pi G a^2 [\rho - \rho_0(t)] , \quad (19)$$

where $\mathbf{v}$ and ϕ are the peculiar velocity and the gravitational potential, respectively. Using the conformal time and the expansion $\theta = \nabla \cdot \mathbf{v}$, the continuity equation becomes

$$\dot{\delta} + (1 + \delta)\theta = 0 . \quad (20)$$

Furthermore, the shear and the rotation are defined through the equation:

$$\nabla_i v_j = \frac{1}{3} \theta \delta_{ij} + \sigma_{ij} + \omega_{ij}, \quad \sigma_{ij} = \sigma_{ji}, \quad \omega_{ij} = -\omega_{ji} . \quad (21)$$

The Euler equation yields

$$\dot{\theta} + \mathcal{H} \theta + \frac{1}{3} \theta^2 + \sigma^{ij} \sigma_{ij} - \omega^{ij} \omega_{ij} = -4\pi G \rho_0 \delta a^2 . \quad (22)$$

Combining the above equations, we obtain

$$\bar{\delta} + \mathcal{H}\dot{\delta} - \frac{4}{3} \frac{\dot{\delta}^2}{(1+\delta)} = (\sigma^{ij}\sigma_{ij} - \omega^{ij}\omega_{ij} + 4\pi G\rho_0 a^2\delta)(1+\delta). \quad (23)$$

When we neglect the shear and the rotation, we will recover the result of the spherical collapse model. Fosalba and Gaztañaga have shown that this approximation is good in the quasi-nonlinear regime. Hence, we will assume the local local approximation from now on. The explicit formula is

$$\delta = \sum_{n=1}^{\infty} \frac{\nu_n}{n!} \delta_l^n, \quad (24)$$

where $\nu_1 = 1, \nu_2 = 34/21, \nu_3 = 682/189, \dots$

Using the local approximation method, we can calculate the hierarchical coefficients S_n . At the tree level approximation, higher order correlation functions can be written as

$$\bar{\xi}_n = S_n \bar{\xi}_2^{n-1}, \quad (25)$$

where S_n is the hierarchical coefficients. The numerical value has been obtained [8] as

$$S_3 = 4.86, S_4 = 45.9, S_5 = 656, S_6 = 12700, \quad (26)$$

if we neglect the smoothing effect. In our preliminary analysis, we have used this value. For the complete analysis, we must take into account the smoothing effects. It should be noted that these values are less dependent on the cosmological models.

It is generally believed that the origin of the cosmic structure is explained by the inflationary scenario. The density field is generated by the quantum fluctuations. It evolves to the present structure. One missing link is the process of the galaxy formation. Here, we model this process by the Poisson model, i.e., each galaxy is distributed by the Poisson process with the probability

$$\delta P = \rho(\mathbf{r})\delta V, \quad (27)$$

where ρ is the random density field and δV is the small volume.

The characteristic function can be written as

$$\psi(t) = \sum_{N=0}^{\infty} P_N e^{Nt} = \langle e^{Nt} \rangle. \quad (28)$$

This quantity is also calculated as

$$\psi(t) = \langle \exp(e^t - 1) \int \rho dV \rangle \quad (29)$$

Thus, we obtain

$$\langle e^{Nt} \rangle = \langle \exp(e^t - 1) \int \rho dV \rangle. \quad (30)$$

This relation provides the infinite sequence of the relations between the cumulant of the discrete process and that of the continuous density field.

Now, it is possible to deduce the moments of number counts as

$$\langle N \rangle = \langle nV \rangle, \quad (31)$$

$$\langle N^2 \rangle = \langle nV + (nV)^2(1 + \bar{\xi}) \rangle, \quad (32)$$

$$\langle N^3 \rangle = \langle nV + 3(nV)^2(1 + \bar{\xi}) + (nV)^3(1 + 3\bar{\xi} + S_3\bar{\xi}^2) \rangle, \quad (33)$$

$$\langle N^4 \rangle = \langle nV + 7(nV)^2(1 + \bar{\xi}) + 6(nV)^3(1 + 3\bar{\xi} + S_3\bar{\xi}^2) + (nV)^4(1 + 6\bar{\xi} + 4S_3\bar{\xi}^2 + 3\bar{\xi}^2 + S_4\bar{\xi}^3) \rangle, \quad (34)$$

$$\langle N^5 \rangle = \langle nV + 15(nV)^2(1 + \bar{\xi}) + 25(nV)^3(1 + 3\bar{\xi} + S_3\bar{\xi}^2) + 10(nV)^4(1 + 6\bar{\xi} + 4S_3\bar{\xi}^2 + 3\bar{\xi}^2 + S_4\bar{\xi}^3) + (nV)^5(1 + 10\bar{\xi} + 10S_3\bar{\xi}^2 + 15\bar{\xi}^2 + 10S_3\bar{\xi}^3 + 5S_4\bar{\xi}^3 + S_5\bar{\xi}^4) \rangle, \quad (35)$$

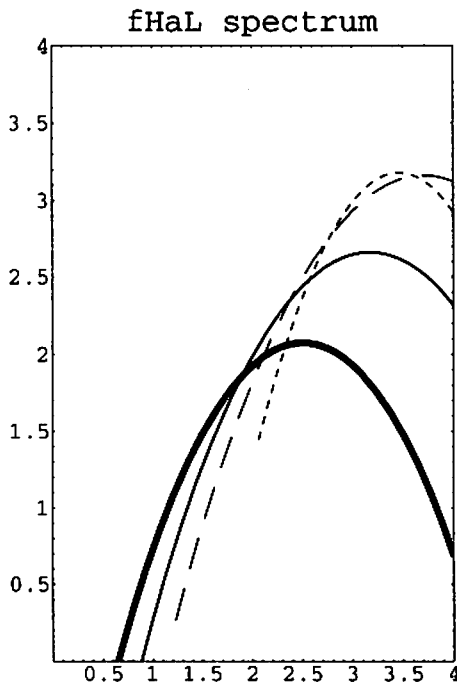


Figure 1: The solid line represents the $f(\alpha)$ spectrum at present calculated by the perturbation theory. It fits the result of CfA data provided the CDM spectrum. Other lines represent the time evolution of the $f(\alpha)$ spectrum.

To calculate the $f(\alpha)$ spectrum, we need to know $\bar{\xi}$ which depends on the cosmological models. It can be written as

$$\bar{\xi} = \sigma_8 \left(\frac{R}{8} \right)^{-n-3}, \quad (36)$$

where σ_8 is determined by the COBE normalization and has the value of the order of unity. The unit of the length is Mpc. The spectral index n is chosen as -1.4 to be consistent with the CDM model. Using the above formula, we can draw the time evolution of the $f(\alpha)$ spectrum. We have assumed the CDM model and set the normalization at present. The shape of the $f(\alpha)$ spectrum shows agreements with that of the CfA data. [9, 10]

A few comments are in order. When we evaluate the $f(\alpha)$ spectrum, the point where we take the differentiation with respect to r is necessarily chosen. As this point, we take the value consistent with the value which is regarded in a scaling region in the analysis of the CfA data. It is needed to clarify to what extent this procedure is valid. We have approximated the variance of the density field of CDM model by a single power at the scale we are interested in. And we have used the COBE normalization of the spectrum. The result would be dependent on this choice. Hence, a systematic and detailed analysis is required to obtain the final conclusion.

4 Summary

We have investigated the quasi-nonlinear evolution of the multifractal statistics. What we have shown is that the perturbation theory can explain the multifractal structure of the universe if we assume the CDM model.

Here, we have taken the cosmological principle as a fundamental principle. Of course, the attitude to regard our universe as multifractal is possible. A physical scenario which leads to such structure is discussed by the author. [11]

To understand if the multifractal $f(\alpha)$ spectrum is really useful, further investigations are necessary.

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Averaging

— From a point of view of an exact solution —

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Abstract

The spherically symmetric dust solution, known as the Lemaitre-Tolman-Bondi solution, with spatial topology S^3 is examined to see on what conditions an “averaging” is justified and what is the meaning of the averaging. In particular, we show that if we think of the total spatial volume V as a functional in the space of solutions, the FLRW solution is an “extremal point” for V for particular variations. This may give a nice meaning of the averaging.

1 Introduction

As we can see stars, galaxies, clusters, and superclusters, our Universe is *not* homogeneous, but has clumpy structures. Nevertheless, one expects that the homogeneous and isotropic universe model, known as the Friedmann-Lemaître-Robertson-Walker (FLRW) model reflects averaged nature of the true Universe over a large scale. In particular, it has been expected that the global expansion of the Universe is well approximated by the FLRW model with the energy distribution given by the volume average. However, this “averaging hypothesis” is never justified in a trivial way [1], since Einstein’s equation is highly nonlinear. In particular, averaging over a (spatial) volume does not commute with the time-evolution in general, so averaged initial data can develop in time in a quite different way from the true data. One is therefore required to clarify the meaning of an averaging and establish the domain of applicability of it. This kind of problem now has been recognized as an important fundamental problem in relativity and cosmology. In spite of many efforts [2, 3, 4, 5, 6, 7, 8], however, the averaging has not been well understood.

Our approach here is quite different from the conventional ones because we use a space of solutions. We are interested in the dynamics of the scale factor a , which is, if the space is closed, equivalent to the dynamics of the total volume V in the sense of $V \propto a^3$. We use the term “averaging” in the restricted sense of “matching with an FLRW solution”. Note that if given a space of solutions P spanned by some arbitrary functions, we can think of V as a functional on it. Our averaging relies on the following specialty of the FLRW solution which shall be shown to be true by giving an explicit example. That is, there exists a subspace $\tilde{P} \subset P$ containing the FLRW solution such that the FLRW solution is an *extremal point* for V with respect to variations in $\tilde{P}$. If there exists such a space $\tilde{P}$, all the solutions in $\tilde{P}$ which are not so far from the FLRW solution virtually manifest the same time-development of V , hence of a , as that of the FLRW solution. This implies that a well-defined averaging procedure exists for the solutions in $\tilde{P}$. On the other hand, for the solutions in $P - \tilde{P}$, any simple averaging, i.e., any matching with an FLRW solution is invalid. We show that there does exist such a $\tilde{P}$, in the case of spatially spherically symmetric and spatially S^3 solutions with dust matter.

2 The model

The metric is written with the synchronous and comoving coordinates as:

$$ds^2 = d\tau^2 - e^{\lambda(\tau, R)} dR^2 - r^2(\tau, R)(d\theta^2 + \sin^2\theta d\varphi^2), \quad (1)$$

where $\lambda(\tau, R)$ and $r(\tau, R)$ are the functions to be determined from the Einstein equation. The general solution to this metric is well known as the Lemaitre-Tolman-Bondi (LTB) solution (see e.g. [9, 10]),

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which possesses three arbitrary functions $f(R)$, $F(R)$, and $\tau_0(R)$. With these the function λ is given by $e^\lambda = r'^2/(1 + f(R))$, where the dash stands for the derivative with respect to R . On the other hand, the function r is given in three separate forms, depending upon whether the arbitrary function $f(R)$ is positive, negative, or zero, and each solution possesses the FLRW limit of negative ($k = -1$), positive ($k = 1$), and flat ($k = 0$) constant curvature, respectively. (Such a limit is achieved when $r(\tau, R)$ separates as $r(\tau, R) = \Phi(\tau)\Psi(R)$ [10].) Since our model is spatially S^3 , the spatial manifold does admit a constant curvature limit and it should be positive. So, we should choose the negative sign of $f(R)$, for which the the function r is given by

$$r = \frac{F}{-2f}(1 - \cos \eta), \quad \tau - \tau_0(R) = \frac{F}{2(-f)^{3/2}}(\eta - \sin \eta). \quad (2)$$

Since each Killing orbit ($R = \text{constant}$) degenerates at time $\tau = \tau_0(R)$, the arbitrary function $\tau_0(R)$ is interpreted as the function of “big bang time”. We just for simplicity assume a “simultaneous big bang”, and put $\tau_0 = \text{constant} = 0$, hereafter.

Now, the local solution of our interest is given by Eq.(2) with $\tau_0(R) = 0$. The remaining task to complete our model is to impose on the local solution the boundary conditions appropriate for the spatial topology S^3 . To understand a spherically symmetric three-sphere, it is good to think of the three-sphere as a sum of two balls, $S^3 \simeq D^3 \cup D^3$. (This is similar to the decomposition of two-sphere into two disks, $S^2 \simeq D^2 \cup D^2$, achieved by cutting the two-sphere along the equator.) A spherically symmetric ball is easy to understand; it has a symmetry center and is foliated about the center by spheres which are Killing orbits. The spherically symmetric three-sphere is therefore the sum of two such balls. The boundaries of the two balls should be identified. An important point is that a spherically symmetric three-sphere has *two* symmetry centers, which are degenerate Killing orbits, as well. We label these points as $R = 0$ and $R = \pi$, so the relevant region of the coordinate R is $[0, \pi]$. The first condition we should impose is therefore

$$r = 0, \text{ at } R = 0 \text{ and } \pi. \quad (3)$$

Furthermore, we must impose some regularity conditions. An efficient way to see this is to calculate the scalar curvature $\mathcal{R}$ and the curvature scalar polynomial $I \equiv \mathcal{R}_{abcd}\mathcal{R}^{abcd}$, where $\mathcal{R}_{abcd}$ is the curvature tensor. Using our local solution, we find $\mathcal{R} = \varepsilon$ and $I = 12(F/r^3)^2 - 8(F/r^3)\varepsilon + 3\varepsilon^2$, where $\varepsilon = F'/(r'r^2)$ is the energy density of dust. Since these scalars must be finite, we have

$$\frac{F}{r^3} < \infty, \quad R \in [0, \pi] \quad (4)$$

and

$$\varepsilon = \frac{F'}{r'r^2} < \infty, \quad R \in [0, \pi]. \quad (5)$$

While condition (4) is necessary to avoid the conical singularity possibly generated at $R = 0$ and π , condition (5) is imposed to avoid [11] the well-known shell-crossing singularity. Finally, we have to impose the following coordinate condition, which is necessary for the coordinates to span the spatial manifold S^3 well;

$$e^\lambda = \frac{r'^2}{1 + f} > 0, \quad R \in [0, \pi]. \quad (6)$$

Without this condition, we possibly have fictitious solutions.

The solution to the four conditions (3)–(6) for $f(R)$ and $F(R)$ is expressed by the following three properties. First, both $f(R)$ and $F(R)$ approach zero at the boundaries $R = 0$ and π , and near these points they behave like [12]

$$f(R) \propto \begin{cases} -R^\alpha & (R \rightarrow 0) \\ -(\pi - R)^\alpha & (R \rightarrow \pi) \end{cases}, \quad (7)$$

and

$$F(R) \propto \begin{cases} R^3 & (R \rightarrow 0) \\ (\pi - R)^3 & (R \rightarrow \pi) \end{cases}, \quad (8)$$

where $\alpha \geq 2$. Second, $f(R)$ takes exactly one extremum at the interior $(0, \pi)$ and the value at the extremum is -1 . Third, $F(R)$ also takes exactly one extremum, and the extremal point is the same as that for $f(R)$.

All the properties for $F(R)$ above is satisfied if we introduce another arbitrary function $\phi(R)$, and put

$$F(R) = \int_0^R (-f')(-f)^{\frac{1}{\alpha}-1} \phi(R') dR', \quad (9)$$

where $\phi(R)$ must be a positive function on $[0, \pi]$ such that it makes $F(\pi) = 0$. Now, the independent arbitrary functions are regarded as $f(R)$ and $\phi(R)$. Let us write $-f(R) = g(R)^\alpha$. In this expression the FLRW limit is achieved when $\alpha = 2$ and $\phi(R) = \text{constant}$, for arbitrary $g(R)$. In fact, $g(R)$ is found to be related to coordinate transformation of type $R \rightarrow \gamma(R)$ [13], so physically *the solution is controlled only by the sole positive function $\phi(R)$ with one real parameter $\alpha \geq 2$* . We shall, hereafter, fix $g(R)$ to $\sin R$, thereby $f(R) = -\sin^\alpha R$. This completes the presentation of our model.

3 Averaging

Recall that the FLRW model (with dust matter) is the one-parameter solution for which the scale factor $a(\tau)$ is given by

$$a(\tau) = a_0(1 - \cos \eta), \quad \tau = a_0(\eta - \sin \eta), \quad (10)$$

where a_0 is the parameter. We have to determine a matched value of a_0 from a given inhomogeneous solution (ϕ, α) . Since the solutions for $\phi(R) = \text{constant}$ with $\alpha = 2$ correspond to the FLRW limits, if we separate $\phi(R)$ into a “background” $\bar{\phi} = \text{constant}$ and the “fluctuation” $\delta\phi(R)$, $\phi(R) = \bar{\phi} + \delta\phi(R)$, we will establish an averaging map $(\phi(R), \alpha) \rightarrow (\bar{\phi}, 2)$. The parameter a_0 is then simply related to the constant $\bar{\phi}$ by $a_0 = \bar{\phi}/3$. We determine $\bar{\phi}$ by requiring

$$E[\phi; \alpha] = E[\bar{\phi}; 2], \quad (11)$$

where $E[\phi; \alpha]$ is the total energy of matter for given ϕ and α , which is conserved;

$$\begin{aligned} E[\phi; \alpha] &= 4\pi \int_0^\pi \epsilon e^{\lambda/2} r^2 dR \\ &= 4\pi \alpha \int_0^\pi \frac{|\cos R|}{\sqrt{1 - \sin^\alpha R}} \sin^2 R \cdot \phi \cdot dR. \end{aligned} \quad (12)$$

From this, we find

$$\bar{\phi} = \frac{\alpha}{\pi} \int_0^\pi \frac{|\cos R|}{\sqrt{1 - \sin^\alpha R}} \sin^2 R \cdot \phi \cdot dR. \quad (13)$$

It is not guaranteed *a priori* that the FLRW solution $(\bar{\phi}, 2)$ selected in this way possesses a similar dynamical property to the original solution (ϕ, α) . In fact, we will see that only the solutions with $\alpha = 2$ possess such a property. As we mentioned, we compare the two solutions through $V(\tau)$, the total volume as a function of time. In fact, the only dynamical content of the FLRW model is the scale factor $a(\tau)$, which can be regarded as the (cubic root of the) total volume. Moreover, considering V instead of a is theoretically advantageous, since it is always well defined for any inhomogeneous spacetime that is spatially closed. [14] Now, note that the volume $V(\tau)$ can be viewed as a functional $V[\phi; \alpha]$ on the space of solutions, possessing its value on real-valued function;

$$V[\phi; \alpha](\tau) = 4\pi \int_0^\pi e^{\lambda/2} r^2 dR. \quad (14)$$

If the inhomogeneous solution (ϕ, α) is not so far from the FLRW solution, we can compare the two solutions by evaluating the “variation” δV at the FLRW point $(\bar{\phi}, 2)$. We first calculate it, fixing α to 2. Varying $V[\phi; 2]$ with respect to ϕ and evaluating at $\phi = \text{constant}$, we find

$$\left. \frac{\delta V}{V} \right|_{\phi = \text{const.}} = \frac{6}{\pi} \frac{\mathcal{A}(\eta)}{(1 - \cos \eta)^2} \int_0^\pi \sin^2 R \cdot \delta\phi \cdot dR, \quad (15)$$

where $\mathcal{A}(\eta) \equiv (1 - \cos \eta)^2 - \sin \eta (\eta - \sin \eta)$. We have factored out the dependence of η , since η depends only on time τ for the FLRW solutions. When $\alpha = 2$, from Eq.(11) and the linearity $E[\phi_1 + \phi_2, \alpha] = E[\phi_1, \alpha] + E[\phi_2, \alpha]$, we have $E[\delta\phi; 2] = 0$, implying

$$I[\delta\phi] \equiv \int_0^\pi \sin^2 R \cdot \delta\phi \cdot dR = 0. \quad (16)$$

This condition is equivalent to requiring that the variation $\delta\phi$ be taken so as to leave the energy E (or equivalently a_0) unchanged. Substituting the last condition into Eq.(15), we find

$$\left. \frac{\delta V}{V} \right|_{\phi=\bar{\phi}} = 0. \quad (17)$$

The FLRW solution is therefore an extremal point of the functional V , provided that the variations are taken in the subspace $\alpha = 2$ with the condition $I[\delta\phi] = 0$.

Eq.(17) implies that the inhomogeneous solution $(\phi, 2)$ is well approximated by the FLRW solution with $a_0 = \bar{\phi}/3$, where

$$\bar{\phi} = \frac{2}{\pi} \int_0^\pi \sin^2 R \cdot \phi \cdot dR. \quad (18)$$

FIG.1 shows an example. In this, profiles of the metric components r and e^λ , those of the energy density ϵ , and the time-development of the total volume V are shown with solid lines for $\phi = 1 + \frac{1}{10}(\frac{1}{3} \sin 5R - \sin 7R)$, and with dashed lines for the averaged solution $\phi = \bar{\phi} = 1$. The time-developments of V for the two solutions coincide so well that the two curves in the figure look like a single one.

We can analytically estimate the accuracy of the coincidence of the two $V(\tau)$ from the second variation $\delta^2 V$. Namely, factoring out, for convenience, a small parameter ϵ from $\delta\phi$ as $\phi = \bar{\phi} + \epsilon\delta\phi$, we can expand $V[\phi](\equiv V[\phi; 2])$ in power of ϵ ;

$$V[\phi] = V_0 + \epsilon\delta V[\delta\phi] + \frac{1}{2}\epsilon^2\delta^2 V[\delta\phi] + \dots, \quad (19)$$

where $V_0 \equiv V[\bar{\phi}]$. Since the first order term now vanishes, the second order term $\frac{1}{2}\epsilon^2\delta^2 V[\delta\phi]$ decides the major difference of the two, i.e., $v \equiv |V - V_0|/V_0 \simeq \frac{1}{2}\epsilon^2\delta^2 V/V_0$. After straightforward calculations, we find

$$\frac{\delta^2 V}{V_0} = -\frac{9}{2\pi^2\bar{\phi}^2} \frac{\mathcal{T}(\eta)}{(1 - \cos \eta)^3} J[\delta\phi], \quad (20)$$

where

$$J[\delta\phi] \equiv \int_0^\pi \frac{(\delta F)^2}{\sin^2 R \cos^2 R} dR, \quad (21)$$

and $\mathcal{T}(\eta) \equiv 15 - 16 \cos \eta + \cos 2\eta - 12\eta \sin \eta + 2\eta^2(1 + 2 \cos \eta)$. δF is the variation of F . By estimating this second order term, we find that the accuracy v is better than 10^{-3} at the expansion phase in the example shown in FIG.1. It is worth noting that the term $\delta^2 V[\delta\phi]$ is always negative, since $\mathcal{T}(\eta) \geq 0$ and $J[\delta\phi] \geq 0$. The true volume $V(\tau)$ is therefore always smaller than the volume $V_0(\tau)$ of the corresponding FLRW model.

We have calculated the variations in the subspace $\alpha = 2$. We can check that the variation with respect to α do *not* vanish, so a solution with $\alpha \neq 2$ will show a different type of dynamical property from any FLRW solution. This is in a sense reasonable, since the matter distribution for a solution with $\alpha \neq 2$ seems inhomogeneous even over the largest scale. At an early time ($\tau \rightarrow 0$), the profile of ϵ is like $\text{const}_1 + \text{const}_2 \times \sin^{\alpha-2} R$, in the case $\phi = \text{const}$. When $\phi \neq \text{const}$, the profile is perturbed, but the one over the largest scale is dominated only by α .

4 Conclusion

We have seen that for the spherically symmetric dust solution with spatial topology S^3 , the space P of solutions is spanned by the positive function $\phi(R)$ and the parameter $\alpha \geq 2$. The (time-development of

the) total volume V has been defined on P as a functional. We have seen that we can characterize the FLRW solution with a_0 as an extremal point in P for the functional V , with respect to variations such that $I[\delta\phi] = 0$ and $\delta\alpha = 0$. The solutions $(\bar{\phi} + \delta\phi, 2)$ for which $I[\delta\phi] = 0$ are therefore well approximated by the FLRW solution $(\bar{\phi}, 2)$, which can be interpreted as the averaged solution for them.

Our view of averaging is apparently applicable to a general case. We should fix a closed spatial manifold M which admits a constant curvature metric, say, S^3 , T^3 , or a closed hyperbolic manifold. There will be the space P of solutions on the spacetime manifold $M \times \mathbb{R}$ for a fixed matter source, say, dust. Then, there will be a subspace $\tilde{P}$ such that the functional V becomes extremal at the FLRW point for variations in $\tilde{P}$. The correspondence to the FLRW point defines the averaging for the solutions in $\tilde{P}$.

Another examples and a relation to observables like the distance-redshift relation [15] will be worth investigating further.

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- [12] Here, we have used freedom of coordinate transformation of type $R \rightarrow \gamma(R)$ to fix the leading power of r at the boundaries to the unity, i.e., $r \propto R$, ($R \rightarrow 0$), and $r \propto \pi - R$, ($R \rightarrow \pi$).
- [13] See note [12]. There, we have used the coordinate transformation of type $R \rightarrow \gamma(R)$ only at the vicinities of the boundaries, so the same freedom remains at the interior $(0, \pi)$. We have exhausted this here.
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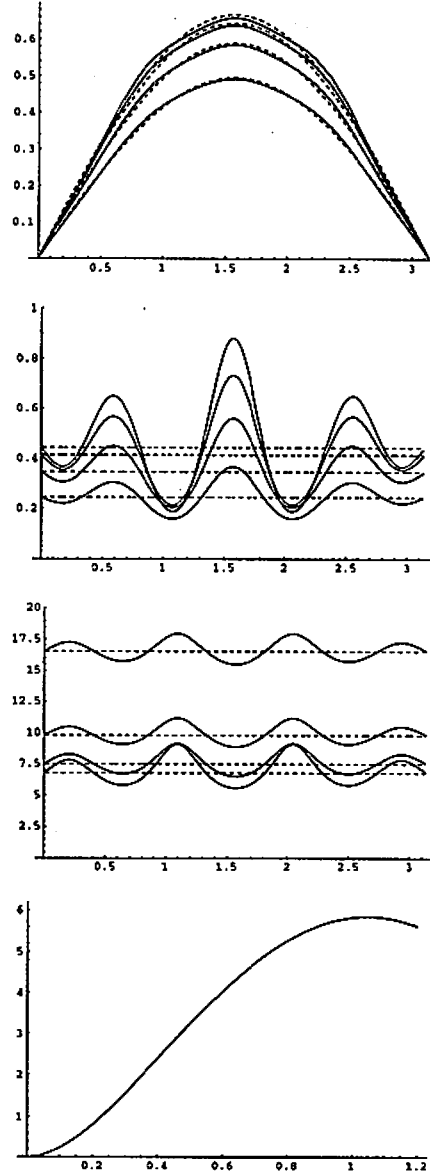


Figure 1: The solution with $\phi = 1 + \frac{1}{10}(\frac{1}{3} \sin 5R - \sin 7R)$ (the solid lines) and the averaged solution $\phi = \bar{\phi} = 1$ (the dashed lines). The time-developments of the total volume V for the two solutions almost coincide.

Topological inflation motivated by M-theory

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Abstract

We consider the possibility that higher-curvature corrections could drive inflation after the compactification to four dimensions. Assuming that the low-energy limit of the fundamental theory is eleven-dimensional supergravity to the lowest order, including curvature corrections and taking the descent from eleven dimensions to four via an intermediate five-dimensional theory, we may obtain a simple model of inflation in four dimensions. The effective degrees of freedom are two scalar fields and the metric. The scalars arise as the large five-dimensional modulus and the self-interacting conformal mode of the metric. The effective potential has a local maximum which admits topological inflation. We show that the model can resolve cosmological problems and provide a mechanism for structure formation with very little fine tuning.

1 Introduction

One of the central problems confronting inflation [1] is the identity of the inflaton, the field responsible for driving inflation, and the manner in which it fits in with unified field theories and/or string theory, notably M theory. The most stringent constraints on inflation arise from the observations of the cosmic microwave background. The naive interpretation of the COBE and other data on fluctuations in the microwave background radiation is that the density of vacuum energy during inflation is $V \sim (10^{16} \text{ GeV})^4$, so that inflation is associated with an energy scale $V^{1/4} \sim 10^{16} \text{ GeV}$. One of the key points in the application of M theory to phenomenology is the reconciliation of the bottom-up calculation of $M_{GUT} \sim 10^{16} \text{ GeV}$ with the string unification scale, which is close to the four-dimensional Planck mass scale $M_4 \sim 10^{19} \text{ GeV}$. This is achieved by postulating a large fifth dimension $\mathcal{R}_5 \gg M_{GUT}^{-1}$, which is not felt by the gauge interactions, but causes the gravitational interactions to rise with energy much faster than in the conventional four dimensions. In this type of scenario, one could expect that inflation should be considered within a five-dimensional framework.

It is now known that ten-dimensional strongly-coupled heterotic string theory is related through dualities to weakly-coupled type I string theories, as well as to eleven-dimensional M theory [2]. In each case, the corresponding thresholds imply the presence of one large scale dimension below the unification point [2, 3]. Within this general five-dimensional framework, two favored ranges for the magnitude of $\mathcal{R}_5$ can be distinguished. One is relatively close to M_{GUT}^{-1} : $\mathcal{R}_5^{-1} \sim 10^{12}$ to 10^{15} GeV , and the other could be as low as $\mathcal{R}_5^{-1} \sim 1 \text{ TeV}$ [4]. The latter is motivated in particular by Scherk-Schwarz models of supersymmetry breaking, in which the gravitino mass $M_{3/2} \sim \mathcal{R}_5^{-1}$. In this latter case, the large dimension is not necessarily the conventional fifth dimension of M theory. Indeed, in models studied in [4] the large dimension may be related to what is normally considered as one of the six “small” dimensions that is conventionally compactified *à la* Calabi-Yau.

In this work, we consider a variant of the Starobinsky model [5] based on R^4 curvature terms, as curvature-squared terms are not known to be present in the action of the full five- or eleven-dimensional theory. We assume throughout that the remaining six dimensions are fixed¹. In a four-dimensional context, Maeda [8] derived the potentials for a general $R + R^n$ theory. For $n \neq 2$, the potential is not flat, but rather shows a peak and a well-defined minimum. We argue that it is possible for this potential to inflate in a so-called topological manner [9]. As we discuss, a fully successful inflationary model of this sort would still require a potential for the dilaton. Whilst this model results in an inflationary stage with a guaranteed exit, the magnitude and spectral index of the resulting density fluctuations force us

¹For recent discussions of fully eleven-dimensional cosmological solutions, see [6, 7].

to consider some possible additional ingredients. Either curvature-squared terms that might appear at the level of the five-dimensional theory or quantum corrections to $T_{\mu\nu}$ ² would be sufficient to provide a self-consistent inflationary model.

Although such a solution is not directly derived from string or M theory, it captures several elements that we expect to be present in the low-energy theory. As such, it represents a novel and motivated possible solution to the problem of inflation in string theory, which may hold some promise.

To model the possibility of curvature-driven inflation, we assume that the five-dimensional action contains a scalar R^4 contribution, and perform a dimensional reduction to four dimensions, ignoring the boundaries. In the context of the theory with walls, this merely means that we assume that the bulk can be foliated by identical and mutually non-interacting copies of the wall. Alternatively, it is clear that this dimensional reduction is identical to the standard Kaluza-Klein reduction on a circle. This approach produces a four-dimensional theory with two scalar fields, an inflaton and a dilaton. The inflaton potential has a maximum, which we will show to be sufficiently flat to support inflation, in a manner resembling the topological inflationary scenario [9].

2 Inflation In Four Dimensions

In this section, we will begin the investigation of the possibility of higher-curvature-driven inflation. We first give details of the dimensional reduction from five dimensions to four dimensions, in order to derive the four-dimensional effective theory. The pure gravity sector of the five-dimensional action which we will consider is

$$S = \int d^5x \sqrt{G_5} \left\{ \frac{M_5^3}{16\pi} R_5 + \alpha M_5^{-3} R_5^4 \right\}, \quad (1)$$

where M_5 is the five-dimensional Planck mass. In the context of the eleven-dimensional theory, our five-dimensional Planck mass is related to the eleven-dimensional Planck mass by $M_5^3 = M_{11}^9 V_6$, where V_6 is the six-volume of the compactified space. We assume that $V_6^{-1/6} \sim M_{11} \sim M_{\text{GUT}}$ in general. The scale of the fifth dimension is then given by $\mathcal{R}_5^{-1} \sim \frac{4}{\alpha} M_{11}^3 M_4^{-2}$ [4]. We note that this effective action is not exactly what one finds after dimensional reduction of the eleven-dimensional supergravity with quartic corrections on an interval S^1/Z_2 , which would contain couplings to two ten-dimensional boundaries with R^2 terms (see, e.g., [7]). Thus, our action (1) is not a direct descendant of a known reduction of M theory on an interval. But such actions may nevertheless arise in some compactifications of the theory, and are simple enough to illustrate our main point.

Let us first outline the reduction procedure we will follow here. We first conformally transform the action (1), following [8], to represent it as a five-dimensional gravity with a minimally coupled self-interacting scalar field. The scalar arises because the conformal mode of the metric becomes dynamical thanks to the R^4 term [10]. Then we dimensionally reduce this action to four dimensions and apply another, four-dimensional, conformal transformation, to put the resulting four-dimensional action in the canonical form. This will produce another scalar field, the compacton (or dilaton) of the four-dimensional theory, which is related to the size of the fifth dimension.

We now give the formulae appropriate for this procedure. The conformal transformation which brings (1) to the Einstein form is

$$\tilde{G}_{AB} = |1 + 64\pi\alpha M_5^{-6} R_5^3|^{\frac{2}{3}} G_{AB}, \quad (2)$$

and the action is [8]

$$\tilde{S}_5 = \int d^5x \sqrt{\tilde{G}_5} \left\{ \frac{M_5^3}{16\pi} \tilde{R}_5 - \frac{1}{2} \tilde{G}^{AB} \partial_A \tilde{\chi} \partial_B \tilde{\chi} - U_5(\tilde{\chi}) \right\}. \quad (3)$$

where the “intermediate” scalar field and its potential are

$$\tilde{\chi} \equiv \frac{2}{\sqrt{3}} \ln |1 + 64\pi\alpha M_5^{-6} R_5^3|, \quad U_5(\tilde{\chi}) = \frac{3M_5^5}{256\pi^{\frac{4}{3}} \alpha^{\frac{1}{3}}} e^{-\frac{\sqrt{3}}{6} \tilde{\chi}} \left(e^{\frac{\sqrt{3}}{2} \tilde{\chi}} - 1 \right)^{\frac{4}{3}}. \quad (4)$$

²The former can also be derived as a quantum correction to the energy momentum tensor.

with $\tilde{\kappa} \equiv \sqrt{(8\pi)/(M_5^3)}$. The standard Kaluza-Klein compactification Ansatz, with the simultaneous conformal transformation of the four-dimensional metric to the canonical form, is

$$d\tilde{s}_5^2 = e^{-\sqrt{\frac{2}{3}}\kappa\phi} g_{\mu\nu} dx^\mu dx^\nu + \frac{M_4^4}{M_5^6} e^{2\sqrt{\frac{2}{3}}\kappa\phi} dz^2, \quad (5)$$

where $g_{\mu\nu}$ is the four-dimensional Einstein-frame metric. The action (3) then reduces to the following four-dimensional action:

$$S_4 = \int d^4x \sqrt{g_4} \left\{ \frac{M_4^2}{16\pi} R_4 - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - U(\phi, \chi) \right\}. \quad (6)$$

where $\chi = M_4 M_5^{-\frac{1}{2}} \tilde{\chi}$. In terms of these fields, the potential is given by

$$U(\phi, \chi) = \frac{3M_4^2 M_5^2}{256\pi^{\frac{4}{3}} \alpha^{\frac{1}{3}}} e^{-\sqrt{\frac{2}{3}}\kappa\phi} e^{-\frac{2\sqrt{3}}{3}\kappa\chi} \left(e^{\frac{\sqrt{3}}{2}\kappa\chi} - 1 \right)^{\frac{4}{3}} \equiv \frac{3M_4^2 M_5^2}{256\pi^{\frac{4}{3}} \alpha^{\frac{1}{3}}} e^{-\sqrt{\frac{2}{3}}\kappa\phi} V(\chi). \quad (7)$$

Before we undertake a detailed investigation of (6), we note that the effective four-dimensional potential (7) is a product of an exponential and a function with a maximum. If we ignore the variation of χ , we recall that in a universe dominated by a scalar field with an exponential potential $V(\phi) \propto e^{-\lambda\kappa\phi}$, and foliated by flat spatial hyperplanes, the expansion of the FRW scale factor $a(t)$ obeys a power law $a(t) \propto t^p$, with the power index given by $p = 2/\lambda^2$ [11].

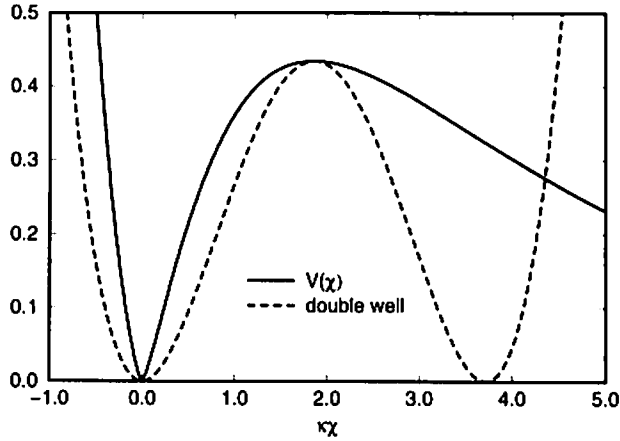


Figure 1: Comparison of the potential $V(\chi)$ produced by a conformal rescaling of the R^4 term in the five-dimensional action and a simple double-well potential.

Next, we ignore the dilaton factor $e^{-\sqrt{\frac{2}{3}}\kappa\phi}$ in the potential and consider the dynamics of the χ field. The χ -dependent factor in the potential, $V(\chi)$, is depicted in Fig. 1. It has a global minimum $V = 0$ at $\chi = 0$, a local maximum at $\chi = \chi_m = 2 \ln 5 / (\sqrt{3}\kappa) = 0.37 M_4$, and diverges as $V(\chi) \propto e^{-\frac{2\sqrt{3}}{3}\kappa\chi}$ for $\chi \rightarrow -\infty$. This is too steep for power-law inflation. For $\chi \rightarrow \infty$, $V(\chi)$ asymptotically approaches zero

with $V(\chi) \propto e^{-\frac{\sqrt{3}}{6}\kappa\chi}$, which may be flat enough for inflation but it will lead to an unphysical universe with a runaway behavior towards the regime of extremely large curvature, in terms of the original five-dimensional description.

As a result, chaotic inflation [12] with a large initial value of χ is impossible here. Nevertheless there may remain the possibility to realize inflationary expansion in this model by using the potential energy around the local maximum, $V(\chi_m)$, as in the topological inflation scenario of Linde and Vilenkin [9]. In this scenario, if the scalar field $\chi(x)$ is randomly distributed initially with a large dispersion, some part of the universe will roll to $\chi = 0$, while in other parts it will run away to infinity. Between any two such regions there will appear domain walls, containing a large energy density, $\rho \sim V(\chi_m)$. If the wall is thicker than the Hubble radius of this energy density, there will exist a sufficiently large quasi-homogeneous region, filled with large potential energy, where inflationary expansion naturally sets in.

The condition for a domain wall to inflate has been investigated numerically in [13] for the case of a simple double-well potential, $V_{dw}(\phi) = \frac{\lambda}{4}(\phi^2 - \eta^2)^2$. There it has been found that a domain wall will undergo inflation if η exceeds a critical value $\eta_{cr} = 0.33M_4$, regardless of the value of λ . When $\eta = \eta_{cr}$, the ratio of the thickness of the wall - characterized by the curvature scale of the potential at the origin, $r_w \equiv (V''_{dw}(\phi = 0))^{-\frac{1}{2}}$, to the horizon $H^{-1} = (\kappa^2 V_{dw}(\phi = 0)/3)^{-\frac{1}{2}}$ is given by $r_w H = 0.48$, and is again independent of λ . An explicit check shows that, in our model, the distance between the potential minimum and the local maximum, $\chi_m = 0.37M_4$, exceeds η_{cr} . Furthermore, the ratio of the characteristic thickness of the wall to the horizon scale is given by $r_w H = (\kappa^2 V(\chi_m)/3V''(\chi_m))^{\frac{1}{2}} = 4/\sqrt{15} \simeq 1.0$, which is larger than the critical case of [13]. In Fig. 1 we have also depicted a double-well potential which has the same global minimum and the local maximum as $V(\chi)$. As is seen there, the latter is much flatter than the former around the local maximum.

Having seen that the potential $V(\chi)$ is flat enough around the local maximum, we now return to the exact form of the potential given in (7), and take the dilaton factor into account. This does not change χ_m nor $r_w H$. However, the rolling dilaton field hampers exponential inflation. Since χ moves slowly near $\chi = \chi_m$, as compared to the dilaton, we practically have an exponential potential

$$U(\varphi, \chi) = \frac{3M_4^2 M_5^2}{256\pi^{\frac{4}{3}}\alpha^{\frac{1}{3}}} e^{-\sqrt{\frac{3}{2}}\kappa\varphi} V(\chi_m) \equiv U_0 e^{-\sqrt{\frac{3}{2}}\kappa\varphi}. \quad (8)$$

The ensuing solution for the FRW scale factor obeys the power law $a(t) \propto t^3$ with $\kappa\varphi(t) = \kappa\varphi(t_i) + \sqrt{6}\ln(t/t_i)$, and is an asymptotic attractor [14] for the scale factor. Since the power is greater than unity, this solution still appears to be slowly inflating. However, when we look at the scalar field, we find $\exp(\sqrt{2/3}\kappa\varphi) = t^2(\pi U_0)/(3M_4^2)$. This evolution of the scalar field is too rapid, as we can easily verify by substituting the solutions back into the five-dimensional metric (5). Indeed, in the original frame the solution is $\bar{a}(\bar{t}) \propto \exp(\sqrt{(4\pi U_0)/(3M_5^3)}\bar{t})$, and $\exp(\sqrt{2/3}\kappa\varphi) \propto \exp(\sqrt{(4\pi U_0)/(3M_5^3)}\bar{t})$. The physical interpretation of this behavior is simple. Substituting $\chi = \chi_m$ in the potential corresponds to adding a positive cosmological constant in five dimensions, and, according to Wald's cosmological no-hair theorem [15], the anisotropic five-dimensional spacetime we are dealing with must approach the de Sitter space due to the effective cosmological constant.

Hence some mechanism is needed to stabilize the size of the fifth dimension. This requires breaking the residual five-dimensional gauge invariance, which equates the fifth direction with the other four in the case discussed above. The problem is linked to that of fixing the v.e.v. of the dilaton, which presumably involves ill-understood non-perturbative phenomena such as supersymmetry breaking and perhaps gaugino condensation [16]. Scenarios for these have been proposed, and it seems quite possible that these may occur at an energy-momentum scale above that of the five-dimensional Kaluza-Klein excitations. In this case, the internal radius may be regarded as fixed, for our purposes here.

After the internal radius is stabilized, we can recover exponential inflation in three spatial dimensions in both conformal frames, as long as χ stays near the local maximum χ_m . In this limit, the potential can be approximated as

$$U(0, \chi) = \frac{3M_4^2 M_5^2}{256\pi^{\frac{4}{3}}\alpha^{\frac{1}{3}}} V(\chi) \sim U(0, \chi_m) - \frac{1}{2}m^2(\chi - \chi_m)^2, \quad m^2 \equiv \frac{5}{16}\kappa^2 U(0, \chi_m), \quad (9)$$

where we have used the relation $V''(\chi_m) = -\frac{5}{16}\kappa^2 V(\chi_m)$. Thus the standard slow-roll solution is

$$\begin{aligned} a(t) &\propto e^{H_m t}, \quad H_m \equiv \sqrt{\frac{\kappa^2}{3} U(0, \chi_m)}, \\ \chi(t) &= \chi_m - (\chi_m - \chi_f) \exp\left(\frac{m^2}{3H_m^2} H_m (t - t_f)\right) = \chi_m - (\chi_m - \chi_f) e^{\frac{5}{16} H_m (t - t_f)}, \end{aligned} \quad (10)$$

where $t < t_f$, and where t_f and χ_f stand for the time and the field amplitude at the end of inflation.

It is important to note that the model we are presently considering does not share the graceful exit problem of typical string-dilaton inflationary models. Generally, as we have described above, inflationary (or de Sitter) solutions with a dilaton coupled to gravity require that the dilaton be fixed. When the dilaton doubles as the inflaton as well, there no means have been found to cancel the vacuum energy which drives inflation. This is different from the graceful exit problem in the pre-big-bang scenarios for which inflation is not driven by vacuum energy density, though these models also suffer from a graceful exit problem [17]. In our present case, these two issues are separated. Although the dilaton still needs to be fixed, inflation is driven by the R^4 -induced potential of the conformal field χ . Since the model we are considering is of the “topological” type, we are sitting on the top of this potential and we are *guaranteed* that inflation will end as the field χ rolls to its minimum.

We can now check the magnitude and spectral index of the induced density fluctuations. The amplitude of a linear curvature fluctuation, Φ_H , [18, 19] on a comoving scale $l = 2\pi/k$ is given by

$$\Phi_H\left(l = \frac{2\pi}{k}\right) = \frac{f H_m^2}{2\pi |\dot{\chi}(t_k)|} = \frac{8f H_m}{5\pi |\chi_m - \chi_f|} e^{\frac{5}{16} H_m (t_f - t_k)}, \quad (11)$$

where $f = 3/5$ ($2/3$) in the matter- (radiation-) dominated era, and t_k is the time when the k mode left the Hubble radius during inflation. The spectral index, n_s , of density perturbation is given by

$$n_s = 1 - \frac{2m^2}{3H_m^2}. \quad (12)$$

We recall that, in the model we are discussing, $m^2/H_m^2 = 15/16$ so that $n_s = 3/8$, which is significantly different from the scale-invariant value $n_s = 1$ and is in disagreement with observations. Furthermore, the large-angle (an)isotropy of cosmic microwave background radiation (CMB) [20] requires $\delta T/T = -\Phi_H/3 = 10^{-5}$ on the comoving scale leaving the Hubble radius about 60 expansion times before the end of inflation. Since we find $|\chi_m - \chi_f| \sim 0.1M_4$, the isotropy of CMB sets the scale of inflation as $H_m \sim 10^{-14}M_4$, which implies the Planck mass in five dimensions must be unacceptably small: $M_5 \sim 10^{-13}\alpha^{\frac{1}{2}}M_4$.

These results appear disconcerting. While our model showed some initial promise, it seems to fail the contact with the observations. A closer scrutiny of the dynamics shows that these problems arise because the potential which is generated by the R^4 term is a bit too steep to produce a satisfactory perturbation spectrum. We could foresee two possible solutions to this problem. The presence of an R^2 term in the five-dimensional action would flatten the potential further, possibly curing the problems with density fluctuations. In addition, the renormalization of the stress energy tensor, as in the original Starobinsky model, could also lower the effective value of m^2/H_m^2 , which was at the root of the problems above.

Here we consider the latter possibility. More generally, quantum corrections to the right-hand side of Einstein’s equation in the absence of matter can be written as [21]

$$\begin{aligned} \langle T_{\mu\nu} \rangle &= \left(\frac{k_2}{2880\pi^2}\right) \left(R_\mu^\rho R_{\nu\rho} - \frac{2}{3}R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^{\rho\sigma}R_{\rho\sigma} + \frac{1}{4}g_{\mu\nu}R^2\right) \\ &\quad + \frac{1}{6}\left(\frac{k_3}{2880\pi^2}\right) \left(2R_{;\mu;\nu} - 2g_{\mu\nu}R_{;\rho}^{\rho} - 2R R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R^2\right), \end{aligned} \quad (13)$$

where k_2 and k_3 are constants that appear in the process of regularization. We recall that k_2 is related to the number of light spin states, which can be very large in variants of string theories based on M theory, as we will discuss below. On the other hand, the coefficient k_3 is independent of the number

of light states. This term is none other but the variation of the R^2 term in the effective action. The theory admits a de Sitter solution which can be found from the 00 component of gravitational equation of motion [22]. Defining $H' = 2880\pi^2/k_2$ and $M^2 = 2880\pi^2/k_3$, and setting the spatial curvature $k = 0$, one finds [23]

$$H^2(H^2 - H'^2) = \frac{H'^2}{M^2}(2\ddot{H}H + 2H^2\dot{H} - \dot{H}^2), \quad (14)$$

where $H = \dot{a}/a$ is the Hubble parameter. The de Sitter solution corresponds to $H = H'$ and of course $\dot{H} = \ddot{H} = 0$.

In order to avoid the overproduction of gravitons there is a *lower* limit on the parameters $k_{2,3}$ [24, 23]: $k_2 \gtrsim 10^{10}$ implying the need for billions of spin degrees of freedom to be present. While this seems like an inordinately large number, it is possible to generate very large numbers of degrees of freedom in theories with extra dimensions, as we will now argue. Although this may not necessarily be the framework for M phenomenology that is eventually adopted, we use the general form of the effective low-energy theory derived from M -theory compactification on a Calabi-Yau manifold to illustrate the discussion of the possibly large magnitude of k_2 . The effective low-energy field theory has the form of a five-dimensional supergravity theory: as such, it contains a graviton supermultiplet, vector supermultiplets and scalar hypermultiplets. We recall that the graviton supermultiplet contains five graviton states, three graviphoton states and eight gravitino states. Each vector supermultiplet contains three vector states, one real scalar and four fermion states, and each scalar hypermultiplet contains four real scalars and four fermion states. The numbers of vector hypermultiplets n_V and scalar hypermultiplets n_H are related to the topological properties of the Calabi-Yau manifold:

$$n_V = n_{11} - 1, \quad n_H = n_{21} + 1. \quad (15)$$

Some of these states have even parity when the fifth dimension is compactified on S_1/Z_2 , and some are odd, but this is not essential for our purpose. We are interested in the number of excited supermultiplets that appear below the effective inflationary scale, which we identify approximately with $10^{16} \text{ GeV} \sim M_{GUT}$. The number of such Kaluza-Klein excitations is given asymptotically by $n_{KK} \sim M_{GUT}\mathcal{R}_5$. Hence we estimate

$$k_2 \sim n_{KK}[16 + 8(n_{11} + n_{21})]. \quad (16)$$

In realistic models, we expect that $n_{11} = 3$ and $n_{21} = n_{11} + 2\chi$, where the Euler characteristic $\chi = 3$. In this case, $k_2 \sim 88n_{KK}$, which exceeds 10^{10} if $n_{KK} \sim M_{GUT}\mathcal{R}_5 \gtrsim 10^8$ ³.

Here we will concentrate on the term proportional to k_2 . Recall that the coefficient k_2 is proportional to the number of degrees of freedom and has been estimated to be $k_2 \sim O(100)\mathcal{R}_5 M_5$. If we ignore the k_3 term – which is actually just the variation of the R^2 term in the action, and has already been considered above – and include our potential (7), (14) becomes

$$H^2 \left(1 - \frac{H^2}{H'^2} \right) = \frac{\kappa^2}{3} U(\phi, \chi), \quad (17)$$

where now $H'^2 = 360\pi M_4^2/k_2$. Near the maximum of the potential at $\chi = \chi_m$, the potential is $V(\chi_m) \simeq 10^{-3}M_4^2 M_5^2$. It is convenient to define $\epsilon = (1 - H^2/H'^2) \simeq m^2/H^2$ where $m^2 \sim 3 \times 10^{-3}M_5^2 M_4^2$ is the curvature of the potential at its maximum. Recall that our problem was related to the fact that $m^2/H^2 \sim 1$.

Using (17), we can now determine a consistent value for ϵ and m^2 and therefore H' . We can then check whether or not the resulting value of k_2 makes any sense with these choices. To obtain the correct magnitude for density fluctuations over the last 60 e-foldings of inflation, we need $H \sim 10^{-5}M_4\epsilon e^{-20\epsilon}$. But recall that $H = m/\sqrt{\epsilon}$, so that we require $M_5 \simeq 2 \times 10^{-3}M_4\epsilon^{3/2}e^{-20\epsilon}$. We see that M_5 is maximized when $\epsilon = 0.075$. In this case, we have $M_5 \sim 10^{-6}M_4 \sim 10^{13} \text{ GeV}$. As commented above, such an estimate is rather too small, but we do not consider this too disastrous, given the present level of sophistication of this class of model.

We should next determine whether or not we can obtain a value of k_2 which is consistent with the desired values of ϵ and M_5 . Substituting k_2 into H' , we find $H' \sim 3M_4/(M_5\mathcal{R}_5)^{1/2}$. For our value of ϵ , we

³There is also the possibility of additional matter and gauge fields associated with D branes in the bulk, but we do not discuss them further here.

need H' to be within 10% of this value ⁴, and we can solve for $\mathcal{R}_5^{-1}$. The result is $\mathcal{R}_5^{-1} \sim 4 \times 10^{-15} M_5$, or about 40 GeV. We note that some estimates of $\mathcal{R}_5$, based on a version of the Scherk-Schwarz approach, could lead one to expect $\mathcal{R}_5^{-1} \sim 100 M_5^3 / M_4^2 \sim 1$ TeV. Therefore, we do not regard such a large value of $\mathcal{R}_5$ as necessarily unmotivated. Finally, we can trivially check that not only do we get sufficient inflation with an acceptable magnitude for density fluctuations, but also find the spectral index to be in agreement with the existing data. For n_s , using $\epsilon = 3/40$, we have that $n_s = 1 - (1/20) = 0.95$, which is a relatively encouraging result, well within the present experimental uncertainties.

3 Conclusion

Many of the problems associated with inflation in string theory can be traced to a rolling dilaton which precludes a (quasi) de Sitter expansion. In this paper we have presented a model based on higher-order curvature terms such as R^4 in the context of a five-dimensional theory. This has some features in common with what may arise in general compactifications of M theory. In this context, the universe passes through an extended five-dimensional phase (down from its initial eleven-dimensional formulation) before it is seen as a four-dimensional space-time with one relatively large extra spatial dimension. We have made use of the conformal properties of the theory to analyze inflation in the familiar four-dimensional Einstein frame. This has enabled us to identify the relevant degrees of freedom, and consider their influence on cosmological dynamics.

We have found that a pure $R + R^4$ theory may provide a potential suitable for topological inflation. However, in this theory the spectral index for density fluctuations is too small, and the magnitude of density fluctuations is too large, unless the unification point is taken at an absurdly low energy scale. We have considered several ways of repairing these problems. One approach may be to include R^2 corrections in the action. Such corrections are not known to arise in superstring theories, but may appear if the supersymmetry breaking scale and the unification scale are close to each other. They can soften the effective inflaton potential further. Alternatively, a similar flattening of the potential may arise as a consequence of a large number of particle degrees of freedom, expected as the Kaluza-Klein modes associated with an extended compact spatial dimension, which could be produced in the course of cosmological evolution. Particle production can likewise raise the spectral index while keeping the magnitude of fluctuations reasonable. This result calls for a rather low unification scale of 10^{13} GeV and a scale $\mathcal{R}_5^{-1} \sim 1$ TeV for the fifth dimension. Given the crudity of this model, these results must be seen as encouraging.

To conclude, we note that our arguments do not depend strongly on the specific form of higher-order corrections in the effective action. We only need some higher-order curvature corrections which produce a local extremum of the effective inflaton potential. It seems plausible that such terms may play a significant role in some region of a highly curved early universe. The usual fine tuning of the coefficients of these terms, which was detrimental for the Starobinsky model, could be avoided here because of the dilaton v.e.v.. In turn, the quadratic corrections may be employed to flatten the potential in order to produce a satisfactory density perturbation spectrum, while not being overconstrained thanks to the fact that inflation is driven by higher corrections. Finally, the graceful exit problem is very easy to solve, due to the inherent instability of the solutions. The inflaton rolls down to the minimum, slowly at first but accelerating along the way. In light of this, we take this model to be a reasonable candidate for standard quasi-exponential inflation at scales close to the unification scale, which might offer some clues how to embed inflation in string theory.

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⁴This is hardly fine tuning by any standards.

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Magnetic Monopole in Bubble

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Abstract

The decay of metastable false vacuum by bubble nucleation in the SO(3) gauge theory is studied. At high temperature there exists a new bubble solution with a magnetic monopole at the center. The decay rate and evolution are analyzed.

1 Introduction

Since the first-order phase transition was formulated in the framework of Euclidean path integral [1, 2], it has attracted much attention mainly in the context of inflationary cosmology [3]. When a spontaneous continuous symmetry is taken into account, an enhancement of the tunneling is expected [4]. However, the uniqueness of the bounce O(3) symmetric solution as a nucleated bubble [5] has never been doubted before three of the authors reported the global *monopole-bubble* solution in the model of a scalar triplet with global SO(3) symmetry [6].

In this paper, we will consider a gauge theory of SO(3) symmetry in the high temperature limit, and show that there exists another gauged *monopole-bubble* solution where a bubble includes a 't Hooft-Polyakov monopole, i.e., a magnetic monopole, at the center. The nucleation rate of the gauged monopole-bubble is lower than that of the bounce, but it is higher than that of global monopole-bubble. Particularly, when the mass of a monopole becomes small in the strong gauged coupling limit, the decay channel through a gauged monopole-bubble with a thick wall is quite considerable. If the size of a nucleated gauged monopole-bubble is smaller than the critical size, the bubble wall starts to shrink and then the monopole at the center disappears. On the other hand, for the opposite case, the bubble wall expands and the monopole survives stably.

The remainder of this paper is organized as follows. In §2 we demonstrate the existence of a new gauged monopole-bubble solution in addition to the known bounce and the global monopole-bubble solution. In §3 we compute the nucleation rates of those bubbles and present the evolution of bubbles. §4 contains some brief concluding remarks.

2 Magnetic Monopole in Bubble

The Euclidean action of the SO(3) gauge theory at finite temperature is [7]

$$S = \int_0^\beta dt_E \int d^3x \left\{ \frac{1}{4} F_{\mu\nu}^a{}^2 + \frac{1}{2} (D_\mu \phi^a)^2 + V \right\}, \quad (1)$$

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where the field strength tensor is $F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + e\epsilon^{abc}A_\mu^b A_\nu^c$, and the covariant derivative is $D_\mu \phi^a \equiv \partial_\mu \phi^a + e\epsilon^{abc}A_\mu^b \phi^c$. In order to describe a first-order phase transition, we choose a ϕ^6 -potential:

$$V(\phi) = \frac{\lambda}{v^2}(\phi^2 + \alpha v^2)(\phi^2 - v^2)^2, \quad (2)$$

where the scalar amplitude ϕ is defined by $\phi = \sqrt{\phi^a \phi^a}$ ($a = 1, 2, 3$), and a parameter α ($0 < \alpha < 1/2$) governs the transition rate from the symmetric false vacuum at $\phi = 0$ to a broken true vacuum at $\phi = v$.

In the high temperature limit, i.e., $\beta \rightarrow 0$, a static electrically-neutral object satisfies

$$(D_i^2 \phi)^a = \frac{\partial V}{\partial \phi^a}, \quad (D_j F_{ij})^a = e\epsilon_{abc}\phi^b (D_i \phi)^c. \quad (3)$$

The ordinary bounce configuration is obtained by the ansatz of the scalar field $\phi^a = (0, 0, f(r))$ with boundary conditions, $df/dr|_{r=0} = 0$ and $f(r = \infty) = 0$. Here one may ask a question whether or not the gauge field affects the bounce configuration. Since the SO(3) gauge symmetry is spontaneously broken to an SO(2) symmetry and the first two components of the scalar field vanish, the gauge field decouples from the bounce solution.

In this paper we are interested in a different bubble solution which is obtainable under the hedgehog ansatz:

$$\phi^a = \hat{r}^a \phi(r), \quad A_i^a = \epsilon^{aij} \hat{r}^j \frac{1 - K(r)}{er}. \quad (4)$$

Substituting (4) into (3), we have

$$\frac{d^2 \phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} - \frac{2K^2}{r^2} \phi = \frac{dV}{d\phi}, \quad \frac{d^2 K}{dr^2} = K \left(\frac{K^2 - 1}{r^2} + e^2 \phi^2 \right). \quad (5)$$

Note that the ordinary bounce solution ϕ_0^a corresponds to $K = 0$ solution where a Dirac monopole in (4) decouples from this solution [1], and that the global monopole-bubble ϕ_{gm}^a is supported by the replacement from $K(r)^2$ in (5) to 1 only when the gauge coupling e vanishes [6]. The first-order transition from a symmetric false vacuum ($\phi = 0$) to a true broken vacuum ($\phi = v$) takes place when $0 < \alpha < 1/2$; possible boundary conditions are read as follows: the well-definiteness of the fields at the origin gives

$$\phi(r = 0) = 0, \quad K(r = 0) = 1, \quad (6)$$

and the spatial infinity should be in the initial setting before transition:

$$\phi(r \rightarrow \infty) = 0, \quad K(r \rightarrow \infty) = 1. \quad (7)$$

Of course, the trivial solution of false symmetric vacuum, i.e., $\phi = 0$ and $K = 1$, is an unwanted one. Therefore, Fig. 1 shows, the nontrivial bubble solution of our interest is: $\phi(r)$ increases from $\phi = 0$ at the beginning, reaches a maximum value ϕ_{max} , and decreases to $\phi = 0$ as r goes to infinity. On the other hand, $K(r)$ starts to decrease from 1, arrives at a minimum, and then grows up to 1 at spatial infinity.

To read the physics of the obtained sphaleron-type monopole-bubble solution, let us examine energy density:

$$T_0^0 = \frac{1}{e^2 r^2} \left[\left(\frac{dK}{dr} \right)^2 + \frac{(K^2 - 1)^2}{2r^2} \right] + \frac{1}{2} \left(\frac{d\phi}{dr} \right)^2 + \frac{K^2}{r^2} \phi^2 + V. \quad (8)$$

For small r we attempt a power series solution:

$$\phi(r) \approx \phi_0 \left[r - \frac{1}{10} (4\kappa_0 - m^2) r^3 + \dots \right], \quad (9)$$

which is formally odd under $r \rightarrow -r$, and

$$K(r) \approx 1 - \kappa_0 r^2 + \frac{1}{10} (3\kappa_0^2 + e^2 \phi_0^2) r^4 + \dots, \quad (10)$$

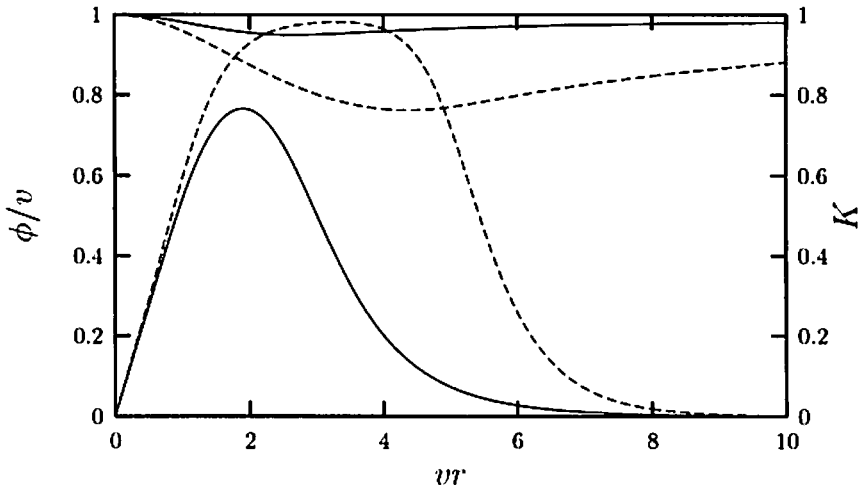


Figure 1: Plots of monopole-bubbles for $\lambda = 1$ and $e = 0.3$. The solid lines correspond to the scalar amplitude $\phi(r)/v$ and the gauge field $K(r)$ of a thin-wall monopole-bubble ($\alpha = 0.15$), and the dashed lines to those of a thick-wall monopole-bubble ($\alpha = 0.35$).

which is formally even under $r \rightarrow -r$. Here m denotes the mass of a scalar particle at the symmetric vacuum, i.e., $m \equiv \sqrt{(d^2V/d\phi^2)|_{\phi=0}} = \sqrt{2\lambda(1-2\alpha)}v$. If κ_0 is negative, then (5) says that $K(r)$ is a monotonic increasing function bounded below 1. It means that any solution with a negative κ_0 cannot satisfy the boundary condition (7) at spatial infinity, and thereby κ_0 must be positive. A characteristic of this new monopole-bubble, which makes it distinguished from the ordinary bounce configuration, is to have a matter lump at the center of the bubble. If we look into the behavior of the energy density for small r by use of the formulas in (9) and (10), it has

$$T_0^0 = \frac{3}{2e^2} \left(4\kappa_0^2 + e^2\phi_0^2 \right) + \left[\left(\frac{m^2}{2} + \lambda v^2(1-2\alpha) \right) \phi_0^2 - 2\kappa_0 \left(\frac{4\kappa_0}{e^2} + 3\phi_0^2 \right) \right] r^2 + \dots \quad (11)$$

The first term in (11) is always positive, which stems from the winding between the 3-dimensional space and the $SO(3)$ internal space. It is an evidence of the formation of an extended object at the center of the bubble. The sign of the second term in (11) determines the monopole size as $1/m_H \sim 1/\sqrt{8\lambda(1+\alpha)}v$. In the case of global monopole-bubbles, κ_0 is zero so that its existence is automatic since the coefficient of the second term is always positive [6]. However, when the gauge coupling is sufficiently large so as to make the coefficient of the second term negative, strong gauge repulsion can sweep the monopole wall away and the energy density is a decreasing function near the origin as shown in the dashed line of Fig. 2.

At the asymptotic region for large r , the scalar field approaches the boundary value in (7) exponentially,

$$\phi \approx \phi_\infty \frac{1+mr}{(mr)^2} e^{-mr}. \quad (12)$$

The gauge field has a long range term because the gauge boson becomes massless in the symmetric phase,

$$K(r) \approx 1 - \frac{\kappa_\infty}{r} + \frac{3\kappa_\infty^2}{4r^2} + \dots, \quad (13)$$

where κ_∞ should also be nonnegative for the same reason as κ_0 is in (10). Outside the bubble wall, the vacuum is in the symmetric phase with $\phi = 0$, and the $SO(3)$ gauge field is massless. Specifically, the

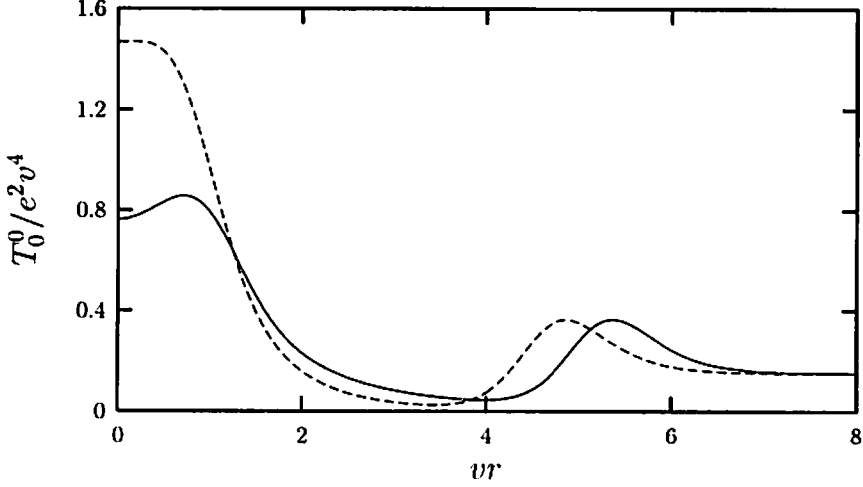


Figure 2: Profiles of the energy density for $\lambda = 1$ and $\alpha = 0.35$. The solid line stands for a monopole-bubble with a monopole wall ($e = 0.1$), and the dashed line for a monopole-bubble without a monopole wall ($e = 0.65$).

magnetic field looks like that of a dipole:

$$F_{ij}^a \sim \frac{1}{e}(\epsilon^{ajk}\hat{r}^i - \epsilon^{aik}\hat{r}^j)\frac{\hat{r}^k}{r}\frac{dK}{dr} \sim \mathcal{O}(1/r^3). \quad (14)$$

The corresponding leading term of the energy density, $T_0^0 - V(0)$, in (8) is of order $1/r^6$, and this fast decay can also be read from Fig. 2. This means that the monopole at the center is screened by the bubble wall.

When the energy gap between the false vacuum $V(0) = \lambda\alpha v^4$ and the true vacuum $V(v) = 0$ is considerably small, the bubble radius becomes much larger than the size of the bubble wall, $1/m_H$. It leads to so-called the thin-wall limit. Let us consider the behavior of the fields near the maximum scalar amplitude $\phi_{\max}$ in this thin-wall limit, where $\phi_{\max}$ has almost the true vacuum expectation value v (see the solid lines in Fig. 1). Near $\phi \approx \phi_{\max}$, the derivative of the scalar field $d\phi/dr$ vanishes, and the slope of the potential $dV/d\phi$ is almost flat. Therefore, if the corresponding value of K at the turning point r_{turn} with $\phi(r_{\text{turn}}) = \phi_{\max}$ is negligibly small, which will turn out to be indeed the case, equation (5) for the scalar amplitude implies that the quadratic derivative of the potential $d^2V/d\phi^2$ is also very small near $\phi_{\max}$ in the thin-wall limit. It means with the vanishing of $d\phi/dr|_{r=r_{\text{turn}}}$ that ϕ stays near $\phi_{\max}$ for a long range of r . For a sufficiently thin-wall bubble, the first term of the RHS of (5) becomes negligible near $\phi \approx \phi_{\max}$ since r_{turn} is large enough. Therefore, K falls to zero exponentially, i.e., $K \sim e^{-e\phi_{\max}r}$ and the scale $e\phi_{\max} \approx ev$ in the exponential is nothing but the mass of gauge boson in the Higgs phase. We can also confirm that ϕ approaches exponentially to the true vacuum expectation value v , i.e., $\phi \approx v - \phi_{\text{turn}}e^{-m_H r}/m_H r$ where ϕ_{turn} is an undetermined constant which will be fixed by boundary conditions (6) and (7).

The region between the monopole and the bubble wall is in a symmetry-broken phase of which the breaking pattern is $\text{SO}(3) \rightarrow \text{SO}(2)$; one of the three internal gauge degrees remains massless and it is to be identified as the photon field described by

$$F_{\mu\nu} = \frac{\phi^a}{\phi}F_{\mu\nu}^a + \frac{1}{e\phi^3}\epsilon_{abc}\phi^a(D_\mu\phi)^b(D_\nu\phi)^c. \quad (15)$$

Inserting the monopole ansatz (4) into the above expression (15), we obtain zero electric field but in contrast the magnetic field with monopole charge $g = 1/e$, i.e., $F_{ij} = \epsilon^{ija}\hat{r}^a/er^2$. Since ϕ approaches to v

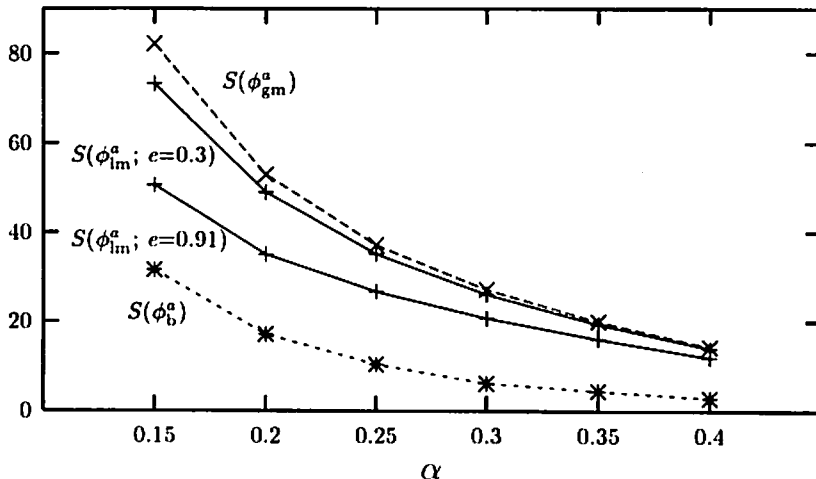


Figure 3: Plots of Euclidean actions $S/v\beta$ versus α for various bubble solutions. The dashed line corresponds to a global monopole-bubble ($e = 0$), two solid lines to gauged monopole-bubbles of $e = 0.3$ and 0.91 , and the dotted line to an ordinary bounce. Here λ is set to be 1.

and K lies about its minimum value near r_{turn} , one can easily notice that the contribution of the magnetic field comes from the first term of (15), and then identify the matter lump produced at the center as a 't Hooft-Polyakov monopole. If we look at the expression of energy density (8) near $\phi \approx \phi_{\text{max}} \sim v$,

$$T_0^0 \sim 1/2e^2r^4 + V(v), \quad (16)$$

the interpretation of the above leading term is nothing but the magnetic energy of the 't Hooft-Polyakov monopole, and its total energy is finite when we do not take into account the contribution from latent heat term $V(v)$. This is different from the global monopole-bubble case, where the global monopole contributes a $1/r^2$ term to the energy density, since this scalar phase term, which would make the energy diverge, is eaten up by the gauge field.

3 Nucleation Rate and Evolution

We begin this section by comparing the nucleation rates of the bounce and of the gauged monopole-bubble. In the semiclassical approximation, the decay rate per unit volume is estimated by the use of the leading exponential factor which is given by the Euclidean action of the bubble solution [1], multiplied by a prefactor determined by zero modes of the fluctuation [2]. When a continuous symmetry or a part of it is spontaneously broken, the number of zero modes increases and then the tunneling rate into vacua is enhanced [4]. In our model the $SO(3)$ gauge symmetry is broken to an $SO(2)$ symmetry and we have two bubble solutions; the relative decay rate between the bounce ϕ_b^a and the gauged monopole-bubble ϕ_{lm}^a is meaningful:

$$\frac{\Gamma_{lm}}{\Gamma_b} \sim \left(\frac{S(\phi_{lm}^a)}{S(\phi_b^a)} \right)^{3/2} \left| \frac{\det'[S''(\phi_{lm}^a)]}{\det'[S''(\phi_b^a)]} \right|^{-1/2} \frac{\int d^3x (\phi_{lm}^a)^2}{\int d^3x (\phi_b^a)^2} e^{-[S(\phi_{lm}^a) - S(\phi_b^a)]}, \quad (17)$$

where primed determinants denote infinite products over the nonzero eigenvalues, which are assumed to be order 1 here.

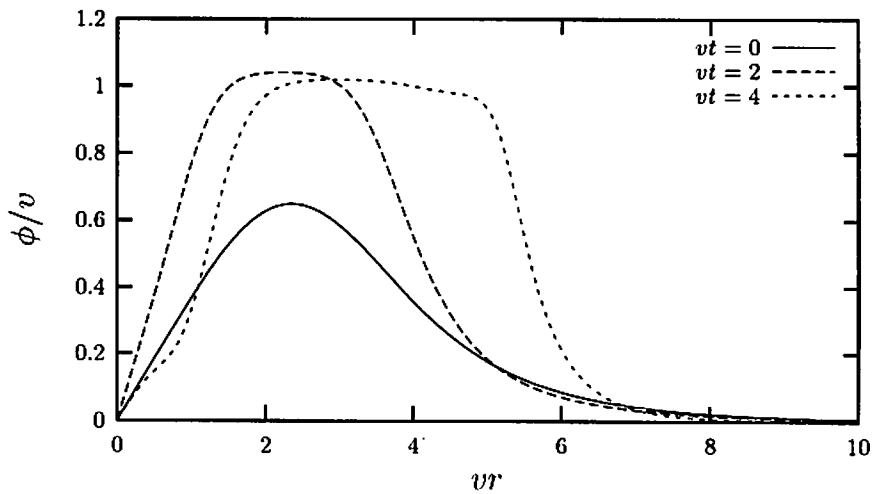
In order to see the leading effect, we plot in Fig. 3 the values of the Euclidean actions for a bounce, gauged monopole-bubbles ($e = 0.3$ and 0.91), and global monopole-bubble for $e = 0$. As expected, the

α	0.2	0.3	0.4
$(\Gamma_{gm}/\Gamma_b) _{e=0}$	1×10^{-15}	6×10^{-9}	1×10^{-4}
$(\Gamma_{lm}/\Gamma_b) _{e=0.3}$	6×10^{-14}	2×10^{-8}	1×10^{-4}
$(\Gamma_{lm}/\Gamma_b) _{e=0.91}$	4×10^{-8}	2×10^{-6}	8×10^{-4}

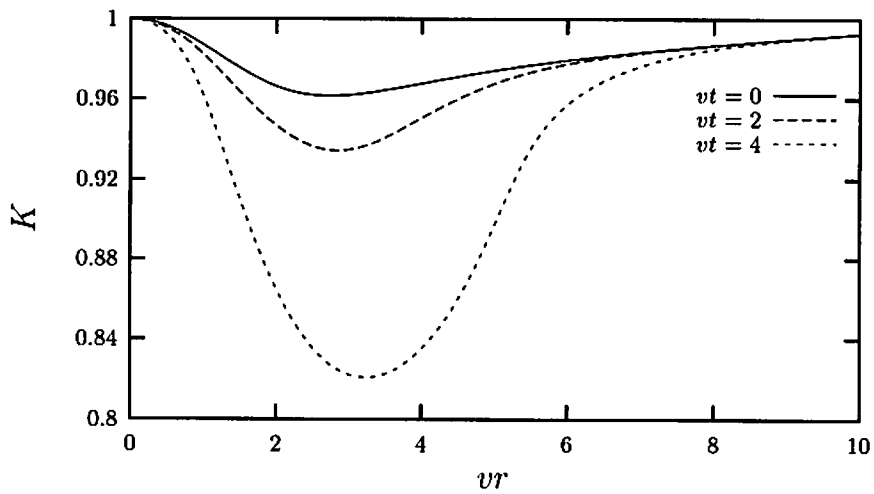
Table 1: The values of the relative decay rates, Γ_{gm}/Γ_b and Γ_{lm}/Γ_b , for various α and $v\beta$ with fixed $e = 1$, $\lambda = 1$, and $v\beta = 1$ when we set the ratio of determinant factors to be 1.

bounce has always the minimum action irrespective of the thickness of the bubble wall (or equivalently the parameter α) (see Fig. 3) so that it forms a dominant decay channel (see Table 1). The difference of the Euclidean actions is almost a constant for a given gauge coupling e , and it is roughly the monopole mass multiplied by inverse temperature. The values of the Euclidean action decrease as the gauge coupling increases. If we recall the mass formula $4\pi v/e$ of BPS monopole of unit winding [8], one may easily notice that this gap decreases in the strong coupling limit. This behavior can easily be understood by rescaling the variables to the dimensionless ones: $\tilde{t}_E = evt_E$, $\rho = evr$, $\phi = v h$, $\tilde{V} = \tilde{\lambda}(h^2 + \alpha)(h^2 - 1)^2$, and $\tilde{\lambda} = \lambda/e^2$. Then the action is rewritten as $S \sim (4\pi v\beta/e) \times (\text{dimensionless energy})$, which reflects the decrease of the monopole mass for strong gauge coupling. If we take into account the enhancement due to the prefactor in (17), we may obtain even an unbelievable relative decay rate, e.g., $\Gamma_{lm}/\Gamma_b > 1$ when $v\beta = 0.1$ and $\alpha = 0.4$. However, since the transition pattern becomes weakly first-order at such high temperature, the above result does not imply a possibility that the decay through a monopole-bubble is dominant decay channel. Instead, it means that in the high temperature limit ($v\beta \sim 1$) the relative production rate is considerable for thick-wall bubbles ($\alpha \approx 0.4$) with strong gauge coupling ($e \sim 1$) (see Table 1). For reference we give the values of the Euclidean action for the global monopole-bubbles when $e = 0$, which are always larger than those of the gauged monopole-bubbles because of their long energy tail proportional to the radius of the global monopole-bubble.

Completion of a first-order phase transition is achieved by the growth and percolation of the nucleated bubbles. The actual process is complicated because our $O(3)$ bounces and gauged monopole-bubbles are generated at high temperature. However, one simple but probably correct way is to analyze the time-dependent field equations with appropriate initial configurations. Since the obtained static solutions do not evolve, we consider some small radial fluctuations to the static solution for an initial configuration. If the bubble radius is smaller than the critical (initial) radius, both the bounce and the gauged monopole-bubble collapse; so does the magnetic monopole at the center. On the other hand, if it is larger than the critical radius, the bubble wall starts to grow and the magnetic monopole remains to be stable with some small damped oscillation. As expected, the terminal velocity of a growing gauged monopole-bubble wall is the light velocity. Figure 4 demonstrates this phenomenon clearly, and it also shows that a gauged monopole-bubble with a thick wall evolves to that of a thin wall.



(a)



(b)

Figure 4: Evolution of a gauged monopole-bubble with a thick-wall: (a) scalar amplitude ϕ/v and (b) gauge field K , where $\lambda = 1$, $e = 0.3$, and $\alpha = 0.4$.

4 Concluding Remarks

We have studied a first-order phase transition in the $SO(3)$ gauge theory, and found a new monopole-bubble solution in the high temperature limit. It is distinguished from the known Euclidean bounce by the production of a magnetic monopole at the center. The production rate of the monopole-bubble is smaller than that of the bounce, but it is considerable for thick-wall cases with strong gauge coupling. When the size of a nucleated bubble is larger than the critical size, the bubble wall starts to expand so that the magnetic monopole remains stable in the bubble. The existence of such a gauged monopole-bubble solution also indicates a possible drastic effect of gauge fields on bubble nucleation.

Acknowledgments

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Oklo Phenomenon

--- Searching for Possible Time-Variability of ‘Constants’ ---

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Abstract

We re-construct Shlyakhter's analysis of the Oklo phenomenon to search for possible time dependence of the strong interaction coupling constant. Based on the new data we almost re-establish the original claim on the upper bound $(10^{-18} - 10^{-19})\text{y}^{-1}$ on the fractional change, still much better than any other results.

It was in 1938, several years after Hubble's discovery of expansion of the universe, when Dirac suggested for the first time that Newton's constant G may not be a true constant but may fall like $\sim t^{-1}$ with the cosmic time t [1]. The suggestion was based on his Large-numbers Hypothesis; huge difference in magnitude between the strengths of the gravitational and the Coulomb interactions among elementary particles is not tolerated at the most fundamental level of the physical laws. Suspecting time variability was also extended to coupling constants of other interactions.

These arguments triggered many phenomenological analyses which have set stringent upper bounds on the possible time-variation of G , α (the fine-structure constant for the electromagnetic coupling), α_s (the strong interaction counterpart of α) and even the one for the weak interaction. Direct experimental studies have been and still are performed, as the measurement techniques are improved. In spite of extensive studies, no evidence for time variability has been reported. Nevertheless active efforts are still under way.

In retrospect, rather few of the physicists seemed to be fully convinced by Dirac's original motivation on which he suggested the time variability. Rather, many of them might have been impressed by the potentially far reaching consequences of the suggestion itself no matter how it had been conceived, and probably also fascinated by physical processes that they discovered to test the idea. More recently, the activity in this direction is fueled by modern versions of the unification program.

It is indicated in fact that coupling constants as well as some of the particle masses subject to our direct observation are rather complicated objects in some way or the other. As an illustration of the idea, we may consider typical theoretical models of unification. At the most fundamental level, they are characterized by coupling constants which are truly constant to begin with. They are generally, however, in higher dimensions, and accommodate many fields other than those observed in the realistic world in 4 dimensions. Among these un(or yet-to-be-)observed fields we find scalar fields that couple to other ordinary matter fields basically as weakly as gravity, but participate in constructing observable masses and coupling constants. In the cosmological context, on the other hand, the scalar fields likely have solutions depending on the cosmic time. It then follows that the coupling constants and masses as we observe may depend on the cosmic time, possibly in a way quite different from what was expected by Dirac. This view seems at least not in contradiction with what we find on G in Table 1 [2],[3]; smaller than Dirac's prediction $\sim 10^{-1}\text{y}^{-1}$ already by nearly 2 orders of magnitude.

Unfortunately we are still away from providing unambiguous theoretical predictions on how these “constants” depend on time. On the other hand, we should expect that knowing how they do (or do not) would give precious clues on what the true unification theory should be like.

As one of the remarkable features noted in Table 1, all the upper bounds are some orders smaller than 10^{-1}y^{-1} which might be still expected “naively.” This by itself needs an explanation as long as we admit time dependence at all. We also notice an outstanding result from “Oklo phenomenon” giving $\sim 10^{-19}\text{y}^{-1}$ is better than others by several orders of magnitude. What is this? How reliable is this? The purpose of the present talk is to answer these questions to begin with. Surprisingly little has been

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known about this superb result. We tried to reconstruct what Shlyakhter did, reaching a conclusion that his claim is essentially confirmed on the basis of the new data obtained from carefully selected samples [4], [5].

Oklo is the name of a place in Republic of Gabon where French Atomic Energy Authority runs a uranium mine (see Figs. 1-3²). In 1972 a mysterious deficit of ^{235}U in the ore from Oklo was reported; 0.7171% vs the world average 0.7207%. It was finally determined that part of ^{235}U was consumed by absorbing neutrons which had been present in "natural reactors" that had been at work in Oklo about 2×10^9 years ago. This Oklo phenomenon turned out to be a rediscovery of what P.K. Kuroda then at Arkansas had predicted in 1956 [7]; a possibility that sustained nuclear reaction could have taken place *naturally* because ^{235}U had been more abundant and water might have served as moderator. This phenomenon has been studied extensively from geological to geochemical points of view [8], still a subject of natural analogue to study long-term behavior of nuclear waste.

In 1976 Shlyakhter then at Leningrad focussed on the anomalous concentration of ^{149}Sm observed at the site of Oklo natural reactors: near 1 % compared with the natural abundance 13.8%. It was known that this amount can be understood in terms of the absorption process



with the cross section measured in *today's* laboratories. This implies that the strength of the nuclear interaction 2 billion years ago should have been the same as today's value within certain accuracy.

He noticed that this process happens to be dominated by a resonance lying as low as $E_{r0} = 97.3\text{meV}$ which is smaller than any energy scales of nuclear physics ($\sim \text{MeV}$ or more) by 7 or so orders of magnitude. A tiny change of the typical energy scale due to a tiny change of the strong interaction coupling constant, would cause a significant fractional change of the resonance energy, on which the cross section would depend sensitively. In a sense there is a two-fold amplification mechanism on possible small change of α_s during the time span of $2 \times 10^9\text{y}$. He assumed that the relevant energy scale is the depth of the one-particle nuclear potential V_0 which is proportional to α_s , hence

$$\frac{\Delta V_0}{V_0} \sim \frac{\Delta \alpha_s}{\alpha_s}. \quad (2)$$

Combining this with a natural assumption for the possible shift ΔE_r of the resonance energy,

$$\Delta E_r = -\Delta V_0, \quad (3)$$

he obtained

$$\frac{\Delta \alpha_s}{\alpha_s} \sim -\frac{\Delta E_r}{V_0}. \quad (4)$$

He estimated the upper bound on the resonance energy as given by

$$|\Delta E_r| \lesssim 20\text{meV}. \quad (5)$$

Using $V_0 = 40\text{MeV}$ and dividing the left-hand side of (4) by $2 \times 10^9\text{y}$, he reached $|\dot{\alpha}_s/\alpha_s| \lesssim 2.5 \times 10^{-19}\text{y}^{-1}$, as shown in Table 1.

This completes a brief outline of Shlyakhter's work which we re-constructed from his very short paper [3] and another unpublished paper [9].³ In what follows, we add some more details including the use of the new data to improve the result. We also include the remarks on Damour and Dyson's work [10] about which we came to know during course of our investigation.

At the center of the analysis we made our own calculation of the thermally averaged cross section of the process (1) anticipating different E_r from today's value, as shown in Fig. 4 [5]. An explanation will be needed for the "effective" cross section σ .⁴ Normally a thermally averaged cross section $\sigma(T)$ is defined by

$$\sigma(T) = [D(T)]^{-1} \int \sigma(E) v(E) \rho(E, T) dE, \quad (6)$$

²Re-drawn from Ref. [6].

³We thank John Barrow for making this paper available to us.

⁴We thank Thibault Damour for his helpful discussion.

where

$$\mathcal{D}(T) = \int v(E)\rho(E,T)dE, \quad (7)$$

with $\rho(E,T)$ the Maxwell-Boltzmann distribution. In most of the analyses of the Oklo phenomenon, however, it is customary [11] to replace the denominator $\mathcal{D}(T)$ by $\hat{\mathcal{D}}(T)$ with the velocity $v(E)$ in (7) replaced by a constant $v_0 = 2200(\text{m/s})$ corresponding to the temperature $T_0 = (m/2k)v_0^2 = 20.4^\circ\text{C}$, so that $\mathcal{D}(T) = v_0$. We have an explicit relationship

$$\hat{\sigma}(T) = \sqrt{\frac{4}{\pi} \frac{T}{T_0}} \sigma(T). \quad (8)$$

The advantage of using this non-standard definition has something to do with the fact that some of the useful absorption cross sections obey the $1/v$ behavior to a good approximation, implying that the effective cross sections are nearly T -independent, hence simplifying practical analyses considerably.

Fig. 4 plots $\hat{\sigma}_{149}$ obtained in this way for the process (1) for the assumed ΔE_r and various values of the reactor temperature T .

The data at earlier years, including those used by Damour and Dyson, were from the samples taken at sites near the surface, hence with unavoidable changes caused by weathering to the extent that the quality of the data is reduced considerably. We rely instead on the samples [12] collected at deep-lying reactor zones (RZs 10, 13, see Fig. 3), also selected carefully with a geologist's expertise to minimize contamination due to elements flowing into from outside after the cessation of the reactor activity.

The result of the new measurements are summarized in Table 2, in which we show the ratios of the related Sm isotopes obtained from five samples. Theoretically these isotopic ratios, which we now observe today as the values at the end of reactor activity unless there is no contamination, are consequences of fission, subsequent neutron capture and of original natural abundance. They are interpreted in terms of chain processes as described by differential equations containing relevant cross sections, which are obtained by fitting the data. We show the result on the effective (thermally averaged) cross section $\hat{\sigma}_{149}$ for the process (1), though readers are advised to refer to Ref. [5] for more details. We obtain

$$\hat{\sigma}_{149} = 91 \pm 6 \text{ kb}. \quad (9)$$

As for the temperature, Shlyakhter simply chose a room temperature $T = 20^\circ\text{C}$, which is obviously unrealistic, while Damour and Dyson assumed a much wider range.

The realistic estimate is, however, rather complicated. The widely accepted estimate based on the observed ratio of $^{176}\text{Lu}/^{175}\text{Lu}$ yields the range $220 - 360^\circ\text{C}$ [12],[13], but sometimes shows a temperature higher than 1000°C , indicating a limitation of the method. In another approach, though less direct, the oxygen isotopic compositions of clay minerals outside the reactor zone are measured, suggesting a core temperature $300 - 350^\circ\text{C}$ [14]. We believe it reasonable to assume the range $200 - 400^\circ\text{C}$. Narrowing this further will not affect the final result very much.

We then proceed to estimate $\Delta E_r = E_r - E_0$, for the assumed temperatures. We here notice that there are two solutions for ΔE_r for a given set of $\hat{\sigma}_{149}$ and T . The right-branch solution in Fig. 4 passes through the area that covers $\Delta E_r = 0$, whereas the left-branch solution extends to the far-negative region of ΔE_r .

If we choose the right-branch solution, we obtain

$$\Delta E_r = 4 \pm 16 \text{ meV}, \quad \text{for } T = 200 - 400^\circ\text{C}. \quad (10)$$

From the left-branch solution, on the other hand, we find

$$\Delta E_r = -97 \pm 8 \text{ meV}, \quad \text{for } T = 200 - 400^\circ\text{C}, \quad (11)$$

which is nonzero beyond 12 standard deviations. Since the level is now even below the threshold, we need more detailed analysis, which, however, seems unlikely to change the result in any substantial manner.

Substituting (10) into (4) with $V_0 = 40\text{MeV}$, we obtain

$$\frac{\alpha_s}{\alpha_9} = (0.5 \pm 2.0) \times 10^{-10} y^{-1}, \quad (12)$$

in agreement with Shlyakhter's result, in spite of the fact that he relied on much less clean data, also assuming too low temperature.

One might argue that the resonance in question is an excited state in a many-particle system and the relevant energy scale should be a neutron separation energy $\sim 8\text{MeV}$. Then the results (12) should be about 5 times larger.

Damour and Dyson pointed out that this type of ambiguity can be avoided if we assume that a nonzero ΔE_r if any is due entirely to the change of the electromagnetic coupling constant α , which causes the change of the Coulomb energy that makes a considerable contribution to the binding energy in medium heavy nuclei. The result for α instead of α_s is obtained in effect by replacing V_0 in (4) by $V_c \sim -1\text{MeV}$. Corresponding to (12) we thus obtain

$$\frac{\dot{\alpha}}{\alpha} = (0.2 \pm 0.8) \times 10^{-17} \text{y}^{-1}, \quad (13)$$

which is about 5 times more stringent than the bound in Ref. [10], reflecting partly smaller scatter of our data due to less contamination in the samples. We also point out that (11) gives

$$\frac{\dot{\alpha}}{\alpha} = (-4.9 \pm 0.4) \times 10^{-17} \text{y}^{-1}, \quad (14)$$

which is found to be comparable with their bound on the negative side; $-6.7 \times 10^{-17} \text{y}^{-1}$. This can be understood because this comes from the "far-negative side" in their plot of $\hat{\sigma}_{149}$, corresponding obviously to our left-branch solution.

We find the same two-fold amplification in the processes involving ^{155}Gd and ^{157}Gd , for which the analysis is quite parallel to that for Sm. Unfortunately, we can expect less accurate results because the absorption processes are more effective to the extent that the residual abundances are much smaller, also affected by contamination much more seriously. Nevertheless we took advantage of the presence of the two resonances each for ^{155}Gd and ^{157}Gd . With the aid of a plausibility assumption that the two resonance energies would shift by nearly the same amount, we find that the Gd data support the conclusion from Sm as long as we choose the right-branch solution. We point out that these analyses require certain amount of contamination in Gd's, whereas it is almost negligible for Sm.

It is also interesting to notice that the Sm result from the left-branch solution is unlikely consistent with the corresponding solution for Gd; ΔE_r for Gd's can be appreciably nonzero only for positive values instead of negative values for Sm.

Summarizing, we conclude that Shlyakhter's "champion result" for the upper bounds on the strong or the electromagnetic interaction coupling constants still stands unchallenged. It should be taken as one of the few valuable guidance in the search for viable unification theories. In this connection we suggest [15] a possible link between small upper bounds for time-variability of the constants on one hand, and a small but nonzero cosmological constant as indicated by recent observations on the other hand, with a scalar field as a common underlying agent.

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Sources	$\dot{G}/G$	$\dot{\alpha}/\alpha$	$\dot{\alpha}_s/\alpha_s$
Primordial nucl synthesis (Hoyle, 1965)	4×10^{-12}	3×10^{-13}	1×10^{-13}
Very long-lived nuclei (Dyson, 1967)			2×10^{-12} 2.5×10^{-10}
Stellar nucl synthesis (Davies, 1972)			
Oklo phenomenon (Shlyakhter, 1976)		3×10^{-13} 5×10^{-16}	
Solar-System exp(Hellings, 1983)			
Time standards, (Godone, 1993)			
Distant QSO, (Drinkwater, 1998)			

Table 1: Upper bounds on the time-variability in units of y^{-1} . See Refs. [2,3].

	Sample				
	SF84-1469	SF84-1480	SF84-1485	SF84-1492	SD.37
$\hat{\phi}t_1$ (1/kb)	0.525	0.798	0.622	0.564	0.780
$N_{144}(t_1)(\%)$	0.1052	0.2401	0.2073	0.1619	0.06909
$N_{147}(t_1)(\%)$	55.34	53.23	54.03	54.81	52.74
$N_{148}(t_1)(\%)$	2.796	3.468	3.079	2.890	4.694
$N_{149}(t_1)(\%)$	0.5544	0.2821	0.4466	0.4296	0.3088
$N_{235}^0(t_1)/^{238}\text{U}$	0.03181	0.02665	0.02971	0.03047	0.02435
$\hat{\sigma}_{149}(\text{kb})$	85.6	96.5	83.8	99.0	89.5

Table 2: Measured isotopic ratios N_x for various isotopes ^{147}Sm and fluence (the time-integrated effective neutron flux $\hat{\phi}$, with t_1 denoting the time of the cessation of the reactors measured from the time of its onset) for five samples and corresponding, calculated values of the cross section $\hat{\sigma}_{149}$. N_{235}^0 means the fraction of ^{235}U from which ^{147}Sm and ^{149}Sm had been produced as fission products.

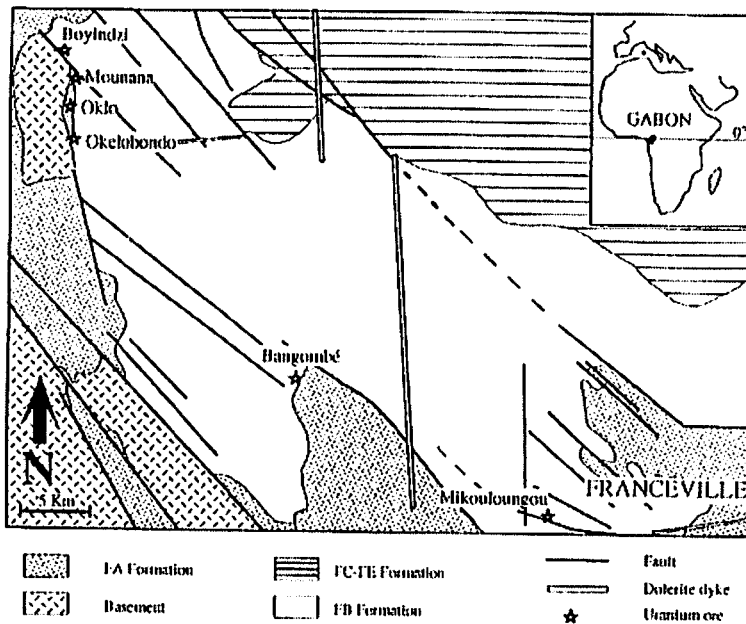


Figure 1: Schematic geological map of the Franceville basin and location of the uranium ore deposits.

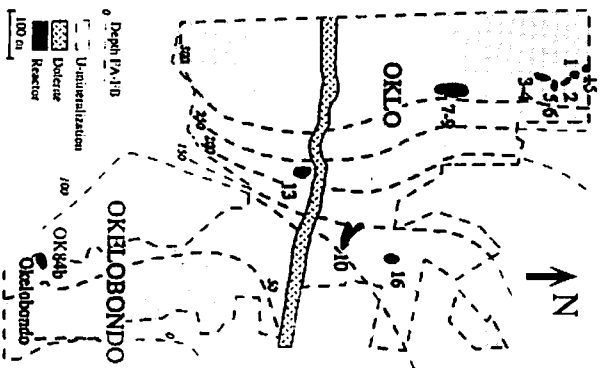


Figure 2: Location of the reactors at Oklo and Okelobondo.

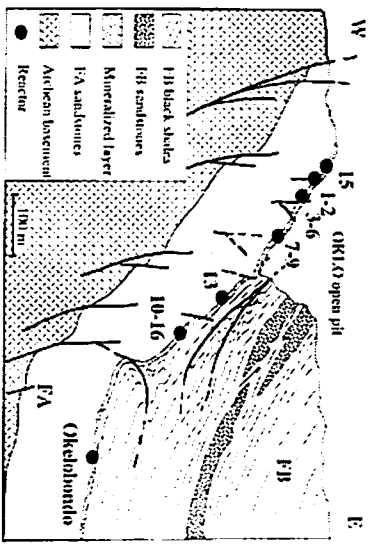


Figure 3: Cross section of the Oklo and Okelobondo deposits with the projection of the reactors at their different levels.

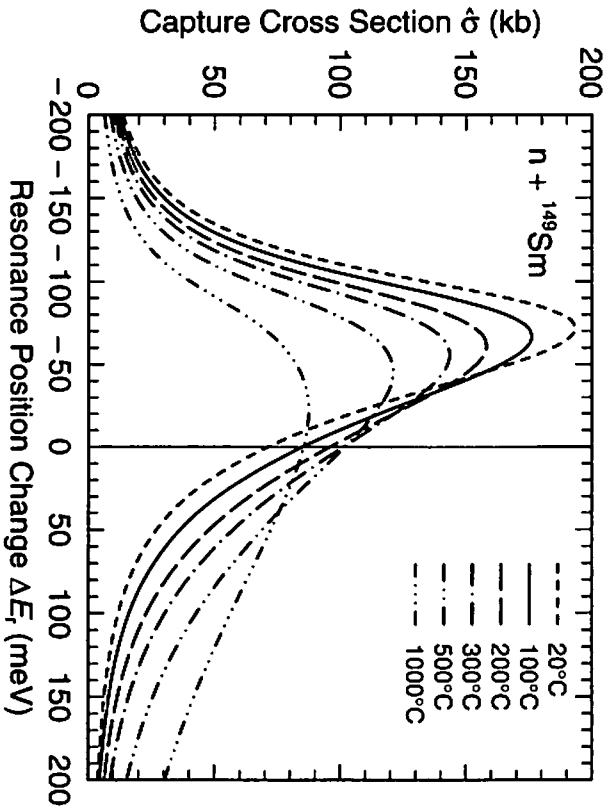


Figure 4: Calculated thermally averaged $\hat{\sigma}_{149}$ for $n + {}^{149}\text{Sm} \rightarrow {}^{150}\text{Sm} + \gamma$ as a function of the resonance-energy-shift ΔE_r and the temperature T .

Evolution of cosmological perturbations in the universe dominated by multiple scalar fields

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Abstract

By efforts of several reserchers, it is recently established that the dynamical behavior of the cosmological perturbation on superhorizon scales is well approximated in terms of that in the long wavelength limit, and that the latter can be constructed from the evolution of corresponding exactly homogeneous universe. Using these facts, we investigate the evolution of the cosmological perturbation on superhorizon scales in the universe dominated by oscillating multiple scalar fields which are generally interacting with each other, and the ratio of whose masses is incommensurable. We point out that all other modes except only one mode well known as the adiabatic growing mode are decaying.

§1 Introduction

The cosmological structures such as the galaxies and the clusters of galaxy, etc, are thought to be produced from the seed perturbations through the growth induced by the gravitational instability. In the inflationary paradigm, the quantum fluctuation of the scalar field driving the inflation called the inflaton plays the role of the seed perturbation. In the slow rolling phase, the seed perturbations produced in the Hubble horizon are stretched into the superhorizon scales by exponential cosmic expansion, and stay out of horizon until the perturbation comes into horizon in the Friedman regime. We can calculate the amplitude of the seed perturbations when the potential of the inflaton is given, and know the evolutionary behavior of the cosmological perturbation in the horizon in the Friedmann universe when the composition of the cosmic matter is given. Therefore in order to obtain the knowledge of the inflationary universe from the statistical informations of the galaxy distribution, we need to know how the amplitudes of perturbations on superhorizon scales grow from the first horizon crossing in the slow rolling phase until the second horizon crossing in the Friedmann regime.

The evolution of the cosmological perturbation on the superhorizon scales are thought to be simple. Because the Bardeen parameter are conserved in a good accuracy, when the equation of state is regular, and the energy of the universe is dominated by a single component. But between the inflationary regime and the Friedmann regime, reheating occurs. During reheating, since the plural scalar fields oscillate rapidly around the local minimum of the potential, and since radiation is generated by the energy transfer from the scalar field oscillation, the equation of state becomes singular periodically, and the entropy modes can appear. Therefore the approximate conservation of the Bardeen parameter is not self-evident, and we need to solve the evolution of the cosmological perturbations without using the conservation of the Bardeen parameter. In this paper, we consider the universe dominated by multiple scalar fields

Recently the strong method of solving the evolution of cosmological perturbations on superhorizon scales is developed. The evolutionary behavior of the cosmological perturbations on superhorizon scales is described by that in the long wavelength limit in a good accuracy. Further the long wavelength limit of the solution of the evolution equation of the cosmological perturbations is generated from the perturbation of solution of the evolution equation of the background homogeneous universe. [4], [5], [6]. By using the

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fact, we investigate the evolution of the cosmological perturbation on superhorizon scales in the universe dominated by the oscillating multiple scalar fields interacting with each other.

During reheating, the scalar fields oscillate rapidly around the local minimum of the potential. As a method of treating dynamical system with rapidly oscillating degrees of freedom, we use the canonical perturbation theory. In the canonical perturbation theory, the hamiltonian is decomposed into an unperturbed part whose solution is already known and an perturbation part, and by constructing the change of dynamical variables by which perturbation part is decreased, the system is made simple enough to be solved. The dynamical degrees of freedom are decomposed into the slowly varying amplitudes of oscillation called action variables which are involutive first integrals of the unperturbed part and the fast increasing phase of oscillation called angle variables. The angle variables are eliminated from the hamiltonian by the canonical transformation.

This paper is organized as follows. In chapter 2, based on the paper, we explain the way to construct a solution of the perturbation equation in the long wavelength limit from a perturbation of a solution of the exactly homogeneous universe. In chapter 3, we write the system of the multiple scalar fields in the homogeneous universe in the hamiltonian form and fix the model which we treat concretely. In chapter 4, we apply the canonical perturbation theory to our system. By using the canonical transformations recurrently, we decrease the power of time variables which the interaction parts contain and eliminate fast oscillation from the evolution equation and extract the slow secular motion. In chapter 5, we investigate the evolution of the multiple scalar fields in the homogeneous universe. We prove the fact that the action variables representing the amplitudes of the scalar field oscillation are perpetually stable. In chapter 6, we investigate the evolution of the cosmological perturbations in the long wavelength limit. The upper bound of evolution of the perturbation variables and the uniqueness of the growing mode are derived. The chapter 7 is devoted to the summary and the conclusion.

Throughout the paper the natural units $c = \hbar = 1$ are adopted and $8\pi G$ is denoted as κ^2 . Further the notations for the perturbation variables adopted in the article [1] are used and their definitions are sometimes omitted except for those newly defined in this paper

§2 Review; Evolution of cosmological perturbation in the long-wavelength limit

The dynamical behavior of cosmological perturbations on superhorizon scales is well approximated by that of cosmological perturbations in the long wavelength limit [7], and the evolution of cosmological perturbations in the $k \rightarrow 0$ limit can be constructed from the perturbation of the exactly homogeneous solution [5]. In this section, we present the results of the paper [5] in case of the universe dominated by multi-component scalar fields. Next we explain two way to construct the perturbation of the exactly homogeneous solution.

We only consider perturbations on a spatially flat ($K = 0$) Robertson-Walker universe throughout the paper. Hence the background metric is given by

$$ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2. \quad (2.1)$$

We consider the universe dominated by multi-component scalar fields, and the energy-momentum tensor is given by

$$T_\nu^\mu = \nabla^\mu \phi \cdot \nabla_\nu \phi - \frac{1}{2} \delta_\nu^\mu (\nabla^\lambda \phi \cdot \nabla_\lambda \phi + 2U). \quad (2.2)$$

In the long wavelength limit, the evolution of the gauge invariant variable Y_i representing the fluctuation of the scalar field ϕ , in the flat time slice are described as

$$\begin{aligned} Y_i &= \chi_i + \frac{\dot{\phi}_i}{H} \int dt \frac{H^2}{2U} W \left(\frac{\dot{\phi}}{H}, \chi \right), \\ W(X_1, X_2) &:= X_1 \cdot \dot{X}_2 - \dot{X}_1 \cdot X_2, \end{aligned} \quad (2.3)$$

where χ_i is a combination of exactly homogeneous perturbation variables described as

$$\chi_i = \delta\phi_i - \frac{\dot{\phi}_i}{H} \frac{\delta a}{a}. \quad (2.4)$$

In the $k = 0$ limit, it is noticed that the equality

$$\frac{a^3 H^2}{U} W \left(\frac{\dot{\phi}}{H}, \chi \right) = \text{const} \quad (2.5)$$

holds.

Corresponding to the way how the homogeneous solution is written, we can construct the exactly homogeneous perturbation in the different two ways. One is the way how it is given as derivative of the homogeneous solution in terms of the solution constants. The homogeneous solution is described as

$$\phi_i = \phi_i(t, C), \quad a = a(t, C), \quad (2.6)$$

or

$$\phi_i = \phi_i(a, C), \quad (2.7)$$

where C is the set of all the solution constants characterizing homogeneous solution completely. χ_i is expressed as

$$\chi_i = \frac{\partial \phi_i(t, C)}{\partial C} - \frac{\dot{\phi}_i}{H} \frac{1}{a} \frac{\partial a(t, C)}{\partial C} \quad (2.8)$$

in the above case, and is expressed as

$$\chi_i = \frac{\partial \phi_i(a, C)}{\partial C} \quad (2.9)$$

in the below case. Noticing the functional relation $a = a(t, C)$, it is seen that the above two expressions are exactly the same. On the other hand, the solution is often expressed as union of the manifold on which the first integral is constant

$$F_n(\phi, p_\phi, a) = \text{constant} \quad (n = 1, \dots, N) \quad (2.10)$$

where p_ϕ is canonical momentum conjugate to ϕ as defined later. In this case, the exactly homogeneous perturbation is produced as

$$\chi_i = \{F_n(\phi, p_\phi, a), \phi_i\} = - \frac{\partial F_n(\phi, p_\phi, a)}{\partial p_{\phi_i}}. \quad (2.11)$$

§3 Evolution equations of corresponding exactly homogeneous universe

We consider the situation where the multiple scalar fields interact with each other

$$S = \int dt (p_a \dot{a} + \sum_i p_{\phi_i} \dot{\phi}_i - N C_H). \quad (3.1)$$

Here N is the lapse function and C_H is the Hamiltonian constraint written as

$$C_H = \Omega a^3 \left(-\frac{\kappa^2}{12} \frac{1}{\Omega^2 a^4} p_a^2 + \frac{1}{2} \frac{1}{\Omega^2 a^6} \sum_i p_{\phi_i}^2 + U(\phi) \right), \quad (3.2)$$

where Ω is the comoving volume of the constant time hypersurface. The potential $U(\phi)$ is decomposed as

$$U(\phi) = \frac{1}{2} \sum_i m_i^2 \phi_i^2 + U_{\text{int}}(\phi). \quad (3.3)$$

where U_{int} is assumed to be a sum of monomials of ϕ of degree not smaller than 3 as in most physical context. We fix the gauge by

$$0 = \psi = a - t, \quad (3.4)$$

and its consistency condition with time evolution determines N as

$$0 = \frac{d\psi}{dt} = -\frac{\kappa^2}{6} \frac{N}{\Omega^2 a^4} p_a - 1. \quad (3.5)$$

For notational simplicity we write a in stead of t henceforth. By using Dirac bracket formalism, the Hamiltonian constraint $0 = C_H$ is changed into a strong equation, and p_a can be removed away. We obtain a reduced action with no constraint

$$S = \int \sum_i p_{\phi_i} d\phi_i - h_a da, \quad (3.6)$$

$$h_a = \frac{2\sqrt{3}}{\kappa} \Omega a^2 \left(\frac{1}{2} \frac{1}{\Omega^2 a^6} \sum_i p_{\phi_i}^2 + U(\phi) \right)^{1/2}. \quad (3.7)$$

We make all physical quantities non-dimensional by dividing them with typical scales as

$$\begin{aligned} \zeta &= \left(\frac{a}{a_0} \right)^{3/2}, \quad \Phi = \frac{\phi}{\phi_0}, \quad P_\phi = \frac{p_\phi}{a_0^3 m_0 \phi_0 \Omega}, \\ \epsilon &= \frac{\sqrt{3}}{2} \kappa \phi_0, \quad S' = \frac{S}{\Omega a_0^3 m_0 \phi_0^2}. \end{aligned} \quad (3.8)$$

From now on, we use t instead of ζ without possibility of confusion.

§4 Integration Method

In this section, we reduce the evolution equation into a simpler form whose perturbed part is sufficiently small by constructing appropriate coordinate transformations. Mathematically this method is known as the method of variation of constants. Since our system is described in the hamiltonian form, we use canonical transformations.

The starting hamiltonian is given by

$$H = \frac{2}{\epsilon} \left(\frac{1}{2} \frac{1}{t^2} \sum_i P_{\phi_i}^2 + \frac{1}{2} t^2 \sum_i \mu_i^2 \Phi_i^2 + t^2 \frac{1}{m_0^2 \phi_0^2} U_{\text{int}}(\phi_0 \Phi) \right)^{1/2}. \quad (4.1)$$

By performing canonical transformation as

$$\xi^i = \frac{1}{\sqrt{2}} \left(\sqrt{\mu_i} t \Phi_i - \frac{1}{\sqrt{\mu_i}} \frac{1}{t} P_{\phi_i} \right), \quad (4.2)$$

$$\eta^i = \frac{1}{\sqrt{2}} \left(-i \sqrt{\mu_i} t \Phi_i + \frac{1}{\sqrt{\mu_i}} \frac{1}{t} P_{\phi_i} \right). \quad (4.3)$$

we obtain as the transformed hamiltonian

$$\begin{aligned} H' &= H + \frac{\partial W}{\partial t} \\ &= \frac{2}{\epsilon} \left(i \sum_i \mu_i \xi^i \eta^i + t^2 \frac{1}{m_0^2 \phi_0^2} U_{\text{int}} \right)^{1/2} \\ &\quad + \frac{i}{2} \frac{1}{t} \left(\sum_i \eta^{i2} + \sum_i \xi^{i2} \right). \end{aligned} \quad (4.4)$$

Since the interaction term U_{int} is written as a sum of monomials of Φ_i ,

$$\Phi_i = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{\mu_i}} \frac{1}{t} \xi^i + i \frac{1}{\sqrt{\mu_i}} \frac{1}{t} \eta^i \right), \quad (4.5)$$

of degree not smaller than 3, we assume that U_{int} is a sum of monomials in terms of $\zeta^i/t = (\xi^i/t, \eta^i/t)$ of power not smaller than 3 with t -independent coefficients;

$$\frac{t^2}{m_0^2 \phi_0^2} U_{\text{int}} = \lambda t^2 \left(\sum_{|k|+|l|=3} + \sum_{|k|+|l|\geq 4} \right) c_{kl} \left(\frac{\xi}{t} \right)^k \left(\frac{\eta}{t} \right)^l, \quad (4.6)$$

where λ is a constant characterizing the strength of interaction. We introduce the action angle variables by

$$\xi^i = \sqrt{J^i} e^{i\theta^i}, \quad \eta^i = -i\sqrt{J^i} e^{-i\theta^i}. \quad (4.7)$$

Since $k \neq l$ under the condition $|k| + |l| = 3$, the first term of the right hand side K_3 ;

$$K_3 = \frac{\lambda}{t} k_3(J, \theta), \quad (4.8)$$

where k_3 is 2π -periodic in $\theta = (\theta^1, \dots, \theta^N)$ and is of power $3/2$ in terms of J , satisfies

$$\frac{1}{(2\pi)^N} \int_0^{2\pi} \dots \int_0^{2\pi} d^N \theta K_3(J, \theta, t) = 0. \quad (4.9)$$

The second term K_4 is written as

$$K_4 = \lambda \sum_{p=2}^{\infty} \frac{1}{t^p} k_{p+2}(J, \theta), \quad (4.10)$$

where k_{p+2} is 2π -periodic in $\theta = (\theta^1, \dots, \theta^N)$ and is of power $(p+2)/2$ in terms of J . The transformed hamiltonian H' is rewritten as

$$\begin{aligned} H &= \frac{2}{\epsilon} (\mu \cdot J + K_3 + K_4)^{1/2} - \sum_i \frac{J^i}{t} \sin 2\theta^i \\ &= \frac{2}{\epsilon} (\mu \cdot J)^{1/2} + \tilde{H}. \end{aligned} \quad (4.11)$$

The interaction part $\tilde{H}$ is decomposed as

$$\tilde{H} = A'_1 + B'_1, \quad (4.12)$$

where

$$\begin{aligned} A'_1 &= \frac{1}{\epsilon} \frac{1}{(\mu \cdot J)^{1/2}} K_4 \\ &+ \frac{2}{\epsilon} (\mu \cdot J)^{1/2} \sum_{n=2}^{\infty} \binom{1/2}{n} \left(\frac{K_3 + K_4}{\mu \cdot J} \right)^n, \end{aligned} \quad (4.13)$$

$$B'_1 = \frac{1}{\epsilon} \frac{1}{(\mu \cdot J)^{1/2}} K_3 - \sum_i \frac{J^i}{t} \sin 2\theta^i. \quad (4.14)$$

Now we assume that λ/ϵ is at most of order unity. By transferring θ -dependent part of A'_1 to B'_1 , we obtain as a starting hamiltonian

$$H = \frac{2}{\epsilon} (\mu \cdot J_1)^{1/2} + A_1(J_1, t) + B_1(J_1, \theta_1, t), \quad (4.15)$$

satisfying

(i) $\mu \cdot J_1$ is bounded as

$$|\mu \cdot J_1| \geq \sigma_1, \quad (4.16)$$

for some positive σ_1 on the domain $(J_1, 1/t) \in D'_1(\rho_1)$ where

$$D'_n(\rho) = \{(J, \frac{1}{t}); |J - J_{n0}| \leq \rho, |\frac{1}{t}| \leq 1 + \rho, |\text{Im} \frac{1}{t}| \leq \rho\}. \quad (4.17)$$

where J_0 is the initial value of J ; $J_0 \equiv J(t = 1)$.

(ii) $t^2 A_1$ is analytic in terms of J_1 , $1/t$ and is real for real J_1 , $1/t$ and the inequality

$$|t^2 A_1| \leq M_1^{(1)} \quad (4.18)$$

holds for some positive $M_1^{(1)}$ on the domain $D_1(\rho_1)$.

(iii) $t B_1$ is analytic in terms of J_1 , θ_1 , $1/t$ and is real for real J_1 , θ_1 , $1/t$ and is 2π -periodic in $\theta_1 = (\theta_1^1, \dots, \theta_1^N)$ and the inequality

$$|t B_1| \leq M_2^{(1)} \quad (4.19)$$

holds for some positive $M_2^{(1)}$ and satisfy

$$\frac{1}{(2\pi)^N} \int_0^{2\pi} \dots \int_0^{2\pi} d^N \theta_1 t B_1(J_1, \theta_1, t) = 0, \quad (4.20)$$

on the domain $D_1(\rho_1)$.

Since rapid oscillation in reheating phase is difficult to treat, we construct change of coordinates in order to eliminate fast phases from the transformed equations of perturbed motion. The canonical perturbation theory is effective for this purpose, and can separate the slow secular motions from the fast oscillations by performing the canonical transformations recurrently, when the frequencies of unperturbed motion are incommensurable.

Proposition1 (Theorem for the canonical transformations)

Suppose that $m \geq 1$. Suppose that the hamiltonian

$$H^{(m)} = \frac{2}{\epsilon} (\mu \cdot J_m)^{1/2} + A_m(J_m, t) + B_m(J_m, \theta_m, t) \quad (4.21)$$

satisfy the following conditions:

(i) For some positive C and d , the inequality

$$|k \cdot \mu| \geq C \frac{1}{|k|^d} \quad (4.22)$$

holds for every integral-valued vector $k = (k_1, \dots, k_N)$.

(ii) $\mu \cdot J_m$ is bounded as

$$|\mu \cdot J_m| \geq \sigma_m, \quad (4.23)$$

for some positive σ_m on the domain $(J_m, 1/t) \in D'_m(\rho_m)$.

(iii) $t^2 A_m(J_m, t)$ is analytic and real for real J_m , $1/t$ and the inequality

$$|t^2 A_m| \leq M_1^{(m)} \quad (4.24)$$

holds for some positive $M_1^{(m)}$ on the domain $(J_m, 1/t) \in D'_m(\rho_m)$.

(iv) $t^m B_m(J_m, \theta_m, t)$ is periodic in $\theta_m = (\theta_1^{(m)}, \dots, \theta_N^{(m)})$. It is analytic and real for real J_m , θ_m , $1/t$ and the inequality

$$|t^m B_m| \leq \epsilon^{m-1} M_2^{(m)} \quad (4.25)$$

holds for some positive $M_2^{(m)}$ on the domain $(J_m, \theta_m, 1/t) \in D_m(\rho_m)$ and satisfy

$$\frac{1}{(2\pi)^N} \int_0^{2\pi} \dots \int_0^{2\pi} d^N \theta_m t^m B_m(J_m, \theta_m, t) = 0. \quad (4.26)$$

For an arbitrary positive δ , there exists ϵ_0 such that, for an arbitrary ϵ satisfying

$$0 < \epsilon < \epsilon_0, \quad (4.27)$$

there exists the canonical transformation induced by the generating function $S_m(J_{m+1}, \theta_m, t)$ satisfying the following condition;

(v) $t^{m-1} S_m(J_{m+1}, \theta_m, t)$ is periodic in $\theta_m = (\theta_1^{(m)}, \dots, \theta_N^{(m)})$. It is analytic and real for real J_{m+1} , θ_m , $1/t$ and the inequality

$$|t^{m-1} S_m| \leq \epsilon^{m-1} L_1^{(m)} \quad (4.28)$$

holds for some positive $L_1^{(m)}$ on the domain $(J_{m+1}, \theta_m, 1/t) \in D_{m+1}(\rho_{m+1})$, where $\rho_{m+1} = \rho_m - \delta$.

The transformed system defined by

$$J_m = J_{m+1} + \frac{\partial S_m}{\partial \theta_m}, \quad (4.29)$$

$$\theta_{m+1} = \theta_m + \frac{\partial S_m}{\partial J_{m+1}}, \quad (4.30)$$

$$H^{(m+1)} = H^{(m)} + \frac{\partial S_m}{\partial t} \quad (4.31)$$

$$= \frac{2}{\epsilon} (\mu \cdot J_{m+1})^{1/2} + A_{m+1}(J_{m+1}, t) + B_{m+1}(J_{m+1}, \theta_{m+1}, t) \quad (4.32)$$

has the following properties;

(vi) $\mu \cdot J_{m+1}$ is bounded as

$$|\mu \cdot J_{m+1}| \geq \sigma_{m+1}, \quad (4.33)$$

for some positive σ_{m+1} on the domain $(J_{m+1}, 1/t) \in D'_{m+1}(\rho_{m+1})$

(vii) $t^2 A_{m+1}(J_{m+1}, t)$ is analytic and real for real J_{m+1} , $1/t$ and the inequality

$$|t^2 A_{m+1}| \leq M_1^{(m+1)} \quad (4.34)$$

holds for some positive $M_1^{(m+1)}$ on the domain $(J_{m+1}, 1/t) \in D'_{m+1}(\rho_{m+1})$.

(viii) $t^{m+1} B_{m+1}(J_{m+1}, \theta_{m+1}, t)$ is periodic in $\theta_{m+1} = (\theta_1^{(m+1)}, \dots, \theta_N^{(m+1)})$. It is analytic and real for real J_{m+1} , θ_{m+1} , $1/t$ and the inequality

$$|t^{m+1} B_{m+1}| \leq \epsilon^m M_2^{(m)} \quad (4.35)$$

holds for some positive $M_2^{(m)}$ on the domain $(J_{m+1}, \theta_{m+1}, 1/t) \in D_{m+1}(\rho_{m+1})$ and satisfy

$$\frac{1}{(2\pi)^N} \int_0^{2\pi} \dots \int_0^{2\pi} d^N \theta_{m+1} t^m B_{m+1}(J_{m+1}, \theta_{m+1}, t) = 0. \quad (4.36)$$

Proof

We consider the canonical transformation induced by the generating function given by

$$S_m(J_{m+1}, \theta_m, t) = \sum_{k \neq 0} S_k(J_{m+1}, t) e^{ik \cdot \theta_m}. \quad (4.37)$$

B_m is decomposed as

$$B_m(J_{m+1}, \theta_m, t) = \sum_{k \neq 0} b_k(J_{m+1}, t) e^{ik \cdot \theta_m}. \quad (4.38)$$

The transformed hamiltonian is

$$\begin{aligned} H^{(m+1)} &= H^{(m)} + \frac{\partial S_m}{\partial t} \\ &= \frac{2}{\epsilon} (\mu \cdot J_{m+1})^{1/2} + A_m(J_{m+1}, t) \\ &\quad + \frac{1}{\epsilon} \frac{1}{(\mu \cdot J_{m+1})^{1/2}} \mu \cdot \frac{\partial S_m(J_{m+1}, \theta_m, t)}{\partial \theta_m} \\ &\quad + B_m(J_{m+1}, \theta_m, t) \\ &\quad + R_1 + R_2 + R_3 + \frac{\partial S_m(J_{m+1}, \theta_m, t)}{\partial t} \end{aligned} \quad (4.39)$$

where $R_1, \dots, R_3$ are defined as

$$R_1 = \frac{2}{\epsilon}(\mu \cdot J_m)^{1/2} - \frac{2}{\epsilon}(\mu \cdot J_{m+1})^{1/2} - \frac{1}{\epsilon} \frac{1}{(\mu \cdot J_{m+1})^{1/2}} \mu \cdot \frac{\partial S_m(J_{m+1}, \theta_m, t)}{\partial \theta_m}, \quad (4.40)$$

$$R_2 = A_m(J_m, t) - A_m(J_{m+1}, t), \quad (4.41)$$

$$R_3 = B_m(J_m, \theta_m, t) - B_m(J_{m+1}, \theta_m, t). \quad (4.42)$$

We determine the generating function S_m so that the leading term of the oscillating part can be eliminated;

$$\begin{aligned} \frac{1}{\epsilon} \frac{1}{(\mu \cdot J_{m+1})^{1/2}} \mu \cdot \frac{\partial S_m(J_{m+1}, \theta_m, t)}{\partial \theta_m} \\ = -B_m(J_{m+1}, \theta_m, t). \end{aligned} \quad (4.43)$$

By comparing the fourier components in the both side, we obtain

$$S_k(J_{m+1}, t) = i\epsilon(\mu \cdot J_{m+1})^{1/2} \frac{1}{(\mu \cdot k)} b_k(J_{m+1}, t). \quad (4.44)$$

Since from (iv), on the domain $(J_{m+1}, 1/t) \in D'_m(\rho_m)$, the inequality

$$|t^m b_k(J_{m+1}, t)| \leq \epsilon^{m-1} M_2^{(m)} \exp(-|k| \rho_{m-1}) \quad (4.45)$$

holds, we evaluate the upper bound of the fourier components S_k ;

$$\begin{aligned} |t^{m-1} S_k(J_{m+1}, t)| \\ \leq \epsilon^m (\mu \cdot J_{m0} + N \mu \rho_m)^{1/2} \frac{M_2^{(m)}}{C} \left(\frac{d}{\epsilon}\right)^d \frac{1}{\delta^d} \\ \exp\{-|k|(\rho_m - \delta)\}. \end{aligned} \quad (4.46)$$

where we use the inequality

$$|k|^d \leq \left(\frac{d}{\epsilon}\right)^d \frac{\exp(|k|\delta)}{\delta^d}, \quad (4.47)$$

for arbitrary positive $|k|, d, \delta$. On the domain $|J_{m+1} - J_{m0}| \leq \rho_m, |\text{Im} \theta_m| \leq \rho_m - 2\delta, t^m S_m$ is bounded as

$$\begin{aligned} |t^m S_m| &\leq \sum_{k \neq 0} |t^m S_k| \exp(|k| |\text{Im} \theta_m|) \\ &\leq \epsilon^m L_1^{(m)}, \end{aligned} \quad (4.48)$$

where $L_1^{(m)}$ is defined as

$$L_1^{(m)} = (\mu \cdot J_{m0} + N \mu \rho_m)^{1/2} \frac{M_2^{(m)}}{C} \left(\frac{d}{\epsilon}\right)^d \frac{1}{\delta^d} \left(\frac{4}{\delta}\right)^N \quad (4.49)$$

In the above evaluation, we use the inequality

$$\begin{aligned} \sum_{k \neq 0} \exp(-|k|\delta) &< [1 + 2 \sum_{k > 0} \exp(-k\delta)]^N \\ &< \left(\frac{4}{\delta}\right)^N, \end{aligned} \quad (4.50)$$

where we assume that $\delta < 1$. On the domain $|J_{m+1} - J_{m0}| \leq \rho_m, |\text{Im} \theta_m| \leq \rho_m - 3\delta, |1/t| \leq 1 + \rho_m, t^m \partial S_m / \partial \theta_m$ is bounded as

$$|t^m \frac{\partial S_m(J_{m+1}, \theta_m, t)}{\partial \theta_m}| \leq \epsilon^m \frac{L_1^{(m)}}{\delta}. \quad (4.51)$$

So the inequality

$$|J_{m0} - J_{m+1} \circ| \leq \epsilon^m \frac{L_1^{(m)}}{\delta} \quad (4.52)$$

holds. Of course, the above two inequalities hold on the domain $|J_{m+1} - J_{m0}| \leq \rho_m - 3\delta$, $|\text{Im}\theta_m| \leq \rho_m - 3\delta$, $|1/\epsilon| \leq 1 + \rho_m$. Then

$$\begin{aligned} |J_{m+1} - J_{m+1} \circ| &\leq |J_{m+1} - J_{m0}| + |J_{m0} - J_{m+1} \circ| \\ &\leq \rho_m - 3\delta + \epsilon^m \frac{L_1^{(m)}}{\delta}. \end{aligned} \quad (4.53)$$

When we assume that ϵ satisfies

$$\epsilon^m \frac{L_1^{(m)}}{\delta} \leq \delta, \quad (4.54)$$

the inequality

$$|J_{m+1} - J_{m+1} \circ| \leq \rho_m - 2\delta \quad (4.55)$$

holds. The inequality (4.48) holds on the domain $(J_{m+1}, \theta_m, 1/t) \in D(\rho_m - 2\delta)$.

Now for an arbitrarily fixed J_{m+1} on the domain $|J_{m+1} - J_{m+1} \circ| \leq \rho_m - \delta$, we consider the mapping $T_{J_{m+1}}$ defined by

$$T_{J_{m+1}}: \theta_m \rightarrow \theta_{m+1} = \theta_m + \frac{\partial S_m}{\partial J_{m+1}}(J_{m+1}, \theta_m, t) \quad (4.56)$$

For an arbitrary θ_m on the domain $|\text{Im}\theta_m| \leq \rho_m - 2\delta$, the inequality

$$\begin{aligned} |\theta_m - \theta_{m+1}| &\leq \left| \frac{\partial S_m}{\partial J_{m+1}}(J_{m+1}, \theta_m, t) \right| \\ &\leq \epsilon^m (1 + \rho_m)^m \frac{L_1^{(m)}}{\delta} \end{aligned} \quad (4.57)$$

holds. Therefore the mapping $T_{J_{m+1}}$ is injective (so diffeomorphic) on the domain $|\text{Im}\theta_m| \leq \rho_m - 2\delta - 4\epsilon^m(1 + \rho_m)^m L_1^{(m)}/\delta$. $T_{J_{m+1}}$ maps an arbitrary θ_m on the domain $|\text{Im}\theta_m| \leq \rho_m - 2\delta - 4\epsilon^m(1 + \rho_m)^m L_1^{(m)}/\delta$ to θ_{m+1} on the domain $|\text{Im}\theta_{m+1}| \leq \rho_m - 2\delta - 3\epsilon^m(1 + \rho_m)^m L_1^{(m)}/\delta$. So on the domain $|J_{m+1} - J_{m+1} \circ| \leq \rho_m - \delta$, $|\text{Im}\theta_{m+1}| \leq \rho_m - 2\delta - 3\epsilon^m(1 + \rho_m)^m L_1^{(m)}/\delta$, the inverse mapping $T_{J_{m+1}}^{-1}$ is defined and diffeomorphic and $|\text{Im}T_{J_{m+1}}^{-1} \theta_{m+1}| = |\text{Im}\theta_m| \leq \rho_m - 2\delta - 4\epsilon^m(1 + \rho_m)^m L_1^{(m)}/\delta$. From now on, we assume that ϵ satisfies

$$3\epsilon^m(1 + \rho_m)^m \frac{L_1^{(m)}}{\delta} \leq \delta. \quad (4.58)$$

Then the canonical transformation $(J_{m+1}, \theta_{m+1}) \rightarrow (J_m, \theta_m)$ induced by the generating function $S_m(J_{m+1}, \theta_m, t)$ is defined and is diffeomorphic on the domain $|J_{m+1} - J_{m+1} \circ| \leq \rho_m - \delta$, $|\text{Im}\theta_m| \leq \rho_m - 3\delta$ and $|\text{Im}T_{J_{m+1}}^{-1} \theta_{m+1}| = |\text{Im}\theta_m| \leq \rho_m - 3\delta + \delta/3 \leq \rho_m - 2\delta$.

Next on the domain $|J_{m+1} - J_{m+1} \circ| \leq \rho_m - 2\delta$, $|\text{Im}\theta_m| \leq \rho_m - 3\delta$, we evaluate the residual part R defined as

$$R = R_1 + R_2 + R_3 + \frac{\partial S_m(J_{m+1}, \theta_m, t)}{\partial t}. \quad (4.59)$$

Since R_1 can be rewritten as

$$R_1 = - \int_{J_{m+1}}^{J_m} \mu \cdot dJ \frac{1}{\epsilon} \left\{ \int_{J_m} \mu \cdot dJ \frac{1}{(\mu \cdot J)^{3/2}} \right\}, \quad (4.60)$$

we evaluate

$$|t^{m+1} R_1| \leq \epsilon^m \mu^2 N^2 (1 + \rho_m)^{3m-1} \frac{L_1^{(m)2}}{\delta^2} \frac{1}{\sigma_m^{3/2}}. \quad (4.61)$$

We evaluate $R_2, R_3, \partial S_m/\partial t$ in the same manner. As a result, R is evaluated as

$$|t^{m+1} R| \leq \epsilon^m M, \quad (4.62)$$

where

$$\begin{aligned} M &= \mu^2 N^2 (1 + \rho_m)^{3m-1} \frac{L_1^{(m)2}}{\delta^2} \frac{1}{\sigma_m^{3/2}} \\ &+ (1 + \rho_m) \frac{M_1^{(m)} L_1^{(m)}}{6\delta^2} + (1 + \rho_m)^{m-1} \frac{M_2^{(m)} L_1^{(m)}}{6\delta^2} \\ &+ (m-1) L_1^{(m)} + (1 + \rho_m) \frac{L_1^{(m)}}{\delta}, \end{aligned} \quad (4.63)$$

on the domain $(J_{m+1}, \theta_{m+1}, 1/t) \in D_{m+1}(\rho_m - 2\delta)$. We decompose R into the fourier components as

$$R(J_{m+1}, \theta_{m+1}, t) = \sum_k R_k(J_{m+1}, t) e^{ik\theta_{m+1}}. \quad (4.64)$$

As for A_{m+1}, B_{m+1} defined as

$$A_{m+1}(J_{m+1}, t) = A_m(J_{m+1}, t) + R_0(J_{m+1}, t), \quad (4.65)$$

$$\begin{aligned} B_{m+1}(J_{m+1}, \theta_{m+1}, t) \\ = \sum_{k \neq 0} R_k(J_{m+1}, t) e^{ik\theta_{m+1}}, \end{aligned} \quad (4.66)$$

we obtain

$$|t^2 A_{m+1}| \leq M_1^{(m+1)}, \quad (4.67)$$

$$|t^{m+1} B_{m+1}| \leq \epsilon^m M_2^{(m+1)}, \quad (4.68)$$

where

$$M_1^{(m+1)} = M_1^{(m)} + (1 + \rho_m)^{m-2} \epsilon^m M, \quad (4.69)$$

$$M_2^{(m+1)} = 2M \quad (4.70)$$

on the domain $(J_{m+1}, \theta_{m+1}, 1/t) \in D_{m+1}(\rho_m - 2\delta)$. We put $\rho_m = \rho_{m-1} - 2\delta$. We complete the proof.

Proof End

As a proposition giving the evaluation of difference between original variables and transformed variables, we present the next one without adding the proof.

Proposition2

Suppose that the generating functions S_m are analytic and the inequalities

$$\begin{aligned} |t^m S_m| &\leq \epsilon^m L_1^{(m)}, \\ |t^m (\frac{\partial}{\partial J_{m+1}})^{\alpha_1} (\frac{\partial}{\partial \theta_m})^{\alpha_2} S_m| &\leq \epsilon^m L_2^{(m)}, \end{aligned} \quad (4.71)$$

holds for $\alpha_1 + \alpha_2 \leq 2$ and for some positive $L_1^{(m)}, L_2^{(m)}$.

As for $m \geq 2$, the inequalities

$$|\delta J_1^i(t) - \delta J_m^i(t)| \leq \frac{\epsilon}{t} C_{11}^{(m)} |\delta J_m(t)| + \frac{\epsilon}{t} C_{12}^{(m)} |\delta \theta_m(t)|, \quad (4.72)$$

$$|\delta \theta_1^i(t) - \delta \theta_m^i(t)| \leq \frac{\epsilon}{t} C_{21}^{(m)} |\delta J_m(t)| + \frac{\epsilon}{t} C_{22}^{(m)} |\delta \theta_m(t)|, \quad (4.73)$$

holds for some positive $C_{11}^{(m)}, C_{12}^{(m)}, C_{21}^{(m)}, C_{22}^{(m)}$, where

$$|\delta J_m(t)| \equiv \max_{1 \leq i \leq N} |\delta J_m^i(t)|, \quad (4.74)$$

$$|\delta \theta_m(t)| \equiv \max_{1 \leq i \leq N} |\delta \theta_m^i(t)|. \quad (4.75)$$

§5 Evolution of the exactly homogeneous universe

In this section, We investigate the evolutionary behavior of the exactly homogeneous universe dominated by scalar fields. In terms of action angle variables, the evolution of the scalar fields is characterized by the perpetual stability of the action variables which we formulate below. We omit the proof.

Proposition3 (Perpetual stability of the action variables)

Suppose that the hamiltonian

$$H = \frac{2}{\epsilon} \left(\frac{1}{2} \frac{1}{t^2} \sum_i P_{\phi_i}^2 + \frac{1}{2} t^2 \sum_i \mu_i^2 \Phi_i^2 + \lambda t^2 U_{\text{int}}(\Phi) \right)^{1/2} \quad (5.1)$$

which satisfy the following conditions;

(i) For some positive C and d , the inequality

$$|k \cdot \mu| \geq C \frac{1}{|k|^d} \quad (5.2)$$

holds for every integral-valued vector $k = (k_1, \dots, k_N)$.

In terms of action angle variables introduced by

$$\Phi_i = \sqrt{\frac{2}{\mu_i}} \sqrt{J_i} \frac{1}{t} \cos \theta_i, \quad (5.3)$$

$$P_{\phi_i} = -\sqrt{2\mu_i} \sqrt{J_i} t \sin \theta_i, \quad (5.4)$$

(ii) $U_{\text{int}}(\Phi)$ is a sum of monomials in terms of Φ of power not smaller than 3. The inequality

$$|\lambda t^2 U_{\text{int}}(\Phi)| \leq \epsilon M \quad (5.5)$$

holds for some positive M on the domain $(J, \theta, 1/t) \in D(\rho)$

(iii) The inequality

$$\left| \frac{1}{2} \frac{1}{t^2} \sum_i P_{\phi_i}^2 + \frac{1}{2} t^2 \sum_i \mu_i^2 \Phi_i^2 \right| = |\mu \cdot J| > \sigma \quad (5.6)$$

holds for some positive σ on the domain $(J, 1/t) \in D'(\rho)$.

There exists some positive ϵ_0 such that for an arbitrary ϵ satisfying

$$0 < \epsilon < \epsilon_0, \quad (5.7)$$

the inequality

$$|J - J_0| \leq \epsilon C \quad (5.8)$$

holds for some positive C .

§6 Evolution of Cosmological Perturbations in the Long Wavelength Limit

We consider the evolution of the cosmological perturbations in the long wavelength limit since it approximates the evolution of the cosmological perturbations on superhorizon scales well [7]. On superhorizon scales, the Bardeen parameter defined by

$$Z = \mathcal{R} - \frac{aH}{k} \sigma_g, \quad (6.1)$$

is conserved in a good accuracy when the equation of state is regular and the energy of the universe is dominated by a single component. On the other hand, in the universe dominated by multi-component scalar fields, the Bardeen parameter is rewritten as

$$Z = -H \frac{\dot{\phi} \cdot Y}{(\dot{\phi})^2}. \quad (6.2)$$

By using Eq.(2.5), we can see that the contribution from the second term in the right hand side of Eq.(2.3) is given by

$$Z \propto \int dt \frac{1}{a^3}. \quad (6.3)$$

So we concentrate the contribution from the first term in the right hand side of Eq.(2.3) henceforth.

$$Z = -H \frac{\dot{\phi} \cdot \delta\phi}{(\phi)^2}. \quad (6.4)$$

The contribution to the Bardeen parameter from the exactly homogeneous perturbation, Eq.(6.4), is rewritten in terms of the action angle variables as

$$Z = \frac{2}{3} \frac{\epsilon}{t} (\mu \cdot J + K)^{1/2} \frac{1}{\sum_i \mu_i J^i (1 - \cos 2\theta^i)} \sum_i \left(\frac{1}{2} \delta J^i \sin 2\theta^i - J^i (1 - \cos 2\theta^i) \delta\theta^i \right). \quad (6.5)$$

As for the fluctuation of the action angle variables, we obtain the evaluation as;

Proposition4 (Evaluation of the perturbation variables)

Under the assumption (i), (ii), (iii) in Proposition3, there exists some positive ϵ_0 such that; for an arbitrary ϵ satisfying

$$0 < \epsilon < \epsilon_0, \quad (6.6)$$

the following inequalities holds.

As for $\delta J^i(t)$, the inequality

$$|\delta J^i(t) - \delta J_0^i| \leq C_1 |\delta J_0| + \epsilon C_2 |\delta\theta_0| \quad (6.7)$$

holds for some positive C_1, C_2 .

When $\delta\theta^i(t)$ is decomposed as

$$\delta\theta^i(t) = \delta\theta_0^i + \frac{\mu_i}{\epsilon} (t-1)V(t) + W^i(t). \quad (6.8)$$

(i) $V(t)$ is i -indendent and the inequality

$$|V(t) - \delta(\frac{1}{\mu \cdot J_0})| \leq \epsilon (C_3 |\delta J_0| + \epsilon C_4 |\delta\theta_0|) \quad (6.9)$$

holds for some positive C_3 and C_4 .

(ii) $W^i(t)$ is i -dependent in general and the inequality

$$|W^i(t)| \leq C_5 |\delta J_0| + \epsilon C_6 |\delta\theta_0| \quad (6.10)$$

holds for some positive C_5 and C_6 .

The decomposition of $\delta\theta^i(t)$ into $V(t)$ and $W^i(t)$ as in (6.8) is not unique. For the purpose explained below, it may be rather convenient to write in the next form.

Proposition4'

Under the same assumption presented in the above proposition, there exists a function $V(t)$ such that

$$\delta\theta^i(t) = \delta\theta_0^i + \frac{\mu_i}{\epsilon} (t-1)V(t) + W^i(t), \quad (6.11)$$

$$|V(t) - V(t = \infty)| \leq \frac{\epsilon^2}{t} (C_1 |\delta J_0| + \epsilon C_2 |\delta\theta_0|), \quad (6.12)$$

$$|W^i(t)| \leq C_3 |\delta J_0| + \epsilon C_4 |\delta\theta_0|, \quad (6.13)$$

for some positive $C_1, \dots, C_4$.

Proof

As $V(t)$ in the proposition, adopt

$$V(t) = \frac{1}{t-1} \int_1^t dt \delta \frac{1}{(\mu \cdot J_3)^{1/2}}. \quad (6.14)$$

Proof End

Using the above results to Eq.(6.5) we can see that for sufficiently large t ,

$$Z \rightarrow -\frac{2}{3}(\mu \cdot J + K)^{1/2} V(t = \infty). \quad (6.15)$$

Therefore for the mode satisfying $V(t = \infty) \neq 0$ which corresponds to the adiabatic growing mode, Z stays constant, and for other $(2N - 1)$ modes satisfying $V(t = \infty) = 0$, Z decays at most in proportion to $1/t$. Because of the factor as

$$\frac{1}{\sum_i \mu_i J^i (1 - \cos 2\theta^i)}, \quad (6.16)$$

all the terms in Eq.(6.5) except (6.15) become boundlessly large when the phase flow $\theta(t)$ approaches $\theta^i = 0$ or π ($i = 1, \dots, N$) boundlessly. The distribution of the singularities in the time axis is probabilistic and can be regarded in the viewpoint of the ergodicity theory.

Proposition5

Under the condition

(i)' If the equality $\mu \cdot k = 0$ holds for some integral-valued vector $k = (k^1, \dots, k^N)$, $k = 0$ holds.
and the assumptions (ii), (iii) in **Proposition3**,
there exists some positive ϵ_0 such that;
for an arbitrary ϵ satisfying

$$0 < \epsilon < \epsilon_0, \quad (6.17)$$

the following statement holds.

As for a definition function $\phi(\theta)$ of the Jordan measurable set $A \subset \prod_{i=1}^N [0, 2\pi]$;

$$\phi(\theta) = \begin{cases} 1 & (\theta \in A) \\ 0 & (\theta \in \prod_{i=1}^N [0, 2\pi] \setminus A) \end{cases} \quad (6.18)$$

the equality

$$\lim_{T \rightarrow \infty} \frac{1}{T-1} \int_1^T dt \phi(\theta(t)) = \frac{1}{(2\pi)^N} \tau(A) \quad (6.19)$$

holds where $\tau(A)$ is a measure of the set A .

The occurrence of singularities is essentially probabilistic and random. It implies that we can find no rule in a sequence of times when Z becomes singular. By using the ergodicity theory, we can only evaluate how long from the initial time to the infinity the system stays in an arbitrary given set in the torus T^N .

§7 Summary and Discussion

We investigate the evolution of the cosmological perturbation in a universe dominated by multiple scalar fields by the canonical perturbation theory. It is concluded that if the ratio of frequencies of the scalar fields are incommensurable, for the adiabatic growing mode Bardeen parameter stay constant and for the other $(2N - 1)$ modes, the Bardeen parameter decays. Although the analysis are confined in the non-resonant case, the elimination of fast phases by the canonical transformation is thought to be still effective also in resonance case. The analysis in resonant case is in future publications.

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Lagrangian perturbations in relativistic cosmology

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Abstract

The Lagrangian description of the fluid flow is useful to treat the non-linear fluctuation in the relativistic cosmology as well as in the Newtonian cosmology. Under the Lagrange condition, the density and the vorticity can be integrated in the straightforward manner. Motivated by this fact, the Lagrangian perturbations in relativistic cosmology has been recently developed in the quite simple manner. The case for the matter with the pressure is also mentioned.

1 Introduction

Understanding the dynamics of fluids in the expanding universe is of great importance in the cosmology. For investigating such a dynamics, the Newtonian treatment is often used as a good approximation for the region $l/L \ll 1$, where l is the scale of fluctuations of fluids and L corresponds to the Hubble radius. In the Newtonian cosmology, the Lagrangian perturbation theory in the dust universe has been developed for non-linear density fluctuations, up to the caustic formation [1, 2, 3, 4]. In order to take into account the relativistic correction to the Newtonian dynamics, the cosmological post-Newtonian approximation has been formulated [5, 6]. Studies beyond the Newtonian approximation may become important because of not only the theoretical interest but also the recent progress in the observational cosmology. For example, the Sloan Digital Sky Survey (SDSS) aims at the deep survey over several hundred Mpc [7], in which it is not clear whether the Newtonian treatment is sufficient for such a large area.

On the other hand, let us consider the evolution of the fluctuations at the early stage. In the early stage, such fluctuations have experienced the horizon-crossing epoch $l/L = 1$. In this era, the linear perturbation theory in the relativistic cosmology has been well developed [8, 9]. The second order perturbation theory has been also developed [10]. However, there is no established way that the Newtonian treatment for the nonlinear density fluctuations can contact with the relativistic perturbation theory assuming the small density fluctuation: Usual perturbation theory in the relativistic cosmology expands both of the metric and material quantities such as the density. Hence, the nonlinear density fluctuation cannot be treated by it. In order to overcome the drawback, perturbative approaches for the nonlinear dynamics have been proposed by Matarrese et al. [11] based on the fluid flow approach [12], by Kasai using the tetrad [13, 14], and by the use of the gradient expansion method [15], which are all restricted within the irrotational dust. In recent, Asada and Kasai have developed the Lagrangian approach for the dust with the rotational motion [16].

In section 2, the integral of the vorticity along the fluid flow is discussed in the framework of the Newtonian theory. In section 3, we show the integrals of the density and the vorticity in the general relativity, in the simple manner using the Lagrangian condition. This fact suggests strongly that the Lagrangian condition allows us to formulate the Lagrangian description in the relativistic cosmology. Under this condition, section 4 presents a perturbative Lagrangian approach. Summary and Discussion are given in section 5. Greek indices run from 0 to 3, and Latin indices from 1 to 3. We use the unit, $c = 1$.

2 Lagrangian description of Newtonian Theory

In this section, we summarize the Lagrangian description in the Newtonian fluid. Basic equations in Newtonian fluid are composed of the continuity equation, the equation of motion and the Poisson equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (1)$$

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$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla \phi, \quad (2)$$

$$\nabla^2 \phi = 4\pi G \rho, \quad (3)$$

where ρ , $\mathbf{v}$ and ϕ are the density of dust matter, the velocity field and the gravitational potential.

Let us introduce the Lagrangian time derivative

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla, \quad (4)$$

and the Lagrangian coordinates $\mathbf{q} \equiv \mathbf{x}(t_0)$, where t_0 means an initial time. The continuity equation (1) means the mass conservation,

$$\rho d^3x = \rho_0 d^3q, \quad (5)$$

which is solved to give in the Lagrangian description,

$$\rho = \frac{\rho_0}{\det(\partial x^i / \partial q^j)}, \quad (6)$$

where ρ_0 denotes the density at t_0 .

We now discuss the vortical part of the fluid flow. The vorticity is defined by

$$\boldsymbol{\omega} \equiv \nabla \times \mathbf{v}. \quad (7)$$

From Eq. (2), we obtain the propagation equation for the vorticity:

$$\frac{d\boldsymbol{\omega}}{dt} + \boldsymbol{\omega} \nabla \cdot \mathbf{v} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{v}. \quad (8)$$

Using Eq. (1), we have the diffusion equation of Beltrami:

$$\frac{d}{dt} \left(\frac{\boldsymbol{\omega}}{\rho} \right) = \left(\frac{\boldsymbol{\omega}}{\rho} \cdot \nabla \right) \mathbf{v}. \quad (9)$$

It is instructive to define the vector components of the vorticity in the Lagrangian frame as follows:

$$\Omega^i \equiv \frac{\partial q^i}{\partial x^j} \omega^j. \quad (10)$$

Then, Beltrami's equation (9) simply reads

$$\frac{d}{dt} \left(\frac{\Omega}{\rho} \right) = 0, \quad (11)$$

which is manifestly solved to give

$$\frac{\Omega}{\rho} = \frac{\Omega}{\rho} \Big|_{t_0}, \quad (12)$$

or

$$\omega^i(\mathbf{q}, t) = \frac{\rho}{\rho_0} \frac{\partial x^i}{\partial q^j} \omega^j(\mathbf{q}, t_0). \quad (13)$$

The result Eq. (13) is known as Cauchy's integral [2].

3 Lagrangian description of General Relativity

Let us consider a simple dust universe, in which the energy momentum tensor is written as

$$T^{\mu\nu} = \rho u^\mu u^\nu. \quad (14)$$

The conservation law $T^{\mu\nu}_{;\nu} = 0$ gives

$$(\rho u^\mu)_{;\mu} = 0, \quad (15)$$

$$u^\mu_{;\nu} u^\nu = 0, \quad (16)$$

which are called as the continuity equation and the geodesic equation, respectively.

The vorticity ω^μ of the fluid flow is defined by [17]

$$\omega^\alpha \equiv \frac{1}{2} \epsilon^{\alpha\mu\nu\rho} u_\mu u_{\nu;\rho}, \quad (17)$$

where $\epsilon^{\alpha\mu\nu\rho}$ denotes the complete anti-symmetric tensor with $\epsilon^{0123} = 1/\sqrt{-g}$ and $g \equiv \det(g_{\mu\nu})$. From the geodesic equation Eq. (16), we obtain the propagation equation for the vorticity [17]:

$$\omega^\mu_{;\nu} u^\nu + u^\nu_{;\nu} \omega^\mu = u^\mu_{;\nu} \omega^\nu. \quad (18)$$

Using Eq. (15), we have

$$\left(\frac{\omega^\mu}{\rho}\right)_{;\nu} u^\nu = u^\mu_{;\nu} \left(\frac{\omega^\nu}{\rho}\right), \quad (19)$$

which may be called as the relativistic Beltrami equation (cf. Eq. (9) in Appendix A).

The Einstein equations are decomposed with respect to the fluid flow as follows:

$$G_{\mu\nu} u^\mu u^\nu = 8\pi G\rho, \quad (20)$$

$$G_{\mu\nu} u^\mu P^\nu_\alpha = 0, \quad (21)$$

$$G_{\mu\nu} P^\mu_\alpha P^\nu_\beta = 0, \quad (22)$$

where P^μ_ν is the projection tensor

$$P^\mu_\nu \equiv \delta^\mu_\nu + u^\mu u_\nu. \quad (23)$$

So far the treatment is fully covariant. In the following, we adopt the Lagrangian condition (e.g. [18]), in which the components of the matter 4-velocity take the values of

$$u^\mu = (1, 0, 0, 0). \quad (24)$$

Under this condition, we immediately have $g_{00} = -1$ and $u_\mu = (-1, g_{0i})$. Furthermore, since the geodesic equation Eq. (16) under the Lagrangian condition simply tells $u_{\mu,0} = 0$, $u_i (= g_{0i})$ are functions of spatial coordinates only:

$$u_i = u_i(\mathbf{x}), \quad \text{and} \quad g_{0i} = g_{0i}(\mathbf{x}). \quad (25)$$

In the Lagrangian description, the continuity equation (15) is simply

$$(\rho\sqrt{-g})_{,0} = 0. \quad (26)$$

Therefore,

$$\rho(\mathbf{x}, t) = \sqrt{\frac{g(\mathbf{x}, t_0)}{g(\mathbf{x}, t)}} \rho(\mathbf{x}, t_0). \quad (27)$$

Under the Lagrangian condition, the determinant of the metric tensor can be expressed as

$$g = -(1 + \gamma^{ij} g_{0i} g_{0j}) \det(g_{ij}), \quad (28)$$

where γ^{ij} is the inverse of the spatial metric g_{ij} . The relativistic Beltrami equation (19) also becomes simply

$$\left(\frac{\omega^\mu}{\rho}\right)_{,0} = 0, \quad (29)$$

which is integrated to give

$$\frac{\omega^\mu}{\rho} = \frac{\omega^\mu}{\rho} \Big|_{t_0}. \quad (30)$$

This is also expressed as

$$\omega^\mu(\mathbf{x}, t) = \sqrt{\frac{g(\mathbf{x}, t_0)}{g(\mathbf{x}, t)}} \omega^\mu(\mathbf{x}, t_0). \quad (31)$$

The ω^0 component is not independent of ω^i . Using the relation $\omega^\mu u_\mu = 0$, we obtain

$$\omega^0 = g_{0i} \omega^i, \quad (32)$$

The result Eq. (30) tells us that the vorticity is coupled to the density enhancement and vice versa. In particular, if the vorticity does not vanish exactly at an initial time, the vorticity will blow up as the density grows larger and larger (i.e. in the collapsing region), even if it has only the decaying mode in the linear perturbation theory. It should also be emphasized that our results Eqs. (27) and (30) in the fully general relativistic treatment precisely correspond to those in the Newtonian case (see Eqs. (6) and (12)).

4 Perturbative Lagrangian approach

In the previous section, we solved exactly the equations for the density and the vorticity. The results Eqs. (27) and (31) show that ρ and ω^i are completely written in terms of the determinant of the metric tensor and their initial values. In this section, we solve the metric perturbatively. We assume that the background is spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe. The extension to the spatially non-flat case must be a straightforward task. The perturbed metric is decomposed into

$$\begin{aligned} g_{0i} &= B_{,i}(\mathbf{x}) + b_i(\mathbf{x}), \\ g_{ij} &= a^2 (\delta_{ij} + 2H_L \delta_{ij} + 2H_T \delta_{ij} + (h_{i,j} + h_{j,i}) + 2H_{ij}), \end{aligned} \quad (33)$$

where B , H_L , and H_T are scalar mode quantities, b_i and h_i are the vector (transverse) mode, and H_{ij} is the tensor (transverse-traceless) mode satisfying

$$b^i{}_{,i} = 0, \quad (34)$$

$$h^i{}_{,i} = 0, \quad (35)$$

$$H^i{}_i = 0, \quad (36)$$

$$H^{ij}{}_{,j} = 0. \quad (37)$$

Raising and lowering indices of the perturbed quantities are done by δ^{ij} and δ_{ij} .

4.1 Residual gauge freedom in the Lagrangian condition

The general gauge transformation to the first order is induced by the infinitesimal coordinate transformation

$$\tilde{x}^\mu = x^\mu + \xi^\mu. \quad (38)$$

The changes due to the gauge transformation are

$$\delta_\xi g_{\mu\nu} = -g_{\mu\nu,\alpha} \xi^\alpha - g_{\mu\alpha} \xi^\alpha{}_{,\nu} - g_{\nu\alpha} \xi^\alpha{}_{,\mu}, \quad (39)$$

$$\delta_\xi u^\mu = \xi^\mu{}_{,\nu} u^\nu - u^\mu{}_{,\nu} \xi^\nu. \quad (40)$$

For the spatially flat background, ξ^μ are decomposed into each mode,

$$\xi^\mu = (T, \delta^{ij} L_{,j} + \ell^i), \quad (41)$$

where the vector mode quantity ℓ^i satisfies $\ell^i{}_{,i} = 0$. In order to maintain the Lagrangian condition $u^\mu = (1, 0, 0, 0)$, we have $\delta_\xi u^\mu = 0$, which leads to $\xi^\mu{}_{,0} = 0$. Therefore,

$$T = T(\mathbf{x}), \quad L_{,i} = L_{,i}(\mathbf{x}), \quad \ell_i = \ell_i(\mathbf{x}). \quad (42)$$

Hence we obtain

$$\bar{B} = B + T(\mathbf{x}), \quad (43)$$

$$\bar{H}_L = H_L - \frac{\dot{a}}{a} T(\mathbf{x}), \quad (44)$$

$$\bar{H}_T = H_T - L(\mathbf{x}), \quad (45)$$

$$\bar{h}_i = h_i - \ell_i(\mathbf{x}), \quad (46)$$

where an overdot ($\dot{}$) denotes $\partial/\partial t$. The vector mode quantity b_i and the tensor mode quantity H_{ij} are gauge-invariant.

4.2 Scalar perturbations

The Einstein equations for the scalar perturbations are

$$\left(H_L + \frac{\dot{a}}{a} B\right)' = 0, \quad (47)$$

$$\ddot{H}_L + 3\frac{\dot{a}}{a}\dot{H}_L = 0, \quad (48)$$

$$\ddot{H}_T + 3\frac{\dot{a}}{a}\dot{H}_T = \frac{1}{a^2}\left(H_L + \frac{\dot{a}}{a} B\right). \quad (49)$$

In order to solve the above equations, we use the residual gauge freedom (see Eq. (43)) to set $B = 0$. Then, from Eq. (47) we have

$$H_L = H_L(\mathbf{x}) \equiv \frac{10}{9}\Psi(\mathbf{x}). \quad (50)$$

Next, from Eq. (49), we obtain two independent solutions for H_T : one is proportional to $t^{2/3}$ and the other to t^{-1} . According to Eq. (45), we may also use the residual gauge freedom for H_T to add any function which does not depend on t . One may choose it as

$$H_T = \left(t^{2/3} - t_0^{2/3}\right)\Psi(\mathbf{x}) + \left(t^{-1} - t_0^{-1}\right)\Phi(\mathbf{x}) \quad (51)$$

so that H_T vanishes at the initial time $t = t_0$. The initial density field $\rho(\mathbf{x}, t_0)$ is also expressed by the metric. When the density contrast $\delta \equiv (\rho - \rho_b)/\rho_b$ is sufficiently small, we obtain from Eq. (20)

$$\delta = -\nabla^2 \left(t^{2/3}\Psi + t^{-1}\Phi\right), \quad (52)$$

where $\nabla^2 = \delta^{ij}\partial_i\partial_j$. Therefore, the initial density can be related with the initial metric perturbation as

$$\rho(\mathbf{x}, t_0) = \rho_b(t_0) \left[1 - \nabla^2 \left(t_0^{2/3}\Psi + t_0^{-1}\Phi\right)\right]. \quad (53)$$

4.3 Vector perturbations

The Einstein equations for the vector perturbations are

$$\nabla^2 \left(\dot{h}_i - \frac{1}{a^2} b_i\right) = 4\left(\frac{\dot{a}}{a}\right)' b_i, \quad (54)$$

$$\ddot{h}_i + 3\frac{\dot{a}}{a}\dot{h}_i - \frac{\dot{a}}{a^3} b_i = 0. \quad (55)$$

Introducing β_i as

$$b_i(\mathbf{x}) = \nabla^2 \beta_i(\mathbf{x}), \quad (56)$$

Eq. (54) is solved to give

$$h_i = -3 \left(t^{-1/3} - t_0^{-1/3}\right) \nabla^2 \beta_i(\mathbf{x}) + \frac{8}{3} \left(t^{-1} - t_0^{-1}\right) \beta_i(\mathbf{x}), \quad (57)$$

where we again used the residual gauge freedom (cf. Eq. (46)) to set $h_i(\mathbf{x}, t_0) = 0$. The $\beta_i(\mathbf{x})$ is directly related to the initial value of the vorticity field. From Eq. (30), we have

$$\omega^i(\mathbf{x}, t_0) = \frac{1}{2\sqrt{-g(\mathbf{x}, t_0)}} \epsilon^{ijk} \nabla^2 \beta_{j,k}, \quad (58)$$

where ϵ^{ijk} is the 3-dimensional Levi-Civita symbol with $\epsilon^{123} = 1$. Particularly when the deviation from the background is sufficiently small, we obtain the first-order expression

$$\omega^i(\mathbf{x}, t_0) \simeq \frac{1}{2a^3(t_0)} \epsilon^{ijk} \nabla^2 \beta_{j,k}. \quad (59)$$

4.4 Tensor perturbations

The equation for the tensor perturbations is

$$\ddot{H}_{ij} + 3\frac{\dot{a}}{a}\dot{H}_{ij} - \frac{1}{a^2}\nabla^2 H_{ij} = 0. \quad (60)$$

This is a homogeneous wave equation in the expanding universe. The solutions are well known and we will not discuss the detail here. See, e.g., [19].

As for the metric, our result is identical to that of the linear perturbation theory (e.g. [8]). However, there is an important difference to be emphasized: In Bardeen's paper, it is essential to linearize the density contrast. On the other hand, our Lagrangian approach does NOT rely on the assumption that *the density contrast should be small*. It is actually an important advantage of the Lagrangian approach that it uses (or extrapolates) the well-known solutions of the linear theory to express the non-linear density contrast.

5 Summary and discussion

Motivated by the fact that the Lagrangian condition enables us to obtain simply the integrals of the density and the vorticity along the fluid flow, the Lagrangian perturbation theory in the general relativistic cosmology has been developed fully using this condition [16]. The main advantage of the present Lagrangian theory is to be amenable to the vorticity, while previous works are limited within the irrotational fluid. In this approach, only the metric is expanded and its behaviors are determined perturbatively through the evolution equations, while the density and the vorticity are calculated non-perturbatively from the integrals along the flow. Seeds of the density and vorticity fluctuations can be related with the metric perturbation at an initial time, only when the constraint equation is solved as the Poisson-like equation. Hence, this simple approach is the first that can describe the dynamics of the dust fluid with the rotational motion as well as the non-linear density fluctuations, up to the caustic formation.

As an illustration, the first order Lagrangian perturbation has been solved explicitly. It is natural that at the first order of the metric perturbation, our result agrees with that of the usual linear theory [8] when we assume the small density contrast. In order to improve the accuracy and clarify the interaction between various modes, an extension to the second order is being investigated by Morita [20]. More comprehensive study of non-linear couplings will be of the subject of future investigation.

Furthermore, it is worthwhile to mention the following point: In the above, we consider the dust as the matter. However, we can take into account the pressure when the equation of state is specified like as $P = \rho/3$. Even in such a case, it is not necessary to expand material quantities a priori, since by using the Hamiltonian constraint, we can always delete the pressure term in the evolution equation.

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Can TAMA300 Detect Gravitational Waves?

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Abstract

Laser interferometers are considered the most promising means for detecting gravitational waves with a possibility of opening a new astronomy in the future, that is, the gravitational-wave astronomy. There are at present several on-going projects in the world for building large-scale (300m to 4km) laser interferometer gravitational wave detectors. TAMA, the Japanese project, is expected to be completed earliest among those projects. Its detector, TAMA300 with an arm length of 300m employs a recycled Fabry-Perot Michelson interferometer illuminated by a 10W laser with a triangular mode cleaner. So far the 10W laser with the mode cleaner was completed, and the Fabry-Perot Michelson interferometer was successfully operated with an auxiliary low power laser. Under the current schedule TAMA300 will start its operation with a reasonable sensitivity by the beginning of 2000. There will be even a slight chance that TAMA300 could detect gravitational waves for the first time in the world, although more realistic objectives of the TAMA project are to establish techniques required for building the next Japanese km-class gravitational wave detector in the near future.

1 Introduction

The detection of gravitational waves was pioneered by Weber in the early 1960s. He developed the first resonant bar detectors and announced that he had detected gravitational waves [1]. Although no convincing evidence for the detection was provided, his pioneering work inspired others to search for gravitational waves. In addition to the subsequent efforts on cooling bar detectors, Weiss and several other researchers had come to the idea of using laser interferometers to detect gravitational waves [2]. Since then several groups over the world have developed laser interferometer detectors and most experimentalists have come to believe that interferometers are the most promising means for detecting gravitational waves.

There are at present several on-going projects in the world for building large-scale (300m to 4km) laser interferometer gravitational wave detectors: LIGO [3] by the United States, VIRGO [4] by France and Italy, GEO [5] by Germany and England, and TAMA [6] by Japan. Although the TAMA project has the smallest detector (TAMA300) in its size among those project, TAMA is expected to be completed earliest; TAMA will start its operation with a reasonable sensitivity by the beginning of 2000. The question is whether TAMA can detect gravitation waves during its observation period for the first time in the world.

In this paper the basics of a laser interferometer gravitational wave detector is reviewed in chapter 2, the detail of the TAMA project is described in chapter 3, the possibility of gravitational wave detection by TAMA 300 is discussed in chapter 4, and finally conclusions are derived in chapter 5.

2 Laser Interferometer Gravitational Wave Detector

2.1 Detection Principle

A Michelson laser interferometer with suspended mirrors is used to detect gravitational waves. As shown in Fig. 1, when gravitational waves with an appropriate polarization passes the interferometer vertically, the lengths of the two arms are modulated differentially. It results in changes in the intensity of the interfering light, which is detected by a photodetector. In the actual interferometer the intensity of the

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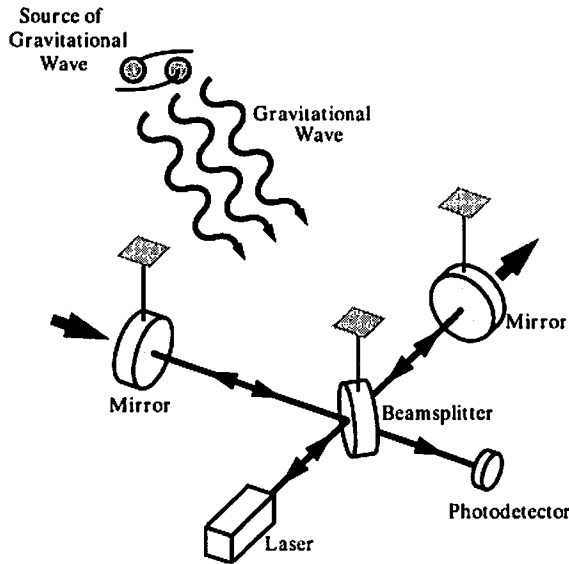


Figure 1: Michelson Interferometer with suspended mirrors for detecting gravitational waves

light at the detector port will be kept dark by controlling the arm length. Therefore the feedback signal to the mirrors should contain gravitational wave signals. The beamsplitter and mirrors should be suspended so that they can free-fall horizontally at frequencies much above their pendulum frequencies.

It should be noted that the length change caused by gravitational waves can be regarded either as the consequence of the force applied to the end mirrors or that of the variation of the apparent speed of light, depending on the coordinate system considered. In the proper reference frame of an observer, where the apparent speed of light is constant, the force is applied to masses by gravitational waves with the amplitude of the force proportional to the mass's coordinate position. Therefore the end mirrors are shaken back and forth by the force, whereas the beamsplitter is sitting at rest. On the other hand, in the TT coordinate system, where any freely-falling mass with no initial velocity does not change its position, the apparent speed of light is modulated by gravitational waves. As a result the optical path difference, measured by such modulated light, is altered. Naturally, the amount of changes in the intensity of the interfering light does not depend on a coordinate system to be considered. It should be also noted that the proper reference frame is only valid when the size of the interferometer is relatively small; when the arm length is larger, the TT coordinate system should be used.

Laser interferometers are ideally suited for detecting gravitational waves, because the length change caused by gravitational waves can be increased by simply scaling up the arm length. Therefore the larger interferometer will provide better sensitivity to gravitational waves. However, the signal would be cancelled if the arm length is too long. For example if the arm length is equal to a half the wavelength of the gravitational wave to be detected, in the TT coordinate system the light experiences modulation of the apparent speed of light positively half the time of its stay in the arm and negatively the rest of the time, resulting in a complete cancellation of the total phase change of the modulated light. It can be derived that the optimum arm length is a quarter the wavelength of the gravitational wave. For example an arm length of 750km is optimum for the 100Hz gravitational wave.

2.2 Noise Sources

Noise sources existing in the interferometer can be classified into two categories: sensor noise and displacement noise. The sensor noise comes in during the sensing process, that includes frequency/intensity noise of the light and shot noise. The displacement noise is the result of directly shaken mirrors, such as seismic noise and thermal noise. Shot noise, seismic noise and thermal noise are regarded as three most fundamental noise sources existing in the interferometer, whereas the other noise sources are considered to be practical noise.

2.2.1 Shot Noise

Shot noise exists because a photon has finite energy. Statistical fluctuations in the photon number in the detection process result in shot noise. In order to reduce the shot noise level, the interferometer should be operated in such a way that the intensity of the light at the detection port is dark. This is because the fluctuations in the photon number are smaller when the intensity of the light at the photodetector is lower.

In addition to this fringe control, it is desirable to have higher incident laser power to reduce the shot noise level by the following reason. Assuming that the interferometer is controlled with a dark fringe at the detection port, the intensity change at the photodetector caused by gravitational waves is proportional to the incident laser power, while the intensity fluctuations caused by shot noise is proportional to square root of the incident laser power; thus the signal to shot noise ratio is improved by square root of the incident laser power. However, note that radiation pressure noise caused by quantum fluctuations of the intensity of the light hitting the mirrors could exceed the sensing shot noise effect if the laser power is too high.

2.2.2 Seismic Noise

Seismic noise can be reduced by isolating the mirrors from the ground motion. Suspending the mirrors to make the freely falling masses also plays a role of isolating the mirrors from the ground mirrors. The isolation factor increases with inversely squared power of frequency above the pendulum frequency of the suspension system. More isolation can be obtained by using a multi-stage pendulum. It should be emphasized that the vertical isolation is crucial, especially the multi-stage pendulum is used. The pendulum is, in principle, only effective for isolation of the horizontal motion. The vertical motion of the mirrors, which is not strongly isolated by the pendulum system, could be turned into the length change of the arm cavity, coupling with the deviation of the beam axis from the local horizontal line or mechanical cross-coupling from the vertical motion to the horizontal motion.

In addition to the isolation by pendulum, very often used are stacks consisting of several metal plates and elastomers on which the suspension system is placed. The stack system is easy to use and has a stable performance, although it is difficult to have a significant isolation below 10Hz.

2.2.3 Thermal Noise

Any mass cannot be exempted from Brownian motion. According to thermodynamics, any mode of a system immersed in another system with a finite temperature is given energy determined by the temperature of the outer system as a result of the energy exchange between the two systems. There are two important modes relevant to the suspended mirrors: the pendulum mode and internal mode. The pendulum frequency is usually much lower than the interesting frequency range of gravitational waves, and the internal mode resonant frequencies are much higher. In any cases, thermal noise at the relevant frequencies can be reduced by making the quality factor of the modes higher. Since a mode with a higher quality factor has more energy concentrated around the resonances, lower thermal motion is engaged off the resonant frequencies. This is because the root-mean-square thermal motion integrated over the frequency should be constant. Therefore our endeavor to reduce thermal noise is usually efforts on increase of the quality factor of the suspended mirrors. It can be realized by reducing any mechanical losses involved in the system.

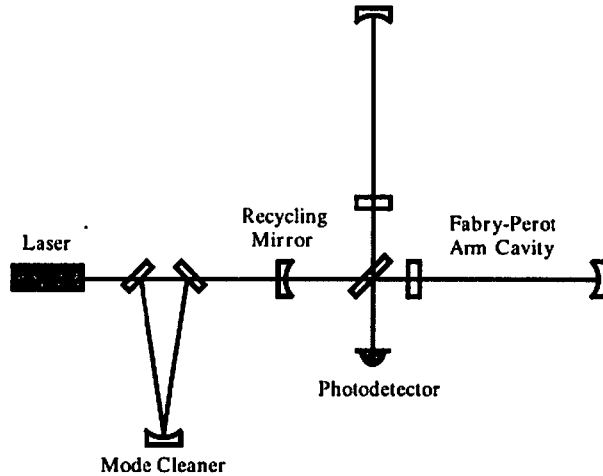


Figure 2: Standard optical configuration which has three elements: Fabry-Perot arm cavity, recycling, and mode cleaner.

2.3 Standard Optical Configuration

The actual interferometer gravitational wave detector is far more complicated than a simple Michelson interferometer. Fig. 2 shows an interferometer, which is currently considered to have a standard optical configuration. It has three optical elements: Fabry-Perot arm cavity, recycling, and mode cleaner.

2.3.1 Fabry-Perot Arm Cavity

Since it is unrealistic to build a 750km interferometer on earth, the idea of using Fabry-Perot arm cavities have been brought up. In the Fabry-Perot arm cavity the light is bounced back and forth effectively by many times when it is on resonance in the cavity. Thus the phase change of the light caused by gravitational waves is amplified by the effective number of bounces. For example, an arm length of 3km with 250 effective bounces is almost equivalent to a simple arm length of 750km in terms of signals caused by gravitational waves. Since most sensor noise do not explicitly depend on the effective number of bounces, the signal to noise ratio of the interferometer can be improved by using the Fabry-Perot arm cavity. However, this advantage does not hold for the displacement noise, since the displacement noise is also amplified by the cavity exactly in the same manner as the gravitational wave signal.

2.3.2 Recycling

Since the interferometer is controlled in such a way that the intensity of the interfering light at the detection port is dark, all the light is reflected back to the laser, and is thrown away unused. The idea of recycling is to reflect this unused light coherently back to the interferometer so that the laser power is effectively increased, which leads better shot noise limit. The extent to which the effective power can be increased depend on the loss of light in the interferometer. The lower loss of the light in the interferometer provides higher recycling gain. The transmission of the recycling mirror should be set nearly equal to the loss of the light in the interferometer to maximize the recycling gain.

2.3.3 Mode Cleaner

The laser geometrical fluctuations, such as beam jitter and pulsation, could make a significant noise in the interferometer coupling with asymmetry of the interferometer. A mode cleaner is used to suppress the laser geometrical fluctuations by cleaning up the transverse mode of the laser light before it enters the interferometer. When the fundamental transverse mode of the light is on resonance in the mode cleaner, other higher-order transverse modes cannot get transmitted through the mode cleaner because of the difference of the resonant condition between different modes. Since the laser geometrical fluctuations are the outcome of mixture between the fundamental mode and higher-order modes with time-varying relative amplitude, the suppression of the higher-order mode results in reduction of the beam geometrical fluctuations.

The mode cleaner also serves as a reference cavity for the coarse frequency stabilization as well as a passive low pass filter for frequency and intensity of the light whose corner frequency is determined by the finesse of the mode cleaner. Incidentally the advantage of using a ring-type mode cleaner is to avoid spurious interference of the light between the near mirrors of the arm cavity and the output mirror of the mode cleaner.

2.4 Control System

In order to operate the recycled Fabry-Perot Michelson interferometer with the mode cleaner properly, all the cavities and Michelson interferometer should be controlled in their lengths as well as alignment.

2.4.1 Length

Length of all the cavities and Michelson interferometer should be controlled so that the light is on resonance in all the cavities and the detection port of the Michelson interferometer is kept dark. Pound-Drever-Hall method and Schnupp modulation are used for this purpose. Both methods employ a phase modulation technique to extract relevant signals corresponding to the cavity length or differential Michelson length.

There are four important lengths in the recycled Fabry-Perot Michelson interferometer: the common-mode arm cavity length, the differential arm cavity length, the recycling cavity length, and the differential Michelson length. It was found that only one phase modulation can extract all the four length signals. As shown in Fig. 3, in order to extract the signals, the light should be picked off before the recycling mirror and between the recycling mirror and the beamsplitter in addition to the ordinary detection port. The output from those photodetectors are demodulated with in-phase and quadrature-phase to produce the four length signals. The signals are fed back to the mirrors and the frequency of the light through appropriate filter/amplifier. The magnet coil system is often used to actuate the mirrors because of its simplicity.

2.4.2 Alignment

Wavefront sensing technique is used to detect angular misalignments of the interferometer mirrors. This is an extension of the length sensing system described above. In the presence of angular misalignments, the length signal has a spatial gradients. This can be detected by using a segmented photodiode and differentiating the demodulated signals from the segments. The angular information of the mirrors in the interferometer can be independently extracted by sensing the spatial gradients at different points. The advantage of the wavefront sensing compared with a conventional optical lever is that the optical lever senses the mirror angle only in the local frame, whereas the wavefront sensing detects the mirror orientation with respect to the incident beam. Another practical advantage of the wavefront sensing is that the phase modulation used for the length sensing can serve a role of modulation for the alignment sensing.

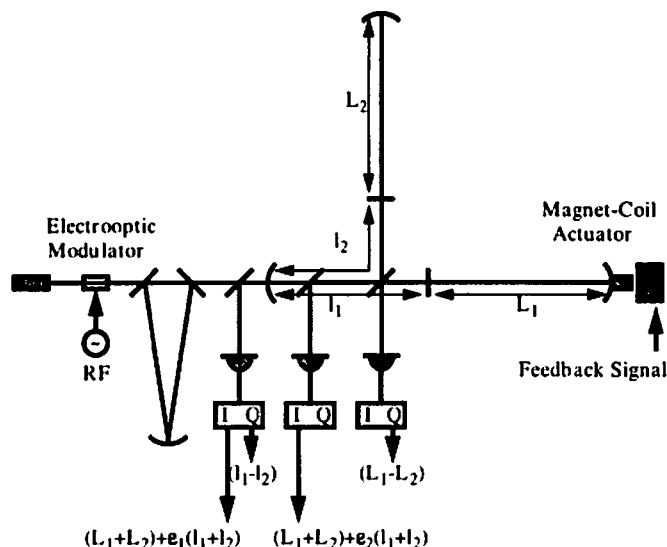


Figure 3: Length signal extraction method for the recycled Fabry-Perot Michelson interferometer

3 The TAMA Project

The TAMA project started in 1995 with the collaborating efforts of National Astronomical Observatory, The University of Tokyo, Institute for Cosmic Ray Research, Institute for Laser Science, High Energy Accelerator Research Organization, Yukawa Institute of Theoretical Physics, and Osaka University. Miyagi University of Education joined the project later. The TAMA project has two objectives: to establish techniques necessary for the future km-class detector and to detect gravitational waves from nearby galaxies if possible at all.

3.1 Specifications of TAMA300

The TAMA project is building a laser interferometer with 300m arm length (TAMA300). It is located on the campus of National Astronomical Observatory in Tokyo (Fig. 4 and Fig. 5). TAMA300 employs a standard interferometer, that is a recycled Fabry-Perot Michelson interferometer illuminated by a 10W laser with a triangular mode cleaner. The finesse of the Fabry-Perot arm cavities is approximately 500. The laser source is an injection-locked Nd:YAG laser with a wavelength of 1064nm. The recycling gain will be about 10. The projected sensitivity for gravitational waves is 3×10^{-21} in terms of the root-mean-square strain at 300Hz with a bandwidth of 300Hz. As shown in Fig. 6, the seismic noise is dominant at low frequencies, the shot noise is dominant at higher frequencies, and the sensitivity is limited by thermal noise at intermediate frequencies.

The most outstanding uniqueness of TAMA300 is its signal extraction scheme. In the standard signal extraction method described in Section 2.3.1, the recycling cavity length signal is always dominated by the common-mode arm cavity length signal, because of the high finesse of the arm cavity. This could cause various complications in the control system such as stability criteria and noise cross-coupling. Ando [7] invented a clever method which circumvents the problem existing in the standard signal extraction method for the recycled Fabry-Perot Michelson interferometer. This new signal extraction scheme requires the reflectance of the recycling mirror to be optimized to the sideband rather than to the carrier. In this



Figure 4: TAMA300 located on the campus of National Astronomical Observatory.



Figure 5: Recycling and mode cleaner chambers of TAMA300

condition, the in-phase demodulated signal from the pick-off port before the recycling mirror can be dominated by the recycling cavity length signal. The principle of this method has been already verified using the 3m prototype at Tokyo University.

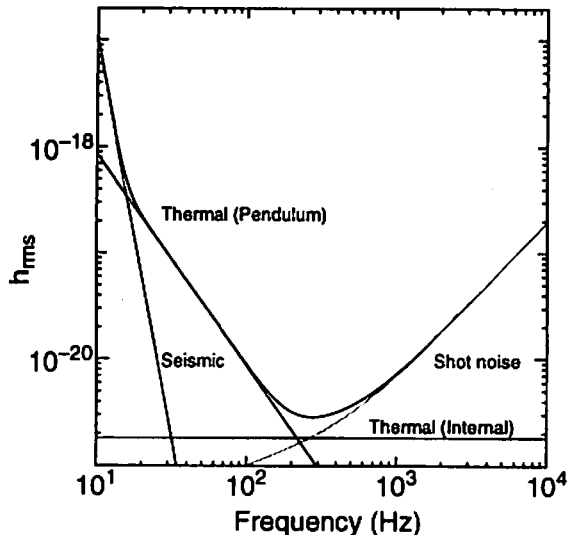


Figure 6: Projected sensitivity of TAMA300

3.2 Schedule, Milestones, and Current Status

In 1997 the facility and the vacuum system of TAMA300 were completed. The 10W Nd:YAG laser using the injection locking technique was developed by Sony Corporation. The ring-cavity mode cleaner was installed and checked in performance with the 500mW Nd:YAG laser. In 1998 the one arm cavity consisting of two suspended mirrors was installed and locked using the 700mW Nd:YAG laser with both mirrors controlled in orientation by the wave front sensing signals. We then installed the other arm cavity, the beam-splitter, and two pick-off mirrors. We successfully locked the Fabry-Perot Michelson interferometer. In parallel with this effort, we have injected the 10W laser to the mode cleaner. In February 1999 we plan to connect the 10W laser and mode cleaner to the Fabry-Perot Michelson interferometer. When we increase the sensitivity to a reasonable level in summer 1999, we will take data of the detector for one month. We will then install the recycling in the interferometer. We expect to have a full sensitivity of TAMA around the beginning of 2000. In the following subsections, some important experimental results obtained so far are introduced.

3.2.1 Sidebands Transmission of Mode Cleaner

The phase modulation required to extract length signals of the recycled Fabry-Perot Michelson interferometer should be imposed on the light before the mode cleaner; otherwise the light cleaned by the mode cleaner would be made dirtier again by an electro-optic modulator for the phase modulation. It is possible to transmit the modulation sidebands through the mode cleaner by making the modulation frequency exactly equal to the free spectral range of the mode cleaner. The problem found in such a system

is that the transmitted light has excess noise beyond the shot noise around the modulation frequency, which would preclude the shot noise limited sensitivity of the whole interferometer. Telada [8] revealed the mechanism of the excess noise by a careful experiment, and demonstrated that the excess noise was satisfactorily suppressed by adjusting the modulation frequency to the free spectral range of the mode cleaner as well as by adjusting the carrier frequency to the resonant frequency of the mode cleaner.

3.2.2 One Arm Interferometer

The one arm cavity was held locking very stably for ten days with only several losses of locking. Both mirrors were controlled in orientation by the wavefront sensing signals. The stability of this one arm interferometer is outstanding with the fact that most prototypes of the laser interferometer gravitational wave detector in the world can be held locking only for several hours at most.

The absolute length of the 300m arm cavity was measured with the accuracy of 1 micrometer using a new technique, which used the phase modulation sideband transmitted through the arm cavity [9]. The result showed that the motion of the 300m arm cavity was dominated by peaks of 20 micrometers which occurred regularly everyday. It turned out that the peaks were caused by pumping the underground water by a hospital nearby.

3.2.3 Fabry-Perot Michelson Interferometer

The Fabry-Perot Michelson interferometer has been locked with the four mirrors controlled in orientation by the wavefront sensing signals. The interferometer was first locked by sensing the two arm cavity lengths independently with the help of the pick-off mirrors located between the beamsplitter and the near mirrors, and then locked by the standard signal extraction method, that is, by sensing the common-mode and differential arm cavity length.

4 Possibility of Detection

The TAMA project has been going very smoothly, which makes it reasonably believable that the aimed sensitivity will be reached by the beginning of 2000. Another question which is relevant to the possibility of detection of gravitational waves by TAMA300 is how much and how often gravitational waves come to the earth. According to the estimate on the neutron star binary coalescence, however, the gravitational wave from the neutron star binary coalescence with a strain detectable by TAMA300 would come to the earth much less than once per a thousand years.

However, Nakamura [10] speculated that if MACHOs are black holes, there should be many black hole binaries in the universe, thus their coalescence rate should be very high. It was estimated that the gravitational wave from the black hole binary coalescence with a strain detectable by TAMA300 would come to the earth once per about twenty years. It means that if TAMA300 is operated for one full year, the possibility of detection is one-twentieth, which is not hopeless but not extremely promising. Nevertheless, the operation of TAMA300, which will take place first in the world, is meaningful, because there might be unexpected strong gravitational wave sources the mechanism of which is still unknown.

5 Conclusions

Among several on-going projects of gravitational wave detection in the world, the TAMA project is the most speedy one and is actually going extremely well. Under the current schedule TAMA300 will start its operation with a reasonable sensitivity by the beginning of 2000. There will be even a slight chance that TAMA300 could detect gravitational waves for the first time in the world. Even without detection, TAMA will establish techniques required for building the next km-class detector which is expected to detect gravitational waves.

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Data Analysis Methods for Laser Interferometric Gravitational Wave Detector

— Study for event search with the TAMA 300 —

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Abstract

We report about event search methods for laser interferometric gravitational wave detector in this paper. Facilities of laser interferometric gravitational wave detector in the world will proceed to realistic observation in next decade. Since gravitational wave events are expected in rare rate or in small amplitude, the search and signal extraction methods are kind of the important tool for detector system. We give not only general descriptions, but also the simulation study for TAMA 300 detector, which is constructing in Japan. The event analysis/extraction methods for the interferometric gravitational wave detector are researching. Especially, methods for binary inspiral waveform are studied very well in principle. On the other hand, to prepare the real data analysis of the interferometer operation, we need more research for the executable procedure with the huge amount of experimental data and study for both systematic and statistical error.

1 Introduction

There are some laser interferometric gravitational wave detector projects in the world as LIGO [1], VIRGO [2], GEO, and TAMA [5]. These are currently going on, and possible to detect gravitational wave event even if a few Mpc away. Now, our researches for gravitational wave detection proceed to the data analysis stage or to prepare it.

The gravitational wave is a propagation of strain of metric perturbation h as solution of wave equation [3].

$$\left(\nabla^2 - \frac{1}{c} \frac{\partial^2}{\partial t^2} \right) h_{\mu\nu} = 0 \quad (1)$$

where, $h_{\mu\nu}$ respects the metric perturbation away from Minkowski space as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (2)$$

The lowest order of a gravitational wave is quadrupole radiation. Consider as wave propagation along z axis, the wave can be represented with two components in transverse and traceless gage as

$$h_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = a \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = a \widehat{h}_+, \quad (3)$$

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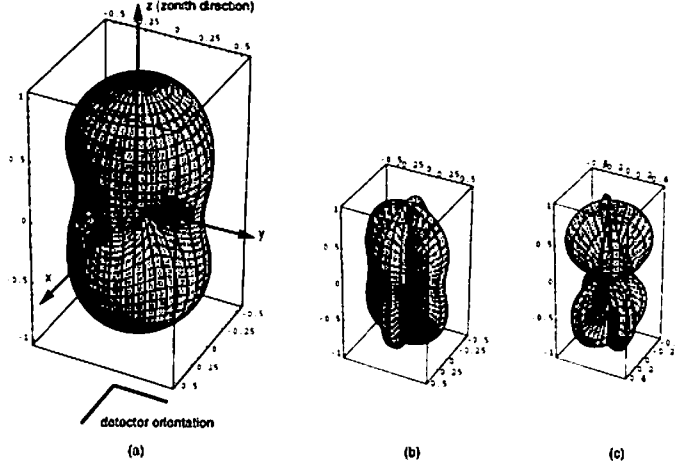


Figure 1: Antenna Pattern of Interferometric Detector

These are antenna sensitivity pattern of the interferometric gravitational wave detector. The surface distance from the origin shows the sensitivity that normalized by it of zenith incident. (a): combined response for two polarization mode, (b): response for one polarization with relative rotation angle (See ref[4]) is 0 rad, (c) same as b) but rotation angle is 1 rad.

and

$$h_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = b \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = b \widehat{h}_x. \quad (4)$$

The linear polarization of $\widehat{h}_+$ and $\widehat{h}_x$ is a basis of the quadrupole wave³.

A derivative of a quadrupole moment of source mass distribution decide the strongest allowed component of gravitational radiation as

$$h_{\mu\nu} = \frac{2G}{Rc^4} \ddot{I}_{\mu\nu}, \quad (5)$$

where R is a distance from the source, c is a speed of light. Since the coupling constant of the gravitation G is weak, a possible gravitational wave source is an astronomical event. Physicists predict various gravitational wave from astronomical object [4]. For the limitation of frequency band due to environments⁴ and instruments, target frequency band of ground-based gravitational wave detectors are higher than a few 10 Hz to lower than 10 kHz. Expected astronomical sources are coalescence of binaries of neutron stars/black holes, supernova, continuous wave from pulsar and others.

The gravitational wave from coalescence of binary stars are roughly 10^{-22} in strain amplitude at frequency of $10 - 10^3$ Hz for $1.4M_\odot$ pairs at 200 Mpc away. Its event rate is expected as 3 events/yr within 200 Mpc [8]. We expect it as a "standard candle" for our research, since we can predict the waveform and amplitude. Type II supernova is another popular expected source. We have been believed the amplitude is comparable with compact binaries. However, using numerical calculation for rotating core in reference [9], the amplitude is expected 10^{-20} at 10 kpc at frequency band $10^2 - 10^3$ Hz. Its rate is said as $3 - 4 \times 10^{-2}$ events/yr/galaxy. Continuous sources are neutron pulsar like as a Crab pulsar. The expected amplitude is strongly depend on the model, and is weak. However, we can enhance the evidence in integration in long time observation. Waveforms of them are different, but the detection principal is similar.

³We can choose circular polarization basis instead of them.

⁴almost seismic motion of the ground

We explain the detection principle in simple. The detection principle is a very precise measurement of the length between two free masses. To use the quadrupole polarization characteristics of the gravitational wave, the laser interferometric detector consists of two perpendicular long path of laser light. This is a scale up of the Michelson interferometer[6]. Moreover, these are not only simple scaling, but also many techniques to enhance the sensitivity. Typical interferometer is a Fabry-Perot-Michelson type as Fig.4. A gravitational wave will move free masses of the detector. Suspended mirrors of the interferometer act as free masses in observation frequency band. The displacements of free masses make a phase difference between returning laser beam light from each main arm. The intensity of the recombined lights on beam splitter will change. A photo-detector measures the intensity of the light on the anti-symmetric (dark) port of the beam splitter. The output photo-current from the detector will be amplified, and will be digitized by ADC(analog-to-digital converter). On-line system acquires the ADC data, and record on mass storage media. Thus, the strain amplitude of metric perturbation of gravitational wave will appear as an amplitude change in a digitized data. The detector response[7] for gravitational waves are Fig.1. It has a Maximum sensitivity at zenith incident. The average response for all sky is $\frac{1}{\sqrt{5}}$ of maximum sensitivity. On the other hand, Due to the instruments of the detectors and due to environments, the data include various kinds of noises.

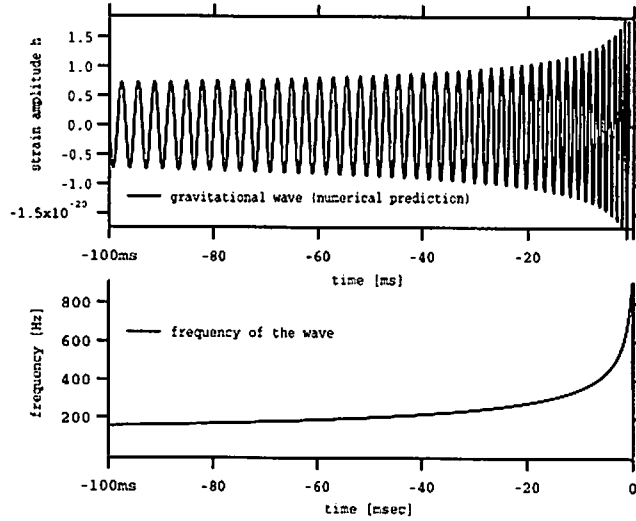


Figure 2: Predicted Gravitational Wave from Binary Coalescence

Post-Newtonian calculation of “chirp” gravitational wave from $1.4M_{\odot} - 1.4M_{\odot}$ stars coalescence at 1 Mpc away. Both Strain $h(t)$ and the frequency of it is shown by numerical calculation. caption text

Since the gravitational event is rare, or is quite small amplitude in expectations [4]. The strain amplitude $h(t)$ of the gravitational wave from coalescence of $1.4M_{\odot}$ neutron star binaries at 1 Mpc away is order of 10^{-20} [11] [12]. Moreover, an amount of data is huge for continuous sampling data. Since we expect not only burst like gravitational wave but also continuous one, we would like to keep all data. For example, the data amount of TAMA is 600 kByte/sec with 20 kHz sampling. Even if only processed strain amplitude $h(t)$, at least 80 kByte/sec in double precision. Therefore, requirements of the data analysis issues of interferometric gravitational wave detector are 1) signal enhancement for small amplitude signals, 2) high noise reduction rate and 3) fast processing for rare event search or for continuous wave search with long time data. The data analysis for gravitational wave event search is generally categorized in two. One is random (stochastic) and short time event like as a burst. Coalescence events of compact binaries are also included this. Other is a continuous wave search.

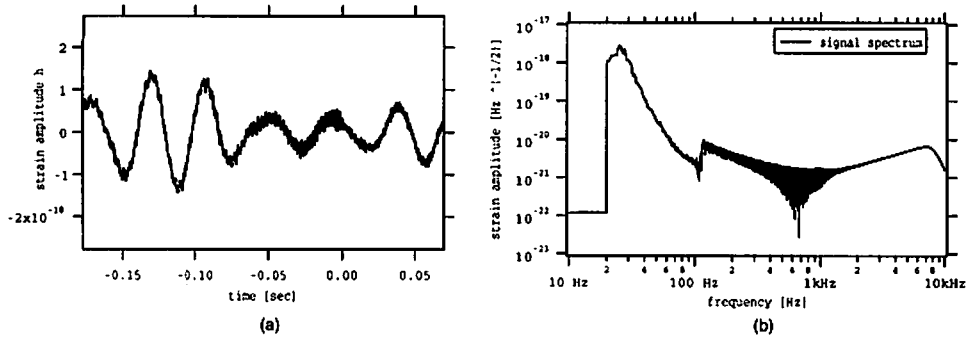


Figure 3: Example of Gravitational Wave on TAMA simulation

Simulated signal with noise on TAMA phase-I operation. (a) is a signal in time domain. To make easy to recognize the signal by human, we reduce the noise by LPF(Low Pass Filter) and by HPF(High Pass Filter) in this figure. The final of the chirp wave is 0 sec in this figure. (b) is a power spectrum of the signal in frequency domain. At low and high frequency, the decrease of the noise amplitude is result of LPF and HPF.

TAMA and Monte-Carlo Simulation Since we want to demonstrate these data analysis method with simulation studies using real detector characteristics, we introduce the TAMA project here. TAMA is a 300 m baseline laser interferometric gravitational wave detector in Mitaka, a suburb of Tokyo, Japan. The project is funded for five years from April 1995 to March 2000. It is a collaboration of universities and institutes. Fig.4 is a schematic view of TAMA detector system. The interferometer is a Fabry-Perot-Michelson type with a recycling mirror. The target sensitivity of the detector is 3×10^{-21} at 300 Hz. It will be possible to detect a gravitational wave from the coalescence of binary neutron stars in our galaxy, or events within 700 kpc. We plan to have continuous observation for the last year of the project. Phase-I operation without power recycling technic will start in summer of 1999. Phase-II operation with recycling and some other developments is planned after it. The fundamental parameters of TAMA are shown in Table 1.

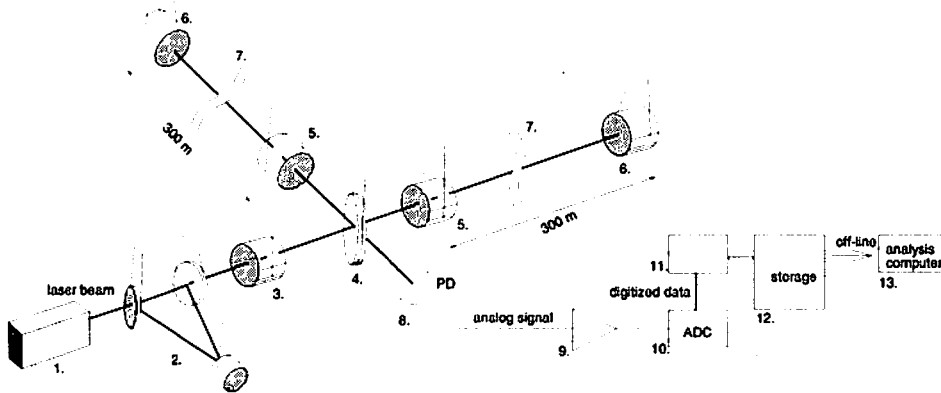


Figure 4: Schematic view of TAMA

1:laser, 2:mode cleaner, 3:recycling mirror, 4:beam splitter, 5:frontal mirror, 6:end mirror, 7:Fabry-Perot cavity of 300 m base line that consists of 5 and 6, 8:photo detector, 9:electronics, 10:ADC, 11:on-line computer for data acquisition, 12:DLT mass storage device, 13:off-line computers for analysis

The noise spectral density of the detector show a characteristics and sensitivity of the detector. We

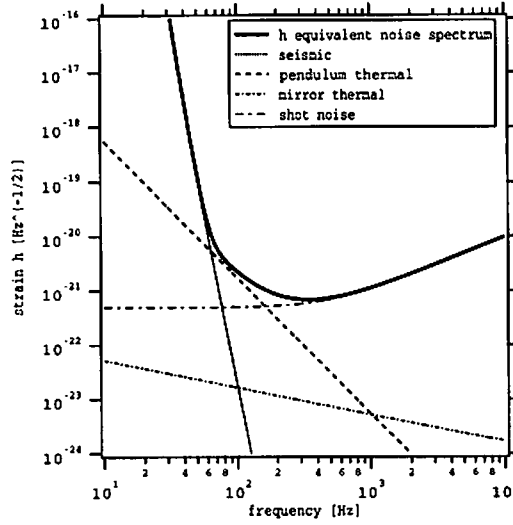


Figure 5: Noise spectrum of TAMA phase-I (expected)

Each lines show contributions of seismic noise, pendulum thermal motion, mirror internal thermal motion, and shot noise due to laser power. The solid line is a total expected noise spectrum reduced to strain amplitude.

give it as a spectrum that is equivalent to the strain amplitude. Fig.5 is an expected noise spectrum⁵ of TAMA phase-I. Dominant noise sources are different in frequency region as shown in the figure. A possible lowest observation frequency is decided by seismic noise and vibration isolation system. A highest frequency is limited by shot-noise and Fabry-Perot cavity transfer function. In TAMA, observation band is roughly from 100 Hz to 5 kHz.

We study with Monte-Carlo simulation using this expected noise spectrum. The assumptions of the noise spectrum are gaussian fluctuate around this expected average in finite time length and random phase. We introduce Monte-Carlo method to archive these effects. We make time domain signal using inverse-FFT(fast fourier transform) of the spectrum which is fluctuate in amplitude and is randomized in phase. The importance of this Monte-Carlo study is to make a realistic detector signal sample according to statistics. We superimpose the calculated gravitational wave signal with the detector response. In any case, we can not get strain of gravitational wave $h(t)$ exactly. We get only $h(t) + \text{noise}$ always. The subtraction of noise is impossible without assumption of gravitational waveform. Also the detector response (transfer function) and noise are strongly depend on the frequency, therefore we must study with realistic reproduction of detector output.

Another simulation approach is to use a real experiment data that may not include gravitational wave instead of Monte-Carlo noise generation as above. This is more realistic study, since it have the detector noise characteristics complete. However, if we want to confirm and to estimate a systematic error due to each process step by step, we need Monte-Carlo like simulation.

We give some examples of data analysis methods on TAMA simulation in following sections.

2 Stochastic Event

Coalescence of compact binary stars is suitable source for first target of gravitational wave detection. Neutron star - neutron star, neutron star - black hole or black hole - black hole binary system radiate its energy as a gravitational wave at roughly twice frequency of Kepler orbit period. And finally, 10^8 years after beginning of binary system, two stars fall into each other. In this death of system, which is called as "coalescence", the system emit strong g gravitational wave [10]. The Kepler frequency will

⁵To compare the gravitational wave, it is reduced equivalent to strain amplitude.

Table 1: Fundamental Parameters of TAMA

Sensitivity	$h_{rms} = 3 \times 10^{-21}$ at 300 Hz (Bandwidth 300 Hz)
Configuration	Fabry-Perot-Michelson with power recycling
Baseline	300 m
Finesse of the arm cavity	520
Light source	Injection locked Nd:YAG laser, 10 W, 1064 nm
Recycling gain	10
Effective input power	30 W

increases according to the falling, gravitational wave frequency will increase, and also amplitude will be same as. This waveform is called as “chirp” wave. The waveform of it is predicted using Post-Newtonian approximation [11] [12]. Fig. 2 is a numerical calculated waveform of $1.4M_{\odot}$ pairs from 1 Mpc away using 2nd order approximation. Mass of each star determines the frequency development of chirp waveform. We can measure the neutron star mass parameter if we detector and extract waveform. Typical data that contains the gravitational wave event is expected as a Fig.3.

Methods to find the chirp waveform are categorized into two kinds.

2.1 Methods with an Assumption of Waveform: for coalescence

One kind of methods is comparison of data with *a priori* waveform. How to compare is different. Matched filtering method is a use a Fourier domain. Resampling method use a characteristic of frequency increase of chirp wave.

Matched Filtering The matched filtering method [13] is a comparison of data to waveforms with variations of parameters. To evaluate the similarity between two waveform $s(t)$ and $h(t)$ as a function of time t , we can make cross-correlation product c as

$$c = \int_{-\infty}^{\infty} h(t - \tau) \cdot s(t) dt, \quad (6)$$

where τ is a time difference. In the data analysis of interferometric detector, $s(t)$ is the observed data and $h(t)$ is a predicted gravitational waveform. We call the predicted waveform as “template”, and call set of templates with parameter variations as “template bank”. Now, if the data include gravitational wave, the data is represented with noise component $n(t)$ as,

$$s(t) = h_{observed} + n(t). \quad (7)$$

We can represent equation 6 in frequency domain by Fourier transform. It make possible to integrate with frequency f instead of t . Since the template waveform is decided by source masses, initial phase and incident angle to the detector⁶, output c can be written as a function of them.

$$\begin{aligned} c(\tau; m_1, m_2, \phi, \dots) &= \int_{-\infty}^{\infty} h(t - \tau) \cdot s(t) dt \\ &= \int_{-\infty}^{\infty} \tilde{h}^*(f) \cdot \tilde{s}(f) e^{-2\pi i f \tau} df \end{aligned} \quad (8)$$

, where $\tilde{s}(f)$ and $\tilde{h}(f)$ is a Fourier transform of $s(t)$ and $h(t)$.

However, noise $n(t)$'s characteristic is not flat in frequency domain. Therefore, the observation must have a weight which depend on noise average at the frequency. We normalize the product by noise spectral density $\tilde{n}(f)$ of interferometer signal. Since a phase of noise will be random, we use square of $\tilde{n}(f)$. Moreover, the data is finite length, and finite sampling frequency. Thus, the Fourier transformation and the integration beyond should be re-written when we calculate for real data.

⁶Remind that we cannot decide incident angle with one detector.

$$c = \int_{f_1}^{f_2} \frac{\tilde{h}^*(f) \cdot \tilde{s}(f)}{|\tilde{n}(f)|^2} e^{-2\pi i f \tau} df. \quad (9)$$

Or, if we include the arrival time variation τ in waveform, τ will not appear explicitly in Fourier domain. It change phases of complex number $\tilde{h}(f)$.

$$c = \int_{f_1}^{f_2} \frac{\tilde{h}^*(f; \tau) \cdot \tilde{s}(f)}{|\tilde{n}(f)|^2} df \quad (10)$$

This is a output value with one template $h(t)$. In analysis for the experiment data, we use measured $|\tilde{n}(f)|$. Data of digitized signal are processed using FFT. Also numerical calculated templates $h(t)$ are converted using FFT.

Since we do not know the gravitational wave source parameter which incident on the detector at first, we need many templates with parameter variations. According to equation 9 or 10 with bank of templates, we found most likely templates as a maximum output

$$c_{max} = \text{Maximum}(c \text{ with various templates}). \quad (11)$$

If data has no chirp evidence, the distribution of the c is a gaussian ,which is normalized to zero average, unity standard variance. Because there is no correlation between noise and chirp wave. C_{max} correspond to the normalized Signal-to-Noise ratio (SNR) with integration from f_1 to f_2 . We can find the gravitational wave evidence as a excess at gaussian tail. Fig.6 shows the normalized SNR (c) distribution and the gravitational waves' c_{max} distribution in Monte-Carlo simulation of TAMA phase-I. The figure demonstrate that 1 Mpc event of typical neutron star coalescence is detectable. Fig.7 shows the c_{max} as a function of distance. The threshold of the event selection must be decide the fake ratio. We set selection criterion as $c_{max} > N\sigma$ with appropriate number N . If we choose lower threshold, we will not miss the gravitational wave events, but also statistical fake with noise will survive.

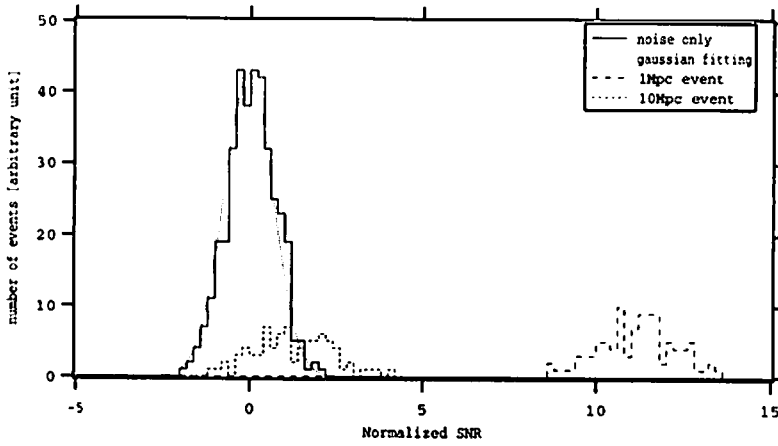


Figure 6: Matched Filter Output

Histogram of normalized SNR c in the text. The distribution that shown with solid line contains no gravitational wave event. Distributions of $1.4M_{\odot}$ pairs coalescence at 10 Mpc and 1 Mpc are also shown.

Certain waveform parameters will decide by c_{max} . Fig.8 is a contour plot of c_{max} output with parameter variations. The answer is $1.4M_{\odot}$ in mass and 0 sec in arrival time. Typically, mass parameter precision with matched filtering method in TAMA phase-I is expected less than 1% for $\frac{\Delta m}{m}$, and 1 msec for arrival time.

The matched filtering is good SNR enhancement, and can decide parameters precisely. Comparing with other method, these are big advantage of it.

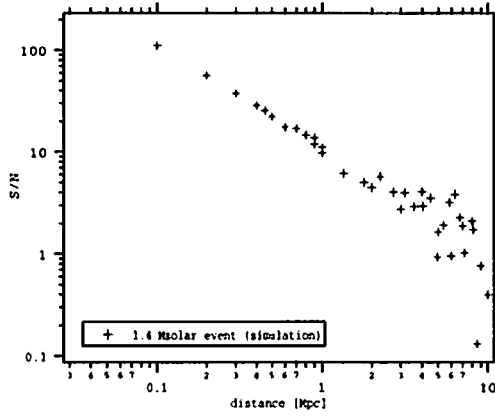


Figure 7: Distance vs. SNR

Distance vs. normalized SNR of matched filtering for $1.4M_{\odot}$ pairs coalescence are shown. If we set the threshold at $\text{SNR} = 10$ as a selection criteria, we can detect events within 1 Mpc with matched filtering method.

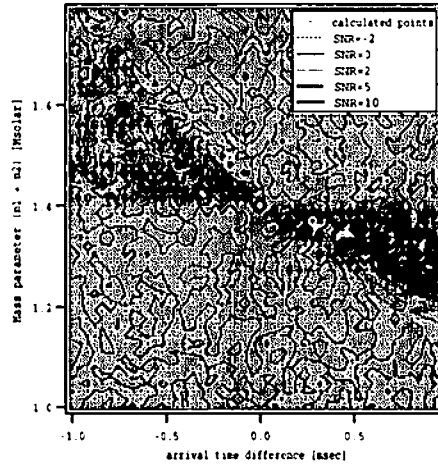


Figure 8: SNR on Lattice of Parameters

The figure shows SNR value for the function of parameters for matched filtering. $1.4M_{\odot}$, 0 sec arrival is a simulated signal. This contour plot suggest the accuracy in parameter estimation.

A disadvantage of the matched filtering is calculation cost. It requires huge size of templates bank, and must be repeat calculation according to data output. A Total CPU cost will up to almost TMips. The bank size is more than 500 GByte. It need to have a fast computing architects not only for CPU speed, but also high speed management of data and template bank. To prepare continuous processing for data of TAMA or future detector, we must find the solution.

Resampling (Smith's Method) The resampling method which is proposed by S. Smith [14] is one of a signal extraction method for "chirp" gravitational wave from binary coalescence. The resampling method is how to change the sampling interval to see a chirp wave as a sinusoidal wave. It is distortion of the time axis. Assuming the arrival time and mass parameters, the frequency of chirp wave at a time will be given. The resampling interval is given as anti-proportional to a frequency of gravitational wave. We use Kepler frequency $f_{orbit}(t)$ which is calculated using second order post-Newtonian approximation in reference [12]. It decide the resampling interval $R_s(t)$ in this analysis.

$$R_s(t) = f_{orbit}(t) / 2\pi f_{cut} \quad (12)$$

where f_{cut} is a cut-off Kepler frequency. The ideal value of f_{cut} is a just before coalescence phase. We chose it 700 Hz in this study. The finite length time series signal $s(t)$ in uniform interval is resampled according to the equation 12. The resampled data points are re-scaled as 50 μ sec interval.

Figure 9 shows an example of FFT spectrum of resampled data $\tilde{S}_R(f)$ with optimal template for simulated signal. The event of the example is 0.1 Mpc away and azimuthal direction of the interferometer. Since the chirp frequency is changing, the spectrum of raw data distributes as a broad spectrum as Fig.3.

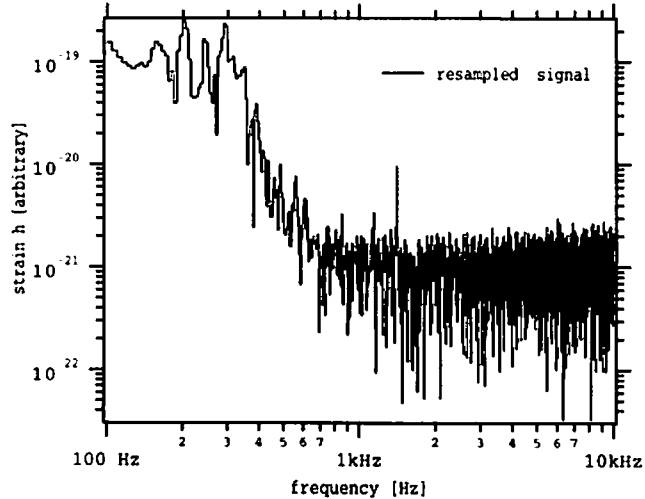


Figure 9: Example of "Resampled" Spectrum
Resampled result of Fig.3. We can find excess at 1.4 kHz due to chirp wave.

On the other hand, resampled signal has a sharp excess at twice of frequency f_{cut} . Because the lowest order term of gravitational wave is twice frequency of Kepler motion. Signal to noise ratio 'SNR' of resampling is defined with height of excess $\tilde{S}_R(2f_{cut})$ and with noise floor average $\langle \tilde{S}_R \rangle$ around excess.

$$SNR = \tilde{S}_R(2f_{cut}) / \langle \tilde{S}_R \rangle \quad (13)$$

Since the resampling event has following advantages comparing with the matched filtering, it is appropriate for a fast on-line triggering method. The signal behavior of resampling method is a quite simple. Chirp wave will appear as narrow peak spectra in FFT spectrum of "resampled" signal. It is important reason to choose first step of roughly tagging of events. The operation of resampling is also simple. The

method is to change the sampling interval according to a frequency of *a priori* gravitational waveform. Thus this resampled waveform will be appeared as a sinusoidal wave which changes only in amplitude. The resampling requires only arrival time and mass parameter for assumption of waveform. No need for variation of incident angle, polarization and final phase of gravitational waves. Therefore, number of templates is smaller comparing with matched filtering method. At last, since a chirp gravitational wave from binary starts has a largest amplitude just before coalescence, the most sensitive frequency of resampling method is at several 100 Hz. Waves in low frequency region (≤ 100 Hz) are not effective for height of signal excess in FFT spectra. It is better for interferometer which have a highest sensitivity over 100 Hz like as TAMA. Moreover, it also means no need to integrate more than a minute. Only a last few seconds of chirp wave contributes to find behavior. Roughly speaking, the evidence enhancement of the resampling is comparable quality of the eye-scan of data. Fig.10 shows the SNR as a function of distance. Fig.11 is a contour plot of SNR with parameter variations.

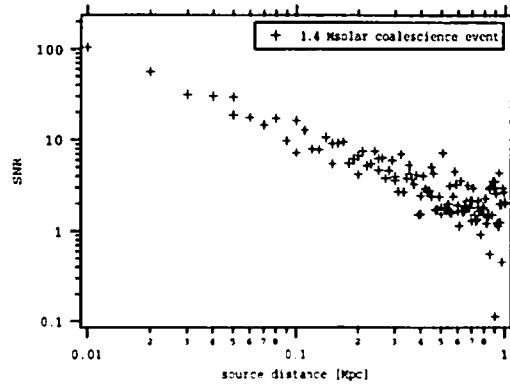


Figure 10: Distance vs. SNR

Distance vs. normalized SNR of resampling for $1.4M_{\odot}$ pairs coalescence are shown.

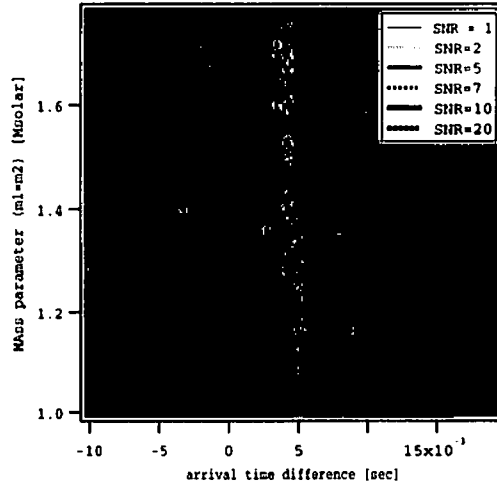


Figure 11: SNR on Lattice of Parameters

The figure shows SNR value for the function of parameters for resampling. $1.4M_{\odot}$, 0 sec arrival is a simulated signal. This contour plot suggest the accuracy in parameter estimation.

On the other hand, the resampling method has some disadvantages comparing with the matched filtering. One is SNR enhancement. Other is a precision of mass parameter. The reasons of these disadvantages are disposal data points in lower frequency region, no phase sensitivity of waveform, and short integration time.

The resampling is appropriate for first level processing such as an on-line selection. It is not suitable for final analysis.

2.2 Methods without an Assumption of Waveform: for coalescence

Another methods use no assumption of gravitational waveform. Wavelet is a typical method. The constraints of the waveform in this kind of methods are loose.

Spectrogram Spectrogram is a repetition of the FFT with every constant time. We can recognize the data in intensity on "time-frequency" plain. The evidence of the chirp wave appear as a line of high intensity as Fig.12. This is quite simple. The evidence is clear, if wave has a enough amplitude. There are commercial hardware products for exclusive use. On the other, we cannot expect drastic SNR enhancement with this method.

Wavelet Wavelet is a more development of the spectrogram, but not use constant time chunk. It cut the data in different length for different frequency band. This will make integration advantage in low frequency region, and time resolution in high frequency region. Fig.13 shows a schematic diagram of the chunk of time domain and resolution of frequency.

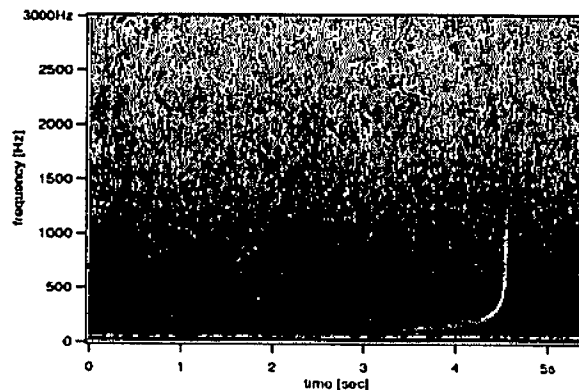


Figure 12: Example of Spectrogram

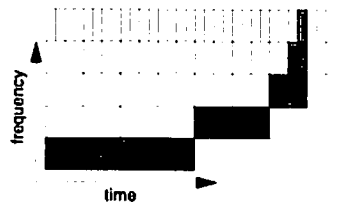


Figure 13: Schematic Diagram of Wavelet

2.3 for Supernova

Another expected source of stochastic is a supernova explosion. Such a “burst like” emission of gravitational wave, we use data analysis method that will not require the waveform. Maybe, the data analysis tool is almost same to methods without an assumption of waveform for binary coalescence. However, criteria for event selection is different due to difference of wave characteristics.

3 Continuous Gravitational Wave Search

We thought “continuous” is a periodic gravitational wave from pulsar et al. as written in section 1. Since the pulsars are found by other astronomical observations, it is a good candidate for survey plan even if small expected amplitude. Since the expected amplitude is small, we need long time integration of data. Constant pulser period and long time integration make narrow frequency bin in FFT. This is a advantage, but it requires Doppler correction for detector relative motion. The detector motion is due to motion of Earth. The observed frequency f of source frequency of f_0 is

$$f = f_0 \left(1 + \frac{\mathbf{v} \cdot \hat{\mathbf{r}}}{c} \right) \quad (14)$$

,where $\mathbf{v}$ is a velocity vector of detector motion relative to the source. $\hat{\mathbf{r}}$ is the direction to the source. For this correction, we must assume the gravitational wave source direction. After correction, the periodic wave can be recognized as sinusoidal wave.

If we do not know the direction of source, possible survey is to analysis data with assumptions of direction in principle.

4 Detector Systematic Errors on Data Analysis

If we will get the physics result after processing these methods, the result contains the error. In another word, the result has a accuracy. How to estimate statistical errors on such a analysis are established in general. We comment about systematic errors on data analysis.

There is a two kind of systematic errors. One is based on the calculation itself due to assumption accuracy, numerical accuracy, et al. For example, in matched filtering methods, prediction of template waveform analytical/numerical induce the error on correlation product c .

Another important origin is detector calibration. We convert from the digitized data of ADC to strain amplitude. It require the detector transfer function of all components that signal will through. For example, Fabry-Perot cavity transfer function of main arm is proportional to

$$\tau_s \text{sinc}(f \tau_s) e^{i\pi f \tau_s}. \quad (15)$$

This term is a frequency response of the cavity. τ_s is a storage time of Fabry-Perot cavity. We studied the case of that a parameter τ_s has a error. To decide the amplitude of the gravitational wave, 10 % error of τ_s make nearly 10 % error in strain amplitude. The amplitude error will propagate when we decide source direction with a few detectors. For mass and arrival time parameters, even if 50 % change of τ_s , $\Delta m/m \approx 0.1$ %, and $\Delta \tau \approx 0.3$ msec in simulation study of TAMA.

Since the interferometric system is so complicated, it is hard to estimate systematic error due each part in simulation. Through the continuous operation, we must check the systematics of the detector step by step.

5 Summary

Many methods for the gravitational wave data analysis were proposed. Some of them are tested. Especially, matched filtering method are well studied in researchers of the world. We induce the Monte-Carlo approach to evaluate detector characteristics in TAMA simulation. We start to study also detector calibration error on physics result of the analysis. With observation data of TAMA, we wish to establish data

analysis methods, software procedures, and evaluation of detector, which include data analysis tasks. We must construct the hardware environment which is enough to achieve them. It is a problem for future plan.

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Gravitational Radiation Reaction

— Method of Regularization —

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Abstract

In this talk, we introduce a new method to extract the singular part from the linear metric perturbation induced by a point particle moving on a black hole space-time. Using this method, we can obtain the radiative evolution of a general orbit of a particle. Here we calculate the regularized radiation reaction force to a particle in a circular orbit around a Schwarzschild black hole [1, 2, 3].

1 Introduction

The theory of general relativity predicts that mass/energy density relate to the curvature of space-time, and deformations in geometry can propagate in space-time and transport energy. This phenomenon of propagation is so-called gravitational wave, and its form tells matter how to move. With the recent developed experimental techniques, gravitational wave detectors are going on within a few years. A gravitational observatory will open up an entirely new observational window onto the universe.

One of the most promising sources of gravitational waves is a coalescing compact binary system. The event rate is estimated to be 3 events per year, and for this reason, most gravitational wave detectors are designed for gravitational waves from binary coalescence.[4] Though the detection becomes possible, it is estimated that the physical information included in gravitational waves is still obscure, due to various sources of noise. For the effective extraction of the signal from observed data, a matched filtering technique is thought to be useful, in which one takes an overlapped integral of data and a theoretical template in order to measure their coincidence. When we could find a template displaying good coincidence with data, we conclude that gravitational waves come from a binary characterized by the parameter for which the template was derived.

Recently the perturbation study of the black hole spacetime has been proposed as an alternative method to study the binary coalescence[5]. It assumes that one compact object of the binary makes a background black hole spacetime and that the effect of the gravity by the other is taken into account as a perturbation on the background. This method is expected to work in a successful manner within the scheme of the linear perturbation, in which only the leading correction by the presence of the companion is computed. In fact, the radiation reaction to the energy and the angular momentum of the particle rotating around a Schwarzschild/Kerr black hole was derived both numerically and analytically in a quite accurate manner[5].

However, at present, the perturbation study cannot be a true alternative of the standard post-Newtonian study of the gravitational waves from a binary coalescence since the systematic method of a non-linear perturbation has not been established yet.

The purpose of this paper is the extension of the present perturbation study so that it can be an efficient tool to investigate the gravitational waves from a binary coalescence. Gravitational waves are generated due to the time-varying distribution of matter. Therefore, the problem is to consider the orbital evolution of the binary, taking into account the effect of radiation reaction.

This radiation reaction problem has long been one of the fundamental theoretical issues in general relativity. Starting from the historical works of Eddington in his 1922 book[6], Chandrasekhar and Esposito[7] discussed the radiation reaction of the self-gravitating fluid emphasizing the importance of the time asymmetric part of the metric appearing in the post-Newtonian expansion, and Burke and Thorne[8]

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found that the leading contribution from the time asymmetric part can be compactly expressed in the form of a resistive potential.

The previous studies of radiation reaction[7, 8] were done under the assumption that the post-Newtonian expansion is valid. Here we consider this problem in the framework of linear perturbation theory in a general spacetime. A problem appears when we consider the point particle limit as one usually do in the linear perturbation study.

In Sec.2, we introduce the matched asymptotic expansion technique to discuss the problem and to construct the metric. We assume that the metric sufficiently far from the particle is approximated by the perturbation on the background spacetime generated by a point particle. We call this the external metric. We also assume that the internal metric that describes the geometry around the particle is represented by a black hole metric of small mass in the lowest order approximation. When the particle moves in the curved background, the internal metric suffers from the tidal distortion. Thus both internal and external metrics are constructed by perturbation. The expansion parameters for the internal and external metrics are, however, different. Assuming the existence of the matching region where both schemes are valid, the internal and external metrics are expanded there as double series regarding the two expansion parameters. Then the terms in these series are labeled by two indices that denote the powers of the two expansion parameters. Equating them order by order, we obtain the matching condition, through which one scheme determines the boundary condition of the other and vice versa. We find the metric perturbation due to a point particle can be divided into two parts, the instantaneous part and tail part, and define the regularized metric, which is the sum of the background metric and the tail part of the metric perturbation. Then, the equation of motion becomes a geodesic equation with respect to the regularized metric. The deviation from the equation of motion becomes a geodesic equation with respect to the background metric is the gravitational radiation reaction force to a point particle.

In Sec.3, we propose a method to derive the explicit components of the gravitational radiation reaction force in the perturbation study of a Schwarzschild space-time, using the scheme of Sec.2 and well-known methods to derive the metric perturbation of a Schwarzschild black hole: the Zerilli-Regge-Wheeler formalism[9]. Here we consider the leading order of the slow motion approximation and calculate the regularized radiation reaction force to a point particle.

2 Matched Asymptotic Expansion

The matched asymptotic expansion is a technique with which the same physical quantities derived in different zones by two different approximation schemes are matched in the overlapping region to obtain an approximate solution valid in the whole region. We first prepare the metrics in both internal and external zones by using different approximation schemes. The internal zone is the region where the self-gravity of the particle dominates while the external zone is the region where the background geometry dominates the full geometry.

2.1 Notion of ‘Point Particle’ and ‘Motion’

Usually, one considers a point particle as a source of perturbation and aims to derive the gravitational radiation reaction force to the particle. Then one may have some questions;

1. Since the perturbed field diverges at the location of the point-like particle, is the approximation scheme of linear perturbation still valid?
2. Does the point-like particle really represent a black hole or a neutron star?
3. If it represents a black hole, the center of it is inside the event horizon, and then how does ‘the motion of the particle’ make sense?

1st is a question on the validity of the linear perturbation, 2nd is on the meaning of the ‘point particle’ and 3rd is on the notion of ‘motion’.

The answer to the 1st question must be no, and one has to take into account the non-linearity of the Einstein equation around the particle. However, unless one clarifies the meaning of the ‘point particle’, one cannot consider the non-linearity.

We first consider the 2nd question. We define the point particle such that when one put the particle on a (quasi-)flat spacetime, the asymptotic form of the full metric of the particle becomes that of a black hole. We use the terminology ‘a monopole (spinning) particle’ to refer to a particle that represents a Schwarzschild (Kerr) black hole asymptotically.

Let us consider a particle put on an exactly flat spacetime. For example, in the case of a monopole particle, it is sufficient that the full metric around the particle is equal to that of a black hole to the order of $O(\mu/r)$ at $r \gg \mu$ if we take into account the effect of the radiation reaction to the first non-trivial order, where r is a circumferential radius.

$$g_{ab}dX^a dX^b = \left(-1 + \frac{G\mu}{r}\right) dt^2 + \left(1 - \frac{G\mu}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 + O((G\mu/r)^2) \quad (1)$$

For simplicity, we don’t consider the case of a spinning particle. This definition is valid for $r \gg G\mu$, and it is possible to extend the definition to the case that one can put the particle on a quasi-flat spacetime. By L , we denote curvature radius of the background spacetime on which one is going to put a particle. If L is sufficiently large, the background is almost flat at $L \gg r$. If $L \gg G\mu$ is satisfied, we can approximately say that the above definition of the particle is applicable at $L \gg r \gg G\mu$. From this extended definition of the point particle, one reaches the idea of ‘matched asymptotic expansion’ so as to define the ‘point particle’ in a mathematically rigorous manner, which we are going to discuss in the next subsection.

The answer to the 2nd question is important in order to construct the metric around the particle. However, it gives us no idea to understand the ‘motion’ in a meaningful manner since the metric at the center of particle is still divergent. So we need to discuss the 3rd question. In order to consider the problem, let us first return to the case that a particle is put on a flat spacetime, when the spacetime is a Schwarzschild (Kerr) black hole by definition. Though the center of the particle is a singularity inside the event horizon, one may intuitively think that the particle continues to stay still at the center. This is because one implicitly assumes an observer moves along a time-like Killing vector in the asymptotic region of the spacetime, $r \gg G\mu$. To this observer, the spacetime seems almost flat and strongly curved only in a narrow region, $r \lesssim G\mu$. If the observer considers himself to stay on a flat spacetime in an approximate manner, the particle looks like a point and the position of the center can be identified on this flat spacetime, then the ‘motion’ of the black hole can be argued in this regular spacetime. The same argument can be applicable even when the particle moves on a curved spacetime. For an observer far from the particle, $r \gg G\mu$, the self-gravity of the particle does not disturb the background spacetime so much, but the full spacetime is strongly curved only at a small region, $r \lesssim G\mu$. Therefore, the position of the particle can be identified on the regular background spacetime and one can consider the ‘motion’ of the particle in a meaningful manner with respect to the background metric.

2.2 Internal, External, Matching Region

In the internal zone, we assume that the metric can be described by that of a black hole plus perturbation. Nameiy, we assume that the particle is represented by a Schwarzschild/Kerr black hole in the lowest order of approximation. In the present case, the perturbation is caused by the tidal effect of the curvature of the spacetime in which the particle travels. As mentioned in the beginning we call this construction of the metric **the internal scheme**. In order to make this scheme valid, the linear extension of the internal zone around the particle must be much smaller than the background curvature scale L . We introduce the coordinate $\{X^a\} = \{T, X^i\}$ ($a = 0, 1, 2, 3$; $i = 1, 2, 3$) for the internal scheme and $|X|$ ($:= \sqrt{X^i X^i}$) is assumed to represent the physical distance scale². Then the internal scheme is valid when

$$|X| \ll L, \quad (2)$$

where L is the length scale of the background curvature.

In the external zone, we expect that the metric is well approximated by the perturbation induced by a point source on a given background spacetime. We call this construction of the metric **the external**

²In this section, we adopt the Minkowskian summation rule on $a, b, \dots$, and the Kronecker summation rule on $i, j, \dots$ over the repeated indices.

scheme. This approximation scheme is valid when the self-gravity of the particle is sufficiently weak, that is,

$$G\mu \ll |X|, \quad (3)$$

where $G\mu$ is the scale of Schwarzschild radius. As the point source is placed within the internal zone, there is no matter source in the external zone. Thus the external metric is given by a vacuum solution of the Einstein equations.

We require that the metrics obtained in both schemes be matched in the overlapping region of both zones, by considering a coordinate transformation between the internal and external metrics. Safely, we may assume the existence of the matching region as long as

$$G\mu \ll L, \quad (4)$$

is satisfied. For definiteness, we set the matching radius at

$$|X| \sim (G\mu L)^{1/2}, \quad (5)$$

in the spatial coordinates of the internal scheme, X^i . Then writing down the metric in the internal scheme, we have two independent small parameters $|X|/L$ and $G\mu/|X|$ in the matching region. The power expansion concerning these two small parameters allows us to consider the matching order by order.

2.3 External Scheme

Here we consider the external metric. As we assume that the gravitational radius of the particle, μ , is small compared with the length scale of the background curvature, L , we approximate $\delta g_{\mu\nu}$ by the linear perturbation induced by a point particle, $h_{\mu\nu}$, in the whole spacetime region except for the vicinity of the world line of the particle.

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + h_{\mu\nu}(x). \quad (6)$$

We take a Green function approach to study the linear perturbation of the metric. Here we introduce the trace-reversed metric perturbation,

$$\psi_{\mu\nu}(x) = h_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)h(x), \quad (7)$$

and we set the harmonic gauge condition,

$$\psi^{\mu\nu}{}_{;\nu}(x) = 0, \quad (8)$$

In this gauge, the linearized Einstein equations become

$$-\frac{1}{2}\psi^{\mu\nu;\xi}{}_{;\xi}(x) - R^\mu{}_\xi{}^\nu{}_\rho(x)\psi^{\xi\rho}(x) = 8\pi GT^{\mu\nu}(x). \quad (9)$$

The calculation is performed in an analogous manner to the case of the scalar and vector perturbations developed by DeWitt and Brehme[10]. We calculate the tensor Green function in a background covariant manner,

Thus we define the tensor Green function $G^{\mu\nu\alpha\beta}(x, z)$ which satisfies

$$G^{\mu\nu\alpha\beta;\xi}{}_{;\xi}(x, z) + 2R^\mu{}_\xi{}^\nu{}_\rho(x)G^{\xi\rho\alpha\beta}(x, z) = -2\tilde{g}^{\alpha(\mu}(x, z)\tilde{g}^{\nu)\beta}(x, z)\frac{\delta^{(4)}(z-x)}{\sqrt{-g}}, \quad (10)$$

where $\tilde{g}_{\mu\alpha}(x, z)$ is the geodetic parallel displacement bi-vector³, g is the determinant of the metric $g_{\mu\nu}(x)$.

³For the latter use, we define the geodetic parallel displacement bi-vector $\tilde{g}_{\mu\alpha}(x, z)$ as

$$\begin{aligned} \tilde{g}_{\mu\alpha;\nu}(x, z)g^{\nu\sigma}(x)\sigma_{;\sigma}(x, z) &= \tilde{g}_{\mu\alpha;\beta}(x, z)g^{\beta\gamma}(z)\sigma_{;\gamma}(x, z) = 0, \\ \lim_{z \rightarrow x} \tilde{g}_{\mu}{}^{\alpha}(x, z) &= \delta_{\mu}{}^{\alpha}. \end{aligned} \quad (11)$$

The symmetric Green function is represented by $\bar{G}^{\mu\nu\alpha\beta}(x, z)$, from which we can obtain the retarded Green function $G_{Ret}^{\mu\nu\alpha\beta}(x, z)$ as

$$\bar{G}^{\mu\nu\alpha\beta}(x, z) = \frac{1}{8\pi} [u^{\mu\nu\alpha\beta}(x, z)\delta(\sigma(x, z)) - v^{\mu\nu\alpha\beta}(x, z)\theta(-\sigma(x, z))], \quad (12)$$

$$G_{Ret}^{\mu\nu\alpha\beta}(x, z) = 2\theta[\Sigma(x), z]\bar{G}^{\mu\nu\alpha\beta}(x, z), \quad (13)$$

where $\sigma(x, z)$ is defined as half the squared geodesic interval,⁴ $\Sigma(x)$ is an arbitrary space-like hypersurface containing x , and $\theta[\Sigma(x), z] = 1 - \theta[z, \Sigma(x)]$ is equal to 1 when z lies in the past of $\Sigma(x)$ and vanishes when z lies in the future. The bi-tensor $u^{\mu\nu\alpha\beta}(x, z)$ is singular at a point particle and $v^{\mu\nu\alpha\beta}(x, z)$ is regular at a point particle.

We assume the energy-momentum tensor of the form,

$$T^{\mu\nu}(x) = \mu \int dT v^\mu(x, T) v^\nu(x, T) \frac{\delta^{(4)}(x - z(T))}{\sqrt{-g}}, \quad (18)$$

$$v^\mu(x, T) = \bar{g}^\mu{}_\alpha(x, z(T)) \dot{z}^\alpha(T), \quad (19)$$

where $\dot{z}^\alpha(T) = dz^\alpha/dT$, μ is the mass of the particle.

Without any further approximation, the metric perturbation due to the point-like particle becomes

$$\begin{aligned} \psi^{\mu\nu}(x) = 2G\mu \left(\left[\frac{1}{\bar{\sigma}(x, z(T))} u^{\mu\nu}{}_{\alpha\beta}(x, z(T)) \dot{z}^\alpha(T) \dot{z}^\beta(T) \right. \right. \\ \left. \left. - \int_{-\infty}^{T_{Ret}(x)} dT \left(v^{\mu\nu}{}_{\alpha\beta}(x, z(T)) \dot{z}^\alpha(T) \dot{z}^\beta(T) \right) \right] \right), \end{aligned} \quad (20)$$

where $T_{Ret}(x)$ is the retarded time of the particle and is a scalar function which is determined by

$$\sigma(x, z(T_{Ret})) = 0, \quad \theta(\Sigma(x), z(T_{Ret})) = 1. \quad (21)$$

Since the retarded time $T_{Ret}(x)$ is not convenient for specifying the field point x around the particle trajectory in the following computations, we introduce a new specification of x as follows. We foliate the spacetime with spacelike 3-surfaces perpendicular to the particle trajectory. Specifically, the 3-surfaces are defined as a one-parameter family of T by the relation, $\sigma_{;\alpha}(x, z(T)) \dot{z}^\alpha(T) = 0$. We denote the value of T of the 3-surface containing the point x by T_x . That is

$$\sigma_{;\alpha}(x, z(T_x)) \dot{z}^\alpha(T_x) = 0, \quad (22)$$

2.4 Internal Scheme

Here we assume that the internal metric that describes the geometry around the particle is represented by a black hole metric of small mass in the lowest order approximation. As the particle moves in the curved background, the internal metric suffers from the tidal distortion.

$$\hat{g}_{ab}(X) = g_{ab}^{P,P}(X) + H_{ab}(X), \quad (23)$$

⁴We define the geodesic interval between x and z as $s(x, z)$. $s(x, z)$ satisfies

$$g^{\mu\nu}(x) s_{;\mu}(x, z) s_{;\nu}(x, z) = g^{\alpha\beta}(z) s_{;\alpha}(x, z) s_{;\beta}(x, z) = \pm 1, \quad (14)$$

$$s(x, x) = 0, \quad (15)$$

where $+$ ($-$) sign is taken when the interval between x and z is spacelike (timelike). Instead of $s(x, z)$,

$$\sigma(x, z) = \pm \frac{1}{2} s^2(x, z), \quad (16)$$

where $+$ ($-$) sign is taken when the interval between x and z is spacelike (timelike). Then we have

$$\begin{aligned} \sigma(x, z) &= \frac{1}{2} g^{\mu\nu}(x) s_{;\mu}(x, z) s_{;\nu}(x, z) = \frac{1}{2} g^{\alpha\beta}(z) s_{;\alpha}(x, z) s_{;\beta}(x, z), \\ \lim_{z \rightarrow x} \sigma(x, z)_{;\mu} &= \lim_{z \rightarrow x} \sigma(x, z)_{;\alpha} = 0. \end{aligned} \quad (17)$$

One finds that $\sigma(x, z)$ is a correspondance of $1/2(x - z)^2$ in curved spacetime, and we consider the expansion of bi-tensors with $\sigma_{;\alpha}(x, z)$.

where $g_{ab}^{P.P.}(X)$ is the metric of a black hole representing the point particle as a black hole and $H_{\mu\nu}(X)$ is the perturbation which is induced by the external curvature.

2.5 Matching

In this section, assuming the existence of the matching region, we consider the matching condition at the matching radius (see detail in [5]). In order to write down the external metric with the internal coordinates, we consider a coordinate transformation from x to $\{X^a\}$ given. Then the internal and external metrics are expanded there as double series regarding the two expansion parameters, and the terms in these series are labeled by two indices that denote the powers of the two expansion parameters. We have two independent small parameters $|X|/L$ and $G\mu/|X|$ in the matching region.

First we consider the expansion of the internal scheme. Recalling that the perturbation in the internal zone is induced by the external curvature which has a characteristic length scale L , the metric can be expanded in powers of $|X|/L$ as

$$\hat{g}_{ab}(X) = {}^{(0)}H_{ab}(X) + \frac{1}{L}{}^{(1)}H_{ab}(X) + \frac{1}{L^2}{}^{(2)}H_{ab}(X) + \cdots, \quad (24)$$

where ${}^{(0)}H_{ab}(X)$ is the unperturbed black hole metric. We expect that ${}^{(1)}H_{ab}(X)$ will be given by the standard linear perturbation of the black hole. We find that ${}^{(1)}H_{ab}(X)$ can be consistently set to zero, which is in accordance with the notion that the spacetime curvature is of $O(1/L^2)$. Thus the standard black hole perturbation theory applies up to ${}^{(2)}H_{ab}(X)$. Further we expand the metric with respect to $G\mu/|X|$ which is also small at the matching radius:

$$\begin{aligned} {}^{(0)}H_{ab}(X) &= \eta_{ab} + G\mu_{(1)}{}^{(0)}H_{ab}(X) + (G\mu)^2{}^{(2)}H_{ab}(X) + \cdots, \\ \frac{1}{L}{}^{(1)}H_{ab}(X) &= \frac{1}{L}{}^{(1)}_{(0)}H_{ab}(X) + \frac{G\mu}{L}{}^{(1)}_{(1)}H_{ab}(X) + \frac{(G\mu)^2}{L}{}^{(1)}_{(2)}H_{ab}(X) + \cdots. \end{aligned} \quad (25)$$

Note that, from the definitions of the expansion parameters, the ${}^{(m)}_{(n)}H_{ab}$ component of the metric behaves as

$${}^{(m)}_{(n)}H_{ab} \sim |X|^{m-n}. \quad (26)$$

The explicit form of the coordinate transformation from the general coordinates of a background metric $\{x^\mu\}$ to the coordinates of the internal scheme $\{X^a\}$ is discussed in Ref.[5]. Assuming the matching can be consistently done, the full metric in the external scheme $\tilde{g}_{\mu\nu}(x)$ is written with the internal coordinates as

$$\tilde{g}_{ab}(X)dX^a dX^b = \tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu. \quad (27)$$

Generally, as the external metric can be expanded by $G\mu/|X|$, we write it as

$$\tilde{g}_{ab}(X) = g_{ab}(X) + G\mu_{(1)}h_{ab}(X) + (G\mu)^2{}_{(2)}h_{ab}(X) + \cdots. \quad (28)$$

Then $G\mu_{(1)}h_{ab}(X)$ can be recognized as the linear perturbation on the background $g_{ab}(X)$. Further we expand it with respect to $|X|/L$ as

$$\begin{aligned} g_{ab}(X) &= {}^{(0)}_{(0)}h_{ab}(X) + \frac{1}{L}{}^{(1)}_{(0)}h_{ab}(X) + \frac{1}{L^2}{}^{(2)}_{(0)}h_{ab}(X) + \cdots, \\ G\mu_{(1)}h_{ab}(X) &= G\mu_{(1)}{}^{(0)}h_{ab}(X) + \frac{G\mu}{L}{}^{(1)}_{(1)}h_{ab}(X) + \frac{G\mu}{L^2}{}^{(2)}_{(1)}h_{ab}(X) + \cdots. \end{aligned} \quad (29)$$

As before,

$${}^{(m)}_{(n)}h_{ab} \sim |X|^{m-n}. \quad (30)$$

For brevity, we call ${}^{(m)}_{(n)}h_{ab}$ or ${}^{(m)}_{(n)}H_{ab}$ the ${}^{(m)}_{(n)}$ -component and the matching condition for them as the ${}^{(m)}_{(n)}$ matching. In the matching region ($|X| \sim (G\mu L)^{1/2}$), the ${}^{(m)}_{(n)}$ -component is of $O((G\mu/L)^{(m+n)/2})$.

The matching condition requires that all the corresponding terms in Eqs. (25) and (29) should be identical. Then the matching condition is given by equating the terms of the same power in $|X|$ in both schemes to desired accuracy. Thus the condition for the $\binom{m}{n}$ matching is

$$\sum_{\substack{m'-n'=m-n \\ m' \leq m}} \frac{(G\mu)^{n'} \binom{m'}{n'}}{L^{m'}} h_{ab} = \sum_{\substack{m'-n'=m-n \\ m' \leq m}} \frac{(G\mu)^{n'} \binom{m'}{n'}}{L^{m'}} H_{ab} + O\left(\frac{(Gm)^{n+1}}{L^{m+1}} |X|^{(m-n)}\right). \quad (31)$$

Equating them order by order, we obtain the matching condition, through which one scheme determines the boundary condition of the other and vice versa. We show the first few orders of the matching in the following table.

	μ^0	μ^1	
$\frac{1}{L^0}$	$\binom{(0)}{(0)} H(X^0)$	$\binom{(0)}{(1)} H(X^{-1})$	
$\frac{1}{L^1}$	$\binom{(1)}{(0)} H(X^1)$	$\binom{(1)}{(1)} H(X^0)$	
$\frac{1}{L^2}$	$\binom{(2)}{(0)} H(X^2)$	$\binom{(2)}{(1)} H(X^1)$	

In the following, we consider the meaning of $\binom{m}{n} H_{ab}$ for the internal scheme. $\binom{(0)}{(0)} H$ is trivially given by the flat Minkowski metric. We set $\binom{(1)}{(0)} H$ and $\binom{(1)}{(1)} H$ to zero to except gauge degrees of freedom. $\binom{(2)}{(0)} H$ is the hypersurface condition to fix the gauge and we consider $\binom{(2)}{(1)} H$ under this condition. For $\binom{(0)}{(1)} H$, since this matching is independent of L , we may regard the background external metric as if it were flat. As is well known, the linear perturbation induced by a point-like particle of mass μ in the flat background spacetime is exactly equal to the asymptotic metric of a Schwarzschild black hole of mass μ in the linear order in μ . One obtains the equations of motion with the $O(G\mu/L^2)$ correction ($\binom{(2)}{(1)} H$) due to the radiation reaction

From the above analysis, we find that the perturbed metric (20) in the external scheme is divided into the two parts:

$$h_{\mu\nu}(x) = h_{(inst)\mu\nu}(x) + h_{(tail)\mu\nu}(x), \quad (32)$$

where $h_{(tail)\mu\nu}(x)$ is the part due to the $v^{\mu\nu\alpha\beta}$ in the Green function while $h_{(inst)\mu\nu}$ is due to the $u_{\mu\nu\alpha\beta}$ term. We show the meaning of the two parts of metric perturbation in the following table.

	<i>Instantaneous part</i>	<i>Tail part</i>
<i>around particle</i>	<i>Singular behavior</i>	<i>Regular behavior</i>
<i>dependence</i>	<i>Local geometry</i>	<i>Global geometry, Past history of the particle</i>

The singular behavior of the perturbed metric in the coincidence limit $x \rightarrow z$ is totally due to $h_{(inst)\mu\nu}(x)$. Thus, we introduce the regularized perturbed metric as

$$\tilde{g}_{(reg)\mu\nu}(x) := g_{\mu\nu}(x) + h_{(tail)\mu\nu}(x), \quad (33)$$

that has no singular behavior any more. Then we find the equations of motion. It is given by

$$\frac{D}{d\tau} v^\alpha(z) = F^\alpha(z) := -\delta\Gamma_{(reg)}{}^\alpha{}_{\beta\gamma} v^\beta(z) v^\gamma(z), \quad (34)$$

to the linear order in $h_{(tail)\mu\nu}$ where

$$\delta\Gamma_{(reg)}{}^\alpha{}_{\beta\gamma} := \frac{1}{2} (h_{(tail)}{}^\alpha{}_{\beta;\gamma} + h_{(tail)}{}^\alpha{}_{\gamma;\beta} - h_{(tail)\beta\gamma}{}^{;\alpha}). \quad (35)$$

This $F^\alpha(z)$ is the **regularized** gravitational radiation reaction force to a point particle.

3 Application

In the previous section, we consider the local behavior of the metric perturbation and derive the equations of motion with the effect of gravitational radiation reaction. (We call this a ‘local’ analysis of the metric perturbation.) However, the tail part of the metric perturbation is left unevaluated, since it depends on the global space-time which we consider and the past history of the particle. Therefore, the result in the previous section alone is of no use for constructing theoretical templates of the gravitational waves. On the other hand, there are well-known methods to derive the metric perturbation of a Schwarzschild space-time, i.e., Zerilli-Regge-Wheeler formalism[9]. (We call this a ‘global’ analysis of the metric perturbation⁵.)

In this section, we consider the bare force to a particle (mass μ) moving along a circular orbit at $r = r_0$ on the equatorial plane $\theta = \pi/2$ in a Schwarzschild black hole

$$ds^2 = - \left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (36)$$

The tangential vector v^α of the orbit has the components

$$v^t = \frac{1}{\sqrt{1 - \frac{3GM}{r_0}}}, \quad v^r = 0, \quad v^\theta = 0, \quad v^\phi = \sqrt{\frac{GM}{r_0^3} \frac{1}{1 - \frac{3GM}{r_0}}}. \quad (37)$$

The orbit is given by

$$\{z^\alpha(T)\} = \{v^t T, r_0, \pi/2, v^\phi T\}, \quad (38)$$

where T is the proper time of the particle. The reason to consider this simplest system is that it is well understood by the previous method of black hole perturbation study, and is useful to verify the new method. In the following subsections, we consider the leading order of the slow motion approximation and calculate the regularized reaction force when the particle is placed at $z^\alpha(T_0)$ by combining the local analysis and the global analysis

3.1 Method of Regularization

We suppose that the particle is placed at a point z^α . The metric perturbation and the instantaneous part of it diverge at $x^\alpha = z^\alpha$. Therefore, we consider deriving them at a point $x^\alpha \neq z^\alpha$. Where do we choose the regularization point x^α ? In general, we can take the point in any place. But we consider the regularization point to perform easily the following calculation of the regularization. This hint is found in the Newton potential with the polar coordinates $\{x^\alpha\} = \{r, \theta, \phi\}$,

$$\begin{aligned} \phi(x) &= \mu \sum_{lm} \frac{1}{(2l+1)} \frac{(r)^l}{(r_0)^{l+1}} Y_{lm}(\theta, \phi) Y_{lm}^*(\theta_0, \phi_0) \\ &\xrightarrow{(\theta=\theta_0, \phi=\phi_0)} \mu \sum_l \frac{(r)^l}{(r_0)^{l+1}} = \frac{1}{r_0 - r}, \end{aligned} \quad (39)$$

where we consider taking the point x^α radially inside the particle at $\{z^\alpha\} = \{r_0, \theta_0, \phi_0\}$; i.e.

$$r < r_0. \quad (40)$$

So we take the regularization point,

$$\{t, r, \theta, \phi\} = \{v^t T, r_0 + \epsilon, \pi/2, v^\phi T\}. \quad (41)$$

Here ϵ is a regularization parameter and $\epsilon < 0$.

⁵We define the bare quantity as a quantity derived by using the metric perturbation

We calculate the regularized reaction force at the regularization point in the following manner

$$F^\alpha(x^\mu) = F^\alpha[h(x^\mu)] - F^\alpha[h(x^\mu)_{(inst)}]. \quad (42)$$

After the regularization of the reaction force, we consider taking the coincidence limit $x^\alpha \rightarrow z^\alpha$ to derive the regularized metric at $x^\alpha = z^\alpha$ under the assumption that the tail part is analytic around the particle.

At this stage, we take the following procedure. One first takes a summation of the Fourier-Harmonic expansion of the metric perturbation, and then extracts the instantaneous part from the sum to derive the tail part.

The problem with this method is that it is difficult to sum a Fourier-Harmonic expansion of the metric perturbation before taking the coincidence limit $x^\alpha \rightarrow z^\alpha$. Our idea to overcome this difficulty is to use the slow motion approximation⁶ of the motion of the particle (i.e. $\omega M \ll \omega r \ll 1$). Thus, we have no need to worry about infinite summation over the Fourier-Harmonic expansion [2]. We expect that it is sufficient to take a sum over only a finite number of mode function when we take into account only finite orders of the slow motion expansion.

On the other hand, it is easy to derive the instantaneous part from the matched asymptotic expansion method. Therefore we can extract the instantaneous part from the metric perturbation in the form of a Taylor expansion. After the extraction, we can derive the regularized part easily. The sufficiently fast convergence of the large- l contribution is expected to make the summation finite, since we have already extracted the singular contribution. We take the coincidence limit $x^\alpha \rightarrow z^\alpha$ for the regularized gravitational radiation reaction force.

Here we note that there is an ambiguity in the identification of the position of the particle. This ambiguity comes from the size of the black hole, therefore, it is of order $G\mu$. Since this ambiguity appears as a gauge freedom of the perturbation⁷, we must consider a gauge problem as well.

3.2 Derivation of $F[h]_{(bare)}$

The bare metric perturbation can be derived from the Zerilli-Regge-Wheeler functions by the Zerilli-Regge-Wheeler formalism[9] in the case of a Schwarzschild background⁸. These functions are investigated in detail in Ref.[11], and the expansion appropriate for the present purpose is the slow motion approximation, in which functions are expanded with Coulomb wave functions.

We use the homogeneous solutions of the Zerilli-Regge-Wheeler equations. The homogeneous solutions satisfying the outgoing wave condition at the future null infinity and those satisfying the ingoing wave condition at the future event horizon are computed to be spherical Bessel functions and spherical Hankel functions of the 1st kind, respectively[12]. The inhomogeneous Zerilli-Regge-Wheeler equations to the leading order of the post-Minkowskian expansion become inhomogeneous spherical Bessel equations.

$$\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + \omega^2 - \frac{l(l+1)}{r^2} \right) \left(\frac{R_{lm}^{(e/m)}(r)}{r} \right) = \frac{1}{r} S_{lm}^{(e/m)}(r), \quad (43)$$

$$S_{lm}^{(e)} = 8(2\pi)^{\frac{3}{2}} G\mu v^t \frac{1}{(l-1)l(l+1)(l+2)} Y_{lm} \left(\frac{\pi}{2}, 0 \right) \delta \left(\omega - m \frac{v^\phi}{v^t} \right) \\ \times \left(-l(l+1) \frac{1}{r_0} \delta(r-r_0) + 2 \frac{d}{dr} \delta(r-r_0) \right), \quad (44)$$

$$S_{lm}^{(m)} = -16(2\pi)^{\frac{3}{2}} G\mu \frac{(v^\phi)^2}{v^t} \frac{1}{(l-1)l(l+1)(l+2)} \frac{\partial^2}{\partial \theta \partial \phi} Y_{lm} \left(\frac{\pi}{2}, 0 \right) \delta \left(\omega - l \frac{v^\phi}{v^t} \right) \\ \times \left(3\delta(r-r_0) + r_0 \frac{d}{dr} \delta(r-r_0) \right). \quad (45)$$

⁶We use 'the slow motion approximation' instead of 'the post-Newtonian approximation' since usually the latter assumes that the gravitational potential is small in addition to the approximation of the former. Contrastingly, we consider the case in which the gravitational potential diverges.

⁷The metric perturbation is obtained by the global analysis in the form of a Fourier-Harmonic expansion, under the Regge-Wheeler gauge condition, and the instantaneous part is obtained by the local analysis under the harmonic gauge condition.

⁸This perturbation is derived from the Teukolsky functions by the method of Chrzanowski[13] in the case of a Kerr background.

The solutions of these equations for $r_0 > r$ with an appropriate boundary condition are

$$R_{lm}^{(e)}(r) = 8i(2\pi)^{\frac{3}{2}} G\mu v^t \frac{1}{(l-1)l(l+1)(l+2)} Y_{lm} \left(\frac{\pi}{2}, 0 \right) \delta \left(\omega - m \frac{v^\phi}{v^t} \right) \\ \times \omega r j_l(\omega r) \left(l(l+1) h_l^{(1)}(\omega r_0) + 2r_0 \frac{d}{dr_0} h_l^{(1)}(\omega r_0) \right), \quad (46)$$

$$R_{lm}^{(m)}(r) = -16i(2\pi)^{\frac{3}{2}} G\mu \frac{(v^\phi)^2}{v^t} \frac{1}{(l-1)l(l+1)(l+2)} \frac{\partial^2}{\partial \theta \partial \phi} Y_{lm} \left(\frac{\pi}{2}, 0 \right) \delta \left(\omega - m \frac{v^\phi}{v^t} \right) \\ \times \omega r j_l(\omega r) \left(-3r_0 h_l^{(1)}(\omega r_0) + r_0^2 \frac{d}{dr_0} h_l^{(1)}(\omega r_0) \right). \quad (47)$$

The post-Newtonian expansion of these functions are obtained by using the expansion formula of spherical Bessel functions and spherical Hankel functions of the 1st kind.

$$j_n(\omega r) = (2\omega r)^n \sum_{m=0}^{\infty} \frac{(-1)^m (n+m)!}{m!(2n+2m+1)!} (\omega r)^{2m} \quad (48)$$

$$h_n^{(1)}(\omega r_0) = j_n(\omega r_0) - \frac{i}{2^n (\omega r_0)^{n+1}} \left[\sum_{m=0}^n \frac{(2n-2m)!}{m!(n-m)!} (\omega r_0)^{2m} + (-1)^n \sum_{m=n+1}^{\infty} \frac{(-1)^m (m-n)!}{m!(2m-2n)!} (\omega r_0)^{2m} \right] \quad (49)$$

We compute the r -component of the bare reaction force at a regular point $\{t, r, \theta, \phi\} = \{v^t T, r, \pi/2, v^\phi T\}$.

$$F_r^{\text{bare}}(r, z^\alpha(T)) \\ = \sum_{lm} \left\{ 8(2\pi)^{\frac{3}{2}} G\mu v^t \frac{1}{(l-1)l(l+1)(l+2)} Y_{lm} \left(\frac{\pi}{2}, 0 \right) \right. \\ \times \left((l^2 + l + 2) h_l^{(1)}(\omega r_0) + 2r_0 \frac{d}{dr_0} h_l^{(1)}(\omega r_0) \right) \\ \times \left(-i2m\omega^2 v^t v^\phi r j_l(\omega r) \right. \\ \left. + \left(i \frac{l^2 + l + 4}{4} \omega ((v^t)^2 + (rv^\phi)^2) - im\omega^2 v^t v^\phi r^2 \right) \frac{d}{dr} j_l(\omega r) \right. \\ \left. + i \frac{1}{2} \omega ((v^t)^2 + (rv^\phi)^2) \frac{d^2}{dr^2} j_l(\omega r) \right) \\ \times Y_{lm} \left(\frac{\pi}{2}, v^\phi T \right) \frac{1}{\sqrt{2\pi}} e^{-i\omega v^t T} \\ - 8(2\pi)^{\frac{3}{2}} G\mu \frac{(v^\phi)^2}{v^t} \frac{1}{(l-1)l(l+1)(l+2)} \frac{\partial}{\partial \theta} Y_{lm} \left(\frac{\pi}{2}, 0 \right) \\ \times \left(-2r_0 h_l^{(1)}(\omega r_0) + r_0^2 \frac{d}{dr_0} h_l^{(1)}(\omega r_0) \right) \\ \times \left((-i2m v^t v^\phi + im^2 \omega r^2 (v^\phi)^2) j_l(\omega r) \right. \\ \left. - i4m v^t v^\phi r \frac{d}{dr} j_l(\omega r) - im v^t v^\phi r^2 \frac{d^2}{dr^2} j_l(\omega r) \right) \\ \left. \times \frac{\partial}{\partial \theta} Y_{lm} \left(\frac{\pi}{2}, v^\phi T \right) \frac{1}{\sqrt{2\pi}} e^{-i\omega v^t T} \right\}_{\omega = m v^\phi / v^t}. \quad (50)$$

Using (48) and (49), the slow motion expansion of F_r^{bare} in the post-Newtonian approximation becomes

$$F_r^{(bare)}(r, z^\alpha(T_0)) = -G\mu \sum_{l=2}^{\infty} (l-1) \frac{(r)^{l-2}}{(r_0)^l} + O(\omega r_0, \omega r). \quad (51)$$

Here we have already calculated the ω and m -summations in the Fourier-Harmonic expansion.

In the same manner, we obtain the t -, θ - and ϕ -components of the bare reaction force, as

$$F^{(bare)}_t(r, z^\alpha(T_0)) = G^4 \mu \frac{16}{5} \frac{M^3}{r_0^4} r + O(\omega r_0, \omega r), \quad (52)$$

$$F^{(bare)}_\theta(r, z^\alpha(T_0)) = 0, \quad (53)$$

$$F^{(bare)}_\phi(r, z^\alpha(T_0)) = -\frac{v^t}{v^\phi} F_t^{bare} + O(\omega r_0, \omega r). \quad (54)$$

The t -, θ - and ϕ -component of the bare force is singularity free as we shall see in the local analysis below. In fact, we shall prove that the t -component of the instantaneous force vanishes and, therefore, we do not need to regularized the t -, θ - and ϕ -component of the bare force. (See (73), (74) and (75).)

3.3 Derivation of $F[h_{(inst)}]$

We next consider the instantaneous force using the external scheme of the matched asymptotic expansion method done in Sec 2. With the trace-reversed metric perturbation

$$\psi^{inst}_{\mu\nu} = h^{inst}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h^{inst}_{\xi}{}^{\xi}, \quad (55)$$

the instantaneous reaction force becomes

$$F^{inst}_\alpha = \left(\frac{1}{2} \psi^{inst}_{\beta\gamma;\alpha} - \psi^{inst}_{\alpha\beta;\gamma} \right) v^\beta v^\gamma + \frac{1}{2} v_\alpha \psi^{inst}_{;\beta} v^\beta - \frac{1}{4} v^2 \psi^{inst}_{;\alpha}, \quad (56)$$

where $\psi^{inst} = g^{\alpha\beta} \psi^{inst}_{\alpha\beta}$. The expression of $\psi^{inst}_{\mu\nu;\xi}$ can be derived in the same manner as we derive Eq. (20), and we have

$$\begin{aligned} \psi^{inst}_{\mu\nu;\xi}(x) = & 2G\mu \bar{g}_{\mu\alpha} \bar{g}_{\nu\beta} \bar{g}_{\xi\gamma} \left(\frac{2}{X^3} \kappa^{-1} v^\alpha v^\beta \sigma^{i\gamma} - 2v^{(\alpha} R^{\beta)}_{\delta}{}^{\gamma}{}_{\epsilon} v^\delta v^\epsilon - 2R^\alpha{}_{\delta}{}^{\beta}{}_{\epsilon} v^\gamma v^\delta v^\epsilon \right. \\ & + \frac{2}{X} v^{(\alpha} R^{\beta)}_{\delta}{}^{\gamma}{}_{\epsilon} v^\delta \sigma^{i\epsilon} + \frac{2}{X} R^\alpha{}_{\delta}{}^{\beta}{}_{\epsilon} \sigma^{i\gamma} v^\delta v^\epsilon \\ & \left. + \frac{2}{X} v^{(\alpha} R^{\beta)}_{\delta\epsilon\zeta} v^\gamma v^\delta \sigma^{i\epsilon} v^\zeta + \frac{2}{3X} v^\alpha v^\beta R^\gamma{}_{\delta\epsilon\zeta} v^\delta \sigma^{i\epsilon} v^\zeta \right) \\ & + O(X). \end{aligned} \quad (57)$$

Here $\sigma(x, z)$ ⁹ is defined as half the squared geodesic interval, $X(x, z) = \sqrt{2\sigma(x, z(T_x))}$, and $\kappa(x, z) = \sqrt{-\ddot{\sigma}(x, z(T_x))}$ respectively in [1].

We compute the explicit components of tensors/bitensors appeared in (57) at a regular point, in the form of an expansion with $r - r_0$.

The background metric is expanded as

$$\begin{aligned} g_{\mu\nu}(r, z) = & \text{diag} \left\{ - \left(1 - \frac{2GM}{r_0} \right) \left(1 + \frac{2GM}{r_0 - 2GM} \frac{r - r_0}{r_0} - \frac{2GM}{r_0 - 2GM} \left(\frac{r - r_0}{r_0} \right)^2 \right), \right. \\ & \left. \left(\frac{r_0}{r_0 - 2GM} \right) \left(1 - \frac{2GM}{r_0 - 2GM} \frac{r - r_0}{r_0} + \frac{2GM r_0}{(r_0 - 2GM)^2} \left(\frac{r - r_0}{r_0} \right)^2 \right), \right. \end{aligned}$$

⁹Here we denote simply x as the point z^α , and z as the point z^α

$$r_0^2 \left(1 + 2 \frac{r-r_0}{r_0} + \left(\frac{r-r_0}{r_0} \right)^2 \right), \\ r_0^2 \left(1 + 2 \frac{r-r_0}{r_0} + \left(\frac{r-r_0}{r_0} \right)^2 \right) \Big\} + O((r-r_0)^3). \quad (58)$$

Since the other relevant bi-tensors in (57) depends on the foliation condition (22), we first consider the foliation parameter T_x . We put the explicit form of half the squared geodetic distance $\sigma(x, z)$ as

$$\begin{aligned} \sigma(x, z) = & \frac{1}{2} g_{\alpha\beta}(z)(x^\alpha - z^\alpha)(x^\beta - z^\beta) + \frac{1}{3!} A_{\alpha\beta\gamma}(z)(x^\alpha - z^\alpha)(x^\beta - z^\beta)(x^\gamma - z^\gamma) \\ & + \frac{1}{4!} A_{\alpha\beta\gamma\delta}(z)(x^\alpha - z^\alpha)(x^\beta - z^\beta)(x^\gamma - z^\gamma)(x^\delta - z^\delta) + O((x-z)^5). \end{aligned} \quad (59)$$

From (17), we obtain

$$A_{\alpha\beta\gamma}(z) = \frac{3}{2} g_{(\alpha\beta;\gamma)}(z), \quad (60)$$

$$\begin{aligned} A_{\alpha\beta\gamma\delta}(z) = & \frac{4}{3} A_{(\alpha\beta\gamma;\delta)}(z) \\ & - g^{\epsilon\eta}(z) (g_{(\alpha\beta|\epsilon}(z) + A_{(\alpha\beta|\epsilon}(z)) (g_{|\gamma\delta);\eta}(z) - A_{|\gamma\delta)\eta}(z)). \end{aligned} \quad (61)$$

Since one does not know the foliation parameter T_x a priori, we compute it by the expansion with a small parameter $T_x - T_0 \sim r - r_0$.

$$\begin{aligned} 0 = & \dot{\sigma}(r, z(T_0)) + \ddot{\sigma}(r, z(T_0))(T_x - T_0) \\ & + \frac{1}{2} \ddot{\sigma}(r, z(T_0))(T_x - T_0)^2 + \frac{1}{3!} \ddot{\sigma}(r, z(T_0))(T_x - T_0)^3 + O((r-r_0)^4) \end{aligned} \quad (62)$$

Though the computation seems complicated, the foliation parameter T_x is found to be simple in the present case.

$$T_x = \frac{r_0 - r - 2GM \log[(r - 2GM)/(r_0 - 2GM)]}{v^t} \quad (63)$$

Half the geodetic distance becomes

$$\begin{aligned} \sigma(r, r_0) = & \frac{r_0}{2(r_0 - 2GM)}(r - r_0)^2 \\ & \times \left\{ 1 - \frac{GM}{r_0 - 2GM} \frac{r - r_0}{r_0} + \frac{GM(8r_0 - GM)}{12(r_0 - 2GM)^2} \left(\frac{r - r_0}{r_0} \right)^2 + O((r - r_0)^3) \right\}. \end{aligned} \quad (64)$$

The components of $\sigma_{;\alpha}$ are

$$\begin{aligned} -\sigma_{;\alpha}(r, r_0) = & \left\{ 0, \right. \\ & -\frac{r_0}{r_0 - 2GM}(r - r_0) \\ & \times \left(1 - \frac{GM}{2(r_0 - 2GM)} \frac{r - r_0}{r_0} - \frac{GM(r_0 + GM)}{6(r_0 - 2GM)^2} \left(\frac{r - r_0}{r_0} \right)^2 + O((r - r_0)^3) \right), \\ & \left. 0, 0 \right\}. \end{aligned} \quad (65)$$

Expansion of the parallel displacement bi-vector can be derived from the following equation ¹⁰.

$$\begin{aligned}\bar{g}^{\mu\alpha}{}_{;\beta}(x, z) &= -\frac{1}{2}\bar{g}^{\mu\gamma}(x, z)R^{\alpha}{}_{\gamma\beta\delta}(z)\sigma^{i\delta}(x, z) + O(s^2), \\ \bar{g}^{\mu\alpha}{}_{;\nu}(x, z) &= -\frac{1}{2}\bar{g}^{\mu\beta}(x, z)\bar{g}_{\nu}{}^{\gamma}(x, z)R^{\alpha}{}_{\beta\gamma\delta}(z)\sigma^{i\delta}(x, z) + O(s^2).\end{aligned}\quad (67)$$

We integrate the equation

$$\begin{aligned}\frac{\partial}{\partial r}\bar{g}_{\mu\alpha}(r, z(T_x)) &= -\Gamma_{\alpha 1}^{\beta}(z(T_x))\bar{g}_{\mu\beta}(r, z(T_x)) \\ &\quad -\frac{1}{2}\bar{g}_{\mu}{}^{\beta}(r, z(T_x))R_{\beta\alpha 1\gamma}(z(T_x))\sigma^{i\gamma}(r, z(T_x)) + O((r-r_0)^2), \\ \lim_{r \rightarrow r_0}\bar{g}_{\mu\alpha}(r, z(T_x)) &= g_{\mu\alpha}(z(T_x)),\end{aligned}\quad (68)$$

$$\lim_{r \rightarrow r_0}\bar{g}_{\mu\alpha}(r, z(T_x)) = g_{\mu\alpha}(z(T_x)), \quad (69)$$

where we integrate (68) on the slice of $T_x = \text{constant}$. We find that $\bar{g}_{\mu\alpha}(r, z(T_x))$ becomes diagonal to the necessary order of expansion.

$$\begin{aligned}\bar{g}_{\mu\alpha}(r, r_0) &= \text{diag}\left\{-\left(1 - \frac{2GM}{r_0}\right)\left(1 - \frac{GM}{r_0 - 2GM}\frac{r-r_0}{r_0} - \frac{GM(2r_0 - 3GM)}{2(r_0 - 2GM)^2}\left(\frac{r-r_0}{r_0}\right)^2\right),\right. \\ &\quad \left.\left(\frac{r_0}{r_0 - 2GM}\right)\left(1 + \frac{GM}{r_0 - 2GM}\frac{r-r_0}{r_0} + \frac{GM(2r_0 - GM)}{2(r_0 - 2GM)^2}\left(\frac{r-r_0}{r_0}\right)^2\right),\right. \\ &\quad \left.r_0^2\left(1 + \frac{r-r_0}{r_0}\right), r_0^2\left(1 + \frac{r-r_0}{r_0}\right)\right\} + O((r-r_0)^3).\end{aligned}\quad (70)$$

The expansion of $\kappa(r, z(T_x))$ becomes

$$\begin{aligned}\kappa(r, r_0) &= 1 - \frac{GM}{6r_0}\left(2(v^t)^2 + \frac{r_0}{r_0 - 2GM}(r_0 v^\phi)^2\right)\left(\frac{r-r_0}{r_0}\right)^2 + O((r-r_0)^3) \\ &= 1 - \frac{GM(2r_0 - 3GM)}{6(r_0 - 2GM)(r_0 - 3GM)}\left(\frac{r-r_0}{r_0}\right)^2 + O((r-r_0)^3).\end{aligned}\quad (71)$$

Putting these expansions into (57) and into (56), it is possible to see the singular behavior of the bare radiation reaction force. The r -component of the instantaneous reaction force does not vanish and shows the singular behavior in the limit $r \rightarrow r_0$, and, therefore, the regularization is necessary to obtain the physically meaningful reaction force. We compute the leading order contribution in the post-Minkowskian expansion,

$$\begin{aligned}F_r^{inst}(r, r_0) &= -G\mu\frac{1}{(r-r_0)^2} + O(\omega r_0, \omega r) \\ &= -G\mu\sum_{n=2}^{\infty}(n-1)\frac{(r)^{n-2}}{(r_0)^n} + O(\omega r_0, \omega r),\end{aligned}\quad (72)$$

with which we can proceed the power-by-power extraction of the instantaneous force.

¹⁰With bi-tensors defined by the background metric, we consider the expansion of bi-tensors around the orbit of a particle in such a form as

$$\begin{aligned}A^{\alpha\beta\cdots\mu\nu\cdots}(x, z) &= \bar{g}_\gamma^\mu(x, z)\bar{g}_\delta^\nu(x, z)\cdots\left(A^{(0)\alpha\beta\cdots\gamma\delta\cdots}(z) + A^{(1)\alpha\beta\cdots\gamma\delta\cdots}{}_{\epsilon_1}(z)\sigma^{i\epsilon}(x, z)\right. \\ &\quad \left.+ \frac{1}{2}A^{(2)\alpha\beta\cdots\gamma\delta\cdots}{}_{\epsilon_1\epsilon_2}(z)\sigma^{i\epsilon_1}(x, z)\sigma^{i\epsilon_2}(x, z) + O(s^3)\right)\end{aligned}\quad (66)$$

for a small geodetic interval between x and z .

3.4 Regularization

We do not need to be regularized the t -, θ - and ϕ -components of the bare force because the t -, θ - and ϕ -components of the instantaneous reaction force vanish, and we have

$$F_t = \lim_{r \rightarrow r_0} F_t^{(bare)}(r, z^\alpha(T_0)), \quad (73)$$

$$F_\theta = \lim_{r \rightarrow r_0} F_\theta^{(bare)}(r, z^\alpha(T_0)), \quad (74)$$

$$F_\phi = \lim_{r \rightarrow r_0} F_\phi^{(bare)}(r, z^\alpha(T_0)). \quad (75)$$

But the r -component has a singular part, and must be regularized by the r component of the instantaneous force. Using the derived equation (51) and (72),

$$\begin{aligned} F_r(z^\alpha(T_0)) &= \lim_{r \rightarrow r_0} \left(F_r[h^{(bare)}(r)] - F_r[h^{(inst)}(r)] \right) \\ &= \lim_{r \rightarrow r_0} \left(-G\mu \sum_{n=2}^{\infty} (n-1) \frac{(r)^{n-2}}{(r_0)^n} - \left(-G\mu \sum_{n=2}^{\infty} (n-1) \frac{(r)^{n-2}}{(r_0)^n} \right) + O(\omega r_0, \omega r) \right) \\ &\rightarrow O(\omega r_0). \end{aligned} \quad (76)$$

Thus, we find that the r -component of the bare force is regularized by extracting the r -component of the instantaneous force, and $O(\omega r_0)$ -terms are the radiation reaction force.

Here it is important to note $F_r[h^{(bare)}(r)] \neq F_r[h(r)]$. When we derive the higher order of the radiation reaction force, we must consider this difference, i.e., gauge problem.

In this paper, we consider the leading order of the slow motion expansion. So the summation in l of the Fourier-Harmonic expansion to the bare force is the same form of its Taylor expansion. However, if we derive the higher order of the slow motion expansion, the log-terms appear in it. Then it is important to do the Taylor expansion and the power-by-power extraction of the instantaneous force from the bare force.

4 Summary and Discussion

In this paper, we have proposed a method to derive explicit components of the regularized reaction force from the linear metric perturbation induced by a point particle moving around a Schwarzschild/Kerr black hole under a slow motion expansion of the motion. The problem is that the (bare) metric perturbation diverges at the particle. For this reason, some regularization prescription is necessary for explicit derivation.

We considered the combination of the external scheme of the matched asymptotic expansion method[5] in the local analysis and [9, 13] in global analysis, taking the Boyer-Lindquist coordinates $\{t, r, \theta, \phi\}$. For the particle at z^α , we derive the bare reaction force consisting of the bare metric perturbation and the instantaneous reaction force consisting of the instantaneous part of the metric perturbation at $x^\alpha \neq z^\alpha$, $r < r_0$. Our regularization method is that to first extract the instantaneous force from the bare force, and then take the coincidence limit, $x^\alpha \rightarrow z^\alpha$.

However, for the instantaneous force, we know the metric perturbation just around the particle, while, for the bare force, the bare metric perturbation is obtained in the form of a Fourier-Harmonic expansion. Furthermore, in the global analysis, the singular behavior of the metric perturbation around the particle is obscure. Hence it is essential here to take a summation over the Fourier-Harmonic expansion of the perturbation in the regularization procedure. In order to make the performance of the summation over the Fourier-Harmonic expansion in the global analysis simpler, we consider the slow motion expansion of the bare reaction force in $\omega r \sim v$ and $2\omega m \sim v^3$. Using this, we can take a summation over Fourier-Harmonic expansion easily, and then perform the Taylor expansion of the bare reaction force. The Taylor expansion of the instantaneous force can easily be obtained, and we proceeded with the power-by-power extraction of the instantaneous force from the bare force, taking the coincidence limit.

Here we note that the bare force is obtained under the Regge-Wheeler gauge or the radiation gauge, and the instantaneous force is obtained under the harmonic gauge condition. We must consider the gauge transformation before the extraction of singular part.

Using the method developed here, the computation of the radiation reaction force to a point particle in the Schwarzschild space-time is in progress[3]. We will compare the energy and angular momentum loss with the balance equation which is derived up to now.

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Gravitational Waves around a Naked Singularity

— Even-Parity Perturbation —

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Abstract

It has been suggested that a naked singularity may be a good candidate for a strong gravitational wave burster. The naked singularity occurs in the generic collapse of an inhomogeneous dust ball. We examine gravitational waves from the naked singularity for the even-parity mode. We solve the wave equation for gravitational waves by numerical integration using the single null coordinate. The result suggests that the naked singularity would be expected as a strong source of the even-parity mode gravitational radiation.

1 Introduction

The challenge to the detection of gravitational waves is one of the most exciting field in modern physics. The success of it will provide a decisive test of general relativity and provide a new window on astrophysical observation and on the Universe. Unfortunately, in spite of a lot of efforts, no one has succeeded the detection of the gravitational waves yet. However the ground-based laser interferometers, such as, LIGO [1], VIRGO [2], TAMA [3] and GEO600 [4] are being constructed and will soon enter a stage of data taking. The strong sources of gravitational waves should be the region where gravity is general relativistic and where the velocities of bulk motion are near the speed of light. Many researchers have theoretically investigated coalescing and colliding black holes and neutron stars as desirable candidates for detection. And also, gravitational waves emitted during a stellar collapse to a black hole has been studied by calculation of perturbations of a spherically symmetric background[5, 6]. Because of the existence of event horizon, the emitted gravitational waves are dominated by the quasi-normal modes of Schwarzschild black hole at the asymptotic region. Some other considerable candidates exist, e.g., supernovae, rapidly rotating neutron stars, relic gravitational waves of early universe, stochastic gravitational waves from cosmic strings, and so on.

Recently, several researchers have shown that the final fate of gravitational collapse is not always a singularity covered by event horizon. For example, in the Lemaitre-Tolman-Bondi (LTB) space-time[7, 8], a naked shell-focusing singularity appears from generic initial data for spherically symmetric configurations of the rest mass density and specific energy of the dust fluid[9, 10, 11, 12]. The matter content in such space-time may satisfy even the dominant energy condition. In this case with small disturbance of the space-time, for the absence of event horizon, very short wave-length gravitational waves, which are created at high density region around a singularity, may propagate to the observer outside dust cloud. If it is true, we can observe the extremely high energy physics which can not achieved by every high energy experiment on the ground. And then we may obtain the information about the physics so-called ‘quantum gravity’. It has been suggested that the naked singularity may emit the considerable gravitational wave radiation by Nakamura, Shibata and Nakao [13]. This suggestion was proposed from the estimate of gravitational radiation from spindle-like naked singularity. They modeled the spindle-like naked singularity formation in gravitational collapse by the Newtonian prolate dust collapse and the sequence of general relativistic momentarily static initial data.

Our purpose is to understand the dynamics and observational meaning (what is observed) of naked singularity formation or in other word, of the observable region with extremely large space-time curvature. Due to the non-linear nature of the problem, the numerical method to solve the Einstein equation will

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play an important role. On the other hand, its singular behavior will prevent accurate numerical analysis at some stage and hence analytic treatment is also important in such a situation. In this article, we investigate the even-parity linear gravitational waves from the collapse of inhomogeneous spherically symmetric dust cloud. Even for the linearized Einstein equation, we have to perform the numerical integration. However in contrast to the numerical simulation of full Einstein equation, the high precision is guaranteed for the numerical integration of the linearized Einstein equation even in the region with extremely large space-time curvature. In this sense, our present analysis might be regarded as an analytic treatment. Further, the linear stability of known examples of naked singularity formation is necessary as a first step to understand the general dynamics near the “naked singularity formation”.

Recently, Iguchi, Nakao and Harada [14] (hereafter paper I) studied odd-parity metric perturbations around a naked singularity of the LTB space-time. In paper I, it was found that propagation of odd-parity gravitational waves would not be affected by collapse of a dust cloud even if there appears a central naked singularity. Also, the evolution equation of $l = 2$ (quadrupole) mode for odd-parity gravitational waves with a source term which originates from a matter perturbation was investigated[15]. It was shown that the gauge invariant variable diverges only at the center and it does not propagate to the outside. There is little possibility that this collapse is a strong source of odd-parity gravitational waves. Then, in this paper, we consider alternative mode, i.e., even-parity gravitational perturbations. We only consider quadrupole mode of gravitational radiations. We numerically integrate the evolution equations of the metric perturbations for the case of marginally bound collapse.

As we know, the LTB space-time is one of the candidate for a counterexample of cosmic censorship hypothesis (CCH), which was introduced by Penrose[16, 17]. The CCH is a very helpful assumption because various theorems on the properties of black holes were proved under this assumption. Here the precise formulation and validity of the cosmic censorship is out of our concern. We only consider the situation in which the extremely high-density and large-curvature region can be seen by some observer from the gravitational collapse. Such a situation can be regarded as a naked singularity in a practical sense since we have not yet had sufficient ability to predict the phenomena beyond Planck scale. However, here it should be noted that even from this practical point of view, the stability of Cauchy horizon against the gravitational-wave perturbations has still an important physical meaning. Since in the system considered here, the mode propagating with speed of light is the gravitational wave only, the instability of the Cauchy horizon means that the gravitational wave brings extremely large space-time curvature along the null hypersurface near the Cauchy horizon from the region near the central singularity. Hence if the Cauchy horizon in this system is unstable, the naked singularity formation of this type might be a strong emitter of gravitational radiation.

The paper is organized as follows: In Sec. 2, the basic equations are developed; in Sec. 3, the numerical results are presented; in Sec. 4, we summarize our results. We adopt the geometrized units, $c = G = 1$. The signature of the metric tensor and sign convention of the Riemann tensor follow Ref.[19].

2 Basic Equations

Here we give a brief introduction to the formalism of Gerlach and Sengupta [24, 25] for even-parity perturbations around the most general spherically symmetric space-time. We consider the general spherically symmetric space-time with a metric

$$g_{\mu\nu}dx^\mu dx^\nu \equiv g_{ab}(x^d)dx^a dx^b + R^2(x^d)\gamma_{AB}(x^D)dx^A dx^B, \quad (1)$$

and stress-energy tensor

$$t_{\mu\nu}dx^\mu dx^\nu \equiv t_{ab}(x^d)dx^a dx^b + \frac{1}{2}t_A^A R^2(x^d)\gamma_{AB}(x^D)dx^A dx^B, \quad (2)$$

where $\gamma_{AB}dx^A dx^B = d\theta^2 + \sin^2\theta d\phi^2$. Lower-case latin indices refer to radial and time coordinate, while capital latin indices refer to θ and ϕ .

The even-parity perturbations are

$$h_{\mu\nu} = \begin{pmatrix} h_{ab}(x^d)Y_{\text{sym}} & h_a(x^d)Y_{;B} \\ K(x^d)R^2\gamma_{AB}Y + G(x^d)Z_{AB} & \end{pmatrix}, \quad (3)$$

for metric and

$$\delta T_{\mu\nu} = \left(\begin{array}{c} \Delta t_{ab}(x^d)Y \\ \text{sym} \end{array} \quad \Delta t^3(x^d)R^2\gamma_{AB}Y + \Delta t^2(x^d)Z_{AB} \right), \quad (4)$$

for matter where $Y \equiv Y_l^m(x^D)$ are the scalar spherical harmonics and $Z_{AB} = Y_{;AB} + \frac{l(l+1)}{2}Y\gamma_{AB}$. For expressional convenience, we introduce

$$v_a \equiv R_{,a}/R, \quad (5)$$

and

$$p_a \equiv h_a - \frac{1}{2}R^2G_{,a}. \quad (6)$$

Here covariant derivatives are distinguished as follows:

$$\gamma_{AB;C} \equiv 0, \quad g_{ab|c} \equiv 0. \quad (7)$$

A set of even-parity gauge-invariant metric perturbations is

$$k_{ab} \equiv h_{ab} - (p_a|_b + p_b|_a) \quad (8)$$

$$k \equiv K + \frac{l(l+1)}{2}G - 2v^a p_a. \quad (9)$$

A set of even-parity gauge-invariant matter perturbations is

$$T_{ab} \equiv \Delta t_{ab} - t_{ab|c}p^c - t_a^c p_{c|b} - t_b^c p_{c|a} \quad (10)$$

$$T_a \equiv \Delta t_a - t_a^c p_c - R^2(t_A^A/4)G_{,a} \quad (11)$$

$$T^3 \equiv \Delta t^3 - (p^c/R^2)(R^2 t_A^A/2)_{,c} + l(l+1)(t_A^A/4)G \quad (12)$$

$$T^2 \equiv \Delta t^2 - (R^2 t_A^A/2)G. \quad (13)$$

The corresponding perturbed Einstein field equations are seven equations Eqs. (3.12) of Ref. [25]. We write down these equations at the case of the LTB space-time. The background marginally bound LTB metric is

$$ds^2 = -dt^2 + R^2 dr^2 + R^2 d\Omega^2. \quad (14)$$

The background stress-energy tensor is

$$\bar{T}^{\mu\nu} = \bar{\rho}\bar{u}^\mu\bar{u}^\nu, \quad (15)$$

where $\bar{\rho}(t, r)$ is the rest mass density and $\bar{u}^\mu$ is the 4-velocity of the dust fluid. The components of the perturbed stress-energy tensor are obtained from the replacement

$$\bar{\rho} \rightarrow \bar{\rho} + \delta\rho Y, \quad \bar{u}^\mu \rightarrow \bar{u}^\mu + \delta u^\mu, \quad (16)$$

where $\delta u^\mu = (0, V_1(x^d)Y, V_2(x^d)Y_A)$. Now we can write down the perturbed Einstein field equations for the background LTB space-time. However it is expected that these equations are still much complicated to solve. Then we focus attention on those components of the linearized Einstein equations with vanishing matter sides. There are only three matter perturbation variables. Therefore four gauge-invariant matter perturbation variables can be vanished. In the LTB case,

$$T_{11} = T_1 = T^3 = T^2 = 0. \quad (17)$$

The others are expressed as follows;

$$T_{00} = \delta\rho(t, r) \quad (18)$$

$$T_{01} = \bar{\rho}V_1(t, r) \quad (19)$$

$$T_0 = \bar{\rho}V_2(t, r). \quad (20)$$

One of these is

$$k_{00} = \frac{1}{R^2}k_{11}. \quad (21)$$

Using this relation and the remaining equations whose r.h.s. are vanished, we get evolution equations for gauge-invariant metric variables as

$$\begin{aligned}
-\ddot{q} + \frac{1}{R^2} q'' &= \frac{4}{R^2} q + \left(\frac{2}{RR'} + \frac{R''}{R^3} \right) q' + 3 \frac{\dot{R}'}{R'} \dot{q} + 4 \left(\frac{\dot{R}}{R} - \frac{\dot{R}'}{R'} \right) \dot{k} \\
&\quad + \frac{2}{R'^3} \left(-\dot{R}'' - \frac{2R'^2 \dot{R}}{R^2} - \frac{R'' \dot{R}}{R} + \frac{2R' \dot{R}'}{R} + \frac{2R'' \dot{R}'}{R'} \right) k_{01} \\
&\quad + \frac{2}{R'^3} \left(-\dot{R}' + \frac{R' \dot{R}}{R} \right) k_{01}',
\end{aligned} \tag{22}$$

$$\ddot{k} = -\frac{2}{R^2} q - \frac{q'}{RR'} + \frac{\dot{R}}{R} \dot{q} - 4 \frac{\dot{R}}{R} \dot{k} + \frac{2}{RR'} \left(-\frac{\dot{R}'}{R'} + \frac{\dot{R}}{R} \right) k_{01}, \tag{23}$$

$$\dot{k}_{01} = -\frac{\dot{R}'}{R'} k_{01} - q', \tag{24}$$

where $q = k - k_{00}$. If we solve these three equations for the some initial conditions and for the adequate boundary conditions, we can obtain the full evolution of the metric perturbations. To investigate the evolution of the matter perturbations, we should solve the field equations of matter perturbations,

$$\left(\frac{\delta \rho}{\bar{\rho}} \right)' = \frac{1}{\bar{\rho} R^2 R'} \left(\frac{R^2 \bar{\rho}}{R'} (k_{01} + V_1) \right)' - \frac{6}{R^2} V_2 - \dot{k} - \frac{3}{2} (\dot{k} - \dot{q}) \tag{25}$$

$$\dot{V}_1 = -\frac{1}{2} (k' - q') \tag{26}$$

$$\dot{V}_2 = -\frac{1}{2} (k - q), \tag{27}$$

which are derived from the linearized conservation equations.

To constrain the boundary conditions in our numerical calculation, we should consider the regularity conditions at the center. The regularity conditions for the metric perturbations at $r = 0$ are as follows,

$$k \sim k_0(t)r^2, q \sim q_0(t)r^4, k_{01} \sim k_0(t)r^3. \tag{28}$$

For the matter perturbations, the regularity conditions at the center are as follows,

$$\delta \rho \sim \delta \rho_0 r^2, V_1 \sim V_{10}(t)r, V_2 \sim V_{20}(t)r^2. \tag{29}$$

Therefore all the variables we should calculate vanish at the center. This fact is very important in the numerical integration of the above differential equations.

3 Numerical Method and Results

We follow numerical techniques employed in paper I to evolve the perturbations, that is, we use the single-null coordinate. The detailed explanations of the single-null coordinate used in our calculation exist in paper I. It was shown that the perturbation variables q, k, k_{01} are all vanish at the center. A careful treatment of the differential equations may be required near the center for adequate propagation through the center. Then we define new variables

$$\tilde{q} = qR'^7/R^4, \quad \tilde{k} = kR'^4/R^2, \quad \tilde{k}_{01} = k_{01}R'^5/R^3. \tag{30}$$

These new variables are not exactly zero at the center and do not diverge according to the central singularity. We rewrite Eqs. (22), (23), and (24) by these new variables. Next we perform coordinate

transformation for Eq. (22) from the synchronous comoving coordinate (t, r) to the single-null coordinate $(u, \tilde{r})$. Then we get following first order differential equations.

$$\frac{1}{\alpha} \frac{d}{du} X = a_1 X + a_2 W + a_3 Z + a_4 \tilde{k} + a_5 \tilde{q}, \quad (31)$$

$$\dot{W} = b_1 X + b_2 W + b_3 Z + b_4 \tilde{k} + b_5 \tilde{q}, \quad (32)$$

$$\dot{Z} = c_1 X + c_2 W + c_3 Z + c_4 \tilde{k} + c_5 \tilde{q}, \quad (33)$$

$$\dot{\tilde{k}} = d_1 X + d_2 W + d_3 Z + d_4 \tilde{k} + d_5 \tilde{q}, \quad (34)$$

$$\partial_{\tilde{r}} \tilde{q} = e_1 X + e_2 W + e_3 Z + e_4 \tilde{k} + e_5 \tilde{q}, \quad (35)$$

$$Z \equiv \tilde{k}_{01} - R R' \tilde{q}. \quad (36)$$

The coefficients $a_1, \dots$ will be represented in Ref. [26]. Eqs. (31) and (35) are integrated along double null grid. Eqs. (32) - (34) are integrated along the timelike directions $r = \text{const}$. The last task that we should do is to integrate numerically these equations for some initial conditions. At first, we select the initial conditions as

$$k = \exp(-\sigma r^2), \quad (37)$$

$$q = 0, \quad (38)$$

$$k_{01} = 0, \quad (39)$$

$$X = 0, \quad (40)$$

$$\dot{W} = 0, \quad (41)$$

on the initial null hypersurface. More detailed consideration for the initial conditions are now under investigation. The numerical results are shown in Figs. 1 - 3. The values of $q/R^4, k/R^2, k_{01}/R^3$ at the constant circumferential radius are plotted in Figs. 1, 2, 3, respectively. The central values of $q/R^4, k/R^2, k_{01}/R^3$ diverge according to power law with power index 14/3, 8/3, 3 respectively. While for the outside values, say at $R = 100M$, two variable $q/R^4, k_{01}/R^3$ diverge according to power law with power index approximately 1, and k/R^2 would not diverge. Outside the cloud q is correspond to the Weyl scalar ψ_4 . Therefore the inhomogeneous dust cloud collapse may be expected as the strong source of the even-parity gravitational waves. The radiated power of gravitational waves should be considered. From the rough estimate, the power of gravitational radiation would be proportional to k^2 . Therefore total energy radiated by the gravitational waves seems to be finite.

4 Summary

We have studied the behavior of the even-parity perturbations in the LTB space-time. For the quadrupole mode, where gravitational waves exist, we have numerically investigated the wave equation on the marginally bound background space-time. The gauge-invariant metric variables q and k_{01} diverge according to power law not only at the center but also outside the cloud. The variable k diverge only at the symmetric center. Then the Weyl scalar ψ_4 , which is related to q outside the dust cloud, also would diverge at the Cauchy horizon. Therefore the Cauchy horizon may be unstable against the even-parity perturbations. The naked singularity formation of this type might be a strong source of gravitational waves. However the total energy radiated by the gravitational waves, which would be estimated by the integration of k^2 , does not seem to diverge. We are now performing the more detailed investigation. The results would be introduced in Ref. [26].

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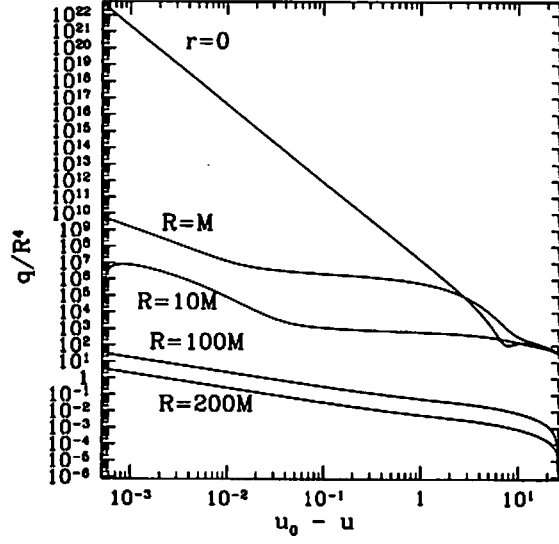


Figure 1: Plots of q/R^4 at constant circumferential radius R . The results of $r = 0$, $R = M$, $R = 10M$, $R = 100M$, and $R = 200M$ are plotted. Here M is total dust mass.

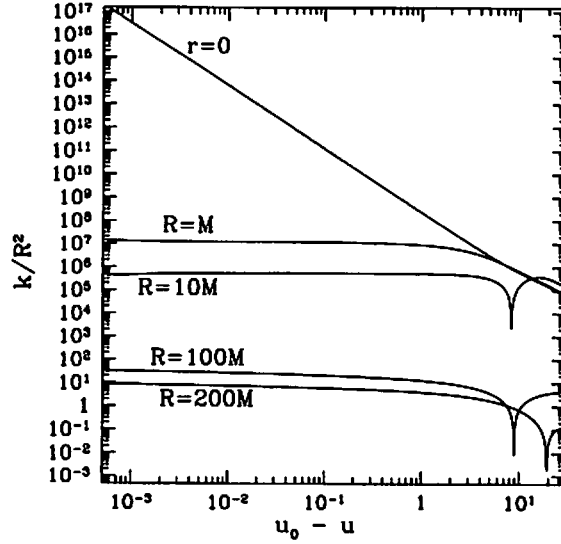


Figure 2: Plots of k/R^2 at constant circumferential radius R . The results of $r = 0$, $R = M$, $R = 10M$, $R = 100M$, and $R = 200M$ are plotted.

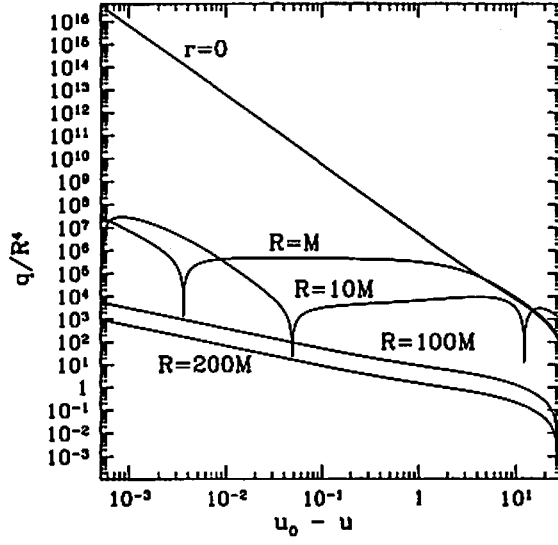


Figure 3: Plots of k_{01}/R^3 at constant circumferential radius R . The results of $r = 0$, $R = M$, $R = 10M$, $R = 100M$, and $R = 200M$ are plotted.

Low frequency gravitational waves from black hole MACHO binaries

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Abstract

The intensity of low frequency gravitational waves from black hole MACHO binaries is studied. First we estimate the gravitational wave background produced by black hole MACHO binaries in the Milky Way halo as well as the cosmological gravitational wave background produced by the extragalactic black hole MACHO binaries. It is found that the cosmological gravitational wave background due to black hole MACHO binaries is larger than the halo background unless an extreme model of the halo is assumed, while it is smaller than the background due to close white dwarf binaries at $\nu_{gw} \lesssim 10^{-2.5}$ Hz if the actual space density of white dwarfs is maximal. This cosmological background due to black hole MACHO binaries is well below the observational constraints from the pulsar timing, quasar proper motions and so on. We find that one year observation by LISA will be able to detect gravitational waves from at least several hundreds of nearby independent black hole MACHO binaries whose amplitudes exceed these backgrounds. This suggests that LISA will be able to pin down various properties of primordial black hole MACHOs together with the results of LIGO-VIRGO-TAMA-GEO network. Furthermore, it may be possible to draw a map of the mass distribution of our halo, since LISA can determine the position and the distance to individual sources consisted of black hole MACHO binaries. Therefore, LISA may open a new field of the gravitational wave astronomy.

1 INTRODUCTION

The observations of gravitational microlensing toward the Large Magellanic Cloud (LMC) have revealed that a significant fraction of the Milky Way halo consists of solar or sub-solar-mass compact objects, which are called massive compact halo objects (MACHOs). The analysis of the first 2.1 years of photometry of 8.5×10^6 stars in the LMC by the MACHO Collaboration [1] suggests that the fraction $0.62^{+0.3}_{-0.2}$ of the halo consists of MACHOs of mass $0.5^{+0.3}_{-0.2} M_{\odot}$ assuming the standard spherical flat rotation halo model.

At present, we do not know what MACHOs are. This is especially because there are strong degeneracies in any microlensing measurements among the mass, the velocity and the distance to a lens object. There are several candidates proposed to explain MACHOs, such as brown dwarfs, red dwarfs, white dwarfs and so on. However, all of them have some theoretical or observational difficulties. Any objects

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clustered somewhere between the LMC and the sun with the column density larger than $25M_{\odot}\text{pc}^{-2}$ can explain the data [2]. They include the possibilities; LMC-LMC self-lensing, the spheroid component, the thick disk, a dwarf galaxy, the tidal debris, warping and flaring of the Galactic disk. However, none of them do not convincingly explain the microlensing events toward the LMC. This means that it is worth considering alternative possibilities that MACHOs are not stellar objects but absolutely new objects such as black holes of mass $\sim 0.5M_{\odot}$ or boson stars with the boson mass of $\sim 10^{-10}\text{eV}$.

In this paper, we consider primordial black holes to explain MACHOs. This possibility is free from observational constraints at present. Theoretically it is still uncertain if primordial black holes (PBHs) are formed in the early universe. Our standpoint here is not to discuss the details of the formation mechanism but to establish the observational signatures of PBHs if they exist. Firstly, electromagnetic radiation from gas accreting to a black hole MACHO (BHMACHO) is too dim to be observed unless the velocity of BHMACHOs is exceptionally small [3]. If PBHs are formed, most of them evolve into binaries through the three body interaction in the early universe at $t \sim 10^{-5}\text{s}$ [4, 5]. Some of these BHMACHO binaries coalesce at or before present epoch and it may be possible to detect the gravitational waves from such coalescing BHMACHO binaries. The event rate of coalescing BHMACHO binaries is estimated as $5 \times 10^{-2} \times 2^{\pm 1}$ events/yr/galaxy, which suggests that we can detect several events per year within 15 Mpc by LIGO-VIRGO-TAMA-GEO network [4, 5].

On the other hand, the other part of BHMACHO binaries still have large separations, and they emit gravitational waves at lower frequencies. These low frequency gravitational waves may be detected by the planned Laser Interferometer Space Antenna (LISA) and Orbiting Medium Explorer for Gravitational Astrophysics (OMEGA), which will be sensitive to gravitational waves in the frequency band of $10^{-4} - 10^{-1}$ Hz. In this paper, we investigate low frequency gravitational waves emitted by BHMACHO binaries, and we argue that LISA will be able to detect gravitational waves from BHMACHO binaries [6, 7].⁴

2 BHMACHO BINARIES

For simplicity, we assume that PBHs dominate the dark matter, i.e., $\Omega = \Omega_{BH}$, where Ω_{BH} is the density parameter of PBHs at present. Furthermore, we assume that all PBHs have the same mass M_{BH} . If PBHs are formed, a part of them evolve into binaries through the three body interaction in the early universe at $t \sim 10^{-5}$ [4, 5]. Assuming that PBHs are distributed randomly at the formation of PBHs, we can obtain the probability distribution function (PDF) for semimajor axis a and eccentricity e of the binary, i.e., the fraction of all BHMACHOs that are binaries with semimajor axis a in the range of da and eccentricity e in the range of de , as

$$f(a, e)dade = \frac{3}{4} \frac{\beta}{(\alpha\bar{x})^{3/2}} \frac{a^{1/2}e}{(1-e^2)^{3/2}} \exp \left[-\frac{\beta}{(1-e^2)^{1/2}} \left(\frac{a}{\alpha\bar{x}} \right)^{3/4} \right] dade, \quad (1)$$

where $\sqrt{1-\beta^2} < e < 1$ and $0 < a < \infty$. Here α and β are constants of order unity, and $\bar{x} = 1.2 \times 10^{16} (M_{BH}/M_{\odot})^{1/3} (\Omega h^2)^{-4/3}$ cm is the mean separation of PBHs at the time of matter-radiation equality. As usual, $H_0 = 100h$ km/s/Mpc is the present Hubble parameter. In this paper, we adopt $\alpha = 0.5$ and $\beta = 0.7$.⁵ Since the Keplerian orbital frequency of a binary, ν_p , is related to the semimajor axis, a , by $2\pi\nu_p = \sqrt{G(M_{BH1} + M_{BH2})/a^3}$, the PDF for a and e can be transformed into that for ν_p and e by

$$f_{\nu,e}(\nu_p, e) = f(a, e) \left| \frac{da}{d\nu_p} \right| = \frac{2a}{3\nu_p} f(a, e). \quad (2)$$

Since a BHMACHO binary radiates gravitational waves, the binary parameters, a and e , change as a function of time due to radiation reaction. Therefore the PDF $f(a, e)$ in Eq. (1) is also a function of time $f(a, e; t_1)$, where t_1 is the time after the formation of binaries. We can compute the initial values of

⁴W. A. Hiscock have also independently carried out a similar study using a more simple model, though he did not consider the individual sources [7].

⁵In [5], $\alpha = 0.4$ and $\beta = 0.8$ are adopted. However, these values are obtained by a least squares fitting without fixing the power indices of a and b . When we use the power indices of the analytical estimates, i.e., 3 for a and 4 for b , it is better to adopt $\alpha = 0.5$ and $\beta = 0.7$. Note that the results of this paper are not affected so much by the detailed choice of these constants.

a and e as a function of a , e and t_1 , using the evolution equations for a and e due to radiation reaction [8]. Therefore it is easy to compute $f(a, e; t_1)$ numerically [6]. As shown in [6], binaries with high orbital frequencies ($\nu_p \gtrsim 10^{-3}$ Hz) are almost in circular orbits at present. In the case $e \ll 1$, an approximate formula for the present PDF for ν_p can be obtained [6] as

$$f_{\nu,t}(\nu_p; t_1) d\nu_p \sim \frac{425}{3552} \left(\frac{t_1}{\bar{t}} \right)^{3/37} \frac{t_0}{t_1} \left(\frac{a}{a_0} \right)^4 \Gamma \left(\frac{58}{37} \right) \frac{d\nu_p}{\nu_p}, \quad (3)$$

where $t_0 = 10^{10}$ yr, $a_0 = 2.0 \times 10^{11} (M_{BH}/M_\odot)^{3/4}$ and $\bar{t} = \beta^7 (\alpha \bar{x}/a_0)^4 t_0$. This expression corresponds to the integral of the PDF $f_{\nu,e,t}(\nu_p, e; t_1)$ over e . Note that $f_{\nu,t}(\nu_p; t_1)$ is proportional to $\nu_p^{-11/3}$.

In this paper, we set $M_{BH} = 0.5 M_\odot$ and $\Omega h^2 = 0.1$ when quantities with M_{BH} or Ωh^2 are evaluated.

3 GRAVITATIONAL WAVE BACKGROUND

3.1 cosmological gravitational wave background

In this model, there are so many BHMACHO binaries that they themselves become stochastic background sources. Let us consider the cosmological background due to the extragalactic BHMACHO binaries [6].

A binary with eccentricity $e \neq 0$ emits gravitational waves at frequencies $\nu_{gw} = p\nu_p$ ($p = 1, 2, \dots$) [9, 10]. The gravitational wave luminosity from a binary which emits gravitational waves at frequency ν_{gw} as the p -th harmonic is given by [9]

$$L^{(p)}(\nu_{gw}, e) = L_0 \nu_{gw}^{10/3} p^{-10/3} g(p, e), \quad (4)$$

where $L_0 = 1.351 \times 10^{44} (M_{BH}/0.5 M_\odot)^{10/3} \text{ erg s}^{-1} \text{ Hz}^{-10/3}$. The function $g(p, e)$ is given by [9].

The spectrum of stochastic gravitational waves is characterized by $\Omega_{GW}(\nu_{gw})$, the ratio of the energy density of the gravitational waves per unit logarithmic frequency to the critical density [11]. To evaluate Ω_{GW} , we use the fact that the gravitational wave energy existing in a unit comoving volume is equal to the red-shifted total gravitational wave energy that is emitted per unit comoving volume until now. Then we obtain

$$\Omega_{GW}(\nu_{gw}) = \frac{1}{\rho_c c^2} \int_{t_f}^{t_0} dt \frac{R/R_0}{(\bar{x} R_0)^3} \mathcal{L}_{GW}(\nu_{gw}; t), \quad (5)$$

where

$$\mathcal{L}_{GW}(\nu_{gw}; t) = L_0 \nu_{gw}^{13/3} \left(\frac{R_0}{R} \right)^{13/3} \sum_{p=0}^{\infty} p^{-13/3} \int de f_{\nu,e,t}(\nu_p, e; t) g(p, e) \quad [\text{erg s}^{-1}], \quad (6)$$

is the gravitational wave luminosity per unit logarithmic frequency at ν_{gw} per BHMACHO and $\rho_c = 3H_0^2/8\pi G$ is the critical density. t_f is the binary formation time at which scale factor R equals to $R_m := (\alpha/\alpha\bar{x})^{3/4}$ [4, 5]. Note that the scale factor is normalized such that $R = 1$ at the time of matter-radiation equality. The relation between t and R is given by the Hubble equation $H^2 := (\dot{R}/R)^2 = 8\pi G\rho/3 - Kc^2/R^2$ where we assume that the energy density is determined by BHMACHOs and the radiation, and the present temperature of the cosmic microwave background is about 2.75 K. We set $h = 0.8$, when quantities with $\rho_c c^2$ or H_0 are evaluated numerically. When we evaluate $\mathcal{L}_{GW}(\nu_{gw}; t)$ in Eq. (6) numerically, we perform the summation up to $p = 10$, since the contribution from higher harmonics is not so large [6]. Since the contribution to the energy density from binaries at higher redshift is smaller [6], we perform the integration in Eq. (5) from $R/R_0 = 1/R_0$ (not R_m/R_0) to $R/R_0 = 1$ to evaluate the energy density of the cosmological background numerically.

The spectral density of the cosmological gravitational wave strain amplitude is given by

$$h_\nu^{cos} = 5.615 \times 10^{-20} \frac{\sqrt{\rho_c c^3 \Omega_{GW}(\nu_{gw})}}{\nu_{gw}^{3/2}} \quad [\text{Hz}^{-1/2}]. \quad (7)$$

When the condition $e \ll 1$ is satisfied, a relation $h_\nu^{cos} \propto \nu_{gw}^{-7/6}$ is satisfied. We have confirmed that the cosmological background is well below the observational constraints from the pulsar timing, quasar proper motions, nucleosynthesis and so on [6].

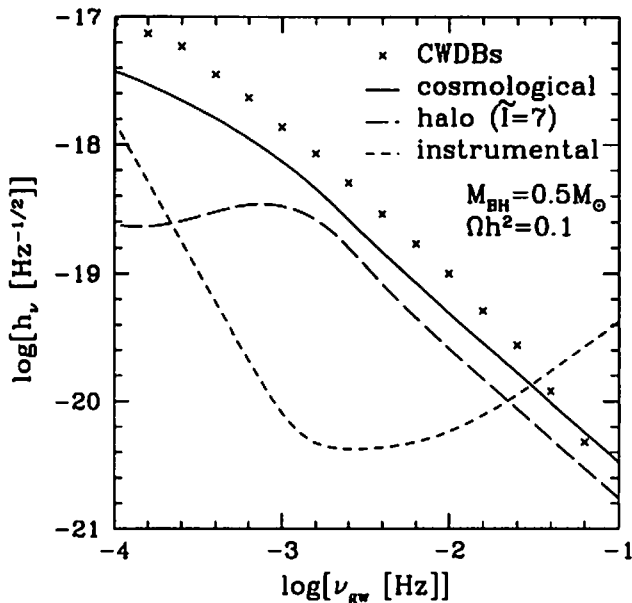


Figure 1: The spectral density h_ν for the cosmological gravitational wave background by the solid line, for the halo background by the dashed line, the CWDBs background by the cross, and the instrumental noise by the dotted line. We assume an isothermal halo to evaluate the halo background.

3.2 other background noises

In a similar manner, we can calculate the gravitational wave background produced by BHMACHO binaries in the Milky Way halo [6]. The halo background depends on the halo model. For simplicity, we assume that the distribution of the number density of MACHOs is spherical symmetric as

$$n(r) = \frac{n_s}{[1 + (r^2/D_a^2)]^\lambda}, \quad (8)$$

where r is the distance from the center of our galaxy, n_s is the density of MACHOs at the galactic center, and D_a is the core radius of the distribution. The optical depth is given by $\tau^{LMC} = 4\pi GM_{BH}/c^2 \times \int_0^{D_s} x(1 - x/D_s)n(r)dx$, where x is the distance from the earth toward a lens, and $D_s = 50$ kpc is the distance to the LMC from the earth. By equating this to the optical depth of MACHOs toward the LMC $\tau^{LMC} \sim 2 \times 10^{-7}$ estimated from gravitational microlensing events [1], we can obtain n_s for fixed λ and D_a . Here the relation between r and x is expressed as $r^2 = D_0^2 - 2D_0x \cos \theta + x^2$, where we adopt $D_0 = 8.5$ kpc as the distance from the galactic center to the earth, and $\cos \theta = 0.153$ as the LMC direction.

In Fig. 1, we summarize the spectral density for the cosmological background, h_ν^{cos} , in Eq. (7) and that for the halo background, h_ν^{halo} , together with the spectral density for Close White Dwarf Binaries (CWDBs), h_ν^{WD} , and that due to the two kinds of instrumental noises, h_ν^{opt} and h_ν^{acc} . When we plot the halo background case, we assume an isothermal halo, i.e., $\lambda = 1$. We obtain the spectral density due to CWDBs, h_ν^{WD} , from Table. 7 in Ref.[12]. The instrumental noises are taken as $h_\nu^{opt} = (4 \times 10^{-11} \text{ m}/\sqrt{\text{Hz}})/(2L) \times \sqrt{1 + (2\pi L\nu_{gw}/c)^2}$ and $h_\nu^{acc} = 6 \times 10^{-15} (\nu_{gw}/10^{-4} \text{ Hz})^{-1/3} \text{ ms}^{-2}/\sqrt{\text{Hz}}/(2L(2\pi\nu_{gw})^2)$ where we adopted $2L = 10^{10}$ m [13]. As we can see from Fig. 1, the cosmological background dominates the halo background for an isothermal halo with this choice of model parameters. If the halo is more concentrated, i.e., λ is larger and R_a is smaller, the halo background can dominate the

cosmological background. However, since such a situation is less interesting for us, we neglect the halo background hereafter.

At first glance, the spectral density due to CWDBs, h_ν^{WD} , dominates the cosmological background due to BHMACHOs, h_ν^{cos} . However, since the expected number of CWDBs at higher frequencies becomes less than one, the confusion noise level due to CWDBs decreases suddenly at a transition frequency $\nu_{gw} \sim 10^{-2.5}$ Hz [14]. Therefore, as for the confusion noise level, the cosmological background due to BHMACHO binaries dominates the background due to CWDBs above the transition frequency. As we will see later, the gravitational wave background in this frequency range almost determines the expected number of individually observable sources. Therefore, we only consider the cosmological background and the instrumental noises. The total noise amplitude is determined by $h_\nu^{tot} = [(h_\nu^{cos})^2 + (h_\nu^{opt})^2 + (h_\nu^{acc})^2]^{1/2}$.

4 INDIVIDUAL SOURCE

To be observed as individual sources, the amplitudes of the gravitational waves from the binaries have to exceed the threshold amplitude $h^{th} = 5h_\nu(\Delta\nu)^{1/2}$ which is determined by the gravitational wave background h_ν and the frequency resolution $\Delta\nu$. Here we assume that the frequency resolution is determined by the observation time T as $\Delta\nu = 1/T$ [15, 11], and we set the signal-to-noise ratio (SNR) as 5.

The amplitude of the gravitational wave at ν_{gw} from a binary with e , p and the distance from the earth, d , is given by $h = 2\sqrt{GF_i}/c^{3/2}\sqrt{\pi}\nu_{gw}$ where $F_i = L^{(p)}(\nu_{gw}, e)/4\pi d^2$. Then, the requirement that the signal exceeds the threshold, $h > h^{th}$, determines the maximum distance to individually observable sources as

$$D_{max}^{(p)}(\nu_{gw}, e) = \frac{\sqrt{G}}{\pi c^{3/2} h^{th}} \sqrt{L_0 g(p, e)} \nu_{gw}^{2/3} p^{-5/3}. \quad (9)$$

As shown in [6], the maximum distance extends beyond $\sim 10^2$ kpc at high frequencies by one year observation. This means that almost all halo BHMACHO binaries radiating gravitational waves at these frequencies can be observed as individual sources.

Then the expected number of individually observable sources per unit logarithmic frequency is given by

$$N(\nu_{gw}) = \sum_{p=1}^{\infty} \int de \mathcal{N}^{(p)}(\nu_{gw}, e) \frac{\nu_{gw}}{p} f_{\nu, e, t}(\nu_p, e; t_0), \quad (10)$$

where

$$\mathcal{N}^{(p)}(\nu_{gw}, e) = \int_{d < D_{max}^{(p)}(\nu_{gw}, e)} n(r) d^3x, \quad (11)$$

and $d = (D_0^2 - 2rD_0 \cos\theta + r^2)^{1/2}$ is the distance between a source and the earth. In Fig. 2, we plot the total expected number of the individual sources per logarithmic frequency, $N(\nu_{gw})$, in Eq. (10) for $T = 1$ yr, $\lambda = 1$ and $D_a = 5$ kpc with the solid line. The total expected number in the range of $\nu_1 < \nu_{gw} < \nu_2$ is $N^{tot} = \int_{\ln \nu_1}^{\ln \nu_2} N(\nu_{gw}) d(\ln \nu_{gw})$. By one year observation, the total expected number N^{tot} in the range of $10^{-4} \text{ Hz} < \nu_{gw} < 10^{-1} \text{ Hz}$ is about 792 for $\lambda = 1$ and $D_a = 5$ kpc.

5 DENSITY PROFILE RECONSTRUCTION

Since LISA can measure distances and positions of BHMACHO binaries [13, 16], it may be possible to obtain the density profile of our halo from the distribution map of BHMACHO binaries, recalling that the BHMACHO binaries trace the mass distribution of the BHMACHO halo. In this section, we demonstrate how well the density profile of our halo can be determined by the observation of low frequency gravitational waves.

The algorithm is roughly as follows. We first consider a density profile of our halo with some parameters, and set parameters as the real ones. Next we distribute BHMACHO binaries according to this profile in our halo, assigning frequencies according to the PDF in Eq. (3) with $t_1 = t_0$. Note that, as long as $\nu_{gw} \gtrsim 10^{-3}$ Hz, it is sufficient to consider the case $e \ll 1$. Then we make an observation in this numerically generated galaxy. The SNR for each BHMACHO binary can be obtained as $h/h_\nu(\Delta\nu)^{1/2}$.

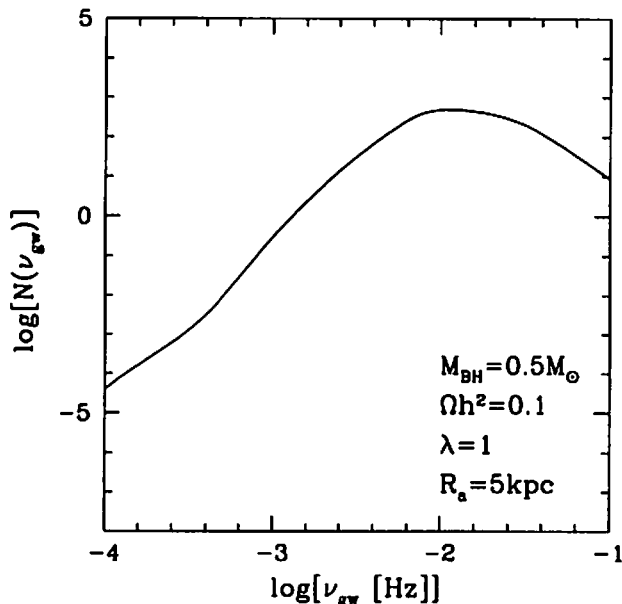


Figure 2: The total expected number of the individual sources per logarithmic frequency $N(\nu_{gw})$ for $T = 1$ yr, $\lambda = 1$ and $D_a = 5$ kpc. The background is due to cosmological BHMACHO binaries and instruments.

We select the BHMACHO binaries whose SNR is above a threshold. Here we take it as 5. Then, after taking into account of the observational error, we can obtain the distribution map of the BHMACHO binaries. We have to note that we cannot use all these binaries to reconstruct the density profile. If we want to determine the density profile within $r < D_{obs}$, we have to use binaries with frequencies $\nu_{gw} > \nu_{obs}$ where ν_{obs} is determined by $D_{mar}^{(2)}(\nu_{obs}, 0) = D_{obs} + D_0$ in Eq. (9). By fitting the distribution map by the density profile, we can compare the reconstructed density profile and the given density profile.

For simplicity, we consider the spherical density profile in Eq. (8), which is expressed by two parameters, the core radius D_a and the power index λ . We take $D_a = 5$ kpc and $\lambda = 1$ as the real ones. The total number of BHMACHOs within $r < D_{halo}$ is given by $N_{total} = \int_{r < D_{halo}} n(r) d^3x$. The total number of BHMACHO binaries with $\nu_1 < \nu_{gw} < \nu_2$ is given by $N_b = N_{total} \int_{\nu_1}^{\nu_2} f_{\nu,t}(\nu_p; t_0) (d\nu_p/d\nu_{gw}) d\nu_{gw}$ with $p = 2$. We distribute these BHMACHO binaries within $r < D_{halo}$. For example, $N_b = 86221$ for $D_{halo} = 500$ kpc, $D_a = 5$ kpc, $\lambda = 1$, $\nu_1 = 4 \times 10^{-3}$, and $\nu_2 = 1 \times 10^{-1}$.

At frequencies above $\sim \nu_{dis} := 10^{-2.23} (T/1\text{yr})^{-6/11} (M_{BH}/0.5M_\odot)^{-5/11}$, LISA can measure distances to BHMACHO binaries since binaries significantly change their orbital frequencies through gravitational wave emission within the observation time T [13]. If the frequency of the gravitational waves from a source changes by $\xi \Delta \nu$ during the observation, the distance to the source is determined by the amplitude of the gravitational waves h , the frequency of the gravitational waves ν_{gw} , and the frequency change ξ as $d = 5c\xi/12\sqrt{10}\pi^2 G^{5/3} T^2 \nu_{gw}^3 h$. The observable parameters, h , ν_{gw} and ξ , contain observational errors as $h(1 \pm \epsilon_1)$, $\nu_{gw}(1 \pm \epsilon_2)$ and $\xi(1 \pm \epsilon_3)$, which can be estimated as $\epsilon_1 = 1/(SNR)$, $\epsilon_2 = 1/\nu_{gw}T$ and $\epsilon_3 = 1/\xi$. Therefore the observational error in the distance is estimated as $d(1 \pm (\epsilon_1 + 3\epsilon_2 + \epsilon_3))$. For simplicity, we assign the observational error to the distance with a uniform probability distribution. The angular resolution of LISA is estimated to be a few degree [16]. For simplicity we assign the observational errors of 3° with a uniform probability distribution.

One experimental realization of this observation is shown in Fig. 3. This histogram shows the number of the observed BHMACHO binaries whose distances from the galactic center are within $[r, r + \delta r]$. We

take $\delta r = 1$ kpc. Here we set $T = 10^3$ yr and $D_{obs} = 200$ kpc, which corresponds to $\nu_{obs} = 5.50 \times 10^{-3}$ Hz. In this realization, the total number of BHMACHO binaries which can be used to determine the density profile, N_{map} , is $N_{map} = 14435$. The fitted parameters turn out to be $D_a = 6.46$ kpc and $\lambda = 1.01$. Therefore the density profile is reconstructed quite well, though this example with $T = 10^3$ yr may be extreme.

Further discussions will be presented in [17].

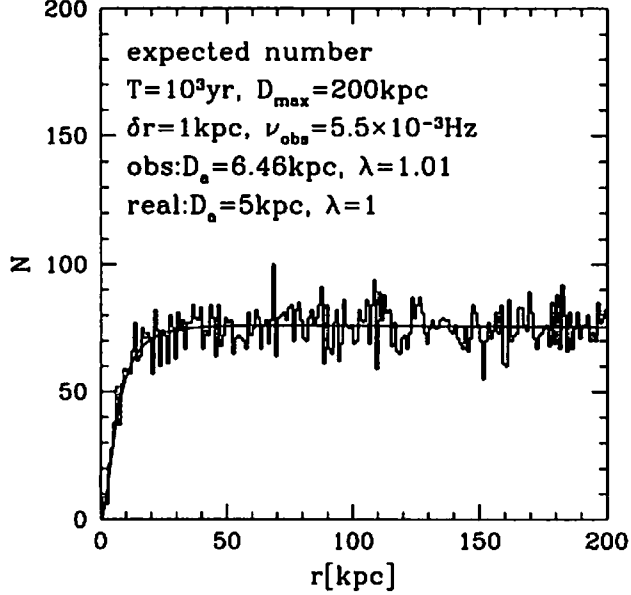


Figure 3: The number of the observed BHMACHO binaries whose distances from the galactic center are within $[r, r + \delta r]$. We take $\delta r = 1$ kpc. Here we set $T = 10^3$ yr and $D_{obs} = 200$ kpc, which corresponds to $\nu_{obs} = 5.50 \times 10^{-3}$ Hz. In this realization, the total number of BHMACHO binaries which can be used to determine the density profile, N_{map} , is $N_{map} = 14435$. The fitted parameters turn out to be $D_a = 6.46$ kpc and $\lambda = 1.01$.

6 SUMMARY

In this paper, we have investigated low frequency gravitational waves emitted by BHMACHO binaries. We have evaluated the gravitational wave background produced by BHMACHO binaries in the Milky Way halo. We have also evaluated the cosmological gravitational wave background produced by the extragalactic BHMACHO binaries, which is larger than the halo background when we assume that the density profile of our halo is isothermal. The root-mean-square strain amplitude of the gravitational wave background due to CWDBs is possibly larger than that due to the extragalactic BHMACHO binaries. However, once the number of CWDBs per frequency resolution bin is taken into account, the cosmological gravitational wave background due to extragalactic BHMACHO binaries will determine

the confusion limit for the detection of individual sources at $\nu_{gw} \gtrsim 10^{-2.5}$ Hz. We have found that the cosmological gravitational wave background produced by BHMACHO binaries is well below the existing observational limits obtained from the pulsar timing, the quasar proper motions, the big bang nucleosynthesis and so on. We have found that one year observation by LISA will be able to reveal at least several hundreds of BHMACHO binaries whose gravitational wave amplitudes exceed the confusion limit when we require the SNR to be greater than 5 for detection. If the confusion limit is determined by the noises due to extragalactic BHMACHO binaries and instruments, the expected number of observable individual BHMACHO binaries will be about eight hundreds. This suggests that LISA will be able to clarify whether MACHOs are primordial black holes or not, together with the results of LIGO-VIRGO-TAMA-GEO network. Furthermore, it may be possible to obtain the density profile of our halo from the distribution map of BHMACHO binaries, since LISA can measure distances and positions of BHMACHO binaries.

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Irrotational and Incompressible Binary Systems in the First-Post Newtonian Approximation

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Abstract

First post-Newtonian (PN) hydrostatic equations for an irrotational fluid are solved for an incompressible binary system. The equilibrium configuration of the binary system is given by the small deformation from the irrotational Darwin-Riemann ellipsoid which is the solution at Newtonian order. It is found that the orbital separation at the innermost stable circular orbit (ISCO) decreases when one increases the compactness parameter M_*/c^2a_* in which M_* and a_* denote the mass and the radius of a star, respectively. If we compare the 1PN angular velocity of the binary system at the ISCO in the unit of $\sqrt{M_*/a_*^3}$ with that of Newtonian order, the angular velocity at the ISCO is almost the same value as in the Newtonian order when one increases the compactness parameter. Also, we do not find the instability point driven by the deformation at the 1PN order where a new sequence bifurcates throughout the equilibrium sequence of the binary system until ISCO.

1 Introduction

At the beginning of the next century, we will have the great instruments for detection of gravitational waves (LIGO/VIRGO/GEO/TAMA). One of the most promising sources of gravitational waves in the sensitive frequency range of these laser interferometers is coalescing binary neutron stars (BNS's). If we have accurate theoretical templates of the inspiraling phase of BNS's, we can extract various information on BNS's from gravitational waves such as their mass and spin[1]. Moreover, when we construct the reliable theoretical templates around the innermost stable circular orbit (ISCO) of BNS's, we can obtain physical information about equation of state of a neutron star, i.e., the relation between the mass and the radius of a neutron star[2].

BNS's evolve due to the radiation reaction of gravitational waves, so that they cannot become equilibrium states. However, the timescale of the orbital decay is much longer than the orbital period of BNS's until the ISCO. Therefore, we can regard the state of the binary system as quasiequilibrium even if the orbit approaches to the ISCO.

In order to obtain the accurate theoretical templates of gravitational waves from irrotational BNS's just prior to the ISCO which is our goal, a general relativistic hydrostatic problem with compressible equation of state has to be numerically solved. Even if we would have such numerical solutions[3, 4, 5, 6, 7], it is necessary to compare them with analytic or semi-analytic ones in order to check the validity of the numerical calculation. Lombardi, Rasio and Shapiro have semi-analytically studied the irrotational BNS's using the energy variational method in the first post-Newtonian (PN) approximation of general relativity[8]. However, they do not solve the velocity fields of the binary system at the 1PN order. They only give the velocity fields of the incompressible fluid at Newtonian order which flows along the perpendicular plane to the rotational axis x_3 . We know that the existence of the x_3 component of the velocity field is important in the irrotational fluid problem[4, 5, 6, 9]. From this fact, we believe the importance of the velocity potential of the 1PN order in the irrotational binary problem.

We have already solved the equilibrium sequence of the corotating binary system[10]. However, the formalism which we have used in these papers is not applicable for the irrotational case. A formalism for constructing quasiequilibrium configuration of an irrotational binary system in general relativity has just been obtained[11, 12, 13, 14, 15]. In this formalism, we have to solve only two hydrostatic equations as for the fluid equations. One of them is the integrated forms of the Euler equation and the other

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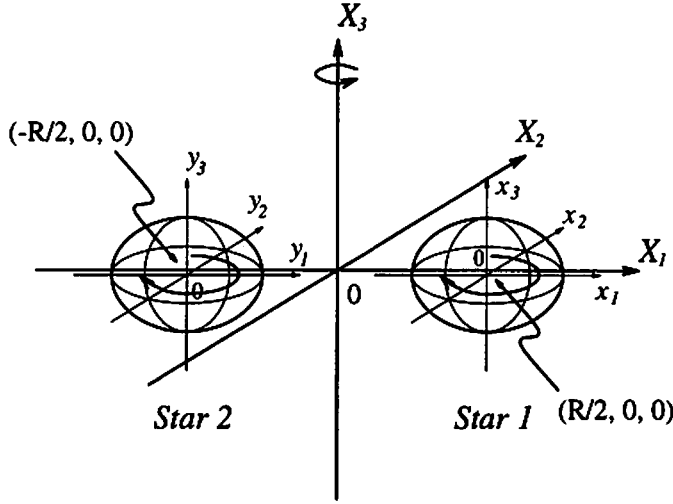


Figure 1: Sketch of the Darwin-Riemann ellipsoid. Three coordinate systems are shown. The center of mass of star 1 locates at the origin of the corotating frame x_i , and that of star 2 locates at the origin of y_i . These two origins of the corotating frames locate at $(R/2, 0, 0)$ and $(-R/2, 0, 0)$ in the coordinate system X_i in which the origin locates at the center of mass of the binary system.

is the Poisson equation for the velocity potential. Thus, the formalism seems to be very tractable for computing equilibrium configuration of realistic irrotational bodies. In order to develop the method for solving the irrotational binary problem consistently, the present author and collaborators recently investigated an irrotational and incompressible star of the 1PN order[9] using the above formalism. Our method is originally introduced by Chandrasekhar[16, 17, 18, 19]. As an extension of this method, we study an irrotational and incompressible binary system in the 1PN approximation of general relativity. We assume that each star in the binary system is constructed by an incompressible and homogeneous fluid. This assumption has a great advantage to solve equations analytically. In the following sections, if one would like to know the calculations and so on in detail, see Ref [20].

Throughout this proceeding, c denotes the light velocity and we use the units in which $G = 1$. Latin indices $i, j, k, \dots$ take values 1 to 3, and δ_{ij} denotes the Kronecker delta. We use I_{ij} and $\mathcal{I}_{ij}$ as the quadrupole moment and its trace free part,

$$\mathcal{I}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} \sum_{k=1}^3 I_{kk} \quad (1)$$

of each star in the binary system.

2 Formulation

Equilibrium configurations of binary systems with non-uniform velocity fields are obtained by solving the Euler, continuity and Poisson equations consistently. Since we consider a binary system constructed by incompressible stars, all the calculations are carried out analytically even in the 1PN case. The procedure is as follows.

(1) We construct a Newtonian equilibrium configuration of the binary system, i.e., the irrotational Darwin-Riemann ellipsoid[21, 22], as a non-perturbed state. For simplicity, we consider only the case in which the directions of the vorticity vectors and the angular velocity vector lie along x_3 -axis, and assume that BNS's are composed of equal mass stars. At this step, we obtain the Newtonian angular velocity Ω_{DR}^2 and the axial ratios a_2/a_1 and a_3/a_1 .

(2) The 1PN corrections for the velocity potential and gravitational potentials are obtained from the 1PN Poisson equations for them.

(3) We calculate the deformation from the Newtonian binary system induced by the 1PN gravity using the Lagrangian displacement vectors introduced by Chandrasekhar[21]. Then, the corrections for the Newtonian quantities due to the deformation of the binary system are estimated.

(4) From the first tensor virial relation at the 1PN order, i.e., the force balance equation at the 1PN order, we calculate the correction for the angular velocity[10].

(5) We substitute all the 1PN corrections obtained in (2), (3) and (4) into the 1PN Euler and continuity equations. Then, the coefficients of the Lagrangian displacement vectors and the 1PN velocity potential are determined from these two equations with the boundary conditions on the stellar surface.

We consider the equilibrium sequences of BNS's of equal masses ($M_1 = M_2 = M$) whose coordinate separation is R .² For simplicity of the calculation, we use three kinds of coordinate systems which are all rotating frames. The first frame is x_i in which the center of mass of a star (star 1) locates at the origin of the coordinate system and the other one (star 2) locates at $(x_1, x_2, x_3) = (-R, 0, 0)$ (see Fig. 1). The second one is y_i in which the center of mass of star 2 locates at the origin and star 1 locates at $(y_1, y_2, y_3) = (R, 0, 0)$. The third one is X_i in which the origin locates at the center of mass of the binary system and the centers of masses of two stars locate at $(X_1, X_2, X_3) = (R/2, 0, 0)$ and $(-R/2, 0, 0)$.

The main purpose of this proceeding is to calculate the 1PN correction for the angular velocity, the velocity potential, the energy, and the angular momentum in the Darwin-Riemann problem. In the calculation, we assume that the parameter a_0/R , in which a_0 is a typical radius of the star, is small and we use it as the expansion parameter. Accordingly, there exist two different types of expansion parameters in this proceeding. One of them is the post-Newtonian parameter and the other is a_0/R .

In the Newtonian order, the angular velocity (Ω_{DR}) becomes[23]

$$\Omega_{\text{DR}}^2 = \frac{2M}{R^3} + \frac{18\mathcal{I}_{11}}{R^5} + O(R^{-7}). \quad (2)$$

Thus, in the 1PN approximation, we can expect that the following types of quantities will be the main terms in the 1PN order:

$$\sim \frac{M}{R^3} \times \frac{M}{a_0 c^2}, \quad \sim \frac{M}{R^3} \times \frac{M}{R c^2}, \quad \sim \frac{M a_0^2}{R^5} \times \frac{M}{a_0 c^2}, \quad \text{and} \quad \sim \frac{M a_0^2}{R^5} \times \frac{M}{R c^2}, \quad (3)$$

where we use the relation $\mathcal{I}_{11} \sim M a_0^2$. We will derive four types of terms shown above.

Since we consider the incompressible fluid, the gravitational potentials inside each star are expressed as the polynomial form of the coordinates x_i . For the purpose of obtaining the 1PN corrections shown above, we need to take into account the coefficients of the terms such as $x_1^{m_1} x_2^{m_2} x_3^{m_3}$ in the gravitational potentials of the 1PN order, where $0 \leq m_1, m_2, m_3 \leq 5$ and $0 \leq m_1 + m_2 + m_3 (\equiv m_t) \leq 5$ up to $O(R^{-5})$ for the case m_t is odd and up to $O(R^{-3})$ for the case m_t is even.

For an irrotational fluid, the relativistic Euler equation can be integrated, and in the 1PN case, it is written as[13]

$$\begin{aligned} \text{constant} = & \frac{P}{\rho} - U + \frac{1}{2} \sum_k (\partial_k \phi_N)^2 - \sum_k \ell^k \partial_k \phi_N \\ & + \frac{1}{c^2} \left[-\frac{P}{\rho} U + \frac{1}{2} U^2 + X - \frac{1}{2} \left(\frac{P}{\rho} + 3U \right) \sum_k (\partial_k \phi_N)^2 - \frac{1}{8} \left(\sum_k (\partial_k \phi_N)^2 \right)^2 \right. \\ & \left. + \sum_k (\partial_k \phi_N)(\partial_k \phi_{\text{PN}}) - \sum_k \ell^k \partial_k \phi_{\text{PN}} - \sum_k \hat{\beta}_k \partial_k \phi_N \right], \end{aligned} \quad (4)$$

where P , ρ , ℓ^k , ϕ_N , ϕ_{PN} , U , X , and $\hat{\beta}_k$, respectively, denote the pressure, the density, the velocity field of the orbital motion, the Newtonian and 1PN velocity potentials, and the last three terms are the

²The coordinate condition in this proceeding is the standard PN one[16].

Newtonian and 1PN potentials, which are derived by solving Poisson equations as

$$\Delta U = -4\pi\rho, \quad (5)$$

$$\Delta X = 4\pi\rho \left[2U + \frac{3P}{\rho} + 2 \sum_k (\partial_k \phi_N)^2 \right], \quad (6)$$

$$\Delta P_k = -4\pi\rho \partial_k \phi_N, \quad (7)$$

$$\Delta X = 4\pi\rho \sum_k (\partial_k \phi_N) X^k. \quad (8)$$

Here $\hat{\beta}_k$ is expressed as

$$\hat{\beta}_k = -\frac{7}{2}P_k + \frac{1}{2} \left(\partial_k X + \sum_l X^l \partial_k P_l \right). \quad (9)$$

We note that using the gravitational potentials, the spacetime line element to the 1PN order can be written as

$$ds^2 = -\alpha^2 c^2 dt^2 + \frac{2}{c^2} \sum_i \hat{\beta}_i dx^i dt + \left(1 + \frac{2U}{c^2} \right) \sum_i dx^i dx^i, \quad (10)$$

where

$$\alpha = 1 - \frac{U}{c^2} + \frac{1}{c^4} \left(\frac{U^2}{2} + X \right) + O(c^{-6}). \quad (11)$$

A characteristic feature of the irrotational fluid is that the continuity equation reduces to a Poisson type equation for a velocity potential ϕ [13, 14], and in the 1PN incompressible case, it becomes

$$\sum_i \rho \partial_i C_i = 0, \quad (12)$$

where

$$C_i = -\ell^i + \partial_i \phi_N + \frac{1}{c^2} \left[-\ell^i \left(\frac{1}{2} \sum_k (\partial_k \phi_N)^2 + 3U \right) - \frac{P}{\rho} \partial_i \phi_N - \hat{\beta}_i + \partial_i \phi_{PN} \right]. \quad (13)$$

The velocity field of Newtonian order of star 1 in the inertial frame is written as $v_i \equiv \partial_i \phi_N$ where

$$\phi_N = F_a \Omega x_1 x_2 + \frac{R}{2} \Omega x_2 \quad (14)$$

and F_a is defined as

$$F_a \equiv \frac{a_1^2 - a_2^2}{a_1^2 + a_2^2}. \quad (15)$$

The pressure at Newtonian order is written as

$$P = P_0 \left(1 - \sum_i \frac{x_i^2}{a_i^2} \right), \quad (16)$$

where P_0 denotes the pressure at the center of the star and is calculated from the scalar virial relation as

$$\frac{P_0}{\rho} = \frac{1}{3} \pi \rho A_0 - \frac{5}{2R^3} I_{11} - \frac{1}{6} F_a (a_1^2 - a_2^2) \Omega_{\text{DR}}^2 + O(R^{-5}). \quad (17)$$

The axial ratios α_2 and α_3 are determined from [22]

$$\begin{aligned} a_1^2 A_1 &= \frac{P_0}{\pi \rho^2} + \frac{M a_1^2}{\pi \rho R^3} + \frac{a_1^2}{2\pi \rho} \Omega_{\text{DR}}^2 F_a (2 - F_a) + O(R^{-5}), \\ a_2^2 A_2 &= \frac{P_0}{\pi \rho^2} - \frac{M a_2^2}{2\pi \rho R^3} - \frac{a_2^2}{2\pi \rho} \Omega_{\text{DR}}^2 F_a (2 + F_a) + O(R^{-5}), \\ a_3^2 A_3 &= \frac{P_0}{\pi \rho^2} - \frac{M a_3^2}{2\pi \rho R^3} + O(R^{-5}). \end{aligned} \quad (18)$$

We can determine the equilibrium sequence of the irrotational Darwin-Riemann ellipsoid from Eqs. (2) and (18).

3 The post-Newtonian angular velocity

The orbital angular velocity $\Omega_{\text{DR}}^2 + \delta\Omega^2/c^2$ is derived using the first tensor virial equation,

$$\int d^3x \frac{\partial P}{\partial x_i} = 0. \quad (19)$$

The angular velocity of the Newtonian and 1PN orders are calculated up to $O(R^{-5})$ as³

$$\Omega_{\text{DR}}^2 = \frac{2M}{R^3} + \frac{18\mathcal{I}_{11}}{R^5} + O(R^{-7}), \quad (20)$$

$$\begin{aligned} \delta\Omega^2 = & \frac{4M\pi\rho}{5R^3}(4 + F_a)A_0 + \frac{9M^2}{2R^4}(2 - 3F_a) + \frac{12\pi\rho}{5R^5}(11 + 3F_a)A_0\mathcal{I}_{11} \\ & + (F_a - 1)\frac{2\Omega_{\text{DR}}}{R}\left(e + \frac{3}{5}a_1^2f + \frac{3}{5}a_2^2g + \frac{1}{5}a_3^2h\right) + \frac{4a_2^2}{5R}(F_a + 1)\Omega_{\text{DR}}f \\ & + (\text{Terms related with deformation of the ellipsoidal figure}) + O(R^{-6}), \end{aligned} \quad (21)$$

where $e \cdots h$ denote the coefficients of the 1PN velocity potential ϕ_{PN} as

$$\phi_{\text{PN}} = (p + qx_1^2 + rx_2^2 + sx_3^2)x_1x_2 + (e + fx_1^2 + gx_2^2 + hx_3^2)x_2 + \text{const.} \quad (22)$$

The terms related with the deformation of the ellipsoidal figure is given by the Lagrangian displacement vectors,

$$\xi_k = \frac{1}{c^2} \sum_{ij} S_{ij} \xi_k^{(ij)}. \quad (23)$$

The explicit forms of $\xi_k^{(ij)}$ are given in Ref. [20].

4 Boundary conditions

We derive equations to determine the coefficients of the 1PN velocity potential and the deformation of the binary system from the continuity equation and integrated Euler equation with the boundary conditions:

$$(A) \quad C_i \frac{\partial S}{\partial x_i} = 0 \quad \text{on} \quad S = 0, \quad (24)$$

$$(B) \quad \frac{P}{\rho} = 0 \quad \text{on} \quad S = 0, \quad (25)$$

where $S(x) = 0$ is an equation of the deformed stellar surface and written by

$$S(x) = \sum_i \frac{x_i^2}{a_i^2} - 1 - \sum_j \xi_j \frac{\partial S_{\text{DR}}}{\partial x_j}. \quad (26)$$

$S_{\text{DR}}(x) = 0$ denotes the stellar surface of the Darwin-Riemann ellipsoid and written by

$$S_{\text{DR}}(x) = \sum_i \frac{x_i^2}{a_i^2} - 1. \quad (27)$$

The condition (A) comes from the constraint that the normal component of the velocity on the surface vanishes. The condition (B) determines the stellar surface.

³In Ref [20], the angular velocity of the 1PN order is presented up to $O(R^{-6})$.

5 The total energy and total angular momentum

We regard the energy minimum point as the innermost stable circular orbit (ISCO). The total energy and total angular momentum of the binary system is defined as[16]

$$E = 2 \int_V d^3x \rho \left[\frac{v^2}{2} - \frac{1}{2}U + \frac{1}{c^2} \left(\frac{5}{8}v^4 + \frac{5}{2}v^2U + \frac{1}{2}\hat{\beta}_i v^i + \frac{P}{\rho}v^2 - \frac{5}{2}U^2 \right) + O(c^{-4}) \right], \quad (28)$$

$$J = 2 \int_V d^3x \rho \left[v_\varphi \left\{ 1 + \frac{1}{c^2} \left(v^2 + 6U + \frac{P}{\rho} \right) \right\} + \frac{\hat{\beta}_\varphi}{c^2} + O(c^{-4}) \right]. \quad (29)$$

6 Numerical results

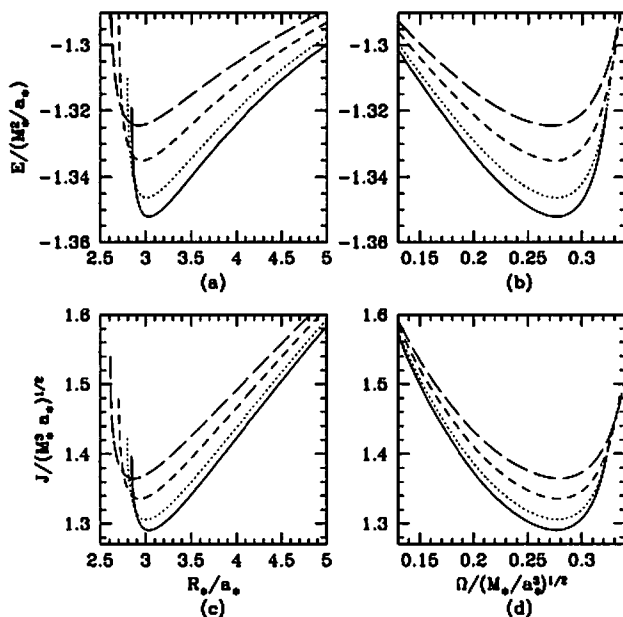


Figure 2: The total energy and angular momentum as functions of the orbital separation and the angular velocity. Solid, dotted, dashed, and long-dashed lines denote the cases of $C_s = 0$ (Newtonian), 0.01, 0.03, and 0.05, respectively.

In Fig. 2, we represent the total energy and total angular momentum as functions of the normalized separation of the binary system R_*/a_* and the normalized orbital angular velocity $\Omega/\sqrt{M_*/a_*^3}$. It is found that even if we increase the compactness parameter $C_s = M_*/c^2 a_*$, the angular velocity at the ISCO in the unit of $\sqrt{M_*/a_*^3}$ has almost the same value as in the Newtonian case, while the orbital separation decreases.

In Fig. 3, we show the relation between the compactness parameter C_s and the angular velocity at the ISCO in the unit of c^3/M_* . Thick and thin solid lines denote the cases of the full calculation and the absence of the 1PN velocity potential. These lines come from our calculation for an incompressible fluid. Dotted and dashed lines denote the “compressible” cases whose polytropic indices are 0.5 and 0.7, respectively. We draw these lines by assuming that the 1PN correction to the angular velocity is the same value as in the incompressible case. Also, we adopt the angular velocity at the ISCO given by Uryū and Eriguchi[5] as that of the irrotational and compressible binary system in the Newtonian theory.

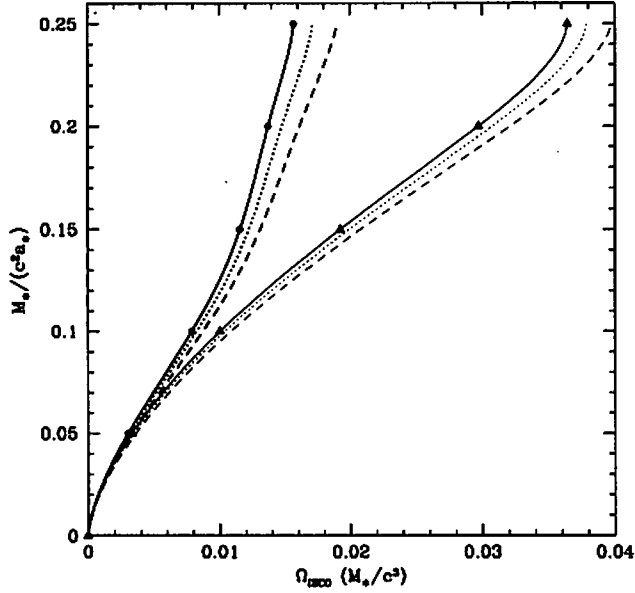


Figure 3: The relation between the angular velocity at the ISCO and the compactness parameter. Thick and thin lines denote the cases of the full calculation and the absence of the 1PN velocity potential, respectively. Filled circles and triangles denote the results of our calculation. Solid, dotted and dashed lines are the fitting lines for $n = 0$ (incompressible case), 0.5 and 0.7 where n denotes the polytropic index.

We can see from Fig. 3 that the angular velocity at the ISCO is overestimated when we neglect the 1PN velocity potential. The qualitative manner of Fig. 3 agrees with that of Lombardi, Rasio and Shapiro[8], but the angular velocities are somewhat overestimated because of the absence of the 1PN velocity potential in their calculation.

7 Summary

Using the scheme developed in the previous paper[9], we have investigated the equilibrium sequences of the irrotational and incompressible binary system in the 1PN approximation. Our results presented in the previous section will be useful to check the numerical code for solving the irrotational binary problem. The conclusion is as follows.

- (1) Due to the 1PN effect, the orbital separation at the ISCO decreases in proportion to the compactness parameter $M_*/c^2 a_*$.
- (2) The orbital angular velocity at the ISCO in the unit of $\sqrt{M_*/a_*^3}$ has almost the same value as in the Newtonian case even if we increase the compactness parameter.
- (3) The ellipsoidal approximation, in which the equilibrium configuration is assumed to have an ellipsoidal figure, gives fairly accurate results throughout the equilibrium sequence.
- (4) It is important to include the velocity potential of the 1PN order.

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The post-Newtonian equations of motion of spinning binary systems to 2.5 post-Newtonian order — the spin-orbit effects at 2.5PN order —

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Abstract

We derive the gravitational field and equations of motion of spinning compact binary systems up to 2.5 post-Newtonian approximation of general relativity. The post-Newtonian gravitational field derived here would be useful for setting up the initial data for numerical evolution of binary black hole spacetimes. On the other hand, we obtain the post-Newtonian spin-orbit interaction terms in the equations of motion of compact binary systems at 1.5 and 2.5 post-Newtonian order. The equations of motion will be used to derive the spin-orbit interaction terms at 2.5 post-Newtonian order in the gravitational radiation emitted to infinity, and to calculate the orbital evolution of the coalescing compact binaries.

1 Introduction

Among the possible sources of gravitational waves, coalescing compact binaries are the most promising candidates for detection by near-future, ground based laser interferometric detectors such as LIGO, VIRGO, GEO600, and TAMA. When a signal of gravitational waves is detected, we will try to extract parameters for binaries, such as masses and spins etc., from inspiral wave forms using the matched filtering technique. In this method, parameters for binaries are determined by cross-correlating the noisy signal from detectors with theoretical templates. If the signal and the templates lose phase with each other by one cycle over $\sim 10^3 - 10^4$ cycles as the waves sweep through the interferometer band, their cross correlation will be significantly reduced. This means that, in order to extract the information optimally, we need to make theoretical templates which are accurate to better than one cycle during an entire sweep through the interferometer's band.

The standard method to calculate inspiraling wave forms from coalescing binaries is the post-Newtonian expansion of the Einstein equations, in which the orbital velocity v of the binaries are assumed to be small compared to the speed of light. Since, for coalescing binaries, the orbital velocity is not so small when the frequency of gravitational waves is in LIGO/VIRGO band, it is necessary to carry the post-Newtonian expansion up to very high order in v . A post-Newtonian wave generation formalism which can handle the high order calculation has been developed by Blanchet, Damour and Iyer[1, 2, 3]. An another formalism is also developed by Will and Wiseman[4] which is based on Epstein-Wagoner's formalism[5]. Based on the first method, gravitational waves at infinity are evaluated to 2.5PN order in the case when the binary is non-spinning.

On the other hand, when spinning bodies are involved, the spin-orbit interaction effects at 1.5PN order and the spin-spin effects at 2PN order in the gravitational wave luminosity are evaluated by Kidder, Will and Wiseman[6]. The next contribution from the spin-orbit effects is expected to appear at 2.5PN

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order, and spin-spin effects at 3PN order. However, in order to evaluate the gravitational waves to such orders, we have to use the equations of motions of spinning binary systems to the same order beyond the Newtonian order. However, the spin-orbit effects have been evaluated only to 1.5PN order. The spin-spin effects have been evaluated only to 2PN order [7].

In this paper, we derive the equations of motions of spinning binary systems to 2.5PN order. The equations of motion of a spinning binary system are divided into two parts, non-spinning parts and spinning parts. Since the non-spinning parts have been well known to 2.5PN order, we have only to work on the spin-orbit interaction terms at 2.5PN order.

Another purpose of this paper is to derive the gravitational field generated by a spinning binary system up to 2.5PN order explicitly. Those field would be useful for setting up the initial data for the three dimensional numerical evolution of a coalescing neutron star and/or black hole binary system.

2 The 2.5PN metric potentials

The Einstein equations in harmonic coordinate are written as

$$\square h^{\mu\nu} = 16\pi[g]T^{\mu\nu} + \Lambda^{\mu\nu}(h), \quad (1)$$

$$h^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu} - \eta^{\mu\nu}, \quad (2)$$

where $\square = \eta^{\mu\nu}\partial_\mu\partial_\nu$, $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, and $\Lambda^{\mu\nu}(h)$ represent the non-linear terms in the Einstein equations. Here it is convenient to define the following densities,

$$\sigma \equiv T^{00} + T^{ii}, \quad (3)$$

$$\sigma_i \equiv T^{0i}, \quad (4)$$

$$\sigma_{ij} \equiv T^{ij}. \quad (5)$$

We define the following potentials which are represented by retarded integrals[8],

$$V = \square_R^{-1} \{-4\pi G\sigma\}, \quad (6)$$

$$V_i = \square_R^{-1} \{-4\pi G\sigma_i\}, \quad (7)$$

$$\dot{W}_{ij} = \square_R^{-1} \{-4\pi G(\sigma_{ij} - \delta_{ij}\sigma_{kk}) - \partial_i V \partial_j V\}, \quad (8)$$

$$\hat{R}_i = \square_R^{-1} \left\{ -4\pi G(V\sigma_i - V_i\sigma) - 2\partial_k V \partial_i V_k - \frac{3}{2}\partial_i V \partial_i V \right\}, \quad (9)$$

$$\begin{aligned} \hat{X} = \square_R^{-1} \left\{ -4\pi G V \sigma_{ii} + 2V_i \partial_i \partial_i V + V \partial_i^2 V \right. \\ \left. + \frac{3}{2}(\partial_i V)^2 - 2\partial_i V_j \partial_j V_i + \dot{W}_{ij} \partial_i^2 V \right\}, \end{aligned} \quad (10)$$

where

$$\square_R^{-1} \{-4\pi f\} \equiv \int \frac{d^3\mathbf{z}}{|\mathbf{x} - \mathbf{z}|} f(\mathbf{z}, t - |\mathbf{x} - \mathbf{z}|). \quad (11)$$

Using the above potentials, the metric is written approximately as

$$g_{00} = -1 + 2V - 2V^2 + 8 \left[\hat{X} + V_i V_i + \frac{V^3}{6} \right] + O(v^6), \quad (12)$$

$$g_{0i} = -4V_i - 8\hat{R}_i + O(v^5), \quad (13)$$

$$g_{ij} = \delta_{ij} (1 + 2V + 2V^2) + 4\dot{W}_{ij} + O(v^5). \quad (14)$$

3 The energy momentum tensor of spinning particles

We use the energy momentum tensor of spinning bodies which are expressed in terms of Dirac's delta function[9, 10],

$$T^{\alpha\beta} = T_{(M)}^{\alpha\beta} + T_{(S)}^{\alpha\beta} + T_{(SA)}^{\alpha\beta},$$

where

$$T_{(M)}^{\alpha\beta}(x) = \sum_{A=1,2} \int d\tau m_A u_A^\alpha(\tau) u_A^\beta(\tau) \frac{\delta^{(4)}(x - y_A(\tau))}{\sqrt{-g}} \quad (15)$$

$$T_{(S)}^{\alpha\beta}(x) = - \sum_{A=1,2} \nabla_\gamma \int d\tau \hat{S}_A^{\gamma(\alpha}(\tau) u_A^{\beta)}(\tau) \frac{\delta^{(4)}(x - y_A(\tau))}{\sqrt{-g}}, \quad (16)$$

$$T_{(SA)}^{\alpha\beta}(x) = - \frac{1}{2} \sum_{A=1,2} \nabla_\gamma \int d\tau \hat{S}_A^{\alpha\beta}(\tau) u_A^\gamma(\tau) \frac{\delta^{(4)}(x - y_A(\tau))}{\sqrt{-g}}, \quad (17)$$

and

$$u_A^\mu \equiv \frac{dy_A^\mu}{d\tau},$$

and τ is the proper time of each body.

In deriving the post-Newtonian potentials, and in deriving the equations of motion, we need to evaluate those potentials at the location of each particles. However, since our energy momentum tensor contain the delta function, those potentials diverge at the position of particles. Thus, we need a regularization procedure in order to remove the infinite self-field of pointlike sources. In this paper, we adopt the Hadamard regularization. We consider a class of function F which admit, when the field point x approaches one of the source point ($r_1 = |x - y_1| \rightarrow 0$), an expansion like

$$F(x) = \sum_{-k_0 \leq k \leq 0} \tau_1^k f_k(n_1) + O(r_1), \quad (18)$$

where k is integer. We define the value of the function F at the position of the particle 1 as follows,

$$(F)_1 \equiv F(y_1, y_1) \equiv \int \frac{d\Omega(n_1)}{4\pi} f_0(n_1, y_1). \quad (19)$$

We also define integrals of the product between F and the delta function $\delta^{(3)}(x - y_1)$ by

$$\int d^3x F(x, y_1) \delta(x - y_1) \equiv (F)_1. \quad (20)$$

Next we consider how we can derive the equations of motion. Let us assume that we already obtained the post-Newtonian metric up to some order needed to obtain the equations of motion. We consider the following form of the conservation of the energy momentum tensor,

$$\partial_\beta (\sqrt{-g} T^{\alpha\beta}) + \Gamma_{\beta\mu}^\alpha \sqrt{-g} T^{\mu\beta} = 0. \quad (21)$$

By performing a spatial volume integral in a region D_1 where only the position of particle 1 is involved, we obtain the equations of motion as

$$m_1 \frac{d}{dt} \left(v_1^i u_1^0 \right)_1 + m_1 \left(\Gamma_{\mu\beta}^i v_1^\mu v_1^\beta u_1^0 \right)_1 + \frac{1}{2} \left(R^i{}_{\beta\mu\nu} v_1^\beta \hat{S}_1^{\mu\nu} \right)_1 = 0, \quad (22)$$

where $v_A^i \equiv u_A^i / u_A^0$. This correspond to a *regularized* Papapetrou's equations of motion of a spinning body.

In order to derive the spin precession equations, we consider the symmetry of the energy momentum tensor. We consider the following integral,

$$\int_{D_1} d^3x \sqrt{-g} (T^{\alpha\beta} - T^{\beta\alpha}) = 0. \quad (23)$$

Then it is straightforward to derive the spin precession equations as

$$\frac{d}{dt} \left(\hat{S}_1^{\alpha\beta} \right)_1 + \left(\Gamma_{\nu\rho}^\alpha \hat{S}_1^{\nu\beta} v_1^\rho \right)_1 + \left(\Gamma_{\nu\mu}^\nu \hat{S}_1^{\alpha\mu} v_1^\beta \right)_1 = 0. \quad (24)$$

4 The spin-orbit terms at 2.5PN order

With preparations in previous sections, it is now straightforward to calculate the post-Newtonian potentials. Those post-Newtonian potentials are inserted into the regularized equations of motion Eq.(22), and the Hadamard regularization are performed.

Although our calculation can be done without assuming the orbit and spin axis of the binary, the equations of motion for the generic orbit with arbitrary spin axis become very complicated. When the spins are aligned with the orbital angular momentum axis, and the orbit is circular, the equations of motion is simplified greatly. Thus, we show only this simplest case.

We define the center of mass of the system, and we express the equations of motion in the relative coordinate. We define the relative acceleration as

$$\mathbf{a} \equiv \frac{d\mathbf{v}_1}{dt} - \frac{d\mathbf{v}_2}{dt}. \quad (25)$$

Then the 1.5PN spin-orbit interaction terms are given by

$$\begin{aligned} \mathbf{a}_{\text{SO}} = & \frac{M^2}{r_{12}^3} \left(6v_{12}(\chi_s + \Delta\chi_a)\mathbf{n}_{12} \right. \\ & \left. - 2v_{12}((2 - \eta)\chi_s + 2\Delta\chi_a)\mathbf{n}_{12} \right), \end{aligned}$$

and the 2.5PN spin-orbit terms are given by

$$\begin{aligned} \mathbf{a}_{\text{PN SO}} = & \frac{M^2}{r_{12}^3} v_{12}^3 5\eta(\chi_s + \Delta\chi_a)\mathbf{n}_{12} \\ & - \frac{M^3}{r_{12}^4} v_{12} (\chi_s(8 + 10\eta + 12\eta^2) + 8\Delta\chi_a(1 + \eta))\mathbf{n}_{12} \end{aligned}$$

where

$$\begin{aligned} \chi_a &= \frac{1}{2} \left(\frac{\mathbf{S}_1}{m_1^2} - \frac{\mathbf{S}_2}{m_2^2} \right), \\ \chi_s &= \frac{1}{2} \left(\frac{\mathbf{S}_1}{m_1^2} + \frac{\mathbf{S}_2}{m_2^2} \right), \end{aligned}$$

and $\chi_a = \pm|\chi_a|$, $\chi_s = \pm|\chi_s|$, $v_{12} = \pm|\mathbf{v}_1 - \mathbf{v}_2|$, $M = m_1 + m_2$, $\eta = m_1 m_2 / M^2$, $\Delta = (m_1 - m_2) / M$. The spin-orbit terms at 1.5PN order agree with the standard results.

These results will be used to calculate of the gravitational radiation emitted to infinity, and to evaluate the orbital evolution of coalescing binaries.

The details of this paper will be published elsewhere [11].

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Evolution of Close Neutron Star Binaries

— Post-Newtonian Self-Gravity and Spin Effects —

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Abstract

We have calculated evolution of neutron star binaries with finite size effects. In the final phase, the stellar spin is modified through the tidal torque. Here, we show that the frequency of innermost stable circular orbit depends on modified stellar spin and the amount of frequency deviation rate is up to 10%.

1 Post-Newtonian Self-Energy

In the previous work [1], to investigate the binary evolution with finite size effects, we used simple stellar model called the affine star model with polytropic equation of state. We assumed Newtonian self-gravity in ref.[1]. For the detail description of this stellar model, see ref.[2]. Assuming the affine star approximation, the internal energy for each star with mass M_o is given by

$$\begin{aligned} U_N &= U_o |\mathbf{q}|^{-1/n} \\ &= k_1 \kappa \rho_{oc}^{1/n} M_o, \end{aligned} \quad (1)$$

and the self-gravitational energy is written by

$$\begin{aligned} \Psi_N &= \frac{1}{2} \Psi_o S_{ik} \int_0^\infty \frac{(\mathbf{S} + x\delta)_{ik}^{-1}}{|\mathbf{S} + x\delta|^{\frac{1}{2}}} dx \\ &= -k_2 G M_o^{5/3} \rho_{oc}^{1/3} f_o, \end{aligned} \quad (2)$$

where U_o, k_1, Ψ_o, k_2 are structure constants depending on the polytropic index n . f_o is dimensionless ratio described by

$$f_o = \frac{A_1 a_1^2 + A_2 a_2^2 + A_3 a_3^2}{2(a_1 a_2 a_3)^{2/3}},$$

where a_1, a_2, a_3 are eigen values of the generalized coordinate q_{ia} , and $f_o = 1$ means a spherically symmetric star. $|\mathbf{q}|$ is determinant of q_{ia} , $S_{ik} = q_{ia} q_{ka}$ and δ_{ik} is kronecker delta function. A_i ($i = 1, 2, 3$) can be calculated by an elliptical integral as

$$A_i = \int_0^\infty \frac{a_1 a_2 a_3}{(a_i^2 + x) \sqrt{(a_1^2 + x)(a_2^2 + x)(a_3^2 + x)}} dx.$$

Recently, using an energy variational method, Lombardi et al. [3] calculated quasi-equilibrium configurations of binary neutron stars modeled as compressible triaxial ellipsoids obeying a polytropic equation of state. We adopt their formulation for the Post-Newtonian (PN) corrections to the internal and self-gravitational energy of the star. Introducing the enclosed rest mass m' and the proper radial coordinate r' , the correction terms are expressed by

$$\begin{aligned} U_{PN} &= \int_0^{M_o} u dm' - U_N + I_1 + I_3 \\ &= -\ell_1 \frac{G}{c^2} \kappa \rho_{oc}^{1/n+1/3} M_o^{5/3}, \end{aligned} \quad (3)$$

$$\begin{aligned}
\Psi_{PN} &= -G \int_0^{M_o} \frac{m'}{r'} dm' - \Psi_N + I_2 + I_4 + I_5 \\
&= -\ell_2 \frac{G^2}{c^2} \rho_{oc}^{2/3} M_o^{7/3},
\end{aligned} \tag{4}$$

where u is the internal energy density and M_o is the rest mass of the star. The central density of the star with radius $R = (u_1 a_2 a_3)^{\frac{1}{3}}$ in Schwarzschild coordinate is

$$\rho_{oc} = \frac{M_o \xi_1}{4\pi R^3 |\theta'_1|} \left\{ 1 + \frac{GM_o}{Rc^2} \frac{1}{n+1} \left[\frac{3\alpha_1}{\xi_1^2 |\theta'_1|^2} + \frac{\beta_1}{\xi_1 |\theta'_1|^2} + \frac{(n-3)(n+1)}{5-n} \right] \right\}, \tag{5}$$

where θ and ξ are the usual Lane-Emden variables for a polytrope, (α_1, β_1) is $(-3.39307, 3.63708)$ for $n = 0.5$ and $(-2.77960, 2.67168)$ for $n = 1.0$. Integral I_k ($k = 1, 2, 3, 4, 5$) are given by

$$\begin{aligned}
I_1 &= -\frac{G}{c^2} \int_0^M u \frac{m}{r} dm, \\
I_2 &= -\frac{G^2}{2c^2} \int_0^M \left(\frac{m}{r} \right)^2 dm, \\
I_3 &= -\frac{G}{c^2} \int_0^M \frac{dm}{r} \int_0^m u dm, \\
I_4 &= \frac{G^2}{c^2} \int_0^M \frac{dm}{r} \int_0^m \frac{m}{r} dm, \\
I_5 &= -\frac{G^2}{c^2} \int_0^M \frac{mdm}{r^4} \int_0^r mr dr.
\end{aligned}$$

The dimensionless coefficients ℓ_1 and ℓ_2 are determined by evaluating these integrals for the chosen family of density profiles. Selecting a family of trial density function based upon the Oppenheimer-Volkoff equation, $(\ell_1, \ell_2) = (0.489198, 0.051391)$ for $n = 0.5$ and $(0.999096, 0.002779)$ for $n = 1.0$. See ref.[3] for detail derivation.

As assuming the affine star model, the internal kinetic energy is given by

$$T_N = \frac{1}{2I_o} p_{ia} p_{ia}, \tag{6}$$

where I_o is structure constant depending on the polytropic index n and p_{ia} is the generalized momenta of the affine star internal motions. Expanding to PN order, the correction terms which include the higher order of internal velocities appear in the kinetic energy parts. Although we discuss the spin effects, we neglect the high order kinematic energy for comparison with Lombardi's results. This is valid approximation for irrotational sequences, but invalid for corotating ones and especially for rapid rotating stars. To construct the model include these velocity terms is future work. Consequently, total self energy of the neutron star is described by

$$E_I = T_N + U_N + \Psi_N + U_{PN} + \Psi_{PN}. \tag{7}$$

2 Orbital Evolution

We have calculated the binary evolution in ref.[1] using Post²-Newtonian (P²N) Hamiltonian formalism without relativistic self-gravity corrections. We showed that the stellar spin was modified through the tidally induced torque and the lag angle became non-zero. We consider that all tidal effects will be suppressed by general relativistic self-gravity but not negligible. So that we can expect that the spin effects in the final phase of orbital evolution change the innermost stable circular orbit (ISCO) frequency. Our final goal is the P²N formalization with P²N self-gravity, but it is difficult to approximate. To maintain the PN accuracy such as in ref.[3], we neglect all P²N effects except the radiation reaction forces and the high order corrections in the hybrid terms. We used hybrid accelerations in ref.[4] for the calculation

of orbital evolution. Furthermore, we include the coupling between the orbital motion and the internal structure which is derived by substituting the effective mass in place of the rest mass in the Newtonian equation of motion or energy. Calculating to PN order, the resulting energy is

$$\begin{aligned} E_{OI} &= E_N(M_o \rightarrow M_{eff}, M'_o \rightarrow M'_{eff}) - E_N(M_o, M'_o) \\ &\simeq \frac{E_{self}}{c^2} \left(\frac{1}{2} \frac{M_o'^2}{M_T^2} v^2 - \frac{GM'_o}{r} \right), \end{aligned} \quad (8)$$

where M_o is the rest mass, $M_{eff} = M_o + E_{self}/c^2$ is the effective mass, and the prime denotes the values for a companion star with the rest mass M'_o . $M_T = M_o + M'_o$ is total mass and $\mu = M_o M'_o / M_T$ is reduced mass, then Newtonian energy function is given by

$$E_N = \frac{1}{2} \mu v^2 - \frac{GM_T \mu}{r}.$$

$E_{self} = E_I - T_N$ is the approximated self energy. We omitted the kinetic part, which is small enough for the slow rotating limit such as the irrotational equilibrium star. In equation (8), the separation r is the harmonic coordinate and v is orbital velocity.

We used the total energy to search the ISCO frequencies, which described by

$$E = E_H + E_T + E_{SO} + E_{SS} + E_{OI} + E'_{OI} + E_I + E'_I, \quad (9)$$

where E_H is the hybrid PN energy term in ref.[4] exclude P²N order corrections, E_T is the lowest order tidal interaction, E_{SO} and E_{SS} are the lowest order spin-orbit and spin-spin interaction energies in ref.[5] respectively. The differences to Lombardi's formalism are the energies which are concerned with the stellar spins. In the irrotational sequence, these terms changed the minimum point of the total energy relative to its values at infinite separation.

3 Results and Discussion

We selected the parameter of polytropic stellar model as $\kappa = 1.35 \times 10^{-10} \text{cm}^8/\text{g}^2/\text{s}^2$ for $n = 0.5$. It means the gravitational mass $M = 0.83, 1.39, 1.74 M_\odot$ and the stellar radius $R = 9.8, 10.3, 10.3 \text{km}$ for the compactness parameter $GM/Rc^2 = 0.125, 0.2, 0.25$ respectively. Additionally, we chose $\kappa = 1.20 \times 10^5 \text{cm}^5/\text{g}/\text{s}^2$ for $n = 1.0$. It means $M = 1.15, 1.52, 1.599 M_\odot$ and $R = 13.6, 11.2, 9.4 \text{km}$ for $GM/Rc^2 = 0.125, 0.2, 0.25$ respectively. After calculating the stellar spin evolution, we substituted the spin values in equation (9) and evaluated the frequency at the minimum point of the binding energy.

We determined the ISCO frequencies f_{ISCO} for the stellar models with $GM/Rc^2 = 0.2$. Figure 1 presents the irrotational sequences in the equal mass case with $n = 0.5$ and $GM/Rc^2 = 0.2$. The frequency shifts of the case with $n = 1.0$ and same compactness are less than the results for $n = 0.5$. Because the tidal deformation become larger as the polytropic index smaller, the spin-up ratio are greater for the $n = 0.5$ stellar models than for the $n = 1.0$ models. Moreover, the frequency deviation along the corotating sequences with same parameter are less than that of irrotational ones. In figure 1, we can estimate the difference between two models. The results are $\Delta f_{ISCO} \sim 30 \text{Hz}$ which are about 10% of the ISCO frequency. The results with other polytropic parameters are not up to 10% of that, ie. $\Delta f_{ISCO}/f_{ISCO} \lesssim 0.1$.

At the realistic orbital shrinking stage, ISCO frequencies are under the influence of stellar spin. Its are small effects for the irrotational and tidally locked equilibrium sequences, but at near the ISCO radius, the magnitude of spin become larger and the spin effects are more important. To confirm the realistic ISCO radius or frequency, the calculations include the high order kinematic corrections and P²N effects are future work.

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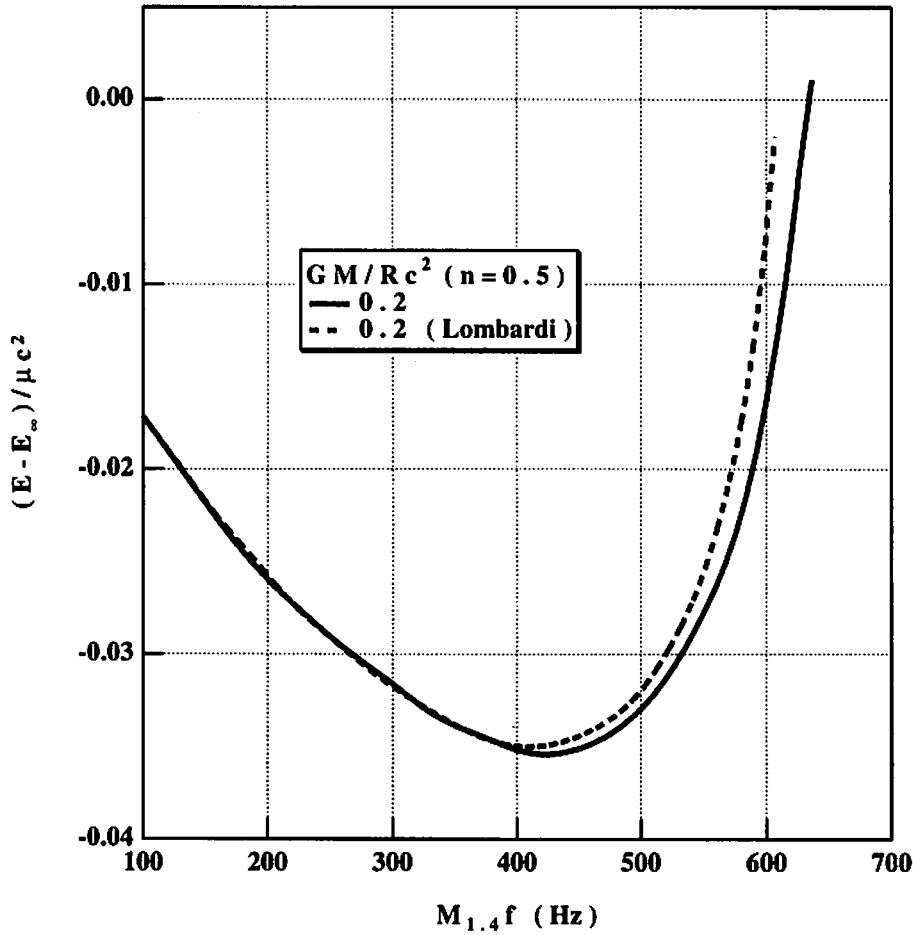


Figure 1: The total energy E relative to its values E_{∞} at infinity, as a function of the orbital frequency f for the case $n = 0.5$, $\kappa = 1.35 \times 10^{-10} \text{ cm}^8/\text{g}^2/\text{s}^2$ and $GM/Rc^2 = 0.2$. The solid line represents our PN results with spin effects. The dashed curve is the sequence corresponding to Lombardi's treatment. $M_{1.4}$ is the gravitational mass normalized by the $1.4M_{\odot}$. The minimum points for each curves denote the ISCO.

Wave optics of gravitationally lensed gravitational waves

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Abstract

We discuss an interesting effect induced by the rotation of a massive object acting as a lens for coherent gravitational radiation. We shall show that the result is a concentric interference pattern which is shifted due to the effect of the angular momentum on the phase of radiation, an effect analogous to the Aharonov-Bohm effect. The possibility of detecting lensed gravitational waves is discussed in the context of upcoming gravitational wave detectors.

1 Introduction

One of the predictions of Einstein's general theory of relativity and one of its early tests concerned light deflection by the sun. A deflecting mass represents the simplest gravitational lens configuration [1] in what has given rise to a whole research field in theoretical and observational astrophysics [2]. Another important prediction of Einstein's theory is the existence of gravitational waves which are emanated from massive objects in non-spherical motion [3].

In this paper we want to discuss gravitational lensing of gravitational waves. The treatment of such a phenomenon demands the use of wave optics rather than geometric optics for the following reasons. Gravitational waves interact only very weakly with matter so that their coherence is preserved over cosmological distances, and in many cases gravitational waves are nearly monochromatic. These properties of gravitational waves are relevant in gravitational lensing for the situation when source, lens and observer are nearly collinear. Since infinitely many rays with almost equal path length intersect at the observer, diffraction and interference should invalidate geometrical optics. This is not the case for sources of electromagnetic radiation: rays originating from points not on but near the optic axis reach the observer at separate times which can exceed the coherence time. In the case of gravitational lensing of gravitational waves, however, the use of wave optics is not only justified but necessary. Along the whole focal line the intensity is amplified by the factor $8\pi^2 m/\lambda$ [4], m being the deflector's mass and λ the radiation's wavelength.

In the case of gravitational lensing by a rotating massive object, thanks to the coherence of gravitational waves, we discover a behavior which has so far been discovered only at the atomic level in two occasions. One is the interference of electron-waves passing either side of a magnetic solenoid, the Aharonov-Bohm effect [5]; the other is the interference of neutron-waves affected by the Coriolis force in the earth's gravitational field, the Colella-Overhauser-Werner (COW) experiment [6]. The phase shift in the Aharonov-Bohm effect is given by the line integral of the vector potential $\int d\vec{l} \cdot \vec{A}$. Similarly one expects that the phase shift of gravitational waves is given by the line integral of the gravitomagnetic potential $\vec{A} = 2(\vec{J} \times \vec{r})/r^3$, with $\vec{J}$ being the angular momentum of a rotating massive object:

$$\int d\vec{l} \cdot \vec{A} = 4(\vec{\xi}/\xi^2) \cdot (\vec{n} \times \vec{J}), \quad (1)$$

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where $\vec{\xi}$ is the impact parameter vector, and $\vec{n}$ is the normalized vector pointing from the source to the lens. We show in Sec. III that this is really the case for the weak field metric.

Gravitational lensing by a Kerr black hole has been discussed previously [7, 8, 9], but none of these computations gave a description of the interference of gravitational waves. Moreover the value used for the Kerr parameter in [9] is overly extreme. In Sec. II we derive the Fresnel-Kirchhoff diffraction formula for the linearized Kerr metric. In Sec. III we use the Fresnel-Kirchhoff diffraction formula adapted to the linearized Kerr metric to compute the interference/diffraction function. In Sec. IV we discuss orders of magnitude for the case of our galaxy, and in Sec. V we summarize our results. We use the convention $c = G = 1$.

2 The Fresnel-Kirchhoff Diffraction Formula

We write the gravitational wave tensor as $h_{ab} = \mathcal{A} e_{ab} e^{iS} = \mathcal{E} e_{ab}$, and treat $\mathcal{E}$ as a scalar wave satisfying the propagation equation

$$\square \mathcal{E} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(g^{\mu\nu} \sqrt{-g} \frac{\partial}{\partial x^\nu} \right) \mathcal{E} = 0. \quad (2)$$

We consider a lens object whose size is much smaller than the Einstein radius ξ_0 , Eq. (19), so we can specialize to a linearized Kerr metric

$$ds^2 = -(1 - 2m/r)dt^2 + (1 + 2m/r)dr^2 - \frac{4am}{r} \sin^2 \theta d\phi dt + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \quad (3)$$

For monochromatic waves of frequency ω , i.e. $\mathcal{E} \propto e^{-i\omega t}$, Eq. (2) becomes

$$\Delta \mathcal{E} + i\omega \frac{4am}{r^3} \frac{\partial \mathcal{E}}{\partial \phi} + (1 + 2m/r)\omega^2 \mathcal{E} = 0 \quad (4)$$

where Δ is the spatial Laplacian. We apply the Kirchhoff integral theorem to Eq. (4), so the wave amplitude at the observer is written as

$$\mathcal{E}_{\text{obs}} = -\frac{1}{4\pi} \oint_{\Sigma_1} [U_2^* \vec{\nabla} \mathcal{E} - \mathcal{E} (\vec{\nabla} U_2)^*] \cdot d\vec{S} \quad (5)$$

where U_2 is a solution of equation (4) which behaves like an incoming spherical waves $\rightarrow e^{-i\omega r_2}/r_2$ when the distance from the observer $r_2 \rightarrow 0$, and Σ_1 is a large closed surface containing the observer.

The disturbance leaving the source S has the form of an outgoing spherical wave $\mathcal{E} \rightarrow A_0 e^{i\omega r_1}/r_1$ as $r_1 \rightarrow 0$, where A_0 is a constant and r_1 is the distance from the source. The Huygens-Fresnel principle is directly traceable to this integral resulting from the scalar wave equation. The gravitational wave originating at S propagates to the surface Σ_1 and from there the superposition of secondary waves sums up to an amplitude at the observer. The surface Σ_1 can be chosen arbitrarily, but to suit our purposes we use a finite plane perpendicular to the line of sight. To close the surface, we attach a sphere from which the contribution to the integral will be negligible because we can blow this spherical portion up to infinity, so that at the time when the disturbance at the observer is considered no contributions from this spherical surface could have reached O. We therefore have to take into account only contributions from the (lens) plane Σ (see [10]).

We shall use the eikonal approximation for the solutions $\mathcal{E}$ and U_2 assuming that the wavelength is much shorter than the curvature radius $(m/\xi_0^3)^{-1/2}$ around the lens object,

$$\mathcal{E} = A_0 e^{iS_1}/r_1 \text{ on the source side of the plane } \Sigma, \quad (6)$$

$$\text{and } U_2 = e^{-iS_2}/r_2 \text{ on the observer side of the plane } \Sigma, \quad (7)$$

where S_1 and S_2 are, using $\omega r \gg 1$, regular solutions of the eikonal equation,

$$g^{ab} S_{,a} S_{,b} = 0, \quad (8)$$

in their respective domains. From equations (5)-(7) we obtain the Fresnel-Kirchhoff diffraction formula:

$$\mathcal{E}_{\text{obs}} = \frac{A_0}{4\pi i} \oint_{\Sigma} \frac{e^{i(S_1+S_2)}}{r_1 r_2} (\vec{\nabla} S_1 - \vec{\nabla} S_2) \cdot \vec{n} dS, \quad (9)$$

where $d\vec{S} = \vec{n} dS$, $\vec{n}$ being the unit normal vector of Σ pointing towards the observer.

In astronomical situations the distances r_1 and r_2 are much larger than the effective scale of the plane of integration which is of the order of the Einstein radius because of the stationary phase approximation (Sec. 3.2). We can therefore replace them by constant distances, i.e. r_1 by the distance from the source to the lens, s , and r_2 by the distance from the lens to the observer, d , and consequently write them in front of the integrand. In the lowest order approximation, $S_1 = \omega r_1$ and $S_2 = \omega r_2$, so that $\vec{n} \cdot \vec{\nabla} S_1 = (\partial S_1 / \partial r_1) \cos(\vec{n}, \vec{r}_1) \approx \omega$ and $\vec{n} \cdot \vec{\nabla} S_2 = (\partial S_2 / \partial r_2) \cos(\vec{n}, \vec{r}_2) \approx -\omega$, because the angles involved are small. Therefore, from Eq. (9) we obtain

$$\mathcal{E}_{\text{obs}} = \frac{\omega A_0}{2\pi i s d} \int_{\Sigma} d^2 \xi e^{i(S_1+S_2)}, \quad (10)$$

where ξ is the two dimensional position vector in the lens plane Σ and dS is approximated by $d^2 \xi$. In the following section we will evaluate the eikonals S_1 and S_2 .

3 Interference and diffraction of gravitationally lensed gravitational waves

3.1 Evaluation of the eikonals S_1 and S_2

With the metric Eq. (3), the eikonal equation, Eq. (8), is written as

$$\begin{aligned} \omega^2(1 + 2m/r) - \omega \frac{4am}{r^3} \frac{\partial S}{\partial \phi} &= (1 - 2m/r) \frac{\partial S}{\partial r} \frac{\partial S}{\partial r} \\ &- \frac{1}{r^2} \frac{\partial S}{\partial \theta} \frac{\partial S}{\partial \theta} - \frac{1}{r^2 \sin^2 \theta} \frac{\partial S}{\partial \phi} \frac{\partial S}{\partial \phi} = 0. \end{aligned} \quad (11)$$

We solve Eq. (11) perturbatively by setting $S = S_0 + \epsilon$, where S_0 is the eikonal in the Euclidean space satisfying $\delta_{ij}(\partial S_0 / \partial x_i)(\partial S_0 / \partial x_j) = \omega^2$, while the correction term ϵ is linear in the mass m and the angular momentum, $\vec{J} = \vec{a}m$, of the lens object.

Let us first consider the case of perfect collinearity among source, lens and observer. Assuming regularity of S_1 and S_2 in their domain, the solutions of (11) are approximately, i.e. up to linear order in m and a , given respectively by

$$\begin{aligned} S_1 = \omega \left[s + \frac{r^2}{2s} - r \cos \theta + m \left\{ 2 \ln \left(\frac{1}{r} - \frac{\cos \theta}{s} \right) \right. \right. \\ \left. \left. - 2 \ln(1 + \cos \theta) + \cos \theta - \frac{2\vec{r}}{r^2} \cdot \frac{\vec{n} \times \vec{a}}{\vec{n} \cdot \hat{r} - 1} \right\} \right], \end{aligned} \quad (12)$$

where $\theta \in [0, \frac{\pi}{2}]$, and

$$\begin{aligned} S_2 = \omega \left[d + \frac{r^2}{2d} + r \cos \theta + m \left\{ 2 \ln \left(\frac{1}{r} + \frac{\cos \theta}{d} \right) \right. \right. \\ \left. \left. - 2 \ln(1 - \cos \theta) - \cos \theta + \frac{2\vec{r}}{r^2} \cdot \frac{\vec{n} \times \vec{a}}{\vec{n} \cdot \hat{r} + 1} \right\} \right], \end{aligned} \quad (13)$$

where $\theta \in [\frac{\pi}{2}, \pi]$, θ being the polar angle measured from the z -axis which points from the lens to the source. In the eikonal approximation, e^{iS_1}/r_1 and e^{-iS_2}/r_2 give respectively an outgoing wave from the source, which is valid in the source side, and an incoming wave to the observer, which is valid in the observer side.

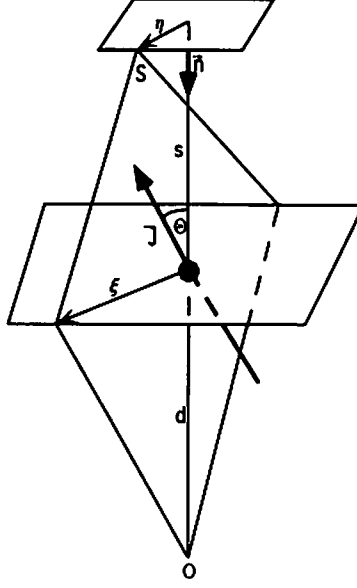


Figure 1: Geometry in case of non-alignment

In general, we expect a displacement $\vec{\eta}$ of the source parallel to the lens plane from the alignment observer-deflector (see Fig. 1). We have

$$S'_1 = \omega \left[s + \frac{|\vec{r} - \vec{\eta}|^2}{2s} - |\vec{r} - \vec{\eta}| \cos \theta \right. \\ \left. + m \left\{ 2 \ln \left(\frac{1}{r} - \frac{\cos \theta}{s} \right) - 2 \ln(1 + \cos \theta) \right. \right. \\ \left. \left. + \cos \theta - \frac{2\vec{r}}{r^2} \cdot \frac{\vec{n} \times \vec{a}}{\vec{n} \cdot \vec{r} - 1} \right\} \right]. \quad (14)$$

We use the eikonal relative to the unlensed case

$$S = S'_1 + S_2 - \omega[(d+s)\vec{n} - \vec{\eta}], \quad (15)$$

to evaluate the Fresnel-Kirchhoff diffraction formula, Eq. (10), since we are not interested in the constant phase factor of $\mathcal{E}$. Furthermore, we do not need the full information of the waves, Eq. (12) and Eq. (13), but only the value at the lens plane where $\theta = \frac{\pi}{2}$ and $\vec{n} \cdot \vec{r} = 0$. The value of S at the lens plane is

$$S = \omega \left[\frac{|(d+s)\vec{\xi} - d\vec{\eta}|^2}{2sd(d+s)} + 4m \ln \left(\frac{1}{\xi} \right) + 4m \frac{\xi}{\xi^2} \cdot (\vec{n} \times \vec{a}) \right]. \quad (16)$$

Here we have replaced $\vec{r}$ by $\vec{\xi}$ defined in the lens plane, and hereafter 2-dimensional vectors in the plane perpendicular to the line of sight are expressed by boldface symbols. The first term is determined by the (Euclidean) geometry of the lens configuration, the second term is due to the influence of the Newton potential, while the third term is a result of the Kerr rotation.

Next, we rescale the coordinates to make them dimensionless according to [2]:

$$\mathbf{x} = \frac{\vec{\xi}}{\xi_0}, \quad (17)$$

and

$$\mathbf{y} = \frac{d}{d+s} \frac{\vec{\eta}}{\xi_0}, \quad (18)$$

where ξ_0 is the Einstein radius defined as

$$\xi_0 \equiv \sqrt{4m \frac{ds}{d+s}}. \quad (19)$$

The eikonal in terms of the dimensionless variables $\mathbf{x}$ and $\mathbf{y}$ is

$$S = 4m\omega \left[\frac{1}{2}|\mathbf{x} - \mathbf{y}|^2 - \ln x + \frac{\mathbf{x}}{x^2} \cdot (\vec{n} \times \frac{\vec{a}}{\xi_0}) \right]. \quad (20)$$

The eikonal S is sometimes called a Fermat potential, the stationarity of S with respect to $\mathbf{x}$ yielding classical paths of geometrical optics.

3.2 Wave amplitude and image positions as stationary points of the eikonal

We evaluate the Fresnel-Kirchhoff formula Eq. (10) using the eikonal obtained in the previous subsection. We define

$$f = 4m\omega, \quad \alpha = \vec{n} \times \frac{\vec{a}}{\xi_0} = \vec{n} \times \frac{\vec{J}}{m\xi_0}, \quad (21)$$

$$T(\mathbf{x}, \mathbf{y}) = \frac{1}{2}|\mathbf{x} - \mathbf{y}|^2 - \ln x + \frac{\mathbf{x}}{x^2} \cdot \alpha. \quad (22)$$

Note that f represents the ratio of the gravitational radius of the lens to the wavelength. The vector α points in the direction perpendicular to the projection of the angular momentum vector onto the lens plane, and its magnitude $|\alpha|$ is of order $\sim \sqrt{m/d}$ which is much smaller than unity. From Eqs. (10) and (20) we see that the wave amplitude at the observer is

$$\mathcal{E}_{\text{obs}}(\mathbf{y}) = \mathcal{E}(O) = \frac{\mathcal{E}_0 f}{2\pi i} \int d^2\mathbf{x} \exp[ifT(\mathbf{x}, \mathbf{y})]. \quad (23)$$

where $\mathcal{E}_0 = \mathcal{A}_0/(d+s)$ is the wave amplitude for the case of no lensing. We note that Eq. (22) can be rewritten as

$$T(\mathbf{x}, \mathbf{y}) = \frac{1}{2}|\mathbf{x} - \mathbf{y}|^2 - \ln |\mathbf{x} - \alpha|, \quad (24)$$

to first order in α . Eq. (24) suggests that we introduce new coordinates

$$\tilde{\mathbf{x}} := \mathbf{x} - \alpha, \quad \tilde{\mathbf{y}} := \mathbf{y} - \alpha. \quad (25)$$

This coordinate transformations are just parallel translations by the constant vector α , and we can change the integral variable of Eq.(23) from $\mathbf{x}$ to $\tilde{\mathbf{x}}$:

$$\mathcal{E}_{\text{obs}}(\tilde{\mathbf{y}}) = \frac{\mathcal{E}_0 f}{2\pi i} \int d^2\tilde{\mathbf{x}} \exp[ifT(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})], \quad (26)$$

where

$$T(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \frac{1}{2}|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}|^2 - \ln \tilde{x}. \quad (27)$$

Therefore it is clear that the whole pattern is just shifted by the amount $\vec{\alpha}$.

As we show in section IV.A, we consider the case where $f \gg 1$, i.e. when the wavelength is much shorter than the gravitational radius of the lens. In this case we can evaluate Eq. (26) using the stationary phase approximation. The phase function $T(\tilde{\mathbf{x}})$ in Eq. (27) has two stationary points $\tilde{\mathbf{x}}_{\pm}$ as solutions of $\vec{\nabla}T(\tilde{\mathbf{x}}) = 0$, ($\vec{\nabla} := \partial/\partial\tilde{\mathbf{x}}$). $\tilde{\mathbf{x}}_+$ is a minimum, and $\tilde{\mathbf{x}}_-$ is a saddle point. Actually these stationary points correspond to the images in the geometric optics approximation, and the condition $\vec{\nabla}T(\tilde{\mathbf{x}}) = 0$ describes Fermat's principle of least time [2]. Expanding $T(\tilde{\mathbf{x}})$ around the stationary points and performing the Gauss integrals, the resulting wave amplitude is given as the superposition of contributions from the two stationary points:

$$\mathcal{E}_{\text{obs}} = \mathcal{E}_0 \sum_{j=\pm} |\mu(\tilde{\mathbf{x}}_j)|^{1/2} \exp[ifT(\tilde{\mathbf{x}}_j) - i\pi n_j/2], \quad (28)$$

where $n_+ = 0$ and $n_- = 1$ correspond respectively to the minimum and to the saddle point, and

$$\mu(\tilde{\mathbf{x}}) := \det[\tilde{\nabla} \otimes \tilde{\nabla} T(\tilde{\mathbf{x}})]^{-1}. \quad (29)$$

When $\tilde{y} \ll 1/f$ the stationary phase approximation is not valid anymore. This case is treated separately below by evaluating the diffraction integral Eq. (26) exactly.

Since the phase function $T(\tilde{\mathbf{x}})$ in Eq. (27) has exactly the same form as that for the non-rotating lens, we can use the familiar formulas for the Schwarzschild lens [2] with $\mathbf{x}$ and $\mathbf{y}$ replaced by $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$. The two stationary points $\tilde{\mathbf{x}}_{\pm}$ are

$$\tilde{x}_{\pm} = \frac{\tilde{y}}{2\tilde{y}} (\tilde{y} \pm \sqrt{\tilde{y}^2 + 4}). \quad (30)$$

The curvatures Eq. (29) at these stationary points are

$$\mu_{\pm}(\tilde{y}) = \frac{1}{2} \pm \frac{\tilde{y}^2 + 2}{2\tilde{y}\sqrt{\tilde{y}^2 + 4}}. \quad (31)$$

The phase difference between them is

$$\Delta T(\tilde{y}) = T(\tilde{\mathbf{x}}_-) - T(\tilde{\mathbf{x}}_+) \quad (32)$$

$$= \frac{\tilde{y}}{2} \sqrt{\tilde{y}^2 + 4} + \ln \frac{\sqrt{\tilde{y}^2 + 4} + \tilde{y}}{\sqrt{\tilde{y}^2 + 4} - \tilde{y}}. \quad (33)$$

Thus, the total wave intensity is magnified by the factor

$$|\mathcal{E}_{\text{obs}}/\mathcal{E}_0|^2 = |\mu_+| + |\mu_-| + 2|\mu_+\mu_-|^{1/2} \sin(f \Delta T) \quad (34)$$

$$= \frac{\tilde{y}^2 + 2 + 2 \sin(f \Delta T)}{\tilde{y}\sqrt{\tilde{y}^2 + 4}}. \quad (35)$$

The third term of this equation expresses the interference of waves reaching from the two images. By exchanging the roles of source and observer, i.e. d and s , we expect by reciprocity an interference pattern to appear on the observer plane. This interference pattern is circular and has the same shape as for the case of a non-rotating lens. The effect of the lens spin is just to shift the whole pattern translationally by the small vector α . In physical units, this shift on the observer plane is

$$\frac{d+s}{s} a \sin \Theta, \quad (36)$$

where Θ is the angle between the angular momentum vector of the lens and the line of sight.

Let us estimate the width of these interference fringes near the caustic where $1/f \lesssim y \ll 1$. From Eq. (35) the condition for constructive interference is $f \Delta T(\tilde{y}) = \pi(2n + \frac{1}{2})$ with the integer n . Since $\Delta T(\tilde{y}) \simeq 2\tilde{y}$ for $\tilde{y} \ll 1$, the distance between adjacent fringes near the center $\tilde{y} = 0$ of the interference pattern is given by $\Delta\tilde{y} = \pi/f$, or by

$$\frac{\pi}{\omega} \sqrt{\frac{d(d+s)}{4ms}}, \quad (37)$$

in physical units on the observer plane.

Since the stationary phase approximation breaks down for $\tilde{y} \ll 1/f$, diffraction effects become important at the center of the interference pattern, where the wave amplitude has its maximum. Performing the diffraction integral Eq.(26) with $\tilde{y} = 0$ yields

$$\mathcal{E}_{\text{obs}} = -i \mathcal{E}_0 f \int_0^\infty d\tilde{x} \tilde{x}^{1-if} e^{if\tilde{x}^2/2} \quad (38)$$

$$= \mathcal{E}_0 (f/2)^{if/2} e^{\pi f/4} \Gamma(1 - \frac{1}{2}if), \quad (39)$$

and the wave intensity at the center of the interference pattern is

$$|\mathcal{E}_{\text{obs}}/\mathcal{E}_0|^2 = \frac{\pi f}{1 - e^{-\pi f}} \simeq 4\pi m\omega, \quad (40)$$

since $f \gg 1$. This is exactly the same as the maximum magnification for the case of a Schwarzschild lens [4].

We conclude that (1) the whole interference pattern is shifted by $\tilde{\alpha}$, which can be understood as a dragging of the gravitational wave by the rotation of the massive object, and (2) all the other features of the circular diffraction/interference pattern are the same as in the case of a non-rotating lens. The analogy to the Aharonov-Bohm effect should now be apparent, the quantity $\tilde{\alpha}$ proportional to the angular momentum of the lens object having a similar effect on the interference of gravitational waves as the magnetic flux has on the interference of electron waves.

4 Orders of magnitude — the case of our galactic center

In the case of large distances of the source from the lens, we can set $s \rightarrow \infty$. In this case the shift will be almost exactly $a \sin \Theta$, and the distance between fringes near the focal line (caustic) will depend only on the wavelength of the gravitational radiation λ and the distance to the lens d . Let us take as a typical wavelength of gravitational radiation $\lambda = 3 \times 10^8 \text{ cm}$, which corresponds to a frequency of 1Hz, and for $d = 8 \text{ kpc} = 2.4 \times 10^{23} \text{ cm}$, i.e. the distance to our center of galaxy, taking $m = 10^6 M_\odot$, which corresponds to a geometrical mass of $3 \times 10^{13} \text{ cm}$, i.e. the mass of the black hole located at the center of our galaxy. For these values the Einstein radius (impact parameter) is $\xi_0 = \sqrt{4md} \approx 5 \times 10^{18} \text{ cm}$; we can easily check that $\alpha = \frac{a}{\xi_0} \sin \Theta$ is less than 6×10^{-6} , i.e. a very small number, the fact we used in the approximation which has lead us to Eq. (24). In this case we have the following distance between fringes located near the caustic:

$$\frac{\lambda}{4} \sqrt{\frac{d}{m}} \approx 0.4 \text{ AU}. \quad (41)$$

The interference pattern we expect is therefore huge. If we assume $a = 0.1m$ and $\Theta = \pi/2$, the pattern will be shifted by

$$a = 3 \times 10^{12} \text{ cm} \approx 0.2 \text{ AU}. \quad (42)$$

The maximum magnification is given by

$$\mu = 8\pi^2 \frac{m}{\lambda} \approx 8 \times 10^6. \quad (43)$$

This gigantic magnification motivates a brief discussion, given in the next section, on the possibility to detect gravitational waves lensed not only by the center of our own galaxy but by the centers of other galaxies as well.

5 Summary and discussion

We found that in astrophysical situation there exists a parallel phenomenon with the Aharonov-Bohm effect or the COW experiment in quantum mechanics, namely, gravitational lensing of gravitational waves by a spinning massive object. The main result of our paper is summarized as follows. Due to the high coherence of gravitational waves, an interference pattern is formed near the caustic on the observer plane. This pattern is circular whether the lens is spinning or not, and the effect of lens spin is to shift the pattern translationally by the amount $a \sin \Theta$ in the observer plane, where a is the Kerr parameter and Θ is the angle between the lens angular momentum vector and the line of sight. This shift occurs in the direction perpendicular to the angular momentum vector projected onto the lens plane, i.e. the direction of dragging, leaving all the other properties of the diffraction/interference pattern unchanged. We conclude that the gravitomagnetic field around a Kerr black hole affects not only the path of the gravitational wave [7, 9] but causes also a phase shift in parallel with the magnetic field or the Coriolis force field in the above mentioned quantum mechanical effects, resulting in a shift of the interference pattern formed near the caustic.

It should be pointed out, however, that in the Aharonov-Bohm effect no force acts upon the electron-waves and we have a pure shift of the interference pattern based on the phase shift resulting from a change in magnetic flux inside the solenoid. On the other hand, the COW experiment is essentially the same

as the effect we described, the Coriolis force acting upon the neutron-waves, causing the interference pattern to shift, just as the gravitational dragging acts upon the gravitational waves.

In real astrophysical situations it is unlikely that we can measure the Kerr parameter of a lensing object from the above shift of the interference pattern for the following reasons. (1) As we found above, except for this shift the overall interference pattern does not depend on the Kerr parameter. Unlike for the case in atomic physics, we cannot prepare a large screen on the observer plane. In other words the sole observable effect would be the misalignment of the lens and the two images, i.e. the stationary points. These are on one line with the lens when the lens is not spinning. But the angle of this misalignment $\sim (\text{Kerr parameter})/(\text{distance to lens})$ is tiny and currently irresolvable. (2) This tiny misalignment may well be due to other reasons such as the external shear from nearby objects surrounding the lens. (3) Detection of lensed waves would also be rare for the laser interferometer detectors under construction such as the advanced LIGO [11].

Einstein wrote in his seminal paper on the gravitational lensing effect of electromagnetic radiation [1] that “there is no hope of observing this phenomenon directly”. Gravitational lensing of electromagnetic radiation, however, has been observed on several occasions in the meantime. Due to the weak interaction and therefore high coherence of gravitational waves, interference of gravitational waves by lensing certainly occurs in our universe even if difficult to detect.

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A Model for Non High Energy Gamma Ray Bursts and Sources of Ultra High Energy Cosmic Rays

— super strongly magnetized milli-second pulsar formed from a (C+O) star and a neutron star (black hole) close binary system —

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Abstract

As a progenitor of NHE (Non High Energy) GRBs (Gamma Ray Bursts), we propose a (C+O) star and a neutron star (black hole) close binary system with separation $\sim 0.2 R_{\odot}$. Since the (C+O) star is tidally locked, the collapsing core should have a spin angular momentum $\sim 6 \times 10^{48}$ cgs, so that a new-born pulsar should be a milli-second pulsar (MSP) ($P \sim 1$ ms). An $\alpha - \Omega$ dynamo in the first 10s after the bounce of the core will generate a superstrong magnetic field ($B \sim 10^{16}$ G). The beam of energy from a super strongly magnetized milli-second pulsar (SSM-MSP) can punch a hole in the supernova ejecta. Then the beam of gamma rays flows out of the ejecta for ~ 10 s with a total energy up to $\sim 2 \times 10^{51}$ erg. If we observe this beam from the lateral direction, the energy of gamma rays should be smaller than 500 keV due to the electron scattering and the total energy of gamma rays should be much smaller than $\sim 2 \times 10^{51}$ erg, which are the very characteristics of NHE-GRBs. In SSM-MSP model of NHE-GRB, GRB events should be associated with Type Ib/c supernova, like GRB980425/SN1998bw. If a SSM-MSP is produced in a fraction ($\sim$ a few %) of Type Ib/c supernovae, the event rate as well as the isotropic and homogeneous distribution of NHE-GRBs can also be explained. According to this model a pulsar with a spin period $P \sim 1.4s(B/10^{16})(t/4\text{month})^{1/2}$ should exist in SN1998bw, so that the search for this pulsar in all the wave length is urgent. SSM-MSPs can accelerate protons up to $\sim 3 \times 10^{22}(P/1\text{ms})^{-2}(B/10^{16}\text{G})\text{eV}$. If a few % ($\sim 10^{50}$ erg) of the beam energy of SSM-MSP in NHE-GRB event ($> 5 \times 10^{51}$ erg) is in the form of high energy protons of energy $\sim 10^{20}$ eV, the flux of the observed UHECRs (Ultra High Energy Cosmic Rays) can be explained. Then it should be possible to find a supernova remnant along the direction of each UHECR. As a natural implication of our model, it is suggested that a NHE-GRB is a part of the progenitor of a soft gamma ray repeater which also has a superstrong magnetic field ($B \sim 10^{16}$ G) and is in a supernova remnant.

1 Introduction

Galama et al.[1] reported the discovery of the very luminous Type Ic supernova SN1998bw in the BeppoSAX Wide Field Camera error box of GRB 980425[2], which occurred within about a day of the gamma ray burst. The duration of GRB 980425 is ~ 30 sec, and the burst fluence is $(4.4 \pm 0.4) \times 10^{-6}$ erg/cm². Since no burst of emission is detected above 300 keV, GRB 980425 belongs to a so-called NHE (Non High Energy) burst[3] which is characterized by a lack of fluence above 300 keV, lower peak flux and an apparent isotropic and homogeneous distribution. A quarter (~ 250 events/y) of GRBs are NHE bursts.[3]

The host galaxy of SN1998bw is a spiral galaxy at a distance of ~ 40 Mpc. The chance probability of the coincidence is $\sim 10^{-4}$. [1] Motivated by the coincidence of Type Ic SN with GRB, Wang and Wheeler[4] examined six recent SN Ib/c for which an outburst epoch can be estimated with some reliability and found that all are correlated in time and space with BATSE gamma ray bursts. Since the joint probability of all six correlations is 1.5×10^{-5} , and no such correlation exists for SN Ia and SN II, Wang and Wheeler [4] proposed that all gamma ray bursts are associated with Type Ib/c supernovae. In this paper I follow the

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proposal by Wang and Wheeler. I assume that all NHE-GRBs are associated with Type Ib/c supernovae. It should be possible to confirm or refute the validity of this assumption in the future observationally. I propose a model in which a (C+O) star and a neutron star(black hole) close binary is a progenitor of NHE gamma ray burst and argue the details of the model.

In §2 a formation scenario of a (C+O) star and a neutron star (black hole) close binary system is discussed. In §3 I discuss a supernova explosion of the (C+O) star which yields a super strongly magnetized milli-second pulsar(SSM-MSP). In §4 I explain how SSM-MSP emits NHE-GRB. Section 5 is devoted to discussions of the astrophysical implications of our SSM-MSP GRB model.

2 Formation of a (C+O) star and a neutron star(black hole) close binary system

Similar to the scenario of formation of double neutron star systems [5, 6] we assume that our (C+O) star-neutron star(black hole) binary is formed as follows[7] :

- 1) Consider a binary system composed of two main-sequence stars 1 and 2 of mass $\sim 25 M_{\odot}$ which has a helium core mass $\sim 8M_{\odot}$,
- 2) Roche lobe overflow of star 1, which becomes a helium star 1,
- 3) the first supernova explosion of helium star 1 to form a neutron star 1,
- 4) Roche lobe overflow of star 2 which leads to a spiral-in of neutron star 1 into star 2 . The system shrinks and finally it consists of a neutron star 1 and a helium star 2. Here we note that if the super Eddington accretion is possible[8] a neutron star 1 becomes a black hole of mass $\sim 2.4M_{\odot}$ in this spiral-in phase[9].

5) During the presupernova evolution of helium star 2, the helium envelope expands as its core contracts. The radii of star 2 with mass $3 \sim 8M_{\odot}$ are $4 \sim 1R_{\odot}$ [10, 6]. So if the Roche radius of star 2 is smaller than $4 \sim 1R_{\odot}$, a spiral-in of neutron star(black hole) 1 occurs again. Now the system consists of (C+O) star 2 and neutron star (black hole) 1. The mass of the (C+O) cores are 6.0 and $3.8 M_{\odot}$ for mass of helium stars 8 and $6M_{\odot}$, respectively [10], while the radius of the (C+O) star is $\sim 0.1R_{\odot}$. Since the radii of helium stars of mass 8 and $6M_{\odot}$ are 1.3 and $1.9 R_{\odot}$, respectively, at the onset of Roche lobe overflow, the separation after the spiral-in of neutron star(black hole) 1 will be significantly smaller than $1R_{\odot}$, say, $0.2R_{\odot}$. Such a (C+O) star in a close binary may be a progenitor of Type Ic supernova such as SN1994I[11].

3 Super Strongly Magnetized Milli-Second Pulsar

Now I discuss an evolution of our close binary system with separation $a \sim 0.2 R_{\odot}$ which consists of a (C+O) star of mass $M_2 \sim 6 M_{\odot}$ and a neutron star(black hole) of mass $M_1 = 1.4(2.4)M_{\odot}$ [7]. There are three important time scales in this system:

1) t_{ev} : the evolution time of the (C+O) star. They are 100 yr and 450 yr for $M_2 = 6$ and $3.8 M_{\odot}$, respectively[10].

2) t_{gw} :coalescing time of the binary due to the emission of gravitational waves which is given by

$$t_{gw} = 4 \times 10^3 yr \left(\frac{M_1}{1.4M_{\odot}} \right)^{-1} \left(\frac{M_2}{6M_{\odot}} \right)^{-1} \left(\frac{M_1 + M_2}{7.4M_{\odot}} \right)^{-1} \left(\frac{a}{1.4 \times 10^{10} cm} \right)^4. \quad (1)$$

3) t_{synch} :the tidal synchronization time which is given by[12]

$$t_{synch} \sim \frac{I\Omega_{orb}}{N}, \quad (2)$$

$$\Omega_{orb} = 1.9 \times 10^{-2} s^{-1} \left(\frac{M_1 + M_2}{7.4M_{\odot}} \right)^{0.5} \left(\frac{a}{1.4 \times 10^{10} cm} \right)^{-1.5}, \quad (3)$$

where I , Ω_{orb} and N are the moment of the inertia of the (C+O) star 2, the orbital angular velocity and the tidal torque, respectively. N is estimated as

$$N = \frac{GM_1^2}{R} \left(\frac{R}{a}\right)^6 \alpha, \quad (4)$$

where R and α are the radius of star 2 and the angle between the orbital separation vector and the tidal bulge, respectively. α is expressed by the viscous damping time t_{vis} as

$$\alpha = \frac{\Omega_{orb} R^3}{GM_2 t_{vis}}. \quad (5)$$

Therefore t_{synch} becomes

$$t_{synch} = \left(\frac{M_2}{M_1}\right)^2 \left(\frac{a}{R}\right)^6 t_{vis} \sim 1800 t_{vis}. \quad (6)$$

Since the convective core exists in the (C+O) star [10], t_{vis} is estimated as

$$t_{vis} = \left(\frac{R}{v_{conv}}\right) \left(\frac{R}{l_m}\right) \quad (7)$$

where v_{conv} and l_m are velocity of convection and the mixing length, respectively. Putting $v_{conv} \sim 10^5 \text{ cm s}^{-1}$ and $l_m \sim 0.2R$, we have

$$t_{synch} \sim 20 \text{ yr} \left(\frac{v_{conv}}{10^5 \text{ cm s}^{-1}}\right)^{-1} \left(\frac{l_m}{0.2R}\right)^{-1} \frac{R}{7 \times 10^9 \text{ cm}} \quad (8)$$

Since we have the inequality as

$$t_{gw} \geq t_{ev} \geq t_{synch}, \quad (9)$$

the (C+O) star 2 is tidally locked in the carbon burning phase and the supernova explosion of star 2 occurs with essentially the same separation a . In tidally locked binary the spin angular velocity is the same as the orbital one so that the collapsing central $1.4M_\odot$ core of star 2 will have the spin angular momentum J estimated as

$$J = I_{core} \Omega_{orb}, \quad (10)$$

where I_{core} is the moment of inertia of the central $1.4M_\odot$ core. Inserting I_{core} of $10^{50} \sim 10^{51} \text{ g cm}^2$ in the carbon burning phase [10], we have

$$J = 2 \times 10^{48 \sim 49} \text{ g cm}^2 \text{ s}^{-1} \left(\frac{M_1 + M_2}{7.4M_\odot}\right)^{0.5} \left(\frac{a}{1.4 \times 10^{10} \text{ cm}}\right)^{-1.5}. \quad (11)$$

This angular momentum is comparable to that of 1 ms pulsar ($\sim 6 \times 10^{48} \text{ g cm}^2 \text{ s}^{-1}$), which means that the new born pulsar after the supernova explosion of star 2 is necessarily a milli-second pulsar.

Duncan and Thompson [13, 14] argued α - Ω dynamo theory for such a rapidly rotating new born pulsar and concluded that such a milli second pulsar may have a superstrong magnetic field of $\sim 10^{16} \text{ G}$ within 10s after the formation. Although Duncan and Thompson considered the accretion induced collapse, the same argument can apply to our (C+O) star. Recently such a superstrong magnetic field ($\sim 10^{15} \text{ G}$) has been identified in a soft gamma ray repeater SGR1806-20 [15] with the age $\sim 10^4 \text{ y}$. This observation supports a model of soft gamma ray repeaters based on super strongly magnetized neutron stars [16, 17]. Therefore even relying only on the observational results, it may be possible to consider a superstrong magnetic field as strong as $\sim 10^{16} \text{ G}$ in a new born milli-second pulsar. Considering both theoretical suggestions and observations of such a superstrong magnetic field, we assume here that after the explosion of our (C+O) star a milli-second pulsar with the superstrong magnetic field ($B \sim 10^{16} \text{ G}$) is formed in $\sim 10 \text{ s}$. Here note that in vacuum there exists the maximum electric field ($E_c = 2\pi m_e^2 c^3 / \hbar e = 4.4 \times 10^{13} \text{ e.s.u.}$) while the maximum magnetic field does not exist since the vacuum with such a strong field is shown to be unstable only for $(E^2 - B^2) > 0$. [18] (See page 677 of reference 18.)

4 Supernova explosion of a (C+O) star and NHE gamma ray bursts

Now star 2 explodes as a Type Ic supernova with the ejecta of mass $M_e = 2 \sim 5M_\odot$. The column density (Σ) of the ejecta at time t from the explosion is given by

$$\Sigma \sim 10^5 \text{ g/cm}^2 \left(\frac{M_e}{M_\odot} \right)^2 E_{51}^{-1} \left(\frac{t}{1 \text{ day}} \right)^{-2} \quad (12)$$

where E_{51} is the explosion energy in the unit of 10^{51} erg. Recently Woosley, Eastman and Schmidt[19] made light curve models for SN1998bw and found that a $6M_\odot$ (C+O) star model with the unusually large explosion energy of $\sim 2 \times 10^{52}$ erg can fit the light curve quite well. They also found that both shock break-out and relativistic shock deceleration in circumstellar material fail to produce a GRB of even the low luminosity ($\sim 10^{48}$ erg) inferred for GRB980425.² This means that the source of gamma rays should be inside the ejecta. Since the time difference between the supernova explosion and GRB 980425 is one day or so, from Eq. (12) the column density at $t \sim 1$ day is so high that any gamma rays emitted from the central new pulsar, neutron star or black hole can not reach the surface of the ejecta unless there is a hole in the ejecta. This reminds us the mysterious spot in SN1987A[21, 22] which appeared 5-7 weeks after the explosion of Type II SN1987A at ~ 0.06 arc-sec away from SN1987A.

To explain mysterious spot in SN1987A, at least four mechanisms were examined to punch a hole in the ejecta[23, 24]. These mechanisms can apply to SN1998bw/GRB980425. They are;

1. Due to the existence of the secondary (a neutron star or black hole in this case) the density in the direction shadowed by the companion will be low[23].
2. Rayleigh-Taylor instability in the ejecta may make a hole[23].
3. The relativistic jets formed at the core collapse punch holes in the ejecta[24].
4. Beams of energy from the super strongly magnetized milli second pulsar may punch holes in the ejecta[24].

In the first mechanism the relativistic plasma from the pulsar gradually inflates a bubble within the remnant. Since the remnant is punctured, the relativistic plasma from the over-pressured bubble would squirt into the surrounding. However the time needed for this mechanism to operate would be much longer than ~ 30 s unless the direction of the hole coincides with the direction of the beam from the pulsar or from the black hole with the accretion disk. In the second mechanism, the chance probability of such a hole would be very low for $t \sim 30$ s. A chance coincidence of the hole and the direction of the beam is also needed.

In the third mechanism the direction of the hole should agree with the direction of the spin of the pulsar or the rotating axis of the black hole. General relativistic numerical simulations[25] have demonstrated that a rapidly rotating core collapses first along its rotation axis. Collapse continues along the equatorial directions, then bounces along the rotation axis to form jets. In one example, a $1.4M_\odot$ stellar core modeled as a $\gamma = 3/4$ polytrope produced jets with mass $0.007M_\odot$, kinetic energy 5×10^{51} erg and average velocity $0.6c$. Since the opening angle of this jet is a few degree, it is possible to punch a hole at the first 10 seconds or so. The total angular momentum in this model $2.4 \times 10^{49} \text{ gcm}^2 \text{ s}^{-1}$ is four times larger than 1ms pulsar. Although this kind of jets may also be formed in MHD rotating core collapses [26], in general, with the increase of the angular momentum the kinetic energy of the explosion decreases since the bounces occur at the lower density and weaker gravitational potential, while SN1998bw has the unusually large kinetic energy. We therefore consider the fourth mechanism mainly in this paper.

For the typical angular momentum in the core of our (C+O) star corresponding to a milli-second pulsar, $\sim 6 \times 10^{48} \text{ gcm}^2/\text{s}$, the collapse is essentially spherically symmetric because the ratio of the

²After the submission of this paper, Nomoto noticed the author the light curve model of SN1998bw by his group.[20] In their model, the mass of the ejecta is $12 \sim 15M_\odot$ with the explosion energy of $2 \sim 5 \times 10^{52}$ ergs. Even if we adopt these values, the arguments in this paper will hold almost the same.

centrifugal force to the gravity is ~ 0.2 even for the neutron star radius. A milli-second pulsar has the rotational energy (E_{rot}) given by ,

$$E_{rot} = 2 \times 10^{52} \text{erg} I_{45} P_{ms}^{-2}, \quad (13)$$

where I_{45} and P_{ms} are the moment of inertia in the unit of 10^{45}gcm^2 and the spin period in milli-second, respectively. The spin period and the luminosity (L) of the pulsar are given by

$$P_{ms} = \sqrt{\left(\frac{P_i}{1ms}\right)^2 + \frac{t}{\tau}}, \quad (14)$$

$$\tau = 5s B_{16}^{-2} I_{45} R_6^{-6}, \quad (15)$$

$$L = 3.8 \times 10^{51} \text{erg/s} B_{16}^2 R_6^6 \left(\left(\frac{P_i}{1ms}\right)^2 + \frac{t}{\tau}\right)^{-2}, \quad (16)$$

where P_i , B_{16} , R_6 and t are the initial spin period, the magnetic field in unit of 10^{16}G , the radius of the neutron star in unit of 10^6cm and the time after the bounce of the core, respectively.

The expansion velocity (V_{exp}) of the ejecta is given by

$$V_{exp} \sim 10^9 \text{cm/s} \left(\frac{M_e}{M_\odot}\right)^{-1/2} E_{51}^{1/2}. \quad (17)$$

For SN1998bw the observed expansion velocity is $\sim 2 \times 10^9 \text{cm/s}$ [27]. This means that $E_{51} \sim 20$ for $M_e \sim 5M_\odot$. This unusually large explosion energy is consistent with a model of light curves[19]. The shock will reach the surface of the (C+O) star ($\sim 10^{10} \text{cm}$) in $\sim 5s$. Since the magnetic field will be amplified to $\sim 10^{16} \text{G}$ within 10s[13], the superstrong magnetic field pulsar begins to operate after or just when the shock breaks out. The solid angle extended by the beam from the pulsar is given by[29]

$$\Delta\Omega = \pi \frac{2\pi R}{P_c} \sim 0.05 \times 4\pi P_{ms}^{-1} \quad (18)$$

Then the energy of $\sim 10^{52} \text{erg}$ would be injected into 5% of the ejecta in $\sim 10s$. If this energy is converted to the kinetic energy of the ejecta within the cone of the beam, the velocity (V_{conc}) would be

$$V_{conc} = 1.4 \times 10^{10} \text{cm/s} \left(\frac{M_e}{M_\odot}\right)^{-1/2} \quad (19)$$

This is much larger than V_{exp} so that a hole is punched out in the ejecta.

If, however, $E_{51} \sim 1$ and $M_e \sim 5M_\odot$ as usual , $V_{exp} \sim 5 \times 10^8 \text{cm/s}$ and the shock will reach the surface in $\sim 20s$ so that the pulsar activity would start before the shock breaks out. Then a part of the energy of the beam would be used to expand the ejecta as a whole so that an effective explosion energy would increase and we may have $V_{exp} \sim 10^9 \text{cm/s}$ finally. In this case V_{conc} will decrease but as far as $V_{conc} > V_{exp} \sim 2 \times 10^9 \text{cm/s}$, we may regard the hole is punched out.³ For this the minimum energy needed is $\sim 2 \times 10^{51} \text{erg}$ so that almost all the rotational energy of the milli-second pulsar might be used to the expansion energy of the ejecta. Note here that the unusually large explosion energy of SN1998bw is the same as the rotational energy of the milli-second pulsar. This may not be a chance coincidence. In any case a hole would be punched out by the beam of the pulsar in the first 10 s or so.

We assume here that at $t \sim 10s$ the hole is punched out. Then the period of the pulsar is $\sim 2ms$ and the rotational energy is $\sim 5 \times 10^{51} \text{erg}$. The energy beam of the pulsar will flow out from the hole. This situation is similar to the millisecond pulsar models of GRBs based on the accretion induced collapse of magnetized white dwarfs [30, 31]. The beam of the energy would be finally converted to the beam of gamma rays [30, 31]. As a whole $\sim 5 \times 10^{51} \text{erg}$ of gamma rays would be emitted in a cone of the solid angle $\sim 0.025 \times 4\pi$. The chance probability that the line of the sight is in the cone is ~ 0.025 . We will observe the beam of the gamma rays, in general, from the lateral direction so that only gamma rays scattered by the thin matter blown off from the ejecta in the first 10s will come along the line of sight .

³We need the detailed numerical simulations on the mechanism of punching a hole in the ejecta. This is underway using finite difference as well as SPH.[28]

In Compton scattering the energy of the scattered photon (ω) is given by

$$\frac{1}{\omega} = \frac{1}{\omega_0} + \frac{1 - \cos\theta}{m_e c^2} \quad (20)$$

where θ is the scattered angle and ω_0 is the incident energy. Even for $\omega_0 \gg m_e c^2$, unless θ is small such that $\omega_0(1 - \cos\theta) \ll m_e c^2$, $\omega < m_e c^2$. That is, the energy of the scattered gamma ray should be smaller than 500keV. This explains the lack of fluence above 300keV for NHE-GRBs. The scattered photon is more or less distributed isotropically, that is, it is observed from almost all the direction and the total energy would be much smaller than $\sim 10^{51}$ erg because only a small fraction of the beamed gamma rays would be scattered by the very low density plasma. This is also consistent with the total energy ($\sim 10^{48}$ erg) of GRB980425.

About a fourth of gamma ray bursts(~ 250 events/y) is NHE-GRBs [3]. The event like GRB980425 will not be observed if it occurred at 20 times farther ~ 800 Mpc. The number of galaxies within 800Mpc is $\sim 2 \times 10^7$ and the event rate of TypeIb/Ic supernova is $\sim 10^{-3}$ /y/galaxy[33] so that the event rate of TypeIb/Ic supernova within 800Mpc is $\sim 2 \times 10^4$ /y. This suggests that only a fraction ($\sim$ a few%) of TypeIb/Ic supernova becomes NHE gamma ray burst which is different from the suggestion by Wang and Wheeler.[4] This is, however, consistent with our model. Depending on the initial separation and initial mass of the binary, only a fraction of the binary yields the super strongly magnetized milli-second pulsar.

After an NHE-GRB event the luminosity of the pulsar is given by

$$L = 1.3 \times 10^{43} \text{erg/s} B_{16}^{-2} I_{45}^2 R_6^{-6} \left(\frac{t}{1\text{day}}\right)^{-2} \quad (21)$$

For SN1998bw, the above luminosity is much smaller than the observed luminosity so that the existence of the pulsar will not affect the light curve of the supernova.

5 Discussion

It is not clear at present that our model can explain HE(High Energy) GRBs and its afterglow also. However if we observe NHE-GRB in our model along the beam direction and if we regard that the energy is emitted isotropically, then the total energy amounts to $\sim 2 \times 10^{53}$ erg, which is comparable to the maximum observed energy of GRBs[32]. We here argue the event rate of NHE-GRBs up to $z \sim 3$. Adopting that $\Omega_M \sim 0.2$ and $\Omega_\Lambda \sim 0.8$ and $H_0 = 65 \text{km/s/Mpc}$, the total number of NHE-GRBs up to $z \sim 3$ is $\sim 10^5$ /y. If NHE-GRBs observed along the cone of the pulsar beam look like HE GRBs, the event rate of HE GRBs would be $\sim 10^3$ /y, which is comparable to the event rate of HE GRBs.

The maximum energy of a proton accelerated by the SSM-MSP is given by [29]

$$E_{\max} = 7.5 \times 10^{21} \text{eV} R_6^3 B_{16}^3 \left(\frac{P}{2\text{ms}}\right)^{-2}. \quad (22)$$

This is larger or comparable to the maximum energy of the cosmic rays so far observed. Such a high energy cosmic ray can propagate only up to the distance of ~ 100 Mpc due to the production of pion through the collision with the microwave background radiation. Then the number density of UHECRs(Ultra High Energy Cosmic Rays $E > 10^{20}$ eV) is estimated as $n_{cr} \sim 10^{-30} \text{cm}^{-3}$. [34] If UHECRs are supplied by NHE-GRBs, the total energy of UHECR per one NHE-GRB-event is estimated as $\sim 10^{50}$ erg. This energy is only 0.05% of the total energy of the milli-second pulsar and 2% of the energy emitted by the pulsar after the hole is punched out in our model. Very recently Agasa[35] team reported possible clustering in position of UHECRs, which may be in favor of GRB model of UHECRs[36]. Since the time broadening of the pulse of UHECRs due to the intergalactic magnetic field is more than 100 y [34], the arrival time of UHECR does not coincide with the GRB event. However in our model there should exist the supernova remnant in the direction of each UHECR if the time broadening is shorter than the maximum age of the observable supernova remnant.

The relation between soft gamma ray repeaters (SGRs) and NHE-GRBs in our model is quite interesting. NHE-GRBs in our model is a strongly magnetized pulsar in the supernova remnant with

the event rate of $10^{-4} \sim 10^{-5}$ events/y/galaxy. This is exactly the same as the characteristics of soft gamma ray repeaters. This suggests that NHE-GRBs in our model consist a part of progenitors of SGRs($\sim 10^{-3}$ events/y/galaxy [15]) .

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Stability of Relativistic Blast Waves and Implications in Gamma Ray Burst

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Abstract

We consider the linear theory of the dynamical instability of an extremely relativistic blast wave in a medium with uniform density. In the Newtonian regime shock wave is governed by the Sedov-Taylor adiabatic solution, and the linear theory of shock instability was considered by Ryu and Vishniac. The extremely relativistic shock wave is governed by the Blandford-McKee adiabatic solution. We obtain the principal frequency eigenvalues as a function of transverse velocity of adiabatic shock wave. We discuss the implications in Gamma-Ray Bursts (GRBs).

1 Introduction

Recent observation shows that GRBs involve immense amount of energy $\sim 10^{52}$ ergs. The “compactness problem” shows that the observed γ -rays must be emitted by a medium with highly relativistic velocities having Lorentz factor $\Gamma \geq 100$. The subsequent evolution of the radiation dominated matter can be described by the expanding blast wave. The observed fluctuations with the small time scale in GRBs are expected to be associated with the complex behavior of shock waves. Two locations are so far proposed as the origin of the irregularities. One is inside the relativistic fluid, which is known as internal shock model. When the shock waves intersect each other, γ ray would be emitted. The complexity of the time profile depends on the internal shock structure. The other is the shock front itself. When the shock front touches ISM clouds, the γ ray is emitted. The observed nature with the rapid variability depends on the fluctuation of the external matter.

The blast wave with the ultra-relativistic speed was considered by Blandford and McKee[5]. They obtained the spherical self-similar solutions, which can be regarded as the relativistic version of the Sedov-Taylor solution.

Ryu and Vishniac[1] examined the dynamical instability of the spherical adiabatic blast wave in the non-relativistic hydrodynamics. They show that the spherical shock becomes overstable under certain conditions. In this paper, we consider the stability of the ultra-relativistic shock waves, which are relevant to the GRBs models.

In section 2 we present the hydrodynamic equations which govern the gas flow motion of the adiabatic shocks and the similarity solutions of unperturbed, ultra-relativistic, adiabatic shocks following Blandford and McKee[5]. In section 3 the perturbation equations and analytic solutions of ultra-relativistic shock wave are derived. In section 4 the boundary conditions at the shock front and the origin are derived. Finally in section 5 we solve the perturbation equations and give the frequency spectra.

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2 Unperturbed shock

The relativistic fluid motion is governed by the conservation laws of the number and energy-momentum. We explicitly write down the equations for the two dimensional flow with the 3-velocity (v_r, v_θ) and the gamma factor γ as

$$\partial_t (W - p) + \frac{1}{r^2} \partial_r (r^2 W v_r) + \frac{1}{r \sin \theta} \partial_\theta (\sin \theta W v_\theta) = 0, \quad (1)$$

$$\partial_t (W v_r) + \frac{1}{r^2} \partial_r (r^2 W v_r^2 + r^2 p) + \frac{1}{r \sin \theta} \partial_\theta (\sin \theta W v_r v_\theta) - \frac{W}{r} v_\theta^2 - 2 \frac{p}{r} = 0, \quad (2)$$

$$\partial_t (W v_\theta) + \frac{1}{r} \partial_r (r W v_r v_\theta) + \frac{1}{r \sin \theta} \partial_\theta (\sin \theta W v_\theta^2 + \sin \theta p) + \frac{2}{r} W v_r v_\theta - \frac{p}{r} \cot \theta = 0, \quad (3)$$

$$\partial_t D + \frac{1}{r^2} \partial_r (r^2 D v_r) + \frac{1}{r \sin \theta} \partial_\theta (\sin \theta D v_\theta) = 0, \quad (4)$$

where

$$W \equiv (e + p) \gamma^2, \quad (5)$$

$$D \equiv n \gamma, \quad (6)$$

and p, e and n are the pressure, the energy density and the number density.

Blandford and McKee[5] obtained the spherical symmetric similarity solution of eqs.(1)-(4). The solution can be described by t and r . Instead of these coordinates, they introduced the gamma factor Γ of the shock front and a new coordinate χ as the similar parameter. That is,

$$\Gamma^2 \propto t^{-m}, \quad (7)$$

$$\chi = [1 + 2(m+1)\Gamma^2](1 - r/t), \quad (8)$$

where m is a constant which specifies the relativistic blast wave. The point explosion corresponds to $m = 3$. The solution inside the shock, which corresponds to the region $\chi > 1$, can be expressed for the radiation dominated fluid $p = e/3$ with large Γ as

$$p = \frac{2}{3} w_1 \Gamma^2 f(\chi), \quad (9)$$

$$\gamma^2 = \frac{1}{2} \Gamma^2 g(\chi), \quad (10)$$

$$D = 2 n_1 \Gamma^2 h(\chi), \quad (11)$$

$$W = \frac{4}{3} w_1 \Gamma^4 f(\chi) g(\chi) \quad (12)$$

$$v_r = 1 - \frac{1}{\Gamma^2 g(\chi)} \quad (13)$$

$$v_\theta = 0 \quad (14)$$

where w_1 and n_1 are the enthalpy and the number density in front of the shock. The functions $f(\chi), g(\chi)$ and $h(\chi)$ are analytic ones. In particular, for $m = 3$ case, corresponding to an "impulsive" injection of energy into the blast wave on a time scale short compared with distance which from center of GRBs to shock front, we have a very simple analytic solution of $f(\chi), g(\chi)$ and $h(\chi)$:

$$f = \chi^{-\frac{17}{12}}, \quad (15)$$

$$g = \chi^{-1}, \quad (16)$$

$$h = \chi^{-\frac{7}{2}}. \quad (17)$$

3 Perturbation equations for spherical shock

The linear perturbation of eqs.(1)-(4) is straightforward. The perturbed variables can be expressed by an appropriate product of the functions as

$$W(\Gamma, \chi) + \delta W(\Gamma, \chi, \theta) = W \{1 + (\Gamma^2)^s g(\chi) (2 \xi_P(\chi) + \xi_R(\chi)) P_l(\cos \theta)\}, \quad (18)$$

$$p(\Gamma, \chi) + \delta p(\Gamma, \chi, \theta) = p \{1 + (\Gamma^2)^s \xi_P(\chi) P_l(\cos \theta)\}, \quad (19)$$

$$D(\Gamma, \chi) + \delta D(\Gamma, \chi, \theta) = D \{1 + (\Gamma^2)^s \xi_D(\chi) P_l(\cos \theta)\}, \quad (20)$$

$$v_r(\Gamma, \chi) + \delta v_r(\Gamma, \chi, \theta) = 1 - \frac{1}{\Gamma^2 g(\chi)} \{1 + (\Gamma^2)^s \xi_R(\chi) P_l(\cos \theta)\}, \quad (21)$$

$$\delta v_\theta(\Gamma, \chi, \theta) = (\Gamma^2)^s \xi_T(\chi) \frac{d P_l(\cos \theta)}{d \theta}, \quad (22)$$

where $P_l(\cos \theta)$ is the Legendre polynomial and s is a constant which will be determined by the boundary conditions. Using these variables, we can split the time and angular parts and have the perturbation equation among the functions $\xi_P, \xi_D, \xi_R, \xi_T$. The explicit form can be written as,

$$(m+1)\left(\chi - \frac{2}{g}\right) \frac{d\xi_D}{d\chi} + m s \xi_D + \frac{4(2-\chi g)\chi g - m(28-60\chi g+30\chi^2 g^2-3\chi^3 g^3)}{(2-\chi g)(4-8\chi g+\chi^2 g^2)} \xi_R \\ + \frac{1}{2}(m+1)\left(\frac{4}{g} - \chi\right) \frac{d\xi_P}{d\chi} - \frac{16+4\chi g-2\chi^2 g^2 - m(28-11\chi g\chi^2 g^2 - s(4-8\chi g+\chi^2 g^2))}{2(4-8\chi g+\chi^2 g^2)} \xi_P = \quad (23)$$

$$(m+1)\left(\chi - \frac{2}{g}\right) \frac{d\xi_R}{d\chi} + \frac{2(4-4\chi g-\chi^2 g^2) + m(-8+8\chi g-\chi^2 g^2 + s(4-8\chi g+\chi^2 g^2))}{4-8\chi g+\chi^2 g^2} \xi_R \\ - \frac{1}{2}(m+1)\left(\chi + \frac{2}{g}\right) \frac{d\xi_P}{d\chi} + \frac{2(2+\chi g)^2 - m(14+5\chi g-\chi^2 g^2 + s(4-8\chi g+\chi^2 g^2))}{2(4-8\chi g+\chi^2 g^2)} \xi_P = \quad (24)$$

$$(m+1)\left(\chi - \frac{2}{g}\right) \frac{d\xi_T}{d\chi} - \frac{4+4\chi g-\chi^2 g^2 - m(8-\chi g-s(4-8\chi g+\chi^2 g^2))}{4-8\chi g+\chi^2 g^2} \xi_T = \quad (25)$$

$$(m+1)\left(\chi - \frac{2}{g}\right) \frac{d\xi_D}{d\chi} + m s \xi_D - \frac{2}{g}(m+1) \frac{d\xi_R}{d\chi} \\ - \frac{4(9m-8) - 4(5m-6)\chi g + 2(m-2)\chi^2 g^2}{(2-\chi g)(4-8\chi g+\chi^2 g^2)} \xi_R + l(l+1) \xi_T = \quad (26)$$

The differential equations (23)-(26) can be solved in analytic way for the $m = 3$ case. The solutions are expressed with four constants, A, B, C and D .

$$\xi_D = A\chi^{p_1} + Bq_1 l(l+1)\chi^{p_2} + Cq_{4-}\chi^{p_+} + Dq_{4+}\chi^{p_-}, \quad (27)$$

$$\xi_R = Bq_2 l(l+1)\chi^{p_2} + Cq_{5+}\chi^{p_+} + Dq_{5-}\chi^{p_-}, \quad (28)$$

$$\xi_P = Bq_3 l(l+1)\chi^{p_2} + 2C\chi^{p_+} + 2D\chi^{p_-}, \quad (29)$$

$$\xi_T = B\chi^{p_2}, \quad (30)$$

where

$$p_1 = \frac{3}{4}s \quad (31)$$

$$p_2 = \frac{7}{6} + \frac{3s}{4} \quad (32)$$

$$p_{\pm} = \frac{-(18s-17) \pm \sqrt{432s^2 - 576s + 289}}{24} \quad (33)$$

and

$$q_1 = -\frac{15s+91}{14(36s^2+57s-7)} \quad (34)$$

$$q_2 = \frac{6s+7}{36s^2+57s-7} \quad (35)$$

$$q_3 = -\frac{7}{36s^2+57s-7} \quad (36)$$

$$q_{4\pm} = \frac{216s^2 - 441s + 136 \pm (9s-8)\sqrt{432s^2 - 576s + 289}}{(3s-4)(36s-17 \pm \sqrt{432s^2 - 576s + 289})} \quad (37)$$

$$q_{5\pm} = \frac{17 \pm \sqrt{432s^2 - 576s + 289}}{12s-16} \quad (38)$$

The second terms of the first equation, the first term of the second equation, and the first term of third equation contain the curvature effect of the spherical geometry.

4 Boundary conditions

Let us consider the boundary conditions at the shock front when the shock is perturbed. For the gas flow through the shock front, we require the conservations of number density flux, momentum flux, and energy flux. In here, the difference between the values of any quantity on the two sides of the surface is denoted by enclosing it in square brackets; for example, $[a] \equiv a_1 - a_2$, and the boundary conditions for a perturbed ultra-relativistic adiabatic shock are

$$[n u^\mu N_\mu] = 0,$$

$$[T^{\mu\nu} U_\mu N_\nu] = 0,$$

$$[T^{\mu\nu} N_\mu N_\nu] = 0,$$

where T , U_s , N_s and u are the energy-momentum tensor of shock wave, the space-like unit vector on shock front, the time-like unit vector on shock front and the ambient gas velocity.

The components of parameter are

$$U_{s\mu} = \{\Gamma (1 + V \Gamma^2 \delta V), \Gamma (V + \Gamma^2 \delta V), \Delta u_s\}, \quad (39)$$

$$N_{s\mu} = \{\Gamma (V + \Gamma^2 \delta V), \Gamma (1 + V \Gamma^2 \delta V), \Delta n_s\}, \quad (40)$$

and

$$u_1{}^\mu = \{1, 0, 0\}, \quad (41)$$

$$u_2{}^\mu = \{\gamma + \Delta\gamma, v\gamma + \gamma \Delta v + v \Delta\gamma, \Delta u_2\}, \quad (42)$$

where V is the velocity of shock front, and u_1 and u_2 are the gas velocity, where the subscript 2 denotes the shocked gas and 1 denotes the unshocked gas. Additionally, Δu_s , Δn_s and Δu_2 are the perturbative value in first order. It should be noted that Δ includes the pure perturbational value δ and the change of slope in radiate direction; $\Delta \equiv \delta + \eta \frac{\partial}{\partial r}$, where η is the perturbation of the position of the shock front. Here, the perturbation of quantities in the ambient gas is neglected, and perturbation variables which include the change of slope with distance are

$$\frac{\Delta D}{D} = (\Gamma^2)^s \left\{ \xi_D + \frac{3^4}{3} \right\}, \quad (43)$$

$$\frac{\Delta W}{W} = (\Gamma^2)^s \left\{ \xi_P - \xi_R + \frac{58}{3} \right\}, \quad (44)$$

$$\frac{\Delta p}{W} = (\Gamma^2)^s \left\{ \frac{\xi_P}{2} + \frac{17}{3} \right\}, \quad (45)$$

$$\Delta v = (\Gamma^2)^s \left\{ -\xi_R + 8 \right\}, \quad (46)$$

$$\frac{\delta V}{V} = (4 - 3s) (\Gamma^2)^s. \quad (47)$$

Especially, for $m = 3$ case, we obtain the normalized boundary conditions of density, pressure, and r -component of velocity perturbations at the shock front in simple form

$$\xi_D = -6(1 + s), \quad (48)$$

$$\xi_P = -\frac{10}{3} - 6s, \quad (49)$$

$$\xi_R = 6s. \quad (50)$$

For the boundary condition at the origin, we require that the fluid should not undergo divergent perturbations (Ref. [1]). With some algebra, we obtain the boundary condition for an adiabatic shock at the origin

$$\delta p \rightarrow 0, \quad (51)$$

from a condition at the origin

$$\nabla \cdot \delta v = 0. \quad (52)$$

It is convenient to change eqs. (51) as

$$\xi_P \rightarrow 0, \quad (53)$$

when, $(\chi \rightarrow \infty)$.

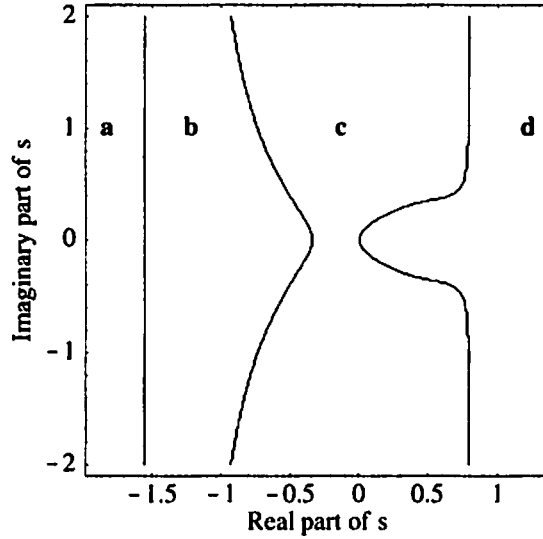


Figure 1: Regions on complex s -plane for the number of non-trivial solutions. The four constants, A,B,C,and D [eq. (27)-(30)] are restricted by the dimensionless perturbation growth rate s for adiabatic ultra-relativistic shocks. In region (a) non-trivial solution are A,B, and D. In region (b) non-trivial solution are A and B. In region (c) there is only trivial solution. In region (d) non-trivial solution is A.

5 Results

From eqs.(27)-(30),(48)-(50), and boundary condition, we restrict s , and determine the coefficient A,B,C, and D for such s .

(region a)

$$A = \frac{9720s^4 - 92394s^3 + 65367s^2 - 6618s - 24407}{7(2592s^3 - 3168s^2 + 330s - 119 + (72s^2 - 138s + 7)\sqrt{432s^2 - 576s + 289})} \quad (54)$$

$$+ \frac{(1242s^3 - 783s^2 + 4170s - 329)\sqrt{432s^2 - 576s + 289}}{7(2592s^3 - 3168s^2 + 330s - 119 + (72s^2 - 138s + 7)\sqrt{432s^2 - 576s + 289})} \quad (55)$$

$$B = \frac{2(36s^2 + 57s - 7)(216s^2 - 135s + 85 - (9s + 5)\sqrt{432s^2 - 576s + 289})}{3l(l+1)(144s^2 - 24s - 105 - 7\sqrt{432s^2 - 576s + 289})} \quad (56)$$

$$C = 0 \quad (57)$$

$$D = -\frac{8(3s - 4)(54s^2 + 30s + 35)}{3(144s^2 - 24s - 105 - 7\sqrt{432s^2 - 576s + 289})} \quad (58)$$

$$(s = -\frac{1}{3})$$

$$A = \frac{611 - 53\sqrt{97}}{33} \quad (59)$$

$$B = 0 \quad (60)$$

$$C = 0 \quad (61)$$

$$D = \frac{17 + \sqrt{97}}{6} \quad (62)$$

$$(63)$$

The adiabatic ultra-relativistic shock wave is governed by this solutions. The implications in GRBs are now under consideration.

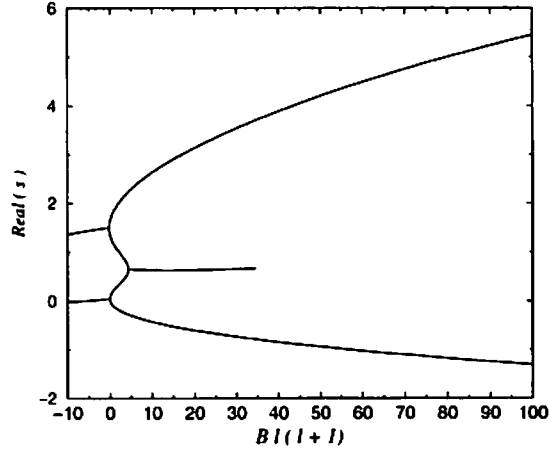


Figure 2: The angular dependence of s in region a.

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R-mode Oscillations and GRR Instability

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Abstract

The axial mode oscillations are examined for relativistic rotating stars with uniform angular velocity. Using the slow rotation formalism and the Cowling approximation, we have derived the equations governing the r-mode oscillations up to the second order with respect to the rotation. In the lowest order, the allowed range of the frequencies is determined, but corresponding spatial function is arbitrary. The spatial function can be decomposed in non-barotropic region by a set of functions associated with the differential equation of the second-order corrections. The equation however becomes singular in barotropic region, and a single function can be selected to describe the spatial perturbation of the lowest order. The frame dragging effect among the relativistic effects may be significant, as it results in rather broad spectrum of the r-mode frequency unlike in the Newtonian first-order calculation.

1 Introduction

In recent years, the r-mode oscillations in rotating stars have been found to be significant implications. The axial oscillations are unstable by the gravitational radiation reaction ([1], [10]). The mechanism of the instability can be understood by the generic argument for the gravitational radiation driven instability, so-called CFS-instability, which is originally examined for the polar perturbations, ([3], [8], [9]). All rotating stars become unstable in the absence of internal fluid dissipation, irrespective of the parity modes. Viscosity however damps out the oscillations, and stabilizes them in general. The polar f-mode instability is believed to act only in rapidly rotating neutron stars ([14]). The axial instability is however found to set in even in much more slowly rotating cases, and to play an important role on the evolution of hot newly-born neutron stars ([15], [2]). The instability carries away most of angular momentum and rotational energy of the stars by the gravitational radiation. The gravitational wave emitted during the spin-down process is expected to be one of the promising sources for the laser interferometer gravitational wave detectors ([20]).

Most of the estimates for the r-mode instability are based on the Newtonian calculations. That is, the oscillation frequencies are determined by inviscid hydrodynamics under the Newtonian gravity, and the gravitational radiation reaction is incorporated by evaluating the (current) multipole moments. Relativity has great influence on stellar structures, redshift in oscillation frequencies, frame dragging, radiations and so on. It is important to explore the relativistic effects, which may not change the general features of the oscillations, but shift the critical angular velocity. The relativistic calculation is not so easy task, even using the linear perturbation method concerning the oscillation amplitude. Both relativity and rotation complicate the problems considerably.

As the first step toward a clear understanding of relativistic corrections, we have examined the r-mode oscillations with two approximations, the slow rotation approximation ([11]) and the Cowling approximation ([5]). The rotation is assumed to be slow, and treated as small perturbation from the non-rotating case. We also assume that the perturbation of the gravity can be neglected in the oscillations. The accuracy of the Cowling approximation are tested in the rotating relativistic system ([7], [13], [26]). The calculations give the same qualitative results and reasonable accuracy of the oscillation frequencies, as in the Newtonian stellar pulsation theory (e.g., [6]). In Section 2, we summarize the basic equations for the inviscid fluid in slowly rotating relativistic stars and the perturbation scheme to solve them. The r-mode solution is given at the lowest order with respect to the rotational parameter in Section 3. The

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mode can not be specified by a single frequency unlike in the Newtonian r-mode oscillation, although the frequency is bounded to a certain range. The spatial function of the mode is arbitrary at this order. In order to determine the radial structure, we need the rotational corrections up to the third order in the background. In Section 4, we include the rotational effects, and derive the equation governing the r-mode oscillations in non-barotropic region. The resultant equation becomes singular in the barotropic case, since the second-order differential term vanishes. In Section 5, we separately consider for the barotropic stars the mode function and the correction to the leading order. In Section 6, we discuss the implication of our results. We use the geometrical units of $c = G = 1$.

2 Pulsation equations of axial mode

We consider a slowly rotating star with a uniform angular velocity $\Omega \sim O(\varepsilon)$. The rotational effects can be treated as the corrections to the non-rotating spherical star. We will take account of the corrections up to the third order in ε . The metric tensor for describing the stationary axisymmetric star is given by ([11],[4], [23])

$$ds^2 = -e^\nu [1 + 2(h_0 + h_2 P_2)] dt^2 + e^\lambda \left[1 + \frac{2e^\lambda}{r} (m_0 + m_2 P_2) \right] dr^2 + r^2 (1 + 2k_2 P_2) \left\{ d\theta^2 + \sin^2 \theta \left[d\phi - \left(\omega + W_1 - W_3 \frac{1}{\sin \theta} \frac{dP_3}{d\theta} \right) dt \right]^2 \right\}, \quad (1)$$

where $P_l(\cos \theta)$ ($l = 2, 3$) denotes the Legendre's polynomial of degree l . The metric functions except $g_{t\phi}$ should be expanded by an even power of ε , while $g_{t\phi}$ by an odd power of ε due to the rotational symmetry, i.e., $\omega \sim O(\varepsilon)$, $(h_0, h_2, m_0, m_2, k_2) \sim O(\varepsilon^2)$ and $(W_1, W_3) \sim O(\varepsilon^3)$. These quantities are functions of the radial coordinate r .

The equilibrium state is assumed to be described with perfect-fluid stress-energy tensor. The 4-velocity of the fluid element inside the star has the components, which are correct to $O(\varepsilon^3)$,

$$U^\phi = \Omega U^t, \quad U^t = (-g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi})^{-1/2}. \quad (2)$$

The pressure and density distributions are subject to the centrifugal deformation, which is the effect of $O(\varepsilon^2)$. These distributions are expressed as

$$p = p_0(r) + \{p_{20}(r) + p_{22}(r)P_2(\cos \theta)\}, \quad (3)$$

$$\rho = \rho_0(r) + \{\rho_{20}(r) + \rho_{22}(r)P_2(\cos \theta)\}, \quad (4)$$

where p_0 and ρ_0 are the values for the non-rotating star, and the quantities in the braces are the rotational corrections. The non-rotating spherical configuration and the rotational corrections in eqs.(1)-(4) are determined by successively solving the perturbed Einstein equations with the same power of ε ([11],[4], [23]).

We consider the linear perturbations of the equilibrium fluid state. We use the Eulerian change of the pressure, density and 4-velocity, which are represented by δp , $\delta \rho$ and δU_ν . The perturbations of the gravitational field are neglected in the Cowling approximation. It is convenient in the following calculation to expand these functions in terms of the spherical harmonics Y_{lm} ,

$$\delta p = \sum_{lm} \delta p_{lm}(t, r) Y_{lm}(\theta, \phi), \quad (5)$$

$$\delta \rho = \sum_{lm} \delta \rho_{lm}(t, r) Y_{lm}(\theta, \phi), \quad (6)$$

$$(\rho + p)\delta U_r = e^{\nu/2} \sum_{lm} R_{lm}(t, r) Y_{lm}(\theta, \phi), \quad (7)$$

$$(\rho + p)\delta U_\theta = e^{\nu/2} \sum_{lm} \left[V_{lm}(t, r) \partial_\theta Y_{lm}(\theta, \phi) - \frac{U_{lm}(t, r)}{\sin \theta} \partial_\phi Y_{lm}(\theta, \phi) \right], \quad (8)$$

$$(\rho + p)\delta U_\phi = e^{\nu/2} \sum_{lm} [V_{lm}(t, r) \partial_\phi Y_{lm}(\theta, \phi) + U_{lm}(t, r) \sin \theta \partial_\theta Y_{lm}(\theta, \phi)], \quad (9)$$

$$\delta U_t = -\Omega \delta U_\phi. \quad (10)$$

With these definitions, the pulsation equations are derived from the conservation laws of the perturbed energy-momentum,

$$\delta T_{\mu;\nu}^\nu = 0, \quad (11)$$

where

$$\delta T_\mu^\nu = (\delta \rho + \delta p) U_\mu U^\nu + [(\rho + p) \delta U_\mu U^\nu + (\rho + p) \delta U^\nu U_\mu] + \delta p \delta_\mu^\nu. \quad (12)$$

Since we assume that the perturbation is adiabatic, the thermodynamic relation between the perturbed pressure and density can be written as

$$\delta p + \xi \cdot \nabla p = \frac{\Gamma p}{p + \rho} (\delta \rho + \xi \cdot \nabla \rho), \quad (13)$$

where Γ is the adiabatic index and ξ is the Lagrange displacement.

We will solve eqs.(11) and (13) by the expansion of ε . In a spherical star, the perturbations decouple into purely polar and purely axial modes for each l and m . A set of the functions $\mathcal{P}_{lm} \equiv (\delta p_{lm}, \delta \rho_{lm}, R_{lm}, V_{lm})$ describes the polar mode, while the function $\mathcal{A}_{lm} \equiv U_{lm}$ describes the axial mode. In the presence of the rotation, the mode will be mixed among the terms with different l , while the mode can still be specified by a single m (e.g., [17], [18]). The coupled equations are schematically expressed as

$$0 = [\mathcal{A}_{lm}] + \mathcal{E} \times [\mathcal{P}_{l\pm 1m}] + \mathcal{E}^2 \times [\mathcal{A}_{lm}, \mathcal{A}_{l\pm 2m}] + \cdots, \quad (14)$$

$$0 = [\mathcal{P}_{lm}] + \mathcal{E} \times [\mathcal{A}_{l\pm 1m}] + \mathcal{E}^2 \times [\mathcal{P}_{lm}, \mathcal{P}_{l\pm 2m}] + \cdots, \quad (15)$$

where the symbol $\mathcal{E}$ means some functions of order ε , and the square bracket formally represents the relation among the axial perturbation function $\mathcal{A}_{lm}$, or the polar perturbation function $\mathcal{P}_{lm}$ appeared therein. We also assume that the time variation of the oscillation is slow and proportional to Ω , i.e., $\partial_t \sim \Omega \sim O(\varepsilon)$. This is true in the r -mode oscillation, as will be confirmed soon. We look for the mode which is described by a single axial function with index (l, m) in the limit of $\varepsilon \rightarrow 0$. That is, the polar part should vanish, while the axial part becomes finite. Hence, the perturbation functions are expanded as $\mathcal{A}_{lm} = \mathcal{A}_{lm}^{(1)} + \varepsilon^2 \mathcal{A}_{lm}^{(2)} + \cdots$, $\mathcal{P}_{lm} = \varepsilon (\mathcal{P}_{lm}^{(1)} + \varepsilon^2 \mathcal{P}_{lm}^{(2)} + \cdots)$. Substituting these functions into eqs.(14) and (15), and comparing the coefficients of $\varepsilon^n (n = 0, 1, 2)$, we have

$$0 = [\mathcal{A}_{lm}^{(1)}], \quad (16)$$

$$0 = [\varepsilon \mathcal{P}_{l\pm 1m}^{(1)} + \mathcal{E} \times \mathcal{A}_{lm}^{(1)}], \quad (17)$$

$$0 = [\varepsilon^2 \mathcal{A}_{lm}^{(2)} + \mathcal{E} \times [\varepsilon \mathcal{P}_{l\pm 1m}^{(1)}] + \mathcal{E}^2 \times [\mathcal{A}_{lm}^{(1)}, \mathcal{A}_{l\pm 2m}^{(1)}] = [\varepsilon^2 \mathcal{A}_{lm}^{(2)} + \mathcal{E}^2 \times \mathcal{A}_{lm}^{(1)}], \quad (18)$$

where we have used $\mathcal{A}_{l\pm 2m}^{(1)} = 0$ and the relation (17) in the last part of (18). Equation (16) represents the axial oscillation at the lowest order, which can be specified by $U_{lm}^{(1)}$. Equation (18) is the second-order form of it, and the term $\mathcal{E}^2 \times \mathcal{A}_{lm}^{(1)}$ can be regarded as the rotational corrections. We will show the explicit forms corresponding to eqs.(16)-(18) in subsequent sections.

3 First-order solution

The leading term of eq.(11) is reduced to

$$\dot{U}_{lm}^{(1)} - im\chi U_{lm}^{(1)} = 0, \quad (19)$$

where

$$\chi = \frac{2}{l(l+1)} \varpi = \frac{2}{l(l+1)} (\Omega - \omega), \quad (20)$$

and a dot denotes time derivative in the co-rotating frame, i.e., $\dot{U}_{lm} = (\partial_t + im\Omega)U_{lm}$. The evolution of the perturbation can be solved by the Laplace transformation,

$$u(s, r) = \int_0^\infty U_{lm}^{(1)}(t, r) e^{-st} dt. \quad (21)$$

The Laplace transformation of eq.(19) is written as

$$(s + im(\Omega - \chi)) u(s, r) - f_{lm}^{(1)}(r) = 0, \quad (22)$$

where $f_{lm}^{(1)}$ describes the initial disturbance at $t = 0$. After solving u and using the inverse transformation, the solution in t -domain is easily constructed as

$$U_{lm}^{(1)}(t, r) = \int f_{lm}^{(1)}(r) \frac{e^{st}}{s + im(\Omega - \chi)} ds = f_{lm}^{(1)}(r) e^{-im(\Omega - \chi)t} H(t), \quad (23)$$

where $H(t)$ is the Heaviside step function. We will consider $t > 0$ region only, so that the function $H(t)$ may well be omitted from now on. For the Newtonian star, $m(\Omega - \chi)$ becomes a constant

$$\sigma_N = \left(1 - \frac{2}{l(l+1)}\right) m\Omega. \quad (24)$$

This is the r-mode frequency measured in the non-rotating frame ([21], [22], [24]). In the relativistic stars, ϖ is monotonically increasing function of r , $\varpi_0 \leq \varpi \leq \varpi_R$. The possible range of the r-mode frequency is spread out. If one regards eq.(23) as the sum of the Fourier component $e^{-i\sigma t}$, then the spectrum is continuous in the range

$$\left(1 - \frac{2}{l(l+1)} \frac{\varpi_R}{\Omega}\right) m\Omega \leq \sigma \leq \left(1 - \frac{2}{l(l+1)} \frac{\varpi_0}{\Omega}\right) m\Omega. \quad (25)$$

This result is the same, even if metric perturbations are considered ([19]). The time-dependence is determined, whereas the radial dependence $f_{lm}^{(1)}$ is arbitrary at this order. The function $f_{lm}^{(1)}$ in eq.(23) is constrained by the equation of motions for the polar part, as will be shown in the subsequent sections. In this manner, the perturbation scheme (16)-(18) is degenerate perturbation.

4 Second-order equation

The pressure and density perturbations are induced by the rotation, while the axial mode in the non-rotating stars is never coupled to them. We may assume that the perturbation can be specified by a single function $U_{lm}^{(1)}$ at the leading order, since general case can be described by the linear combination. The axial part induces the polar parts with index $(l \pm 1, m)$ according to the perturbation scheme (17). It is straightforward to solve $\delta p_{l \pm 1, m}$, and $\delta \rho_{l \pm 1, m}$ from two components of eq.(11) in terms of the first-order corrections and $U_{lm}^{(1)}$. The explicit results are

$$\delta p_{l \pm 1, m} = 2S_\pm \varpi U_{lm}^{(1)}, \quad (26)$$

$$\delta \rho_{l \pm 1, m} = -4S_\pm \frac{e^{-\nu/2}}{\nu'} (e^{\nu/2} \varpi U_{lm}^{(1)})' + 2T_\pm \frac{e^\nu}{r^2 \nu'} (r^2 \varpi e^{-\nu})' U_{lm}^{(1)}, \quad (27)$$

where $S_+ = \frac{l}{l+1} Q_+$, $S_- = \frac{l+1}{l} Q_-$, $T_+ = l Q_+$, $T_- = -(l+1) Q_-$, $Q_+ = \sqrt{\frac{(l+1)^2 - m^2}{(2l+1)(2l+3)}}$ and $Q_- = \sqrt{\frac{l^2 - m^2}{(2l-1)(2l+1)}}$. We here denote a derivative with respect to r by a prime. The lowest order form of eq.(13) is also decoupled into the equation of each l, m component,

$$\delta \dot{p}_{l \pm 1, m} - C^2 \delta \dot{\rho}_{l \pm 1, m} = AC^2 e^\nu \left(e^{-\lambda} R_{l \pm 1, m} - \frac{3im\xi_2}{r^2} Q_\pm U_{lm}^{(1)} \right), \quad (28)$$

with $C^2 = \frac{\Gamma p_0}{p_0 + \rho_0}$ and $A = \frac{\rho'_0}{p_0 + \rho_0} - \frac{p'_0}{\Gamma p_0}$.

In eq.(28), the displacement ξ_2 represents the quadrupole deformation of the stationary star, and is related to the quantities of $O(\varepsilon^2)$ as $\xi_2 = -\frac{2}{\nu'}(h_2 + \frac{1}{3}\varpi^2 r^2 e^{-\nu})$. The region for $A < 0$ is convectively unstable, while that for $A > 0$ is stably stratified. We here assume $A \neq 0$ in the following calculations, but will separately consider the case $A = 0$ in Section 5. From eq.(28), the function $R_{l\pm 1m}$ is solved by $U_{lm}^{(1)}$ for the region $A \neq 0$. The function $V_{l\pm 1m}$ describing the horizontal motion is calculated from the remaining component of eq.(11). The explicit form is given by

$$[l'(l' + 1)V_{l'm}]_{l'=l\pm 1} = S_{\pm}(v_2 + l(l+1)r^2\Omega e^{-\nu}\dot{U}_{lm}^{(1)}) + T_{\pm}(v_1 + \frac{1}{2}r^2\varpi e^{-\nu}\dot{U}_{lm}^{(1)}) + l(l+1)Q_{\pm}v_0, \quad (29)$$

where

$$v_2 = 4e^{-(3\nu+\lambda)/2} \left[\frac{r^2 e^{\lambda/2}}{A\nu'} \left((e^{\nu/2}\varpi\dot{U}_{lm}^{(1)})' + \frac{\nu'}{2C^2}(e^{\nu/2}\varpi\dot{U}_{lm}^{(1)}) \right) \right]' - \left(\frac{4r^2 e^{-3\nu/2}}{\nu'} \right) (e^{\nu/2}\varpi\dot{U}_{lm}^{(1)})', \quad (30)$$

$$v_1 = 2e^{-(3\nu+\lambda)/2} \left[\frac{e^{(3\nu+\lambda)/2}}{A\nu'} (r^2\varpi e^{-\nu})' \dot{U}_{lm}^{(1)} \right]' - \left[\frac{2e^{-\nu/2}}{\nu'} (r^2\varpi e^{-\nu/2})' + \Omega r^2 e^{-\nu} \right] \dot{U}_{lm}^{(1)}, \quad (31)$$

$$v_0 = \frac{3}{2} \frac{\xi_2}{\varpi} \dot{U}_{lm}^{(1)\nu} - \left[\frac{r^2 e^{-\nu}}{2} (\varpi + 2\Omega) - \frac{3}{2} \frac{e^{\lambda} m_2}{\varpi r} - \frac{3}{2} e^{-(\nu+\lambda)/2} \left[e^{(\nu+\lambda)/2} \frac{\xi_2}{\varpi} \right]' \right] \dot{U}_{lm}^{(1)}. \quad (32)$$

In eq.(29), we have used eq.(19) to simplify it.

These corrections in the polar functions affect the axial parts with indices $(l \pm 2, m)$ and (l, m) . We are interested in the term with index (l, m) as the corrections to the leading equation. In deriving the axial equation with these corrections, the terms up to $O(\varepsilon^3)$ in the background field also affect it. We include both corrections to the axial equation and have the following form,

$$0 = \dot{U}_{lm}^{(2)} - im\chi U_{lm}^{(2)} + \mathcal{L}[\dot{U}_{lm}^{(1)}], \quad (33)$$

where $\mathcal{L}$ is the Sturm-Liouville differential operator defined by

$$\mathcal{L}[\dot{U}_{lm}^{(1)}] = 8c_3\varpi e^{(-\lambda/2-\nu)}(\rho_0 + p_0) \left[\frac{r^2 e^{(\lambda-\nu)/2}}{A\nu'(\rho_0 + p_0)} (e^{\nu/2}\varpi\dot{U}_{lm}^{(1)})' \right]' - (F + G)\dot{U}_{lm}^{(1)}, \quad (34)$$

$$F = -4c_3\varpi^2 e^{(-\lambda/2-\nu)} \left(\frac{r^2 e^{\lambda/2} \rho'_0}{A p'_0} \right)' - \frac{4c_2\varpi^2 e^{(-\lambda-3\nu)/2}}{\rho_0 + p_0} \left[\frac{e^{(\lambda+3\nu)/2}}{A\nu'\varpi} (\varpi r^2 e^{-\nu})' (\rho_0 + p_0) \right]' + \left(\frac{2c_1 e^{\nu}}{A\nu' r^2} \right) [(r^2\varpi e^{-\nu})']^2, \quad (35)$$

$$G = -8c_3\varpi^2 e^{(-\lambda/2-\nu)} \left(\frac{r^2 e^{\lambda/2}}{\nu'} \right)' + \frac{4c_2}{\nu' r^2} (\varpi^2 r^4 e^{-\nu})' - 3c_1 \left[r e^{-\lambda/2} \left(\frac{e^{\lambda/2} \xi_2}{r} \right)' - \frac{3}{2} \frac{\varpi'}{\varpi} \xi_2 - k_2 + \frac{e^{\lambda}}{r} m_2 + \frac{5W_3}{\varpi} \right] - \frac{3m^2}{l(l+1)} \left[\frac{\xi_2}{r} + \frac{1}{2} \frac{\varpi'}{\varpi} \xi_2 + k_2 - \frac{5W_3}{\varpi} \right] - \left(\frac{W_1}{\varpi} + \frac{6W_3}{\varpi} \right), \quad (36)$$

$$c_n = \frac{l+1}{l^n} Q_-^2 + (-1)^{n-1} \frac{l}{(l+1)^n} Q_+^2. \quad (37)$$

In order to solve eq.(33), we introduce a complete set of functions as for the operator $\mathcal{L}$,

$$\mathcal{L}[y_{\kappa}] + \kappa y_{\kappa} = 0. \quad (38)$$

where $-\kappa$ is the eigenvalue and the eigenfunction $y_\kappa(r)$ is characterized by κ , e.g., the number of the nodes. The eigenvalue is real number of $O(\varepsilon^2)$, since $\mathcal{L}$ is the Hermitian operator of $O(\varepsilon^2)$. The eigenvalue problem is solved with appropriate boundary conditions. For example, the function should satisfy the regularity condition at the center and the Lagrangian pressure should vanish at the stellar surface. Certain initial data $f_{lm}^{(1)}$ in eq.(23) can be decomposed by the set. We may restrict our consideration to a single function labeled by κ , since the general case is described by discrete sum or integration over a certain range. By putting $y_\kappa = im\chi f_{lm;\kappa}^{(1)}$, we have $\mathcal{L}[\dot{U}_{lm}^{(1)}] = \mathcal{L}[im\chi f_{lm;\kappa}^{(1)} e^{-im(\Omega-\chi)t}] = -im\kappa\chi f_{lm;\kappa}^{(1)} e^{-im(\Omega-\chi)t}$, where $(e^{-im(\Omega-\chi)t})'$ gives the first order correction and is neglected here. Using $f_{lm;\kappa}^{(1)}$, we can integrate eq.(33) with respect to t , and have the function $U_{lm}^{(2)}(t, r)$ of $O(\varepsilon^2)$ as $U_{lm}^{(2)} = (im\kappa\chi t f_{lm;\kappa}^{(1)} + f_{lm}^{(2)})e^{-im(\Omega-\chi)t}$, where the function $f_{lm}^{(2)}$ of $O(\varepsilon^2)$ is unknown at this order. The sum of the first and second order forms is approximated as

$$U_{lm}^{(1)} + U_{lm}^{(2)} = \left[(1 + im\kappa\chi t) f_{lm;\kappa}^{(1)} + f_{lm}^{(2)} \right] e^{-im(\Omega-\chi)t} \quad (39)$$

$$= \left[f_{lm;\kappa}^{(1)} + f_{lm}^{(2)} \right] e^{-im(\Omega-(1+\kappa)\chi)t}, \quad (40)$$

where we have exploited the freedom of $f_{lm}^{(2)}$ to eliminate the unphysical growing term in eq.(39). The value κ originated from the fixing of the initial data becomes evident for large t , since the accumulation of small effects from the higher order terms is no longer neglected. As a result, the frequency should be adjusted with the second-order correction to be a good approximation even for slightly large t , as in eq.(40). This renormalization of the frequency is closely related to treating t as strained coordinate in the perturbation method. (See e.g. [12].) In this way, the specification of the initial data at the leading order has influence on the second-order correction κ .

5 Oscillations in barotropic stars

The structure of the neutron stars is almost approximated to be barotropic. The pulsation equation is quite different from that of non-barotropic case, as in the Newtonian pulsation theory. The relation (28) for the case $A = 0$ is replaced by

$$\delta p_{l\pm 1m} = C^2 \delta \rho_{l\pm 1m} = \frac{\rho'_0}{\rho'_0} \delta \rho_{l\pm 1m} = -\frac{\nu'(p_0 + \rho_0)}{2\rho'_0} \delta \rho_{lm}. \quad (41)$$

In the last part of eq.(41), we have used hydrostatic equation of the non-rotating star. In this case, the function $R_{l\pm 1m}$ is never determined through eq.(28) unlike in $A \neq 0$ case, but rather we have two restrictions to a single function $U_{lm}^{(1)}$. These conditions are never satisfied simultaneously unless for $m = \pm l$, in which one condition is trivial, $\delta p_{l-1m} = \delta \rho_{l-1m} = 0$ due to $Q_- = 0$. The other condition for $m = \pm l$ becomes $(e^{\nu/2} \varpi U_{lm}^{(1)})' + \left(\frac{\nu'}{2C^2} - \frac{l+1}{2} \frac{(\varpi r^2 e^{-\nu})'}{\varpi r^2 e^{-\nu}} \right) e^{\nu/2} \varpi U_{lm}^{(1)} = 0$. Substituting the form (23) into this and neglecting the higher order term due to $(e^{-im(\Omega-\chi)t})'$, we have the same differential equation for $f_{lm}^{(1)}$. The integration with respect to r results in

$$\begin{aligned} U_{lm}^{(1)}(t, r) &= f_{lm;\kappa}^{(1)} e^{-im(\Omega-\chi)t} \\ &= \left[N_0(\rho_0 + p_0) r^{l+1} e^{-\nu} (\varpi e^{-\nu})^{(l-1)/2} \right] e^{-im(\Omega-\chi)t}, \end{aligned} \quad (42)$$

where N_0 is a normalization constant. In this way, the function of the lowest order is determined. The corresponding the 3-velocity at $t \rightarrow +0$ is given by

$$\dot{\xi}_\phi = \delta v_\phi = N_0 r^{l+1} (\varpi e^{-\nu})^{(l-1)/2}. \quad (43)$$

As shown by [10], the canonical energy of the perturbation is negative for the Lagrange displacement ξ_ϕ , and the solution therefore means unstable if the gravitational radiation reaction sets in.

We now specify $R_{l\pm 1m}$ of $O(\varepsilon^2)$ to proceed to the pulsation equation with the second order corrections. The function $X_{l\pm 1m}$ describing the radial motion is introduced as $R_{l\pm 1m} = X_{l\pm 1m} + \frac{3im\varepsilon^2 \ell_2}{r^2} Q_\pm U_{lm}^{(1)}$. The

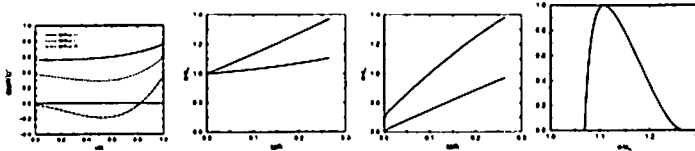


Fig.1

Fig.2

Fig.3

Fig.4

function $X_{l\pm 1m}$ is arbitrary at this order, but we have two special cases. One is $X_{l\pm 1m} = 0$, which is the limiting case for eq.(28) in a sense. The other is chosen so as to vanish the Lagrangian change of the pressure within the entire star. The condition corresponds to $X_{l\pm 1m} = 4S_{\pm} \omega e^{\lambda-\nu} \dot{U}_{lm}^{(1)}/\nu'$. For the first choice, the Lagrangian change of the pressure never vanishes at the surface unless ρ_0 vanishes there. The second one is the usual way treated in the Newtonian r-mode, as shown by [16]. With the second choice of $X_{l\pm 1m}$ and eq.(42), the pulsation equation for $m = \pm l$ leads to

$$\dot{U}_{lm}^{(2)} - im\chi U_{lm}^{(2)} - G\dot{U}_{lm}^{(1)} = 0, \quad (44)$$

where G is defined in eq.(36). This equation is solved for $U_{lm}^{(2)}$ as in Section 4. The solution up to $O(\epsilon^2)$ is written as

$$U_{lm}^{(1)} + U_{lm}^{(2)} = \left[f_{lm;\bullet}^{(1)} + f_{lm}^{(2)} \right] e^{-im(\Omega - (1+G)\chi)t}, \quad (45)$$

where the function $f_{lm}^{(2)}$ of $O(\epsilon^2)$ is unknown at this order. This expression (45) is formally the same as in eq.(40) for the non-barotropic case with $\kappa = G$. The second-order correction in the frequency however depends on the position owing to the particular choice of the function $f_{lm;\bullet}^{(1)}$.

We show the second-order rotational correction G for $l = m = 2$ mode in Fig.1. We adopt the polytropic stellar model with index $n = 1$.

In Fig.2, we show the frequency range of the r-mode oscillations in the first-order rotational calculation. The frequency is a single value σ_N , given by eq.(24) in the Newtonian limit.

In Fig. 3, the allowed range is shown including the second-order corrections for the extreme case $\Omega^2 = M/R^3$.

We examine the effect on the spectrum of the gravitational waves emitted by the r-mode oscillations. We for simplicity neglect all relativistic corrections except in the frequency, and estimate the spectrum by the Newtonian radiation theory. The dimensionless gravitational amplitude $h(t)$ at infinity is determined by evaluating the time variation of the current multipole moment, $h(t) \sim d^l S_{lm}/dt^l$ ([25]). The current multipole moment S_{lm} for $l = m = 2$ mode and the velocity (43) is given by $S_{22} = N \int \rho_0 (\omega e^{-\nu})^{1/2} e^{-2i(\Omega-\chi)t} r^6 dr$, where the normalization N is also included the constant from the integration over angular parts. The Fourier component $h(\sigma)$ can be expressed as $h(\sigma) = \int h(t) e^{+i\sigma t} dt \propto \int \rho_0 (\omega e^{-\nu})^{1/2} (\Omega - \chi)^2 \delta(\sigma - 2(\Omega - \chi)) r^6 dr$. The spectrum has a finite line breadth as shown in Fig.4. The spectrum is non-zero only for $\sigma = \sigma_N$ in the Newtonian treatment, but is broad in the relativistic one. The width at the half maximum is $\Delta\sigma/\sigma_N \sim 0.1$.

6 Discussion

In this paper, we have calculated the r-mode oscillations in the relativistic rotating stars, neglecting the gravitational perturbations. The evolution can be described by oscillatory solutions which are neutral, i.e., never decay or grow in the absence of the dissipation. The oscillation is described not by a single frequency, but by the frequencies of a broad range, unlike in the Newtonian case. The reason is that the local rotation rate depends on the position due to the frame dragging effect even for uniform rotation. The r-mode oscillation frequency forms a continuous spectrum within a certain range. This property is distinguished from the well-known stellar oscillation modes such as the polar f-, p-modes, in which the

spectrum of frequency is discrete. The r-mode frequencies lie in the unstable region for the gravitational radiation reaction instability, but it will be an important issue whether or not the different spectrum of the unstable modes leads to different growth, e.g., in the non-linear regime.

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Constructing hyperbolic systems in the Ashtekar formulation of general relativity

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Abstract

Hyperbolic formulation of equations of motion is an essential technique for proving well-posedness of the Cauchy problem of a system, and is also helpful for stable long time evolution in numerical applications. We present three kinds of hyperbolic systems in the Ashtekar formulation of general relativity for Lorentzian vacuum spacetime. We show several (I) weakly hyperbolic, (II) diagonalizable hyperbolic and (III) symmetric hyperbolic systems, with discussions of required gauge conditions and reality conditions. In order to construct a first-order symmetric hyperbolic system, we add terms from the constraint equations to the evolution equations with appropriate combinations, which is the same technique used by Iriondo, Leguizamón and Reula. However our system is different from theirs in the points that we primarily use Hermiticity of a characteristic matrix of the system to define our system *symmetric*, and in the way of treating reality conditions.

1 Introduction

Hyperbolic formulation of the Einstein equation is the one of the main research areas in general relativity [1]. This formulation is used in the proof of the existence, uniqueness and stability (well-posedness) of the solutions of the Einstein equation by analytical methods [2]. So far, several first order hyperbolic formulations are proposed; some of them are flux conservative [3], symmetrizable [4], or symmetric hyperbolic system [5, 6]. The recent interest in hyperbolic formulation arises from its application to numerical relativity. One of the expected advantages is the existence of the characteristic speeds of the system, with which we may treat the numerical boundary with appropriate condition. Some numerical tests have been reported along this direction [7].

Recently, Iriondo, Leguizamón and Reula (ILR) [8, 9] discuss a symmetric hyperbolic system in the Ashtekar formulation [10] of general relativity. Ashtekar's formulation has many advantages in the treatment of gravity. By using his special pair of variables, the constraint equations which appear in the theory become low-order polynomials, and the theory has the correct form for gauge theoretical features. These suggest possibilities for treating a quantum description of gravity nonperturbatively. The classical applications of the Ashtekar's formulation has also been discussed by several authors. For example, we [12] discussed the reality conditions for metric and triad and proposed new set of variables from the point of Lorentzian dynamics. We [13] also showed an example of passing degenerate point in 3-space by losing reality condition locally.

In this *proceedings* ³, we show several hyperbolic systems in the Ashtekar formulation of general relativity for Lorentzian vacuum spacetime. We categorize hyperbolic systems into three levels: (I) system has all real eigenvalues, (II) system is diagonalizable, and (III) symmetric hyperbolic system, and discuss gauge conditions and reality conditions required by each systems.

ILR says, in their first short paper [8], that they present a symmetric hyperbolic system in the Ashtekar formulation. However, we do not think that their discussion is clear in the following three points. First, they used *anti*-Hermiticity of the principal symbol for defining their system *symmetric*. We, however, think that this does not derive Hermiticity of the characteristic matrix [J below in eq. (12), B in [8]], since they do not define their vector k_a explicitly. We rather use Hermiticity of the characteristic matrix primarily to construct a *symmetric hyperbolic* system. Second, they did not mention the consistency of their system with the reality conditions which are crucial in the study of the Lorentzian dynamics in the Ashtekar variables. Third, they did not discuss the characteristic structure of the system, which should be shown in the normal hyperbolic formulations. Our discussion covers all these matters. We cleared these points in our *Letter* [14], and we again include our discussion in §5.

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ILR, in their second paper [9], expand their previous discussion [8] into how we can handle reality conditions during evolution. They impose metric is real-valued (metric reality condition), and propose to use a freedom of internal rotation to set up their soldering form so as to it forms anti-Hermitian, which is their basis to characterize a system *symmetric*. However, since they do not count a new gauge freedom, *triad lapse* $\mathcal{A}_0^a$ ([12] and will be mentioned in §2), for dynamical evolutions in the Ashtekar formulation, their discussion of consistency with reality condition, especially its secondary condition, remain incomplete. As we will describe in §5, our symmetric hyperbolic system requires reality condition on triad (triad reality condition), and we show how gauge variables will be constrained from its secondary condition.

The construction of this paper is as follows. After giving a brief review of Ashtekar's variables and reality conditions in §2, we define three kinds of hyperbolic systems in §3. The following §4 and §5 are devoted for constructing hyperbolic systems. Summary is in §6.

2 Ashtekar's formulation

The key feature of Ashtekar's formulation of general relativity [10] is the introduction of a self-dual connection as one of the basic dynamical variables. Let us write⁴ the metric $g_{\mu\nu}$ using the tetrad, e_μ^I , and define its inverse, E_I^μ , by $g_{\mu\nu} = e_\mu^I e_\nu^J \eta_{IJ}$ and $E_I^\mu := e_\nu^\mu g^{\nu\rho} \eta_{\rho I}$. We define $SO(3, \mathbb{C})$ self-dual and anti self-dual connections ${}^\pm \mathcal{A}_\mu^a := \omega_\mu^{ab} \mp (i/2)\epsilon_{bc} \omega_\mu^{bc}$, where ω_μ^{IJ} is a spin connection 1-form (Ricci connection), $\omega_\mu^{IJ} := E^{I\nu} \nabla_\mu e_\nu^J$. Ashtekar's plan is to use only a self-dual part of the connection ${}^+ \mathcal{A}_\mu^a$ and to use its spatial part ${}^+ \mathcal{A}_i^a$ as a dynamical variable. Hereafter, we simply denote ${}^+ \mathcal{A}_\mu^a$ as $\mathcal{A}_\mu^a$.

The lapse function, N , and shift vector, N^i , both of which we treat as real-valued functions, are expressed as $E_0^\mu = (1/N, -N^i/N)$. This allows us to think of E_0^μ as a normal vector field to Σ spanned by the condition $t = x^0 = \text{const.}$, which plays the same role as that of Arnowitt-Deser-Misner formulation. Ashtekar treated the set $(\mathcal{A}_i^a, \tilde{E}_a^i)$ as basic dynamical variables, where $\tilde{E}_a^i$ is an inverse of the densitized triad defined by $\tilde{E}_a^i := e E_a^i$ where $e := \det e_\mu^a$ is a density. This pair forms the canonical set.

In the case of pure gravitational spacetime, the Hilbert action takes the form

$$S = \int d^4x [(\partial_t \mathcal{A}_i^a) \tilde{E}_a^i + (i/2) N \tilde{E}_a^i \tilde{E}_b^j F_{ij}^c \epsilon^{ab} - \Lambda N \det \tilde{E} - N^i F_{ij}^a \tilde{E}_a^j + \mathcal{A}_0^a \mathcal{D}_i \tilde{E}_a^i], \quad (1)$$

where $N := e^{-1} N$, Λ is the cosmological constant, $\mathcal{D}_i \tilde{E}_a^i := \partial_i \tilde{E}_a^i - i \epsilon_{ab}^c \mathcal{A}_i^b \tilde{E}_c^i$, and $\det \tilde{E}$ is defined to be $\det \tilde{E} = (1/6) \epsilon^{abc} \tilde{E}_{ijk} \tilde{E}_a^i \tilde{E}_b^j \tilde{E}_c^k$, where $\epsilon_{ijk} := \epsilon_{abc} e_\mu^a e_\nu^b e_\rho^c \epsilon_{ijk}^\mu$ and $\tilde{\epsilon}_{ijk} := e^{-1} \epsilon_{ijk}$ ⁵. Varying the action with respect to the non-dynamical variables N , N^i and $\mathcal{A}_0^a$ yields the constraint equations,

$$C_H := (i/2) \epsilon^{ab} \tilde{E}_a^i \tilde{E}_b^j F_{ij}^c - \Lambda \det \tilde{E} \approx 0, \quad (2)$$

$$C_{Mi} := -F_{ij}^a \tilde{E}_a^j \approx 0, \quad (3)$$

$$C_{Ga} := \mathcal{D}_i \tilde{E}_a^i \approx 0, \quad (4)$$

where $F_{\mu\nu}^a := (d\mathcal{A}^a)_{\mu\nu} - (i/2) \epsilon_{bc}^a (\mathcal{A}^b \wedge \mathcal{A}^c)_{\mu\nu} = \partial_\mu \mathcal{A}_\nu^a - \partial_\nu \mathcal{A}_\mu^a - i \epsilon^{abc} \mathcal{A}_\mu^b \mathcal{A}_\nu^c$ is the curvature 2-form. The equations of motion for the dynamical variables $(\mathcal{A}_i^a$ and $\tilde{E}_a^i)$ are

$$\partial_t \mathcal{A}_i^a = -i \epsilon^{ab} N \tilde{E}_b^j F_{ij}^c + N^j F_{ji}^a + \mathcal{D}_i \mathcal{A}_0^a + e \Lambda N e_i^a, \quad (5)$$

$$\partial_t \tilde{E}_a^i = -i \mathcal{D}_j (\epsilon^{cb} N \tilde{E}_c^j \tilde{E}_b^i) + 2 \mathcal{D}_j (N^{[j} \tilde{E}_a^{i]}) + i \mathcal{A}_0^b \epsilon_{ab}^c \tilde{E}_c^i, \quad (6)$$

where $\mathcal{D}_j X_a^{ji} := \partial_j X_a^{ji} - i \epsilon_{ab}^c \mathcal{A}_j^b X_c^{ji}$, for $X_a^{ij} + X_a^{ji} = 0$.

In order to construct metric variables from the variables $(\mathcal{A}_i^a, \tilde{E}_a^i, N, N^i)$, we first prepare tetrad E_I^μ as $E_0^\mu = (1/eN, -N^i/eN)$ and $E_a^\mu = (0, \tilde{E}_a^i/e)$. Using them, we obtain metric $g^{\mu\nu}$ such that $g^{\mu\nu} := E_I^\mu E_J^\nu \eta^{IJ}$. Notice that in general the right hand side of this equation is not real. To ensure the metric is real-valued, we need to impose real lapse and shift vectors together with two conditions (metric reality condition);

$$\text{Im}(\tilde{E}_a^i \tilde{E}^{ja}) = 0, \quad (7)$$

$$\text{Re}(\epsilon^{abc} \tilde{E}_a^i \tilde{E}_b^j \tilde{E}_c^k) = 0 \quad (8)$$

⁴We use $\mu, \nu = 0, \dots, 3$ and $i, j = 1, \dots, 3$ as spacetime indices, while $I, J = (0), \dots, (3)$ and $a, b = (1), \dots, (3)$ are $SO(1, 3)$, $SO(3)$ indices respectively. We raise and lower $\mu, \nu, \dots$ by $g^{\mu\nu}$ and $g_{\mu\nu}$ (Lorentzian metric); $I, J, \dots$ by $\eta^{IJ} = \text{diag}(-1, 1, 1, 1)$ and η_{IJ} ; $i, j, \dots$ by γ^{ij} and γ_{ij} (3-metric). We use volume forms ϵ_{abc} ; $\epsilon_{abc} \epsilon^{abc} = 3!$.

⁵When $i = 1, j = 2$ and $k = 3$, we have $\epsilon_{ijk} = e$, $\tilde{\epsilon}_{ijk} = 1$, $\epsilon^{ijk} = e^{-1}$, $\tilde{\epsilon}^{ijk} = 1$.

where the latter comes from the secondary condition of reality of the metric $\text{Im}\{\partial_t(\tilde{E}_a^i \tilde{E}^{ja})\} = 0$ [11], and we assume $\det \tilde{E} > 0$ (see [12]). For later convenience, we also prepare stronger reality conditions. These conditions are

$$\text{Im}(\tilde{E}_a^i) = 0, \quad (9)$$

$$\text{Im}(\partial_t \tilde{E}_a^i) = 0 \quad (10)$$

and we call them the “primary triad reality condition” and the “secondary triad reality condition”, respectively. Using the equations of motion of $\tilde{E}_a^i$, the gauge constraint (2)-(4), the metric reality conditions (7), (8) and the primary condition (9), we see that (10) is equivalent to [12]

$$\text{Re}(\mathcal{A}_0^a) = \partial_i(N) \tilde{E}^{ia} + (1/2e) e_i^b N \tilde{E}^{ja} \partial_j \tilde{E}_b^i + N^i \text{Re}(\mathcal{A}_i^a) = \partial_i(N) E^{ia} + N^i \text{Re}(\mathcal{A}_i^a). \quad (11)$$

From this expression we see that the secondary triad reality condition restricts the three components of “triad lapse” vector $\mathcal{A}_0^a$. Therefore (11) is not a restriction on the dynamical variables ($\mathcal{A}_i^a$ and $\tilde{E}_a^i$) but on the slicing, which we should impose on each hypersurface. Thus the secondary triad reality condition does not restrict the dynamical variables any further than the secondary metric condition does.

Throughout this paper, we basically impose metric reality condition. We assume that initial data of ($\tilde{E}_a^i, \mathcal{A}_i^a$) for evolution are solved so as to satisfy all three constraint equations and metric reality condition (7) and (8). Practically, this is obtained by, for example, solving ADM constraints and transform a set of initial data to Ashtekar’s notation.

3 Definitions of hyperbolic systems

3.1 Three definitions

We start by defining hyperbolic systems, which are used in this paper.

Definition 1 We assume a certain pair of (complex) variables u_α ($\alpha = 1, \dots, n$) forms a first-order (quasi-linear) partial differential equation system,

$$\partial_t u_\alpha = J^{l\beta}_\alpha(u) \partial_l u_\beta + K_\alpha(u), \quad (12)$$

where J (characteristic matrix) and K are functions of u but do not include any derivatives of u . We say that the system (12) is

- (I). **weakly hyperbolic**, if all the eigenvalues of the characteristic matrix are real [16].
- (II). **diagonalizable hyperbolic**, if the characteristic matrix is diagonalizable and has all real eigenvalues [17].
- (III). **symmetric hyperbolic**, if the characteristic matrix is a Hermitian matrix [6, 18].

Here we state each definition more concretely. We treat $J^{l\beta}_\alpha$ as a $n \times n$ matrix when l is fixed. Therefore, we say λ^l is an eigenvalue of $J^{l\beta}_\alpha$ when the eigen-equation, $\det(J^{l\beta}_\alpha - \lambda^l \delta^\beta_\alpha) = 0$, is satisfied. The eigenvectors, p^α , are given by solving $J^{l\alpha}_\beta p^\beta = \lambda^l p^\alpha$.

The weakly hyperbolic system, (I), is obtained when J is *real spectral*, that is, when this eigen-equation can be divided by n real first-degree factors. For any single equation system, the Cauchy problem under weak hyperbolicity is not, in general, C^∞ well-posed, while it is solvable in the class of the real analytic functions and in some suitable Gevrey classes, provided that the coefficients of the principal part are sufficiently smooth.

The diagonalizable hyperbolic system, (II), is obtained when J is *real diagonalizable*, that is, when there exists complex regular matrix P^l such that $((P^l)^{-1})^\alpha_\gamma J^{l\gamma}_\delta P^{l\delta}_\beta$ is real diagonal matrix for every l . This system is often used as a model in the studies of well-posedness in coupled linear hyperbolic system, with a purpose of generalization of the known facts in the symmetric hyperbolic system. In order to define the symmetric hyperbolic system, (III), we need to declare an inner product $\langle u|u \rangle$ to judge that $J^{l\beta}_\alpha$ is Hermitian. In other words, we are required to define the way of lowering index α of u^α . We say $J^{l\beta}_\alpha$ is Hermitian with respect to this index rule, when $J^{l\beta}_\alpha = \bar{J}^l_{\alpha\beta}$, where overhead bar denotes complex conjugate.

Any Hermitian matrix is real diagonalizable and the eigenvalues of real diagonalizable matrix are all real; therefore (III) $\in$ (II) $\in$ (I). We also note that there are other definitions of hyperbolicity⁶, but these three (I)-(III) are the only systems that we treat in this paper.

The symmetric system gives us the energy integral inequalities, which are the primary tools for studying well-posedness of the system. As was discussed by Geroch [19], most physical systems are expressed as symmetric hyperbolic systems.

⁶For example, we can say that the system is *strictly hyperbolic* or *effectively hyperbolic* if all eigenvalues of the characteristic matrix are real and distinct (and non-zero for the latter). These definitions are stronger than (II), but no inclusion relation with (III).

3.2 Characteristic matrix for Ashtekar's variables

We shall see how the above definitions of hyperbolic systems can be applied for Ashtekar's equations of motion (6) and (5). Since both dynamical variables $\tilde{E}_a^i$ and $\mathcal{A}_i^a$ have 9 components each (spatial index: $i = 1, 2, 3$ and SO(3) index: $a = (1), (2), (3)$), the combined set of variables, $u^\alpha = (\tilde{E}_a^i, \mathcal{A}_i^a)$, has 18 components. Ashtekar's formulation itself is in the first-order (quasi-linear) form in the sense of (12), but not a symmetric hyperbolic form.

We start by writing the principal part of the Ashtekar's evolution equations as

$$\partial_t \begin{bmatrix} \tilde{E}_a^i \\ \mathcal{A}_i^a \end{bmatrix} \cong \begin{bmatrix} A^{i \ b i}_{a \ j} & B^{i \ j}_{a \ b} \\ C^{i a b}_{i \ j} & D^{i a}_{b \ j} \end{bmatrix} \partial_t \begin{bmatrix} \tilde{E}_b^j \\ \mathcal{A}_j^b \end{bmatrix}, \quad (13)$$

where $\cong$ means that we extracted only the terms which appear in the principal part of the system, and we name its components as A, B, C and D for later convenience.

The eigen-equation of characteristic matrix becomes

$$\det \begin{pmatrix} A^{i \ b i}_{a \ j} - \lambda^i \delta_a^b \delta_j^i & B^{i \ j}_{a \ b} \\ C^{i a b}_{i \ j} & D^{i a}_{b \ j} - \lambda^i \delta_b^a \delta_j^i \end{pmatrix} = 0. \quad (14)$$

If $B^{i \ j}_{a \ b}$ and $C^{i a b}_{i \ j}$ vanish, the spectrum of characteristic matrix is composed of that of A and that of D . Then, the characteristic matrix is diagonalizable if A and D are diagonalizable.

The lowering rule of α of u^α is naturally defined as follows. Corresponding inner product takes the form $\langle u|u \rangle := u_\alpha \bar{u}^\alpha$. According to this rule, we say the characteristic matrix is a Hermitian when

$$0 = A^{i a b i j} - \bar{A}^{i b a j i}, \quad (15)$$

$$0 = D^{i a b i j} - \bar{D}^{i b a j i}, \quad (16)$$

$$0 = B^{i a b i j} - \bar{C}^{i b a j i}. \quad (17)$$

4 Constructing hyperbolic systems with original equations of motion

In this section, we consider which form of hyperbolicity applies to the original equations of motion (5) and (6) under metric reality condition (§4.1) or under triad reality condition (§4.2).

4.1 Under metric reality condition (system Ia and IIa)

As the first approach, we take the equations of motion (5) and (6) with metric reality conditions (7) and (8). The principal term of (5) and (6) become

$$\partial_t \tilde{E}_a^i \cong [-i \epsilon^{bc}{}_a N \delta_j^i \tilde{E}_c^j - i \epsilon^{cb}{}_a N \tilde{E}_c^j \delta_j^i + N^i \delta_j^i \delta_a^b - N^i \delta_j^i \delta_a^b] (\partial_t \tilde{E}_b^j),$$

$$\partial_t \mathcal{A}_i^a \cong [+i \epsilon^{a \ c}{}_b N \tilde{E}_c^j \delta_j^i - i \epsilon^{a \ c}{}_b N \tilde{E}_c^j \delta_j^i + N^i \delta_b^a \delta_j^i - N^j \delta_b^a \delta_j^i] (\partial_t \mathcal{A}_j^b)$$

where $\cong$ means that we extracted the terms which contribute to characteristic matrix. The terms in the notation of (13) become

$$A^{i a b i j} = -i \tilde{N} \epsilon^{a b c} \tilde{E}_c^i \gamma^{i j} + i \tilde{N} \epsilon^{a b c} \tilde{E}_c^i \gamma^{i j} + N^i \delta^{a b} \gamma^{i j} - N^i \delta^{a b} \gamma^{i j}, \quad (18)$$

$$B^{i a b i j} = C^{i a b i j} = 0, \quad (19)$$

$$D^{i a b i j} = +i \tilde{N} \epsilon^{a b c} \tilde{E}_c^j \gamma^{i i} - i \tilde{N} \epsilon^{a b c} \tilde{E}_c^j \gamma^{i j} + N^i \delta^{a b} \gamma^{i j} - N^j \delta^{a b} \gamma^{i i}. \quad (20)$$

We get the 18 eigenvalues of the characteristic matrix, all of which are independent from the choice of a triad: 0 (multiplicity=6), N^i (4), $N^i \pm N \sqrt{\gamma^{ii}}$ (4 each) where we do not take the sum in γ^{ii} (hereafter for eigenvalues and related discussions). Therefore we can say that this system constitutes weakly hyperbolic, (I).

We note that this system is not (II) in general, because this is not diagonalizable, for example, when $N^i = 0$. We classify this system to (I), and call this *system Ia*, hereafter.

We can prove that the necessary and sufficient conditions to make this system diagonalizable, (II), are that the gauge conditions

$$N^i \neq 0 \text{ nor } \pm N \sqrt{\gamma^{ii}}, \quad \text{and } \gamma^{ii} > 0 \quad (21)$$

where the last one is the positive definiteness of γ^{II} ⁷. The components of the characteristic matrix are the same as *system Ia*, so all eigenvalues are equivalent with *system Ia*. We can also show that this system is not Hermitian hyperbolic. Therefore we classify the system $[Ia + (21)]$ to real diagonalizable hyperbolic, (II), and call this set as *system IIa*. However, we will show in the next section that real diagonalizable hyperbolic system can also be constructed with less strict gauge conditions by modifying right-hand-side of equations of motion (*system IIb*).

4.2 Under triad reality condition (system Ib)

Next, we consider systems of the original equations of motion (5) and (6), with the triad reality condition. Since this reality condition requires the additional (11) as the secondary condition (that is, to preserve the reality of triad during time evolution), in order to be consistent with this requirement and to avoid the system becoming second order in fundamental variables, we need to set $\partial_t N = 0$. This fixes a real part of *triad lapse* gauge as $\text{Re}(\mathcal{A}_0^a) = \text{Re}(\mathcal{A}_i^a N^i)$. We naturally define its imaginary part as $\text{Im}(\mathcal{A}_0^a) = \text{Im}(\mathcal{A}_i^a N^i)$. Namely the triad lapse is fixed as $\mathcal{A}_0^a = \mathcal{A}_i^a N^i$. This gauge restriction does not affect principal part of the evolution equation for $\tilde{E}_a^i$, but requires us to add the term

$$\mathcal{D}_i \mathcal{A}_0^a \cong \partial_i \mathcal{A}_0^a = \partial_i (\mathcal{A}_j^a N^j) \cong N^j (\partial_i \mathcal{A}_j^a) = N^j \delta_a^b \delta_i^l (\partial_l \mathcal{A}_j^b)$$

to the right-hand-side of the equation of $\mathcal{A}_i^a$. That is, we need to add $N^j \delta_a^b \delta_i^l$ to D_{bi}^{la} ^j in (20),

$$D^{labij} = +i\epsilon^{abc} N \tilde{E}_c^j \gamma^{li} - i\epsilon^{abc} N \tilde{E}_c^l \gamma^{ji} + N^l \delta^{ab} \gamma^{ji},$$

and other components of the characteristic matrix remain the same [(18) and (19)]. We get the set of eigenvalues of this system as 0 (multiplicity =3), N^l (7), $N^l \pm N\sqrt{\gamma^{ll}}$ (4 each). Therefore the system is again, (I). This system is not real diagonalizable because D^l is not. So we classify this system to (I) and call this set as *system Ib*. We note that this system is not real diagonalizable for any choice of gauges. Therefore we can not construct system (II) using the same technique of constructing *system IIa*. However, as we will show in the next section, the system becomes diagonalizable (and symmetric) hyperbolic under the triad reality condition if we modified the equations of motion.

5 Constructing symmetric hyperbolic system

From the study in the previous section, we found that a set of original equations of motion in the Ashtekar formulation constitute weakly hyperbolic, (I), or diagonalizable hyperbolic, (II), systems under appropriate gauge conditions, but we also found that we can not obtain symmetric hyperbolic system, (III). In this section, we show that (III) is obtained if we modify equations of motion. We begin describing our approach without considering reality conditions, but soon we will show that triad reality condition is required for making characteristic matrix Hermitian.

We first prepare the constraints (2)-(4) as

$$C_H \cong -i\epsilon_b^{cd} \tilde{E}_c^j \tilde{E}_d^l (\partial_l \mathcal{A}_j^b), \quad (22)$$

$$C_{Mk} \cong [-\delta_k^l \tilde{E}_b^j + \delta_k^j \tilde{E}_b^l] (\partial_l \mathcal{A}_j^b), \quad (23)$$

$$C_{Ga} \cong \partial_l \tilde{E}_a^l. \quad (24)$$

We apply the same technique used by ILR to modify the equation of motion of $\tilde{E}_a^i$ and $\mathcal{A}_i^a$ by adding the constraints which weakly produce $C_H \approx 0$, $C_{Mk} \approx 0$, and $C_{Ga} \approx 0$. We also assume triad lapse $\mathcal{A}_0^a$ as

$$\partial_i \mathcal{A}_0^a \cong T^{lab}{}_{ij} \partial_l \tilde{E}_b^j + S^{la}{}_{bi}{}^j \partial_l \mathcal{A}_j^b, \quad (25)$$

where T and S are parameters which do not include derivatives of the fundamental variables. This assumptions is general for our purpose of studying the principal part of the system.

One natural way to construct symmetric hyperbolic system is to keep $B = C = 0$ and modify A and D terms in (13), so that we modify (6) using C_G and modify (5) using C_H and C_M . That is, we add the following terms in

⁷Suppose that (21) is satisfied. Then 0, N^l , $N^l \pm \sqrt{\gamma^{ll}}$ are four distinct eigenvalues and we see $\text{rank}(J^l) = 12$, $\text{rank}(J^l - N^l I) = 14$, $\text{rank}(J^l - (N^l \pm N\sqrt{\gamma^{ll}})I) = 14$. Therefore the characteristic matrix is diagonalizable. Conversely suppose that $N^l = 0$ or $N^l = \pm N\sqrt{\gamma^{ll}}$. Then we see the characteristic matrix is not diagonalizable in each case.

the equations of motion:

$$\text{modifying term to } \partial_t \tilde{E}_a^i = P^{iab} C_G^b \cong P^{iab} \partial_j \tilde{E}_b^j = (P^{iab} \delta_j^i)(\partial_t \tilde{E}_b^j) \quad (26)$$

$$\begin{aligned} \text{modifying term to } \partial_t \mathcal{A}_i^a &= +\mathcal{D}_i \mathcal{A}_0^a + Q_i^a C_H + R_i^{ja} C_{Mj} \\ &\cong [+S^{ia}{}_{bi}{}^j - iQ_i^a \epsilon^{bcd} \tilde{E}_c^j \tilde{E}_d^i - R_i^{ia} \tilde{E}_b^j + R_i^{ja} \tilde{E}_b^i](\partial_t \mathcal{A}_j^b) + T^{iab}{}_{ij} \partial_t \tilde{E}_b^j, \end{aligned} \quad (27)$$

where P, Q and R are parameters and will be fixed later. Note that we truncated $\mathcal{A}_0^a$ in (26), while it remains in (27), since only the derivative of $\mathcal{A}_0^a$ effects the principal part of the system. The terms in (13) become

$$A^{(ab)ij} = -i\epsilon^{bca} \tilde{N} \gamma^{ij} \tilde{E}_c^i - i\epsilon^{cba} \tilde{N} \tilde{E}_c^i \gamma^{ij} + N^i \gamma^{ij} \delta^{ab} - N^i \gamma^{ij} \delta^{ab} + P^{iab} \gamma^{ij}, \quad (28)$$

$$B^{(ab)ij} = 0, \quad C^{(ab)ij} = T^{(ab)ij}, \quad (29)$$

$$\begin{aligned} D^{(ab)ij} &= +i\epsilon^{abc} \tilde{N} \tilde{E}_c^j \gamma^{li} - i\epsilon^{abc} \tilde{N} \tilde{E}_c^i \gamma^{lj} + N^l \delta^{ab} \gamma^{ji} - N^j \delta^{ab} \gamma^{li} + S^{(ab)ij} \\ &\quad - iQ^{ai} \epsilon^{bcd} \tilde{E}_c^j \tilde{E}_d^l - R^{ia} \tilde{E}^{jb} + R^{ja} \tilde{E}^{ib}. \end{aligned} \quad (30)$$

The condition (17) immediately show $T^{(ab)ij} = 0$. The condition (15) is written as

$$\begin{aligned} 0 &= -i\epsilon^{bda} \tilde{N} \gamma^{ij} \tilde{E}_d^i - i\epsilon^{adb} \tilde{N} \gamma^{li} \tilde{E}_d^j - 2i\epsilon^{dba} \tilde{N} \text{Im}(\tilde{E}_d^i) \gamma^{ij} \\ &\quad - N^i \gamma^{ij} \delta^{ab} + N^j \gamma^{li} \delta^{ba} + P^{iab} \gamma^{lj} - \bar{P}^{jba} \gamma^{li} := \dagger^{(ab)ij}. \end{aligned} \quad (31)$$

By contracting $\dagger^{(ab)ij}$ with $\epsilon_{abc} \gamma_{ij}$, we get $\epsilon_{abc} \gamma_{ij} \dagger^{(ab)ij} = 4\tilde{N} \text{Im}(\tilde{E}_c^i) - i6\tilde{N} \text{Im}(\tilde{E}_c^i) + 2\epsilon^{ab} \text{Re}(P^{iab})$. This suggests that we should impose $\text{Im}(\tilde{E}_c^i) = 0$, in order to get $\dagger^{(ab)ij} = 0$. This means that triad reality condition is required for making characteristic matrix Hermitian.

5.1 Under triad reality condition (system IIIa)

We, therefore, assume triad reality condition hereafter in this section. In order to be consistent with the secondary triad reality condition (11) during time evolution, and in order to avoid the system becomes second order, we need to specify the lapse function as $\partial_t N = 0$. This lapse condition reduces

$$\text{Re}(\mathcal{A}_0^a) = N^i \text{Re}(\mathcal{A}_i^a), \quad (32)$$

$$\partial_i \text{Re}(\mathcal{A}_0^a) = N^j \partial_i \text{Re}(\mathcal{A}_j^a). \quad (33)$$

By comparing these with real/imaginary components of (25), we obtain

$$\text{Re}(S^{ia}{}_{bi}{}^j) = N^j \delta_i^l \delta^{ab}, \quad \text{Im}(S^{ia}{}_{bi}{}^j) = 0.$$

This means that we found S is determined as

$$S^{ia}{}_{bi}{}^j = N^j \delta_i^l \delta^{ab}. \quad (34)$$

Thus S and $T^{(ab)ij} = 0$ decides the form of the triad lapse as

$$\mathcal{A}_0^a = \mathcal{A}_i^a N^i + \text{non-dynamical terms} \quad (35)$$

in result. ILR does not discuss consistency of the system with reality condition (especially with secondary reality condition). However, since ILR assume $\mathcal{A}_0^a = \mathcal{A}_i^a N^i$, we think that ILR also needs to impose similar restricted lapse condition in order to preserve reality of the system.

By multiplying γ_{li} to real/imaginary part of $\dagger$ (15), and taking symmetric/anti-symmetric components on the indices ab , we get

$$P^{iab} = N^i \delta^{ab} + i\tilde{N} \epsilon^{abc} \tilde{E}_c^i. \quad (36)$$

The rest of our effort is finished when we specify parameters Q and R . Under triad reality condition, (16) becomes

$$\begin{aligned} 0 &= i\epsilon^{abc} \tilde{N} \tilde{E}_c^j \gamma^{li} + i\epsilon^{bac} \tilde{N} \tilde{E}_c^i \gamma^{lj} - N^j \delta^{ab} \gamma^{li} + N^i \delta^{ba} \gamma^{lj} + S^{(ab)ij} - \bar{S}^{(ba)ji} \\ &\quad - iQ^{ai} \epsilon^{bcd} \tilde{E}_c^j \tilde{E}_d^l - i\bar{Q}^{bj} \epsilon^{acd} \tilde{E}_c^i \tilde{E}_d^l - R^{ia} \tilde{E}^{jb} + R^{ja} \tilde{E}^{ib} + \bar{R}^{jb} \tilde{E}^{ia} - \bar{R}^{ib} \tilde{E}^{ja}. \end{aligned} \quad (37)$$

We found that the combination of the choice

$$Q^{ai} = e^{-2} \tilde{N} \tilde{E}^{ia}, \quad R^{ia} = ie^{-2} \tilde{N} \epsilon^{acd} \tilde{E}_d^i \tilde{E}_c^l \quad (38)$$

system	Eqs of motion	reality condition	gauge conditions required	first order	all real eigenvals	diagonalizable	sym. matrix
Ia	original	metric	-	yes	yes	no	no
Ib	original	triad	$\mathcal{A}_0^a = \mathcal{A}_i^a N^i, \partial_i N = 0$	yes	yes	no	no
IIa	original	metric	$N^i \neq 0, \pm N \sqrt{\gamma^{ii}} (\gamma^{ii} \neq 0)$	yes	yes	yes	no
IIb	modified	metric	$\mathcal{A}_0^a = \mathcal{A}_i^a N^i$	yes	yes	yes	no
IIIa	modified	triad	$\mathcal{A}_0^a = \mathcal{A}_i^a N^i, \partial_i N = 0$	yes	yes	yes	yes

satisfies the condition (37).

The final equations of motion are summarized as

$$A^{labij} = i\epsilon^{abc} N \tilde{E}_c^i \gamma^{ij} + N^l \gamma^{ij} \delta^{ab}, \quad B^{labij} = C^{labij} = 0, \quad (39)$$

$$D^{labij} = iN(\epsilon^{abc} \tilde{E}_c^j \gamma^{li} - \epsilon^{abc} \tilde{E}_c^i \gamma^{lj}) \\ - e^{-2} \tilde{E}^{ia} \epsilon^{bcd} \tilde{E}_c^j \tilde{E}_d^i - e^{-2} \epsilon^{acd} \tilde{E}_d^i \tilde{E}_c^l \tilde{E}^{jb} + e^{-2} \epsilon^{acd} \tilde{E}_d^i \tilde{E}_c^j \tilde{E}^{lb}) + N^l \delta^{ab} \gamma^{ij}. \quad (40)$$

We note that in the modifying process of characteristic matrix (39), (40) with parameter (36) and (38), we do not use any facts from imposing triad reality condition. To sum up, we get the symmetric hyperbolic system (III) by modifying equations of motion restricting gauge: $\mathcal{A}_0^a = \mathcal{A}_i^a N^i, \partial_i N = 0$ and assuming triad reality condition. We name this set *system IIIa*. The eigenvalues of this system N^i (multiplicity = 6), $N^i \pm \sqrt{\gamma^{ii}} N$ (5 each), and $N^i \pm 3\sqrt{\gamma^{ii}} N$ (1 each). These speeds are again independent from the choice of (real) triad.

5.2 Under metric reality condition (system IIb)

Using this technique, we can also show another version of diagonalizable hyperbolic system. Since parameters S and T specify *triad lapse*, a gauge variable for time evolutions, it is possible to change our interpretation that we take the evolution of the system within *metric* reality condition. Of course, the characteristic matrix is not Hermitian anymore, but from the fact that we do not use triad reality condition in the process of modifying characteristic matrix with parameters (36),(38) nor in the process of deriving eigenvalues, this system has the same components in its characteristic matrix and has the same eigenvalues.

To summarize we gain another diagonalizable hyperbolic system by modifying the equations of motion using terms from constraint equations, with characteristic matrix (39)-(40) under metric reality condition. The eigenvalues are the same as of *system IIIa*, and this system is restricted only by a condition on triad lapse, $\mathcal{A}_0^a = \mathcal{A}_i^a N^i$, and not on lapse and shift vector like *system IIa*. We call this system *system IIb*.

6 Discussion

We have shown several hyperbolic systems available in the Ashtekar formulation of general relativity, together with discussions of required gauge conditions and reality conditions. We summarize each features in the Table above.

The original dynamical equations in the Ashtekar formulation are classified as weakly hyperbolic system. If we further assume a set of gauge condition, (21), then the system can be a diagonalizable hyperbolic system.

We finally succeeded to present a symmetric hyperbolic system by modifying right-hand-side of dynamical equations using terms of constraint equations with appropriate combinations, which is the same technique used by Iriondo, Leguizamón and Reula. This formulation requires triad reality condition, which we suspect that Iriondo *et al* implicitly assuming in their system and we also suspect that they do not consider its secondary condition. requires to impose a constant lapse function for the evolving system. As we denoted in §5, in order to keep the system first order and to be consistent with the secondary triad reality condition, the lapse function is assumed to be constant. Shift vectors and triad lapse $\mathcal{A}_0^a$ should have a relation (35). This can be interpreted that shift is free and triad lapse is determined. This gauge restriction sounds tight, but this arises from our general assumption of (25).

There might be a possibility to improve the situation by renormalizing shift and triad lapse terms into left-hand-side of equations of motion like the case of general relativity [4]. Or this might be because our system is constituted by Ashtekar's original variables. We are now trying to release this gauge restriction and/or to simplify the characteristic speeds by other gauge possibilities and also by introducing new dynamical variables. This effort will be reported elsewhere.

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Recent status of research on neutrino oscillations

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Abstract

In this talk I have reviewed some of recent topics on neutrino oscillations.

1 Solar neutrino problem

Solar neutrinos have been observed by experiments with targets such as gallium (GALLEX · SAGE), chlorine (Homestake) and water (Kamiokande · Superkamiokande), and the following results have been obtained[1].

Experiment	measured flux	SSM(BP98)	ratio exp/BP98	threshold energy
Homestake	$2.56 \pm 0.16 \pm 0.16$	7.7 $\begin{smallmatrix} +1.2 \\ -1.0 \end{smallmatrix}$	0.33 ± 0.029	0.814 MeV
Kamiokande	$2.80 \pm 0.19 \pm 0.33$	5.15 $\begin{smallmatrix} +1.0 \\ -0.7 \end{smallmatrix}$	0.54 ± 0.07	7.5 MeV
Gallex	$77.5 \pm 6.2 \begin{smallmatrix} +4.3 \\ -4.7 \end{smallmatrix}$	129 $\begin{smallmatrix} +8 \\ -6 \end{smallmatrix}$	0.60 ± 0.06	0.233 MeV
SAGE	66 $\begin{smallmatrix} +7.8 \\ -8.1 \end{smallmatrix}$	129 $\begin{smallmatrix} +8 \\ -6 \end{smallmatrix}$	0.52 ± 0.06	0.233 MeV
Superkamiokande	$2.44 \pm 0.05 \begin{smallmatrix} +0.09 \\ -0.07 \end{smallmatrix}$	5.15 $\begin{smallmatrix} +1.0 \\ -0.7 \end{smallmatrix}$	0.474 ± 0.020	6.5 MeV

Solar neutrino problem is twofold: a) All the experiments obtained smaller numbers of events than theory predicts; b) The ratio of (experimental value)/(theoretical prediction) depends on the threshold energy of the experiment. The most promising explanation for this phenomena at present is neutrino oscillations. The popular way to account for the solar neutrino deficit is to have $\Delta m^2 \simeq 10^{-10} \text{eV}^2$ with the vacuum oscillation solutions or $\Delta m^2 \simeq 10^{-5} \text{eV}^2$ with the MSW solutions which requires the matter effect. On the other hand, people have also discussed energy independent solutions, where $\Delta m_{ij}^2 \gg 10^{-5} \text{eV}^2$ and $\sin^2(\Delta m_{ij}^2 L/4E) \rightarrow 1/2$ are satisfied for all i, j . At present this scenario gives a fit to data at about 3σ confidence level[1] and may be still acceptable. One motivation to consider this scheme is to account for the solar neutrino deficit, the atmospheric neutrino anomaly and the LSND result in the three flavor framework. If the solar neutrino problem is explained by the energy independent solution and if we give up the LSND result while sticking to the three flavor neutrino scenario, then it is possible that all the Δm_{ij}^2 is of order of $10^{-3} \text{eV}^2 \sim 10^{-2} \text{eV}^2$ and in this case we might be able to observe large CP violation effects in long baseline experiments. This possibility should be studied further in future.

There have been a few articles for non-experts on the solar neutrino problem [2, 3, 4].

2 Atmospheric neutrino problem

Atmospheric neutrinos are $\nu_\mu + \bar{\nu}_\mu$ and $\nu_e + \bar{\nu}_e$ which are produced in the collisions of the primary cosmic rays and nucleons in the atmosphere, and we expect from a naive argument that the ratio of the fluxes of two kinds of neutrinos is approximately 2. So far several experiments such as NUSEX, Frejus, IMB, Kamiokande, Soudan2, Superkamiokande have been performed. NUSEX and Frejus have concluded that the ratio was the same as the theoretical prediction, but IMB, Kamiokande, Soudan2, Superkamiokande have reported that $(\# \text{ of } \nu_\mu + \bar{\nu}_\mu)/(\# \text{ of } \nu_e + \bar{\nu}_e)$ is rather close to 1. It is not so clear how to interpret the results of NUSEX and Frejus, but as far as four other experiments are concerned, there is difference between observations and theory, and this is called atmospheric neutrino problem. In particular, in the so-called multi-GeV data ($1 \text{ GeV} \lesssim E_\nu \lesssim 100 \text{ GeV}$) of Kamiokande and Superkamiokande, while the value of the ratio $(\# \text{ of } \nu_\mu)/(\# \text{ of } \nu_e)$ is close to the theoretical prediction for neutrinos which travel for a short length after they are produced above the detector, the same ratio is significantly smaller

than the theoretical value for neutrinos which travel across the Earth (zenith angle dependence). This phenomenon can be interpreted in terms of neutrino oscillations and it has been reported that the best fit value is $\Delta m^2 \sim \text{several} \times 10^{-3} \text{eV}^2$. Unlike the case of solar neutrinos, the zenith angle dependence of the flux ratio ($\#$ of ν_μ)/($\#$ of ν_e) cannot be explained by a constant suppression factor, and it seems that the zenith angle dependence in the atmospheric neutrino data forces us to have the condition $\text{several} \times 10^{-4} \text{eV}^2 \lesssim \Delta m^2 \lesssim \text{several} \times 10^{-2} \text{eV}^2$.

There have been a few articles for non-experts on atmospheric neutrinos[4, 5, 6].

3 Reactor and accelerator experiments

Reactor experiments are disappearance experiments of $\bar{\nu}_e$ produced in reactors, where people measure how many neutrinos are converted from $\bar{\nu}_e$ into something else at a given distance. So far all the reactor experiments gave negative results. Bugey[7] gave the strongest bound on $\sin^2 2\theta$, and CHOOZ[8] the strongest bound on Δm^2 .

Among all the accelerator experiments on neutrino oscillations, LSND[9], which looks at a channel of $\bar{\nu}_\mu \rightarrow \bar{\nu}_e$, is the only experiment which reports affirmative result. If one combines this result with all other experiments (except KARMEN2 which will be explained later), one gets $0.3 \text{eV}^2 \lesssim \Delta m^2 \lesssim 2.3 \text{eV}^2$. At the conference of Neutrino98 in June 1998, however, KARMEN2[10] reported negative results in the same channel. Conclusions from the KARMEN2 data seem to vary, depending on what kind of criterion of the statistics[11], so we have to wait until KARMEN2 group gain more statistics. All other accelerator experiments have obtained negative results. KARMEN2[10] and E776[12] gave strong bound on the same channel $\nu_\mu \rightarrow \nu_e$ as LSND, E531[13] gave strong bounds on $\nu_\mu \rightarrow \nu_\tau$ and $\nu_e \rightarrow \nu_\tau$, CDHSW[14] and CFRR[15] gave strong bound on disappearance of $\nu_\mu \rightarrow \nu_\mu$.

4 Hot dark matter

This subject is not directly related to neutrino oscillations, but HDM (hot dark matter) is one of the objects which might be suggesting neutrino mass. It has been suggested that mixed dark matter scenarios, in which both CDM (cold dark matter) components (heavy particles such as axions) and HDM components (neutrinos) exist, account for the spectrum of density fluctuations in the structure formation of the universe. If this assertion is established, then it follows that neutrinos have mass of order a few eV. [16] is one of the recent references which claim that the mixed dark matter scenario explains the observations well. Other people claim that the scenario with CDM (+cosmological constant) gives a better fit, or some other would say that it is impossible to determine neutrino masses from cosmological observations due to large systematic errors, so it may be premature to conclude on neutrino masses by this kind of argument only.

5 Sterile neutrino ν_s

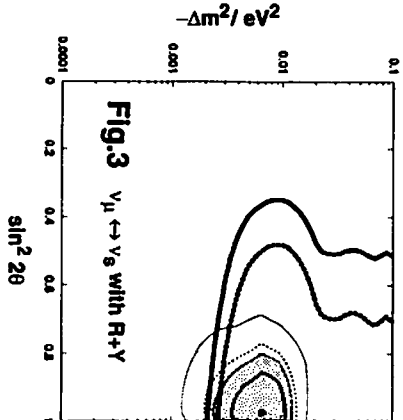
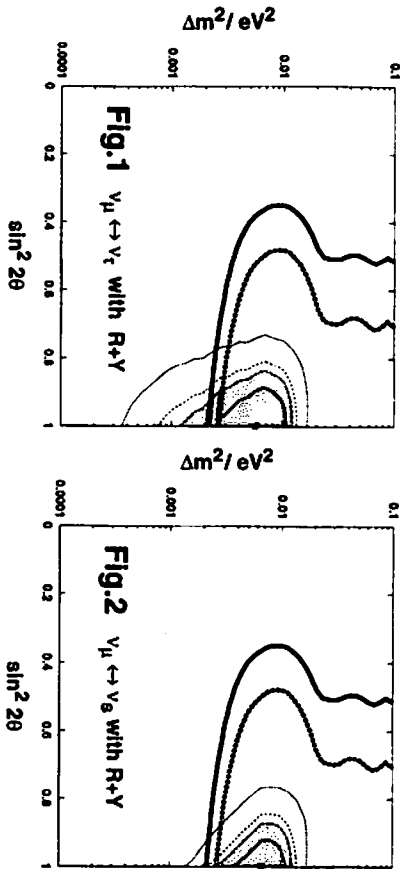
ν_s is a particle with spin 1/2 which is singlet with respect to the standard model gauge group, and it interacts with other neutrinos (active neutrino) only through mass terms. The difference of neutrino oscillations with and without ν_s is that ν_s has neither charged current nor neutral current interactions in matter. In general neutrino oscillations in matter can be described by a Schrödinger equation

$$i \frac{d}{dx} \begin{pmatrix} \nu_\alpha(x) \\ \nu_\beta(x) \end{pmatrix} = \left[U \text{diag}(0, \frac{\Delta m^2}{2E}) U^{-1} + \text{diag}(A_{\alpha\beta}(x), 0) \right] \begin{pmatrix} \nu_\alpha(x) \\ \nu_\beta(x) \end{pmatrix}$$

where

$$U \equiv \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

is the MNS mixing matrix[17]. ν_e has both charged current and neutral current interactions, ν_μ and ν_τ have neutral current interactions, and ν_s has no interaction, so that the potential between ν_e and ν_μ or



K2K 90%CL — SK atm 414 days
 K2K 99%CL --- best fit
 (ν_μ ↔ ν_μ disappearance) 3σ
 90%CL 1σ

ν_τ is given by

$$A_{\alpha\alpha}(x) = \sqrt{2}G_F N_e(x) \quad \text{for } \alpha = \mu, \tau,$$

the one between ν_μ and ν_τ is

$$A_{\mu\tau}(x) = 0,$$

and the one between ν_μ or ν_τ and ν_s is

$$A_{\alpha s}(x) = -\frac{1}{\sqrt{2}}G_F N_n(x) \quad \text{for } \alpha = \mu, \tau,$$

where $N_e(x)$ and $N_n(x)$ stand for the density of electrons and neutrons, respectively. Therefore, there is difference of neutrino oscillations $\nu_\mu \leftrightarrow \nu_\tau$ and $\nu_\mu \leftrightarrow \nu_s$ in the atmospheric neutrinos, due to the matter effect in the Earth. The reason that ν_s have been often discussed recently is mainly because Liu et al.[18] claimed that the zenith angle dependence of the upward going mu data by the MACRO group is consistent with theoretical prediction based on $\nu_\mu \leftrightarrow \nu_s$. Historically people have believed for some time that $\nu_\mu \leftrightarrow \nu_s$ cannot be used for the atmospheric neutrino anomaly. This is because we get $\Delta m^2 \sin^4 2\theta \lesssim 10^{-4} \text{eV}^2$ due to the constraint from the Big Bang Nucleosynthesis, which tells us that ν_s should not have been in thermal equilibrium in order for the theoretical prediction of ^4He to reproduce the observational value, and therefore the $\nu_\mu \leftrightarrow \nu_s$ scenario in the atmospheric neutrino anomaly with $\Delta m^2 \sim 10^{-2} \text{eV}^2$, $\sin^2 \theta \sim 1$ contradicts the BBN constraint[19]. However, it was pointed out later on by Foot and Volkas[20] that for a certain range of the oscillation parameters neutrino oscillations

themselves create asymmetry between ν and $\bar{\nu}$, and the asymmetry prevents ν_s from oscillating into active neutrinos so that ν_s would not have been brought into equilibrium. This is how people started investigation on $\nu_\mu \leftrightarrow \nu_s$ seriously as a scenario to account for the atmospheric neutrino anomaly. Analysis has been performed for Superkamiokande contained events data for 414 days with scenarios $\nu_\mu \leftrightarrow \nu_\tau$ and $\nu_\mu \leftrightarrow \nu_s$, and the allowed region for each scenario is given in Fig. 1 - 3[21]. In this analysis, as variables we have adopted the double ratio $R \equiv \frac{(N_\mu/N_\tau)|_{\text{osc}}}{(N_\mu/N_\tau)|_{\text{no-osc}}}$ and the up-down asymmetry $Y_\alpha \equiv (N_\alpha(\cos \Theta < -0.2)/N_\alpha(\cos \Theta > 0.2))|_{\text{osc}}/(N_\alpha(\cos \Theta < -0.2)/N_\alpha(\cos \Theta > 0.2))|_{\text{no-osc}}$ ($\alpha=e,\mu$), where $N_\alpha(\cos \Theta < -0.2)$, $N_\alpha(\cos \Theta > 0.2)$ stand for the upward and downward events numbers, respectively, and our results differ a little from those by Superkamiokande group. In the figures we have also plotted the region which can be probed in the $\nu_\mu \leftrightarrow \nu_\mu$ disappearance channel by K2K experiment which will start from January 1999. In any of the scenarios given in the figures, a fit to the data is good, and we conclude that both $\nu_\mu \leftrightarrow \nu_\tau$ and $\nu_\mu \leftrightarrow \nu_s$ are possible schemes as far as Superkamiokande contained events are concerned.

6 Three flavor neutrino oscillations

We know that there are at least kinds of three neutrinos so it is in principle necessary to discuss neutrino oscillations among three flavors. We start with the relation between the flavor eigenstates and the mass eigenstates

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix},$$

and we assume without loss of generality that $m_1^2 < m_2^2 < m_3^2$, where m_1^2, m_2^2, m_3^2 are the mass squared for the mass eigenstates. If there are only three kinds of neutrinos, then there are only two independent Δm_{ij}^2 , since $\Delta m_{31}^2 = \Delta m_{32}^2 + \Delta m_{21}^2$. In this case it is impossible to account for the solar neutrino deficit, the atmospheric neutrino anomaly and LSND (the only nontrivial possibility is to take the smaller Δm_{ij}^2 as the scale of atmospheric and the larger as that of LSND and to try to explain the solar neutrino problem with the energy independent solution; It turns out, however, that the main oscillation channel in the atmospheric neutrinos in this case is $\nu_\mu \leftrightarrow \nu_e$ and therefore the zenith angle dependence of the atmospheric neutrino data cannot be explained). So we have to give up an effort to explain LSND and (i) take the smaller Δm_{ij}^2 as the scale $\Delta m_\odot^2$ of the solar neutrino problem and the larger as Δm_{atm}^2 of the atmospheric neutrino anomaly, or (ii) take all Δm_{ij}^2 as Δm_{atm}^2 and account for the solar neutrino problem by the energy independent solution. In either case of (i) or (ii) in order for neutrinos to have a few eV mass, we have to add overall constant to each neutrino species, yielding so-called almost degenerate neutrino scenario. If neutrinos are of Majorana type, then severe constraints from neutrinoless double beta decay almost excludes the possibility of the almost degenerate neutrino scenario[22]. If neutrinos are of Dirac type, then it is possible that neutrinos have masses of order a few eV, since there is no constraint in this case. However, theorists tend to prefer neutrinos of Majorana type, to explain small neutrino mass in terms of the seesaw mechanism[23], so from that point of view neutrino mass suggested from HDM is not desirable.

(i) If $\min \Delta m_{ij}^2 \sim \Delta m_\odot^2$, $\max \Delta m_{ij}^2 \sim \Delta m_{\text{atm}}^2$, then $\Delta m_\odot^2$ and Δm_{atm}^2 have a large hierarchy, so we assume $\Delta m_{\text{atm}}^2 \simeq \Delta m_{32}^2 \equiv m_3^2 - m_2^2 \gg \Delta m_{21}^2 \equiv m_2^2 - m_1^2 \simeq \Delta m_\odot^2$. Under this condition the oscillation probability is given by

$$P(\bar{\nu}_e \rightarrow \bar{\nu}_e) = 1 - \sin^2 2\theta_{13} \sin^2 \left(\frac{\Delta m_{32}^2 L}{4E} \right),$$

so if $\Delta m_{\text{atm}}^2 > 2 \times 10^{-3} \text{eV}^2$ then either $\theta_{13} \simeq 0$ or $\theta_{13} \simeq \pi/2$ has to be satisfied. On the other hand, the oscillation probability in the three flavor framework is given by

$$P^{(3)}(\nu_e \rightarrow \nu_e; A(x)) = c_{13}^4 P^{(2)}(\nu_e \rightarrow \nu_e; c_{13}^2 A(x)) + s_{13}^4,$$

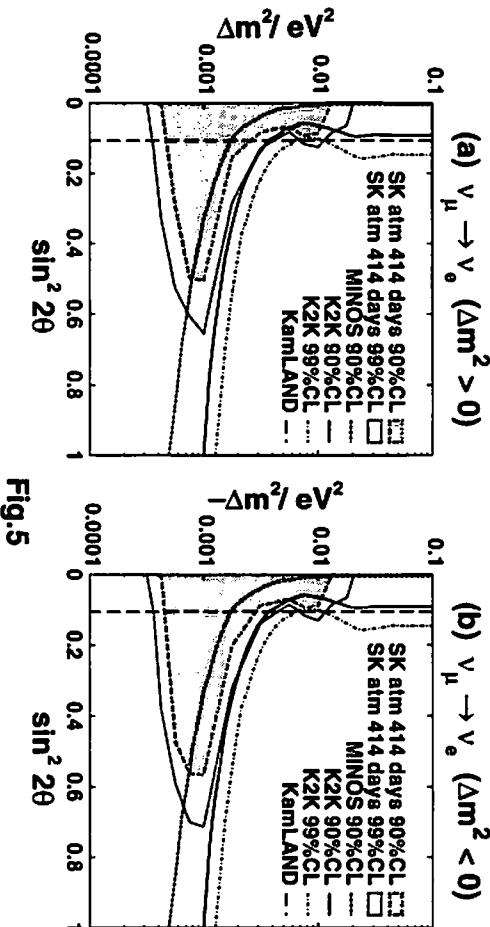


Fig.5

where $A(x)$ stands for the matter effect. To account for the solar neutrino deficit, s_{13} cannot be too large, so it follows that $\theta_{13} \ll 1$, and the MNS mixing matrix U becomes

$$U \simeq \begin{pmatrix} c_{\odot}/\sqrt{2} - c_{\odot}/\sqrt{2} & c_{\odot}/\sqrt{2} - s_{\odot}/\sqrt{2} & 0 \\ s_{\odot}/\sqrt{2} - c_{\odot}/\sqrt{2} & -c_{\odot}/\sqrt{2} - s_{\odot}/\sqrt{2} & 1/\sqrt{2} \end{pmatrix},$$

which indicates that the solar neutrino problem is explained by oscillations half of which is $\nu_e \rightarrow \nu_\mu$ and the other is $\nu_e \rightarrow \nu_\tau$, and that the atmospheric neutrino anomaly is accounted for by oscillations of 100% $\nu_\mu \rightarrow \nu_\tau$. On the other hand, if $\Delta m_{am}^2 < 2 \times 10^{-3} \text{eV}^2$, then θ_{13} can be fairly large. From the three flavor analysis of the Superkamiokande atmospheric neutrino data[24, 25], we see that $\theta_{13} \sim \pi/4$ is allowed at largest. In this case the probability

$$P(\nu_\mu \rightarrow \nu_e) = s_{23}^2 \sin^2 2\theta_{13} \sin^2 \left(\frac{\Delta m_{32}^2 L}{4E} \right)$$

of appearance of ν_e can be relatively large and there is a chance in long baseline experiments to observe ν_e . Fig. 5 (a) and (b) are the allowed regions which are obtained by projecting the results of three flavor analysis of the Superkamiokande atmospheric neutrino data onto the two flavor channel $\nu_\mu \rightarrow \nu_e$, where $s_{23}^2 \sin^2 2\theta_{13}$ corresponds to $\sin^2 2\theta$ in the two flavor framework. The regions which can be reached in the $\nu_\mu \rightarrow \nu_e$ channel by K2K and KamLAND are also given in Fig. 5 (a) and (b).

(ii) If $\Delta m_{ij}^2 \sim \Delta m_{am}^2$ for all i, j , then a fit to the solar neutrino data and therefore to the total data is naturally worse, since the solar neutrino deficit is explained by the energy independent solution. This possibility is probably allowed at 3σ CL, but quantitative analysis has not been done yet.

7 Neutrino oscillations among (3+1) neutrinos

If the solar neutrino deficit, the atmospheric neutrino anomaly and the LSND result turn out to be all due to neutrino oscillations, then it appears to be impossible to explain these three anomalies within the three flavor framework. So we introduce another neutrino, called sterile neutrino, to account for all the three anomalies. It has been shown[26] that if we postulate all the constraints of the reactor and the accelerator experiments and if we assume that there has been no asymmetry between ν and $\bar{\nu}$, then the 4×4 MNS mixing matrix becomes almost block diagonal with two 2×2 matrices, where $\nu_e \leftrightarrow \nu_\mu$ explains the solar neutrino problem, $\nu_\mu \leftrightarrow \nu_\tau$ the atmospheric anomaly, and the small mixing between ν_e and ν_μ the LSND result. As was mentioned earlier, if there was asymmetry between ν and $\bar{\nu}$, then it is possible

to explain the solar neutrino problem by $\nu_e \leftrightarrow \nu_\tau$, the atmospheric anomaly by $\nu_\mu \leftrightarrow \nu_s$. In all of these cases we can add overall constant to the masses of all neutrinos to obtain three components of HDM if they are of Dirac type, but if they are of Majorana type we cannot have HDM due to the constraints of neutrinoless double beta decay experiments.

On the other hand, if we give up explaining LSND by neutrino oscillations, then we can have Δm^2 of the HDM scale instead of that of LSND. In such a scenario, we either (a) explain the solar neutrino problem by $\nu_e \leftrightarrow \nu_s$ and the atmospheric neutrino anomaly by $\nu_\mu \leftrightarrow \nu_\tau$, or (b) explain the solar neutrino problem by $\nu_e \leftrightarrow \nu_\tau$ and the atmospheric neutrino anomaly by $\nu_\mu \leftrightarrow \nu_s$. Since ν_s should not have been in thermal equilibrium anyway, if neutrinos are of Majorana type, we have two HDM components in case of (a) and one HDM component in case of (b) because of the constraints of neutrinoless double beta decay experiments. It has been claimed in the theory of the structure formation of the universe that two or three components of HDM is better to account for observations than one component does. There is no restriction on the numbers of HDM components if neutrinos are of Dirac type.

In scenarios with the LSND or HDM scale described above, the oscillation probability of $\nu_\mu \leftrightarrow \nu_e$ is strongly suppressed for large value of Δm^2 because of the constraints from the reactor experiments (the strongest is Bugey). It seems even more difficult, therefore, to observe appearance of ν_e in long baseline experiments than in the three flavor case.

Most general analysis of (3+1) neutrino mixing scenarios which takes into account the asymmetry between ν and $\bar{\nu}$ has not been done, and it is one of the important problems in this field in the future.

Scanned images of the transparencies of Neutrino98 in June 1998 are available at

<http://www-sk.icrr.u-tokyo.ac.jp/nu98/scan/>,

and those of the satellite symposium at Tokyo Metropolitan University are at

http://musashi.phys.metro-u.ac.jp/era_index.htm.

Transparencies of the author's talk at the workshop "neutrino mixings and study of unification models" on August 2, 1998 are available at

<http://www2.yukawa.kyoto-u.ac.jp/~kunitomo/SI98.html>

which are almost the same as the one on September 21, 1998.

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Discreteness of matter distribution and gravitational lens

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Abstract

The relation between the angular diameter distance and redshift (d_A - z relation) in a spherically symmetric dust-shell universe is studied. This model has large inhomogeneities of matter distribution on small scales. We have found that the relation agrees with that of an appropriate Friedmann-Lemaître(FL) model if we set a “homogeneous” expansion law and a “homogeneous” averaged density field. This will support the averaging hypothesis that a universe looks like a FL model in spite of small-scale fluctuations of density field, if its averaged density field is homogeneous on large scales. We show that we should use rest mass of shells in defining the averaged density. We also clarify the difference from the other recent work which states that the d_A - z relation does not obey a FL-like relation under mild density inhomogeneities in a spherically symmetric system.

1 Introduction

We usually regard that a FL model is a large-scale “average” of a locally inhomogeneous universe (averaging hypothesis). Even though the observational data are consistent with the picture that our universe is described well by a RW metric, we are not sure how to derive the background FL model from the inhomogeneous universe by any averaging procedure, or how the non-linear inhomogeneities on small scales affect large-scale behavior of the universe[1]. Although one can derive a background FL model from observations of the nearby galaxies with any rule of averaging one likes, it is uncertain whether this background FL model agrees with the FL model based on the highly homogeneous universe at earlier stages (or its time evolution). The discrepancies might appear, for example, in the estimate of the density of baryonic matter, the density parameter, the age of our universe, and so on. These still remain a non-trivial question to which we have to give a clear answer.

Averaging problem has been often studied from the viewpoint of the so-called back-reaction problem, i.e., how the small-scale inhomogeneities affect the global dynamics when averaged on larger scales. Using the perturbation formalism and volume averaging, the back-reaction problem has been studied by several authors [2, 3, 4, 5] (see Sec.I of [6] for the overview of these articles). Apart from the problem whether the volume averaging is appropriate or not, one factor which makes the discussion of averaging unclear is the fact that there exists no natural choice of time slice in an inhomogeneous universe. This leads to the ambiguity in the definition of averaged expansion rate or averaged density. Actually, there are even apparent discrepancies in the statements of the papers above.

One possible prescription to avoid this difficulty is to treat observable quantities which we can give a clear definition. From this viewpoint, observational effects on the CMBR of small-scale inhomogeneities have been studied [7, 8]. These are significant in the sense that the back reactions on the observable quantities were discussed, but the cases where we determine the cosmological parameters by observing an inhomogeneous universe have not been studied.

In Sugiura *et al.*(1998; hereafter paper I), we investigated the distance-redshift relation in a spherically symmetric dust-shell model, and compare it with that of a FL model[6]. We focused on a highly (locally) inhomogeneous model; We shook ourselves from perturbations on FL backgrounds, and prepared pressureless fluid (dust) distributed in discrete shells. We discussed the relation between the behavior of the d_A - z relation and the conditions on the initial time slice concerning the mass density and the expansion rate. We found that the distance-redshift relation observed at the center obeys a FL-like relation, even when there exist only several shells in the initial horizon scale, if the following conditions are satisfied:

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the expansion law is homogeneous and the density, averaged on larger scales than the inhomogeneity scale, is scale-independent (i.e., large-scale homogeneity of density field) on the initial space-like hypersurface. Here the averaged density is defined simply by dividing the mass of shells by the volume on the hypersurface.

Our analysis was limited to spatially-flat cases (Einstein de-Sitter model). There is a claim, however, that the Einstein de-Sitter solution is in a special position as a solution of the Einstein equation; it is a kind of fixed point under renormalization group flow defined by the scaling properties of the Einstein equation [9]. In this paper, we treat non-flat cases and show that the similar statements do hold. We also answer some questions unsolved in Paper I. In particular, the previous study of spatially flat cases could not answer which mass should be used in defining the average density, proper mass which appears in the energy-momentum of dust shell, or gravitational mass which specifies the parameter in Schwarzschild space-time; they are the same in spatially flat models.

A similar theme was treated by Mustapha *et al.*[10]. They investigated d_A - z relation in spherically symmetric space time filled with dust (Lemaitre-Tolmann-Bondi solution) and found that the d_A - z relation does not obey a FL like relation under “mild” density inhomogeneities. This seems to be contradictory to our results; we will clarify the difference between their results and ours.

The organization of this paper is as follows. In the next section, we summarize the basic equations for the dynamics of a dust-shell universe, distance to the shells from the center, and redshift of the shells measured by an observer at the center. This is an extension of the treatment in Paper I. We give our results and discussion on d_A - z relation and averaged density in section 3, followed by concluding remarks in section 4.

2 Equation of motion of dust-shell

We put a number of spherically symmetric shells whose centers are common at $r = 0$. The innermost shell is called the first shell, the next one is called the second shell, and so on. The region enclosed by the $(i - 1)$ th shell and i th shell is called the i -th region. Each shell is infinitesimally thin, characterized by the surface stress-energy tensor S^{ab} which is given by

$$S^{ab} \equiv \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{+\epsilon} T^{ab} dx, \quad (1)$$

where x is a Gaussian coordinate ($x = 0$ on the shell) in the direction normal to the shell.

Since each region between the shells is vacuum, the space-time is described by the Schwarzschild geometry by the Birkhoff's theorem. The line element in the i -th region is written in the form

$$ds_i^2 = - \left(1 - \frac{2Gm_{gi}}{r} \right) dt^2 + \left(1 - \frac{2Gm_{gi}}{r} \right)^{-1} dr^2 + r^2 d\Omega^2, \quad (2)$$

where the parameter m_{gi} will be referred to as a gravitational mass ($m_{g1} = 0$), and $d\Omega$ is the line element of a unit sphere.

We first derive the expansion law of the i th shell. Following[11, 12, 13, 14], we obtain the equation for the circumferential radius r_i (the “expansion law” of the dust shell):

$$\left(\frac{dr_i}{d\tau} \right)^2 = \frac{2G\bar{M}_i}{r_i} + \left\{ \left(\frac{M_-(i)}{m_s(i)} \right)^2 - 1 \right\} + \frac{G^2 m_s^2(i)}{4r_i^2}, \quad (3)$$

where $\bar{M}$ and M_- are defined by

$$\bar{M}_i \equiv \frac{m_{gi} + m_{gi+1}}{2}, \quad (4)$$

$$M_-(i) \equiv m_{gi+1} - m_{gi}, \quad (5)$$

and τ is the proper time of the shell. We have also introduced the rest mass $m_s(i)$ given by

$$m_s(i) = 4\pi s_i r_i^2 \quad (6)$$

where s_i is the surface energy density of the i th shell; $s_i = -S_a^a(i)$, the parameters in equation (3) and the initial hypersurface. We first rewrite equation (3) in the form corresponding to the Hubble equation of FL models. We denote the initial circumferential radius of the i th shell by x_i , i.e.,

$$r_i = x_i \quad \text{on initial hypersurface.} \quad (7)$$

We define ρ_i by

$$\bar{M}_i \equiv \frac{4}{3}\pi\rho_i x_i^3, \quad (8)$$

and k_i by

$$\left(\frac{M_-(i)}{m_s(i)}\right)^2 \equiv E_i^2 \equiv 1 - k_i x_i^2. \quad (9)$$

Here $E_i(>0)$ is the specific energy. Then the expansion law of the dust-shell is written as

$$\left(\frac{1}{r_i} \frac{dr_i}{d\tau}\right)^2 = \frac{8}{3}\pi G \rho_i \left(\frac{x_i}{r_i}\right)^3 - k_i \left(\frac{x_i}{r_i}\right)^2 + \frac{G^2 m_s^2(i)}{4r_i^4}. \quad (10)$$

A dust-shell universe is specified if we set the parameters contained in equation (10), i.e., ρ_i , k_i , x_i , and an initial hypersurface. When we increase the number of the shells to infinity with ρ_i and k_i being finite and independent of i (we will mention this limit as “large N limit”), the dust-shell universe approaches a FL model if we take an appropriate initial hypersurface. Then the parameters ρ_i and k_i agree with the ordinary energy density and curvature in the Hubble equation.

3 Results and Discussion

3.1 Setting the parameters of dust-shell model

As mentioned in Sec.III the choice of the parameters and the initial hypersurface determines the behavior of a dust-shell universe. Since we are interested in cases which have a FL limit, we set the parameters so that they approach a FL model in the large N limit.

For most cases described below, we set the mass distribution of shells as

$$\rho_i = \rho_c \text{ (independent of } i), \quad (11)$$

at the initial slice. We also set the “curvature” k_i in equation (10) to be independent of i ;

$$k_i = k_c. \quad (12)$$

Using ρ_c and k_c , we define parameters H_{shell} and r_H by

$$H_{\text{shell}}^2 \equiv r_H^{-2} \equiv \frac{8\pi G}{3}\rho_c - k_c. \quad (13)$$

Further, we define

$$\Omega \equiv \frac{8\pi G}{3}\rho_c / H_{\text{shell}}^2. \quad (14)$$

In terms of FL models, H_{shell} , r_H and Ω may be regarded as the “Hubble constant”, “Hubble horizon radius”, and the “density parameter.”

For x_i , we put

$$x_i(\tau_{\text{init}}) = i\Delta x \quad (15)$$

with a constant interval Δx

$$\Delta x \equiv \frac{r_H}{N_H} \quad (16)$$

where N_H is some positive integer.

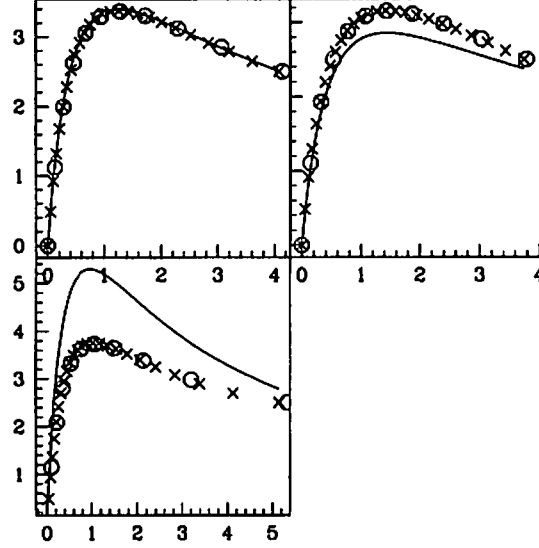


Figure 1: Angular diameter distance-redshift relation in dust-shell universes for model A. Data points are shown by cross ($\times$) and circle ($\circ$). The number of shells within the initial Hubble horizon N_H is 4 ($\circ$) and 10 ($\times$). The total number of shells N_T is taken to be $2.5 \times N_H$. The solid line shows the d_A - z relation in a flat FL model with $H_i = H_{\text{shell}}$. The redshift of the initial hypersurface is identified with the redshift of the outermost shell for the case $N_T = 25$, i.e., $z_1 = z(i = 25)$. As shown in Paper I, the deviation between the flat FL model and the dust-shell universe is almost unrecognizable.

3.2 Distance-redshift relation

We fix the expansion law to be homogeneous [eqs.(11) and (12)], try some choices of initial time slice, and study which time slice makes the distance-redshift relation behave like that of a FL model. We will consider which physical meaning that choice have, especially in terms of averaged density. We have already analysed spatially flat cases (i.e., $k_c = 0$) in Paper I, and found that the distance-redshift relation similar to that of a (spatially flat) FL model when we choose time slice so that the averaged density is homogeneous. The averaged density $\bar{\rho}(i)$, defined around the central point up to radius r_i , is obtained by dividing the mass contained within r_i by the three-volume $\text{Vol}(i)$ on the hypersurface up to r_i . We may call the density field homogeneous when $\bar{\rho}(i)$ is independent of i .

Let us study homogeneous density cases, as Paper I. Now we have two kinds of mass in defining the averaged density; rest mass $m_s(i)$ defined in eq.(6) and gravitational mass m_g , which determines the dynamics of its outer shells.

We first take the volume element so that $\bar{\rho}(i)$ using gravitational mass is homogeneous. The averaged density is defined as²

$$\bar{\rho}(i) \equiv \left\{ \frac{1}{2} E_i m_s(i) + \sum_{j \leq i-1} E_j m_s(j) \right\} / \sum_{j \leq i} \text{Vol}(j) = \frac{1}{2} (m_{g_i} + m_{g_{i+1}}) / \sum_{j \leq i} \text{Vol}(j). \quad (17)$$

We plot the d_A - z relation of dust-shell models in Fig.1(case A). The parameters are $\Omega = 1.0, 0.9$ and 1.1 , $H_{\text{shell}} = 1.0$, and the number of shells within the initial Hubble radius N_H is chosen to be 4 and 10. We

²For the i th shell, we add only half of its mass.

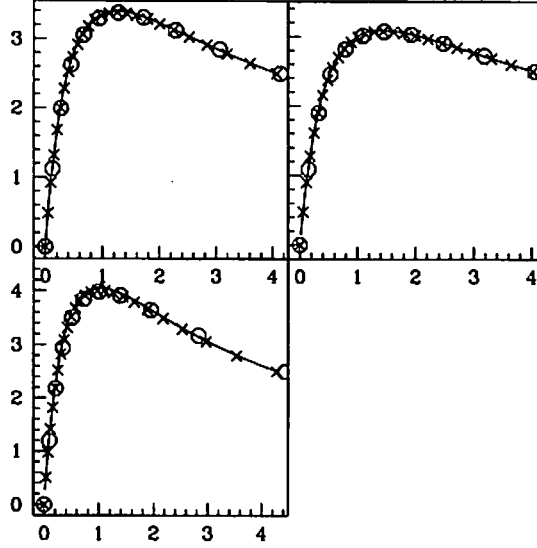


Figure 2: Angular diameter distance-redshift relation in the dust-shell universe for case B. Compare with Fig.1

also plot the d_A - z relation in FL models. It is determined by the initial Hubble parameter H_1 , initial density parameter Ω_1 , and the redshift of the initial time slice z_1 . Here these parameters are set using the parameters of the corresponding shell model as $H_1 = H_{\text{shell}}$, $\Omega_1 = \Omega$, and z_1 equal to the redshift of the outermost shell for $N_H = 10$. In flat cases, the data of the dust-shell models agree with the FL relation as seen in Paper I. In non-flat cases, however, the FL models chosen in this way do not approximate the shell models. Even when we increase the number density of the shells, no improvement is achieved. Thus, using gravitational mass in averaged density is inappropriate.

Next, we try the choice using the rest mass in defining the averaged density;

$$\bar{\rho}(i) \equiv \left\{ \frac{1}{2} m_s(i) + \sum_{j \leq i-1} m_s(j) \right\} / \sum_{j \leq i} \text{Vol}(j) \quad . \quad (18)$$

In flat case, this choice is the same with case A. The d_A - z relations for open and closed models are plotted in Figs.2. These plots show that the relations agree quite well with FL models. Thus, we should use the proper mass in defining the averaged density. We note that due to the amplification of inhomogeneity closed universe shows slight deviation one can notice when compared with the other cases.

For comparison, we try the orthogonal slice (in a FL model, the simultaneous hypersurface is orthogonal to the trajectory of matter). Fig. 3 shows the d_A - z relation in this model. We can see a mild deviation from FL models

3.3 Discussion

In paper I, we checked that the choice of constant interval $\Delta x = r_H/N_H$ is not essential by trying some patterns of the initial circumferential radius of the shells. We confirmed that the results are unchanged. We also tried another interesting choice; the expansion law is homogeneous (characterized by ρ_c), and

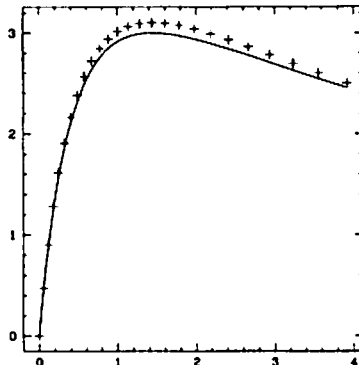


Figure 3: Angular diameter distance-redshift relation in dust shell universe for case C (open model with $\Omega = 0.9$). We see a mild deviation.

the averaged density is also homogeneous, but each quantity differs; $\rho_c \neq \bar{\rho}$. This model did not obey a FL-like relation. Considering all of these results, we concluded in Paper I that the d_A - z relation in a dust-shell universe looks like a flat FL universe, when the expansion law resembles the flat FL model, and the behavior of averaged density field is scale-independent when we increase the scale of averaging, and the averaged density agrees with ρ_c . The discreteness nature of the shell model plays no significant role in observing the distance-redshift relation. However, our analysis was limited to spatially-flat cases (Einstein de-Sitter model). In this paper, we extend the non-spatially-flat cases, and confirm that the previous statement remains valid. (We checked the variations of time-slice described above do not change the statement.) Moreover, the previous study of spatially flat cases could not answer which mass should be used in defining the average density, proper mass which appears in the energy-momentum of dust shell, or gravitational mass which specifies the parameter in Schwarzschild space-time; the latter includes the gravitational potential energy and the kinetic energy of a shell. In spatially flat models, they balance and the two masses agree with each other. From the results of this paper, now it becomes clear that the rest mass should be used in defining the averaged density. This might at first sound strange since the geometry of each region is determined by the gravitational mass which includes the kinetic energy and the potential energy. Where has their information gone? Reexamining the Hubble equation of FL models and the expansion law of the dust-shell models, one notices that it is contained in the curvature term. In eq.(9), the curvature term is expressed as the ratio of the rest mass to the difference of gravitational mass between the inside region and outside region. Thus, the curvature gives the ratio of the total energy to the rest mass.

Mustapha *et al.*[10] treated a similar theme; they constructed an exact inhomogeneous model and its FL approximation, and compared the area distance-redshift and density-redshift relations in the two. They examined Lemaître-Tolman-Bondi spherically symmetric dust solutions, since exact integrations of the field equations are available for the past light cone of observers at the central position. They found that the d_A - z relation does not obey a FL like relation under “mild” density inhomogeneities. Their results apparently contradict to ours, but this is not surprising. As pointed out by Linder[15], one cannot directly compare to the FL situation in terms of the observable, the redshift, when the inhomogeneous model has different dynamics; the redshift to the source will differ if the observer and/or the source is in a dynamically significant inhomogeneity. Thus, different dynamics implies different light behavior regardless of similarities in the large scale volume averages.

In our model, on the other hand, the dynamics (expansion law for the circumferential radius) is homogeneous when there are enough number of shells; the effect on expansion rate of density inhomogeneities is small (of order N_H^{-2}). Fixing the expansion law like FL, we studied the connection between the d_A - z relation and the averaged density. We have found that if we make the averaged density (which is defined

using the ordinary three-volume element and the rest mass) homogeneous, the d_A - z relation agrees well with that of a FL model. That is, there exists a strong connection among the homogeneous averaged density (on the initial spacelike hypersurface), the homogeneous expansion law (on the initial spacelike hypersurface), and the FL like distance-redshift relation, in spite of the discrete nature (local inhomogeneity of matter distribution) of the dust-shell model. This will support the “averaging hypothesis” that a universe behaves like a FL model in spite of small-scale fluctuations of density field, if its averaged density field is homogeneous on large scales. This implies that even if there exist large wall-like structures, our universe is approximated by a FL model on larger scales if the walls satisfy the above conditions. At the same time, our work, together with the other work mentioned above, makes the meaning of the terms in the averaging hypothesis clear, such as the large-scale homogeneity of averaged density. This will lead to a deeper understanding of the averaging hypothesis, though the geometry is restricted to spherically symmetric cases.

4 Summary

We have studied the behavior of d_A - z relation in a spherically symmetric dust-shell universe where the mass distribution is discrete. We have compared the relation of dust-shell universe with that of FL models, and discussed the relation with the behavior of averaged density. We have seen that the d_A - z relation observed at the center agrees well with that of a flat FL model if the following conditions are satisfied: (i) the expansion law of the circumferential radius of the shells resembles the Hubble equation of a spatially flat FL model, (ii) the behavior of averaged density around the observer at the center is scale-independent as we increase the scale on which we take the average, and (iii) the averaged density agrees with the energy density of the FL model. In defining the averaged density, we take the rest mass of each shell and divide it by the three-volume of the initial hypersurface. We have also seen that the choice of the initial hypersurface relates the expansion law to the averaged density.

The effect of discreteness of mass distribution appears in the equation of motion of each dust shell. This effect becomes smaller as we increase the number density of shell, though we found the positive curvature have tendency to enhance the inhomogeneity effect. We conclude the discreteness of matter distribution itself is of no significance in spherically symmetric system in discussing the observed quantities such as d_A and z , as long as the expansion law and the averaged density field is homogeneous in the sense described above. This supports the averaging hypothesis that a universe is described by a FL model if the universe is homogeneous when the density is averaged on some scale larger than the scale of the inhomogeneities.

For general inhomogeneities, the gravitational lensing effect on distant sources due to the inhomogeneities in the matter distribution of the universe has been studied using various methods[16-27]. In these cases, the d_A - z relation can have a large scatter. Our work, however, implies that, if we average the luminosity of the sources of the same redshift over the whole sky and calculate the distance-redshift relation with the averaged luminosity, it should agree with the FL relation.

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Analysis of Asymmetry in Microlensing-Induced Light Curves

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Abstract

We discuss distortion in microlensing-induced light curves which are considered as ones of single-point-mass lenses at a first glance. As factors of the distortion, we consider close binary and planetary system, which are the limiting cases of two-point-mass lenses, and the gravitational potential is regarded as the sum of the single-point-mass and the corrections. In order to quantitatively estimate the asymmetric features of such distorted light curves, we propose a cut-off dependent skewness, and show how to discriminate the similar light curves with it. We also examine as the distortion the general relativistic effect of the frame dragging, but the effect coincides with the close binary case in the light curves.

1 Introduction

Gravitational microlensing [1] is one of important probes of studying the nature and distribution of mass in the Galaxy. Chang & Refsdal [2] and Gott [3] suggested that even though multiple images by the lensing are unresolved, the time variation of the magnitude of the source can still be detected if the lens moves relative to the source. Such light curves caused by the microlensing can be distinguished from ones of intrinsically variable sources, because the change of magnitude by lensing is achromatic whereas intrinsically variable stars change their colors in general. Furthermore, the microlensing-induced light curves can be distinguished from the magnification by bursts, which are likely to appear in the sheer shapes. As is well known, the microlensing by a point mass has the time-symmetric light curves, provided that the lens and the source have constant relative transverse velocity. Hence, most of events are expected to have almost time-symmetric light curves. If the relative velocity is not constant, then of course the time-asymmetric light curves will be detected. Gould [4] predicted the parallax effect due to the orbital motion of the Earth, and the effect was indeed detected by Alcock et al. [5]. However, apart from the parallax effect, other factors such as the non-spherical gravitational potential of the lens would also produce the distortion from the time-symmetric light curves. The effect may be regarded as the higher-order correction to the point-mass lens. The aim of this paper is to evaluate the effect of the intrinsic nature of the lenses, which may slightly distort the light curves. We exclude significantly peculiar light curves such as double peaks, in which the direct information will be available without any detailed analyses. We rather restrict our consideration to the light curves which are regarded as ones of single-point-mass lenses at a first glance, that is, almost time-symmetric light curves. Since the asymmetric part contains additional information about massive astrophysical compact halo objects (MACHOs), this subject is very important to understanding the nature of MACHOs.

One of the important factors to induce the time-asymmetric manifestation is the binary system of the lensing objects, in which the contribution from both objects to the Newtonian gravitational potential is no longer spherical. The discussion is simplified by considering two-point-mass lenses. In particular, since we consider the almost time-symmetric light curves, we hit on two situations: close binary and planetary system. In the close binary, the separation distance between the two point masses is much less than the Einstein radius of the total mass, so that the approximated light curve is described by the total mass and some corrections to it. In the planetary system, the light curve is expected to be described by the contribution from larger mass and some corrections from the smaller one. Another factor for the time-asymmetric manifestation is the asymmetry of the lens object itself. The multipole moment of the lens deviates from gravitational potential of the point mass. The gravitational potential

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due to the quadrupole never vanishes for the two-point-mass case. Therefore, such a correction term in the gravitational potential can be regarded as the limiting case of the binary. Instead, we shall consider general relativistic effects of dragging of inertia frames due to a rotating object as another factor, which can not be expressed as corrections in the Newton-like potential. While the Newton-like potential corresponds to the gravitoelectric field, this effect results from the gravitomagnetic one. In this case, the corrections up to the post-Newtonian order are sufficient. Other post-Newtonian potentials also affect the light curves, but are neglected here.

This paper is organized as follows. In §2, we discuss the two situations by the two-point-mass lenses and obtain the microlensing-induced light curves slightly deviated from the time-symmetric ones. In §3, we discuss the post-Newtonian corrections due to the rotation and obtain the time-asymmetric light curves as well. We shall provide a certain tool to estimate the time-asymmetric features quantitatively in §4. Finally, we give a summary and discussion in §5.

2 Two-point-mass lenses

We investigate two extreme situations in two-point-mass lenses in order. (The general computational details including critical lines and caustics can be found in Ref. [6].) First, we consider a situation where the distance l between the two point masses is much less than the Einstein radius r_E corresponding to the total mass. Next, we consider a situation where the mass M_1 is much smaller than the other mass M_2 .

2.1 Close binary

We consider the lens plane onto which the positions of two point masses M_1 and M_2 are projected. (See Fig. 1.) We introduce an angular coordinate system (θ_x, θ_y) , in which two point masses lie on the θ_x -axis and the origin is chosen as the geometrical center of them. We also define an angular coordinate system (β_x, β_y) on the source plane, corresponding to the lens plane. We express the distance between the observer and the lens plane, the lens plane and the source plane, and the observer and the source plane by D_L , D_{LS} , and D_S , respectively.

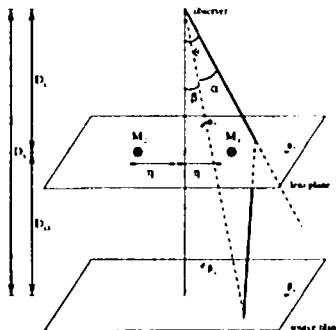


Figure 1: Geometry of gravitational lensing considered in §2.1.

If we denote the angular separation between the source and the image by α , then α is given by

$$\alpha = \frac{4GM_1}{c^2} \frac{D_{LS}}{D_L D_S} \frac{\theta - \eta}{|\theta - \eta|^2} + \frac{4GM_2}{c^2} \frac{D_{LS}}{D_L D_S} \frac{\theta + \eta}{|\theta + \eta|^2}, \quad (1)$$

where $\eta = (\eta, 0)$ is the angular separation of the mass M_1 (or M_2) from the optical axis. Therefore, the lens equation is written in the form

$$\beta = \theta - \alpha. \quad (2)$$

We normalize it by the Einstein radius for the total mass, $\theta_E = \left(\frac{4G(M_1+M_2)}{c^2} \frac{D_L s}{D_L D_S} \right)^{1/2}$. Introduce normalized quantities, $\tilde{\beta} \equiv \beta/\theta_E$, $\tilde{\theta} \equiv \theta/\theta_E$, $\tilde{\eta} \equiv \eta/\theta_E$, the lens equation is given by

$$\tilde{\beta} = \tilde{\theta} - \mu_1 \frac{\tilde{\theta} - \tilde{\eta}}{|\tilde{\theta} - \tilde{\eta}|^2} - \mu_2 \frac{\tilde{\theta} + \tilde{\eta}}{|\tilde{\theta} + \tilde{\eta}|^2}, \quad (3)$$

where μ_1 and μ_2 are defined as $\mu_1 = M_1/(M_1 + M_2)$, $\mu_2 = M_2/(M_1 + M_2)$.

The separation distance on the projected plane is $l = D_L \cdot 2\eta$, and the Einstein radius for this system $r_E = D_L \cdot \theta_E$. The ‘close binary’ in the gravitational lens means $l \ll r_E$. This condition in astronomical situation is expressed as $l \ll 10^{14} (M/M_\odot)^{1/2} (D/10\text{kpc})^{1/2} \text{cm}$, where we have chosen the typical astronomical distances as the scale of the Galactic halo, $D_{LS} \sim D_L \sim D_S \sim D$. Therefore, the range of applicability is not so severe. The condition of the close binary $l \ll r_E$ is mathematically expressed as $\tilde{\eta} \ll 1$.

Under this condition, we expand the right-hand side of the lens equation with respect to $\tilde{\eta}$. Up to the first order in $\tilde{\eta}$, we have

$$\tilde{\beta}_x = \tilde{\theta}_x - \frac{\tilde{\theta}_x}{\tilde{\theta}_x^2 + \tilde{\theta}_y^2} - \tilde{\eta}(\mu_1 - \mu_2) \frac{\tilde{\theta}_x^2 - \tilde{\theta}_y^2}{(\tilde{\theta}_x^2 + \tilde{\theta}_y^2)^2}, \quad \tilde{\beta}_y = \tilde{\theta}_y - \frac{\tilde{\theta}_y}{\tilde{\theta}_x^2 + \tilde{\theta}_y^2} - \tilde{\eta}(\mu_1 - \mu_2) \frac{2\tilde{\theta}_x \tilde{\theta}_y}{(\tilde{\theta}_x^2 + \tilde{\theta}_y^2)^2}, \quad (4)$$

where the last term in each right-hand side shows the deviation from a single-point-mass lens. The first-order corrections vanish for equal mass case, identical $\mu_1 = \mu_2$. It is therefore convenient to express the combination $\tilde{\eta}(\mu_1 - \mu_2)$ as one small parameter $\epsilon \equiv \tilde{\eta}(\mu_1 - \mu_2)$.

The inversion of Eq. (4), i.e., solving $\tilde{\theta}$ by $\tilde{\beta}$, is possible, but the general form is quite messy. However, under the condition $\epsilon \ll 1$, the approximate solution, i.e., the first-order solution in ϵ , is given in the form

$$\tilde{\theta} = \tilde{\theta}_0 + \epsilon \tilde{\theta}_1, \quad (5)$$

where $\tilde{\theta}_0$ and $\tilde{\theta}_1$ denote the zero-order and the first-order solutions, respectively. Substituting Eq. (5) into Eq. (4), we can find such solutions. The zero-order solution is given by

$$\tilde{\theta}_0 = \frac{1}{2} \left(\tilde{\beta} \pm \tilde{\beta} \sqrt{1 + \frac{4}{|\tilde{\beta}|^2}} \right). \quad (6)$$

Using these zero-order solution, the first-order solution is written in the form

$$\tilde{\theta}_{1x} = \frac{\tilde{\theta}_{0x}^2 - \tilde{\theta}_{0y}^2 - 1}{(\tilde{\theta}_{0x}^2 + \tilde{\theta}_{0y}^2)^2 - 1}, \quad \tilde{\theta}_{1y} = \frac{2\tilde{\theta}_{0x}\tilde{\theta}_{0y}}{(\tilde{\theta}_{0x}^2 + \tilde{\theta}_{0y}^2)^2 - 1}. \quad (7)$$

Next, we turn our attention to the derivation of magnification. The magnification M is given by

$$M = M_+ + M_- = \left| \det \left(\frac{\partial \tilde{\beta}_i}{\partial \tilde{\theta}_j} \right)_+ \right|^{-1} + \left| \det \left(\frac{\partial \tilde{\beta}_i}{\partial \tilde{\theta}_j} \right)_- \right|^{-1}, \quad (8)$$

where subscript (+) and (−) correspond to the solutions with a plus sign and with a minus sign, respectively, in Eq. (6). The inverse of the Jacobian $\det \left(\frac{\partial \tilde{\beta}_i}{\partial \tilde{\theta}_j} \right)$ is calculated, up to the first order in ϵ , in the following way:

$$\left[\det \left(\frac{\partial \tilde{\beta}_i}{\partial \tilde{\theta}_j} \right) \right]^{-1} = \left(1 - \frac{1}{(\tilde{\theta}_{0x}^2 + \tilde{\theta}_{0y}^2)^2} \right)^{-1} \left[1 + \epsilon \frac{4\tilde{\theta}_{0x}}{(\tilde{\theta}_{0x}^2 + \tilde{\theta}_{0y}^2 + 1)^2 (\tilde{\theta}_{0x}^2 + \tilde{\theta}_{0y}^2 - 1)} \right]. \quad (9)$$

Therefore, using Eqs. (6), (8) and (9), we can derive the magnification as a function of the source position:

$$M = M(\beta_x, \beta_y; \epsilon). \quad (10)$$

The time variation is produced by the relative change of the positions. The time variation of magnitude Δm is then given by

$$\Delta m = \Delta m(t) = 2.5 \log_{10} M(\beta_x(t), \beta_y(t); \epsilon). \quad (11)$$

From this, we can derive the microlensing-induced light curves, slightly deviated from time-symmetric ones. An example of the light curves is shown in Fig. 2 for $\epsilon = 0.1$, $\tilde{\beta}_x = t/t_0$ (t_0 is the time taken to cross the Einstein radius) and $\tilde{\beta}_y = \text{const.}$ The solid line denotes the light curve with the correction for the deviation from a single-point-mass lens, while the dashed line corresponds to the single-point-mass lens.

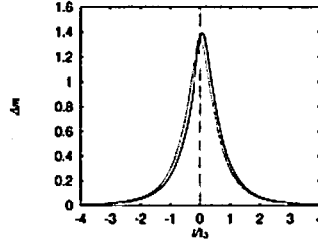


Figure 2: Microlensing-induced light curves with (solid) and without (dashes) correction for the deviation from a single-point-mass lens. The light curve with the correction is plotted for the small parameter of $\epsilon = 0.1$. The relative motion of the source to the lens is assumed to be described by $\tilde{\beta}_x = t/t_0$ (t_0 is the time taken to cross the Einstein radius) and $\tilde{\beta}_y = 0.3$.

2.2 Planetary system

In this subsection, we discuss a case that the mass M_1 is much smaller than the other mass M_2 , that is, the object with smaller mass is regarded as the planet. Under this situation, it will be convenient to chose the origin of the (θ_x, θ_y) system to be on the mass M_2 . Furthermore, let the angular separation between the mass M_1 and M_2 be η . (See Fig. 3.) Then the normalized lens equation is given by

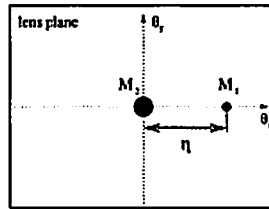


Figure 3: A lens plane considered in §2.2. The angular separation of the two point masses is denoted by η .

$$\tilde{\beta} = \bar{\theta} - \frac{\bar{\theta}}{|\bar{\theta}|^2} - \mu \frac{\bar{\theta} - \tilde{\eta}}{|\bar{\theta} - \tilde{\eta}|^2}, \quad (12)$$

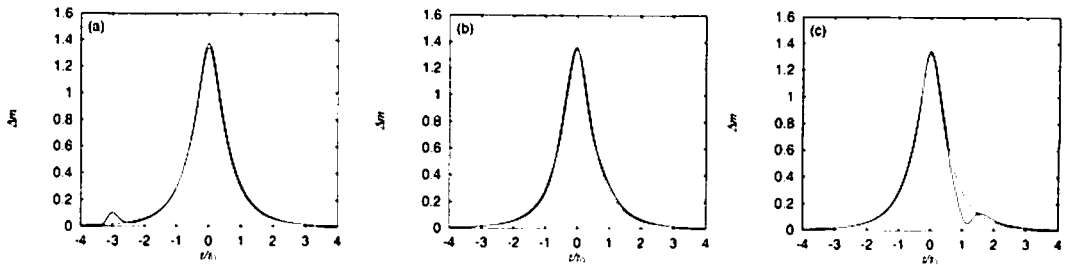


Figure 4: Microlensing-induced light curves with (solid) and without (dashes) correction for the deviation from a single-point-mass lens. The light curves with the correction are plotted for the small parameter of $\mu = 0.05$ and for the angular separation of (a) $\tilde{\eta} = 0.3$, (b) $\tilde{\eta} = 1.0$, and (c) $\tilde{\eta} = 1.7$. The relative motion of the source to the lens is assumed to be described by $\beta_x = t/t_0$ (t_0 is the time taken to cross the Einstein radius) and $\beta_y = 0.3$.

where $\mu = M_1/M_2 (\ll 1)$, and we have used for the normalization the Einstein radius of the mass M_2 , $\theta_E = \left(\frac{4GM_2}{c^2} \frac{D_{LS}}{D_L D_S} \right)^{\frac{1}{2}}$. As in the previous subsection, under the situation $\mu \ll 1$, we approximate the solutions of the lens equation up to the first order in μ . Using the zero-order solution (6), we obtain the first-order solution

$$\tilde{\theta}_{1x} = \frac{\left[\left(\tilde{\theta}_{0x}^2 + \tilde{\theta}_{0y}^2 \right)^2 - \left(\tilde{\theta}_{0x}^2 - \tilde{\theta}_{0y}^2 \right) \right] \left(\tilde{\theta}_{0x} - \tilde{\eta} \right) - 2\tilde{\theta}_{0x}\tilde{\theta}_{0y}^2}{\left[\left(\tilde{\theta}_{0x}^2 + \tilde{\theta}_{0y}^2 \right)^2 - 1 \right] \left[\left(\tilde{\theta}_{0x} - \tilde{\eta} \right)^2 + \tilde{\theta}_{0y}^2 \right]}, \quad (13)$$

$$\tilde{\theta}_{1y} = \frac{\tilde{\theta}_{0y} \left[\left(\tilde{\theta}_{0x}^2 + \tilde{\theta}_{0y}^2 \right)^2 + \left(\tilde{\theta}_{0x}^2 - \tilde{\theta}_{0y}^2 \right) - 2\tilde{\theta}_{0x} \left(\tilde{\theta}_{0x} - \tilde{\eta} \right) \right]}{\left[\left(\tilde{\theta}_{0x}^2 + \tilde{\theta}_{0y}^2 \right)^2 - 1 \right] \left[\left(\tilde{\theta}_{0x} - \tilde{\eta} \right)^2 + \tilde{\theta}_{0y}^2 \right]}. \quad (14)$$

Furthermore, we can calculate the magnification by using Eq. (8) and the inverse of the Jacobian $\det \left(\frac{\partial \tilde{\theta}_i}{\partial \theta_j} \right)$. Therefore, we can also obtain the microlensing-induced light curves

$$\Delta m = \Delta m(t) = 2.5 \log_{10} M(\beta_x(t), \beta_y(t); \mu). \quad (15)$$

Some examples of the light curves with different $\tilde{\eta}$ values are shown in Fig. 4 for $\mu = 0.05$, $\beta_x = t/t_0$, and $\beta_y = \text{const}$. As seen in these figures, the light curves with $\tilde{\eta}$ far away from 1 tend to have double peaks. The second peak corresponds to the effect of the planet of the mass $0.05M_2$. The ratio of the height at the peaks is almost determined by the mass ratio μ . Although we have restricted ourselves to the case of $\mu \ll 1$ in order to exclude the peculiar light curves, the light curves with double peaks are still obtained under some configurations. This is because the correction term $\left[(\theta_{0x} - \tilde{\eta}) + \theta_{0y}^2 \right]^{-1}$ becomes effective when θ_{0x} is equal to $\tilde{\eta}$, which corresponds to β far away from the origin if $\tilde{\eta}$ is not in the neighborhood of 1. (The more detailed discussion of planetary-binary lensing with dramatic features can be seen in Ref. [7].) On the other hand, when $\tilde{\eta}$ is nearly equal to 1, the almost time-symmetric light curves are obtained as seen in Fig. 4(b).

3 Rotating objects

Several relativistic effects also causes corrections to the point-mass lens. We only consider the dragging effect of inertia frames arising from a rotating object, since other spherical post-Newtonian terms never induce asymmetry in light curves. This additional effect is described with the spin angular momentum $\mathbf{J}$. We consider the projection of angular momentum of the rotating object onto the lens plane and define

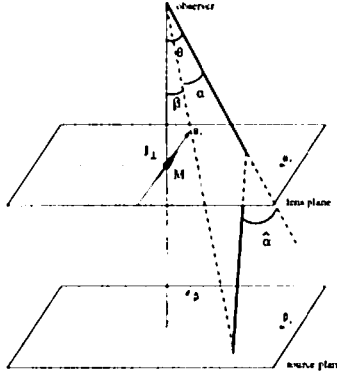


Figure 5: Geometry of lensing by a rotating object considered in §3.

the (θ_x, θ_y) coordinate system as the θ_y -axis is oriented parallel to the projected angular momentum $J_\perp$ (see Fig. 5). The deflection angles $(\hat{\alpha}_x, \hat{\alpha}_y)$ are written as [8]

$$\hat{\alpha}_x = \frac{4GM}{c^2 D_L} \frac{\theta_x}{\theta_x^2 + \theta_y^2} + \frac{4GJ_\perp}{c^3 D_L^2} \frac{\theta_x^2 - \theta_y^2}{(\theta_x^2 + \theta_y^2)^2}, \quad \hat{\alpha}_y = \frac{4GM}{c^2 D_L} \frac{\theta_y}{\theta_x^2 + \theta_y^2} + \frac{4GJ_\perp}{c^3 D_L^2} \frac{2\theta_x \theta_y}{(\theta_x^2 + \theta_y^2)^2}. \quad (16)$$

Hence, we have $\alpha = \frac{D_{LS}}{D_S} \hat{\alpha}$. Therefore, using the normalized quantities by the Einstein radius, the lens equation becomes

$$\tilde{\beta}_x = \tilde{\theta}_x - \frac{\tilde{\theta}_x}{\tilde{\theta}_x^2 + \tilde{\theta}_y^2} - \gamma \frac{\tilde{\theta}_x^2 - \tilde{\theta}_y^2}{(\tilde{\theta}_x^2 + \tilde{\theta}_y^2)^2}, \quad \tilde{\beta}_y = \tilde{\theta}_y - \frac{\tilde{\theta}_y}{\tilde{\theta}_x^2 + \tilde{\theta}_y^2} - \gamma \frac{2\tilde{\theta}_x \tilde{\theta}_y}{(\tilde{\theta}_x^2 + \tilde{\theta}_y^2)^2}, \quad (17)$$

where $\gamma = \frac{1}{\theta_E} \cdot \frac{J_\perp}{McD_L}$. It is interesting that this lens equation is the same as that in the two-point-mass lenses for the close binary $l \ll r_E$, with the correspondence $\gamma \leftrightarrow \epsilon$. Hence, the same asymmetric light curves are obtained. Furthermore, it will be impossible to distinguish such two different corrections. However, the parameter γ is quite small. For example, we consider a Kerr black hole. The angular momentum is $J \sim GM^2/c$, so that we have $\gamma \sim (GMD_S/(c^2 D_L D_{LS}))^{1/2} \ll 1$.

4 Quantitative estimate

In order to estimate asymmetry in the light curves quantitatively, we now introduce a notion of the skewness in statistics [9]. In statistics, the skewness for any distribution function $f(t)$ is defined as

$$\text{skewness} = \frac{1}{N} \int_{-\infty}^{\infty} \left(\frac{t - \mu}{\sigma} \right)^3 f(t) dt, \quad (18)$$

where N is the normalization factor, μ the mean, and σ the standard deviation. However, there is a problem in utilizing the skewness exactly as it is. To see this, we consider a single-point-mass lens. The

light curve is then given by $\Delta m = 2.5 \log_{10} \left[\left| \frac{\tilde{\theta}_+^2}{\tilde{\theta}_+^2 - 1} \right| + \left| \frac{\tilde{\theta}_-^2}{\tilde{\theta}_-^2 - 1} \right| \right]$. If we set $\tilde{\beta}_x = t/t_0$ and $\tilde{\beta}_y = \text{const.}$ and

consider the case that the time t approaches infinity, then we have $\tilde{\theta}_+ \rightarrow \tilde{\beta} = (t/t_0, \text{const.})$, $\tilde{\theta}_- \rightarrow 0$. It follows that at large t , $\Delta m \simeq \frac{2.5}{\ln 10} \cdot t^{-4}$. From this, we find that the integral $\int_{\mu}^{\infty} t^n \Delta m(t) dt$ diverges if $n \geq 3$. Therefore, we can not use Eq. (18) itself. Nevertheless, since the actually observed light curves necessarily have noises, it seems that the integral to infinity is meaningless. Under this consideration, we will define the skewness for the restricted region of a light curve $\Delta m > \lambda \Delta m_{\max}$, where $\Delta m_{\max}$ is the maximum value. We have introduced a cut-off $\lambda (0 < \lambda < 1)$, and the usual skewness corresponds

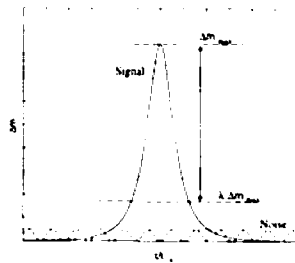


Figure 6: Illustration of discussion in §4. Schematic light curve with noise.

to $\lambda = 0$. The cut-off will naturally appear in the actual data, e.g., the region $\Delta m < \lambda \Delta m_{\max}$ may be meaningless due to the noise level (see Fig. 6). In microlensing-induced light curves, the bottom level is constant, but the maximum value may have some uncertainty as $(1 \pm \lambda) \Delta m_{\max}$.

We shall apply this tool to the almost identical light curves, which are caused from quite different physical situations. That is, the close binary and the planetary system. The skewness of these light curves are respectively displayed as a function of $1/\lambda$ in Fig. 7 and Fig. 8. Figure 7 shows the skewness of the light curve given in Fig. 2, while Fig. 8 corresponds to Fig. 4(b). The skewness with respect to different values of λ fully shows the asymmetric features of the light curves. The absolute values of the skewness depend on the small parameters (ϵ or μ), but the shapes are quite different in these two figures. For example, in Fig. 7 the skewness has the maximum at $1/\lambda \sim 2$, while in Fig. 8 the maximum appears at $1/\lambda \sim 7$. Therefore, we may discriminate the underlying cases for the asymmetry by the λ -dependent skewness for good signal-to-noise ratio ~ 10 .

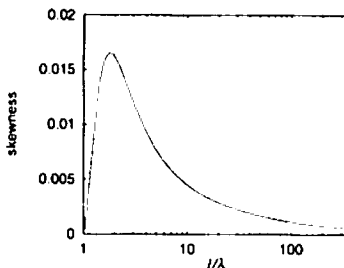


Figure 7: The skewness of a light curve by a two-point-mass lens with $l \ll r_E$. The light curve is calculated for the small parameter of $\epsilon = 0.1$ and the relative motion of $\tilde{\beta}_x = t/t_0$ and $\tilde{\beta}_y = 0.3$.

5 Summary and discussion

We have studied distortion in microlensing-induced light curves which are considered as ones of single-point-mass lenses at a first glance. In particular, we have attributed factors inducing the distortion to lenses itself and considered two sorts of corrections: corrections due to deviation from the Newtonian gravitational potential $\phi = -GM/r$ and corrections due to general relativistic effects of dragging of inertia frames arising from a rotating object. For simplicity, we have discussed two extreme situations in two-point-mass lenses for the corrections of the potential; one is the close binary where $l \ll r_E$, and the other the planetary system where $M_1 \ll M_2$. Moreover, we have considered the corrections up to the post-Newtonian order for the effect of dragging of inertia frames. From this, we found the same time-asymmetric light curves as in the two-point-mass lenses where $l \ll r_E$. Furthermore, we have introduced the cut-off dependent skewness and estimated the asymmetry in the light curves quantitatively. In

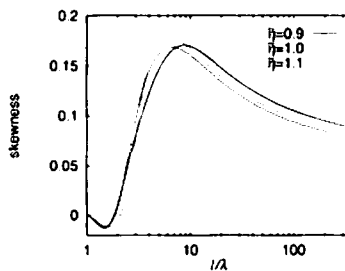


Figure 8: The skewness of light curves by a two-point-mass lens with $M_1 \ll M_2$. The light curves are calculated for the small parameter of $\mu = 0.05$ and the relative motion of $\tilde{\beta}_x = t/t_0$ and $\tilde{\beta}_y = 0.3$. The skewness is plotted for the light curves with different values of $\tilde{\eta}$, which is in the neighborhood of 1.

particular, we showed the clear difference in the skewness for almost similar light curves.

Here is a comment on the additional factor for the asymmetry in the binary system. In this paper, we have assumed that the lens systems in two-point-mass are fixed, but each star constituting the binary revolves around the center of mass. Therefore, the rotational effect may also appear. However, if the rotation period $T \sim (l^3/(G(M_1 + M_2)))^{1/2}$ of the binary is much larger than the typical time scale t_0 of a microlensing event, our consideration of the fixed lens systems would be appropriate. For the extreme case $T \ll t_0$, the time-averaged gravitational potential which is projected onto the lens plane may be regarded approximately as that of a single-point-mass lens or a fixed two-point-mass lens with $l \ll r_E$ if the lens is compact. Therefore, more complicated variations of light curves, corresponding the phase, are expected, only if $T \sim t_0$.

It is very interesting whether the time-asymmetric features of light curves discussed in this paper are actually detected or not by a MOA project and so forth, which are in progress.

Acknowledgements

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Wave optics in gravitational lensing ¹

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Abstract

The diffraction formula for the lensed wave amplitude is derived using the path-integral formulation. We reduce this formula to the geometric optics result in the short wavelength limit, to state clearly the conditions on which the wave effects are non-negligible in gravitational lensing.

1 Introduction

Gravitational lensing of astronomical objects causes magnification of their observed images. Since geometric optics is an excellent approximation in most astrophysical situations to compute this magnification, the wave optics, which is more fundamental, is discarded in many literatures on gravitational lensing.

In this paper we would like to review the wave aspects of gravitational lens theory[1]-[10]. The main purpose is to state clearly the conditions on which the wave effects are non-negligible. They are negligible in a usual situation as the optical lensing of quasars by galaxies, but are non-negligible in such extreme cases as femtolensing of gamma-ray bursts[11, 12], gravitational lensing of gravitational waves[13], scintillation of radio sources by electron density fluctuation[14], etc. More importantly, the geometric optics approximation breaks down near the lens mapping singularity (caustic), where this approximation gives an infinite brightness for a point source. This artifact of infinity should be remedied by the wave optics theory.

Since the wave optics is more fundamental than the geometric optics, it is possible to construct the gravitational lens theory on the basis of the former and with no reference to the latter. So we try to make this paper self-contained and readable without a prior background on the lens theory. We use the units $c = 1$.

2 Derivation of diffraction integral through the path integral approach

In this section we derive a formula (diffraction integral; eq.[8]) to calculate the amplitude of lensed waves through the path integral approach.²

2.1 Monochromatic waves from a point source

We consider a situation of figure 1; a point source of radiation emits spherical, monochromatic waves of frequency ω , which propagate through a gravitational lens and reach a distant observer. The space-time metric

$$ds^2 = -(1 + 2U)dt^2 + (1 - 2U)d\vec{r}^2 \quad (1)$$

is Minkowskian plus a small disturbance due to the gravitational potential $U(\vec{r})$ ($\ll 1$) of the lensing object. Influence of this potential is localized within a small region around the lens compared with the large distances between source, lens and observer. The propagation equation $\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = 0$ of the wave amplitude $\phi(\vec{r}, t) = \tilde{\phi}(\vec{r})e^{-i\omega t}$ is

$$(\nabla^2 + \omega^2)\tilde{\phi} = 4\omega^2 U\tilde{\phi} \quad (2)$$

¹This paper is the summary of our recent work to appear in PTP suppl.

²To our knowledge this approach is new to the gravitational lens theory.

to first order in U . We take the spherical coordinates (r, θ, φ) with origin at the source and polar-axis pointing toward the lens. The observer is located at $\vec{r}_0 = (r_0, \theta_0, \varphi_0)$ where $\theta_0 \ll 1$. Since the waves reaching the observer should be confined in the region of $\theta \ll 1$, we can set $\sin \theta \simeq \theta$ in the equations below, and regard $\theta = \theta(\cos \varphi, \sin \varphi)$ as a two-dimensional vector on a flat plane. Without the lensing object, the wave amplitude would be $\tilde{\phi}_0(r) = A e^{i\omega r}/r$. We define the amplification factor $F(\vec{r}) = \tilde{\phi}(\vec{r})/\tilde{\phi}_0(r)$ of wave amplitude due to the lensing. Equation (2) is rewritten in terms of F as

$$\frac{\partial^2 F}{\partial r^2} + 2i\omega \frac{\partial F}{\partial r} + \frac{1}{r^2} \nabla_\theta^2 F = 4\omega^2 U F, \quad (3)$$

where $\nabla_\theta^2 = \partial^2/\partial\theta^2 + \theta^{-1}\partial/\partial\theta + \theta^{-2}\partial^2/\partial\varphi^2$. Assuming that $\omega/|\partial \ln F/\partial r| \sim (\text{scale on which } F \text{ varies})/(\text{wavelength}) \gg 1$, we neglect the first term $\partial^2 F/\partial r^2$ compared with the second term (eikonal approximation). Then eq.(3) looks like the Schrödinger equation with the “time” coordinate r , the particle “mass” ω and the “time dependent potential” $2\omega U(r, \theta)$. The corresponding Lagrangian which yields the classical equation of motion is

$$L(r, \theta, \dot{\theta}) = \omega \left[\frac{1}{2} r^2 |\dot{\theta}|^2 - 2U(r, \theta) \right], \quad (4)$$

where $\dot{\theta} = d\theta/dr$. From the path integral formulation of quantum mechanics[15], the solution of eq.(3) is formally written as

$$F(\vec{r}_0) = \int \mathcal{D}\theta(r) \exp \left\{ i \int_0^{r_0} dr L[r, \theta(r), \dot{\theta}(r)] \right\}. \quad (5)$$

These notations mean; i) Choose a particular function $\theta(r)$ representing a path from source at the origin to observer at $\vec{r}_0 = (r_0, \theta_0)$; ii) Evaluate the integral in the curly bracket along this particular path. The result is the “phase” as a functional of $\theta(r)$; iii) Sum up these phases for all possible functions $\theta(r)$. These procedures become more concrete when one divides the distance r_0 between source and observer

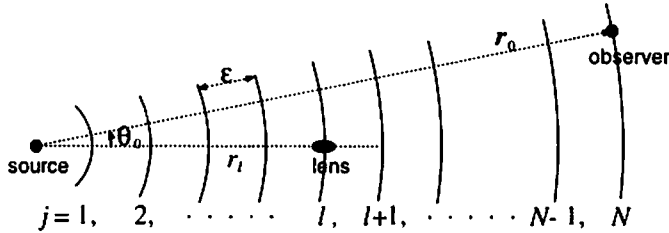


Figure 1:

into infinitesimally small “time” steps $\epsilon = r_0/N$ by a infinitely large integer N , and specify the path $\theta(r)$ by the set of coordinates $\theta_j = \theta(r_j)$ ($j = 1, \dots, N$) on the j -th sphere of radius $r_j = j\epsilon$ (figure 1). Let the lens be located on the l -th sphere (lens plane), so that $r_l = l\epsilon$ is the distance between source and lens. Here we introduce the “thin lens” approximation, whose formal content is to substitute $\frac{1}{2}\delta(r - r_l)\hat{\psi}(\theta)$ for $U(r, \theta)$ in eq.(5), where

$$\hat{\psi}(\theta) = 2 \int_0^{r_0} dr U(r, \theta). \quad (6)$$

The integral in this equation is performed for fixed θ . Thus eq.(5) is rewritten more concretely as

$$F(\vec{r}_0) = \left[\prod_{j=1}^{N-1} \int \frac{d^2\theta_j}{A_j} \right] \exp \left\{ i\omega \left[\epsilon \sum_{j=1}^{N-1} \frac{r_j r_{j+1}}{2} \left| \frac{\theta_j - \theta_{j+1}}{\epsilon} \right|^2 - \hat{\psi}(\theta_l) \right] \right\}, \quad (7)$$

where the normalizations $A_j = 2\pi i\epsilon/(\omega r_j r_{j+1})$ are such that $F = 1$ if $\hat{\psi} = 0$. Equation (6) does not at all mean that the lens is infinitesimally thin, but states that those paths which contribute to the phase integral (eq.[5]) are well approximated by a constant vector $\theta(r) \simeq \theta_l$ within the tiny region of $U(\vec{r}) \neq 0$

compared with the huge distances r_0 and r_l . In appendix it is shown that eq.(7) reduces straightforwardly to the desired formula

$$F(\vec{r}_0) = \frac{\omega}{2\pi i} \frac{r_l r_0}{r_{l0}} \int d^2\theta_l \exp\left\{i\omega \left[\frac{r_l r_0}{2r_{l0}} |\theta_l - \theta_0|^2 - \hat{\psi}(\theta_l) \right] \right\}, \quad (8)$$

where $r_{l0} = r_0 - r_l$ is the distance between lens and observer. Thanks to the thin lens approximation (eq.[6]) we could restrict the possible forms of functions $\theta(r)$ within those paths which go straight from source to a “deflection point” θ_l and again go straight from θ_l to observer. The first term is the path length difference between a straight path from source to observer and a deflected path penetrating the lens plane through θ_l . The second term is the time delay caused by the gravitational field around the lens.

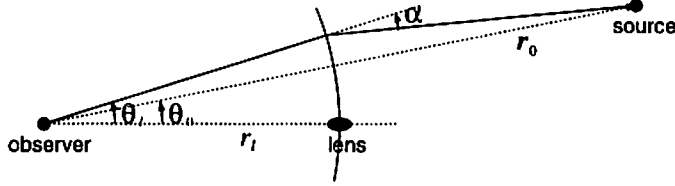


Figure 2:

Let us interchange the roles of source and observer, namely set another coordinate system (figure 2) in which origin is at the *observer*, polar-axis is pointing toward the lens, and the source is located at $\vec{r}_0 = (r_0, \theta_0)$. If one repeats the above procedure in this coordinate system, then the time delays in the two coordinate systems are identical up to order θ_0^2 . Therefore eq.(8) also expresses the lensed wave amplitude for the *source* position $\vec{r}_0$ if r_l is the distance between observer and lens, and r_{l0} is that between lens and source.

We rewrite eq.(8) in more compact notations. Introducing a characteristic angle scale θ_* and defining dimensionless quantities

$$\mathbf{x} = \theta_l / \theta_*, \quad \mathbf{y} = \theta_0 / \theta_*, \quad w = \frac{r_l r_0}{r_{l0}} \theta_*^2 \omega, \quad (9)$$

$$\psi(\mathbf{x}) = \frac{r_{l0}}{r_l r_0} \theta_*^{-2} \hat{\psi}(\theta_* \mathbf{x}), \quad T(\mathbf{x}, \mathbf{y}) = \frac{1}{2} |\mathbf{x} - \mathbf{y}|^2 - \psi(\mathbf{x}), \quad (10)$$

one obtains

$$F(w, \mathbf{y}) = \frac{w}{2\pi i} \int d^2x \exp[iw T(\mathbf{x}, \mathbf{y})]. \quad (11)$$

This is the diffraction integral formula expressing the amplification factor of lensed wave amplitudes as a function of frequency w and the source (or observer) position $\mathbf{y}$. The amplification of wave intensity is $|F|^2$. When the lensing object is spherically symmetric, then $\psi(\mathbf{x})$ depends only on $x = |\mathbf{x}|$ and eq.(11) becomes

$$F(w, \mathbf{y}) = -i w e^{i w y^2 / 2} \int_0^\infty dx x J_0(w x y) \exp\{i w [\frac{1}{2} x^2 - \psi(x)]\}, \quad (12)$$

where J_0 is the Bessel function of zeroth order.

Although the metric in eq.(1) does not take account of the expansion and geometry of our universe, all the formulae in this paper are applicable to cosmological situations, if we use the angular diameter distance for r_0 , r_l and r_{l0} , replace ω with $\omega(1+z)$ where z is the lens redshift, and assume that the wavelength is much smaller than the horizon scale. Also the scalar wave analysis in this paper is true for waves of any helicity, if the rotation of polarization direction (such as the Faraday rotation of electromagnetic wave) is negligible. Throughout this paper we use the scaled length and time as defined in eq.(9).

2.2 Non-monochromatic waves from a source of finite size

Let $\tilde{\phi}_0(w, \mathbf{y})$ be the amplitude of unlensed waves which are emitted at a point $\mathbf{y}$ on the source plane and with frequency w . The observed wave amplitude at time τ is the superposition of the lensed one

$F(w, \mathbf{y}) \tilde{\phi}_0(w, \mathbf{y})$ over $\mathbf{y}$ and w :

$$\phi(\tau) = \int d^2\mathbf{y} \int dw F(w, \mathbf{y}) \tilde{\phi}_0(w, \mathbf{y}) e^{-i\omega\tau}. \quad (13)$$

When the source is observed through a band pass filter, then $\tilde{\phi}_0(w, \mathbf{y})$ in this equation represents the source spectrum at $\mathbf{y}$ times the filtering function of frequency. Since $\phi(\tau)$ is real, $\phi_0^*(w, \mathbf{y}) = \tilde{\phi}_0(-w, \mathbf{y})$. Actually $\tilde{\phi}_0$ should also depend on time because of the fluctuation in the source activity. Assuming that the source activity is stationary in time, this fluctuation in $\tilde{\phi}_0$ would be a stochastic process. So we define the correlation function of observed wave amplitude as

$$C(\tau) = \frac{\langle \phi(\tau') \phi(\tau' + \tau) \rangle}{\langle |\phi(\tau')|^2 \rangle_{F=1}}, \quad (14)$$

where $\langle \dots \rangle$ denotes averaging over ensemble of stochastic functions $\tilde{\phi}_0(w, \mathbf{y})$. Because of the stationarity of the source activity, $\langle \dots \rangle$ also reads an integral with respect to τ' over sufficiently long time $\gg 1/w$. Assuming moreover that different parts of the source are radiating incoherently, one has $\langle \tilde{\phi}_0(w, \mathbf{y}) \tilde{\phi}_0^*(w', \mathbf{y}') \rangle = \hat{I}(w, \mathbf{y}) \delta(w - w') \delta^2(\mathbf{y} - \mathbf{y}')$, where $\hat{I}(w, \mathbf{y})$ is the source surface brightness (times the filtering function). Therefore one obtains

$$C(\tau) = \int d^2\mathbf{y} \int dw |F(w, \mathbf{y})|^2 \hat{I}(w, \mathbf{y}) \cos(\omega\tau), \quad (15)$$

where $\hat{I}(w, \mathbf{y}) = \hat{I}(w, \mathbf{y}) / [\int d^2\mathbf{y} \int dw \hat{I}(w, \mathbf{y})]$. The amplification of wave intensity due to lensing is given by $C(0)$.

3 Geometric optics limit

In this section we see how the formulae derived in the previous section reduce to the “geometric optics” result in the short wavelength limit, and discuss the conditions on which the wave effects are non-negligible.

3.1 Monochromatic waves from a point source

Let us imagine the two dimensional surface $T(\mathbf{x}) = \frac{1}{2}|\mathbf{x} - \mathbf{y}|^2 - \psi(\mathbf{x})$ (the time delay function; eq.[10]) above the $\mathbf{x}$ -plane for a fixed $\mathbf{y}$. For example $\psi(\mathbf{x}) = \ln|\mathbf{x}|$ for a point mass lens model. In the short wavelength limit $w \rightarrow \infty$, eq.(11) is an integral of a rapidly oscillating function, so the value of F is contributed from regions near the stationary points of this surface. The equation $\partial T(\mathbf{x}) = 0$ for the stationary points is

$$\mathbf{y} = \mathbf{x} - \partial\psi(\mathbf{x}) \quad \text{or} \quad \theta_0 = \theta_l - \frac{r_{l0}}{r_0} \alpha(\theta_l), \quad (16)$$

where $\partial = \partial/\partial\mathbf{x}$ and $\alpha(\theta_l) = r_l^{-1} \partial\hat{\psi}/\partial\theta_l$. These stationary points correspond to the “images” in geometric optics, and the condition $\partial T(\mathbf{x}) = 0$ just expresses the Fermat’s principle of least time. In fact eq.(16) is the “lens equation”[8] which determines the positions of images $\mathbf{x}$ (or θ_l) for a given source position $\mathbf{y}$ (or θ_0) with the deflection angle α of null geodesics (cf. figure 2). For a sufficiently “strong” lens potential $\psi(\mathbf{x})$, the number of images is more than one. We expand $T(\mathbf{x})$ around the j -th image $\mathbf{x}_j(\mathbf{y})$ as

$$T(\mathbf{x}) = T(\mathbf{x}_j) + \frac{1}{2} \sum_{ab} \bar{x}_a \bar{x}_b \partial_a \partial_b T(\mathbf{x}_j) + \frac{1}{6} \sum_{abc} \bar{x}_a \bar{x}_b \bar{x}_c \partial_a \partial_b \partial_c T(\mathbf{x}_j) + \dots, \quad (17)$$

where $\bar{x} = \mathbf{x} - \mathbf{x}_j$ and $abc\dots$ are the indices of two dimensional quantities. If the frequency w is so large as to satisfy

$$w|\partial^2 T|^3 \gg |\partial^3 T|^2, \quad w|\partial^2 T|^2 \gg |\partial^4 T|, \quad \dots \quad (18)$$

then the third and higher order terms of this expansion can be neglected in calculating eq.(11) near $\mathbf{x}_j$. Assuming for the moment that all the images satisfy eq.(18), one obtains using the Gauss integral³

$$F = \sum_j |\mu(\mathbf{x}_j)|^{1/2} \exp[iw T(\mathbf{x}_j) - i\pi n_j], \quad (19)$$

where

$$\mu(\mathbf{x}) = 1 / \det[\partial_a \partial_b T(\mathbf{x})] = 1 / \det[\delta_{ab} - \partial_a \partial_b \psi(\mathbf{x})], \quad (20)$$

and $n_j = 0, 1/2, 1$ when $\mathbf{x}_j$ is a minimum, saddle, maximum point of $T(\mathbf{x})$, respectively. Equation (19) says that the observed wave is a superposition of waves from each image, with the amplitude ratio $|\mu(\mathbf{x}_j)|^{1/2}$ and the phase shift $wT(\mathbf{x}_j) - \pi n_j$. The amplification of wave intensity is

$$|F|^2 = \sum_j |\mu_j| + 2 \sum_{j < k} |\mu_j \mu_k|^{1/2} \cos(w T_{jk} - \pi n_{jk}), \quad (21)$$

where $\mu_j = \mu(\mathbf{x}_j)$, $T_{jk} = T_k - T_j = T(\mathbf{x}_k) - T(\mathbf{x}_j)$ is the arrival time difference between k -th and j -th images, and $n_{jk} = n_k - n_j$. Note that in geometric optics, the magnification factor of image size for an infinitesimal source is given as the Jacobian of the lens mapping $\mathbf{y} \mapsto \mathbf{x}$ in eq.(16)[8]:

$$\mu(\mathbf{x}) = 1 / \det[\partial \otimes \mathbf{y}] = 1 / \det[\delta_{ab} - \partial_a \partial_b \psi(\mathbf{x})]. \quad (22)$$

From the conservation of surface brightness in geometric optics[16], this image magnification is equal to the amplification of source intensity. Thus the first term of eq.(21) coincides with the “geometric optics” result, where $|\mu_j|$ is the magnification factor for the j -th image. The second term expresses the “interference” between the images which produces a fringe pattern of wave intensity on the source (or observer) plane. Equation (19) is the semi-classical approximation in quantum mechanics.

3.2 Diffraction near lens mapping singularity

Geometric optics predicts that the image magnification $\mu(\mathbf{x})$ for a point source diverges when the image is located near critical curve, which is a curve on the $\mathbf{x}$ -plane defined by $\det[\partial_a \partial_b T(\mathbf{x})] = 0$ (cf. eq.[20]). Caustic is the corresponding curve on the $\mathbf{y}$ -plane mapped by eq.(16). Namely, when a point source is located near caustic, some of the images are near critical curve and have infinitely large magnifications. Of course this divergence is an artifact of the semi-classical approximation in eq.(19) because $|\partial^2 T|$ for these images is too small to satisfy eq.(18). Thus the wave amplitude is generally written as a sum $F = F_c + F_d$ of two terms. The second term F_d , diffraction part, is given by performing the diffraction integral (eq.[11]) exactly around these images near the critical curve. This term describes the diffraction phenomenon in the sense that the non-geodesic paths contribute significantly to the wave amplitude. The first term F_c , semi-classical part given by eq.(19), is the contribution from other normal images.

Equation (18) roughly says that the diffraction effect is unimportant when the time delay T_{jk} between images is much larger than the wave period $1/w$. This fact is also seen by the following simple consideration. Regarding the double image in gravitational lensing as the double-slit of the Young’s interference experiment, the angle of diffraction is $\theta \sim \lambda/d$ where λ is wavelength and d is distance between two images. On the other hand the deflection angle of gravitational lensing is $\alpha \sim d/r_l$ where r_l is distance from observer to lens. Then the condition $\theta \ll \alpha$ is equivalent to say that the time delay between images $\sim r_l \alpha^2 / c$ is much larger than λ/c .

3.3 Non-monochromatic waves from a source of finite size

Let Δy and Δw be the typical sizes of the region where the source surface brightness I is non-zero, and let $\mathbf{y}$ and w be the centers of this region. For example, Δy is the radius of a circular source whose center is at $\mathbf{y}$, and Δw is the width of band pass filter whose central frequency is w . For definiteness we assume

³ using $\int_{-\infty}^{\infty} dx e^{ix^2} = \sqrt{\pi} e^{i\pi/4}$ after diagonalizing $\partial_a \partial_b T$

that the source is sufficiently far from caustic so that the semi-classical approximation (eq.[21]) is valid in whole this region. Then substituting eq.(21) into (15) results in

$$C(\tau) = \text{Re}\{e^{-i\omega\tau}[C_0(\tau) + C_1(\tau) + C_2(\tau)]\}, \quad (23)$$

where

$$C_0(\tau) = \sum_j \int d^2\mathbf{y}' |\mu_j(\mathbf{y}')| \tilde{I}(\tau, \mathbf{y}'), \quad (24)$$

$$C_{1,2}(\tau) = \sum_{j < k} \int d^2\mathbf{y}' |\mu_j(\mathbf{y}') \mu_k(\mathbf{y}')|^{1/2} e^{\mp i[w T_{j,k}(\mathbf{y}') - \pi n_{j,k}]} \tilde{I}[\tau \pm T_{j,k}(\mathbf{y}'), \mathbf{y}'], \quad (25)$$

and $\tilde{I}(\tau, \mathbf{y}') = \int_0^\infty dw' I(w', \mathbf{y}') e^{-i(w' - \omega)\tau}$. The terms C_1 and C_2 arise from the interference between images, but the integral in eq.(25) tend to erase the fringe pattern produced by the interference. If the source size Δy is much larger than the typical fringe width $1/|w dT_{j,k}/dy|$, or if the coherence time $1/\Delta\omega$ is much smaller than the typical time delay $|\tau \pm T_{j,k}(\mathbf{y})|$, then this oscillatory integral makes C_1 and C_2 much smaller than C_0 . Conversely, when the source size is comparable to or smaller than the fringe width, then we can always choose such a value of τ as to make the interference term C_1 or C_2 observable. In the short wavelength limit $w \rightarrow \infty$, the fringe width goes to zero and therefore the interference terms vanish. Geometric optics is to approximate $C(\tau) = \text{Re}[e^{-i\omega\tau} C_0(\tau)]$ even when the source is located near or straddling on a caustic. In this case eq.(24) remains finite though some of $\mu_j(\mathbf{y})$ are divergent there.

In order to observe the interference term C_1 and C_2 , not only the source size but also the observer size (or diameter of telescope) must be smaller than the fringe width. Otherwise the further integration of eq.(25) over $\mathbf{y}$ smears out the interference pattern. Using the same argument as that given at the end of section 3.2, this fringe width on the observer plane is of order $\sim \lambda/\alpha$ where λ is wavelength and α is deflection angle. On the other hand, in order to resolve the multiple image formed by gravitational lensing, the observer size must be larger than $\sim \lambda/\alpha$. Therefore it is impossible to observe both the interference pattern and the multiple image simultaneously, like the uncertainty principle in quantum mechanics.

This paper will be continued in PTP supplement to show the examples of wave effects for some lens models[6, 7, 8, 9] and to present the numerical method of performing the diffraction integral[10].

A

The first $l-1$ integrals from $j=1$ to $l-1$ in eq.(7) are the Gauss integrals and give unity, so $j=1$ under $\prod$ and $\sum$ can be replaced by $j=l$. We use the following identity (proven by induction)

$$\sum_{j=l}^{N-1} r_j r_{j+1} |\boldsymbol{\theta}_j - \boldsymbol{\theta}_{j+1}|^2 = \epsilon \frac{r_l r_0}{r_{l0}} |\boldsymbol{\theta}_l - \boldsymbol{\theta}_0|^2 + \sum_{j=l+1}^{N-1} r_j^2 \frac{r_{l,j+1}}{r_{lj}} |\boldsymbol{\theta}_j - \mathbf{u}_{lj}|^2, \quad (26)$$

where $r_0 = r_N$, $r_{lj} = r_j - r_l$ and $\mathbf{u}_{lj} = (\epsilon r_l \boldsymbol{\theta}_l + r_{lj} r_{j+1} \boldsymbol{\theta}_{j+1}) / (r_j r_{l,j+1})$. Thus eq.(7) becomes

$$\begin{aligned} F(\tilde{r}_0) &= \int \frac{d^2\boldsymbol{\theta}_l}{A_l} \exp\left\{i\omega \left[\frac{r_l r_0}{2r_{l0}} |\boldsymbol{\theta}_l - \boldsymbol{\theta}_0|^2 - \hat{\psi}(\boldsymbol{\theta}_l)\right]\right\} \\ &\times \left[\prod_{j=l+1}^{N-1} \int \frac{d^2\boldsymbol{\theta}_j}{A_j} \right] \exp\left[\frac{i\omega}{2\epsilon} \sum_{j=l+1}^{N-1} r_j^2 \frac{r_{l,j+1}}{r_{lj}} |\boldsymbol{\theta}_j - \mathbf{u}_{lj}|^2\right]. \end{aligned} \quad (27)$$

Performing the Gauss integrals in the second line of this equation results in

$$\text{2nd line of (27)} = \prod_{j=l+1}^{N-1} \frac{1}{A_j} \frac{2\pi i \epsilon}{\omega r_j^2} \frac{r_{lj}}{r_{l,j+1}} = \prod_{j=l+1}^{N-1} \frac{r_{j+1}}{r_j} \frac{r_{lj}}{r_{l,j+1}} = \frac{r_0}{r_{l+1}} \frac{\epsilon}{r_{l0}}. \quad (28)$$

Substituting this into eq.(27) yields eq.(8).

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Parametric Amplification of Density Perturbation in the Oscillating inflation

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Abstract

We study the evolution of cosmological perturbation in the *oscillating inflation*, proposed by Damour and Mukhanov. In this model, the accelerated expansion takes place during the coherently oscillating stage and there is a possibility of the parametric amplification of the adiabatic long-wavelength perturbation. We analytically show that the curvature perturbation can be amplified by the parametric resonance and the instability band becomes very broad in the oscillating inflation. We also numerically examined this issue and found that the parametric resonance is strongly suppressed for the perturbation on large scales comparable to the Hubble horizon. Therefore, in the single field case of the oscillating inflation, there would be no significant imprint of the parametric resonance on the adiabatic perturbation spectrum.

1 Introduction

Dynamics of the coherently oscillating scalar field plays important roles in the early stage of the universe. Specifically, the significance of parametric resonance is recently found in the reheating process[1]. Due to the coherent oscillation of the background inflaton field, the light boson field or the fluctuation of the inflaton itself is amplified through the non-linear interaction or the self-interaction. The efficiency of the parametric resonance could have the important implication for GUT scale baryogenesis.

From the viewpoint of the structure formation in the early universe, the influence of parametric resonance on the metric perturbation during the coherently oscillating stage after inflation is investigated by several authors[2, 3, 4]. The important conclusion of these analyses is that unless the iso-curvature perturbation plays an important role, the scalar-type metric perturbation on super-horizon scales behaves like as the perfect fluid, in which no parametric amplification appears. This indicates that the metric perturbation prevents the fluctuations of the matter field from the enormous amplification by the parametric resonance.

However, there remains a possibility that the growth of density perturbation by the parametric resonance can occur on well inside the Hubble horizon. As usual, the universe expands decelerately during the coherently oscillating stage and the parametric resonance inside the Hubble horizon cannot affect the density perturbation on super-horizon scale. However, Damour and Mukhanov [5] recently considered the model with the non-convex type potential. In this model, remarkably, the accelerated expansion can take place during the oscillating stage. Liddle and Mazumdar called it *oscillating inflation* and confirmed this fact numerically [6].

As was suggested by Damour and Mukhanov, it could be expected that the quantum fluctuation is enormously amplified due to the very broad parametric resonance. This implies that the adiabatic density perturbation with the large amplitude might be produced on super-horizon scales in course of the accelerated expansion. Therefore, there exists a possibility that the cosmological objects such as the primordial black holes may be formed by the large amplitude of the perturbation.

In this article, to explore the possibility suggested by Damour and Mukhanov, we pay an attention to the evolution of the cosmological perturbation in the oscillating inflation. We describe the model in Sec.2. The parametric resonance of metric perturbation is discussed in Sec.3. We show that the efficiency of the parametric amplification on the spectrum of the curvature perturbation depends on the energy scale of the oscillating inflation and the growth factor. Analyzing the evolution equation for perturbation analytically and numerically, we evaluate the growth factor in section 4. The concluding remarks are presented in the last section 5.

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2 Oscillating inflation model

Let us consider the minimally coupled scalar field ϕ in the flat FRW universe. The homogeneous background equations become

$$3H^2 = \frac{1}{\tilde{M}_{pl}^2} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) \right), \quad \ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0, \quad (1)$$

where H is the Hubble parameter and $\tilde{M}_{pl}$ is the reduced Planck mass given by $\tilde{M}_{pl} = \sqrt{3/8\pi} M_{pl}$. The evolution of the scalar field depends on the shape of the potential $V(\phi)$. We shall consider the model of *oscillating inflation* proposed by Damour and Mukhanov[5], in which the potential is given by

$$V(\phi) = \frac{A}{q} \left[\left(\frac{\phi^2}{\phi_c^2} + 1 \right)^{q/2} - 1 \right]. \quad (2)$$

When the effective mass of the inflaton is larger than the Hubble parameter, i.e. $|V_{,\phi\phi}| \gg H^2$, the inflaton field shows oscillatory behavior. For the large amplitude $\phi \gg \phi_c$, taking the time average during a period of oscillation yields the virial theorem $\langle \dot{\phi}^2 \rangle \simeq q \langle V(\phi) \rangle$. Using this relation, the inflaton field can be regarded as the perfect fluid matter with the equation of state $P = (\gamma_{osc} - 1)\rho$, where γ_{osc} is the effective adiabatic index given by $\gamma_{osc} = 2q/(q + 2)$. Thus the time dependence of the scale factor, the amplitude of the inflaton field and the inflaton potential are obtained[5]:

$$a(t) \propto t^{(q+2)/3q}, \quad \bar{\phi}(t) \propto a^{-6/(q+2)}, \quad V(\bar{\phi}) \propto a^{-6q/(q+2)}, \quad (3)$$

where $\bar{\phi}$ is the amplitude of the scalar field evaluated at the time that $\dot{\phi} = 0$ holds. Eq.(3) implies that if the potential index q is smaller than unity, the universe can experience the accelerated expansion even in the oscillating phase. When the scalar field approaches to the critical value $\bar{\phi} \simeq \phi_c$, the inflation ends. Hereafter, we shall restrict our attention on the inflationary phase with the potential index $q < 1$.

3 Cosmological perturbation

We now investigate the cosmological perturbation during the oscillating inflation. We shall ignore the interaction with the other scalar fields which lead to the inflaton decay in the reheating process[1]. In Ref.[2, 3], the well-behaved perturbed quantity during the oscillating stage is given as $Q = \delta\phi - (\dot{\phi}/H)\mathcal{R}$, where $\delta\phi$ and $\mathcal{R}$ are the perturbation of scalar field and spatial curvature, respectively. The quantity Q is the gauge-invariant variable and the evolution equation is simply given by

$$\ddot{Q} + 3H\dot{Q} + \left[\left(\frac{k}{a} \right)^2 + V_{,\phi\phi} + 6\tilde{M}_{pl}^{-2} \left(\frac{V(\phi)}{H} \right) \right] Q = 0. \quad (4)$$

If we neglect the third term in the bracket, the above equation is just reduced to the equation for perturbation $\delta\phi$ ignoring the metric perturbation. We thus understand that the effect of gravitational interaction to the scalar field perturbation is encoded in the third term in the bracket.

In equation (4), there exists the exact solution in the long-wavelength limit. For $k \rightarrow 0$, we have

$$Q = c_1 \frac{\dot{\phi}}{H} + c_2 \frac{\dot{\phi}}{H} \int \frac{dt}{a^3} \frac{H^2}{\dot{\phi}^2}, \quad (5)$$

where c_1 and c_2 are the integration constants. The solution proportional to the coefficient c_1 is named as the growing mode and the one proportional to c_2 as the decaying mode. For the growing mode, the curvature perturbation on comoving slice defined by

$$\mathcal{R}_c = \frac{H}{\dot{\phi}} Q \quad (6)$$

remains constant in time. In this article, matching the amplitude of quantum fluctuation Q produced during the oscillating inflation inside the horizon with that of the long wavelength growing mode, we will evaluate the power spectrum of the curvature perturbation $\mathcal{P}_{\mathcal{R}}(k)$.

We focus on the evolution of the quantity Q on sub-horizon scales. In equation (4), the time dependence of the terms in the bracket is estimated by using the results (3):

$$V_{,\phi\phi}(\bar{\phi}) \propto a^{-6(\frac{q-2}{q+2})}, \quad \left(\frac{V(\phi)}{H} \right) \Big|_{\phi=\bar{\phi}} \simeq \frac{V_{,\phi}\bar{\phi}}{H} \Big|_{\phi=\bar{\phi}} \propto a^{-6(\frac{q-1}{q+2})}, \quad (7)$$

which are valid for $\bar{\phi} \gg \phi_c$. During the oscillating inflation, the term $V_{,\phi\phi}$ becomes dominant compared to the third term in the bracket, as opposed to the situation in Ref.[2, 3]. We then introduce the following variables:

$$Q = \bar{\phi}(t)\hat{Q}(\eta), \quad d\eta = g(t)dt \quad ; \quad g(t) \equiv \frac{\sqrt{A}}{\bar{\phi}} \left(\frac{\bar{\phi}}{\phi_c} \right)^{q/2} \propto a^{-3(\frac{q-2}{q+2})}. \quad (8)$$

Ignoring the third term in the bracket, the evolution equation (4) is rewritten as the normal form :

$$\hat{Q}_{,\eta\eta} + \left[\left(\frac{k}{ga} \right)^2 + \hat{V}_{,\hat{\phi}\hat{\phi}}(\bar{\phi}) + \dots \right] \hat{Q} = 0, \quad (9)$$

where $\bar{\phi}$ is defined by $\phi(t) = \bar{\phi} \bar{\phi}(\eta)$ and satisfies the equation of motion

$$\bar{\phi}_{,\eta\eta} + \frac{d\bar{V}}{d\bar{\phi}} = 0 \quad ; \quad \bar{V}(\bar{\phi}) = \frac{1}{q} \left[\left(\bar{\phi}^2 + \frac{\phi_c^2}{\bar{\phi}^2} \right)^{q/2} - \left(\frac{\phi_c}{\bar{\phi}} \right)^q \right], \quad (10)$$

with the initial condition $\bar{\phi}(\eta_i) = 1$. The quantity $\hat{V}_{,\hat{\phi}\hat{\phi}}$ in (9) is obtained by differentiating twice the effective potential $\bar{V}$ with respect to $\bar{\phi}$. Since we have $\bar{V} \simeq \bar{\phi}^q/q$, $\bar{\phi}$ oscillates rapidly during the oscillating inflation. Treating the time dependence of the term $\bar{V}$ adiabatically, the period of oscillation $\bar{\phi}$ is estimated as

$$T = 4 \int_0^1 \frac{d\bar{\phi}}{\sqrt{2(q^{-1} - \bar{V})}} \simeq \sqrt{\frac{8\pi}{q}} \frac{\Gamma(\frac{1}{q})}{\Gamma(\frac{1}{q} + \frac{1}{2})}, \quad (11)$$

in terms of the time parameter η . For the range $0.01 \leq q \leq 1$, we have $T \simeq 5$.

According to the Floquet theorem, presence of the oscillating term in (9) implies that there exists the following solution for the equation (9):

$$\hat{Q}(\tau) = e^{\mu\tau} P(\tau) \quad ; \quad P(\tau + \pi) = P(\tau), \quad (12)$$

where we define the new time parameter $\tau \equiv (2\pi/T)\eta$. The periodic function $P(\tau)$ is bounded and has a finite amplitude. The characteristic exponent μ takes either a imaginary or a real number, which depends on the shape of $\hat{V}_{,\hat{\phi}\hat{\phi}}$ and the parameter k/ga . The parametric resonance means that the characteristic exponent has a real number.

We can see that the parametric resonance can affect the curvature perturbation $\mathcal{R}_c$ produced during the oscillating inflation as follows. In the slow-rolling inflation, the amplitude of quantum fluctuation is given by $Q \simeq |\delta\phi| \sim H/2\pi$. On the other hand, the amplification of quantum fluctuation can appear in the oscillating inflation due to the parametric resonance. We have

$$\left| \frac{\hat{Q}_*}{\hat{Q}_0} \right| \simeq e^{\mu\Delta\tau}, \quad (13)$$

where subscripts $(_0)$ and $(_*)$ denotes the quantities evaluated at the beginning of oscillating inflation and at the time when the wavelength of the quantum fluctuation exceeds the Hubble horizon scale ($k = a_* H_*$), respectively. Here, $\Delta\tau$ denotes the time interval during which the quantum fluctuation is amplified by the parametric resonance. Therefore, for the modes $a_0 Q_0 \rightarrow \exp(i \int dt/a)/\sqrt{2k}$ inside the horizon initially, the power spectrum of curvature perturbation (6) can be roughly estimated by matching the amplitude of short wavelength fluctuation with that of the long wavelength solution (5). The result becomes

$$\mathcal{P}_{\mathcal{R}}^{1/2}(k) \equiv \sqrt{\frac{k^3}{2\pi^2}} \langle |\mathcal{R}_c|^2 \rangle_{ens}^{1/2} \simeq \frac{\gamma_{osc}^{-1}}{2\pi} \left(\frac{\bar{\phi}_*}{\bar{\phi}_0} \right)^{(4-q)/(2+q)} \left(\frac{V_*^{1/4}}{\bar{M}_{pl}} \right)^2 e^{\mu\Delta\tau}, \quad (14)$$

which is evaluated at the horizon-crossing time. $\langle \cdots \rangle_{ens}$ denotes the ensemble average. In the derivation of the expression (14), we have naively replaced the ensemble average $\langle \dot{\phi}^2 \rangle_{ens}$ with the time average $\langle \dot{\phi}^2 \rangle$ and used the virial theorem. Eq.(14) shows that the appearance of exponential factor $e^{\mu \Delta \tau}$ may give the significant contribution to the spectrum of curvature perturbation, although the amplitude of $\mathcal{P}_{\mathcal{R}}$ is always suppressed by the energy scale of the oscillating inflation $V_*^{1/4} \equiv V^{1/4}(\bar{\phi}_*)$.

4 Parametric Amplification during oscillating inflation

In the previous section, we have seen that the effect of parametric resonance can lead to the amplification of the curvature perturbation, which may lead to $\mathcal{P}_{\mathcal{R}} \sim 1$. In this section, to explore the efficiency of the parametric amplification, we analytically and numerically investigate the growth factor $e^{\mu \Delta \tau}$.

4.1 Analytic estimation

Let us evaluate the characteristic exponent μ from the reduced equation (9). Using the solution (12) obtained from the Floquet theorem, we have the following formula[9]:

$$\mu = \frac{1}{\pi} \log \left[\sqrt{F^2} + \sqrt{F^2 - 1} \right] \quad ; \quad F = 1 + 2\tilde{Q}'_1(\tau = \frac{\pi}{2})\tilde{Q}_2(\tau = \frac{\pi}{2}), \quad (15)$$

where the prime denotes the differentiation with respect to the time τ . $\tilde{Q}_1$ and $\tilde{Q}_2$ are the solutions satisfying the initial conditions $\tilde{Q}_1 = 1$, $\tilde{Q}'_1 = 0$ and $\tilde{Q}_2 = 0$, $\tilde{Q}'_2 = 1$ at $\tau = 0$, respectively.

Although it is difficult to obtain the solutions $\tilde{Q}_1$ and $\tilde{Q}_2$ for our complicated potential $\tilde{V}_{\tilde{\phi}\tilde{\phi}}$, we can get the approximate expression of the characteristic exponent. The effective mass of the oscillating inflaton field becomes negative at the shallow wings of the potential and positive at the minimum. Thus we replace the periodic function $\tilde{V}_{\tilde{\phi}\tilde{\phi}}$ with the *step-wise* function given by

$$\tilde{V}_{\tilde{\phi}\tilde{\phi}} \longrightarrow \tilde{V}_{\tilde{\phi}\tilde{\phi}}(\tilde{\phi} = 0) \quad \text{for } |\tau| < \tau_1, \quad \tilde{V}_{\tilde{\phi}\tilde{\phi}}(\tilde{\phi} = 1) \quad \text{for } \tau_1 < |\tau| < \frac{\pi}{2}, \quad (16)$$

where we choose

$$\tau_1 = \frac{2\pi}{T} \int_0^{\tilde{\phi}_1} \frac{d\tilde{\phi}}{\sqrt{2(q^{-1} - \tilde{V})}} \quad ; \quad \tilde{\phi}_1 = \sqrt{\frac{3}{1-q}} \frac{\phi_c}{\bar{\phi}}.$$

At $\tilde{\phi} = \tilde{\phi}_1$, we have $\tilde{V}_{\tilde{\phi}\tilde{\phi}} = 0$. The approximation (16) enables us to obtain the analytic solutions $\tilde{Q}_1$ and $\tilde{Q}_2$. Then the quantity F is evaluated as follows:

$$F = \frac{1}{2} \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha} \right) \sin(2\alpha\tau_1) \sin(2\beta(\tau_1 - \frac{\pi}{2})) + \cos(2\alpha\tau_1) \cos(2\beta(\tau_1 - \frac{\pi}{2})), \quad (17)$$

where

$$\alpha = \frac{T}{2\pi} \left[\left(\frac{k}{ga} \right)^2 + \tilde{V}_{\tilde{\phi}\tilde{\phi}}(\tilde{\phi} = 0) \right]^{1/2}, \quad \beta = \frac{T}{2\pi} \left[\left(\frac{k}{ga} \right)^2 + \tilde{V}_{\tilde{\phi}\tilde{\phi}}(\tilde{\phi} = 1) \right]^{1/2}.$$

In Fig.1, we plot the characteristic exponent μ as a function of k/ga . The solid line shows the approximation evaluated from (15) and (17). The parameters are chosen as $\phi_c = 10^{-6} \tilde{M}_{pl}$ and $q = 0.1$. As for the quantity $\tilde{\phi}$, we treat it as constant and set $\tilde{\phi} = q \tilde{M}_{pl}/\sqrt{6}$, corresponding to the initial value of the oscillating inflation [6]. The circles show the characteristic exponent obtained from the substitution of the numerical solutions of equations (9) and (10) into the formula (15). We see that the approximation predicts the numerical result reasonably well. The crucial observation is that the instability band is very broad and the characteristic exponent can be of the order of unity.

To evaluate the growth factor $e^{\mu \Delta \tau}$, we further need the period $\Delta \tau$. It is given by

$$\Delta \tau = \frac{2\pi}{T} \int dt g(t) = \sqrt{\frac{\pi}{72}} q(q+2) \frac{\Gamma(\frac{1}{q} + \frac{1}{2})}{\Gamma(\frac{1}{q})} \left(\frac{\tilde{M}_{pl}}{\bar{\phi}_*} - \frac{\tilde{M}_{pl}}{\bar{\phi}_0} \right), \quad (18)$$

where the integral is evaluated from the time at which the oscillating inflation start to the horizon-crossing time. Eq.(18) is valid for the case $q > 0$. Rewriting the above equation with the relation $k = a_* H_*$, the wave number dependence of the growth factor $e^{\mu \Delta \tau}$ shows that the characteristic peak appears on the spectrum of curvature perturbation (14). Because of the rapid oscillation of the inflaton field, the period $\Delta \tau$ effectively becomes very large. Therefore, even if we get the rather small value of the characteristic exponent $\mu = 0.01$, for example, the contribution of the growth factor to the curvature perturbation becomes significantly large. For $\phi_* = 10^{-5} \bar{M}_{pl}$ and $q = 0.1$, we have $e^{\mu \Delta \tau} = 10^{182}$!

4.2 Numerical result

The analytic estimation in the previous subsection overestimates the growth factor. Rigorously, we must take into account the time dependence of $\tilde{\phi}$ and k/ga . Furthermore, remember that the previous result comes from the analysis ignoring the metric perturbation. This also leads to the overestimation of the growth factor. However, there still exists a possibility of the parametric amplification of the curvature perturbation spectrum. To clarify the influence of parametric resonance, we numerically solve the equations (1) and (4).

Fig.2 shows the growth factor $e^{\mu \Delta \tau_c}$ for the index $q = 0.1$, i.e, the ratio of the amplitude $\tilde{Q}_c$ evaluated at the end of oscillating inflation to the amplitude $\tilde{Q}_0$ at the beginning of inflation by varying the parameter ϕ_c . For each value ϕ_c , we start to calculate the background evolution by setting the value $\phi_0 = q \bar{M}_{pl} / \sqrt{6}$ corresponding to the final value of the slow-rolling inflation. As for the fluctuations, the initial conditions $\tilde{Q}'_0 = 0$ are chosen. Although we have checked the evolution for the various initial conditions $\tilde{Q}_0 \propto \cos(k \int dt/a)$, corresponding to the vacuum state in the short wavelength limit, the maximal amplitude can be obtained from the initial condition $\tilde{Q}'_0 = 0$. Fig.2 shows that the amplification for the fluctuations occurs by the effect of parametric resonance. It is obvious that the maximal amplitude depends on the parameter ϕ_c . For the smaller value of ϕ_c , the amplitude of the fluctuation becomes larger, which comes from the fact that the period of inflation represented by the e-folding number $N_c = \log_e(a_c/a_0)$ depends on the critical value ϕ_c [5, 6]. When the oscillating inflation takes place for a longer time, the effect of parametric resonance works out more efficiently and the fluctuations are significantly amplified. In Fig.2, the three different lines are depicted. The solid line is the fluctuation with the wavelength ten times smaller than the radius of the Hubble horizon at the end of oscillating inflation, i.e, $k = 10a_c H_c$. The dashed line has the wavelength $k = 2a_c H_c$ and the dotted line corresponds to the fluctuation whose wavelength just reaches at the Hubble radius after oscillating inflation. It is remarkable that the modes inside the Hubble horizon are largely amplified, while the amplification of the long-wave mode comparable to the Hubble horizon is strongly suppressed. Though the dotted line apparently takes the growth factor $\sim 10^7$, it can be recognized that the parametric resonance becomes ineffective for the long-wave modes.

To see the efficiency of the parametric amplification explicitly, we plot in Fig.3 the time evolution of the fluctuations using the quantity Q from the beginning to the end of oscillating inflation. We set the parameters $q = 0.1$ and $\phi_c = 10^{-5} \bar{M}_{pl}$. The horizontal axis represents the cosmic time normalized by the factor $\sqrt{A}/\bar{M}_{pl}$. Each figure shows the time evolution with the different wavelength under the same initial conditions $Q_0 = 1.0$ and $Q'_0 = 0$: (a) $k = 10a_c H_c$; (b) $k = 2a_c H_c$; (c) $k = a_c H_c$. These figures clearly reveal that the growth of the perturbation is much sensitive to the wavelength of the fluctuations. The parametric amplification works efficiently inside the horizon, however, the amplitude Q becomes constant for the wavelength near the Hubble horizon.

We can understand these behavior as follows. As was described in Sec.3, there exists the exact solution for the variable Q in the limit $k \rightarrow 0$. For the growing mode, the amplitude of the long wavelength solution becomes nearly constant, which can be deduced from (3). This behavior can also be obtained in the case of $k \neq 0$, when the term $(k/a)^2$ in equation (4) is negligible compared to the terms $V_{\phi\phi}$ and $\bar{M}^{-2}(V/H)$, which gives the condition $k/a \ll H$ [3]. Fig.3 indicates that the term induced by the gravitational perturbation plays a crucial role even for the modes comparable to the Hubble horizon. This is consistent with the result of the paper [2] and [3]. Therefore, the spectrum of the curvature perturbation produced during oscillating inflation on super-horizon scales has the rather small amplitude compared with the analytic prediction.

5 Concluding remarks

In this article, we investigated the cosmological perturbation in the oscillating inflation. Analytical estimation shows that the perturbation can be amplified by the broad band parametric resonance, however, we numerically found that the amplification of the curvature perturbation is strongly suppressed on large scales comparable to the Hubble horizon.

Now discuss the cosmological implication of the fluctuations produced during the oscillating inflation. According to Ref.[7], the physical length of the fluctuation produced during the oscillating inflation is evaluated at the present time:

$$\lambda_{phy} \lesssim 10^{1+0.43\alpha_*} \text{ cm}; \quad \alpha_* = N_* + \log_e \left(\frac{10^{16} \text{ GeV}}{V_*^{1/4}} \right). \quad (19)$$

where λ_{phy} denotes the present wavelength. N_* denotes the e-folding number of the oscillating inflation evaluated at the Hubble crossing time. On the other hand, if the amplitude $\mathcal{R}_c$ produced during the inflation becomes of the order of unity, the fluctuations can experience the gravitational collapse. Assuming that the primordial black holes are formed at the horizon re-entry time during the radiation epoch, we can evaluate the typical mass of a black hole using the equation (19). We obtain

$$M_{BH} \lesssim 10^{1+0.86\alpha_*} \text{ g}, \quad (20)$$

which is evaluated at the formation epoch. α_* is the same quantity as defined in equation (19).

As is shown by Liddle and Mazumdar [6], a period of oscillating inflation is brief. This can be checked in our numerical calculation. For the critical value $\phi_c = 10^{-6} M_{pl}$, we obtain the small value of the e-folding number $N_* = 3.5$. Hence, the expressions (19) and (20) show that the energy scale of the oscillating inflation should be rather low to have the cosmologically interesting scale for the black hole mass and the physical size of the fluctuations. Asaka, Kawasaki and Yamaguchi [8] investigated the inflation model to resolve the cosmological moduli problem, in which the oscillating inflation occurs at the energy scale $V_*^{1/4} \simeq 10 \text{ GeV}$. In this energy scale, if we have the e-folding number $N_* \simeq 3$, it is possible to get $M_{BH} \sim 1 M_\odot$, as a candidate for explaining the recent observation of the massive compact halo objects, although the physical length is still small, $\lambda_{phy} \sim 1 \text{ pc}$.

However, the amplitude of the vacuum fluctuation becomes extremely small in the inflation at low energy scale. We saw in Sec.3 that the spectrum of curvature perturbation $\mathcal{P}_R$ given by (14) contains the suppression factor $(V_*^{1/4}/M_{pl})^2$, which takes $\sim 10^{-36}$ in the above case. Furthermore, our results show that the gravitational interaction plays a crucial role for the parametric amplification of the fluctuations. To obtain the relevant growth factor $e^{\mu\Delta\tau}$ to compensate the suppression factor, the long period of oscillating inflation is necessary, which requires the very fine-tuning of the model parameter ϕ_c . Therefore, we conclude that the influence of the parametric resonance on the cosmological perturbation produced during oscillating inflation might not be imprinted on the universe observed as long as the adiabatic mode of the perturbation is dominant.

The conclusion might be changed if we consider the multi-field system, because influence of iso-curvature mode cannot be ignored in general. Efficiency of the parametric resonance in the multi-field system should be studied from the extension of our single field model to the hybrid-type inflation, which is a more realistic inflation model motivated by the particle physics[7]. The cosmological implication of the hybrid-type oscillating inflation is important and must be explored.

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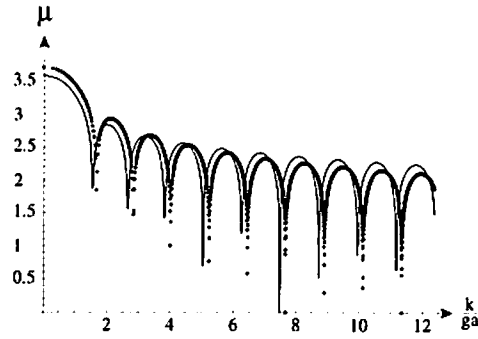


Fig.1

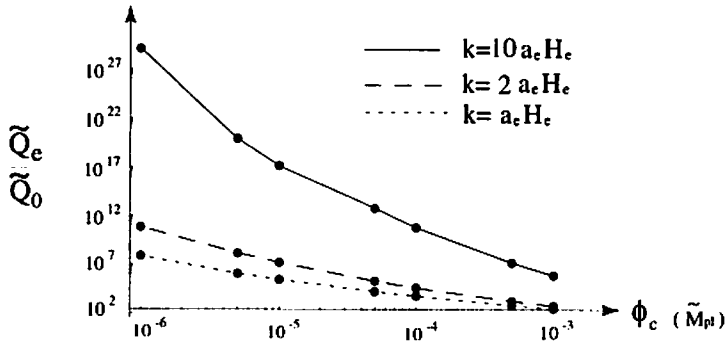


Fig.2

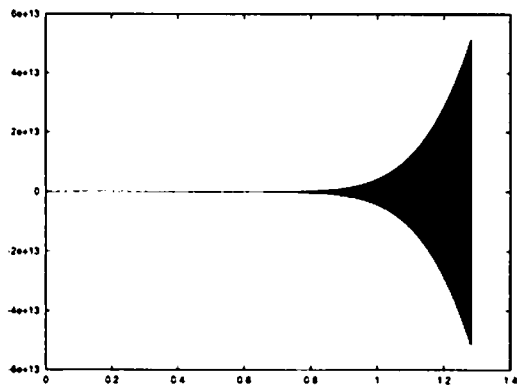


Fig.3a

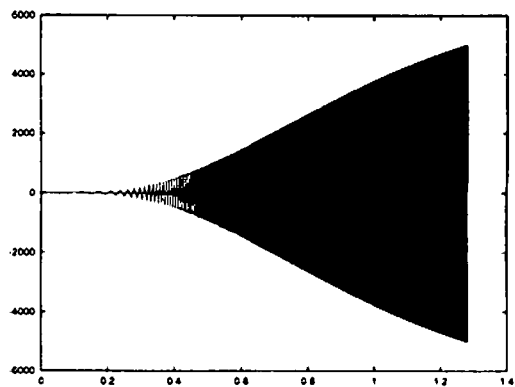


Fig.3b

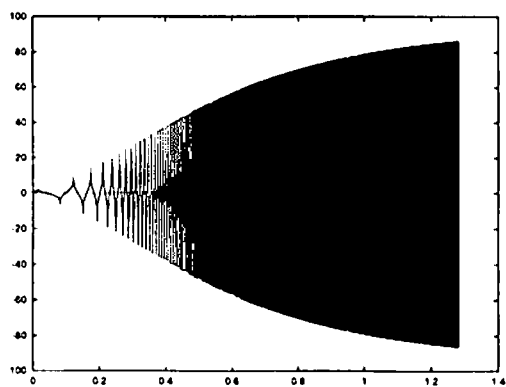


Fig.3c

Asymptotically Schwarzschild Spacetimes

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Abstract

It is shown that if an asymptotically flat spacetime is asymptotically stationary, in the sense that $\mathcal{L}_{\xi}g_{ab}$ vanishes at the rate $\sim t^{-3}$ for asymptotically timelike vector field ξ^a , and the energy-momentum tensor vanishes at the rate $\sim t^{-4}$, then the spacetime is an asymptotically Schwarzschild spacetime. This gives a new aspect of the uniqueness theorem of a black hole.

1 Introduction

There are many astrophysical phenomena that are best explained by black holes, for example, active galactic nuclei and X-ray binaries. The analysis on the phenomena are done assuming that those black holes are described by the Kerr spacetimes. This is because the uniqueness theorem of a black hole guarantees that a spacetime which is stationary, vacuum and asymptotically flat is uniquely the Kerr spacetime, and we believe that when gravitational collapse takes place and a black hole is formed, the spacetime around it becomes vacuum and accordingly stationary. (For the details on the uniqueness theorem of a black hole, see [1].) However, one may argue that such a spacetime does not become *exactly* vacuum nor *exactly* stationary: the spacetime becomes *asymptotically* vacuum, and accordingly *asymptotically* stationary at a certain rate of the time. In this context, a more adequate “uniqueness theorem” is the one that states an asymptotically stationary, vacuum and flat spacetime is uniquely an asymptotically Schwarzschild spacetime. This is what we show in this paper. (One may find it peculiar that an asymptotically *stationary* spacetime is an asymptotically Schwarzschild spacetime. However, if we define *asymptotically* Schwarzschild spacetimes as a class of spacetimes that asymptotically approach the exact Schwarzschild spacetime, the spacetimes comprise a wide class of spacetimes, including the Kerr spacetime, which is stationary. See the end of sec.4 for the details.)

To show the theorem, we use the notion of the asymptotic flat spacetime, first introduced by Ashtekar and Romano[2] at spacelike infinity and succeedingly developed in our previous study[3] at timelike infinity. We investigate the asymptotic behavior of the gravitational field at the future timelike infinity, because we would like to know whether the gravitational field approaches asymptotically that of the Schwarzschild spacetime at the *late time* when the spacetime becomes asymptotically stationary. The standard definition of the asymptotic flatness[5] is based on the conformal completion method, which is used to obtain the Penrose diagram. This method is useful in that spacelike and timelike infinities can be simultaneously treated with null infinity, and thus that investigation concerning the global causal structure can be done. However, the method compresses the spacelike and timelike infinities, which possess rich 3-manifold information, down to points, and this compression results in a complicated differential structure at these points and makes it difficult to obtain the comprehensive picture of the behavior of gravitational fields in general asymptotically flat spacetimes. In contrast, the completion method introduced by [2] for

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defining the asymptotic flatness at spacelike infinity leaves the infinity as a 3-manifold. As a result, the complicated differential structure in the former treatment can be avoided. Subsequently, in our previous study[3], we applied the method to timelike infinity and clarified that the method leads to a definite picture of hierarchy in the asymptotic behavior of the gravitational field and the symmetry, and that it is suitable to discuss such a notion as an “asymptotically Schwarzschild spacetime”. (Perng also investigates hierarchy in the asymptotic gravitational field at spacelike infinity[6], although it does not directly correspond to the hierarchy discussed in [3].) In this paper, we further investigate the hierarchy and prove the theorem stating that an asymptotically stationary, vacuum and flat spacetime is uniquely an asymptotically Schwarzschild spacetime.

The plan of this paper is the following. Sec.2 is devoted to preliminaries. The definition of the asymptotic flatness at timelike infinity is recalled, and some of its useful consequences are summarized for the subsequent discussion. In sec.3, the first order asymptotic structure is explored, that is, the one order higher structure than the basic asymptotic structure that all the asymptotic flat spacetimes possess. Then, in sec.4, we introduce the notion of asymptotic stationarity and investigate how the condition that an asymptotic flat spacetime be asymptotically stationary, constrain the asymptotic structure. The investigation leads to the proof of the main theorem.

Throughout the paper, we follow the notation of Wald[7].

2 Preliminaries and the Zero-th Order Asymptotic Structure

In this section, we recall the definition of asymptotic flatness at timelike infinity of [3] that will be used in the main proof and fix the notation.

DEFINITION: A physical spacetime $(\hat{\mathcal{M}}, \hat{g}_{ab})$ is said to possess an *asymptote at future timelike infinity* $\check{r}^+$ to order n (ATI- n) for a non-negative integer n , if there exists a manifold $\mathcal{M}$ with boundary $\mathcal{H}$, a smooth function Ω defined on $\mathcal{M}$, and an imbedding Ψ of an open subset $\hat{\mathcal{F}}$ in $\hat{\mathcal{M}}$ to $\mathcal{M} - \mathcal{H}$ satisfying the following conditions:

- (1) $\check{r}^+ := \partial\mathcal{F} \cap (\mathcal{M} - \Psi(\hat{\mathcal{M}}))$ is not empty and $\check{r}^+ \subset I^+(\mathcal{F})$ where $\mathcal{F} := \Psi(\hat{\mathcal{F}})$;
- (2) $\Omega \neq 0$ and $\nabla_a \Omega \neq 0$, where $\hat{=}$ denotes the equality evaluated on $\check{r}^+$;
- (3) $n^a := \Omega^{-4} \Psi^* \hat{g}^{ab} \nabla_b \Omega$ and $q_{ab} := \Omega^2 (\Psi^* \hat{g}_{ab} + \Omega^{-4} F^{-1} \nabla_a \Omega \nabla_b \Omega)$ admit smooth limits to $\check{r}^+$ with q_{ab} having signature $(+++)$ on $\check{r}^+$, where $F := -\mathcal{L}_n \Omega$; and
- (4) $\lim_{\check{r}^+} \Omega^{-(2+n)} T_{\hat{\mu}\hat{\nu}} \hat{= 0}$ where $\hat{T}_{\hat{\mu}\hat{\nu}} := \Psi^*[(\hat{e}_\mu)^a (\hat{e}_\nu)^b \hat{T}_{ab}]$ in which $\{(\hat{e}_\mu)^a\}$ and $\hat{T}_{ab}$ are a tetrad and the physical energy-momentum tensor of $(\hat{\mathcal{M}}, \hat{g}_{ab})$, respectively.

Henceforth, we use a tetrad consisting of a unit vector field $\hat{n}^a$ that is normal to the Ω -const. surfaces and a triad $\{(\hat{e}_i)^a\}_{i=1,2,3}$ of the metric $\hat{q}_{ab} := \hat{g}_{ab} + \hat{n}_a \hat{n}_b$ on the Ω -const. surface. We denote the timelike components with the subscript 0 and the spacelike component with capital-Roman-letter subscript, e.g., $\hat{A}^{\hat{b}} := \hat{n}_a \hat{A}^a$ and $\hat{A}^{\hat{i}} := (\hat{e}_i)_a \hat{A}^a$. If a tensor $A^{a\dots b}_{c\dots d}$ admits a smooth limit to $\check{r}^+$, it is useful to define the n -th order term of $A^{a\dots b}_{c\dots d}$ as

$${}^{(0)}A^{a\dots b}_{c\dots d} := \lim_{\check{r}^+} A^{a\dots b}_{c\dots d} \quad \text{and} \quad {}^{(n)}A^{a\dots b}_{c\dots d} := \lim_{\check{r}^+} \Omega^{-n} (A^{a\dots b}_{c\dots d} - \sum_{\ell=0}^{n-1} {}^{(\ell)}A^{a\dots b}_{c\dots d} \Omega^\ell) \quad \text{for } n \geq 1. \quad (1)$$

This definition of the n -th order terms of a tensor implies that in the vicinity of $\check{r}^+$, $A^{a\dots b}_{c\dots d}$ can be expanded as

$$A^{a\dots b}_{c\dots d} = \sum_{n=0}^{\infty} {}^{(n)}A^{a\dots b}_{c\dots d} \Omega^n. \quad (2)$$

Since all the equations appearing in the following discussion are those on $\mathcal{M}$, unless it may cause ambiguity, we omit hereafter Ψ^* in front of the tensors defined on $\hat{\mathcal{M}}$ for brevity.

Before we examine the properties of ATI- n spacetimes, we introduce some valuable tensors. First, the projection operator with respect to the Ω -const. surface can be introduced as

$$q^a_b := \sum_{i=1,2,3} (\hat{e}_i)^a (\hat{e}_i)_b = \delta^a_b + F^{-1} n^a \nabla_b \Omega.$$

This operator admits a smooth limit to $\check{\gamma}^+$ by virtue of the definition of an ATI- n spacetime. Second, note that the above definition of an ATI- n spacetime implies that $\check{\gamma}^+$ is a 3-submanifold of $\mathcal{M}$ with an imbedding, say Π . Hence, if a tensor field $A_{c\dots d}^{a\dots b}$ is tangential to $\check{\gamma}^+$, or $A_{c\dots d}^{a\dots b} := q^a_e \dots q^b_f q^g_c \dots q^h_d A_{g\dots h}^{e\dots f}$, it is useful to consider the tensor field $\Pi^* A_{c\dots d}^{a\dots b}$ defined on $\check{\gamma}^+$. Hereafter, we denote such a tensor field in boldface, i.e., $\mathbf{A}_{c\dots d}^{a\dots b} := \Pi^* A_{c\dots d}^{a\dots b}$, and say that $A_{c\dots d}^{a\dots b}$ induces $\mathbf{A}_{c\dots d}^{a\dots b}$ on $\check{\gamma}^+$.

Now we explore the consequence of the definition of an ATI- n spacetime for $n = 0$. Solving the Einstein equation under the fall-off condition on the energy-momentum tensor, $\lim_{\check{\gamma}^+} \Omega^{-2} \hat{T}_{\hat{\mu}\hat{\nu}} = 0$, it is found that

$$F \doteq 1 \quad \text{and} \quad q_{ab} \doteq h_{ab} \quad (3)$$

in an ATI-0 spacetime, where $h_{ab} = (d\chi)_a(d\chi)_b + \sinh^2 \chi [(d\theta)_a(d\theta)_b + \sin^2 \theta (d\phi)_a(d\phi)_b]$ is the 3-metric of the unit timelike 3-hyperboloid. (For the details of the derivation, see [3].) Because eq.(3) is a gravitational structure common to all the ATI-0 spacetimes, we call it the *zero-th order asymptotic structure*. Using conformal time $\eta := \ln \Omega$, these results imply

$$\hat{g}_{ab} = {}^{(0)}\hat{g}_{ab} + \Omega^{(1)}\hat{g}_{ab} + O(\Omega^2) \quad \text{where} \quad {}^{(0)}\hat{g}_{ab} = (e^{-\eta})^2 [-(d\eta)_a(d\eta)_b + h_{ab}] \quad (4)$$

in which ${}^{(n)}\hat{g}_{ab}$ is defined by

$${}^{(n)}\hat{g}_{ab} := \sum_{\mu,\nu} (\hat{e}_\mu)_a (\hat{e}_\nu)_b {}^{(n)}g_{\mu\nu} \quad (5)$$

with a function $g_{\mu\nu} := (\hat{e}_\mu)^a (\hat{e}_\nu)^b \hat{g}_{ab}$ that admits a smooth limit to $\check{\gamma}^+$ and thus is expanded in the manner described in eq.(1). ${}^{(0)}\hat{g}_{ab}$ is a metric of the Milne universe and is equivalent to the metric of a Minkowski spacetime, $\hat{g}_{ab}^{\text{MIN}}$. In other words, eq.(4) tell us that an ATI-0 spacetime is an asymptotically Minkowski spacetime:

$$\hat{g}_{ab} = \hat{g}_{ab}^{\text{MIN}} + O(\Omega) \quad (6)$$

Hence, it is no surprise that the leading terms of the Riemann tensor asymptotically vanishes in such a spacetime. The leading order term of the trace part of the Riemann tensor asymptotically vanishes by virtue of the fall-off condition on the energy-momentum tensor, and that of the traceless part, or the Weyl tensor $\hat{C}_{ambn}$, also vanishes, which can be calculated from the above results on n^a and q_{ab} . (See [3, 4] for the derivations.)

3 the First Order Asymptotic Structure

In this section, we derive the first order asymptotic structure, that is, a structure which is possessed by all the ATI-1 spacetimes but not by ATI-0 spacetimes. The point is that the first order asymptotic structure may be considered as *the perturbation to the Milne universe*, eq.(4), where Ω plays the role of a small parameter of the perturbation. Hence, we can apply the technique of the cosmological perturbation[8, 9] to derive the first order asymptotic structure.

The energy-momentum tensor of an ATI-1 spacetime satisfies a stronger fall-off condition, $\lim_{\check{\gamma}^+} \Omega^{-3} \hat{T}_{\hat{\mu}\hat{\nu}} = 0$, than that of an ATI-0 and thus the behavior of asymptotic gravitational fields is constrained stronger. In other words, ATI-1 spacetimes possess richer asymptotic gravitational structure. The structure can be derived by solving the Einstein equation under the condition $\lim_{\check{\gamma}^+} \Omega^{-3} \hat{T}_{\hat{\mu}\hat{\nu}} = 0$. To obtain the equation, we first decompose the first-order metric ${}^{(1)}\hat{g}_{ab}$ as

$${}^{(1)}\hat{g}_{ab} = (e^{-\eta})^2 \left[{}^{(1)}F(d\eta)_a(d\eta)_b - 2 {}^{(1)}\beta_a(d\eta)_b - 2 {}^{(1)}\psi h_{ab} + 2 {}^{(1)}\chi_{ab} \right] \quad (7)$$

where ${}^{(1)}\beta_a$ and ${}^{(1)}\chi_{ab}$ are tangential to the Ω -const. surfaces, i.e., $q_a{}^b {}^{(1)}\beta_b = {}^{(1)}\beta_a$ and $q_a{}^c q_b{}^d {}^{(1)}\chi_{cd} = {}^{(1)}\chi_{ab}$; and ${}^{(1)}\chi_{ab}$ is traceless, i.e., $q^{ab} {}^{(1)}\chi_{ab} = 0$. Next, we choose the Poisson gauge, in which ${}^{(1)}\beta_a$ is transverse and ${}^{(1)}\chi_{ab}$ is transverse-traceless. In this gauge, the Einstein equation induces on $\check{\gamma}^+$ the following set of differential equations in an ATI-1 spacetime satisfying $\lim_{\check{\gamma}^+} \Omega^{-3} \hat{T}_{\hat{\mu}\hat{\nu}} = 0$:

$$3 {}^{(1)}F + 2 \Delta {}^{(1)}\psi \doteq 0 \quad (8)$$

$$D_a ({}^{(1)}F + 2 {}^{(1)}\psi) + \frac{1}{2} (\Delta - 2) {}^{(1)}\beta_a^T \doteq 0 \quad (9)$$

$$(h_{ab} \Delta - D_a D_b) ({}^{(1)}F + 2 {}^{(1)}\psi) + 2 D_a ({}^{(1)}\beta_b^T) + 2 (\Delta + 3) {}^{(1)}\chi_{ab}^{TT} \doteq 0 \quad (10)$$

where D_a is the derivative operator associated with the metric h_{ab} of $\mathbb{R}^+$; $\Delta := D^a D_a$; and the subscripts T and TT denote that the quantities are transverse and transverse-traceless, respectively.

First, we derive the scalar ${}^{(1)}F$. Subtracting eq.(8) from the divergence of eq.(9), we obtain

$$(\Delta - 3){}^{(1)}F \doteq 0. \quad (11)$$

The general solution of the above equation can be derived by first expanding the function ${}^{(1)}F$ as

$${}^{(1)}F(\chi, \theta, \phi) \doteq \sum a_{\ell m} {}^{(1)}F^{\ell m}(\chi, \theta, \phi) \quad \text{where} \quad {}^{(1)}F^{\ell m}(\chi, \theta, \phi) := T_0^\ell(\chi) Y^{\ell m}(\theta, \phi) \quad (12)$$

in which the summation is taken over $\ell \in \mathbb{Z}$ and $|m| \leq |\ell|$, and $Y^{\ell m}(\theta, \phi)$ are the 2-dimensional spherical harmonics. Substituting eq.(12) into eq.(11), it is found that T_0^ℓ is given by

$$T_0^\ell(\chi) = \mathcal{P}_2^\ell(\chi) \quad (13)$$

where the functions $\mathcal{P}_n^\ell(\chi)$ is defined by

$$\mathcal{P}_n^\ell(\chi) := \frac{1}{\sqrt{\sinh \chi}} P_{n-\frac{1}{2}}^{\ell+\frac{1}{2}}(\cosh \chi); \quad (14)$$

and $P_\nu^\mu(z)$ is an associated Legendre function. (Note here that unconventional choice of ℓ is taken. It is related to the conventional choice ℓ_c by $\ell = -\ell_c - 1$, and thus it is $\ell \leq -1$ terms that are regular. The reason for the choice is to have the coefficient a_{00} in eq.(12) correspond to the mass of a spacetime. See eq.(38).)

Second, we derive the scalar ${}^{(1)}\psi$. Adding eq.(11) to eq.(8), we obtain

$$\Delta({}^{(1)}F + 2{}^{(1)}\psi) \doteq 0. \quad (15)$$

Let A be the solution of $\Delta A \doteq 0$. Then, ${}^{(1)}\psi$ may be given by ${}^{(1)}\psi \doteq -{}^{(1)}F/2 + A/2$. Noting that, under a gauge transformation generated by $\xi^a = \Omega T(\partial_\eta)^a + \Omega L^a$, the quantities are transformed as

$$\begin{aligned} {}^{(1)}\bar{F} &\doteq {}^{(1)}F & {}^{(1)}\bar{\beta}_a &\doteq {}^{(1)}\beta_a + D_a T - L_a \\ {}^{(1)}\bar{\psi} &\doteq {}^{(1)}\psi + T - \frac{1}{3} D^a L_a & {}^{(1)}\bar{\chi}_{ab} &\doteq {}^{(1)}\chi_{ab} + D_{(a} L_{b)} - \frac{1}{3} h_{ab} D_m L^m \end{aligned} \quad (16)$$

we perform gauge transformation generated by $\xi^a = \Omega T(\partial_\eta)^a$ where $T \doteq -A/2$. This transformation preserves the Poisson gauge, and simplifies ${}^{(1)}\psi$ as ${}^{(1)}\psi \doteq -{}^{(1)}F/2$.

Next, we consider the vector field ${}^{(1)}\beta_a^T$. The substitution of the above results on ${}^{(1)}F$ and ${}^{(1)}\psi$ into eq.(9) yields

$$(\Delta - 2){}^{(1)}\beta_a^T \doteq 0. \quad (17)$$

Hence, together with the fact ${}^{(1)}\beta_a^T$ is transverse, we see that the gauge transformation generated by $\xi^a = \Omega L^a$ where $L_a \doteq {}^{(1)}\beta_a^T$ preserves the Poisson gauge, and results in ${}^{(1)}\beta_a^T \doteq 0$.

Finally, we derive the tensor field ${}^{(1)}\chi_{ab}^{TT}$. Substituting the above results into eq.(10), we obtain

$$(\Delta + 3){}^{(1)}\chi_{ab}^{TT} \doteq 0. \quad (18)$$

To solve this equation, we consider the spherical harmonic expansion again. As generally done, we decompose ${}^{(1)}\chi_{ab}^{TT}$ into the even (or electric-type) parity part ${}^{(1)}\chi_{ab}^{TT(+)}$ and the odd (or magnetic-type) parity part ${}^{(1)}\chi_{ab}^{TT(-)}$: ${}^{(1)}\chi_{ab}^{TT} \doteq {}^{(1)}\chi_{ab}^{TT(+)} + {}^{(1)}\chi_{ab}^{TT(-)}$. In the present case, the even parity part that satisfies eq. (18) is found to be [10]

$${}^{(1)}\chi_{ab}^{TT(+)} \doteq (D_a D_b - h_{ab}) X \quad (19)$$

where X is a function that satisfies $(\Delta - 3)X \doteq 0$. On the other hand, the odd parity part is found to be [11]

$${}^{(1)}\chi_{ab}^{TT(-)} \doteq \sum_{\ell \neq 0} b_{\ell m}^{(-)} {}^{(1)}\chi_{ab}^{TT(-)\ell m} \quad (20)$$

where the summation is taken over $\ell \in \mathbb{Z}$ for $\ell \neq 0$, $|m| \leq |\ell|$; and

$${}^{(1)}\chi_{\chi\chi}^{TT(-)\ell m} \doteq 0, \quad {}^{(1)}\chi_{\chi A}^{TT(-)\ell m} \doteq T_1^\ell(\chi) \epsilon_A{}^B \mathcal{D}_B Y^{\ell m} \quad \text{and} \quad {}^{(1)}\chi_{AB}^{TT(-)\ell m} \doteq T_2^\ell(\chi) \epsilon^C{}_{(A} \mathcal{D}_{B)} \mathcal{D}_C Y^{\ell m} \quad (21)$$

in which functions $T_1^\ell(\chi)$ and $T_2^\ell(\chi)$ are given by

$$T_1^\ell(\chi) \doteq \mathcal{P}_0^\ell(\chi) \quad \text{and} \quad T_2^\ell(\chi) \doteq \frac{\sinh^2 \chi}{(\ell-1)(\ell+2)} \left[\partial_\chi + 2 \frac{\cosh \chi}{\sinh \chi} \right] \mathcal{P}_0^\ell(\chi) \quad \text{for } \ell \neq 1, -2. \quad (22)$$

(Here, there is no need to derive $T_2^\ell(\chi)$ for $\ell = 1$ or $\ell = -2$ since $\epsilon^C{}_{(A} \mathcal{D}_{B)} \mathcal{D}_C Y^{\ell m}$ vanishes for these values of ℓ .) Next, consider the gauge transformation generated by $\xi^a = \Omega T(\partial_\eta)^a + \Omega L^a$ where $T \doteq -X$ and $L_a \doteq -D_a X$. This transformation leaves ${}^{(1)}\beta_a$ and ${}^{(1)}\psi$ unchanged and kills the even parity part of ${}^{(1)}\chi_{ab}^{TT}$, and thus

$${}^{(1)}\chi_{ab}^{TT} \doteq \sum_{\ell \neq 0} b_{\ell m}^{(-)} {}^{(1)}\chi_{ab}^{TT(-)\ell m}. \quad (23)$$

To summarize, the solutions of the Einstein equation (8)-(10) is given by

$$\begin{aligned} {}^{(1)}\bar{F} &\doteq \sum a_{\ell m} {}^{(1)}F^{\ell m}, & {}^{(1)}\bar{\beta}_a &\doteq 0, \\ {}^{(1)}\bar{\psi} &\doteq -\frac{1}{2} \sum a_{\ell m} {}^{(1)}F^{\ell m} & \text{and} & \quad {}^{(1)}\bar{\chi}_{ab} \doteq \sum_{\ell \neq 0} b_{\ell m}^{(-)} {}^{(1)}\chi_{ab}^{TT(-)\ell m} \end{aligned} \quad (24)$$

in the suitably chosen Poisson gauge. This is the structure of the gravitational field common to all the ATI-1 spacetimes, and thus we call it *the first order asymptotic structure* of an ATI-1 spacetime.

In such a spacetime, the electric and the magnetic parts of the induced Weyl tensor on $\mathfrak{I}^+$ are written in terms of ${}^{(1)}\bar{F}$ and ${}^{(1)}\bar{\chi}_{ab}$, respectively. (See [4] for the details.)

4 Asymptotic Stationarity

In this section, we introduce the notion of *asymptotic stationarity* of ATI- n spacetimes, and prove that an ATI-1 spacetime that is asymptotically stationary to order 2 must be an asymptotically Schwarzschild spacetime. The plan of the proof is: 1) we derive, in the Poisson gauge, the reduced first asymptotic structure of an ATI-1 spacetime that is asymptotically stationary to order 2 in the lemma; 2) we perform a suitable gauge transformation as to show explicitly that such an asymptotic structure approaches asymptotically the Schwarzschild metric in the theorem.

A Killing vector field $\hat{\xi}^a$ is a vector field with respect to which the Lie derivative of the metric vanishes, $\mathcal{L}_{\hat{\xi}} \hat{g}_{ab} = 0$. This fact motivates us to define an asymptotic Killing vector field and its order as follows.

DEFINITION: An ATI- m spacetime $(\mathcal{M}, \hat{g}_{ab})$ is said to admit an *asymptotic Killing field* $\hat{\xi}^a$ to order n if

$$\lim_{\rightarrow \mathfrak{I}^+} \Omega^{-n} (\mathcal{L}_{\hat{\xi}} \hat{g})_{\hat{\mu}\hat{\nu}} = 0 \quad (25)$$

where $(\mathcal{L}_{\hat{\xi}} \hat{g})_{\hat{\mu}\hat{\nu}} \doteq \Psi^*(\hat{e}_\mu)^a (\hat{e}_\nu)^b \mathcal{L}_{\hat{\xi}} \hat{g}_{ab}$.

Next, we consider how to define asymptotic stationarity, using this notion of an asymptotic Killing vector field. A vector field $\hat{\xi}^a$ is said to be a *stationary Killing vector field* in an asymptotically flat spacetime, if $\hat{\xi}^a$ is a timelike Killing field and satisfies $\hat{g}_{ab} \hat{\xi}^a \hat{\xi}^b = -1$ at infinity. Hence, we define its asymptotic correspondence as follows.

DEFINITION: A vector field ξ^a is said to be an *asymptotic stationary Killing vector field* to order n of an ATI- m spacetime if $\hat{\xi}^a$ is admitted as an asymptotic Killing vector field to order n in the ATI- m spacetime and satisfies

$$\hat{g}_{ab} \hat{\xi}^a \hat{\xi}^b \doteq -1 \quad (26)$$

on $\mathfrak{I}^+$.

It is important to note that the definition of the asymptotic stationary Killing vector field implies that the leading term of $\hat{\xi}^{\hat{\mu}} = \hat{\xi}^a (\hat{e}_a)_a$ is of the order Ω^0 . Hence, $\xi^{\hat{\mu}} := \hat{\xi}^{\hat{\mu}}$ admits a smooth limit to $\mathfrak{I}^+$ and can be expanded as

$$\xi^{\hat{\mu}} = \sum_{n=1}^{\infty} {}^{(n)}\xi^{\hat{\mu}} \Omega^n. \quad (27)$$

(Here, ${}^{(n)}\xi^{\hat{0}}$ and ${}^{(n)}\xi^{\hat{1}}$ correspond to ${}^{(n+2)}\xi^{\hat{0}}$ and ${}^{(n+1)}\xi^{\hat{1}}$ in [3]. We do not use the notation of [3] for $\hat{\xi}^a$, because the above notation reflects the nature of the completion more intrinsically.) Simple calculation shows that the function $(\mathcal{L}_{\hat{\xi}}\hat{g})_{\hat{\mu}\hat{\nu}}$ admits a smooth limit to $\mathfrak{I}^+$, and thus can be expanded in the manner described by eq.(1).

Now we are ready to prove the following lemma.

LEMMA: An AT1-1 spacetime is asymptotically stationary to order 2 if and only if $a_{\ell m} \equiv 0$ for $\ell \neq 0$ and $b_{\ell m} \equiv 0$.

Proof of only if: If an AT1-1 spacetime is asymptotically stationary to order 2,

$${}^{(n)}(\mathcal{L}_{\hat{\xi}}\hat{g})_{\hat{\mu}\hat{\nu}} \equiv 0 \quad \text{for } n \leq 2, \quad (28)$$

$$\hat{g}_{ab}\hat{\xi}^a\hat{\xi}^b \equiv -1 \quad (29)$$

hold. First, we note that eq.(29) is equivalent to

$$-({}^{(0)}\xi^{\hat{0}})^2 + {}^{(0)}\xi^a {}^{(0)}\xi_a \equiv -1 \quad (30)$$

where ${}^{(0)}\xi^a := {}^{(0)}\xi^{\hat{K}}(e_K)^a$ and $\{(e_i)_a\}_{i=1,2,3}$ is a triad of h_{ab} . Second, we simplify the $n \leq 1$ part of eq.(28). In an AT1-1 spacetime, ${}^{(0)}(\mathcal{L}_{\hat{\xi}}\hat{g})_{\hat{\mu}\hat{\nu}}$ all vanish, and the $O(\Omega^1)$ terms of $(\mathcal{L}_{\hat{\xi}}\hat{g})_{\hat{\mu}\hat{\nu}}$ are given by

$${}^{(1)}(\mathcal{L}_{\hat{\xi}}\hat{g})_{\hat{0}\hat{0}} \equiv 0, \quad {}^{(1)}(\mathcal{L}_{\hat{\xi}}\hat{g})_{\hat{0}\hat{i}} \equiv \left[{}^{(0)}\xi_a - D_a {}^{(0)}\xi^{\hat{0}} \right] (e_i)^a, \quad \text{and} \quad {}^{(1)}(\mathcal{L}_{\hat{\xi}}\hat{g})_{\hat{i}\hat{j}} \equiv \left[D_{(a} {}^{(0)}\xi_{b)} - {}^{(0)}\xi^{\hat{0}} h_{ab} \right] (e_i)^a (e_j)^b.$$

Hence, the $n \leq 1$ part of eq.(28) is equivalent to

$${}^{(0)}\xi_a - D_a {}^{(0)}\xi^{\hat{0}} \equiv 0 \quad \text{and} \quad D_{(a} {}^{(0)}\xi_{b)} - {}^{(0)}\xi^{\hat{0}} h_{ab} \equiv 0. \quad (31)$$

Solving eqs.(30)(31) simultaneously, we find that

$${}^{(0)}\xi^{\hat{0}} \equiv \cosh \chi \quad \text{and} \quad {}^{(0)}\xi^{\hat{K}}(e_K)^a \equiv \sinh \chi (\partial_{\chi})^a. \quad (32)$$

Next, we consider the $n = 2$ part of eq.(28). Let us choose the Poisson gauge in which the first order asymptotic structure takes the form eq.(24) in an AT1-1 spacetime. Because

$$\begin{aligned} {}^{(2)}(\mathcal{L}_{\hat{\xi}}\hat{g})_{\hat{0}\hat{0}} &\equiv -{}^{(1)}\xi^{\hat{0}} + \frac{1}{2} \left({}^{(0)}\xi^{\hat{0}} + {}^{(0)}\xi^m D_m \right) {}^{(1)}F \quad \text{and} \\ {}^{(2)}(\mathcal{L}_{\hat{\xi}}\hat{g})_{\hat{0}\hat{i}} &\equiv \left[2{}^{(1)}\xi_a - D_a {}^{(1)}\xi^{\hat{0}} + \left(D_a {}^{(0)}\xi^{\hat{0}} - {}^{(0)}\xi_a \right) {}^{(1)}F - 2{}^{(0)}\xi_m {}^{(1)}\chi_a^m \right] (e_i)^a, \end{aligned} \quad (33)$$

the $n = 2$ part of eq.(28) for $\hat{\mu} = \hat{0}$ is equivalent to

$${}^{(1)}\xi^{\hat{0}} \equiv \frac{1}{2} \left({}^{(0)}\xi^{\hat{0}} + {}^{(0)}\xi^m D_m \right) {}^{(1)}F \quad \text{and} \quad {}^{(1)}\xi^{\hat{K}}(e_K)^a \equiv \frac{1}{2} \left[D_a {}^{(1)}\xi^{\hat{0}} - \left(D_a {}^{(0)}\xi^{\hat{0}} - {}^{(0)}\xi_a \right) {}^{(1)}F + 2{}^{(0)}\xi_m {}^{(1)}\chi_a^m \right]. \quad (34)$$

Using eqs.(32)(34), we find that ${}^{(2)}(\mathcal{L}_{\hat{\xi}}\hat{g})_{\hat{i}\hat{j}}$ are given by

$${}^{(2)}(\mathcal{L}_{\hat{\xi}}\hat{g})_{\hat{i}\hat{j}} \equiv 2 \sinh \chi \left[{}^1\mathcal{L}_{ab} {}^{(1)}F + {}^2\mathcal{L}_{ab} {}^{(1)}\chi_{b|m} \right] (e_i)^a (e_j)^b \quad (35)$$

where the derivative operators ${}^1\mathcal{L}_{ab}$ and ${}^2\mathcal{L}_{ab}$ are given by

$${}^1\mathcal{L}_{ab} := \frac{3}{\tanh \chi} (D_a D_b - h_{ab}) + (\partial_{\chi})^m D_{(a} D_{b)} D_m - (d\chi)_{(a} D_{b)}, \quad \text{and} \quad {}^2\mathcal{L}_{ab} := 2(d\chi)_p h_a^{lp} h_b^{mp} D_m, \quad (36)$$

respectively. By the definition of ${}^{(1)}F$ and ${}^{(1)}\chi_{ab}$, ${}^{(2)}(\mathcal{L}_{\xi}\hat{g})_{ij}$ vanishes if and only if ${}^1\mathcal{L}_{ab}{}^{(1)}F$ and ${}^2\mathcal{L}_{(a}{}^s{}^{(1)}\chi_{b)s}$ vanish independently. First, consider ${}^1\mathcal{L}_{ab}{}^{(1)}F \neq 0$. Using eq.(24), we find that the $\ell = 0$ term of the RHS of the equation always vanishes, but each of the $\ell \neq 0$ terms vanish only when $a_{\ell m} = 0$. (See [4] for the details of the calculation.) In other words, ${}^1\mathcal{L}_{ab}{}^{(1)}F \neq 0$ implies $a_{\ell m} \neq 0$ for $\ell \neq 0$. In the similar manner, ${}^2\mathcal{L}_{(a}{}^s{}^{(1)}\chi_{b)s} \neq 0$ implies $b_{\ell m} \neq 0$. $\square$

Proof of if: If $a_{\ell m} \neq 0$ for $\ell \neq 0$ and $b_{\ell m} = 0$ in an ATI-1 spacetime, the vector field $\hat{\xi}^a$ whose $O(\Omega^0)$ and $O(\Omega^1)$ terms are given by eqs.(32)(34) satisfy ${}^{(n)}(\mathcal{L}_{\xi}\hat{g})_{\mu\nu} \neq 0$ for $n \leq 2$ and $\hat{g}_{ab}\hat{\xi}^a\hat{\xi}^b \neq -1$. Hence, $\hat{\xi}^a$ is an asymptotically stationary Killing vector field to order 2. $\square$

The fact that $a_{\ell m} \neq 0$ for $\ell \neq 0$ and $b_{\ell m} \neq 0$ means that in such an ATI-1 spacetime, the first asymptotic structure takes the simple form

$${}^{(1)}F \neq a_{00}{}^{(1)}F^{00}, \quad {}^{(1)}\psi \neq -\frac{1}{2}a_{00}{}^{(1)}F^{00}, \quad {}^{(1)}\beta \neq 0 \quad \text{and} \quad {}^{(1)}\chi_{ab} \neq 0. \quad (37)$$

Theorem: An ATI-1 spacetime which is asymptotically stationary to order 2 is an asymptotically Schwarzschild spacetime with mass a_{00} in the sense that the metric takes the form

$$\hat{g}_{ab} = \hat{g}_{ab}^{\text{SCH}} + O(r^{-2}) + O(t^{-2}) \quad \text{where} \quad \hat{g}_{ab}^{\text{SCH}} := -\left(1 - \frac{2a_{00}}{r}\right)(dt)_a(dt)_b + \left(1 - \frac{2a_{00}}{r}\right)^{-1}(dr)_a(dr)_b + r^2(d\sigma)_{ab} \quad (38)$$

in which $t > r$.

Proof: From the lemma, in an ATI-1 spacetime which is asymptotically stationary to order 2, $a_{\ell m} = 0$ for $\ell \neq 0$ and $b_{\ell m} = 0$ hold. Thus, the first order term of the metric ${}^{(1)}\hat{g}_{ab}$ takes the form

$${}^{(1)}\hat{g}_{ab} = (e^{-\eta})^2 \left[a_{00}{}^{(1)}F^{00}(d\eta)_a(d\eta)_b + a_{00}{}^{(1)}F^{00}h_{ab} \right]. \quad (39)$$

Under a gauge transformation generated by $\xi^a = \Omega T(\partial_\eta)^a + \Omega D^a L$ where

$$T : \neq a_{00}(2\chi \cosh \chi + \sinh \chi) \quad \text{and} \quad L : \neq -4a_{00} \sinh \chi + T, \quad (40)$$

${}^{(1)}\hat{g}_{ab}$ transforms as

$${}^{(1)}\hat{g}_{ab} \mapsto (e^{-\eta})^2 \left[a_{00}{}^{(1)}F^{00}(d\eta)_a(d\eta)_b - 8a_{00} \cosh \chi (d\eta)_{(a}(d\chi)_{b)} + a_{00}{}^{(1)}F^{00}(d\chi)_a(d\chi)_b \right]. \quad (41)$$

Then, the change of variables, $t = \Omega^{-1} \cosh \chi$ and $r = \Omega^{-1} \sinh \chi$, leads us to ${}^{(0)}\hat{g}_{ab} + \Omega {}^{(1)}\hat{g}_{ab} = \hat{g}_{ab}^{\text{SCH}} + O(\Omega^2)$. Hence, eq.(38) holds. $\square$

It is important to note here that asymptotically Schwarzschild spacetimes, that are defined in eq.(38), comprise the Kerr spacetime also. This can be understood by writing the Kerr metric $\hat{g}_{ab}^{\text{KER}}$ with the coordinates (Ω, χ) where $t = \cosh \chi$ and $r = \sinh \chi$:

$$\hat{g}_{ab}^{\text{KER}} = \hat{g}_{ab}^{\text{SCH}} + \Omega^2 {}^{(2)}\hat{g}_{ab} + O(\Omega^3) \quad (42)$$

where

$$\begin{aligned} {}^{(2)}\hat{g}_{ab} = & (4m^2 - a^2 \sin^2 \theta) \left((d\eta)_{(a}(d\eta)_{b)} - 2 \frac{(d\chi)_{(a}(d\eta)_{b)}}{\tanh \chi} \right) + 4ma \frac{\sin^2 \theta}{\tanh \chi} (d\phi)_{(a}(d\eta)_{b)} \\ & + \frac{4m^2 - a^2 \sin^2 \theta}{\tanh \chi} (d\chi)_a(d\chi)_b - 4am \sin^2 \theta^2 (d\phi)_{(a}(d\chi)_{b)} + a^2 (d\sigma)_{ab} - a^2 \sin^2 \theta (d\theta)_a(d\theta)_b. \end{aligned} \quad (43)$$

In other words, the Kerr spacetime is a special spacetime of asymptotically Schwarzschild spacetimes, which possesses a particular second-order asymptotic structure, i.e., ${}^{(2)}\hat{g}_{ab}$ given by eq.(43).

5 Summary and Remarks

In this paper, we have proved that an asymptotically flat spacetime as defined in definition 1 of sec.2 is an asymptotically Schwarzschild spacetime in the sense of eq.(38), if the energy-momentum of the spacetime falls off at the rate faster than $O(\Omega^3)$ and the spacetime is asymptotically stationary to order 2 in the sense that $(\mathcal{L}_{\hat{\xi}}\hat{g})_{\hat{\mu}\hat{\nu}}$ falls off at the rate faster than $O(\Omega^2)$ for the asymptotically timelike vector $\hat{\xi}^a$, $\hat{g}_{ab}\hat{\xi}^a\hat{\xi}^b \doteq -1$.

Finally, we give a remark. Although we have solved the Einstein equation (8)-(10) and obtained the first order asymptotic structure, we did not impose any physically-suitable boundary conditions on the solutions. Hence, the obtained first order asymptotic structure may include those that are *unphysical*. For example, a solution that describes an incoming gravitational wave from future null infinity or an outgoing wave from the event horizon. In other words, *physically acceptable* gravitational fields around a black hole may be obtained only after a suitable boundary condition are imposed on the solutions. The derivation of the physical first order asymptotic structure may be profitable because it may be possible to show that an ATI-1 spacetime with such a physical structure is intrinsically asymptotically stationary to order 1, and thus is an asymptotically Schwarzschild spacetime. This result is anticipated because we expect that a spacetime with a black hole that becomes vacuum also becomes stationary, due to the nature of the black hole. This is an important point that should be clarified.

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Another way for black holes to light up?

— A Generalization of Corley's result —

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Abstract

Corley analytically derived the thermal spectrum for the non-covariant model consisting of a scalar field in two-dimensional black hole with higher spatial derivative terms. The higher spatial derivative terms are introduced to mimic the modification due to some unknown physics which might exist beyond some critical scale. The effects of unknown physics are modeled so as to be suppressed for the waves which have a wavelength much longer than the critical scale, k_0^{-1} . In introducing these terms, which break the general covariance, the rest frame of freely-falling observers was employed as the special reference frame in previous works. Here we give an extension of the Corley's work by allowing a more general choice of the reference frame. We show that the resulting spectrum of created particles again becomes the thermal one at the Hawking temperature even if the choice of the reference frame is generalized. Using the technique of the matched asymptotic expansion, we also show that the correction to the thermal radiation stays of order k_0^{-2} or higher.

1 Introduction

The thermal radiation from a black hole was predicted by Hawking [1], which phenomenon is widely known as the Hawking radiation. This spectrum was derived on the basis of quantum field theory in curved space. The appearance of the Hawking radiation is thought to be responsible for the existence of wave packets which left the past null infinity with very high frequency, propagated through the collapsing body just before the horizon formed, and underwent a large redshift on the way out to the future null infinity. Here arises a question if one can safely apply the quantum field theory in curved space to such a system that includes the infinitely high frequency regime. There may exist some unknown physics which invalidates the use of the standard quantum field theory in curved space. One of such possibilities is that the spacetime is not continuous, and it reveals its discrete nature at such high frequencies. To take account of the effect of possible modification of theory in the high frequency regime, Unruh proposed a simple toy model[2]. In his model, the dispersion relation of fields at high frequencies are modified. In doing so, one must assume the existence of a reference frame because the concept of high frequencies can never be a covariant one. That is, the Unruh's model manifestly breaks the general covariance. To our surprise, even with this drastic change of theory, the spectrum observed at the future infinity was turned out to be kept unaltered[2, 3, 4]. Here, in this paper, we consider a generalization of this model.

One may think that it is not fruitful to study in detail such a toy model that violates the general covariance at the moment when we have no evidence suggesting this radical possibility. But, we also have an alternative motivation to study this toy model even if we could believe that the general covariance is an exact symmetry. In the most of literature, the Hawking radiation was studied in the framework of non-interacting quantum fields in curved space. However, when we consider a realistic model, it will be necessary to consider fields with interaction terms. The evolution of interacting fields in the background that is forming a black hole is a very interesting issue but to study it is very difficult. Hence, as a first step, it will be interesting to take partly into account the interaction between the quantum fields and the matter which is forming a black hole. Then, it will be natural to introduce a modified dispersion relation associated with the rest frame of the matter remaining around the black hole. In this sense, the Unruh's model does not require that the fundamental theory violates the general covariance.

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In order to introduce the modified dispersion relation we need to specify one special reference frame. In the previous works[2, 3, 4], the rest frame of freely-falling observers with vanishing velocity at infinity was employed as the special reference frame. In this case, the thermal spectrum of the Hawking radiation was shown to be reproduced. However, it is still uncertain whether the same thing remains true even when we adopt another reference frame as the special reference frame. In this paper, we give a generalization of previous works[2, 3, 4] by allowing a more general choice of the reference frame.

In most parts, we follow the work done by Corley[4], who analyzed a modification of the Unruh's original model, and showed analytically that the thermal spectrum at the Hawking temperature is reproduced for this modified model. To solve the wave equation analytically, he used the technique of the asymptotic matching of two approximate solutions which are obtained by using the method of Fourier transform and the WKB approximation. In this paper, we show that the resulting spectrum of created particle stays thermal one at the Hawking temperature even if we generalize the choice of the special reference frame. Furthermore, by a systematic use of the technique of the so-called matched asymptotic expansion, we also evaluate how small the leading correction to the thermal radiation is.

The remainder of this paper is as follows. First we introduce a generalization of Unruh's model in Sec.2. We also review what quantities need to be evaluated in computing the spectrum of particle creation in our model. In Sec.3 we describe the method to obtain a solution of the field equation, which is necessary to compute the spectrum. In Sec.4 we end with some conclusion. We use units with $\hbar = c = G = 1$.

2 Model

Following Ref.[2, 3, 4], we consider a massless scalar field propagating in a spacetime given by

$$ds^2 = -d\tilde{t}^2 + (d\tilde{x} - \tilde{v}(\tilde{x})d\tilde{t})^2, \quad (1)$$

where $\tilde{v}(x)$ is a function which goes to a constant at $\tilde{x} \rightarrow \infty$ and satisfies $\tilde{v}(\tilde{x}) \geq -1$ for $\tilde{x} \geq 0$. The equality is satisfied at $\tilde{x} = 0$. If we set $\tilde{v}(x) = -\sqrt{2M/(\tilde{x} + 2M)}$, this metric represents a 2-dimensional counterpart of a Schwarzschild spacetime, with the event horizon at $\tilde{x} = 0$. In this coordinate system, the unit vector that is perpendicular to the $\tilde{t}$ -constant hypersurfaces is given by $\tilde{u}_\alpha := \partial_\alpha \tilde{t}$, and the differentiation in this direction is given by $\tilde{u}^\alpha \partial_\alpha = (\partial_{\tilde{t}} + \tilde{v} \partial_{\tilde{x}})$. We denote the unit outward pointing vector normal to $\tilde{u}_\alpha$ by $\tilde{s}^\alpha$. In Ref.[2, 3, 4], in order to examine the effect on the spectrum of the Hawking radiation due to a modification of theory in the high frequency regime, they investigated a system defined by the modified action of a scalar field,

$$S = \frac{1}{2} \int d^2 \tilde{x} \sqrt{-g} g^{\alpha\beta} \mathcal{D}_\alpha \phi^* \mathcal{D}_\beta \phi, \quad (2)$$

where the differential operator $\mathcal{D}$ is defined by $\tilde{u}^\alpha \mathcal{D}_\alpha = \tilde{u}^\alpha \partial_\alpha$, $\tilde{s}^\alpha \mathcal{D}_\alpha = \hat{F}(\tilde{s}^\alpha \partial_\alpha)$. If we set $\hat{F}(z) = z$, the action (2) reduces to the usual one. Since we are interested in the effect caused by the change in the high frequency regime, we consider the case that $\hat{F}(z)$ differs from z only for large z .

In the above model, the dispersion relation for the scalar field manifestly breaks the general covariance, and there is a special reference frame specified by $\tilde{u}$. One can easily show that this reference frame is associated with a set of freely falling observers. As noted in Introduction, in Ref. [2, 3, 4], it was shown that the spectrum of the Hawking radiation is reproduced in this model. Here we investigate a more generalized model which allows us to adopt other reference frames as the special reference frame.

From the technical difficulty, however, we restrict our attention to stationally reference frames which satisfy $\mathcal{L}_{\partial/\partial \tilde{t}} u^\alpha = 0$, where $\mathcal{L}_{\partial/\partial \tilde{t}}$ is the Lie derivative in the direction of $\partial/\partial \tilde{t}$. Hence, $\mathcal{L}_{\partial/\partial \tilde{t}} s^\alpha = 0$ also holds. Let us introduce a new time coordinate, t , which satisfies $\partial_\alpha t = \Omega^{-1}(\tilde{x}) u_\alpha$, where $\Omega(\tilde{x})$ is independent of $\tilde{t}$. Then the condition $\mathcal{L}_{\partial/\partial \tilde{t}} u^\alpha = 0$ implies $dt = d\tilde{t} - f(\tilde{x})d\tilde{x}$, where $f(\tilde{x})$ is a function of $\tilde{x}$ and is related to Ω by $\Omega^{-2} = (1 - \tilde{v}f)^2 - f^2$. By using this new coordinate t , the metric (1) is written as

$$ds^2 = \Omega^2(x) [-dt^2 + (dx - v(x)dt)^2]. \quad (3)$$

Then, by using the new coordinate system, we find $u^\alpha \partial_\alpha = \frac{1}{\Omega} (\partial_t + v \partial_x)$, $s^\alpha = \frac{1}{\Omega} \partial_x$, $g^{\alpha\beta} = -u^\alpha u^\beta + s^\alpha s^\beta$, and $\sqrt{-g} = \Omega^2(x)$.

Now, we find that to generalize the choice of the special reference frame introduced to set the dispersion relation is equivalent to generalize the metric form from the one given in (1) to the one given in (3) replacing $\tilde{u}$ and $\tilde{s}$ with u and s in the defining equations of the differential operator $\mathcal{D}$. If we set $\Omega^2 \equiv 1$, the models are reduced to the original one discussed in Ref.[4].

Here, we should mention one important relation for the later use. The temperature of the Hawking radiation is determined by the surface gravity κ defined by $\kappa := d\tilde{v}/d\tilde{x}|_{\tilde{x}=0}$. Surprisingly, we can verify that the surface gravity κ can also be represented as $\kappa = dv/dx|_{x=0}$ by a straightforward calculation.

Then, let us derive the field equation by taking the variation of the action. Assuming that $\hat{F}(z)$ is an odd function of z , we obtain

$$(\partial_t + \partial_x v)(\partial_t + v\partial_x)\phi = \Omega \hat{F}\left(\frac{1}{\Omega}\partial_x\right) \Omega \hat{F}\left(\frac{1}{\Omega}\partial_x\right)\phi. \quad (4)$$

To proceed further calculations, we need to assume a specific dispersion relation. Following Ref. [4], we adopt here

$$\hat{F}(z) = z + \frac{1}{2k_0^2}z^3, \quad (5)$$

where k_0 is a constant. Since the model should be arranged to differ from the ordinary one only in the high frequency regime, the critical wave number k_0 must be chosen to be sufficiently large. With this choice of $\hat{F}$, neglecting the terms that are inversely proportional to the fourth power of k_0 , the field equation becomes

$$(\partial_t + \partial_x v)(\partial_t + v\partial_x)\phi = \left[\partial_x^2 + \frac{1}{2k_0^2} \left(\partial_x^2 \frac{1}{\Omega} \partial_x \frac{1}{\Omega} \partial_x \right) + \frac{1}{2k_0^2} \left(\partial_x \frac{1}{\Omega} \partial_x \frac{1}{\Omega} \partial_x^2 \right) \right] \phi. \quad (6)$$

To evaluate the spectrum of Hawking radiation, we need to get a solution of the above field equation with appropriate boundary condition. Owing to the time translation invariance, we do not have to solve the partial differential equation (6) directly. By setting $\phi(t, x) = e^{-i\omega t}\psi(x)$, Eq. (6) reduces to an ordinary differential equation,

$$\left[(-i\omega + \partial_x v)(-i\omega + v\partial_x) - \left\{ \partial_x^2 + \frac{1}{2k_0^2} \left(\partial_x^2 \frac{1}{\Omega} \partial_x \frac{1}{\Omega} \partial_x \right) + \frac{1}{2k_0^2} \left(\partial_x \frac{1}{\Omega} \partial_x \frac{1}{\Omega} \partial_x^2 \right) \right\} \right] \psi = 0. \quad (7)$$

From this ordinary differential equation, the asymptotic form of the general solution of the above equation at $x \rightarrow \infty$ is easily obtained as a superposition of plane waves as

$$\psi(x) = \sum_{l=1}^4 c_l(\omega) e^{ik_l(\omega)x}. \quad (8)$$

As shown in [3], two of the wave numbers have large absolute values, and the other two have small absolute values. For each pair, one is positive and the other is negative. By such a local analysis, however, the coefficients, $c_l(\omega)$, determining the weights of the respective partial waves are not determined. To determine them, we need the solution of the field equation that satisfies the boundary condition representing no ingoing wave into the event horizon. The required solution is obtained in the next section.

Finally, we present the formula to evaluate the expectation value of the number of particles in the state of an outgoing wave packet naturally defined at the future null infinity. For a wave packet which is peaked around a frequency ω , the expectation value of the number of created particles is given by[3]

$$N(\omega) = \frac{|\omega'(k_-)v_g(k_-)c_-^2(\omega)|}{|\omega'(k_{+s})v_g(k_{+s})c_{+s}^2(\omega)|}, \quad (9)$$

where k_- and k_{+s} are the large negative wave number and the small positive wave number, respectively. The negative wave number k_- corresponds to the ingoing negative frequency wave packet. Here $v_g(k) := \partial\omega/\partial k$ is the group velocity measured by a static observer, and $\omega'(k) := \omega - v(\infty)k$ is the frequency measured by the observers associated with the special reference frame.

3 Solving the field equation

3.1 WKB solution

Since the potentials, Ω and v , change slowly at far from the event horizon, we can use there the WKB approximation to find approximate solutions of the field equation (7). We construct a solution in the form, $\psi = \exp[i \int dx k(x)]$, where the wave number $k(x)$ is a function of x . Keeping the terms up to the first order differentiations of $v(x)$ and $1/\Omega(x)$, we obtain

$$\left(\frac{1}{k_0\Omega}\right)^2 k^4 - (1-v^2)k^2 - 2v\omega k + \omega^2 = i \frac{d}{dx} \left[2 \left(\frac{1}{k_0\Omega}\right)^2 k^3 - (1-v^2)k - v\omega \right] + \frac{3k'^2 + 4kk'' - ik'''}{k_0^2\Omega^2} + \frac{4}{k_0^2\Omega} \left(\frac{d}{dx} \frac{1}{\Omega} \right) [3kk' - ik''], \quad (10)$$

where, $(')$ denotes a derivative with respect to x . We solve this equation for k perturbatively, expecting that the terms including x -differentiations, i.e., the terms on the right hand side, give small corrections. As a result, the four solutions for wave numbers are obtained as

$$k_{\pm} = \pm k_0\Omega\sqrt{1-v^2} + \frac{i}{2} \left\{ \frac{d}{dx} \ln \left| (\pm k_0\Omega(1-v^2)^{3/2} + 4v\omega) \right| \right\} + \frac{v(1+r)\omega}{1-v^2} (1 + O(\epsilon^2))$$

$$k_{\pm s} = \pm \frac{\omega}{1 \pm v} (1 + O(\epsilon^2)), \quad (11)$$

where $\epsilon(x) := \sqrt{\omega^2/k_0^2\Omega^2(1-v^2)^3}$. The explicit form of the terms of $O(\epsilon)$ in $k_{\pm}$ are not given completely. The remaining terms are all real, and they are represented by r . Substituting this wave numbers, we obtain the WKB solutions,

$$\psi_{\pm}(x) \approx (k_0\Omega)^{-\frac{1}{2}} (1-v^2)^{-\frac{3}{4}} \exp \left[\pm i \int dx k_0\Omega\sqrt{1-v^2} \right]$$

$$\times \left(1 \pm \frac{4v\omega}{k_0\Omega} (1-v^2)^{-3/2} \right)^{-1/2} \exp \left[i\omega \int dx \frac{v(1+r)}{1-v^2} \right] (1 + O(\epsilon^2)),$$

$$\psi_{\pm s}(x) \approx \exp \left(\pm i\omega \int dx \frac{1}{1 \pm v} \right) (1 + O(\epsilon^2)). \quad (12)$$

These approximate solutions are valid in the region satisfying $\epsilon \ll 1$. That is, the WKB approximation breaks down near the horizon. To show the boundary of this region more explicitly, we expand $v(x)$ and $1/\Omega(x)$ near the horizon as $v(x) = -1 + \kappa x + \dots$ and $1/\Omega(x) = 1/\Omega_0 + \dots$, respectively. Assuming $\omega \sim \kappa$, we obtain the lower limit on x as

$$x > x_1 \approx \left(\frac{1}{k_0^2\Omega_0^2\kappa} \right)^{\frac{1}{3}}, \quad (13)$$

for the region where the WKB approximation is valid.

3.2 Solution near the event horizon

As we saw in the preceding subsection, the WKB approximation cannot be applied near the horizon, where the boundary conditions for the wave function are imposed. In this section, we construct an approximate solution of the field equation (6) which is valid near the horizon. For this purpose, following the method used in Ref. [4], we approximate Eq. (6) by expanding $v(x)$ and $1/\Omega(x)$ in powers of x . Solving this approximate equation, we obtain a solution which is valid near the horizon. For definiteness, we restrict our consideration to the region limited by $|x| < x_2$.

We first expand $v(x)$ and $1/\Omega(x)$ as

$$v(x) = -1 + \kappa x + \frac{1}{2}\kappa_1^2 x^2 + \dots,$$

$$\frac{1}{\Omega(x)} = \frac{1}{\Omega_0} + \frac{1}{\Omega_1} x + \dots. \quad (14)$$

Since we are interested in the region with small $|x|$, we neglect the higher order terms of $O(|x|^3)$ in $v(x)$ and those of $O(|x|^2)$ in $1/\Omega(x)$. Define

$$\delta := \max_{|x| < x_2, i=0,1,2} \bar{\delta}_i(x), \quad (15)$$

with

$$\bar{\delta}_0 := \kappa x, \quad \bar{\delta}_1 := \frac{\kappa_1^2 x}{\kappa}, \quad \bar{\delta}_2 := \frac{\Omega_0}{\Omega_1} x, \quad (16)$$

Assuming that x_2 is sufficiently small satisfying the condition, $\delta \ll 1$, we find that the upper bound of the region of validity, x_2 , should be taken to satisfy

$$x_2 \ll \text{Min}(1/\kappa, \kappa/\kappa_1^2, \Omega_1/\Omega_0). \quad (17)$$

Substituting the truncated expressions (14) into Eq. (6), we obtain

$$\begin{aligned} L[\psi(x)] := & \frac{1}{k_0^2} \left[\frac{1}{\Omega_0^2} + \frac{2}{\Omega_0 \Omega_1} x \right] \psi^{(4)} + \frac{4}{k_0^2} \frac{1}{\Omega_0 \Omega_1} \psi''' + (2\kappa x - \kappa^2 x^2 + \kappa_1^2 x^2) \psi'' \\ & + 2 \left[-(i\omega - \kappa) + (\kappa(i\omega - \kappa) + \kappa_1^2 x) \right] \psi' - i\omega(i\omega - \kappa - \kappa_1^2 x) \psi = 0. \end{aligned} \quad (18)$$

where we neglect the terms containing $\delta_i^2(x)$ ($i = 0, 1, 2$). We solve this equation by moving to the momentum space. Let us introduce $\hat{\psi}(s)$ by

$$\psi(x) = \int_C ds e^{sx} \hat{\psi}(s), \quad (19)$$

where the contour of integration C is determined so that the wave function $\psi(x)$ satisfy the boundary condition that the wave function exponentially decay inside the horizon ($x < 0$). Then Eq.(18) is transformed into the equation for $\hat{\psi}$

$$\frac{1}{k_0^2 \Omega_0^2} s^4 \hat{\psi} - 2\kappa \partial_s s^2 \hat{\psi} - 2(i\omega - \kappa) s \hat{\psi} = \dots \quad (20)$$

All the terms newly added in the present work are on the right hand side in this equation, although their explicit form is suppressed here. Taking these terms as small, we solve the above equation iteratively. After substitution of thus obtained $\hat{\psi}$ into Eq.(19), we evaluate the contour integral by taking the contour C as described in Ref.[4]³ For $x < 0$, we can evaluate this integration by using steepest descents and we can verify that the wave function satisfies the required boundary condition. When we consider $\psi(x)$ for $x > 0$, the place of saddle points for steepest descents moves. Hence, we deform this contour so that it runs through the saddle points. Then, the contour integral are turned out to be divided into three parts, which can be evaluated separately. Two of them are evaluated by using steepest descents, and the remaining integral is expressed by using a gamma function. Here we skip the detailed evaluations and show the resulting formula. Up to $O(\epsilon\delta)$, contour integrals are evaluated as

$$\psi(x) = \psi_1(x) + \psi_2(x) + \psi_3(x), \quad (21)$$

where

$$\begin{aligned} \psi_{1,2}(x) &\approx e^{\pm i\pi\omega/2\kappa} \frac{1}{\sqrt{k_0 \Omega_0}} (k_0 \Omega_0)^{-\frac{i\pi}{2}} \sqrt{2\pi\kappa} (2\kappa x)^{-\frac{1}{2} - \frac{i\pi}{2}} \exp \left(\pm i \frac{2}{3} k_0 \Omega_0 \sqrt{2\kappa x}^{\frac{3}{2}} + W_{1,2}(\bar{\epsilon}, \bar{\delta}) + O(\epsilon^2, \delta^2) \right), \\ \psi_3(x) &\approx -2 \sinh(\pi\omega/\kappa) \Gamma(-i\omega/\kappa) x^{i\omega/\kappa} \exp \left(-\frac{i\omega}{2\kappa} \bar{\delta}_1 + O(\epsilon^2, \delta^2) \right), \end{aligned} \quad (22)$$

where

$$W_{1,2} = \pm i \frac{\sqrt{2}}{\bar{\epsilon}_2} \left(-\frac{1}{10} \bar{\delta}_0 + \frac{1}{10} \bar{\delta}_1 - \frac{2}{5} \bar{\delta}_2 \right)$$

³If we are careful enough, we find that we need to modify the contour C corresponding to addition of new terms. However, we skip this delicate issue because we do not have enough space to discuss it here.

$$\begin{aligned} \pm i \frac{\tilde{\epsilon}_2}{\sqrt{2}} & \left[\left(-\frac{41}{48} - \frac{i\omega}{\kappa} - \frac{1}{4} \left(\frac{i\omega}{\kappa} \right)^2 \right) + \left(\frac{11}{64} + \frac{1}{4} \frac{i\omega}{\kappa} + \frac{1}{16} \left(\frac{i\omega}{\kappa} \right)^2 \right) \tilde{\delta}_0 \right. \\ & \left. + \left(-\frac{11}{64} + \frac{3}{4} \frac{i\omega}{\kappa} + \frac{15}{16} \left(\frac{i\omega}{\kappa} \right)^2 \right) \tilde{\delta}_1 + \left(-\frac{5}{16} - \frac{i\omega}{\kappa} - \frac{3}{4} \left(\frac{i\omega}{\kappa} \right)^2 \right) \tilde{\delta}_2 \right] \end{aligned} \quad (23)$$

3.3 asymptotic matching

	ϵ^{-1}	ϵ^0	ϵ^1	ϵ^2
δ^0	$x^{3/2}$	x^0	$x^{-3/2}$	x^{-3}
δ^1	$x^{5/2}$	x^1	$x^{-1/2}$	x^{-2}
δ^2	$x^{7/2}$	x^2	$x^{1/2}$	x^{-1}
δ^3	$x^{9/2}$	x^3	$x^{3/2}$	x^0

To obtain the flux of the created particle observed in the asymptotic region, where $v(x)$ is essentially constant, we now propagate the solution (22) by patching onto WKB solutions which are valid in the region distant from the horizon. To explain this matching process, we present a table here. In the WKB approximation, we considered an expansion of the solution with respect to ϵ , and calculated the corrections up to $O(\epsilon)$, though the explicit form is not presented in (12). Namely, the terms corresponding to the first three columns in the above table are obtained in the WKB approximation. Substituting (14), we expand this expression also with respect to δ up to $O(\delta)$ to obtain the first 2×3 elements in the table. The final expression to this order becomes

$$\begin{aligned} \psi_{\pm}(x) &= \frac{1}{\sqrt{k_0 \Omega_0}} (2\kappa x)^{-\frac{1}{2}} \exp \left(\pm i \frac{2}{3} k_0 \Omega_0 \sqrt{2\kappa} x^{\frac{3}{2}} + W_{1,2} + O(\epsilon^2, \delta^2) \right), \\ \psi_{+s} &= x^{\frac{i\omega}{\kappa}} \left(1 - \frac{i\omega}{2\kappa} \tilde{\delta}_1 \right), \end{aligned} \quad (24)$$

where $W_{1,2}$ is the same that is shown in Eq.(23). On the other hand, in the construction of a solution near the horizon, we first expand the equation with respect to δ to obtain an equation which correctly gives the terms corresponding to the first two lines in the above table. Then, with further expansion with respect to ϵ , we obtained the final expression, which also contains the terms corresponding to the first 2×3 elements in the table.

Such two fold expansion in both schemes is valid only in the region $x_1 < x < x_2$, where $\tilde{\epsilon}(x)$ and $\tilde{\delta}_i(x)$ are small. As long as k_0 is taken to be sufficiently large, this overlapping region always exists. Since both expressions obtained by using the above two different schemes are approximate solutions of the same equation, they must be identical if we take an appropriate superposition of the four independent WKB solutions. This is directly verified by comparing the expressions (12) and (21). In fact, we find

$$\psi(x) = \sqrt{2\pi\kappa} (e^{\frac{\pi\omega}{2\kappa}} \psi_+ + e^{-\frac{\pi\omega}{2\kappa}} \psi_-) - 2 \sinh \left(\frac{\pi\omega}{\kappa} \right) \Gamma \left(-\frac{i\omega}{\kappa} \right) \psi_{+s}. \quad (25)$$

From this expression, we can read the coefficients c_i . Substituting these coefficients into the expression (9), we obtain the expectation value of the number of created particles as

$$N(\omega) = \frac{1}{e^{2\pi\omega/\kappa} - 1} (1 + O(\omega^2/k_0^2)). \quad (26)$$

The spectrum of Hawking radiation is reproduced even if we generalize the choice of the special reference frame introduced to set the dispersion relation.

Finally, we discuss the reason why we can say that the possible deviation from the thermal flux in our model is at most $O(\omega^2/k_0^2)$. Let us look at the table again. For each element in the table we have

presented the powers of x of that the corresponding terms possess. As we mentioned above, to proceed matching, we need to take an appropriate superposition of the four independent WKB solutions. This choice of constant coefficients must be determined by the matching of the x -independent elements. The leading order of the x -independent elements consists of the terms of $O(\epsilon^0 \delta^0)$. One may notice that the second lowest one consists of the terms of $O(\epsilon^2 \delta^3) = O(\omega^2/k_0^2)$. This fact tells us that the possible deviation from the thermal radiation starts from this order. In fact, we have already confirmed this deviation numerically by assuming an appropriate model of $v(x)$ and $\Omega(x)$.

4 Conclusion

We studied the particle creation in a model which is a generalization of the Unruh's toy model. In this model, the field equation for a scalar field is modified by introducing a non-standard dispersion relation. To do so, we necessarily violates the general covariance. This radical change of theory is originally motivated by the possible existence of the effect due to the unknown physics at Planck scale. Here, however, we pointed out a new point of view. Even if we believe the general covariance is exact, it is also meaningful to study this model as an effective theory which takes into account the interaction between various fields.

In the original model, the dispersion relation is modified on the basis of the freely falling observers. In our present work, we generalized the choice of the reference frame with respect to which we set the non-standard dispersion relation. Extending the analytic method developed by Corley[4], we have shown that the thermal spectrum of radiation from a black hole up to $O(\omega/k_0)$ even if we do not take the freely falling frame as the special reference frame. The deviation from the exact thermal spectrum appears from $O(\omega^2/k_0^2)$, where ω is the typical energy scale of the radiation ($\approx$ Hawking temperature) and k_0 is that of the modified dispersion relation.

Of course, we do not think that the model that we studied here is good enough to modelize the radiation from a black hole. It is still an open question if a possible modification at high frequency regime may change the nature of the Hawking radiation. Thus, further studies on this subject are necessary.

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Cluster Expansion Approach to Self-Gravitating Systems

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Abstract

We study statistical mechanics of the self gravitating system applying the cluster expansion method originally developed in solid state physics. By summing infinite series of diagrams, we derive the finite complex free energy whose imaginary part is related to the relaxation time of the system. Summation of another series yields two-point correlation function whose correlation length is essentially given by the Jeans wavelength of the system.

1 Introduction

Variety of structures found in the present Universe are thought to be seeded by quantum fluctuations in the very early Universe and evolved through the gravitational interaction in the expanding Universe aided by yet unknown dark matter. All these ingredients, the initial condition (quantum fluctuations), the force (gravity), and the environment (cosmic expansion and dark matter), should be indispensable for the structure formation.

Among these ingredients, we can study the gravity in the most intrinsic way without any assumption in principle. Therefore, in this paper, we would like to study fundamental properties of the self-gravitating system (SGS) from a view point of statistical mechanics. We specify the SGS as a mass distribution which consists of discrete mass points mutually interacting through the force of Newtonian gravity. The contact collisions of the ingredients will be neglected.

The SGS is well known to be unstable and eventually collapses. Thus it is widely believed that there is no thermal equilibrium in SGS. However relevant structures would be observed in the intermediate stage of its evolution well before the collapse. Therefore we may have a room to construct statistical mechanics of SGS if we restrict the stage of the evolution. Actually there have been many studies dealing with statistical mechanics of SGS[1, 2, 3, 4]. For example, an approach by de Vega *et al.*[5] considering canonical/grand canonical ensemble of SGS in order to explain the scaling relation in the interstellar medium seems interesting. In the thermodynamic limit, thermodynamic quantities of SGS diverge and SGS becomes unstable. De Vega *et al.* introduced another thermodynamic limit and studied the stable region of SGS in their work. We would like to circumvent this divergence allowing the imaginary part in thermodynamic objects. The appearance of the imaginary part apparently suggests the failure of the

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ordinary *static* thermodynamics, while it provides us *dynamical* information of the system such as the decay strength.

We shall construct statistical mechanics of SGS after the model of classical electron-gas system on the uniform ion background. This system has the same square-inverse law of force but with opposite signature to SGS: $F_{\text{electron-gas}} = +e^2/r^2$, $F_{\text{SGS}} = -Gm^2/r^2$. Contrary to the ordinary belief, the natural correspondence $e^2 \leftrightarrow -Gm^2$ is quite suggestive, where e is the charge of electron and G is the Newton's constant. We analytically continue e^2 to $-Gm^2$ with the phase rotation from 0 to π through the upper half complex plain and identify the temperature T as the velocity dispersion of SGS $\langle \tilde{v}^2 \rangle$. Then for example, the Debye wavelength λ_D corresponds to the Jeans wavelength λ_J , and the plasma frequency $\omega_P = \sqrt{4\pi ne^2/m}$ is related to the inverse of the free fall time, τ_{FF}^{-1} . We believe that these correspondences are not merely appearance. Actually, we will show that the free energy of SGS can be calculated in the same way as the classical electron-gas system by applying the cluster expansion method [6, 7, 8]. Moreover n -point correlation functions of SGS are similarly obtained.

2 Cluster Expansion Method

We consider a non-relativistic gas system of N particles with the same mass m in a box of size L . Since the number N is enormous, we have to study the system statistically. We work in the canonical ensemble for the technical convenience. If the SGS we consider is not a closed system and exchanges energy with other components such as dark matter, this is a reasonable choice. If the SGS we consider is a closed system, we should use the micro-canonical ensemble and the use of canonical ensemble is an approximation. If the statistical mechanics of SGS is entirely different from the ordinary one, we have to reconsider the nature of ensemble. This issue is fairly tough and therefore we will postpone the argument to our future intensive work.

The Hamiltonian of SGS is

$$H = \sum_{i=1}^N \frac{\vec{p}_i^2}{2m} + \sum_{1 \leq i < j \leq N} \frac{-Gm^2}{|\vec{x}_i - \vec{x}_j|}. \quad (1)$$

The gas of particles is in thermal equilibrium at temperature T . Fixing the particle number N , the volume $V = L^3$, and the temperature $T = (k_B \beta)^{-1}$, where k_B is the Boltzmann's constant, the partition function of the system is given by

$$Z(u) = \text{Tr} \exp \left[-\beta H + \sum_{i=1}^N u(\vec{x}_i) \right], \quad (2)$$

where $u(\vec{x}_i)$ is the source function and Tr means the whole integration on the phase space of SGS. After the simple Gaussian integration on the momentum, we obtain

$$Z(u) = \frac{(2\pi m \beta^{-1})^{\frac{3N}{2}} V^N}{N! h^{3N}} \underbrace{\int_{V^N} \frac{d^N \vec{x}}{V^N} \exp \left[\sum_{i=1}^N u(\vec{x}_i) + \sum_{1 \leq i < j \leq N} \varphi_{ij} \right]}_{e^{W(u)}}, \quad (3)$$

where $\varphi_{ij} = \beta G m^2 / |\vec{x}_i - \vec{x}_j|$ and h is the Plank constant. We have introduced the above source function for convenience of calculation: By using functional derivative with respect to the source $u(\vec{x}_i)$, k -point correlation functions are given by

$$\begin{aligned} G(\vec{x}_1, \dots, \vec{x}_k) &= \frac{\delta^k \ln Z}{\delta u(\vec{x}_1) \dots \delta u(\vec{x}_k)} \\ &= \frac{1}{Z} \frac{1}{(N-k)!} \int d\vec{r}_{k+1} \dots d\vec{r}_N \exp \left(\sum_i u(\vec{r}_i) + \sum_{1 \leq i < j \leq N} \phi_{ij} \right). \end{aligned} \quad (4)$$

In the canonical ensemble, the velocity dispersion $\langle v^2 \rangle$ of the system is automatically related with the temperature as $\frac{1}{2}k_B T \times 3 = \frac{1}{2}m \langle v^2 \rangle$.

For the calculation of thermodynamic quantities, we introduce the logarithm of the average of the configurational sum setting $u \rightarrow 0$. We denote this quantity as W_0 :

$$\begin{aligned} W_0 &= W(0) \\ &= \ln \left\langle \exp \left(\sum_{1 \leq i < j \leq N} \varphi_{ij} \right) \right\rangle, \end{aligned} \quad (5)$$

where $\langle \dots \rangle = \frac{1}{V^N} \int \dots \int d^N \vec{x}$. The cumulant expansion of this is possible:

$$W_0 = \left\langle \exp \left(\sum_{1 \leq i < j \leq N} \varphi_{ij} \right) - 1 \right\rangle_C = \sum_{\nu_1 \dots \nu_{\#pair}} \frac{\langle \varphi_1^{\nu_1} \varphi_2^{\nu_2} \dots \varphi_{\#pair}^{\nu_{\#pair}} \rangle_C}{\nu_1! \dots \nu_{\#pair}!}, \quad (6)$$

where $\langle \dots \rangle_C$ is the cumulant and $1, 2, \dots, \#pair = N(N-1)/2$ are all the possible pairs of N -particles. The ordinary momentum expansion is the series expansion of $\exp(W_0)$ in terms of the interaction φ_{ij} while the cumulant expansion is the series expansion of W_0 . If a cumulant reduces to multiple integration pieces which are mutually independent, then it vanishes.

It is convenient to use graphical representations for W_0 . An each term in Eq.(6) corresponds to a graph which consists of vertices and lines. Each line terminates at two distinct vertices. A vertex $\vec{x}_i$ simply represents an integration position and a line connecting $\vec{x}_i$ and $\vec{x}_j$ represents an interaction φ_{ij} (Fig.1).



Figure 1: A line represents the interaction between two particles at $\vec{x}_i$ and $\vec{x}_j$.

All the elements in a graph are multiplied with each other and integrated over all vertices (Fig.2).

This is the *cluster expansion method*. All the graph which contributes to W_0 should be connected, because we are dealing with the cumulant expansion. Moreover all the graph which contributes to W_0 should not include the point which divides the graph into multiple disconnected pieces when the graph is cut at this point. These points are called articulation points. This is because the translational invariance of the system makes the integration of the articulation point trivial and the rest of the graph breaks into multiple integration pieces which are mutually independent.

By expanding with respect to the number density $n = N/V$, W_0 is reduced to the following form [6, 7, 8]:

$$W_0 = \sum_{k=1}^{\infty} \frac{n^{k+1}}{(k+1)!} \sum_{\substack{\text{connected} \\ \text{irreducible}}} \int_{V^{k+1}} \prod_{1 \leq i < j \leq k+1} \left[\frac{\varphi_{ij}^{\nu_{ij}}}{\nu_{ij}!} \right] d\vec{x}_1 \dots d\vec{x}_{k+1}, \quad (7)$$

where ν_{ij} is the number of interaction lines between the vertices i and j . The "irreducible" means that the graph does not include any articulation points.



Figure 2: W_0 is the sole body of possible configurations. This can be represented as the sum of all possible graphs.

3 Complex Free Energy

Let us now calculate the free energy for SGS by using the cluster expansion method. We shall include all the higher orders in n for calculating W_0 . Here we choose a series of graphs which are the lowest order of φ_{ij} in each set of graphs containing k internal vertices. This is equivalent to a sum of all graph which has the topology of a ring (Fig.3). This approximation will not be valid for short distances where φ_{ij} grows without bound.

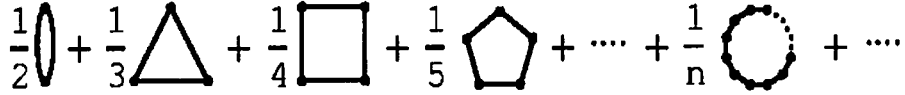


Figure 3: A series of cluster expansion by ring graphs. A symmetric factor is written in the left side of each graph.

A ring graph which contains k vertices in Eq.(7) corresponds to

$$n^k \int \varphi_{12} \varphi_{23} \cdots \varphi_{k1} d\vec{x}_1 \cdots d\vec{x}_k = \int_{|\vec{q}| > 2\pi/L} \frac{d\vec{q}}{(2\pi)^3} \left[n \tilde{\varphi}(\vec{q}) \right]^k, \quad (8)$$

where $\tilde{\varphi}(\vec{q}) = \frac{4\pi\beta Gm^2}{q^2}$ is the Fourier transform of $\varphi_{ij}(x)$. There are $(k-1)!/2$ terms of this kind. Therefore,

$$\begin{aligned} W_0 &= \frac{V}{2} \int_{|\vec{q}| > 2\pi/L} \frac{d\vec{q}}{(2\pi)^3} \sum_{k=2}^{\infty} \frac{[n \tilde{\varphi}(\vec{q})]^k}{k} \\ &= \frac{V}{4\pi^2} \int_{2\pi/L}^{\infty} dq \cdot q^2 \left[-\frac{\kappa^2}{q^2} - \ln \left(1 - \frac{\kappa^2}{q^2} \right) \right] \\ &= \frac{1}{12\pi^2} \left[2\pi\kappa^2 L + \kappa^3 L^3 \ln \left| \frac{\kappa L + 2\pi}{\kappa L - 2\pi} \right| + 8\pi^3 \ln \left| 1 - \frac{\kappa^2 L^2}{4\pi^2} \right| + i\pi(\kappa^3 L^3 + 8\pi^3 \theta(\kappa L - 2\pi)) \right], \quad (9) \end{aligned}$$

where $\kappa^2 = 4\pi Gm^2 \beta n = 12\pi^2 / \lambda_J^2$ with $\lambda_J = \sqrt{\pi \langle v^2 \rangle / (Gmn)}$ beeing the Jeans wave length and θ is the Heaviside function. For the integration in Eq.(9), we analytically continue e^2 to $-Gm^2$ from the

above complex e^2 plane by the reason under Eq.(10). We notice that the imaginary part appears for $L > 2\pi/\kappa = \lambda_J/\sqrt{3}$. This originates from the negative argument in the logarithm for small wavenumber q . The appearance of it apparently indicates the Jeans instability. The thermodynamic limit $N \rightarrow \infty$, $V \rightarrow \infty$ with fixed number density n , yields the free energy:

$$\begin{aligned} F &= -\frac{1}{\beta} \ln Z(0) \\ &= -\frac{N}{\beta} \left[\frac{3}{2} \ln \left(\frac{2\pi m}{\beta h^2} \right) + 1 - \ln n - \frac{i\kappa^3}{12\pi n} \right]. \end{aligned} \quad (10)$$

The above choice of analytical continuation guarantees the positivity of the imaginary part of F : The system is truly dissipative rather than anti-dissipative.

Finiteness of W_0 or F , before and after taking the thermodynamic limit, is remarkable. This is because we only considered the ring graphs allowing complexity of the partition function. Therefore the instability of SGS, which is ordinary thought to be the origin of divergence in thermodynamic quantities, will be represented in the imaginary part of the free energy.

In general, the imaginary part of the free energy is related with the decay strength Γ of the system [9]

$$\Gamma = \frac{\omega\beta}{\pi} \Im F, \quad (11)$$

where $-\omega^2$ is the negative eigenvalue at the saddle point dividing the metastable region from the stable region(Fig.4).

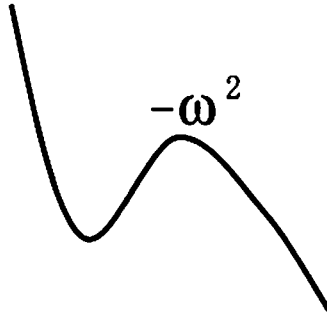


Figure 4: The schematic graph of the metastable potential for an order parameter. The local minimum is the metastable state, which eventually decays toward right through the saddle point, whose curvature is $-\omega^2$.

In our case in SGS, we do not have an explicit order parameter and its metastable potential. Though the complete correspondence between SGS and the general metastable state has not yet been established, it is tempting to identify ω in Eq.(11) with the inverse of the free fall time, i.e. $\omega^2 = -4\pi Gmn = -(\tau_{FF})^{-2}$, since they are related with each other through the analytical continuation $e^2 \rightarrow -Gm^2$. Then we find the decay strength Γ as

$$\Gamma = \frac{4\sqrt{3}NG^2m^2n}{\langle v^2 \rangle^{3/2}} \quad (12)$$

which is essentially the inverse of the binary relaxation time [3, 10],

$$\tau_{bc} = \frac{\langle v^2 \rangle^{3/2}}{32\pi^2 G^2 m^2 n \ln(N/2)} \quad (13)$$

except $\ln N$ is replaced by N .

It is worth noting that the imaginary part of the free energy ($\Im F \propto n^{3/2}$) is related to the fluctuation of the system through the fluctuation-dissipation theorem. In this context, it seems interesting to notice that the number 3 in the above $n^{3/2}$ comes from the spatial dimensionality and 2 from the inverse-square law of the gravitational force. This reminds us of the Holtsmark distribution of the gravitational force acting in the uniform self-gravitating system [10] or the stable distribution of index 3/2 [11].

4 Two-point Correlation Function

Let us now turn our attention to the correlation functions. One-point correlation function is simply a number: $G_1(\vec{x}) = N$. The normalized two-point correlation function which is usually used in astrophysics is

$$\xi(r) = \frac{G(\vec{x}_1, \vec{x}_2)}{G(\vec{x}_1)G(\vec{x}_2)}, \quad (14)$$

where $r = |\vec{x}_1 - \vec{x}_2|$. A similar cluster expansion for $G(\vec{x}_1, \vec{x}_2)$ would be possible,

$$G(\vec{x}_1, \vec{x}_2) = N(N-1) \sum_{k=0}^{\infty} \frac{n^k}{k!} \sum_{\text{connected}} \int_{V^k} \prod_{1 \leq i < j \leq k+2} \left[\frac{\varphi_{ij}^{\nu_{ij}}}{\nu_{ij}!} \right] d\vec{x}_3 \cdots d\vec{x}_{k+2}. \quad (15)$$

This time, reducible graphs *do* contribute since the translational invariance is lost once we fix the positions $\vec{x}_1$ and $\vec{x}_2$. We shall include all the higher orders of n as in the previous calculation. Therefore, among each set of graphs which contain k internal vertices (excluding both $\vec{x}_1$ and $\vec{x}_2$), we choose the lowest order skeleton graph in φ_{ij} . This is equivalent to a sum of all the graph which has the topology of a chain shown in Fig.5.

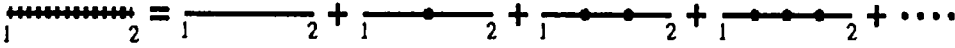


Figure 5: The series of graphs which have the topology of chain.

This yields the renormalized interaction (line) $\bar{\varphi}_{12}$ as,

$$\begin{aligned} \bar{\varphi}_{12} &= \varphi_{12} + \int \varphi_{13} \varphi_{32} d\vec{x}_3 + \int \varphi_{13} \varphi_{34} \varphi_{42} d\vec{x}_3 d\vec{x}_4 + \cdots \\ &= \int \frac{d\vec{q}}{(2\pi)^3} \frac{\bar{\varphi}(\vec{q})}{1 - n\bar{\varphi}(\vec{q})} \exp(i\vec{q}(\vec{x}_1 - \vec{x}_2)) \\ &= \frac{2Gm^2\beta}{\pi r} \int_0^\infty d\lambda \frac{\lambda \sin \lambda}{\lambda^2 - \kappa^2 r^2} \\ &= \beta Gm^2 \frac{\cos(\kappa r)}{r}. \end{aligned} \quad (16)$$

Since we have analytically continued e^2 to $-Gm^2$ from the above complex e^2 plane in Eq.(9), we should choose the pole at $-\kappa r$ for the λ integration. We can further sum over multi-lines of these chain graphs as shown in Fig.6.

This another summation yields

$$\hat{\varphi}_{12} = \bar{\varphi}_{12} + \frac{1}{2!} \bar{\varphi}_{12}^2 + \frac{1}{3!} \bar{\varphi}_{12}^3 + \frac{1}{4!} \bar{\varphi}_{12}^4 + \cdots$$

$$\text{Chain graph}_{12} = \text{Chain graph}_{12} + \frac{1}{2!} \text{Chain graph}_{12}^{(1)} + \frac{1}{3!} \text{Chain graph}_{12}^{(2)} + \frac{1}{4!} \text{Chain graph}_{12}^{(1,2)} + \dots$$

Figure 6: Summation over multi-lines of chain graphs yields the two point correlation function except the mass renormalization which will be discussed below.

$$= \exp[\bar{\varphi}_{12}] - 1. \quad (17)$$

Moreover, we should consider the mass renormalization for the external vertex. In principle, this is given by sum of all the graph which consists of a cluster around the external vertex $\vec{x}_1$. Since infinite summation of this class of graphs at present is technically impossible, we phenomenologically introduce the effective mass. Since the gravitational attractive force balances with the stirring force arising from the velocity dispersion at the length scale κ^{-1} , the mass inside of this scale is thought to behave collectively. Thus we simply estimate the effective mass $m^* = \frac{4\pi}{3}(\kappa^{-1})^3 nm$ which should be replaced with the mass at $\vec{x}_1$ or at $\vec{x}_2$ but not both (Fig.7).

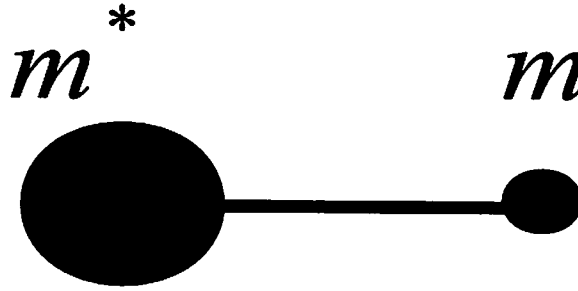


Figure 7: The mass renormalization is sufficient either at $\vec{x}_1$ or at $\vec{x}_2$ since the two-point correlation is the conditional probability.

Strictly speaking, m^* is merely a parameter in the present phenomenological argument. By replacing m at $\vec{x}_1$ in Eq.(17) with m^* , the two-point correlation function becomes

$$G(\vec{x}_1, \vec{x}_2) = N^2 \hat{\varphi}_{12} \Big|_{m(\text{at } \vec{x}_1) \Rightarrow m^*(\text{at } \vec{x}_1)}. \quad (18)$$

Finally we obtain the normalized two-point correlation function:

$$\begin{aligned} \xi(r) &= \frac{G(\vec{x}_1, \vec{x}_2)}{G(\vec{x}_1)G(\vec{x}_2)} \\ &= \exp\left(\frac{1}{3\kappa r} \cos(\kappa r)\right) - 1. \end{aligned} \quad (19)$$

The scaling property is manifest: Small scale SGS with large κ has the same correlation function as that of large SGS with small κ . This function $\xi(r)$ has interesting features. As a function of $s = \kappa r$, it has the unique inflection point at $s = 0.440$ when plotted in the Log-Log graph. The slope at $s = 0.44$ is -1.667 and the magnitude there is 0.986 . Therefore, the inflection point $r_c \approx 0.44\kappa^{-1} \approx 0.04\lambda_J$ can be regarded as the correlation length of SGS.

This two-point correlation function is shown in Fig.8 with typical observational results[12, 13] on top of it. However this comparison with observation is not intended as the demonstration of our approach but simply as an illustration.

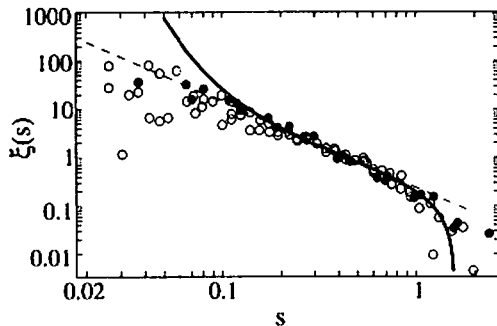


Figure 8: A Log-Log graph of two-point correlation function: $\xi(s) = \exp(\cos(s)/(3s)) - 1$ (solid line), $(s/0.45)^{-1.77}$ (broken line), observational data of galaxies [12] (open circle), and observational data of clusters of galaxies [13] (filled circle). These correlation lengths of the observational data are rescaled. The error bars ignored here are found in [13].

The observed correlation seems not to exclude our correlation function except small scale region where the interaction φ exceeds one and our approximation is no longer valid. The comparison with the data for clusters of galaxies should not be taken seriously. Actually, an extended object such as the cluster of galaxies, in the strict sense, could hardly be regarded as a point object which consists of N-body.

From the observational data[12, 13], the correlation lengths of galaxies and of clusters of galaxies directly read off as $r_c^{galaxy} \approx 6.2h^{-1} \text{ Mpc}$ and $r_c^{cluster} \approx 15h^{-1} \text{ Mpc}$ respectively. Since $r_c \approx 0.44\kappa^{-1}$ for the two-point correlation function we obtained, we rescaled the observational data: $r \rightarrow s = \kappa r$ with $\kappa = 1/14h[\text{Mpc}^{-1}]$ for galaxies and $\kappa = 1/34h[\text{Mpc}^{-1}]$ for clusters of galaxies, where $h = H_0/(100\text{kms}^{-1}\text{Mpc}^{-1})$ with H_0 being the Hubble constant at the present time. These values of κ correspond to $\lambda_j^{galaxy} \approx 152h^{-1} \text{ Mpc}$ and $\lambda_j^{cluster} \approx 370h^{-1} \text{ Mpc}$ from the relation given immediately after Eq.(9). On the other hand, typical Jeans lengths calculating from the standard observations for galaxies and for clusters of galaxies are respectively

$$\lambda_j^{galaxy} \approx 123 \text{ Mpc} \left[\left(\frac{\langle v^2 \rangle}{(500 \text{ kms}^{-1})^2} \right) \left(\frac{2 \times 10^{44} g}{m} \right) \left(\frac{3.7 \times 10^{-2} h^3 \text{ Mpc}^{-3}}{n} \right) \right]^{-1/2}, \quad (20)$$

and

$$\lambda_j^{cluster} \approx 450 \text{ Mpc} \left[\left(\frac{\langle v^2 \rangle}{(1000 \text{ kms}^{-1})^2} \right) \left(\frac{2 \times 10^{47} g}{m} \right) \left(\frac{1.1 \times 10^{-5} h^3 \text{ Mpc}^{-3}}{n} \right) \right]^{-1/2}. \quad (21)$$

Here, we have used typical values for the normalization though there are some amount of ambiguity especially in the mass estimation which comes from unknown dark matter contributions. The above values of the Jeans lengths are not so far from the above values through two-point correlation function we obtained.

5 Summary and Outlook

Studying the canonical ensemble of the self-gravitating system (SGS), we obtained the complex free energy by summing an infinite series of graph in the cluster expansion method for SGS, Eq.(10). The imaginary part of the free energy yields the decay strength of SGS, Eq.(12). Similar summation of an infinite series of graph yields the universal two-point correlation function Eq.(19) which scales essentially with the Jeans wavelength. The correlation length is linearly proportional to the mean separation of ingredients.

Several points presented in this paper need further investigation. We would like to report them in the near future.

1) The choice of the canonical ensemble for SGS should be justified. We have worked in the canonical ensemble only for technical convenience. If the SGS we consider is a closed system, we should use the micro-canonical ensemble with given energy. This partition function or the entropy would be the Laplace transform, with respect to the inverse temperature β , of the canonical partition function. We would like to study whether similar cluster expansion is possible for this micro-canonical ensemble.

2) Validity of the ring- and chain- approximations should be further clarified. In these approximations, only special type of graphs are summed. For this purpose, studying much simpler models, in which the cluster expansion method can be easily compared with other approximations, may be advantageous. We are now studying the modified Konishi- Kaneko model with the Hamiltonian

$$H = \sum_{i=1}^N \frac{p_i^2}{2} - \kappa \sum_{1 \leq i < j \leq N} \cos(\theta_i - \theta_j), \quad (22)$$

where θ_i is the angle ($0 \leq \theta_i < 2\pi$) and $p_i = \dot{\theta}_i$, and one-dimensional gravity — the periodic sheet model with the Hamiltonian

$$H = \sum_{i=1}^N \frac{p_i^2}{2} - \kappa \sum_{1 \leq i < j \leq N} |\theta_i - \theta_j|. \quad (23)$$

3) The binary relaxation time Eq.(13) and our decay time given by Eq.(12) are different with each other by the factor $N/\ln(N)$. This would be generally enormous in practical astronomical applications and the observational relevance of our decay time scale Γ^{-1} is not clear. However, it is important that the formula of Γ^{-1} comes from a clear calculation method; the canonical ensemble of SGS with the ring approximation. Our decay time scale seems to reflect the collective property of the system, contrary to the binary relaxation time scale which reflects a simple compilation of two body interactions which yields the typical mixing time scale.

4) The relation between the imaginary part of thermodynamic quantities and the instability dynamics should be further clarified. We are attempting now much profound calculation on the complex free energy.

5) We would like to check the relevance of our results in observational cosmology. For this purpose, we have to take into account the effects of the cosmic expansion, the existence of dark matter and the effect of specific initial conditions of density fluctuations.

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The second law of black hole thermodynamics in quadratic gravity

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Abstract

We consider the second law of black hole thermodynamics in quadratic gravity, i.e., a gravitational theory whose Lagrangian includes the square of the Ricci tensor and the square of the scalar curvature, in addition to the usual Einstein-Hilbert Lagrangian and a matter Lagrangian. By using the equivalence of the black hole entropy in such a gravitational theory to that in the Einstein frame and Hawking's area theorem, we derive the condition for the second law to hold in quasi-stationary processes in terms of matter fields. We also argue that it is satisfied for physically realistic matter fields which satisfy the dominant energy condition.

1 Introduction

The number of the microscopic states of a black hole have been counted in the context of quantum gravity and it has been shown [1, 2, 3, 4] to coincide in some limits with the macroscopic black hole entropy derived in the Einstein theory. Together with such microscopic investigations, it will be important to clarify, from the macroscopic points of view, the thermodynamical properties of a black hole corrected by the effects of quantum gravity. It is expected that the effective theories of quantum gravity will provide important information of the macroscopic properties of large black holes compared to the Planck scale.

However, black hole thermodynamics was originally formulated within the Einstein theory by using the Einstein equation [6], and hence it is not trivial whether the analogous laws to the usual thermodynamics hold in the generalized gravitational theory, such as the effective theories which include higher curvature corrections [7, 8]. On the other hand, since the gravitational field equation is not necessary to show that a black hole emits Hawking radiation with the temperature $T = \kappa/2\pi$, where κ is the surface gravity of the black hole [5], it is implied that a black hole in the generalized gravitational theory also has thermodynamical properties.

In fact, the zeroth law of black hole thermodynamics has been proved without using the gravitational field equation [9, 10], and the thermodynamical variables have been defined in the generalized theory such that they satisfy the first law [11, 12]. However, the second and the third laws have not been established yet. While the general proof of the third law is still absent even in the Einstein theory, the second law will be crucial if a black hole has thermodynamical properties and hence the physically meaningful entropy in the generalized theory.

One of the reasons why the analyses of the second law in the generalized theory have not been developed so far is that the physical meaning of the black hole entropy defined in Ref.[11, 12] is not clear, and then definition of the entropy in a dynamical black hole is still ambiguous. The definition in Ref.[11, 12] relies on the existence of a bifurcation surface of the horizon, and has no ambiguity if the horizon is bifurcate, such as the case of a stationary black hole with the nonzero temperature. However, a dynamical black hole will not have bifurcate horizons in general, and ambiguity remains in the dynamical case.

One possible approach to analyze the second law will be then to consider quasi-stationary processes where an initial stationary black hole spacetime is mildly perturbed and settles down to another stationary black hole spacetime. Increase of the black hole entropy in quasi-stationary processes has been discussed in Ref.[13], but without considering the effects of higher curvature corrections. For a sufficiently large

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black hole, the effects of higher curvature corrections on entropy increase seem small in ordinary cases, but those may become important in some circumstances where the leading contribution is small. Furthermore, it will be necessary to clarify such effects explicitly to explore the property of the black hole entropy in the generalized gravitational theory.

Another point which makes the analyses of the second law in the generalized theory difficult is that we cannot apply Hawking's area theorem [14]. This is because the black hole entropy in the generalized theory is not proportional to the horizon area of the black hole in general, and the energy condition doesn't lead to the null convergence condition directly through the gravitational field equation. If the gravitational theory can be transformed into the Einstein frame, however, we may analyze the second law by applying Hawking's area theorem in the Einstein frame. Actually, in the previous work [15], we have shown that the thermodynamical variables of a stationary black hole in the gravitational theory, which can be transformed into the Einstein frame, are the same as those in the Einstein frame.

In this paper, we consider quasi-stationary processes in quadratic gravity, i.e., a gravitational theory whose Lagrangian includes the square of the Ricci tensor and the square of the scalar curvature, in addition to the usual Einstein-Hilbert Lagrangian and a matter Lagrangian, which is a special case of the gravitational theory considered in Ref.[15]. By using the equivalence of the thermodynamical variables and applying Hawking's area theorem in the Einstein frame, we analyze the second law of black hole thermodynamics.

In the Einstein theory, we can show that the null convergence condition is satisfied by using the Einstein equation if the null energy condition on matter fields is imposed, which, as well as the cosmic censorship conjecture, leads to the second law by Hawking's area theorem. The purpose of this paper is to derive the condition for the second law to hold in terms of the matter fields and investigate whether it is satisfied or not when we impose the energy condition of matter fields.

The notations and units we use in what follows are the same as those used in Ref.[15], and we focus on four-dimensional spacetimes.

2 Black hole entropy and transformation of theory

In this section, we review the definition of the black hole entropy and the transformation of the gravitational theory into the Einstein frame.

In Ref.[12], the first law of black hole thermodynamics in the generalized gravitational theory has been established for an arbitrary perturbation from a regular background spacetime of a black hole, for which the following assumptions are made: (1) It is stationary and axisymmetric, and hence it possesses two Killing vectors $\xi_{(t)}^\mu$ and $\xi_{(\varphi)}^\mu$ associated with the respective symmetry. (2) It is asymptotically flat. (3) The event horizon of the black hole is a bifurcate Killing horizon.

Definition of the thermodynamical variables is based on the covariance of the Lagrangian density 4-form $\epsilon_{\mu\nu\rho\sigma} L$ under a diffeomorphism, where L is a scalar function of the field variables which we denote as $\psi_{(i)}$ collectively. The variation of $\epsilon_{\mu\nu\rho\sigma} L$ is described as

$$\delta(\epsilon_{\mu\nu\rho\sigma} L) = \epsilon_{\mu\nu\rho\sigma} E^{(i)} \delta\psi_{(i)} + \epsilon_{\mu\nu\rho\sigma} \nabla_\beta \Theta^\beta[\psi_{(i)}, \delta\psi_{(i)}], \quad (1)$$

where $E^{(i)} = 0$ is the field equation for $\psi_{(i)}$. $\Theta^\beta[\psi_{(i)}, \delta\psi_{(i)}]$ in the totally divergent term, which is a functional of the field variables $\psi_{(i)}$ and their variations $\delta\psi_{(i)}$, has ambiguity since we can add an arbitrary total divergence. When we identify the variation $\delta\psi_{(i)}$ with a general coordinate transformation $\mathcal{L}_\zeta \psi_{(i)}$ induced by an arbitrary vector field ζ^μ , we obtain

$$\nabla_\beta (\zeta^\beta L) = -E^{(i)} \mathcal{L}_\zeta \psi_{(i)} + \nabla_\beta \Theta^\beta[\psi_{(i)}, \mathcal{L}_\zeta \psi_{(i)}]. \quad (2)$$

Then the vector field $J^\beta[\psi_{(i)}, \zeta^\rho, \mathcal{L}_\zeta \psi_{(i)}] \equiv \Theta^\beta[\psi_{(i)}, \mathcal{L}_\zeta \psi_{(i)}] - \zeta^\beta L$ is divergence-free, $\nabla_\beta J^\beta = 0$, when the field equations, $E^{(i)} = 0$, are satisfied. In addition, we can find an antisymmetric potential $Q^{\beta\alpha}[\psi_{(i)}, \zeta^\rho, \nabla_\lambda \zeta^\rho]$ of the current $J^\beta[\psi_{(i)}, \zeta^\rho, \mathcal{L}_\zeta \psi_{(i)}]$ such that $J^\beta = \nabla_\alpha Q^{\beta\alpha}$. $\Theta^\beta[\psi_{(i)}, \delta\psi_{(i)}]$ and $Q^{\beta\alpha}[\psi_{(i)}, \zeta^\rho, \nabla_\lambda \zeta^\rho]$ are calculated in a straightforward manner for a given action.

When we consider perturbations of the field variables $\delta\psi_{(i)}$, which satisfy the linearized field equations, and set ζ^μ to be the Killing vector $\xi^\mu \equiv \xi_{(t)}^\mu + \Omega_H \xi_{(\varphi)}^\mu$, where Ω_H is the angular velocity of the black hole,

the symplectic current 3-form $\omega_{\mu\nu\rho}$, which is given by the exterior derivative of

$$\delta \left(\frac{1}{2} \epsilon_{\beta\alpha\mu\nu} Q^{\beta\alpha} [\psi_{(i)}, \xi^\rho, \nabla_\lambda \xi^\rho] \right) + \epsilon_{\beta\alpha\mu\nu} \xi^{[\beta} \Theta^{\alpha]} [\psi_{(i)}, \delta\psi_{(i)}] , \quad (3)$$

identically vanishes. Thus, integrating $\omega_{\mu\nu\rho}$ over a Cauchy surface which connects the bifurcation surface, H , of the Killing horizon with the spatial infinity, we find the relation

$$\begin{aligned} & \delta \int_H \frac{1}{2} \epsilon_{\beta\alpha\mu\nu} Q^{\beta\alpha} [\psi_{(i)}, \xi^\rho, \nabla_\lambda \xi^\rho] dx^\mu dx^\nu \\ &= \delta \int_\infty \frac{1}{2} \epsilon_{\beta\alpha\mu\nu} Q^{\beta\alpha} [\psi_{(i)}, \xi^\rho, \nabla_\lambda \xi^\rho] dx^\mu dx^\nu + \int_\infty \epsilon_{\beta\alpha\mu\nu} \xi^{[\beta} \Theta^{\alpha]} [\psi_{(i)}, \delta\psi_{(i)}] dx^\mu dx^\nu . \end{aligned} \quad (4)$$

The entropy S of a stationary black hole is then defined as

$$\frac{\hbar\kappa}{2\pi} S \equiv \int_H \frac{1}{2} \epsilon_{\beta\alpha\mu\nu} Q^{\beta\alpha} [\psi_{(i)}, \xi^\rho, \nabla_\lambda \xi^\rho] dx^\mu dx^\nu , \quad (5)$$

where κ is the surface gravity of the black hole. In the case of the Einstein theory, Eq.(5) gives $S = A/4\hbar$, where A is the area of the black hole. The right hand side of Eq.(4) corresponds the variations of the mass and the angular momentum, but we will not show their definitions here.

In the integral over the bifurcation surface, i.e., the left hand side of Eq.(4), the term proportional to ξ^μ does not contribute, which includes an ambiguous term due to the ambiguity in definition of Θ^β , because $\xi^\mu = 0$ on that surface. Existence of the bifurcation surface has been proved when the black hole horizon is the Killing horizon with the non-zero surface gravity [9, 10], but it is not expected to exist in a dynamical black hole spacetime. Then, the term proportional to ξ^μ does not vanish in that case, and hence ambiguity remains in definition of the entropy of a dynamical black hole.

Next, we mention the transformation of the gravitational theory into the Einstein frame [16]. We focus on a class of the generalized gravitational theory whose action is written as

$$I = \frac{1}{16\pi} \int d^4x \sqrt{-g} f(g^{\alpha\beta}, R_{\alpha\beta}, \phi, \nabla_\alpha \phi) , \quad (6)$$

where Lagrangian f is an arbitrary function of the metric $g^{\alpha\beta}$, the Ricci tensor $R_{\alpha\beta}$, a scalar field ϕ and its derivative.

To transform the action (6) into the Einstein frame, we first define “new metric” $\bar{g}^{\mu\nu}$ by

$$\bar{g}^{\mu\nu} \equiv \frac{\partial(\sqrt{-g}f)}{\partial R_{\mu\nu}} \frac{1}{\sqrt{-\det(\partial(\sqrt{-g}f)/\partial R_{\alpha\beta})}} = \frac{\sqrt{-g}}{\sqrt{-\bar{g}}} \frac{\partial f}{\partial R_{\mu\nu}} . \quad (7)$$

Since $\bar{g}^{\mu\nu}$ is a function of $g^{\alpha\beta}$, $R_{\alpha\beta}$, ϕ and $\nabla_\alpha \phi$, it is written as

$$\bar{g}^{\mu\nu} = \bar{g}^{\mu\nu}(g^{\alpha\beta}, R_{\alpha\beta}, \phi, \nabla_\alpha \phi) . \quad (8)$$

For quadratic gravity which we consider in this paper (see Eq.(15) below), an inequality

$$\det \frac{\partial^2 f}{\partial R_{\mu\nu} \partial R_{\rho\sigma}} \neq 0 \quad (9)$$

holds, and hence we can solve Eq.(8) for $R_{\mu\nu}$ as

$$R_{\mu\nu} = R_{\mu\nu}(g^{\alpha\beta}, \bar{g}^{\alpha\beta}, \phi, \nabla_\alpha \phi) . \quad (10)$$

Then an action $\bar{I}$ in the Einstein frame defined as

$$\begin{aligned} \bar{I} &= \frac{1}{16\pi} \int d^4x \sqrt{-\bar{g}} \left[\bar{R}(\bar{g}^{\alpha\beta}, \partial_\gamma \bar{g}^{\alpha\beta}, \partial_\gamma \partial_\lambda \bar{g}^{\alpha\beta}) + \bar{g}^{\mu\nu} \left(\bar{\Gamma}_{\rho\sigma}^\rho \bar{\Gamma}_{\mu\nu}^\sigma - \bar{\Gamma}_{\mu\sigma}^\rho \bar{\Gamma}_{\rho\nu}^\sigma \right) \right. \\ &\quad \left. - \bar{g}^{\mu\nu} R_{\mu\nu}(g^{\alpha\beta}, \bar{g}^{\alpha\beta}, \phi, \bar{\nabla}_\alpha \phi) + \frac{\sqrt{-g}}{\sqrt{-\bar{g}}} f(g^{\alpha\beta}, R_{\alpha\beta}(g^{\rho\sigma}, \bar{g}^{\rho\sigma}, \phi, \bar{\nabla}_\rho \phi), \phi, \bar{\nabla}_\alpha \phi) \right] \end{aligned} \quad (11)$$

gives the same field equations as those derived from the original action I . In the action $\bar{I}$, the independent variables are $g^{\mu\nu}$, $\bar{g}^{\mu\nu}$ and ϕ , and $\bar{\Gamma}_{\mu\nu}^\rho$ is defined by

$$\bar{\Gamma}_{\mu\nu}^\rho \equiv \frac{1}{2} g^{\rho\sigma} (\bar{\nabla}_\mu g_{\nu\sigma}^{-1} + \bar{\nabla}_\nu g_{\mu\sigma}^{-1} - \bar{\nabla}_\sigma g_{\mu\nu}^{-1}) , \quad (12)$$

where $\bar{\nabla}_\mu$ is the covariant derivative associated to the new metric $\bar{g}^{\mu\nu}$. Although the scalar curvature $\bar{R}$ in $\bar{I}$ is given by the derivatives of the new metric $\bar{g}^{\mu\nu}$ in the usual way, the Ricci tensors $R_{\mu\nu}$ in the third and the fourth terms are given by the relation (10).

In Ref.[15], we have shown that the thermodynamical variables defined as above are the same between the original frame (6) and the Einstein frame (11) under the assumptions: (1) The spacetimes in two frames are asymptotically flat. (2) The transformation into the Einstein frame is regular. (3) The temperatures of the black holes in two frames are non-zero. Especially, the entropy is shown to be the same between these two frames, because the position of the horizon and the surface gravity are the same, and the integrand in Eq.(5) is calculated as

$$\frac{1}{2} \epsilon_{\beta\alpha\mu\nu} Q^{\beta\alpha} = -\frac{1}{16\pi} \epsilon_{\alpha\beta\mu\nu} \frac{\partial f}{\partial R_{\alpha\gamma}} \nabla_\gamma \xi^\beta , \quad (13)$$

in the original frame (6) and

$$\frac{1}{2} \bar{\epsilon}_{\beta\alpha\mu\nu} \bar{Q}^{\beta\alpha} = -\frac{1}{16\pi} \bar{\epsilon}_{\alpha\beta\mu\nu} \bar{g}^{\alpha\gamma} \bar{\nabla}_\gamma \xi^\beta , \quad (14)$$

in the Einstein frame (11), respectively, where $\bar{g}^{\mu\nu}$ is given by Eq.(7), the components of the volume elements $\epsilon_{\alpha\beta\mu\nu}$ and $\bar{\epsilon}_{\alpha\beta\mu\nu}$ are $\sqrt{-g}$ and $\sqrt{-\bar{g}}$ respectively, and $\nabla_\nu \xi^\mu = \bar{\nabla}_\nu \xi^\mu$ on the bifurcation surface. In what follows, we proceed based on this equivalence of the entropy between two frames.

3 The second law in quasi-stationary processes

As we mentioned above, there remains ambiguity in definition of the black hole entropy in the dynamical case. This ambiguity makes the analyses of the second law difficult. Then it will be important to establish the second law in quasi-stationary processes where an initial stationary axisymmetric black hole spacetime is mildly perturbed and settles down to another stationary axisymmetric black hole. In this section, we analyze the second law in such processes by applying the equivalence of the entropy of a stationary axisymmetric black hole.

The gravitational theory we consider here is described by an action

$$I = \frac{1}{16\pi} \int dx^4 \sqrt{-g} [R + a R_{\mu\nu} R^{\mu\nu} + b R^2 + L_{\text{matter}}] , \quad (15)$$

where the constants a and b are order unity times the Planck length squared. L_{matter} is the Lagrangian of matter fields which does not include the derivatives of the metric, and hence it does not contribute to the black hole entropy. The action of quadratic gravity may include the square of the Riemann tensor $R^{\mu\nu\rho\lambda} R_{\mu\nu\rho\lambda}$, but it is integrated into a surface term in four dimensions by using Gauss-Bonnet theorem. Since the following analysis is based on the entropy of a stationary black hole and surface terms do not contribute to the entropy of a stationary black hole, we will not consider $R^{\mu\nu\rho\lambda} R_{\mu\nu\rho\lambda}$ in what follows.

Since the action (15) has the form of Eq.(6), it is transformed into the Einstein frame, and the entropy of a stationary axisymmetric black hole in the original frame (15) is the same as that in the Einstein frame. The spacetime in the original frame is stationary and axisymmetric before and after the quasi-stationary processes. Hence the entropy of a black hole in such regions of spacetime in the original frame is the same as that in the corresponding regions in the Einstein frame, and thus we can analyze change of the entropy during quasi-stationary processes in the Einstein frame, instead of analyzing in the original frame.

In the Einstein frame, since the black hole entropy is given by $\bar{S} = \bar{A}/4\hbar$, where $\bar{A}$ is the horizon area of the black hole, we can apply Hawking's area theorem [14]. Here we assume the cosmic censorship

conjecture, and then we can show by Hawking's theorem that the second law holds if the null convergence condition $\bar{R}_{\mu\nu}\bar{k}^\mu\bar{k}^\nu \geq 0$ is satisfied for an arbitrary null vector $\bar{k}^\mu$ in the Einstein frame.

The Ricci tensor in the Einstein frame $\bar{R}_{\mu\nu}$ is given by the identity

$$\bar{R}_{\mu\nu}(\bar{g}^{\alpha\beta}, \partial_\gamma \bar{g}^{\alpha\beta}, \partial_\lambda \bar{g}^{\alpha\beta}) = R_{\mu\nu}(g^{\alpha\beta}, \partial_\lambda g^{\alpha\beta}, \partial_\gamma \partial_\lambda g^{\alpha\beta}) + \Gamma_{\rho\sigma}^\rho \Gamma_{\mu\nu}^\sigma - \Gamma_{\mu\sigma}^\rho \Gamma_{\nu\rho}^\sigma + \nabla_\rho \Gamma_{\mu\nu}^\rho - \nabla_\mu \Gamma_{\nu\rho}^\rho, \quad (16)$$

in terms of the tensors in the original frame, where $\Gamma_{\mu\nu}^\rho$ is defined by

$$\Gamma_{\mu\nu}^\rho \equiv \frac{1}{2} \bar{g}^{\rho\sigma} (\nabla_\mu \bar{g}_{\nu\sigma}^{-1} + \nabla_\nu \bar{g}_{\mu\sigma}^{-1} - \nabla_\sigma \bar{g}_{\mu\nu}^{-1}). \quad (17)$$

The gravitational field equation in the Einstein frame reduces to Eq.(16) when we substitute the definition of $\bar{g}^{\mu\nu}$ (Eq.(7)), and hence the right hand side of Eq.(16) corresponds to $\bar{T}_{\mu\nu} - 1/2 \bar{g}_{\mu\nu}^{-1} \bar{T}$, where $\bar{T}_{\mu\nu}$ is the energy-momentum tensor and $\bar{T}$ is its trace in the Einstein frame.

Since a and b are order unity times the Planck length squared, the terms in Eq.(16) which include a and b will be sufficiently small for a large black hole compared to the Planck scale. Hence, in what follows, we consider only linear terms in a and b , and neglect higher order terms to evaluate $\bar{R}_{\mu\nu}\bar{k}^\mu\bar{k}^\nu$.

From Eq.(7) and (15), $\bar{g}^{\mu\nu}$ is given as

$$\bar{g}^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{1}{\sqrt{-\det(P^{\alpha\beta})}} P^{\mu\nu}, \quad (18)$$

where $P^{\mu\nu}$ is defined as

$$P^{\mu\nu} \equiv g^{\mu\nu} + 2aR^{\mu\nu} + 2bg^{\mu\nu}R. \quad (19)$$

By substituting Eq.(18) and (19) into Eq.(17), we find

$$\begin{aligned} \Gamma_{\mu\nu}^\rho = & -a \left[\nabla_\mu R_\nu^\rho + \nabla_\nu R_\mu^\rho - \nabla^\rho R_{\mu\nu} - \frac{1}{2} \delta_\nu^\rho \nabla_\mu R - \frac{1}{2} \delta_\mu^\rho \nabla_\nu R \right. \\ & \left. + \frac{1}{2} g_{\mu\nu}^{-1} \nabla^\rho R \right] + b \left[\delta_\nu^\rho \nabla_\mu R + \delta_\mu^\rho \nabla_\nu R - g_{\mu\nu}^{-1} \nabla^\rho R \right] + \mathcal{O}(a^2, ab, b^2), \end{aligned} \quad (20)$$

to the first order in a and b . Since $\Gamma_{\mu\nu}^\rho$ doesn't include the zeroth order terms in a and b , we neglect the bilinear terms of $\Gamma_{\mu\nu}^\rho$ in Eq.(16) and obtain

$$\begin{aligned} \bar{R}_{\mu\nu} = & R_{\mu\nu} - 2aR_{\mu\alpha}R_\nu^\alpha + (2aR^{\alpha\beta}R_{\alpha\mu\beta\nu} - a\nabla_\mu \nabla_\nu R + a\nabla^\alpha \nabla_\alpha R_{\mu\nu} - 2b\nabla_\mu \nabla_\nu R) \\ & - g_{\mu\nu}^{-1} \left(\frac{a}{2} + b \right) \nabla_\alpha \nabla^\alpha R + \mathcal{O}(a^2, ab, b^2). \end{aligned} \quad (21)$$

This equation is rewritten, by using the gravitational field equation in the original frame

$$\begin{aligned} R_{\mu\nu} - g_{\mu\nu}^{-1} \left(\frac{1}{2} R + \frac{a}{2} R_{\alpha\beta} R^{\alpha\beta} + \frac{b}{2} R^2 - \frac{a}{2} \nabla^\alpha \nabla_\alpha R - 2b\nabla^\alpha \nabla_\alpha R \right) \\ + 2aR^{\alpha\beta} R_{\alpha\mu\beta\nu} + 2bR_{\mu\nu}R - a\nabla_\mu \nabla_\nu R + a\nabla^\alpha \nabla_\alpha R_{\mu\nu} - 2b\nabla_\mu \nabla_\nu R = T_{\mu\nu}, \end{aligned} \quad (22)$$

as

$$\begin{aligned} \bar{R}_{\mu\nu} = & -2aR_{\mu\alpha}R_\nu^\alpha - 2bR_{\mu\nu}R + T_{\mu\nu} \\ & + g_{\mu\nu}^{-1} \left\{ \frac{R}{2} + \frac{a}{2} R_{\alpha\beta} R^{\alpha\beta} + \frac{b}{2} R^2 - (a + 3b) \nabla^\alpha \nabla_\alpha R \right\} + \mathcal{O}(a^2, ab, b^2), \end{aligned} \quad (23)$$

where $T_{\mu\nu}$ is the energy-momentum tensor in the original frame.

A null vector in the Einstein frame $\bar{k}^\mu$ may not be a null vector in the original frame, in general. However, $\bar{k}^\mu$ is expressed by a linear combination of non-orthogonal null vectors k^μ and n^μ in the original frame, as

$$\bar{k}^\mu = k^\mu + c n^\mu, \quad (24)$$

where k^μ and n^μ are normalized by

$$g_{\mu\nu}^{-1} k^\mu k^\nu = 0, \quad g_{\mu\nu}^{-1} n^\mu n^\nu = 0, \quad g_{\mu\nu}^{-1} k^\mu n^\nu = -1, \quad (25)$$

and the constant c is assumed to be first order in a and b . Since $\bar{g}_{\mu\nu}^{-1} \bar{k}^\mu \bar{k}^\nu = 0$, and $\bar{g}_{\mu\nu}^{-1}$ is written as

$$\bar{g}_{\mu\nu}^{-1} = \frac{\sqrt{-\bar{g}}}{\sqrt{-g}} (g_{\mu\nu}^{-1} - 2aR_{\mu\nu} - 2bg_{\mu\nu}^{-1}R) + \mathcal{O}(a^2, ab, b^2), \quad (26)$$

the constant c is calculated as

$$c = -aR_{\mu\nu}k^\mu k^\nu + \mathcal{O}(a^2, ab, b^2). \quad (27)$$

From Eq.(23), (24), (25) and (27), we obtain

$$\begin{aligned} \bar{R}_{\mu\nu} \bar{k}^\mu \bar{k}^\nu &= T_{\mu\nu} k^\mu k^\nu - 2aR_{\mu\alpha} R_\nu^\alpha k^\mu k^\nu - 2bRR_{\mu\nu} k^\mu k^\nu \\ &\quad - 2aT_{\alpha\beta} k^\alpha n^\beta R_{\mu\nu} k^\mu k^\nu + aRR_{\mu\nu} k^\mu k^\nu + \mathcal{O}(a^2, ab, b^2), \end{aligned} \quad (28)$$

which does not include any derivatives of the curvature tensors, while both Eq.(21) and (22) do. If the right hand side of Eq.(28) is positive definite, the second law holds, when we assume the cosmic censorship conjecture in the original frame which ensures the cosmic censorship in the Einstein frame because of regularity of the transformation. In order to investigate whether the energy condition leads the positivity of Eq.(28), we now rewrite the curvature tensors in the right hand side of Eq.(28) in terms of the energy-momentum tensor $T_{\mu\nu}$ in the original frame by using again the gravitational field equation (22) in the original frame. Since there are no zeroth order terms in a and b which include the curvature tensors in Eq.(28), we only have to substitute the zeroth order part of Eq.(22)

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}^{-1}T + \mathcal{O}(a, b), \quad (29)$$

into Eq.(28). Then it becomes

$$\bar{R}_{\mu\nu} \bar{k}^\mu \bar{k}^\nu = [1 + (a + 2b)T - 2aT_{\alpha\beta} k^\alpha n^\beta] T_{\mu\nu} k^\mu k^\nu - 2aT_\mu^\alpha T_{\alpha\nu} k^\mu k^\nu + \mathcal{O}(a^2, ab, b^2). \quad (30)$$

Positivity of the right hand side of Eq.(30) is the null convergence condition in the Einstein frame, and thus the condition in terms of matter fields for the second law to hold in the original frame. By comparing with the corresponding equation in the Einstein theory

$$R_{\mu\nu} k^\mu k^\nu = T_{\mu\nu} k^\mu k^\nu, \quad (31)$$

we see that the term proportional to $T_\mu^\alpha T_{\alpha\nu} k^\mu k^\nu$ and the correction to the coefficient of $T_{\mu\nu} k^\mu k^\nu$ arise due to the quadratic curvature corrections. In particular, since the former is proportional to a , it results from the square of the Ricci tensor $R_{\mu\nu} R^{\mu\nu}$.

Although the null convergence condition results from the null energy condition in the Einstein theory, it is not sufficient in the present case because it doesn't provide information of $T_\mu^\alpha T_{\alpha\nu} k^\mu k^\nu$ which appears in Eq.(30). Here we assume that matter fields satisfy the dominant energy condition. Then, $T_{\mu\nu} k^\mu k^\nu \geq 0$, and $T_\mu^\alpha k^\mu$ is a timelike vector and so $T_\mu^\alpha T_{\alpha\nu} k^\mu k^\nu \leq 0$. In addition, the inside of the bracket in the right hand side of Eq.(30) will be positive, since it includes unity and the other terms are first order in a and b . Hence, the right hand side of Eq.(30) is positive in the case of $a \geq 0$. This seems to be the case, even when $a < 0$, since the first term will dominate in the usual cases because it includes the zeroth order terms. However, although the energy condition requires that $T_{\mu\nu} k^\mu k^\nu$ is greater than zero, it may be sufficiently small and comparable with the second term. To clarify this point, we consider two types of energy-momentum tensors, into which physically realistic energy-momentum tensors are classified [14],

$$T_{\mu\nu} = \rho t_\mu t_\nu + P_1 x_\mu x_\nu + P_2 y_\mu y_\nu + P_3 z_\mu z_\nu, \quad (32)$$

and

$$T_{\mu\nu} = \rho t_\mu t_\nu + \rho t_\mu z_\nu + \rho z_\mu t_\nu + \rho z_\mu z_\nu, \quad (33)$$

where $\{t^\mu, x^\mu, y^\mu, z^\mu\}$ is the appropriately chosen tetrad frame and ρ is required to be positive by the energy condition. For these types of energy-momentum tensors, we can show

$$2\rho T_{\mu\nu} k^\mu k^\nu \geq -T_{\alpha\mu} T_\nu^\alpha k^\mu k^\nu, \quad (34)$$

for an arbitrary null or timelike vector k^μ . Then, when $a < 0$, we find, from Eq.(30) and (34),

$$\bar{R}_{\mu\nu} \bar{k}^\mu \bar{k}^\nu \geq [1 + (a + 2b)T - 2aT_{\alpha\beta} k^\alpha n^\beta + 4a\rho] T_{\mu\nu} k^\mu k^\nu, \quad (35)$$

the right hand side of which is positive if the energy condition $T_{\mu\nu} k^\mu k^\nu \geq 0$ is imposed.

From these, we see that the null convergence condition in the Einstein frame is satisfied to the first order in a and b , if the dominant energy condition is imposed in the original frame, and hence the second law in the original frame holds in quasi-stationary processes if we assume the cosmic censorship conjecture.

4 Summary and discussion

We consider the second law of black hole thermodynamics in quadratic gravity. A black hole spacetime is assumed to vary by quasi-stationary processes, where it is stationary and axisymmetric initially, then is mildly perturbed and settles down to another stationary axisymmetric black hole. By applying the equivalence of the entropy of a stationary axisymmetric black hole and Hawking's area theorem in the Einstein frame, we analyzed the null convergence condition in the Einstein frame to investigate the second law. We found that the coefficient of $T_{\mu\nu} k^\mu k^\nu$ is corrected and the term proportional to $T_\mu^\alpha T_{\alpha\nu} k^\mu k^\nu$ arises in the expression of the null convergence condition, and that it is satisfied to the first order of the quadratic curvature corrections, if we impose the dominant energy condition in the original frame. Hence the second law in the original frame holds in quasi-stationary processes to the first order of those corrections.

Quadratic curvature terms may be regarded as the first order corrections to the Einstein theory in the effective theory of quantum gravity. Since such a effective theory will describe physics adequately only for sufficiently larger black holes than the Planck scale and the effects of the higher curvature corrections will be small for large black holes, the analysis to the first order of the higher curvature corrections seems reasonable. However, from more fundamental point of view, it will be interesting to analyze the higher order corrections in order to investigate whether the second law is a peculiar property in the Einstein theory and its simple generalization [13, 17, 18], or the universal property in the gravitational theory. From this point of view, it is also interesting to investigate the second law in other gravitational theories.

The second law is meaningful in dynamical processes, and hence it seems necessary to extend the analyses into the general dynamical black hole spacetimes, not only in quasi-stationary processes. However, as we mentioned above, definition of the entropy of a dynamical black hole has not been established yet. The result of this paper may provide a foundation to establish the definition of the dynamical entropy and analyze the second law in dynamical processes.

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Semiclassical consistency of black hole thermodynamics

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Abstract

We investigate two closely related candidates of the statistical-mechanical origin of the black hole entropy: the entanglement entropy and the brick wall model. We analyze semiclassical consistency of these two models in details. It is shown that the brick wall model does provide a perfectly self-consistent description of a configuration which is indistinguishable from a black hole to outside observers, and which accounts for the Bekenstein-Hawking entropy purely as thermal entropy of quantum fields at the Hawking temperature (i.e. in the Hartle-Hawking state), providing one accepts the ad hoc but plausible ansatz a Planck-length cutoff near the horizon. After that, it is suggested that the principle of maximum of entanglement entropy combined with the small backreaction condition may provide a new derivation of the Hawking radiation.

1 Introduction

It is well known that black holes have many properties analogous to those of thermodynamics. Those are as a whole called *black hole thermodynamics*. In particular, four laws of black holes [1] combined with the generalized second law [2, 3, 4, 5] make up a main framework of the black hole thermodynamics. In these laws, *black hole entropy* defined as follows plays an important role.

$$S_{BH} = \frac{k_B}{4} \frac{A}{l_{pl}^2}, \quad (1)$$

where $l_{pl} = \sqrt{\hbar G/c^3}$ is the Planck length and A is area of black hole horizon. The formula (1) for black hole entropy is often called *Bekenstein-Hawking formula* since the concept of black hole entropy was first introduced by Bekenstein [6] as a quantity proportional to the horizon area and the proportionality coefficient was fixed by Hawking's discovery of thermal radiation from a black hole [7] (see arguments below). He showed that a black hole radiates thermal radiation with temperature given by

$$k_B T_{BH} = \frac{\hbar \kappa}{2\pi c}, \quad (2)$$

where κ is the surface gravity of a background black hole. This thermal radiation and its temperature are called *Hawking radiation* and *Hawking temperature*, respectively.

Thermodynamics has a well-established microscopic description: the quantum statistical mechanics. In the thermodynamical description, information on each microscopic degree of freedom is lost, and only macroscopic variables are concerned. However, the number of all microscopic degrees of freedom is implemented in a macroscopic variable: entropy S is related to the number of all consistent microscopic states N as

$$S = k_B \ln N. \quad (3)$$

In analogy, it is expected that there might be a microscopic description of the black hole thermodynamics, too. In particular, it is widely believed that the black hole entropy might be related to a number of microscopic states. Since the microscopic description seems to require a quantum theory of gravity, detailed investigations of the black hole entropy should contribute a lot toward construction of the theory of quantum gravity. This is one among the several reasons why the origin of the black hole entropy needs to be understood at the fundamental level.

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In this paper, we investigate two closely related candidates of the statistical-mechanical origin of the black hole entropy: the entanglement entropy and the brick wall model. We analyze semiclassical consistency of these two models in details.

2 The entanglement entropy and thermodynamics

Now let us review the concept of the entanglement entropy. We consider a Hilbert space $\mathcal{F}$ constructed from two Hilbert spaces $\mathcal{F}_1$ and $\mathcal{F}_2$ as

$$\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2. \quad (4)$$

From a pure density matrix $\rho = |\phi\rangle\langle\phi|$ on $\mathcal{F}$ we can define reduced density matrices ρ_1 and ρ_2 by $\rho_{1,2} = \text{Tr}_{2,1}\rho$, where $\text{Tr}_{1,2}$ represents a partial trace over $\mathcal{F}_{1,2}$, respectively, and $|\phi\rangle$ is an element of $\mathcal{F}$ with unit norm. The *entanglement entropy* is defined by $S[\rho_1]$ or $S[\rho_2]$, where $S[\cdot]$ is the von Neumann entropy. We can denote these two entropies by the same symbol S_{ent} since $S[\rho_1] = S[\rho_2]$ (see Appendix of Ref. [8] for a proof).

In the context of black hole thermodynamics the entanglement entropy of matter fields on a black hole background is regarded as a strong candidate for black hole entropy [9, 10, 11, 12]. In this case the Hilbert space $\mathcal{F}$ is a space of all quantum states of matter fields and the direct product structure (4) of $\mathcal{F}$ is obtained from a division of a spacelike hypersurface into two regions: one inside and one outside of a boundary surface. Due to the above mentioned symmetric property $S_{ent} = S[\rho_1] = S[\rho_2]$, it is expected that S_{ent} for a pure state is proportional to the area $A = 4\pi r_0^2$ of the boundary. Thus,

$$S_{ent} \simeq N\mathcal{N}_S \left(\frac{r_0}{a} \right)^2, \quad (5)$$

is expected, where N is the number of fields, $\mathcal{N}_S$ is a numerical constant of order unity, and a is a cutoff length of the theory introduced in order to make the expression finite and dimensionless as entropy should be. In particular, in Ref. [12], the cutoff length a is introduced so that the quantum field theory in Schwarzschild spacetime is discretized with the lattice spacing a and the spatial boundary is set to be a timelike surface at the proper distance $n_B a$ to the horizon, where n_B is a positive integer. In Ref. [12], the behavior (5) of S_{ent} was confirmed with $\mathcal{N}_S \simeq 0.3$. So the entanglement entropy for a pure state has a property similar to black hole entropy, provided that the boundary is set to be close to a black hole horizon and the cutoff length is of Planck order.

Moreover, in Ref. [8, 12], it was shown that concepts of entanglement energy and entanglement temperature can be introduced for matter fields in a black hole background and that their behavior is the same as for the energy and temperature of the black hole. Namely, entanglement thermodynamics constructed above the so called Boulware ground state [20] has the following properties [12, 13].

$$\begin{aligned} E_{ent} &\simeq N\mathcal{N}_E \frac{r_0}{a^2}, \\ T_{ent} &\simeq \mathcal{N}_T T_{BH}, \end{aligned} \quad (6)$$

where $\mathcal{N}_E$ is a numerical factor of order unity, and $\mathcal{N}_T = 2\pi\mathcal{N}_E/\mathcal{N}_S$. We call the structure organized by $(S_{ent}, E_{ent}, T_{ent})$ the *entanglement thermodynamics*.

The behavior shown by (5) and (6) is expected to hold also for states different from the Boulware state, provided that the definition of entanglement quantities are properly modified. For other states, of course, the numerical factors $\mathcal{N}_S$ and $\mathcal{N}_E$ will be different from those for the Boulware state.

3 The brick wall model

The entanglement interpretation seems to be implicit in, and is certainly closely related to a pioneering calculation done by Gerard 'tHooft [14] in 1985. He considered the statistical thermodynamics of quantum fields in the Hartle-Hawking state (i.e. having the Hawking temperature T_{BH} at large radii) propagating on a fixed Schwarzschild background of mass M . To control divergences, 'tHooft introduced a "brick wall"—actually a static spherical mirror at which the fields are required to satisfy Dirichlet or Neumann

boundary conditions—with radius a little larger than the gravitational radius $2M$. He found, in addition to the expected volume-dependent thermodynamical quantities describing hot fields in a nearly flat space, additional wall contributions proportional to the area. These contributions are, however, also proportional to α^{-2} , where α is the proper altitude of the wall above the gravitational radius, and thus diverge in the limit $\alpha \rightarrow 0$. For a specific choice of α (which depends on the number of fields, etc., but is generally of order l_{pl}), 'tHooft was able to recover the Bekenstein-Hawking formula with the correct coefficient.

However, this calculation raises a number of questions which have caused many, including 'tHooft himself, to have reservations about its validity and consistency.

- (a) S_{BH} is here obtained as a one-loop effect, originating from thermal excitations of the quantum fields. Does this material contribution to S_{BH} have to be *added* to the zero-loop Gibbons-Hawking contribution which arises from the gravitational part of the action and already by itself accounts for the full value of S_{BH} ? [15]
- (b) The ambient quantum fields were assumed to be in the Hartle-Hawking state. Their stress-energy should therefore be bounded (of order M^{-4} in Planck units) near the gravitational radius, and negligibly small for large masses. However, 'tHooft's calculation assigns to them enormous (Planck-level) energy densities near the wall.
- (c) The integrated field energy gives a wall contribution to the mass

$$\Delta M = \frac{3}{8}M \quad (7)$$

when α is adjusted to give the correct value of S_{BH} . This suggests a substantial gravitational back-reaction [14] and that the assumption of a fixed geometrical background may be inconsistent [15, 16, 17].

Our main purpose in this section is to point out that these difficulties are only apparent and easy to resolve [18]. The basic remark is that the *brick-wall model strictly interpreted does not represent a black hole*. It represents the exterior of a starlike object with a reflecting surface, compressed to nearly (but not quite) its gravitational radius. The ground state for quantum fields propagating around this star is not the Hartle-Hawking state [19] but the Boulware state [20], corresponding to zero temperature, which has a quite different behavior near the gravitational radius.

We wish to study the thermodynamics of hot quantum fields confined to the outside of a spherical star with a perfectly reflecting surface whose radius r_1 is a little larger than its gravitational radius r_0 . To keep the total field energy bounded, we suppose the system enclosed in a spherical container of radius $L \gg r_1$.

It will be sufficiently general to assume for the geometry outside the star a spherical background metric of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2. \quad (8)$$

This includes as special cases the Schwarzschild, Reissner-Nordström and de Sitter geometries, or any combination of these.

Into this space we introduce a collection of quantum fields, raised to some temperature T_∞ at large distances, and in thermal equilibrium. The local temperature $T(r)$ is then given by Tolman's law [21],

$$T(r) = T_\infty f^{-1/2} \quad (9)$$

and becomes very large when $r \rightarrow r_1 = r_0 + \Delta r$. We shall presently identify T_∞ with the Hawking temperature T_H of the horizon $r = r_0$ of the exterior metric (8), continued (illegitimately) into the internal domain $r < r_1$.

Characteristic wavelengths λ of this radiation are small compared to other relevant length-scales (curvature, size of container) in the regions of interest to us. Near the star's surface,

$$\lambda \sim \hbar/T = f^{1/2} \hbar/T_\infty \ll r_0. \quad (10)$$

Elsewhere in the large container, at large distances from the star,

$$f \simeq 1, \quad \lambda \simeq \hbar/T_\infty \sim r_0 \ll L. \quad (11)$$

Therefore, a particle description should be a good approximation to the statistical thermodynamics of the fields. (Equivalently, one can arrive at this conclusion by considering the WKB solution to the wave equation.)

For particles of rest-mass m , energy E , 3-momentum p and 3-velocity v as viewed by a local stationary observer, the energy density ρ , pressure P and entropy density s are given by the standard expressions

$$\begin{aligned} \rho &= \mathcal{N} \int_0^\infty \frac{E}{e^{\beta E} - \epsilon} \frac{4\pi p^2 dp}{h^3}, \\ P &= \frac{\mathcal{N}}{3} \int_0^\infty \frac{vp}{e^{\beta E} - \epsilon} \frac{4\pi p^2 dp}{h^3}, \\ s &= \beta(\rho + P). \end{aligned} \quad (12)$$

Here, as usual,

$$E^2 - p^2 = m^2, \quad v = p/E, \quad \beta = T^{-1}; \quad (13)$$

ϵ is $+1$ for bosons and -1 for fermions and the factor $\mathcal{N}$ takes care of helicities and the number of particle species. The total entropy is given by the integral

$$S = \int_{r_1}^L s(r) 4\pi r^2 dr / \sqrt{f}, \quad (14)$$

where we have taken account of the proper volume element as given by the metric (8). The factor $f^{-1/2}$ does not, however, appear in the integral for the gravitational mass of the thermal excitations [22] (It is canceled, roughly speaking, by negative gravitational potential energy):

$$\Delta M_{\text{therm}} = \int_{r_1}^L \rho(r) 4\pi r^2 dr. \quad (15)$$

The integrals (14) and (15) are dominated by two contributions for large container radius L and for small $\Delta r = r_1 - r_0$:

- (a) A volume term, proportional to $\frac{4}{3}\pi L^3$, representing the entropy and mass-energy of a homogeneous quantum gas in a flat space (since $f \simeq 1$ almost everywhere in the container if $L/r_0 \rightarrow \infty$) at a uniform temperature T_∞ . This is the result that would have been expected, and we do not need to consider it in detail.
- (b) Of more interest is the contribution of gas near the inner wall $r = r_1$, which we now proceed to study further. We shall find that it is proportional to the wall area, and diverging like $(\Delta r)^{-1}$ when $\Delta r \rightarrow 0$.

Because of the high local temperatures T near the wall for small Δr , we may insert the ultrarelativistic approximations

$$E \gg m, \quad p \simeq E, \quad v \simeq 1$$

into the integrals (12). This gives

$$P \simeq \frac{1}{3}\rho \simeq \frac{\mathcal{N}}{6\pi^2} T^4 \int_0^\infty \frac{x^3 dx}{e^x - \epsilon} \quad (16)$$

in Planck units ($\hbar = 2\pi\hbar = 2\pi$). The purely numerical integral has the value $3!$ multiplied by 1 , $\pi^4/90$ and $\frac{7}{8}\pi^4/90$ for $\epsilon = 0, 1$ and -1 respectively, and we shall adopt $3!$, absorbing any small discrepancy into $\mathcal{N}$. (This normalization is related to that in the previous section as $\mathcal{N} = N\pi^4/90$.) Then, from (12),

$$\rho = \frac{3\mathcal{N}}{\pi^2} T^4, \quad s = \frac{4\mathcal{N}}{\pi^2} T^3 \quad (17)$$

in terms of the local temperature T given by (9).

Substituting (17) into (14) gives for the wall contribution to the total entropy

$$S_{\text{wall}} = \frac{4\mathcal{N}}{\pi^2} 4\pi r_1^2 T_\infty^3 \int_{r_1}^{r_1+\delta} \frac{dr}{f^2(r)}, \quad (18)$$

where δ is an arbitrary small length subject to $\Delta r \ll \delta \ll r_1$. It is useful to express this result in terms of the proper altitude α of the inner wall above the horizon $r = r_0$ of the exterior geometry (8). (Since (8) only applies for $r > r_1$, the physical space does not, of course, contain any horizon.) We assume that $f(r)$ has a (simple) zero for $r = r_0$, so we can write

$$f(r) \simeq 2\kappa_0(r - r_0), \quad \kappa_0 = \frac{1}{2}f'(r_0) \neq 0 \quad (r \rightarrow r_0), \quad (19)$$

where κ_0 is the surface gravity. Then

$$\alpha = \int_{r_0}^{r_1} f^{-1/2} dr \quad \Rightarrow \quad \Delta r = \frac{1}{2}\kappa_0\alpha^2, \quad (20)$$

and (18) can be written

$$S_{\text{wall}} = \frac{\mathcal{N}}{90\pi\alpha^2} \left(\frac{T_\infty}{\kappa_0/2\pi} \right)^3 \frac{1}{4} A \quad (21)$$

in Planck units, where $A = 4\pi r_1^2$ is the wall area.

Similarly, we find from (15) and (17) that thermal excitations near the wall contribute

$$\Delta M_{\text{them,wall}} = \frac{\mathcal{N}}{480\pi\alpha^2} \left(\frac{T_\infty}{\kappa_0/2\pi} \right)^3 AT_\infty \quad (22)$$

to the gravitational mass of the system.

The wall contribution to the free energy

$$F = \Delta M - T_\infty S \quad (23)$$

is

$$F_{\text{wall}} = -\frac{\mathcal{N}}{1440\pi\alpha^2} \left(\frac{T_\infty}{\kappa_0/2\pi} \right)^3 AT_\infty. \quad (24)$$

The entropy is recoverable from the free energy by the standard prescription

$$S_{\text{wall}} = -\partial F_{\text{wall}}/\partial T_\infty. \quad (25)$$

(Observe that this is an “off-shell” prescription [23, 24]: the geometrical quantities A , α and, in particular, the surface gravity κ_0 are kept fixed when the temperature is varied in (24).)

Following ‘tHooft [14], we now introduce a crude cutoff to allow for quantum-gravity fluctuations by fixing the wall altitude α so that

$$S_{\text{wall}} = S_{BH}, \quad \text{when} \quad T_\infty = T_{BH}, \quad (26)$$

where the Bekenstein-Hawking entropy S_{BH} and Hawking temperature T_{BH} are defined to be the *purely geometrical* quantities defined by (1) and (2) in terms of the wall’s area A and redshifted acceleration (= surface gravity) κ_0 . From (26) and (21), restoring conventional units for a moment, we find

$$\alpha = l_{Pl} \sqrt{\mathcal{N}/90\pi}, \quad (27)$$

so that α is very near the Planck length if the effective number $\mathcal{N}$ of basic quantum fields in nature is on the order of 300.

It is significant and crucial that the normalization (27) is *universal*, depending only on fundamental physics, and independent of the mechanical and geometrical characteristics of the system.

With α fixed by (27), the wall's free energy (23) becomes

$$F_{wall} = -\frac{1}{16} \left(\frac{T_\infty}{T_{BH}} \right)^3 AT_\infty. \quad (28)$$

This “off-shell” formula expresses F_{wall} in terms of three independent variables: the temperature T_∞ and the geometrical characteristics A and T_{BH} . From (28) we can obtain the wall entropy either from the thermodynamical Gibbs relation (25) (with T_∞ set equal to T_{BH} *after* differentiation), or from the Gibbs-Duhem formula (23) which is equivalent to the statistical-mechanical definition $S = -\text{Tr}(\rho \ln \rho)$. Thus the distinction [23, 24] between “thermodynamical” and “statistical” entropies disappears in this formulation, because the geometrical and thermal variables are kept independent.

The wall's thermal mass-energy is given “on-shell” ($T_\infty = T_{BH}$) by

$$\Delta M_{therm,wall} = \frac{3}{16} AT_{BH} \quad (29)$$

according to (22) and (27). For a wall skirting a Schwarzschild horizon, so that $T_{BH} = (8\pi M)^{-1}$, this reduces to ‘tHooft’s result (7).

As already noted, thermal energy is not the only source of the wall’s mass. Quantum fields outside the wall have as their ground state the Boulware state, which has a negative energy density growing to Planck levels near the wall. On shell, this very nearly cancels the thermal energy density (17); their sum is, in fact, the Hartle-Hawking value:

$$(T_\mu^\nu)_{therm, T_\infty=T_{BH}} + (T_\mu^\nu)_B = (T_\mu^\nu)_{HH}, \quad (30)$$

which remains bounded near horizons, and integrates virtually to zero for a very thin layer near the wall. The total gravitational mass of the wall is thus, from (22) and (27),

$$\begin{aligned} (\Delta M)_{wall} &= (\Delta M)_{therm,wall} + (\Delta M)_{B,wall} \\ &= \frac{3}{16} AT_{BH} ((T_\infty/T_{BH})^4 - 1), \end{aligned} \quad (31)$$

which vanishes on shell. For a central mass which is large in Planck units, there is no appreciable back-reaction of material near the wall on the background geometry (8).

We may conclude that many earlier concerns [14, 15, 16] were unnecessary: ‘tHooft’s brick wall model does provide a perfectly self-consistent description of a configuration which is indistinguishable from a black hole to outside observers, and which accounts for the Bekenstein-Hawking entropy purely as thermal entropy of quantum fields at the Hawking temperature (i.e. in the Hartle-Hawking state), providing one accepts the ad hoc but plausible ansatz (27) for a Planck-length cutoff near the horizon [18].

4 Semiclassical consistencies

It has been shown that the brick wall model is a consistent semiclassical description of black hole entropy: thermal excitations raised to the Hawking temperature above the Boulware state explains the black hole entropy; the positive divergence in thermal energy is canceled by the negative divergence in the vacuum energy of the Boulware state. Namely, the following simultaneous equations have a solution:

$$\begin{aligned} S_{wall} &= S_{BH}, \\ T_\infty &= T_{BH}, \\ (\Delta M)_{therm,wall} + (\Delta M)_{B,wall} &= 0. \end{aligned} \quad (32)$$

It has been found that the last equation is equivalent to the second one. Thus, the solution is Eq. (27) with the on-shell condition $T_\infty = T_{BH}$.

Our question now is whether the entanglement thermodynamics above the Boulware ground state is a consistent semiclassical description of black hole thermodynamics or not. Here let us assume that the

sum of the vacuum energy and the entanglement energy contributes to gravitational energy. Hence, the question is restated as whether the following simultaneous equations hold or not.

$$\begin{aligned} S_{ent} &= S_{BH}, \\ T_{ent} &= T_{BH}, \\ E_{ent} + (\Delta M)_{B,wall} &= 0. \end{aligned} \quad (33)$$

Since the cutoff length α in section 3 seems to be related to the cutoff length a in section 2 as

$$\alpha \simeq n_B a, \quad (34)$$

a solution of Eqs. (33) is given by

$$\begin{aligned} a &\simeq \sqrt{\frac{N\mathcal{N}_S}{\pi}} l_{pl}, \\ \mathcal{N}_E &\simeq \frac{\mathcal{N}_S}{2\pi}, \\ n_B &\simeq \sqrt{\frac{\pi^4}{90} \cdot \frac{1}{240\mathcal{N}_S}}. \end{aligned} \quad (35)$$

Unfortunately, since $\mathcal{N}_S = 0.3$, the r.h.s of the last equation is less than unity. Hence the last equation does not hold for positive integer value of n_B . Even if n_B would be allowed to be less than unity, the value of n_B given by the last equation would not be consistent with the second equation for all our definitions of entanglement energy [13]. This discrepancy suggests that our model of entanglement thermodynamics suffers from a strong backreaction near horizon. This might mean that our choice of the pure state (the Boulware state) would be wrong. However, we strongly expect that there is a state for which the consistency condition (35) holds. It will be worthwhile to analyze whether Eqs. (35) are satisfied for the Hartle-Hawking state or not.

Next, we shall combine a variational principle in entanglement thermodynamics with the third equation in the semiclassical consistency conditions. The principle of minimum of entanglement free energy introduced in Ref. [25] determines a quantum state to be

$$|\phi\rangle = \frac{1}{\sqrt{Z}} \sum_{n=1}^N \left[e^{-E_n/2T_{ent}} |n\rangle_1 \otimes |n\rangle_2 \right] \quad (36)$$

up to a unitary transformation in one of two subspaces, where T_{ent} is the entanglement temperature, $Z = \sum_{n=1}^N e^{-E_n/T_{ent}}$. Here E_n and $|n\rangle_2$ ($n = 1, 2, \dots, N$) are eigenvalues and orthonormalized eigenstates of the normal-ordered Hamiltonian of the subsystem 2. In Ref. [25], it was also mentioned that, if we maximize entanglement entropy with entanglement energy fixed, the state (36) is obtained. In this case, entanglement temperature should be determined so that the entanglement energy coincides with the fixed value. On the other hand, the third equation in the semiclassical consistency conditions (33) does fix entanglement energy, provided that we do not change $(\Delta M)_{B,wall}$. We shall call this condition the small backreaction condition. Hence, the principle of maximum of entanglement entropy combined with the small backreaction condition determines a state of a quantum field near horizon to be (36) up to a unitary transformation in one of two Hilbert subspaces, provided that a suitable regularization scheme is introduced. This state has exactly the same form as those appearing in the thermo field dynamics of black holes [26] and the quantum field theory on a collapsing black hole background [27], provided that the entanglement temperature T_{ent} can be set to be the Hawking temperature. In this case, since the entanglement energy is the same as thermal energy in the brick wall model because of the small backreaction condition, and since the entanglement entropy is expected to have the almost same value as the Bekenstein-Hawking entropy, the entanglement temperature also seems to coincide with the Hawking temperature up to a numerical factor of order unity. Therefore, it seems that the principle of maximum of entanglement entropy combined with the small backreaction condition may provide a new derivation of the Hawking radiation. Further investigations on this point will be valuable.

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Hawking Radiation and Mode Conversion

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Abstract

Hawking radiation[1] emerges from very high frequency modes which are over Planckian scale near the event horizon. A black hole radiation model, whose dispersion relation was modified to cut off the high energy modes[3] [4] [5], have been introduced to study this problem. The origin of the black hole radiation of this model is different from that of Hawking radiation. This origin of radiation is called Mode Conversion. The spectrum of the flux of the mode conversion model was calculated perturbatively in two dimension up to lowest orders of the surface gravity and the energy of a massless scalar field. The result showed this spectrum was a thermal[4]. Numerical calculation[3], however, showed the deviation from thermal spectrum should arise. In this article, we calculate the spectrum up to next orders and result that the difference do not occur from next order corrections. Thus we need to analyze the spectrum with a method other than the perturbation to obtain the deviation. It should be a kind of a full order calculation about the energy of the matter field.

1 Introduction

About a hundred years ago, two important theories were established, quantum mechanics and the theory of general relativity. The former idea has been applied to field theories and made many triumphs to understand the wide range of phenomena. But these phenomena have no relation with the gravity. On the other hand, the latter has given us insights into the phenomena having deep relation with the gravity. It has changed drastically our way to understand space and time. But it is classical. We have still no idea to treat the gravity in Planckian scale.

It seems not clear whether the unification of the quantum mechanics and the general relativity will be accomplished in the context of the idea of quantum physics. Today, however, we do not have any idea superior to the quantum idea in treating Planckian scale physics. So it is useful to study how quantum and/or semi-classical effects will be expected in the presence of the gravity.

First proposal of semi-classical gravity effect is Hawking radiation[1], that a black hole may radiate a particle flux of a thermal spectrum, whose temperature is $\kappa/2\pi$ where κ is the surface gravity. This indicates that black holes may evaporate. Hawking radiation is concluded in treating the metric as classical but matter fields on spacetime as quantum. And it seems that Hawking radiation have some relations with other physics; the super string theory, the black hole thermodynamics, the early universe and so on. Hawking radiation attracts many physicists' attention. So, today, the validity of existence of Hawking radiation is actually a interesting issue.

There is a problem of the validity of existence of Hawking radiation, that there exists the very high frequency modes in the radiation at the event horizon. These high frequencies should be over Planckian scale, since, if we traced back in time the radiation to near the event horizon, the radiation should undergo very high blueshift. Existence of such modes indicates that the semi-classical treatment may break down, and may be led to the non-validity of existence of Hawking radiation.

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In the studies of this problem, a model of a black hole radiation[3] [4] [5] have been introduced. In this model, the EOM of the matter field was modified to have a dispersion relation arising the cut off the high frequency modes with respect to the free falling observer. (This model was motivated by the dumb hole[2].) The origin of radiation of this model is different from that of the original Hawking radiation. This type of origin of radiation is called Mode Conversion, which is already known in plasma physics. Hereafter we call this model the mode conversion model.

The spectrum of the flux of two dimensional mode conversion model was calculated perturbatively up to lowest orders of the surface gravity and the energy of a massless scalar field. The result showed this spectrum was thermal[4]. On the other hand, a numerical calculation showed the spectrum of the mode conversion model should deviate from a thermal spectrum[3].

The purpose of this article is to proceed with the calculation of the spectrum of the mode conversion model up to next order perturbation. In section 2, the mode conversion model is reviewed. Section 3 is for the calculation up to next order perturbation. Lastly, a summary is briefly stated in section 4.

2 Mode conversion model

We use Planck units, $\hbar = c = G = 1$, and restrict our discussion in two dimension for simplicity hereafter.

2.1 Back ground spacetime

We consider the static spacetime with the metric of the form

$$ds^2 = -dt^2 + (dx - v(x) dt)^2, \quad (1)$$

where $v(x)$ is assumed to have the properties; $v(x)$ is a function of only x , negative valued $v(x) < 0$ and monotone increasing $\partial_x v(x) > 0$. It is obvious that the killing vector field of this spacetime is ∂_t .

To see the global structure of this spacetime, transform the coordinate from (t, x) to (τ, x) with the definition $d\tau = t + \int v/(1-v)dx$. In this coordinate the form of the metric becomes

$$ds^2 = -(1-v^2)d\tau^2 + \frac{dx^2}{1-v^2}. \quad (2)$$

This spacetime has a event horizon at x satisfying $v(x) = -1$. Note that the world line of $dx - v(x) dt = 0$ has three properties; Its tangent vector is normal to the surface, $t = \text{const}$. It is a geodesic, i.e. the world line of the free falling observer. The proper time of this free falling observer is t of the coordinate of (1), that is, v is the free fall velocity of this observer in the coordinate of (1).

2.2 Modification of the EOM of a massless scalar field

Define the vector $\mathbf{u}$ as a 4-velocity of the free falling observer and $\mathbf{s}$ as a unit space-like vector which is outward-pointing and orthogonal to $\mathbf{u}$. Then $\mathbf{u}$ and $\mathbf{s}$ are expressed in the coordinate of (1) as

$$\mathbf{u} = \partial_t + v(x) \partial_x \quad (3)$$

$$\mathbf{s} = \partial_x. \quad (4)$$

We aim to modify the EOM of a massless scalar field, Φ , with respect to the free falling system of the orthonormal basis $(\mathbf{u}, \mathbf{s})$. To do this modification, define the derivatives of Φ along $\mathbf{u}$ and $\mathbf{s}$ as

$$\mathbf{u}(\Phi) = u^\mu \partial_\mu \Phi = (\partial_t + v(x) \partial_x) \Phi \quad (5)$$

$$\mathbf{s}(\Phi) = \hat{F}(s^\mu \partial_\mu) \Phi = \hat{F}(\partial_x) \Phi, \quad (6)$$

where $\hat{F}(\partial_x)$ is a function of the differential operator, ∂_x , and we require $\hat{F}(-\partial_x) = -\hat{F}(\partial_x)$.

The action of this modified matter system is

$$S = \frac{1}{2} \int dt dx \left[|(\partial_t + v(x) \partial_x) \Phi|^2 - |\hat{F}(\partial_x) \Phi|^2 \right] \equiv \int L dt, \quad (7)$$

where L is the lagrangian. This leads the EOM of Φ to be

$$(\partial_t + v(x) \partial_x)(\partial_t + \partial_x v(x)) \Phi = \hat{F}^2(\partial_x) \Phi. \quad (8)$$

Hereafter we specify the form of $\hat{F}(\partial_x)$ as

$$\hat{F}^2(\partial_x) = \partial_x^2 + \alpha \partial_x^4, \quad (9)$$

where $\alpha = 1/k_c^2$ and k_c should be the cut off scale of the wave number of the modes of Φ .

2.3 Quantization, mode conversion and the spectrum of the radiation

For the purpose of understanding the mode conversion, it is useful to set

$$\Phi(t, x) = \exp \left(-i\omega t + i \int k(x) dx \right). \quad (10)$$

Note that ω is a frequency with respect to killing vector, ∂_t , since

$$\partial_t \Phi = -i\omega \Phi. \quad (11)$$

This means that, because ω is a eigenvalue ($= \text{const.}$) of Φ , ω of each mode of a scalar field are conserved. And Φ of the form of (10) can be thought as a mode function having a “killing frequency” ω .

We require that, at least far from the event horizon, $v(x)$ is approximately constant enough to require WKB approximation, that is, $\partial_x v(x)/k(x) \ll 1$ and $\partial_x k(x)/k^2(x) \ll 1$. Then Φ is expressed as

$$\Phi \simeq \exp(-i\omega t + ikx), \quad (12)$$

and the EOM (8) becomes

$$(\omega - vk)^2 = F^2(k), \quad (13)$$

where $F(k)$ is a function of k taking the form as $F^2(k) = k^2 - \alpha k^4$. So the wave number, k , of Φ satisfies the equations,

$$\begin{cases} \omega' &= -vk + \omega \\ \omega' &= \pm F(k) = \pm \sqrt{k^2 - \alpha k^4} \end{cases}. \quad (14)$$

And note that ω' of (14) is a frequency with respect to the free falling observer, because with WKB approximation

$$\mathbf{u}(\Phi) = -i(\omega - vk)\Phi = -i\omega'\Phi. \quad (15)$$

The inner product defined from the action (7) is

$$\begin{aligned} (\Phi_1 \Phi_2) &= i \int_{t=\text{const.}} dx [\Phi_1^* (\partial_t + v \partial_x) \Phi_2 - \Phi_2 (\partial_t + v \partial_x) \Phi_1^*] \\ &= (\omega'_1 + \omega'_2) \int_{t=\text{const.}} dx \Phi_1^* \Phi_2. \end{aligned} \quad (16)$$

So we define the positive and negative frequency modes of Φ according not to the sign of the killing frequency but to the sign of the “free fall frequency” ω' . That is, Φ is a positive frequency mode for $\omega' > 0$ and is a negative mode for $\omega' < 0$.

The formalism of quantization of Φ is simple. Define a canonical momentum

$$\Pi = \frac{\delta L}{\delta(\partial_t \Phi)} = (\partial_t + v \partial_x) \Phi, \quad (17)$$

and require the equal time canonical commutation relation for the Hermitian operators $\hat{\Phi}$ and $\hat{\Pi}$,

$$[\hat{\Phi}(t, x), \hat{\Pi}(t, y)] = i\delta(x - y). \quad (18)$$

From this commutation relation, the creation and annihilation operators, $\hat{a}$ and $\hat{a}^\dagger$ respectively, are obtained as

$$\begin{aligned} \hat{a} &= (U_+ \hat{\Phi}) \\ \hat{a}^\dagger &= -(U_- \hat{\Phi}) \end{aligned} \quad (19)$$

where U_+ and U_- are normalized positive and negative (free fall) frequency modes respectively. And the vacuum state, $|0\rangle$, is defined by, $\hat{a}|0\rangle = 0$.

From above preparations, we can see what the mode conversion is. We consider the observer rests at infinity where $-1 < v(x \rightarrow \infty) = \text{const.} \leq 0$. This observer receives a flux radiated from a black hole. We assume the flux is a wave packet constructed by positive (free fall) frequency modes having positive killing frequency, and trace it back in time toward the event horizon. The group velocity of this wave packet with respect to the rest observer is expressed as

$$V_g = \frac{d\omega(k)}{dk}, \quad (20)$$

where $\omega(k)$ is the killing frequency spectrum of this wave packet. Then we obtain from (14)

$$V_g - v(x) = \pm \frac{dF(k)}{dk} \left(= \frac{d\omega'(k)}{dk} \right). \quad (21)$$

This packet is outgoing at the moment of observation, since $V_g < 0$. We call this moment the final time. And we set the representative value of the wave number of this packet as k_{+s} . (see figure 1)

In this process of tracing back, ω of each mode in this packet are conserved and $-v(x) (= |v(x)|)$ increases gradually. And the packet reaches the position, where the free fall speed of a free falling observer, $|v|$, is equal to the group velocity of the packet with respect to the free falling observer, $d\omega'/dk$, that is, $V_g = 0$. At this position, the outgoing packet that traced back in time converts to three ingoing wave packets. This is the Mode Conversion. (see figure 1) Note that WKB breaks down there because the position is not far from the horizon, and there is no rigid reason to keep only one mode corresponding to one of the solutions of (14). This naive picture may make the physics of the mode conversion clear that, in the inverse-evolution of time, the packet that traced back is just like the ingoing wave on the fluid which flows out of the horizon with the velocity $|v(x)|$ and the group velocity of the packet come to be zero at the position of the mode conversion to be scattered back[2].

Three ingoing wave packets which produced by the mode conversion reach again the rest observer at last in tracing them back in time. We call this moment the initial time. And we set the representative wave numbers of these ingoing packets as k_- , k_{-s} and k_+ in increasing order. Of course $V_g > 0$ for them. (see figure 1)

It should be pointed out that the mode conversion raises a ingoing negative (free fall) frequency packet of k_- from the outgoing positive (free fall) frequency packet of k_{+s} . This means that the Bogoljubov transformation of the mode functions of $\hat{\Phi}$ occurs and the vacuum expectation value of the number operator is not zero, that is, $\langle 0 | \hat{a}^\dagger \hat{a} | 0 \rangle \neq 0$, where $|0\rangle$ is the vacuum state at the initial time and $\hat{a}$ and $\hat{a}^\dagger$ are the operators at the final time. There exists a radiation from a black hole.

For the last of this section 2, show the form of the spectrum of the radiation, $\langle 0 | \hat{a}^\dagger \hat{a} | 0 \rangle$, with the assumption that $\omega(k)$ is sharply peaked around the mean value of k [3]. (We think $k_{\pm s}$ and $k_{\pm}$ as these mean values.) This is

$$N(\omega) = \langle 0 | \hat{a}^\dagger \hat{a} | 0 \rangle = \left| \frac{\omega'(k_-) V_g(k_-) c_-^2(\omega)}{\omega'(k_{+s}) V_g(k_{+s}) c_{+s}^2(\omega)} \right|, \quad (22)$$

where $\omega = \omega(k_-) = \omega(k_{+s})$. (Note that ω is killing frequency conserved during the time evolution.) $c_-(\omega)$ and $c_{+s}(\omega)$ are the amplitudes of the mode functions of k_- and k_{+s} at infinity, respectively.

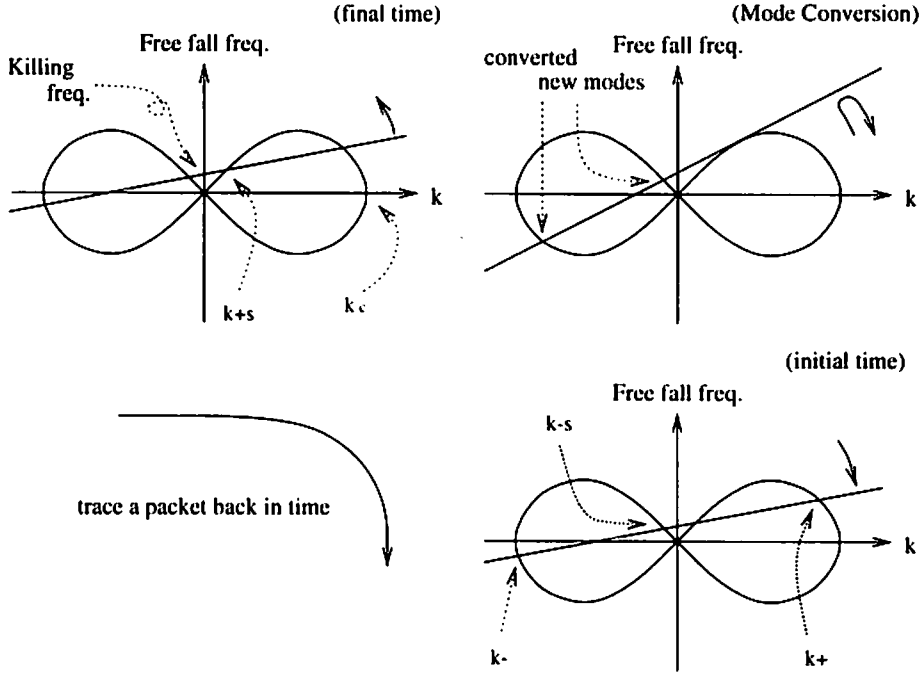


Figure 1: Mode Conversion (understood with (14))

3 Calculation of the spectrum of BH radiation

We solve EOM (8) perturbatively and calculate the spectrum $N(\omega)$ of the mode conversion model.

3.1 The plan to calculate the spectrum

The form of $N(\omega)$ is obtained with considering the wave packet propagation. And the packet is assumed to have a sharp peak in its killing frequency spectrum, $\omega(k)$. Thus we consider only the mode function having the wave number at the peak of $\omega(k)$. This is the representative mode of the packet, which has one killing frequency (conserved during time evolution) while comprises some free fall frequencies (not conserved during time evolution).

We set the form of this mode function as

$$\Phi(t, x) = \exp(-i\omega t) \phi(x), \quad (23)$$

where ω is the representative value of killing frequency of the packet. Then EOM (8) becomes

$$\alpha\phi^{(4)} + (1 - v)\phi'' + 2v(i\omega - v')\phi' + (\omega^2 + i\omega v')\phi = 0. \quad (24)$$

The solution of this ordinary differential equation is understood as a superposition of space parts of some (free fall frequency) modes of k_+ , k_{+s} , k_{-s} and k_- (see figure 1), which have same killing frequency. This is the representative mode function mentioned at the beginning of this section. And we adopt the boundary condition of this ordinary differential equation as ; B.C. : *The mode inside the event horizon dumps with decreasing x .* And we take the origin of spatial coordinate x at the location of the event horizon, $v(x = 0) = -1$. We assume for $v(x)$ that the value of v at the observer, which is $v_0 (= v(x \rightarrow \infty))$ satisfying $-1 < v_0 \leq 0$ by definition, is not zero. And we assume ω is so small that $\omega'(k_-)$ at the observer ($x \rightarrow \infty$) is of negative value.

The plan of calculation consists of three steps. First step is to solve (24) near the horizon, but it should be not too near, because, while we use Laplace transformation in this step, we approximate the Laplace integral by the method of steepest descent contour (SDC) to obtain the modes corresponding to k_+ and k_- , that is, it is required that x is large enough to let SDC be valid. And we make use of the integral representation of gamma function to obtain the modes corresponding to k_{+s} . The contour of Laplace integral should be chosen as it satisfies the B.C. (The mode corresponding to k_- , does not appear under the B.C.[4].) Second step is to solve (24) far from the horizon. We use WKB approximation, and match the WKB solutions with the solutions of the first step. And it must be checked whether the overlap region of SDC and WKB exists or not. Third step is to calculate $N(\omega)$ at infinity using (22).

3.2 Calculation of the spectrum

3.2.1 Small parameters

Three small parameters should arise in the following analysis. These parameters are

$$\begin{cases} \omega/s_0 \ll 1 & : \text{required in obtaining the Laplace mode } \tilde{\phi}(s) \text{ of } \phi(x) \text{ approximately.} \\ \kappa_1 x \ll 1 & : \text{required in considering Laplace transformation near but not too near the horizon.} \\ 1/s_0 x \ll 1 & : \text{required by WKB and the validity of } \phi_{+s} \text{ in Laplace integrating.} \end{cases}$$

where $s_0 = \sqrt{\kappa_1 x / \alpha}$, $v(x) \simeq -1 + \kappa_1 x + \kappa_2 x^2$ and $O(\kappa_2) = O(\kappa_1^2)$. And we take the terms of the orders up to $O(\kappa_1 x)$, $O(\omega/s_0)$ and $O((1/s_0 x)^0)$ in the approximations in following sections. Note that the calculation of [4] was carried out by taking the orders up to $O((\kappa_1 x)^0)$, $O((\omega/s_0)^0)$ and $O((1/s_0 x)^{-1})$.

3.2.2 First step; Laplace integral near but not too near horizon

We transform $\phi(x)$ (Laplace transformation),

$$\phi(x) = \int_c ds e^{sx} \tilde{\phi}(s). \quad (25)$$

Expand $\tilde{\phi}(s)$ perturbatively as

$$\tilde{\phi}(s) \simeq \tilde{\phi}_0(s) + \tilde{\phi}_1(s), \quad (26)$$

where $|\tilde{\phi}_1/\tilde{\phi}_0| < 1$. With expanding $v(x) \simeq -1 + \kappa_1 x + \kappa_2 x^2$, where $\kappa_1 x \ll 1$, and taking the terms up to $O((\kappa_1 x)^2)$ in (24), we obtain

$$\tilde{\phi}_0'(s) - \frac{B(s)}{2\kappa_1 f(s)} \tilde{\phi}_0(s) = 0 \quad (27)$$

$$\tilde{\phi}_1'(s) - \frac{B(s)}{2\kappa_1 f(s)} \tilde{\phi}_1(s) = \left(\frac{\kappa_2}{\kappa_1} - \frac{\kappa_1 s^2}{2f(s)} \right) \tilde{\phi}_0''(s) + \frac{C(s)}{\kappa_1 f(s)} \tilde{\phi}_0'(s), \quad (28)$$

where $f(s) = s^2 + i\omega s$, $B(s) = \alpha s^4 - \kappa_1 \beta f'(s)$, $C(s) = \kappa_2 f'(s) - \kappa_1^2 s$ and $\beta = 1 + i\omega/\kappa_1$. We expand the right-hand side of (28) according to the section (3.2.1), and integrate these equations to obtain

$$\tilde{\phi}_0 = R_0 s^{-\beta} \left(1 + i \frac{\omega}{\kappa_1} \right)^{-\beta/2} \exp \left[\frac{\alpha}{2\kappa_1} \left(\frac{s^3}{3} - i \frac{\omega s^2}{2} - \omega^2 s \right) \right] \equiv g(s) \exp[h(s)], \quad (29)$$

$$\tilde{\phi}_1 = \epsilon \left(\frac{\alpha^2 s^5}{20\kappa_1^2} + \frac{\alpha(2-\beta)s^2}{2\kappa_1} + \frac{\beta(1-\beta)}{s} \right) \tilde{\phi}_0 \equiv D(s) \tilde{\phi}_0, \quad (30)$$

where $\epsilon = \kappa_2/\kappa_1 - \kappa_1/2$, and R_0 is normalization constant. Then we renormalize $\tilde{\phi}_1$ into $\tilde{\phi}_0$ to obtain

$$\tilde{\phi}(s) \simeq g(s) \exp[h(s) + D(s)]. \quad (31)$$

The contour of (25) corresponding to the B.C. consist of three pieces[4]. At last we obtain $\phi(x)$ as

$$\phi(x) = \phi_1(x) + \phi_2(x) + \phi_3(x), \quad (32)$$

where

$$\phi_1(x) = \left[1 - \frac{3}{4}\varepsilon x - i\frac{\omega}{s_{0-}}\beta \right] \exp \left[\frac{\varepsilon}{5}s_{0-}x^2 + i\frac{\omega}{2}\left(1 + \frac{\varepsilon}{\kappa_1}\right)x \right] \times R_1 \sqrt{\frac{2\pi\kappa_1}{\alpha}} s_{0-}^{-1/2-\beta} \exp \left[\frac{2}{3}s_{0-}x \right] \quad (33)$$

$$\phi_2(x) = \text{replace } R_1 \rightarrow R_2, s_{0-} \rightarrow s_{0+} \text{ in } \phi_1(x) \quad (34)$$

$$\phi_3(x) = \left[1 - i\frac{\omega\kappa_2}{\kappa_1^2}x \right] R_3 \sinh \left(\frac{\pi\omega}{\kappa_1} \right) \Gamma \left(-i\frac{\omega}{\kappa_1} \right) x^{i\omega/\kappa_1}, \quad (35)$$

where $R_{1,2,3}$ are phase constants and $s_{0\pm} = \pm i\sqrt{2\kappa_1 x/\alpha}$, and $\phi_{1,2}$ are obtained by SDC and ϕ_3 is obtained by applying the integral representation of gamma function. Note that the upper limit of the region where these approximate solutions are valid is estimated that $x \ll 1/\kappa_1$ ($1/\kappa_1$ is very large), and that x should be large enough to let SDC be valid.

3.2.3 Second step; matching WKB with SDC

We set

$$\phi(x) = \exp \left[i \int^x k(x') dx' \right], \quad (36)$$

and perturb $k(x) = k_0(x) + (1/\lambda)k_1(x) + \dots$ in considering the scaling $x \rightarrow \lambda x$, then obtain from (24)

$$O(1) : \alpha k_0^4 - (1 - v^2)k_0^2 - 2\omega v k_0 + \omega^2 = 0 \quad (37)$$

$$O\left(\frac{1}{\lambda}\right) : k_1 = \frac{i}{2} \frac{d}{dx} \ln \left[\sqrt{\alpha} (2\alpha k_0^3 - (1 - v^2)k_0 - \omega v) \right]. \quad (38)$$

In solving (37) we further approximate k_0 perturbatively up to $O(\omega^2)$, and obtain (as $\lambda \rightarrow 1$)

$$\begin{aligned} k_{\pm}(x) &\simeq k_{0\pm}(x) + k_{1\pm}(x) \\ &= \pm \sqrt{\frac{1-v^2}{\alpha}} + \frac{\omega v}{1-v^2} \mp \frac{\omega^2 \sqrt{\alpha}}{2} \frac{1+2v^2}{(1-v^2)^{5/2}} \\ &\quad + \frac{d}{dx} \frac{1}{i} \ln \left[(1-v^2)^{-3/4} \left(1 \mp \frac{2\omega v}{\alpha} \left(\frac{1-v^2}{\alpha} \right)^{-3/2} - \frac{\omega^2}{4\alpha^2} (-5+2v^2) \left(\frac{1-v^2}{\alpha} \right)^{-3} \right) \right] \end{aligned} \quad (39)$$

$$k_{\pm s}(x) \simeq \pm \frac{\omega}{1 \pm v(x)} + O(\omega^3). \quad (40)$$

Then we obtain the mode functions with expanding $v(x) \simeq -1 + \kappa_1 x + \kappa_2 x^2$ and approximate according to the section (3.2.1),

$$\begin{aligned} \phi_{\pm}(x) &\simeq \left[1 - \frac{3}{4}\varepsilon x - i\frac{\omega}{s_{0-}}\beta \right] \exp \left[\frac{\varepsilon}{5}s_{0-}x^2 + i\frac{\omega}{2}\left(1 + \frac{\varepsilon}{\kappa_1}\right)x \right] \times \\ &\quad Q_{\pm} \alpha^{-3/4} \exp \left(\pm \frac{\pi\omega}{2\kappa_1} \right) s_{0\pm}^{-1/2-\beta} \exp \left[\frac{2}{3}s_{0\pm}x \right] \end{aligned} \quad (41)$$

$$\phi_{+s}(x) \simeq \left[1 - i\frac{\omega\kappa_2}{\kappa_1^2}x \right] Q_{+s} x^{i\omega/\kappa_1}, \quad (42)$$

where $Q_{\pm, +s}$ are phase constants.

The region where these WKB solutions are valid is estimated by $|k_1/k_0| \ll 1$ to lead $(\alpha/\kappa_1)^{1/3} \ll x$. Thus the overlap region of SDC and WKB exists in the region $(\alpha/\kappa_1)^{1/3} \ll x \ll 1/\kappa_1$.

Consequently we obtain the approximate solution of (24) through (32) in the overlap region,

$$\phi(x) = P \alpha^{1/4} \sqrt{2\pi\kappa_1} \left[i \exp \left(-\frac{\pi\omega}{2\kappa_1} \right) \phi_{-}(x) + \exp \left(\frac{\pi\omega}{2\kappa_1} \right) \phi_{+}(x) \right] + 2 \sinh \left(\frac{\pi\omega}{\kappa_1} \right) \Gamma \left(-i\frac{\omega}{\kappa_1} \right) \phi_{+s}(x), \quad (43)$$

where P is a phase constant.

3.2.4 Third step; calculation of $N(\omega)$ at ∞

We estimate $\phi_{\pm,+s}$ and $\omega'v_g(k_{-,+s})$ at infinity. Take the limit of (39) and (40) as $v \rightarrow v_0$ ($x \rightarrow \infty$), and obtain the amplitudes of ϕ_- 's and $\phi_{+,s}$'s from (43) to be

$$\left| \frac{c_-}{c_{+,s}} \right|^2 = \sqrt{\alpha} \omega (1 - v_0^2)^{-3/2} \left[1 + 4\omega \sqrt{\alpha} \frac{v_0}{(1 - v_0^2)^{3/2}} + \frac{\omega^2 \alpha}{2} \frac{5 + 6v_0^2}{(1 - v_0^2)^3} \right] \frac{1}{\exp(2\pi\omega/\kappa_1) - 1}, \quad (44)$$

and obtain by (14) and (21)

$$\omega'(k_-)v_g(k_-) \simeq \frac{(1 - v_0^2)^{3/2}}{\sqrt{\alpha}} \left[1 - 4\omega \sqrt{\alpha} \frac{v_0}{(1 - v_0^2)^{3/2}} - \frac{\omega^2 \alpha}{2} \frac{5 - 2v_0^2}{(1 - v_0^2)^3} \right] \quad (45)$$

$$\omega'(k_{+,s})v_g(k_{+,s}) \simeq \omega. \quad (46)$$

Then the form of $N(\omega)$ to be

$$N(\omega) \simeq \left[1 - 12\omega^2 \alpha \frac{v_0^2}{(1 - v_0^2)^3} \right] \frac{1}{\exp(2\pi\omega/\kappa_1) - 1}, \quad (47)$$

but we should approximate it according to the section (3.2.1), so finally we obtain the spectrum of the mode conversion model to be

$$N(\omega) \simeq \frac{1}{\exp(2\pi\omega/\kappa_1) - 1}. \quad (48)$$

This is a thermal spectrum of temperature $\kappa_1/2\pi$, where κ_1 corresponds to the surface gravity, that is, this spectrum is the thermal flux of same temperature as the original Hawking radiation.

4 Summary

We have proceeded with the calculation of the spectrum of the mode conversion model up to higher orders than that of [4]. We had expected that the deviation of the spectrum from a black body would be derived, since numerical calculation of $N(\omega)$ have already showed the deviation had to arise[3]. The result of our analysis, however, could not show us it.

There expected to be two types of the deviation. One of them is the change of temperature. Another is a prefactor like a gray body factor of the original Hawking radiation. If we expect the former type of deviation, the change on the mode function should appear as the change of the exponents of $s_{0\pm}$ and x in (33), (34), (35), (41) and (42). Especially for (41) and (42), these exponents are determined by the second term of (39) and the first term of (40) respectively. In pushing forward the perturbation of k_0 up to finite order of ω , it seems hard to expect that a term proportional to $1/(1 - v^2)$ or $1/(1 + v)$ will arise as a higher order term.

Thus, for the purpose of deriving the deviation of the spectrum, it may be required to obtain the WKB and SDC solutions by a full order calculation of ω or, at least, the order of ω may be set larger than that assumed in this article. We are about to analyze the spectrum by resetting the order of ω .

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Does the generalized second law hold in the form of time derivative expression?

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Abstract

We investigate whether the generalized second law is valid, using two dimensional model of gravity. A time derivative expression of the generalized second law is formulated and shown to be invalid for eternal black holes. The results might make the generalized second law questionable and then might suggest that the analogy between the ordinary thermodynamics and the black hole thermodynamics breaks down.

1 Introduction

One of the most interesting developments in black hole physics is a discovery of the analogy between certain laws of black hole mechanics and the ordinary laws of thermodynamics [1]. Bekenstein [2] conjectured that the total entropy never decrease in any process, where the total entropy is the sum of the black hole entropy and the ordinary thermodynamic entropy of the matter outside the black hole. This is known as the generalized second law (GSL) of thermodynamics and it is important to check the validity of this conjecture because the validity strongly supports that the ordinary laws of thermodynamics can apply to a self-gravitating quantum system containing a black hole.

Frolov and Page [3] proved the GSL for an eternal black hole by assuming that (i) the process in the investigation is quasistationary which means that the changes to the holes are sufficiently small compared with the corresponding background quantities, that (ii) the state of matter fields on the past horizon $\mathcal{H}^-$ is a thermal state with the Hawking temperature, (iii) initial set of radiation modes on the past horizon $\mathcal{H}^-$ and that on the past null infinity $\mathcal{I}^-$ are quantum mechanically uncorrelated, and that (iv) the Hilbert space and the Hamiltonian of modes at $\mathcal{H}^-$ are identical to those of modes at $\mathcal{H}^+$.

But these assumptions have several problems as follows; (1) the physical reason to justify the assumption (ii) is not clear, (2) for the black hole formed by a gravitational collapse, the assumptions (iii) and (iv) break down due to the correlation between modes at $\mathcal{I}^+$ and modes at $\mathcal{I}^-$ located after the horizon formation and the violation of time reversal symmetry, respectively. So we do not think that their proofs can be applied to realistic black holes formed by gravitational collapse.

The GSL for a black hole formed by gravitational collapse was studied by Sorkin [4] and Mukohyama [5], making use of the nondecreasing function in a Markov process. The proof finally come to showing that the matter fields in the black hole background have a stationary canonical distribution with temperature equal to that of a black hole and the canonical partition function remains a constant.

But there are also some problems in their proofs. Mukohyama showed that the canonical distribution with temperature equal to the black hole is stationary by calculating the transition matrix between states at $\mathcal{I}^+$ and states at the portion of $\mathcal{I}^-$ after the formation of the event horizon $\mathcal{H}^+$. If the spectrums of the Hamiltonians of the both regions are identical, we can show invariance of the partition function and complete the proof. He took the equivalence of the spectrums for granted and concluded that the GSL

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also holds in collapsing cases. However it is doubtful in collapsing cases whether the equivalence holds. Further, in collapsing cases, the assumption (i) can not be justified in general.

Since Sorkin argued any process occurring between two adjacent time slices, the assumption (i) is valid. He concluded that the canonical distribution of matter fields are stationary because the Hamiltonian does not change between the two adjacent time slices, thanks to the time translation invariance in the background. His assertion seems to be derived by implying that we take the Killing time slices and the Killing Hamiltonian. However, if we take the Killing time slices, there is no energy flux across the event horizon. It means that we cannot see evaporating black hole by the Killing time slices.

In this paper, in order to satisfy the assumption (i), we consider the change of the total entropy in the infinitesimal time evolution for two dimensional dilaton gravity and argue the validity of the GSL. Unlike Sorkin, we use the time slices that are across the horizon not at the bifurcate point to avoid no evaporating situation.

Our program is as follows. First, we write the change of total entropy between two adjacent time slices in terms of quantities of matter (i.e., energy and entropy of matter) using the assumption (i). Thus, all we have to do are to calculate energy-momentum tensor and entanglement entropy of the matter field. These are obtained easily for conformal fields in two dimensional spacetime. Finally, using the Callan-Giddings-Harvey-Strominger (CGHS) model [6], we will demonstrate that the GSL does not hold for conformal vacuum state in a two dimensional eternal black hole in general situations. Although our toy model is two dimensional model, it is worth to investigate this toy model because there also exists the same black hole physics in two dimensional spacetime as in four dimensional one.

This paper is organized as follows. In Sec.2 we review the CGHS model and we formulate a time derivative form of the GSL. Since we express the change of the total entropy in terms of physical quantities of matters, our task is to calculate time evolution of matter fields in a fixed black hole background. We consider two cases, the Hartle-Hawking state and the Unruh one. In Sec.3 we calculate the change of the energy of matter fields for these two states. In Sec.4 the entanglement entropy is obtained for the two states. Then, it is shown that the time derivative version of the GSL does not hold in the case of an eternal black hole background. This is summarized in terms of the free energy in Sec.5. In Sec.6 we summarize and discuss of our results.

2 Review of the CGHS model and the GSL

2.1 Review of the CGHS model

In this subsection, we summarize the CGHS model. The classical action of the CGHS model [6] is

$$S = S_G + S_M , \quad (1)$$

$$S_G = \frac{1}{2\pi} \int_V d^2x \sqrt{-g} e^{-2\phi} [R + 4(\nabla\phi)^2 + 4\lambda^2] , \quad (2)$$

$$S_M = \sum_{i=1}^N \frac{-1}{2} \int_V d^2x \sqrt{-g} (\nabla f_i)^2 , \quad (3)$$

where ϕ , f_i and R are the dilaton field, the N conformal massless scalar fields and the Ricci curvature, respectively. A constant λ is the cosmological constant.

The CGHS model can be viewed as the low energy effective theory that governs the S-wave modes propagating on the background of a magnetically charged dilaton black hole in the four dimensional dilaton gravity. Further the action resembles to the two dimensional effective action derived from four dimensional Einstein-Hilbert action by truncating all but the spherically symmetric modes assuming that the metric and matter fields have spherical symmetry.

It is known [6, 7] that the CGHS model has a static black hole solution as a vacuum solution ($T_M^{\mu\nu} = 0$)

$$ds^2 = -\xi^2(r)dt^2 + \frac{dr^2}{\xi^2(r)} , \quad (4)$$

$$\xi^2(r) = 1 - \frac{\pi M_{ADM}}{2\lambda} e^{2\phi(r)} , \quad (5)$$

$$\phi(r) = -\lambda r .$$

The position of the horizon is specified by r with $\xi(r) = 0$. Then the ADM mass of the black hole M_{ADM} can be expressed in terms of the value ϕ_H of the dilaton field at the event horizon as ⁵

$$M_{ADM} = \frac{2\lambda}{\pi} e^{-2\phi_H} . \quad (6)$$

Semiclassically, the black hole has a certain Hawking temperature. This can be confirmed by the standard analysis; the Euclidean method [7] or the method of the Bogolubov transformation [8]. In the present case, this temperature is given by

$$T_{BH} = \frac{\lambda}{2\pi} , \quad (7)$$

which is independent of the black hole mass.

Moreover, the thermodynamical nature of this solution had been investigated by Frolov [7], and shown to satisfy the three laws of black holes mechanics in this two dimensional model as in “standard” four dimensional black hole physics. In particular, we can derive an expression of the black hole entropy S_{BH} using Wald’s Noether charge method [9] or York’s method [10];

$$S_{BH} = 2e^{-2\phi_H} , \quad (8)$$

and can show that S_{BH} is a nondecreasing function of time in classical level.

Finally, we must note several points about our following calculation. CGHS [6] considered the semiclassical correction to this classical theory. In order to justify the semiclassical treatment, they assumed the number N of the scalar fields is very large. We perform our calculation with $N = 1$ for simplicity, but we can always get the result for N species by multiplying our result by N .

2.2 The GSL under the quasistationary approximation

Before examining whether the GSL holds in the CGHS model, we formulate a precise statement of the GSL in the quasistationary approximation.

As stated in the introduction, we foliate our black hole spacetime by spacelike time slices that are across the event horizon but do not cross one another at the horizon. We take two adjacent time slices among them to consider a quasistationary process (to justify the assumption (i) at the introduction) and consider the change of total entropy between two time slices.

Under this situation satisfying the assumption (i), by making use of the first law of black hole mechanics ($\Delta S_{BH} = \Delta E_{BH}/T_{BH}$), we can rewrite the change of total entropy in terms of quantities of the matter fields alone.

$$\Delta S_{total} = \Delta S_M + \Delta S_{BH} \quad (9)$$

$$= \Delta S_M + \beta_{BH} \Delta E_{BH} \quad (10)$$

$$= \Delta S_M - \beta_{BH} \Delta E_M , \quad (11)$$

where we use the definition of the inverse temperature $\beta_{BH} \equiv T_{BH}^{-1}$ in the second line, and the energy conservation law ($\Delta E_{BH} = -\Delta E_M$) is used in the third line.

Thus, all we have to do are to calculate the change of energy and entropy of matter fields between two adjacent time slices in the eternal black hole background Eq.(4). These are obtained easily for massless conformal fields in two dimensional spacetime, using the energy-momentum tensor $T_{\mu\nu}^M$ and entanglement entropy $S_M = -\text{Tr}[\hat{\rho}_{out} \ln \hat{\rho}_{out}]$ of the matter fields. Where $\hat{\rho}_{out}$ is the density matrix of matter fields outside the horizon.

Further if we define the free energy of the matter fields F_M by $F_M \equiv E_M - \beta_{BH}^{-1} S_M$, we can write

$$\Delta S_{total} = -\beta_{BH} \Delta F_M , \quad (12)$$

and to prove the GSL in the quasistationary approximation is equivalent to show that the free energy F_M is a monotonically decreasing function in time. So we will be concerned with examining the change of free energy of matter fields in the fixed black hole background.

⁵We get this result from the definition of the quasi-local mass and this is double the Frolov’s definition. But this difference does not effect our result at all and we do not care about this difference.

3 The calculation of energy-momentum tensor

For two dimensional conformal fields, we can obtain the energy-momentum tensor $T_{\mu\nu}^M$ by the energy-momentum conservation law $\nabla_\nu T_M^{\mu\nu} = 0$ and the trace anomaly, which is for a scalar field,

$$T_M = \frac{R}{24\pi} . \quad (13)$$

The CGHS black hole, Eq.(4) can be rewritten as follows;

$$ds^2 = -\xi^2 dx^+ dx^- = -\Omega_{HH}^2 dU dV , \quad (14)$$

$$\Omega_{HH}^2 = \left[\frac{\pi M_{ADM}}{2\lambda} - \lambda^2 UV \right]^{-1} , \quad (15)$$

where $x^\pm = t \pm r_*$ and $r_* = \int dr/\xi$. The null coordinate (U, V) is a Kruskal like one which is regular at the horizon $\mathcal{H}^\pm$ given by $U = -e^{-\lambda x^-}/\lambda$ and $V = e^{\lambda x^+}/\lambda$.

Then, $T_{\mu\nu}^M$ for a conformal scalar field is written as [11]

$$\begin{aligned} T_{++}^M &= -\frac{\lambda M_{ADM}}{96} [\Omega_{HH}^2 - \lambda^2 UV \Omega_{HH}^4] + t_+|_{\mathcal{I}^-}(x^+) , \\ T_{--}^M &= -\frac{\lambda M_{ADM}}{96} [\Omega_{HH}^2 - \lambda^2 UV \Omega_{HH}^4] + t_-|_{\mathcal{I}^+}(x^-) , \\ T_{+-}^M &= \frac{\lambda M_{ADM}}{48} \lambda^2 UV \Omega_{HH}^4 , \end{aligned} \quad (16)$$

where $t_+|_{\mathcal{I}^-}$ ($t_-|_{\mathcal{I}^+}$) coincides the value of T_{++}^M at $\mathcal{I}^-$ (T_{--}^M at $\mathcal{I}^+$) and these are fixed by boundary conditions.

Due to the existence of the timelike Killing vector $\xi^\mu = (\partial/\partial t)^\mu$, the function $E_M = -\int_\Sigma d\Sigma^\mu T_{\mu\nu}^M \xi^\nu$ is a function of the boundary of Σ . We consider two adjacent time slices (Σ_0 and Σ_1), each of which starts from a point on the future horizon $\mathcal{H}^+$ and extends to the spatial infinity. The change of the energy between these time slices can be given by

$$\Delta E_M = \left(-\int_{\Sigma_1} d\Sigma^\mu T_{\mu\nu}^M \xi^\nu \right) - \left(-\int_{\Sigma_0} d\Sigma^\mu T_{\mu\nu}^M \xi^\nu \right) \quad (17)$$

$$= -\int_{\text{on } \mathcal{H}^+}^{x^+ + \Delta x^+ > x'^+ > x^+} dx'^+ T_{\mu\nu}^M \xi^\mu \xi^\nu . \quad (18)$$

Thus, we get finally

$$\frac{dE_M(x^+)}{dx^+} = -T_{++}^M|_{\mathcal{H}^+} . \quad (19)$$

Next, we evaluate the quantity $T_{++}^M|_{\mathcal{H}^+}$ for two vacuum states; the Hartle-Hawking one and the Unruh one. The Hartle-Hawking state is in equilibrium with the black hole and its energy-momentum tensor is given by

$$T_{++}^M = T_{--}^M = \frac{\lambda^2}{48\pi} [\Omega_{HH}^2 \lambda^2 UV]^2 . \quad (20)$$

And then, we obtain the change of the energy for the Hartle-Hawking state as

$$\frac{dE_{HH}(x^+)}{dx^+} = -T_{++}^M|_{\mathcal{H}^+} = 0 . \quad (21)$$

This result is expected one from the fact that the energy flow lines of the Hartle-Hawking state are along the orbits of the Killing vector because the Hartle-Hawking state is stationary state with respect to the Killing time.

On the other hand, the Unruh state represents, on the eternal black hole, a state which is in a gravitationally collapsing spacetime. Its energy-momentum tensor is written by

$$T_{++}^M = -\frac{\lambda M_{ADM}}{96} [\Omega_{HH}^2 - \lambda^2 UV \Omega_{HH}^4] , \quad (22)$$

$$T_{--}^M = \frac{\lambda^2}{48\pi} [\Omega_{HH}^2 \lambda^2 UV]^2 . \quad (23)$$

Since $T_{++}^M|_{\mathcal{H}^+} = -\lambda^2/(48\pi)$ at the future event horizon, we obtain the result

$$\frac{dE_U(x^+)}{dx^+} = +\frac{\lambda^2}{48\pi} = +\frac{\pi}{12} (T_{BH})^2 . \quad (24)$$

This represents the energy density of a right-moving one dimensional boson gas with the temperature T_{BH} and can be interpreted as the outgoing energy flux due to the Hawking radiation.

4 The calculation of the entanglement entropy

4.1 Entanglement entropy in flat spacetime

First, we consider the Minkowski vacuum of a massless conformal scalar field in flat two dimensional spacetime and calculate the entanglement entropy. Since two dimensional spacetime is conformally flat, we can easily generalize this result to any curved spacetime.

The metric of the flat spacetime is written by

$$ds^2 = -dU dV . \quad (25)$$

We want to compute the entanglement entropy of the Minkowski vacuum when the boundary of the accessible region is given by the point P_0 , (U_0, V_0) and the point P_1 , (U_1, V_1) . Making use of the Poincaré invariance of the Minkowski vacuum and the fact that the entanglement entropy is a function of the boundary of the accessible region, we can easily find that the entanglement entropy of the region between P_0 and P_1 is equal to one of the region between Q_0 , $(t, x) = (0, 0)$ and Q_1 , $(t, x) = (0, D)$ where t and x are the Minkowski coordinates, and $D^2 = |(V_1 - V_0)(U_1 - U_0)|$.

To proceed this calculation, we need to decompose the Minkowski vacuum into a superposition state of tensor products that are product of the states live in the Hilbert space associated with the accessible region and ones live in the Hilbert space associated with the inaccessible region. Such decomposition is given by Fiola, et.al. [12], following Unruh's calculation [13].

The result is appeared by

$$S = \frac{1}{6} [(\ln D - \ln \epsilon_0) + (\ln D - \ln \epsilon_1)] , \quad (26)$$

where ϵ_0 and ϵ_1 are the ultraviolet cutoff at P_0 (or Q_0) and P_1 (or Q_1), respectively. Since $\ln D - \ln \epsilon_0$ and $\ln D - \ln \epsilon_1$ are "volume" in terms of the Rindler coordinate between $Q'_0 = (0, \epsilon_0)$ and Q_1 , and between Q_0 and $Q'_1 = (0, D - \epsilon_1)$, respectively, we can interpret this result as sum of the thermal entropies of one dimensional photon gases with the temperature $T = 1/(2\pi)$ confined to the regions between Q'_0 and Q_1 , and between Q_0 and Q'_1 .

Next, we examine the change of entropy between two adjacent time slices. We are interested in the case that the Q_1 is fixed at $x = D$ and Q_0 moves along the null line $U = 0$. Then $Q_0 = (V/2, V/2)$ and

$$S = \frac{1}{6} \left[\left\{ \ln \left(D - \frac{V}{2} \right) - \ln \epsilon_0 \right\} + \left\{ \ln \left(D - \frac{V}{2} \right) - \ln \epsilon_1 \right\} \right] , \quad (27)$$

we get

$$\frac{dS}{dV} \Big|_{U=0} = -\frac{1}{6} \frac{1}{D - V/2} < 0 . \quad (28)$$

where the proper lengths $\epsilon_{0,1}$ is fixed.

This decrease of the entanglement entropy can be understood as a result of the decrease of the volume of the accessible region.

4.2 Entanglement entropy in curved spacetime

Next, we will generalize the result in the previous subsection to the case of curved spacetime. When we take the state as conformal vacuum, the result in flat spacetime can be used.

Since any two dimensional spacetime is conformally flat, we can write the metric as

$$ds^2 = \hat{\Omega}^2(-d\hat{t}^2 + d\hat{x}^2) \quad (29)$$

$$= -\hat{\Omega}^2 d\hat{U} d\hat{V} . \quad (30)$$

Assuming that we take the conformal vacuum associated with the time $\hat{t}$ and the accessible region is the interval $\hat{D}_0 \leq \hat{x} \leq \hat{D}_1$ on the constant time slice $\hat{t} = \text{constant}$, the entanglement entropy is given by

$$S_{\hat{\Omega}} = \frac{1}{6} \ln \left(\frac{\hat{D}_1 - \hat{D}_0}{\hat{\epsilon}_0} \right) + \frac{1}{6} \ln \left(\frac{\hat{D}_1 - \hat{D}_0}{\hat{\epsilon}_1} \right) \quad (31)$$

$$= \frac{1}{6} \ln \hat{\Omega}_0 + \frac{1}{6} \ln \hat{\Omega}_1 + \frac{1}{3} \ln (\hat{D}_1 - \hat{D}_0) - \frac{1}{3} \ln \epsilon , \quad (32)$$

In the second line, we rewrite the short distance cutoffs $\hat{\epsilon}_{0,1}$ in the unphysical spacetime in terms of proper lengths, that is $\epsilon_{0,1} = \hat{\Omega}_{0,1} \hat{\epsilon}_{0,1}$ and set $\epsilon_0 = \epsilon_1 = \epsilon$. We apply the above result to the Hartle-Hawking state and the Unruh one, and move $\hat{D}_0$ along the future horizon $\mathcal{H}^+$ with the proper length ϵ fixed.

Before doing this, we arrange our notation;

$$ds^2 = -\Omega_{HH}^2 dU dV = -\Omega_U^2 dx^+ dU , \quad (33)$$

$$\Omega_U^2 \equiv \Omega_{HH}^2 \frac{dV}{dx^+} = \Omega_{HH}^2 e^{\lambda x^+} . \quad (34)$$

For the Hartle-Hawking state, we substitute $\hat{\Omega} = \Omega_{HH}$ in Eq.(32) and use the notations $D_{0,1}^{HH}$ for $\hat{D}_{0,1}$. Using the relation $dD_0^{HH} = dV_0/2 = \exp(\lambda x_0^+) dx_0^+/2$ on $\mathcal{H}^+$, we get

$$\left. \frac{dS_{HH}}{dx_0^+} \right|_{\mathcal{H}^+} = -\frac{1}{12} \frac{\exp(\lambda x_0^+)}{D_1^{HH} - D_0^{HH}} < 0 . \quad (35)$$

This result is against our intuition and shows that the entropy S_{HH} is decreasing as time elapses, because of the decrease of the volume of the accessible region.

For the Unruh state, substituting $\hat{\Omega} = \Omega_U$, and using the notations $D_{0,1}^U$ for $\hat{D}_{0,1}$, we get

$$\left. \frac{dS_U}{dx_0^+} \right|_{\mathcal{H}^+} = \frac{\lambda}{12} - \frac{1}{12} \frac{1}{D_1^U - D_0^U} \quad (36)$$

$$= \frac{\pi}{6} T_{BH} - \frac{1}{12} \frac{1}{D_1^U - D_0^U} , \quad (37)$$

where we use the relation $dD_0^U = dx_0^+/2$ on $\mathcal{H}^+$.

As the interpretation of Eq.(24), the first term in Eq.(37) can be understood as the entropy density of one dimensional photon gas with temperature T_{BH} and can be interpreted as the entropy production rate due to the Hawking radiation. This term always gives a non-negative contribution to the change of the entropy. But as with the case of the Hartle-Hawking state, since the second term gives a negative contribution, the R.H.S. of Eq.(37) can not have a definite sign.

We will comment about fixing the proper length ϵ as the black hole evolves. We think it is natural to fix the short distance cutoff in the physical spacetime, because the first term of Eq.(32) has a natural interpretation as the entropy production by the Hawking radiation.

5 Our result

In this section, we summarize our result in terms of the change of the free energy. Using the relations $\beta_{BH} = 2\pi/\kappa = 2\pi/\lambda$ and the definition $F_M \equiv E_M - \beta_{BH}^{-1} S_M$, the change of the free energy is given by;

for the Hartle-Hawking state

$$\frac{dF_{HH}}{dx_0^+} = \frac{dE_{HH}}{dx_0^+} - \frac{\lambda}{2\pi} \frac{dS_{HH}}{dx_0^+} = + \frac{\lambda}{24\pi} \frac{\exp(\lambda x_0^+)}{D_1^{HH} - D_0^{HH}} > 0, \quad (38)$$

for the Unruh state

$$\frac{dF_U}{dx_0^+} = \frac{dE_U}{dx_0^+} - \frac{\lambda}{2\pi} \frac{dS_U}{dx_0^+} = - \frac{\lambda^2}{48\pi} + \frac{\lambda}{24\pi} \frac{1}{D_1^U - D_0^U}. \quad (39)$$

Where, if we take the limit that our accessible region extends to the spatial infinity, i.e., $D_1^{HH, U} \rightarrow \infty$, we obtain the result for the GSL. However, provided that we hold the accessible region finite, the above results show that the free energy (total entropy) does not necessarily decrease (increase) in time and the time derivative form of the GSL does not hold for the eternal black hole background. We recognized that this decrease of the total entropy was due to the decrease of the volume of the accessible region.

6 Summary and Discussion

In this paper, we examined the infinitesimal time change of the GSL for an eternal black hole in the CGHS model and proved that it does not hold for general time evolution in eternal black hole.

We considered the time derivative change of total entropy to satisfy the assumption (i) of Sec.1 and to avoid the Killing time slice which gives a trivial situation, we took the time slice which do not have time translation invariance. That is, we take the time slice that penetrates inside the black hole but avoids the singularity and determined the quantum state on it.

Under this time foliation, we examine its validity for two states; the Hartle-Hawking state and the Unruh state. We get the result that the free energy (total entropy) increases (decreases) as time evolves for the Hartle-Hawking state against our intuition, either situation is possible for the Unruh state.

The change of entropy for matter fields comes from two parts. One is associated with ultraviolet divergent term which represents short-distance correlations between fluctuations near the horizon, and the other associated with infrared divergent term which contains long-distance correlations between the modes far inside and far outside the horizon. The ultraviolet divergent part can be interpreted as the entropy production due to the Hawking radiation and this always gives non-negative contribution to the change of the total entropy. On the other hand, the infrared divergent part gives negative contribution due to the decreasing of volume outside the event horizon of black hole.

However, this behavior of the infrared divergent term is difficult to understand intuitively. Because it seems usual that the total entropy does not change for Hartle-Hawking state in eternal black hole background, which describe the thermal equilibrium situation between the black hole and the surrounding matter fields and it is believed that entropy production does not occur in this situation.

Two choice is possible for this problem: one is to redefine the total entropy to satisfy the GSL and interpret its procedure physically, and another is to recognize our result as a truth and think that the analogy between the ordinary thermodynamics and the black hole mechanics breaks down.

If we take the former opinion, a method to avoid the violation of the GSL is to subtract the infrared divergent part by some physical reason and define the new total entropy,

$$S_{total}^* = S_{total} - \frac{N}{6} \ln \left(\frac{\hat{D}}{\epsilon} \right), \quad (40)$$

where $\hat{D}$ is the volume of our accessible region in unphysical spacetime in which the conformal vacuum is defined and ϵ is a proper length of short-distance cutoff. If we subtract the infrared divergent part as above, we obtain the desired result; $dS_{HH}^* = 0$ for the Hartle-Hawking state and $dS_U^* > 0$ for the Unruh state. But the physical interpretation of this procedure is not clear ⁶.

⁶One possible choice is as follows; the result (32) indicates that there must exist a fundamental scale ϵ which plays the role of ultraviolet regulator and characterizes the size of the elementary state of the quantum gravitational system in phase space (Remember the ultraviolet catastrophe problem in classical field theory), while $\hat{D}$ and ϵ was originally introduced artificially. Thus we should fix the physical scale ϵ and drop the unphysical cutoff $\hat{D}$ which would not appear when we consider massive particles (Fixed mass m of the field enters the expression instead of D in this case). In fact, some people consider that the appearance of ultraviolet divergent term ϵ implies the necessity of introducing a fundamental cut-off scale l_P into the theory and that it is important from a information theoretical point of view.

At the last part of this paper, we will state here the relation between our work and previous ones.

Our result does not contradict the result of Frolov and Page [3]. Because they compare the total entropy of past null infinity $\mathcal{I}^-$ with that of future null infinity $\mathcal{I}^+$ for eternal black hole. In this situation, our result reproduce their result; $\bar{d}S_{HH} = 0, \bar{d}S_U = \frac{\pi}{3}T_{BH} > 0$, where $(\bar{d} \equiv S|_{\mathcal{I}^+} - S|_{\mathcal{I}^-})$. Note that our result reversed in white hole region.

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The entanglement entropy for the Gaussian states in terms of the correlation function

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Abstract

We give an expression of the entanglement entropy of a real scalar field with Gaussian states, in terms of the correlation function.

1 Introduction

Usually in field theories, we are interested in the S-matrix, effective potential and the Feynman Green function, which are, in a sense, quantities over the spacetime. Then, we concentrate into the input-output relation (such as S-matrix) or the static nature of a theory such as mass spectrum, rather than the dynamical aspects of the theory, and obtain a wealth of knowledge. However, in the case of no stable vacuum, physical quantities calculated by the standard technique are associated with the transition amplitude ($\langle out | \dots | in \rangle$), rather than expectation values ($\langle in | \dots | in \rangle$). Therefore, the complex numbers often appear and we must give those pathological complex numbers any physical interpretations.

In the cosmology, since we have dynamical background fields, we must understand just the dynamics of the theory. For example, investigation of the inflation models driven by scalar fields is to solve the dynamics of the inflaton.

Further, to understand statistical aspects of the nature, we introduce any coarse graining and need to calculate the reduced density matrix at any time, which is associated with the coarse graining. For example, the (von Neuman) entropy of the system in interest at any time is defined by the reduced density matrix at the time, as

$$S(t) = -\text{Tr} [\hat{\rho}_{red}(t) \ln \hat{\rho}_{red}(t)] \quad (1)$$

(we often call it the entanglement entropy).

In the black hole spacetime, we can access degrees of freedom only outside the event horizon. Therefore, it is natural to concentrate into the behavior of the degrees of freedom outside the event horizon, that is, the dynamics of them. And then, it may be useful to investigate the (non-)equilibrium thermodynamics of matters outside the horizon as the first step toward extension of the black hole thermodynamics.

For the cases which are essentially dynamical, it is useful to develop techniques that are suitable for investigating the dynamical and statistical aspects of the theory. As a method suitable for the purpose, we know the closed time path (CTP) formalism (for example, see reference [1]). By the CTP formalism, we can get the retarded Green function, the n-points correlation functions and so on. And then, we can calculate the entropy, e.t.c., in principle, since the knowledge of the correlation functions is equivalent to one of the density matrix. Since we already have various approximation methods, such as the perturbation technique, for calculating the correlation functions, it may be convenient to give explicit expressions of various physical quantities in terms of the correlation functions.

The purpose of this paper is to express the (entanglement) entropy in terms of the correlation functions. In this paper, we will fulfill the purpose for the case that physical states in investigation are Gaussian. Although the assumption of Gaussian states is, of course, artificial for non-linear systems, it is important to give procedures for artificial Gaussian states as the zeroth order approximation or as practice.

This paper is organized as follows: In Sec.2, we define the characteristic function of a density matrix and relate it to the correlation function, for a system with one degree of freedom. In Sec.3, we calculate

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the von Neuman entropy for a Gaussian state, making use of the fact that the thermal state of a linear system is a Gaussian state. In Sec.4, we extend the results for the system with one degree of freedom to one with many degrees of freedom and mention how to obtain the characteristic function of a reduced density matrix. In Sec.5, we apply the obtained results to a scalar field theory. Sec.6 is devoted to comments.

2 The characteristic and correlation function for Gaussian state

2.1 the characteristic and correlation function

For simplicity, we begin with a system with one degree of freedom, the phase space of which is extended by the coordinate q and the conjugate momentum p ($-\infty < q, p < \infty$). For example, we may imagine a harmonic oscillator,

$$H = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2. \quad (2)$$

When we quantize the system, observables become operators, which are distinguished by the caret “ $\hat{}$ ”, for instance, $\hat{q}$, $\hat{p}$ and so on.

A quantum state of the system in consideration at the time t is specified by the density matrix $\hat{\rho}(t)$ that is a hermitian operator and satisfies the properties

$$\text{Tr}\hat{\rho}(t) = 1, \quad (3)$$

$$S_{lin}(t) := \text{Tr}\hat{\rho}^2(t) \leq 1, \quad (4)$$

where we call S_{lin} the linear entropy and the equality of the latter equation holds in pure states. Eqs.(3) and (4) mean the normalization and positivity of probability, respectively.

We introduce the characteristic function (CF) of the state $\hat{\rho}(t)$, the information of which is equivalent to one of $\hat{\rho}$. The definition of the CF is given by

$$\rho(k, \Delta | t) := \text{Tr}[\exp\{+i(k\hat{q}_S + \Delta\hat{p}_S)\} \hat{\rho}(t)], \quad (5)$$

$$= \text{Tr}[\exp\{+i(k\hat{q}_H(t) + \Delta\hat{p}_H(t))\} \hat{\rho}(0)], \quad (6)$$

where the indices “S” and “H” mean that the operator with either index is in the Schrödinger or Heisenberg picture, respectively. The CF is related to the coordinates representation of the density matrix and the Wigner representation by

$$\rho(k, \Delta | t) = \int dq \rho(q + \Delta/2, q - \Delta/2 | t) \exp(+ikq), \quad (7)$$

$$= \int dq dp W(q, p | t) \exp[+i(kq + \Delta p)], \quad (8)$$

where

$$\rho(q + \Delta/2, q - \Delta/2 | t) := \langle q + \Delta/2 | \hat{\rho}(t) | q - \Delta/2 \rangle, \quad (9)$$

$$W(q, p | t) := \int \frac{d\Delta}{2\pi} \rho(q + \Delta/2, q - \Delta/2 | t) \exp(-i\Delta p). \quad (10)$$

Thus, the CF is nothing but the Fourier transformation of the density matrix, we can have all the information of the state in terms of the CF.

It is convenient to use the cumulant expansion of the CF that is the expansion of the logarithm of the CF with respect to k and Δ .

$$\begin{aligned} \ln \rho(k, \Delta | t) &= ik \langle \hat{q}_H(t) \rangle + i\Delta \langle \hat{p}_H(t) \rangle \\ &\quad - \frac{1}{2} (k \quad \Delta) \begin{pmatrix} C_q(t) & C_{qp}(t) \\ C_{qp}(t) & C_p(t) \end{pmatrix} \begin{pmatrix} k \\ \Delta \end{pmatrix} + O(k^3), O(\Delta^3), \end{aligned} \quad (11)$$

where $\langle \dots \rangle = \text{Tr}[\dots \hat{\rho}(0)]$. Using the relation between the coordinate and the momentum, $p = \hat{L}_t q$ (for example, the operator $\hat{L}_t$ for Eq.(2) is given by $m d/dt$), the cumulant functions at the second order are given by

$$C_q(t) := \langle \hat{q}_H^2(t) \rangle - \langle \hat{q}_H(t) \rangle^2 = \lim_{s \rightarrow t} G_C(t, s), \quad (12)$$

$$C_{qp}(t) := \frac{1}{2} \langle \{ \hat{q}_H(t), \hat{p}_H(t) \} \rangle - \langle \hat{q}_H(t) \rangle \langle \hat{p}_H(t) \rangle = \lim_{s \rightarrow t} \hat{L}_s G_C(t, s), \quad (13)$$

$$C_p(t) := \langle \hat{p}_H^2(t) \rangle - \langle \hat{p}_H(t) \rangle^2 = \lim_{s \rightarrow t} \hat{L}_t \hat{L}_s G_C(t, s), \quad (14)$$

where the correlation function G_C is defined by

$$G_C(t, s) := \frac{1}{2} \langle \{ \hat{q}_H(t), \hat{q}_H(s) \} \rangle - \langle \hat{q}_H(t) \rangle \langle \hat{q}_H(s) \rangle. \quad (15)$$

It is easy to check the above equations by using a derivative formula for any operator $\hat{X}$,

$$\delta(\exp \hat{X}) = \int_0^1 da \exp \left[(1-a)\hat{X} \right] \delta \hat{X} \exp \left(a\hat{X} \right). \quad (16)$$

The normalization and positivity conditions of the state, Eqs.(3) and (4) are rewritten, in terms of the CF, by

$$\text{Tr} \hat{\rho}(t) = \rho(k = 0, \Delta = 0 | t) = 1, \quad (17)$$

$$S_{lin}(t) = \int \frac{dk d\Delta}{2\pi} |\rho(k, \Delta | t)|^2 \leq 1. \quad (18)$$

2.2 Gaussian state

We call the states of which the CF has cumulant functions only up to the second order, Gaussian states. Therefore, the CF of a Gaussian state is written by

$$\ln \rho(k, \Delta | t) = -\frac{1}{2} \begin{pmatrix} k & \Delta \end{pmatrix} C(t) \begin{pmatrix} k \\ \Delta \end{pmatrix} + i \begin{pmatrix} k & \Delta \end{pmatrix} \begin{pmatrix} \langle \hat{q}_H(t) \rangle \\ \langle \hat{p}_H(t) \rangle \end{pmatrix}, \quad (19)$$

where the covariance matrix of the CF is given by the correlation function as

$$C(t) := \begin{pmatrix} C_q(t) & C_{qp}(t) \\ C_p(t) & C_{pp}(t) \end{pmatrix} = \lim_{i \rightarrow t} \begin{pmatrix} 1 & \hat{L}_i \\ \hat{L}_i & \hat{L}_i \hat{L}_i \end{pmatrix} G_C(t, s). \quad (20)$$

By Eq.(18), we have a constraint

$$\det [2C(t)] = 4 [C_q(t)C_p(t) - C_{qp}^2(t)] \geq 1, \quad (21)$$

which is nothing but the uncertainty principle [2].

We can easily obtain the coordinate representation of the Gaussian density matrix,

$$\begin{aligned} \rho(q + \Delta/2, q - \Delta/2 | t) &= \sqrt{\frac{A(t)}{2\pi}} \exp [i\Delta \langle \hat{p}_H(t) \rangle] \\ &\quad -\frac{1}{2} \begin{pmatrix} q - \langle \hat{q}_H(t) \rangle & \Delta \end{pmatrix} \begin{pmatrix} A(t) & B(t) \\ B(t) & A(t)D^2(t)/4 \end{pmatrix} \begin{pmatrix} q - \langle \hat{q}_H(t) \rangle \\ \Delta \end{pmatrix} \end{aligned} \quad (22)$$

$$A(t) := [C_q(t)]^{-1}, \quad B(t) := -i[C_q(t)]^{-1} C_{qp}(t), \quad D(t) := 2\sqrt{C_q(t)C_p(t) - C_{qp}^2(t)}. \quad (23)$$

In addition, the Wigner function is given by

$$W(q, p | t) = \sqrt{\frac{\det C^{-1}(t)}{(2\pi)^2}} \exp \left[-\frac{1}{2} \begin{pmatrix} q - \langle \hat{q}_H(t) \rangle & p - \langle \hat{p}_H(t) \rangle \end{pmatrix} C^{-1}(t) \begin{pmatrix} q - \langle \hat{q}_H(t) \rangle \\ p - \langle \hat{p}_H(t) \rangle \end{pmatrix} \right]. \quad (24)$$

Thus, as we know from Eqs.(12)-(14), the covariance matrix shows the extension of the Wigner function in the phase space.

3 The von Neuman entropy for the Gaussian states

We want to calculate the von Neuman entropy for Gaussian states, where the von Neuman entropy is defined by

$$S(t) := -\text{Tr}[\hat{\rho}(t) \ln \hat{\rho}(t)] \geq 0, \quad (25)$$

and the last equality holds in pure states. Hereafter, we call the von Neuman entropy merely the entropy.

First, we mention that the entropy is independent of the cumulant functions at the first order, $\langle \hat{q}_H(t) \rangle$ and $\langle \hat{p}_H(t) \rangle$. This fact shows that the entropy is related to the extension of the distribution in the phase space.

To show that, we introduce a new function without the first cumulant functions as

$$\rho'(k, \Delta) := \rho(k, \Delta | t) \exp[-i(k \langle \hat{q}_H(t) \rangle + \Delta \langle \hat{p}_H(t) \rangle)], \quad (26)$$

which has the same cumulant functions that the CF has, except for the first order cumulant functions. The new function $\rho'(k, \Delta)$ is the CF of the new density matrix $\hat{\rho}'$ that is related to the old density matrix unitarily as

$$\hat{\rho}' = \hat{U} \hat{\rho}(t) \hat{U}^\dagger, \quad (27)$$

$$\hat{U} := \exp(-i \langle \hat{p}_H(t) \rangle \hat{q}_S) \exp(i \langle \hat{q}_H(t) \rangle \hat{p}_S). \quad (28)$$

Because the entropy is invariant with respect to unitary transformations, the new density matrix has the same entropy that the old one has, that is, the entropy is independent of the first order cumulant functions. Thus, we will often neglect the first order cumulant functions below.

Since Eq.(25) includes the logarithm of the operator, we need to diagonalize the density operator $\hat{\rho}(t)$. To bypass this procedure temporarily, we make use of the fact that a thermal state of a linear system is a Gaussian state. The thermal state $\hat{\rho}_\beta = \exp(-\beta \hat{H})/Z(\beta)$ with the inverse temperature β of a linear system, of which Hamiltonian is given by

$$\hat{H} = \frac{1}{2m} (p - fq)^2 + \frac{1}{2} m \omega^2 q^2, \quad (29)$$

becomes in the coordinate representation as

$$\rho_\beta(q + \Delta/2, q - \Delta/2) = \sqrt{\frac{A_\beta}{2\pi}} \exp\left[-\frac{1}{2} \begin{pmatrix} q & \Delta \end{pmatrix} \begin{pmatrix} A_\beta & B_\beta \\ B_\beta & A_\beta D_\beta^2/4 \end{pmatrix} \begin{pmatrix} q \\ \Delta \end{pmatrix}\right], \quad (30)$$

$$A_\beta = 2m\omega \tanh\left(\frac{\beta\omega}{2}\right), \quad B_\beta = -if, \quad D_\beta = \coth\left(\frac{\beta\omega}{2}\right). \quad (31)$$

And then, from the canonical partition function,

$$Z(\beta) = \left\{ 2 \sinh\left(\frac{\beta\omega}{2}\right) \right\}^{-1}, \quad (32)$$

we can obtain the entropy by means of the standard method

$$\begin{aligned} S_\beta &= -\text{Tr}[\hat{\rho}_\beta \ln \hat{\rho}_\beta] = \beta^2 \frac{\partial}{\partial \beta} \left(-\frac{1}{\beta} \ln Z(\beta) \right) \\ &= \frac{D_\beta + 1}{2} \ln\left(\frac{D_\beta + 1}{2}\right) - \frac{D_\beta - 1}{2} \ln\left(\frac{D_\beta - 1}{2}\right). \end{aligned} \quad (33)$$

Therefore, we can obtain the entropy of the Gaussian state Eq.(19) as

$$S(t) = \frac{D(t) + 1}{2} \ln\left(\frac{D(t) + 1}{2}\right) - \frac{D(t) - 1}{2} \ln\left(\frac{D(t) - 1}{2}\right), \quad (34)$$

where $D(t)$ is given by Eq.(23).

4 The CF of reduced density matrix

4.1 the CF of a system with many degrees of freedom

We can easily extend the results obtained in the previous sections to a system with N degrees of freedom, which is specified by the coordinate and momentum variables q^μ and p_μ ($\mu = 1, 2, \dots, N$).

Correspondingly introducing $\vec{k} = (k_\mu)$ and $\vec{\Delta} = (\Delta^\nu)$, we give the CF of the system with N degrees of freedom as

$$\begin{aligned} \rho(\vec{k}, \vec{\Delta} | t) &:= \text{Tr} [\exp \{ +i (k_\mu \hat{q}_\mu^S + \Delta^\nu \hat{p}_{\nu S}) \} \hat{\rho}(t)] , \\ &= \text{Tr} [\exp \{ +i (k_\mu \hat{q}_\mu^H(t) + \Delta^\nu \hat{p}_{\nu H}(t)) \} \hat{\rho}(0)] . \end{aligned} \quad (35)$$

The CF of a Gaussian state is written by

$$\rho(\vec{k}, \vec{\Delta} | t) := \exp \left[-\frac{1}{2} (k_\mu \quad \Delta^\nu) C(t) \begin{pmatrix} k_\mu \\ \Delta^\nu \end{pmatrix} + i (k_\mu \quad \Delta^\nu) \left(\begin{pmatrix} \hat{q}_\mu^H(t) \\ \hat{p}_\nu^H(t) \end{pmatrix} \right) \right] , \quad (37)$$

where

$$\begin{aligned} C(t) &:= \begin{pmatrix} [C_q(t)]^{\mu\nu} & [C_{qp}(t)]^\mu_\nu \\ [C_{qp}(t)]^\nu_\mu & [C_p(t)]_{\mu\nu} \end{pmatrix} , \\ &= \lim_{s \rightarrow t} \begin{pmatrix} 1 & \hat{L}_s \\ \hat{L}_t & \hat{L}_t \hat{L}_s \end{pmatrix} \mathcal{G}_C(t, s) . \end{aligned} \quad (38)$$

Similarly to one dimensional case, the operator $\hat{L}_t$ represents the relation between the coordinates and momentums as

$$p_\mu = \left[\hat{L}_t \right]_{\mu\nu} q^\nu , \quad (40)$$

and the correlation function $\mathcal{G}_C$ are given by

$$[\mathcal{G}_C(t, s)]^{\mu\nu} = \frac{1}{2} \langle \{ \hat{q}_\mu^H(t), \hat{q}_\nu^H(s) \} \rangle - \langle \hat{q}_\mu^H(t) \rangle \langle \hat{q}_\nu^H(s) \rangle . \quad (41)$$

The entropy of the system with N degrees of freedom is also given by the equation similar to Eq(34),

$$S(t) = \text{tr} \left[\frac{\mathcal{D}(t) + 1}{2} \ln \left(\frac{\mathcal{D}(t) + 1}{2} \right) - \frac{\mathcal{D}(t) - 1}{2} \ln \left(\frac{\mathcal{D}(t) - 1}{2} \right) \right] , \quad (42)$$

$$\mathcal{D}(t) := 2 \left[C_q(t) C_p(t) - C_q(t) {}^T C_{qp}(t) C_q^{-1}(t) C_{qp}(t) \right]^{1/2} , \quad (43)$$

where $\mathcal{D}$ is $N \times N$ -matrix and tr is trace operation with respect to the matrix. Since $\mathcal{D}$ is a matrix, we need to diagonalize $\mathcal{D}$ in order to calculate the entropy.

4.2 the CF of reduced density matrix

We find how to get the reduced CF of reduced density matrix from the whole CF. For simplicity, we consider a system with two degrees of freedom.

Assuming that the system 1 is accessible for a observer and the system 2 is inaccessible, the available information for the observer is included in the reduced density matrix

$$\hat{\rho}_1(t) := \text{Tr}_2 [\hat{\rho}(t)] . \quad (44)$$

Then we calculate the whole CF with arguments $k_2 = \Delta^2 = 0$,

$$\begin{aligned} &\rho(k_1, \Delta^1 ; k_2 = \Delta^2 = 0 | t) \\ &= \text{Tr} \left[\exp \{ i (k_1 \hat{q}_S^1 + \Delta^1 \hat{p}_{1S}) \} \hat{\rho}(t) \right] \\ &= \text{Tr}_1 \left[\exp \{ i (k_1 \hat{q}_S^1 + \Delta^1 \hat{p}_{1S}) \} \text{Tr}_2 (\hat{\rho}(t)) \right] \\ &= \rho_1(k_1, \Delta^1 | t) . \end{aligned} \quad (45)$$

Thus we can easily obtain the reduced CF by setting the arguments of the inaccessible degrees of freedom in the whole CF vanishing.

Given the whole CF, we calculate the entropy of the reduced density matrix as follows: (1) in the whole CF, put the arguments of the inaccessible degrees of freedom zero, (2) read out the reduced covariance matrices and (3) follow the procedures Eqs.(42) and (43).

5 Extension to the field theory

We consider a real scalar field $\phi(x)$ in $(D + 1)$ -dimensional spacetime. Its conjugate momentum is given by

$$\pi(x) = \hat{L}_z \phi(x) = \sqrt{\hbar(x)} n^\mu \nabla_\mu \phi(x), \quad (46)$$

where n^μ is the future directed normal vector of the spatial hypersurface Σ .

We regard the scalar field as infinite degrees of freedom system which has one degree of freedom at each spatial point. Then, by the relation $\mu \rightarrow \vec{x}$ and replacing $\hat{q}_S^\mu, \hat{q}_H^\mu(t), k_\mu, \dots$ with $\hat{\phi}_S(\vec{x}), \hat{\phi}_H(\vec{x}|t) = \hat{\phi}_H(x)$, $k(\vec{x})$ and so on, the whole CF for a Gaussian state of the scalar field is written by

$$\ln \rho[k, \Delta|t] = i \left(k \quad \Delta \right) \circ \left(\left\langle \hat{\phi}_H(t) \right\rangle \right) - \frac{1}{2} \left(k \quad \Delta \right) \circ C(t) \circ \begin{pmatrix} k \\ \Delta \end{pmatrix}, \quad (47)$$

$$C(\vec{x}, \vec{y}|t) := \begin{pmatrix} C_\phi(\vec{x}, \vec{y}|t) & C_{\phi\pi}(\vec{x}, \vec{y}|t) \\ C_\pi(\vec{x}, \vec{y}|t) & \hat{L}_z \end{pmatrix} = \lim_{t_x, t_y \rightarrow t} \begin{pmatrix} 1 & \hat{L}_z^x \\ \hat{L}_x & \hat{L}_x \hat{L}_y \end{pmatrix} \mathcal{G}_C(x, y), \quad (48)$$

where $T\mathcal{A}(\vec{x}, \vec{y}|t) = \mathcal{A}(\vec{y}, \vec{x}|t)$, and for $f(\vec{x}|t), g(\vec{x}|t), \mathcal{A}(\vec{x}, \vec{y}|t)$ and $\mathcal{B}(\vec{x}, \vec{y}|t)$,

$$[f \circ g](t) = \int_\Sigma d^D x f(\vec{x}|t) g(\vec{x}|t), \quad (49)$$

$$[\mathcal{A} \circ f](\vec{x}|t) = \int_\Sigma d^D y \mathcal{A}(\vec{x}, \vec{y}|t) f(\vec{y}|t), \quad (50)$$

$$[\mathcal{A} \circ \mathcal{B}](\vec{x}, \vec{y}|t) = \int_\Sigma d^D z \mathcal{A}(\vec{x}, \vec{z}|t) \mathcal{B}(\vec{z}, \vec{y}|t), \quad (51)$$

and so on.

The correlation function is given by

$$\mathcal{G}_C(x, y) := \frac{1}{2} \left\langle \left\{ \hat{\phi}_H(x), \hat{\phi}_H(y) \right\} \right\rangle - \left\langle \hat{\phi}_H(x) \right\rangle \left\langle \hat{\phi}_H(y) \right\rangle, \quad (52)$$

and for $\langle \hat{\phi}_H \rangle = 0$ as usual, it becomes $\mathcal{G}_C = G^{(1)}/2$.

Similarly to Eqs.(42) and (43), the entropy is calculated by

$$S(t) = \text{tr} \left[\frac{\mathcal{D}(t)+1}{2} \ln \left(\frac{\mathcal{D}(t)+1}{2} \right) - \frac{\mathcal{D}(t)-1}{2} \ln \left(\frac{\mathcal{D}(t)-1}{2} \right) \right], \quad (53)$$

$$\mathcal{D}(\vec{x}, \vec{y}|t) := 2 \left[C_\phi \circ C_\pi - C_\phi \circ T C_{\phi\pi} \circ C_\phi^{-1} \circ C_{\phi\pi} \right]^{1/2}(\vec{x}, \vec{y}|t). \quad (54)$$

When we are accessible only to the restricted region Ω on Σ , the reduced CF for us is obtained by restrict the supports of $k(\vec{x})$ and $\Delta(\vec{x})$ into Ω . And then, in various calculations such as the entanglement entropy, what to do is only to replace the range of integration in “inner product” $\circ$ with the accessible region Ω .

6 Comments

Eqs.(48), (52)-(54) are main results. These only give a formal expression of the (entanglement) entropy, and we need to diagonalize the infinite dimensional matrix $\mathcal{D}$ in order to explicit calculations of

the entropy. In general, to implement that is formidable task and few available example are known [3]. The development of explicit diagonalization or any approximations is further investigation.

At last, it is worth to make a few comments. Any coherent state has the same (entanglement) entropy that the original vacuum has, provided that the accessible regions for both are same. A coherent state is defined by

$$| \alpha \rangle := \hat{U}_\alpha | 0_a \rangle , \quad \hat{U}_\alpha := \exp \left[\sum_k \left(\alpha_k \hat{a}_k^\dagger - \alpha_k^* \hat{a}_k \right) \right] , \quad (55)$$

where $\hat{a}_k | 0_a \rangle = 0$. And then, by $\hat{U}_\alpha^\dagger \hat{\phi}_H(x) \hat{U}_\alpha = \hat{\phi}_H(x) + \sum_k [\alpha_k u_k(x) + c.c.]$ where $u_k(x)$ is mode function associated with $\hat{a}_k$, the CF of the coherent state has the same cumulant functions that the CF of the vacuum has, except for the first order cumulant functions. Because the first order cumulant functions do not influence on the entropy, as commented in Sec.3, the both have same entanglement entropy.

On the other hand, the entanglement entropy of a squeezed vacuum is generally different from one of the original vacuum, though the entropy of the whole region is same. As $\hat{\phi}_H(x) = \sum_k [v_k(x) \hat{b}_k + h.c.]$, $\hat{b}_k | 0_b \rangle = 0$ and $| 0_b \rangle = \hat{U} | 0_a \rangle$, the reduced CF of b-vacuum for the accessible region to be Ω is given by

$$\rho_b[k, \Delta; t] = \text{Tr} \left[\exp \left(ik \circ \hat{\phi}_H + i\Delta \circ \hat{\pi}_H \right) | 0_b \rangle \langle 0_b | \right] , \quad (56)$$

$$= \text{Tr} \left[\hat{U}^\dagger \exp \left(ik \circ \hat{\phi}_H + i\Delta \circ \hat{\pi}_H \right) \hat{U} | 0_a \rangle \langle 0_a | \right] , \quad (57)$$

$$=: \text{Tr} [\exp (ik \circ \hat{\phi}_H + i\Delta \circ \hat{\pi}_H^\varphi) | 0_a \rangle \langle 0_a |] , \quad (58)$$

where $\circ$ is inner product over the accessible region Ω and

$$\hat{\varphi}_H(x) := \hat{U}^\dagger \hat{\phi}_H(x) \hat{U} = \left[J_2 \circ \hat{\phi}_H - J_1 \circ \hat{\pi}_H \right] (x) , \quad (59)$$

$$J_1(x, y) := i \sum_k [v_k(x) u_k(y) - v_k^*(x) u_k^*(y)] , \quad (60)$$

$$J_2(x, y) := \hat{\mathcal{L}}_y J_1(x, y) . \quad (61)$$

This shows that the entanglement entropy of the accessible degrees of freedom $\hat{\phi}_H(\vec{x} | t)$ ($\vec{x} \in \Omega$), e.t.c. for b-vacuum is equivalent to one of the accessible degrees of freedom $\hat{\varphi}_H(\vec{x} | t)$ ($\vec{x} \in \Omega$), e.t.c. for a-vacuum. Because $\hat{\varphi}_H$ is related to the original variables $\hat{\phi}_H$ and $\hat{\pi}_H$ by non-local manner, the division of the whole Hilbert space by which we can access the degrees of freedom $\hat{\varphi}_H(\vec{x} | t)$ ($\vec{x} \in \Omega$) is different from the original division. In general, different divisions of the whole Hilbert space give different entanglement entropies for the same state.

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Averaging Procedure in Terms of the Spectral Representation

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Abstract

We study the averaging problem in cosmology in connection with the problem of introducing a suitable measure of closeness between spaces. First, we introduce a space of all spaces equipped with the spectral distance, which is a measure of closeness between spaces defined by means of the spectra of the Laplacian. This metric space will be a suitable arena for spacetime physics. We discuss on the averaging problem with the help of this space.

1 Introduction

In cosmology we try to get a comprehensive picture of our universe as a function of time. The real structure of the universe is, however, too complicated to allow us its complete description. It is also doubtful how meaningful such a complete, too detailed description would be, even if possible. Rather, from the viewpoint of cosmology, it is more important to grasp the coarse grained picture of the universe as a whole. Therefore it is inevitable to prepare some 'format of recognition', namely cosmological models, to construct cosmology.

In this way some averaging procedure, or mapping from the real space to a model space ('model-fitting'), plays one of the essential roles in cosmology. Now the problem is that such an averaging is not a trivial procedure and its property has not been well understood. The typical difficulty is that it is not easy to analyze the influence of the averaging on the effective dynamics of the universe. To state symbolically, the mapping from reality to a model and the dynamical evolution described by the Einstein equations are not commutable with each other [1].

Hence the averaging procedure itself should be analyzed and understood quantitatively; otherwise we would make serious mistakes in the interpretations of the observational data: Typically mistakes in the determination of the cosmological parameters and matter contents in the universe[1].

The aim of this brief report is to investigate the averaging procedure from a new angle, namely from the viewpoint of the spectral representation. Here the *spectral representation* means a scheme of representing local and global geometrical structures of space in terms of the spectra of some elliptic operator defined on the space, e.g. the Laplacian [2].

2 The spectral distance between spaces

In the process of averaging or model-fitting procedure, a suitable model geometry should be assigned to the real geometry. Independent of the details of such a procedure, this process includes an approximation of the global as well as local geometry of the real space. Now the concept of 'neighborhood' is essential in talking about whatsoever related to 'approximation'. As the first step for analyzing the averaging problem, thus, we face with the problem of defining 'closeness' between spaces [2, 3]. Once such a concept of closeness is established, then, the problem of finding the best-fitted model of reality can be reformulated as the problem of picking up the 'closest' model for the real space among a set of all model spaces $\{\textit{models}\}$.

Now let us first recollect the attempts in defining 'closeness' between spaces in the field of Riemannian geometry.

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We have the Gromov-Hausdorff distance between two metric spaces X and Y , $d_{GH}(X, Y)$ [4]. It is defined by means of isometric embeddings of X and Y into another metric space Z . Roughly speaking, we search for the optimal embeddings such that X and Y overlap 'as close as possible' in Z . Then it is possible to define a distance between X and Y by means of the metric of Z in a rather standard manner.

Another concept related to 'closeness' is a norm of Riemannian manifolds due to Petersen [5]. This norm measures how close a Riemannian manifold is to the Euclidean space, but it does not provide us with the distance between two manifolds.

Though these tools play a central role in the convergence theory of Riemannian geometry [5], it seems too abstract and complicated to be directly applied to spacetime physics.

Then, we focus on another measure of closeness between spaces suitable for spacetime physics. We can introduce the measure of closeness in terms of the spectra of some elliptic operator [2].

Let $\mathcal{G} = (\Sigma, h)$ and $\mathcal{G}' = (\Sigma', h')$ be two compact Riemannian manifolds without boundaries.² Setting the eigenvalue problem on each manifold

$$\Delta f = -\lambda f$$

with the Dirichlet boundary condition, for instance, the set of eigenvalues (numbered in increasing order) is obtained; $\{\lambda_m\}_{m=0}^{\infty}$ for $\mathcal{G}$ and $\{\lambda'_n\}_{n=0}^{\infty}$ for $\mathcal{G}'$. Let us call such a numbered set of eigenvalues *spectra* for brevity. Now we can introduce a measure of closeness between $\mathcal{G}$ and $\mathcal{G}'$ as

$$d_N(\mathcal{G}, \mathcal{G}') = \sum_{n=1}^N f\left(\frac{\lambda_n}{\lambda'_n}\right),$$

where f is a suitably chosen function which is semi-positive definite and increasing. We note that λ_n with the larger number n reflects the smaller scale behavior of the geometry. Therefore the cut-off number N determines up to which scale $\mathcal{G}$ and $\mathcal{G}'$ are compared. In other words the differences of the two geometries of order $o(1/\sqrt{\lambda_N})$ are neglected. Therefore the spectral measure of closeness is suitable for analyzing the scale-dependent behavior of geometry. Here we remember that such a quantitative description of the scale-dependent geometry is very essential for developing spacetime physics [3].

Now there are several possibilities for the choice of f . Its detailed form is decided according to which features of spatial geometry we are interested in. Among several possibilities, however, there is one especially interesting choice for f , which is $f_1(x) = \frac{1}{2} \ln \frac{1}{2}(\sqrt{x} + 1/\sqrt{x})$. In this choice³ we get [2]

$$d_N(\mathcal{G}, \mathcal{G}') = \frac{1}{2} \sum_{n=1}^N \ln \frac{1}{2} \left(\sqrt{\frac{\lambda_n}{\lambda'_n}} + \sqrt{\frac{\lambda'_n}{\lambda_n}} \right).$$

This measure of closeness⁴ clearly satisfies

(I) $d_N(\mathcal{G}, \mathcal{G}') \geq 0$, and $d_N(\mathcal{G}, \mathcal{G}') = 0 \Leftrightarrow \mathcal{G} \sim \mathcal{G}'$, where $\sim$ means equivalent in the sense of isospectral manifolds.

(II) $d_N(\mathcal{G}, \mathcal{G}') = d_N(\mathcal{G}', \mathcal{G})$

but it does not satisfy the triangle inequality [2],

(III) $d_N(\mathcal{G}, \mathcal{G}') + d_N(\mathcal{G}', \mathcal{G}'') \not\geq d_N(\mathcal{G}, \mathcal{G}'')$.

However, it turns out that the breakdown of the triangle inequality is only a mild one in the following sense [6]: A universal positive constant a can be chosen such that $\bar{d}'_N(\mathcal{G}, \mathcal{G}') := d_N(\mathcal{G}, \mathcal{G}') + a$ satisfies the triangle inequality. Here a is independent of $\mathcal{G}$ and $\mathcal{G}'$.

Indeed there is another option for f as $f_2(x) := \log_2(\sqrt{x} + 1/\sqrt{x})^{\frac{2c}{\sqrt{x}}}$. Here c is an arbitrary positive constant. Note that $f_2 = \frac{2c}{N \ln 2} (f_1 + \frac{1}{2} \ln 2)$. Then a modified measure of closeness becomes

$$\bar{d}_N(\mathcal{G}, \mathcal{G}') = \sum_{n=1}^N \log_2 \left(\sqrt{\frac{\lambda_n}{\lambda'_n}} + \sqrt{\frac{\lambda'_n}{\lambda_n}} \right)^{\frac{2c}{\sqrt{x}}}.$$

²They are spaces and not spacetimes.

³For the derivation and physical interpretation of this measure, see Ref.[2].

⁴We will see below that it is justified to treat d_N as a distance. Until then, however, let us call it a 'measure of closeness' for safety.

The new measure $\bar{d}_N$ satisfies

(I') $\bar{d}_N(\mathcal{G}, \mathcal{G}') \geq c$, and $\bar{d}_N(\mathcal{G}, \mathcal{G}') = c \Leftrightarrow \mathcal{G} \sim \mathcal{G}'$, where $\sim$ means equivalent in the sense of isospectral manifolds.

(II') $\bar{d}_N(\mathcal{G}, \mathcal{G}') = \bar{d}_N(\mathcal{G}', \mathcal{G})$

(III') $\bar{d}_N(\mathcal{G}, \mathcal{G}') + \bar{d}_N(\mathcal{G}', \mathcal{G}'') \geq \bar{d}_N(\mathcal{G}, \mathcal{G}'')$.

In this case the triangle inequality holds ((III')) but the lower bound of $\bar{d}_N$ is $c(> 0)$ and not zero ((I')). Note that $\bar{d}_N = \frac{2c}{N \ln 2} d_N + c$. Thus, the precise form of (III) is

$$d_N(\mathcal{G}, \mathcal{G}') + d_N(\mathcal{G}', \mathcal{G}'') + \frac{N}{2} \ln 2 \geq d_N(\mathcal{G}, \mathcal{G}'') .$$

Looking at (I)-(III) and (I')-(III'), we see that the measures of closeness introduced here are generalizations of an ordinary distance.⁵

Now let *Riem* be a space of all $D-1$ -dimensional, compact Riemannian manifolds without boundaries.⁶ Then we can consider a space $\mathcal{G}_N := (Riem, d_N) / \sim$, where $/ \sim$ implies the identification of isospectral manifolds. Here we understand that a point-set topology is introduced through the open balls defined by d_N . It is not a simple task to investigate the mathematical structures of $\mathcal{G}_N$ because of the property (III). Significantly enough, however, it turns out that the topology of $\mathcal{G}_N$ is equivalent to the topology of $(Riem, \bar{d}_N) / \sim$; Here $\bar{d}_N$ is another modification of d_N which fully satisfies axioms of a distance[6]. (Thus one can investigate a metric space $(Riem, \bar{d}_N) / \sim$ instead of studying $\mathcal{G}_N$ directly.) In this way it is justified to treat $\mathcal{G}_N$ as a metric space. Thus it turns out that $\mathcal{G}_N$ is not only a Hausdorff space but also a normal space.⁷ Furthermore it seems to satisfy the second countability axiom also. These observations mean that one can construct analysis theory on $\mathcal{G}_N$, which makes $\mathcal{G}_N$ a basic arena for spacetime physics [6].

Based on these arguments, we call from now on d_N a *spectral distance*.

3 Averaging procedure in the spectral representation

There are several subtle problems to be clarified in the averaging procedure in cosmology. For instance,

- (a) How to incorporate the spatial diffeomorphism invariance of the theory.
- (b) How to incorporate the scale-dependent geometrical structures of a space [3].
- (c) How to incorporate the apparatus dependence of the observational data
- (d) How to fix the ambiguity in the choice of time-slicing of a given spacetime.

One sees that the spectral representation provides a suitable language for formulating and analyzing these problems in the averaging procedure.

Let us now sketch how the averaging procedure looks from the viewpoint of the spectral representation.

Suppose we need to decide which model spacetime $(\mathcal{M}_*, g_*)$ among a set of model spacetimes $\{\text{models}\}$ most reflects 'reality', i.e. a given spacetime $(\mathcal{M}, g)$. Then this selection procedure can be formulated as to

Pick up $(\mathcal{M}_*, g_*) \in \{\text{models}\}$ s.t. $D_{A,A}((\mathcal{M}_*, g_*), (\mathcal{M}, g))$ gives the minimum.

⁵ It is desirable to construct the theory of a generalized metric space which is defined by generalized axioms of distance [6]:

- (a) $d(p, q) \geq 0$, and $d(p, q) = 0 \Leftrightarrow p = q$.
- (b) $d(p, q) = d(q, p)$.
- (c) $d(p, q) + d(q, r) + c \geq d(p, r)$, where $c > 0$ is a universal constant independent of p, q and r .

⁶ We let D be the dimension of spacetime.

⁷ For the basics of point set topology, see Ref. [7]

Here $D(\cdot, \cdot)$ is a suitable integral of the spectral distance between two spaces over a time interval, which will be described in more detail below; The suffix A is the label for an elliptic operator, symbolizing which apparatus is used in observation; The suffix Λ indicates the cut-off scale of the spectral distance. (Λ corresponds to the cut-off number N in the previous section.)

Now let us describe in more detail the model-fitting procedure mentioned above.

[1] *Choice of time-slicing*

For definiteness we consider a portion of a spacetime $(\mathcal{M}, g)$ bounded by two non-intersecting spatial sections Σ_0 and Σ_1 of $(\mathcal{M}, g)$. Let us denote this portion as $(\mathcal{M}, g)_{(0,1)}$ for brevity. ($(\mathcal{M}, g)$ is regarded as ‘reality’.) Let $Slice(\mathcal{M}, g)_{(0,1)}$ be a set of all possible way of slicing $(\mathcal{M}, g)_{(0,1)}$.

Now a slice $s \in Slice(\mathcal{M}, g)_{(0,1)}$ represents a parameterized set of spatial geometries $\{(\Sigma, h(\beta))\}_{0 \leq \beta \leq 1}$. Let $(\mathcal{M}, g)_{(0,1),s}$ denote this set for brevity. In the same way, let $(\mathcal{M}', g')_{(0,1),s'}$ be a portion of a model spacetime $(\mathcal{M}', g')$ bounded by two spatial sections Σ'_0 and Σ'_1 with a time-slicing $s' \in Slice(\mathcal{M}', g')_{(0,1)}$. Here $(\mathcal{M}', g')_{(0,1),s'}$ can be looked as a parameterized set of spatial geometries $\{(\Sigma', h'(\beta))\}_{0 \leq \beta \leq 1}$. Then $(\Sigma, h(\beta))$ and $(\Sigma', h'(\beta))$ can be compared by means of the spectral distance $d_{A,\Lambda}((\Sigma, h(\beta)), (\Sigma', h'(\beta)))$.

Now the most suitable time-slicing of $(\mathcal{M}, g)_{(0,1)}$ w.r.t. (with respect to) the model $(\mathcal{M}', g')_{(0,1),s'}$ can be formulated as

$s_0 \in Slice(\mathcal{M}, g)_{(0,1)}$ s.t.

$$D_{A,\Lambda}((\mathcal{M}, g)_{(0,1),s}, (\mathcal{M}', g')_{(0,1),s'}) \\ := \int_0^1 [d_{A,\Lambda}((\Sigma, h(\beta)), (\Sigma', h'(\beta)))] d\mu(\beta)$$

gives the minimum.

Here $\mu(\beta)$ is some suitable integral measure for β . In this way one time-slicing s_0 of $(\mathcal{M}, g)_{(0,1)}$ is picked up w.r.t. A (apparatus), Λ (scale) and the model $(\mathcal{M}', g')_{(0,1),s'}$.

[2] *Assignment of a model to reality*

We can continue the same procedure for every model spacetime $\in \{models\}$ to pick up the best-fitted model $(\mathcal{M}_*, g_*)_{(0,1)}$ and, w.r.t. it, the time-slicing s_* of $(\mathcal{M}, g)_{(0,1)}$. Then one can regard $(\mathcal{M}_*, g_*)_{(0,1)}$ to be the cosmological counterpart of $(\mathcal{M}, g)_{(0,1)}$ w.r.t. (A, Λ) . In this way the spectral representation naturally leads us to the concept of apparatus- and scale-dependent effective evolution of the universe.

As another variation of the above procedure it may be useful to make use of the conformal geometry. In the step [1], the geometrical information of $(\Sigma, h(\beta))$ can be decomposed into the information of the spatial volume and the information of the conformal geometry normalized as *spatial volume* = 1:

$$(\Sigma, h(\beta)) = (V(\beta), (\Sigma, \bar{h}(\beta))) ,$$

and in the same manner, $(\Sigma', h'(\beta)) = (V'(\beta), (\Sigma', \bar{h}'(\beta)))$. Thus instead of $D_{A,\Lambda}$ introduced above, it is possible to use, for instance,

$$\bar{D}_{A,\Lambda}((\mathcal{M}, g)_{(0,1),s}, (\mathcal{M}', g')_{(0,1),s'}) \\ := \int_0^1 [d_{A,\Lambda}((\Sigma, \bar{h}(\beta)), (\Sigma', \bar{h}'(\beta))) + (V(\beta) - V'(\beta))^2] d\mu(\beta) ,$$

and afterwards it goes in the same way.

4 Spectral distance between two similar universes

Now the spectral distance and the space $\mathcal{G}_N$ being at hand, we can step into the averaging problem or the model-fitting problem. As one of the easiest cases, we can investigate the case in which the reality $\mathcal{G}' = (\Sigma, h')$ and the model $\mathcal{G} = (\Sigma, h)$ are geometrically very similar to each other with the same global topology. (Note that we are talking about spatial geometries.) Let us estimate the spectral distance between them.

To state more precisely, we study the case in which

$$\delta h_{ab} := h'_{ab} - h_{ab}$$

can be treated as a tensor field on the model space $\mathcal{G}$. It is useful to decompose δh_{ab} into the trace-part and the trace-free part:

$$\delta h_{ab} = \phi h_{ab} + \tilde{\delta} h_{ab} \quad .$$

Furthermore we decompose the scalar function ϕ as $\phi = \sum_{m=0}^{\infty} \phi_m f_m$ where f_m is the m th eigenfunction of Δ on $\mathcal{G}$. Now one can show that

$$d_N(\mathcal{G}', \mathcal{G}) = \sum_{n=1}^N \left(\frac{\delta \lambda_n}{\lambda_n} \right)^2 + O(\delta h^4) \quad ,$$

where $\delta \lambda_n$ is the difference $\lambda'_n - \lambda_n$ neglecting $o(\delta h)$.

It can be shown that⁸

$$\frac{\delta \lambda_n}{\lambda_n} = - \left\{ \langle \tilde{\delta} h_{ab} \rangle_n + \sum_{m=0}^{\infty} \left(1 + \frac{D-3}{4} \frac{\lambda_m}{\lambda_n} \right) \phi_m \langle m \rangle_n \right\} \quad ,$$

where we have introduced the notations as

$$\begin{aligned} \langle A \rangle_n &:= \int_{\Sigma} f_n A f_n \sqrt{h} \quad , \\ \langle A_{ab} \rangle_n &:= \frac{1}{\lambda_n} \int_{\Sigma} \partial^a f_n A_{ab} \partial^b f_n \sqrt{h} \quad , \end{aligned}$$

and $\langle m \rangle_n := \langle f_m \rangle_n$.

Thus we can roughly estimate $d_N(\mathcal{G}', \mathcal{G})$ as

$$d_N(\mathcal{G}', \mathcal{G}) \sim \sum_{n=1}^N \langle \tilde{\delta} h_{ab} \rangle_n^2 + \sum_{n=1}^N \langle \phi \rangle_n \langle \tilde{\delta} h_{ab} \rangle_n + \sum_{n=1}^N \langle \phi \rangle_n^2 \quad .$$

Clearly we can investigate the behavior of $d_N(\mathcal{G}', \mathcal{G})$ in detail for the quantitative study of the validity of replacing $\mathcal{G}'$ by the model $\mathcal{G}$, though we do not go further here.

5 Spacetime picture from the viewpoint of the spectral representation

We discussed the averaging problem in cosmology in connection with the problem of defining a suitable distance between spaces. Then we focused on the spectral distance, which provides us with the concept of ‘neighborhood’ in the space of all spatial geometrical structures. In terms of the spectral distance, we sketched the model-fitting procedure quantitatively, which naturally led us to the concept of the apparatus- and scale-dependent effective evolution of the universe.

We can even foresee a new picture of ‘space’ suggested by the spectral representation. One may regard the geometrical information of a space as a collection of all spectra such as

$$Space = \bigcup_i (\mathcal{D}_i, \{\lambda_n^{(i)}\}_{n=0}^{\infty})$$

⁸We let D be the dimension of spacetime.

where $\mathcal{D}_i$ denotes an elliptic operator and $\{\lambda_n^{(i)}\}_{n=0}^\infty$ are its spectra. The index i runs over all possible elliptic operators. Any observation selects out a subclass of elliptic operators related to the apparatus so that only a small portion of the whole geometrical information is obtained by a single observation. For instance it is well known that the spectra of the Laplacian are not enough to fully distinguish Riemannian manifolds in some cases ('isospectral manifolds'). There is no surprise for such cases. A single observation extracts only a small portion of the whole geometrical information of a space. In some cases, such incomplete information may not be enough to distinguish manifolds. Then one only needs to resort to other elliptic operators to make further distinctions. It is also tempting to regard the spectral information more fundamental than the concept of (Riemannian) manifolds. Further developments of the scheme of the spectral representation will reveal whether such a viewpoint of spacetime is valid.

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Vorticity-density relation in second-order relativistic Lagrangian dynamics

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Abstract

We extensively formulate Lagrangian perturbation theory up to second order in the framework of general relativity, including vortical effect of the fluid flow. Our formulation is based on the so-called Lagrange condition, which allows us to solve equations for energy density and vorticity exactly. This means that the energy density and the vorticity can be written exactly in terms of the metric tensor in the Lagrange condition. We then solve the metric perturbatively up to second order, giving care to mode mixing at the second-order level. We also discuss non-locality which appears in the second-order perturbation solutions.

1 Introduction

It is significant to investigate gravitational instability in the expanding universe from the viewpoint of probing large-scale structure formation in the universe. A first step to study gravitational instability is linear perturbation theory of the Friedmann-Lemaître-Robertson-Walker (FLRW) universe [1]. In the linear perturbation theory, non-linearity of gravitation is neglected by definition. But non-linear effect becomes essential at later stage of the structure formation, where density fluctuation is as large as unity. Starting from such motivation, development of non-linear approximations of gravitational instability has been worked in recent years. One of the celebrated approximations is Zel'dovich approximation [2], which was proposed as an extrapolation of the linear theory using Lagrangian coordinates. The Zel'dovich approximation has been established in the framework of perturbation theory in Lagrangian description of hydrodynamics in Newtonian cosmology. Especially Buchert [3] showed that the Zel'dovich approximation can be regarded as a subclass of the first-order perturbative solutions in the framework. He also pointed out that in the Lagrangian description, vorticity of the fluid flow is coupled to the density enhancement and it will blow up as the density field develops singularities. This consequence cannot be obtained from usual linear perturbation theory because it only predicts that the vorticity decays as the universe expands [1]. (Note that this is not a contradiction because the Lagrangian perturbation theory reproduces results of usual linear perturbation theory in the limit of the linear perturbation.) This fact is confirmed by another work by Barrow and Saich [4]. They considered perturbations of an exact solution with one-dimensional inhomogeneity (which is often called Zel'dovich's pancake solution), and found that vortical perturbations couple to the background inhomogeneity so that the vorticity parallel to the pancake will grow as gravitational collapse occurs. Moreover, Sasaki and Kasai [5] solved the Lagrangian perturbation with the vortical effect up to third order, based on Buchert's approach.

General relativistic formulation of the Lagrangian perturbation theory has been recently attempted by several authors, including myself [6, 7, 8, 9, 10]. Such relativistic treatments may become important in understanding results of forthcoming cosmological observation such as Sloan Digital Sky Survey. In these approaches, however, the vortical effect did not taken into account. Reconsidering Buchert's formulation, it is natural to re-formulate relativistic Lagrangian perturbation theory to include the vortical effect. Asada and Kasai [11] have pointed out this possibility and made the re-formulation with first-order perturbative solutions in the framework of general relativity. In order to improve accuracy and clarify the interaction between various perturbation modes, it is necessary to develop higher-order Lagrangian perturbations.

In this article, we develop relativistic Lagrangian perturbation theory with the vortical effect up to second order. As in the Newtonian case, equations for energy density and vorticity are solved exactly. And then only the metric tensor should be solved perturbatively. We solve the metric perturbation up

to second order, giving attention to mode mixing that will arise at the second-order level. In addition, higher-order perturbation solutions are generally non-local. We then also mention non-locality of the second-order solutions obtained here.

2 Basic equations in relativistic Lagrangian approach

First, we summarize a general relativistic formulation of Lagrangian perturbation theory. We consider cosmological models which contain pressureless matter (dust) with energy density ρ and four-velocity u^μ . Then the energy-momentum tensor is

$$T^{\mu\nu} = \rho u^\mu u^\nu. \quad (1)$$

The conservation law $T^{\mu\nu}_{;\nu} = 0$ gives the continuity equation $(\rho u^\mu)_{;\mu} = 0$ and the geodesic equation $u^\mu_{;\nu} u^\nu = 0$. From the geodesic equation, we obtain an equation for vorticity in the form

$$\omega^\mu_{;\nu} u^\nu + u^\nu_{;\nu} \omega^\mu = u^\mu_{;\nu} \omega^\nu. \quad (2)$$

Here vorticity of the fluid flow is defined by

$$\omega^\mu \equiv \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} u_\nu u_{\lambda;\rho}, \quad (3)$$

where $\epsilon^{\mu\nu\lambda\rho}$ is the complete anti-symmetric tensor with $\epsilon^{0123} = 1/\sqrt{-g}$ ($g \equiv \det(g_{\mu\nu})$).

If we impose the so-called Lagrange condition $u^\mu = (1, 0, 0, 0)$ as a coordinate condition, the continuity equation and the vorticity equation become respectively

$$\dot{\rho} + \rho \frac{\dot{\sqrt{-g}}}{\sqrt{-g}} = 0, \quad \dot{\omega}^\mu + \omega^\mu \frac{\dot{\sqrt{-g}}}{\sqrt{-g}} = 0, \quad (4)$$

where an overdot ($\dot{}$) denotes $\partial/\partial t$. These equations can be solved exactly as

$$\rho = \rho_0(\mathbf{x}) \frac{\sqrt{-g_0(\mathbf{x})}}{\sqrt{-g(t, \mathbf{x})}}, \quad \omega^\mu = \omega_0^\mu(\mathbf{x}) \frac{\sqrt{-g_0(\mathbf{x})}}{\sqrt{-g(t, \mathbf{x})}}, \quad (5)$$

where a subscript 0 denotes an initial value of each quantity.

Next let us write down the Einstein equations in the Lagrange condition. We adopt the 3+1 formulation and decompose the metric tensor as

$$\begin{aligned} ds^2 = g_{\mu\nu} dx^\mu dx^\nu &= -N^2 dt^2 + {}^3g_{ij} (-N^i dt + dx^i) (-N^j dt + dx^j) \\ &= (-N^2 + N^i N_i) dt^2 - 2N_i dt dx^i + {}^3g_{ij} dx^i dx^j, \end{aligned} \quad (6)$$

where N is the lapse function, N^i is the shift vector, and $N_i \equiv {}^3g_{ij} N^j$. In the Lagrange condition, $g_{00} = -1$, i.e. $N^2 = 1 + N^i N_i$ should hold. And from the geodesic equation, we have $\dot{N}_i = 0$, i.e. $N_i = N_i(\mathbf{x})$. The Einstein equations read

$${}^3R^i_i + (K^i_i)^2 - K^i_j K^j_i = 16\pi G \rho N^2, \quad (7)$$

$$K^j_{i||j} - K^j_{j||i} = 8\pi G \rho N N_i, \quad (8)$$

$$\dot{K}^i_j + N(K^k_k K^i_j + {}^3R^i_j) - N^{\parallel i}_{||j} + N^k K^i_{j||k} + N^k_{||j} K^i_k - N^i_{||k} K^k_j = 4\pi G \rho N (\delta^i_j + 2N^i N_j), \quad (9)$$

where ${}^3R^i_j$ and $||$ denote the Ricci tensor and the covariant derivative, respectively, with respect to the three-metric ${}^3g_{ij}$, and

$$K^i_j \equiv \frac{1}{2N} {}^3g^{ik} (\dot{{}^3g}_{jk} + N_{k||j} + N_{j||k})$$

is the extrinsic curvature. We rewrite the above Einstein equations for convenience. Introducing the following quantities:

$$V^i_j \equiv K^i_j - \frac{\dot{a}}{a} \delta^i_j, \quad \mathcal{R}^i_j \equiv a^2 {}^3R^i_j, \quad \mathcal{N}_i \equiv \frac{1}{a} N_i, \quad a = a(t) : \text{scale factor},$$

we obtain from Eq. (8),

$$V^j_{|j} - V^j_{j|} = 8\pi G a N \mathcal{N}_i \rho, \quad (10)$$

and from Eqs. (7), (8), and (9),

$$\begin{aligned} \dot{V}^i_j + \left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a} \right)^2 \right] \delta^i_j + N \left[\left(V^k_k + 3 \frac{\dot{a}}{a} \right) \left(V^i_j + \frac{\dot{a}}{a} \delta^i_j \right) + \frac{1}{a^2} \mathcal{R}^i_j \right] - \frac{1}{a^2} \mathcal{N}^i_{|j} \\ + \frac{1}{a} \left(\mathcal{N}^k V^i_{j|k} + \mathcal{N}^k_{|j} V^i_k - \mathcal{N}^i_{|k} V^k_j - \mathcal{N}^i V^k_{j|k} + \mathcal{N}^i V^k_{k|j} \right) \\ = \frac{1}{4N} \left[\frac{1}{a^2} \mathcal{R}^k_k + (V^k_k)^2 - V^k_\ell V^\ell_k + 4 \frac{\dot{a}}{a} V^k_k + 6 \left(\frac{\dot{a}}{a} \right)^2 \right] \delta^i_j, \end{aligned} \quad (11)$$

where $|$ denotes the covariant derivative with respect to the conformally transformed three-metric $\gamma_{ij} \equiv a^{-2} g_{ij}$. Note that the quantities V^i_j and $\mathcal{N}^i$ represent deviation from the FLRW universe and will be treated as small quantities in the perturbative approach.

3 Perturbative approach

Here we solve Eqs. (10) and (11) perturbatively. We take the zeroth-order solution as the spatially flat FLRW universe with energy density $\rho_b(t)$. Then the zeroth-order part of Eq. (11) gives

$$\frac{\ddot{a}}{a} + \frac{1}{2} \left(\frac{\dot{a}}{a} \right)^2 = 0, \quad \text{i.e.} \quad a = t^{2/3}. \quad (12)$$

Equation (7) gives $\rho_b = 1/(6\pi G t^2)$.

Next we consider the first-order perturbation solutions. Since $N^2 = 1 + \mathcal{N}^i \mathcal{N}_i$, we may put $N = 1$ in the first-order calculation. Thus Eqs. (10) and (11) become

$$V^j_{|j(1)} - V^j_{j(1)} = 8\pi G \rho_b a \mathcal{N}_{i(1)}, \quad (13)$$

$$\dot{V}^i_{j(1)} + 3 \frac{\dot{a}}{a} V^i_{j(1)} + \frac{1}{a^2} \left(\mathcal{R}^i_{j(1)} - \frac{1}{4} \mathcal{R}^k_{k(1)} \delta^i_j \right) = 0, \quad (14)$$

where a subscript (1) denotes first-order smallness in the perturbation. Considering the metric perturbations

$$\gamma_{ij} = \delta_{ij} + \gamma_{ij(1)} = \delta_{ij} + h_{ij}, \quad \mathcal{N}_i = \mathcal{N}_{i(1)} = \frac{1}{a} b_i(x), \quad (15)$$

Eqs. (13) and (14) read

$$\dot{h}^j_{i,j} - \dot{h}^j_{j,i} + \frac{1}{a^2} (b^j_{i,j} - b^j_{j,i}) = \frac{8}{3} t^{-2} b_i, \quad (16)$$

$$\ddot{h}_{ij} + 3 \frac{\dot{a}}{a} \dot{h}_{ij} + \frac{1}{a^2} (h^k_{i,jk} + h^k_{j,ik} - h^k_{ij,k} - h^k_{k,ij}) - \frac{1}{2a^2} (h^k_{\ell,k}{}^\ell - h^k_{k,\ell}{}^\ell) \delta_{ij} + \frac{\dot{a}}{a^3} (b_{i,j} + b_{j,i}) = 0. \quad (17)$$

A conventional way to study cosmological perturbations is decomposition of the metric perturbations into scalar, vector, and tensor modes:

$$h_{ij} = 2H_L \delta_{ij} + 2H_T{}_{,ij} + (h_{i,j} + h_{j,i}) + 2H_{ij}^{TT}, \quad b_i = B_{,i} + b_i^T, \quad (18)$$

where $h_i{}^i = 0$, $H_{ij}^{TT,j} = 0$, $H_i^{TT,i} = 0$, and $b_i^T{}^i = 0$. In fact, the scalar function B can be set to zero by using a residual gauge freedom in the Lagrange condition. (See Appendix B of Asada & Kasai [11].) Substituting Eq. (18) into Eq. (17), we obtain

$$H_L = \frac{10}{9} \Psi(x), \quad H_T = t^{2/3} \Psi(x) + t^{-1} \Phi(x), \quad (19)$$

$$h_{i,j} + h_{j,i} = 3t^{-1/3} (b_{i,j}^T + b_{j,i}^T) + t^{-1} (\tilde{b}_{i,j} + \tilde{b}_{j,i}), \quad (20)$$

$$\ddot{H}_{ij}^{TT} + 3\frac{\dot{a}}{a}\dot{H}_{ij}^{TT} - \frac{1}{a^2}\nabla^2 H_{ij}^{TT} = 0, \quad (21)$$

where ∇^2 denotes $(\cdot)_{,i,i}$. Relation between b_i^T and $\bar{b}_i$ is obtained from Eq. (16) as

$$\nabla^2 \bar{b}_i = -\frac{8}{3}b_i^T \equiv -\frac{8}{3}\nabla^2 \beta_i(\mathbf{x}). \quad (22)$$

The first-order solution is, therefore,

$$h_{ij} = \frac{20}{9}\Psi\delta_{ij} + 2t^{2/3}\Psi_{,ij} + 2t^{-1}\Phi_{,ij} + 3t^{-1/3}\nabla^2(\beta_{i,j} + \beta_{j,i}) - \frac{8}{3}t^{-1}(\beta_{i,j} + \beta_{j,i}) + 2H_{ij}^{TT}, \quad (23)$$

$$b_i = \nabla^2 \beta_i(\mathbf{x}). \quad (24)$$

The scalar functions Ψ and Φ and the vector function β_i are related to the energy density and the vorticity, respectively, at an initial time $t = 1$. From Eq. (7) and (3), we find

$$\rho_0(\mathbf{x}) = \rho_{b0}(1 - \nabla^2(\Psi + \Phi)), \quad (25)$$

$$\omega_0^i(\mathbf{x}) = \frac{1}{2}\epsilon^{ijk}\nabla^2\beta_{j,k}. \quad (26)$$

(Note that ω^0 is found to be a higher-order quantity because $\omega^0 = -N_i\omega^i$, which follows $\omega^\mu u_\mu = 0$.)

Let us proceed to obtain the second-order solutions. For simplicity, we neglect the terms $2t^{-1}\Phi_{,ij}$ and $2H_{ij}^{TT}$, which are included in the first-order solution. Then we start from the following form of the metric:

$$\gamma_{ij} = \delta_{ij} + h_{ij} + f_{ij}, \quad \mathcal{N}_i = \frac{1}{a}(b_i + c_i), \quad (27)$$

where f_{ij} and c_i are second-order quantities, $b_i = \nabla^2\beta_i$, and

$$h_{ij} = \frac{20}{9}\Psi\delta_{ij} + 2t^{2/3}\Psi_{,ij} + 3t^{-1/3}\nabla^2(\beta_{i,j} + \beta_{j,i}) - \frac{8}{3}t^{-1}(\beta_{i,j} + \beta_{j,i}). \quad (28)$$

The second-order metric perturbation f_{ij} consists of three parts: f_{ij}^s , f_{ij}^c , and f_{ij}^v , where f_{ij}^s is a quadratic part of the first-order scalar mode, f_{ij}^c denotes couplings of the scalar and vector modes, and f_{ij}^v is a vector-vector part. Since the f_{ij}^s is the second-order solution in the case of an irrotational dust universe, it has been already obtained in previous works [8, 12, 13]. Here we only present the final form of the f_{ij}^s without deriving it. It is

$$f_{ij}^s = t^{2/3}\psi_{ij}(\mathbf{x}) + t^{4/3}\varphi_{ij}(\mathbf{x}) + 2\chi_{ij}(t, \mathbf{x}), \quad (29)$$

where

$$\psi_{ij} = \frac{10}{9}(-6\Psi_{,i}\Psi_{,j} - 4\Psi\Psi_{,ij} + \Psi^{,k}\Psi_{,k}\delta_{ij}), \quad (30)$$

$$\varphi_{ij} = \frac{1}{7}\left[19\Psi_{,ik}\Psi_{,jk} - 12\nabla^2\Psi\Psi_{,ij} + 3\left((\nabla^2\Psi)^2 - \Psi^{,k}\Psi_{,k}\right)\delta_{ij}\right], \quad (31)$$

and χ_{ij} is a transverse-traceless part ($\chi_{ij}^{,j} = 0$, $\chi^i_i = 0$) which satisfies the following wave equation

$$\ddot{\chi}_{ij} + 3\frac{\dot{a}}{a}\dot{\chi}_{ij} - \frac{1}{a^2}\nabla^2\chi_{ij} = S_{ij}, \quad (32)$$

with the source term

$$S_{ij} = \frac{3}{7}\left[\zeta_{k,i,j}^k + \nabla^2(\zeta_{ij}^k\delta_{ij} - 4\zeta_{ij})\right]; \quad \zeta_{ij} \equiv \frac{1}{2}(\nabla^2\Psi\Psi_{,ij} - \Psi_{,ik}\Psi_{,jk}). \quad (33)$$

Equation (32) can be solved as the following non-local form [13]:

$$\chi_{ij} = -t^{4/3}\Delta^{-1}S_{ij} - \frac{28}{9}t^{2/3}\Delta^{-1}\Delta^{-1}S_{ij} - \frac{280}{81}\Delta^{-1}\Delta^{-1}\Delta^{-1}S_{ij}, \quad (34)$$

where Δ^{-1} is the inverse Laplacian.

Now let us turn our attention to the part f_{ij}^c , which reflects existence of the vorticity most. Instead of detailed calculation, we present a broad procedure to obtain f_{ij}^c . The procedure is the following:

1. Start from Eq. (10) at the second-order level. It reads

$$f^{c,j}_{i,j} - f^{c,j}_{j,i} = t^{-4/3}(c^{j,j}_{i,j} - c^{j,j}_{i,j}) + \frac{8}{3}t^{-2}c_i + O(\Psi \cdot \beta_i). \quad (35)$$

Integrating this equation, we find

$$f^{c,j}_{i,j} - f^{c,j}_{j,i} = -3t^{-1/3}(c^{j,j}_{i,j} - c^{j,j}_{i,j}) - \frac{8}{3}t^{-1}c_i + \alpha_i(x) + F_i, \quad (36)$$

where α_i is an integration constant which can be eliminated by the residual degree of freedom of gauge, and F_i denotes a term of couplings of Ψ and β_i .

2. Consider trace of Eq. (11). It gives

$$\tilde{f}^c + 3\frac{\dot{a}}{a}f^c + \frac{1}{2a^2}(f^{c,i,j}_{j,i} - f^{c,i}_{i,i}) + \frac{4}{3}t^{-7/3}c^i_{,i} = S^i_i, \quad (37)$$

where $f^c \equiv f^{c,i}_{i,i}$ and S_{ij} is a second-order source term due to couplings of Ψ and β_i . Substituting Eq. (36) into Eq. (37), we have

$$\tilde{f}^c + 3\frac{\dot{a}}{a}f^c = S^i_i - \frac{1}{2}t^{-1/3}F^i_{i,i}. \quad (38)$$

Thus we immediately find a solution for f^c .

3. Consider Eq. (11), not only trace part. To second order, it becomes

$$\tilde{f}^c_{ij} + 3\frac{\dot{a}}{a}f^c_{ij} + \frac{1}{a^2}(f^{c,k}_{i,j,k} + f^{c,k}_{j,i,k} - f^{c,k}_{i,j,k} - f^{c,k}_{j,i,k}) - \frac{1}{2a^2}(f^{c,k,k}_{i,i} - f^{c,k}_{i,i}) + \frac{\dot{a}}{a^3}(c_{i,j} + c_{j,i}) = S_{ij}. \quad (39)$$

Using Eq. (36), this is rewritten as

$$\begin{aligned} \tilde{f}^c_{ij} + 3\frac{\dot{a}}{a}f^c_{ij} - \frac{1}{a^2}\nabla^2 f^c_{ij} = S_{ij} - t^{-4/3}(F_{i,j} + F_{j,i} + f^c_{i,j} + f^c_{j,i} - \frac{1}{2}F^k_{,k}\delta_{ij}) \\ + 3t^{-5/3}(2c^k_{i,j,k} - \nabla^2 c_{i,j} - \nabla^2 c_{j,i}) + 2t^{-7/3}(c_{i,j} + c_{j,i} - \frac{2}{3}c^k_{,k}\delta_{ij}). \end{aligned} \quad (40)$$

Equation (40) can be solved by the Green function method.

In this way, we obtain the solution for f^c_{ij} in the form

$$f^c_{ij} = t^{1/3}\mu_{ij}(x) + t^{-1/3}\nu_{ij}(x) + t^{-1}\lambda_{ij}(x), \quad (41)$$

where μ_{ij} , ν_{ij} , and λ_{ij} are spatial functions written in terms of $\Psi \cdot \beta_i$ and c_i . They are represented in non-local form in general. For example,

$$\begin{aligned} \nabla^2 \mu_{ij} = 6\nabla^2(\Psi_{,ik}\nabla^2\beta^k - 2\Psi_{,k}\nabla^2\beta^k_{,i,j}) \\ + 6(\Psi^k_{,i}\nabla^2\beta^j_{,k} - \nabla^2\Psi_{,k}\nabla^2\beta^k - 3\Psi_{,k}\nabla^2\nabla^2\beta^k + 5\Psi_{,k}\nabla^2\beta^k)_{,ij}. \end{aligned} \quad (42)$$

Note that the trace μ^i_i is written in local form:

$$\mu^i_i = 6(\Psi^i_{,j}\nabla^2\beta^j_{,i} - 5\Psi_{,i}\nabla^2\nabla^2\beta^i + 5\Psi_{,i}\nabla^2\beta^i). \quad (43)$$

A vector-vector part f^v_{ij} , which should not be so effective, will not be considered in this article. Complete results including a solution for f^v_{ij} will be reported in a forthcoming paper [14].

4 Non-locality of the solutions

Higher-order perturbation solutions are generally non-local. Actually in the Newtonian Lagrangian perturbation theory, a solution up to second order is [5]

$$\mathbf{x} = \mathbf{q} + \mathbf{p}_{(1)}(\mathbf{q}, t) + \mathbf{p}_{(2)}(\mathbf{q}, t), \quad (44)$$

where $\mathbf{x}$ and $\mathbf{q}$ are Eulerian and Lagrangian coordinates, respectively, and $\mathbf{p}$ represents deviation from the background FLRW universe and has the form

$$\mathbf{p}_{(1)} = t^{2/3} \nabla_{\mathbf{q}} \Psi^{(1)}(\mathbf{q}) + t^{-1/3} \boldsymbol{\beta}^{(1)}(\mathbf{q}); \quad \nabla_{\mathbf{q}} \cdot \boldsymbol{\beta}^{(1)} = 0, \quad (45)$$

$$\mathbf{p}_{(1)} = t^{4/3} \nabla_{\mathbf{q}} \Psi^{(2a)}(\mathbf{q}) + t^{-2/3} \nabla_{\mathbf{q}} \Psi^{(2b)}(\mathbf{q}) + t^{1/3} \boldsymbol{\beta}^{(2)}(\mathbf{q}); \quad \nabla_{\mathbf{q}} \cdot \boldsymbol{\beta}^{(2)} = 0, \quad (46)$$

with

$$\nabla_{\mathbf{q}}^2 \Psi^{(2a)} = \frac{3}{14} \left[\Psi_{,ij}^{(1)} \Psi_{,ij}^{(1)} - (\nabla_{\mathbf{q}}^2 \Psi^{(1)})^2 \right], \quad \nabla_{\mathbf{q}}^2 \Psi^{(2b)} = \frac{3}{4} \beta_{i,j}^{(1)} \beta_{j,i}^{(1)}, \quad (47)$$

$$\left[\nabla_{\mathbf{q}} \times \boldsymbol{\beta}^{(2)} \right]_i = -3 \epsilon_{ijk} \Psi_{,j\ell}^{(1)} \beta_{\ell,k}^{(1)}. \quad (48)$$

(We neglect contribution of a first-order longitudinal decaying mode proportional to t^{-1} .) In order to know all about the second-order spatial dependency, we must solve the above Poisson-type equations (47). In this sense, higher-order solutions are non-local.

Second-order solutions in relativistic approach are also, of course, non-local. However, the solutions obtained in the previous section are easier to treat because they are explicitly divided into local and non-local parts. (See Eqs. (29), (30), (31), (34), and (42).) Hence by our formulation, it may be easier to see non-local effect of gravity in examining feature of non-linear growth of perturbations.

5 Summary

We have developed the Lagrangian perturbation theory with vortical effect in general relativity. The metric perturbations are solved in the Lagrange condition up to second order. Especially the couplings of the first-order scalar and vector modes have been considered. Non-locality which appears in the second-order solutions has been also mentioned. Complete results and more detailed discussions will be given in a forthcoming paper [14].

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Trans-Fast MHD Accretions onto a Black Hole and the Activity of AGNs

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Abstract

The magnetohydrodynamical (MHD) accretion injected from a plasma source with low-velocity must pass through the fast/slow magnetosonic point and the Alfvén point, and then falls into the black hole. We present the critical conditions of the MHD accretion in a stationary and axisymmetric black-hole magnetosphere. The main result is the discovery of two separate regimes: magnetically dominated MHD accretion and fluid dominated MHD accretion. For a weak magnetic field limit, the former has no solution, while the latter coincident with the hydrodynamical case. Transition between these two regimes is discontinuous.

1 Introduction

To explain the activity of active galactic nuclei (AGNs) and compact X-ray sources, we consider a black hole magnetosphere in the center of these objects, which is composed of a massive black hole with surrounding plasma and magnetic field. The magnetic field is originated from an accretion disk rotating around the massive black hole and its corona; and the black hole is immersed in this magnetic field; the accreting plasma is provided from the disk surface or the corona.

In this paper, we assume a stationary and axisymmetric magnetosphere, and consider ideal magnetohydrodynamical (MHD) flows starting with a slow initial velocity, which is at least less than the slow magnetosonic wave speed. The stationary and axisymmetric relativistic hot MHD flow equation has been formulated by Camenzind [1, 2, 3] and Takahashi et al. [4]. In § 2 we summarize the basic equations of the MHD flows.

The differential equation of motion along a magnetic field line has three type critical points, where the poloidal velocity equals to one of the Alfvén wave speed and the fast and slow magnetosonic wave speeds, and the equation becomes singular at these points. To accrete onto the black hole, the ejected inflow from the plasma source must pass through these critical points (Fig. 1): the slow magnetosonic point, the Alfvén point and the fast magnetosonic point, in order. From the regularity conditions at these critical points, the flow's five physical quantities are restricted. In § 3, we derive the critical conditions at the fast and slow magnetosonic points, which is denoted in terms of the location of the fast or slow magnetosonic point, the sound velocity at the the fast or slow magnetosonic point and the location of the Alfvén point.

Though we consider the general relativistic plasma flow, the significance of the critical conditions at the Alfvén point and the fast and slow magnetosonic point is essentially similar to that of a Newtonian wind model by Weber and Davis [5]. In a zero-temperature limit (cold limit) on plasma flow, the Alfvén point and the fast magnetosonic point are available in the accretion solutions, and the regularity condition of the trans-fast MHD flow was discussed by Takahashi [6]. The main purpose of this paper is to examine the “thermal effects” on a MHD plasma streaming in a black hole magnetosphere. We can see the slow magnetosonic point, which is added on the previous cold MHD flow solutions, and find *two* kind of trans-fast MHD flow solutions for the accretion onto a black hole. In the case of the thermal pressure dominated flow limit, one of the solutions is coincide with the hydrodynamical flow solution [7], while other solution becomes unphysical.

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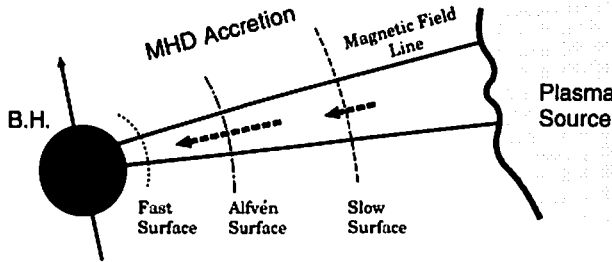


Figure 1: MHD accretion onto a black hole. The flow injected from a plasma source (e.g., the disk surface or the corona) passes through the slow magnetosonic point, the Alfvén point and the fast magnetosonic point, in order, and then reaches the event horizon.

2 Basic Equations of MHD Flows

We present basic equations of a stationary and axisymmetric hot ideal MHD flows. To determine the configuration of magnetic field lines and the poloidal velocity of MHD flows streaming along each magnetic field lines, we must solve what is called the Bernoulli equation along magnetic field lines and the magnetic force-balance equation, self-consistently. These equations are derived from the equation of motion for relativistic MHD plasma

$$T^{\mu\nu}_{;\nu} = \left[(\rho + P)u^\mu u^\nu - P g^{\mu\nu} + \frac{1}{4\pi} \left(F^{\mu\delta} F_\delta{}^\nu + \frac{1}{4} g^{\mu\nu} F^2 \right) \right]_{;\nu} = 0, \quad (1)$$

the conservation law of particle number $(nu^\mu)_{;\mu} = 0$, the ideal MHD condition $u^\mu F_{\mu\nu} = 0$ and Maxwell's equations. Here, ρ , P and n are the total energy density, the pressure of plasma and the proper particle density. The electromagnetic field tensor $F^{\mu\nu}$ satisfies Maxwell equations and u^μ is the four-velocity of plasma. The background metric is written by the Boyer-Lindquist coordinate with $c = G = 1$ units.

The flow with the above assumptions streams along a magnetic field line, which is expressed by a magnetic stream function $\Psi = \Psi(r, \theta)$ as $\Psi = \text{constant}$; Ψ is basically the toroidal component of the vector potential. To obtain the magnetic field line geometry, we should treat the relativistic force-balance equation [8]. However, it is very hard to solve this equation with the Bernoulli equation. So, we only solve the Bernoulli equation under given magnetic field lines.

Stationary and axisymmetry require the existence of five constants of motion (e.g., [9, 10]). These conserved quantities are the total energy $E(\Psi)$, the total angular momentum $L(\Psi)$, the angular velocity of the field line $\Omega_F(\Psi)$ and the particle flux through a flux tube $\eta(\Psi)$, which are given by

$$E = \mu u_t - \frac{\Omega_F}{4\pi\eta} B_\phi, \quad (2)$$

$$L = -\mu u_\phi - \frac{1}{4\pi\eta} B_\phi, \quad (3)$$

$$\Omega_F = -\frac{F_{t\phi}}{F_{\phi r}} = -\frac{F_{t\theta}}{F_{\phi\theta}}, \quad (4)$$

$$\eta = \frac{nu_p}{B_p}, \quad (5)$$

where B_ϕ is the toroidal component of the magnetic field, u_p is the poloidal component of the velocity defined by $u_p^2 \equiv -u_A u^A$ ($A = r, \theta$), B_p is the poloidal component of the magnetic field seen by a lab-frame observer

$$B_p^2 \equiv -\frac{1}{\rho_u^2} [g^{rr}(\partial_r \Psi)^2 + g^{\theta\theta}(\partial_\theta \Psi)^2], \quad (6)$$

and $\rho_w^2 \equiv g_{t\phi}^2 - g_{tt}g_{\phi\phi}$. For a polytropic equation of state, the relativistic specific enthalpy μ is written by (see [2]),

$$\mu = m_p \left[1 + \mu_{\text{inj}} \left(\frac{n}{n_{\text{inj}}} \right)^{\Gamma-1} \right] = m_p \left[1 + \mu_{\text{inj}} \left(\frac{u_p^{\text{inj}} B_p}{u_p B_p^{\text{inj}}} \right)^{\Gamma-1} \right], \quad (7)$$

where

$$\mu_{\text{inj}} \equiv \frac{\Gamma}{\Gamma-1} \frac{P_{\text{inj}}}{n_{\text{inj}} m_p}, \quad (8)$$

and m_p is the rest mass of the particle; the quantities labeled by “inj” is specified at a point injected plasma from the plasma source (i.e., the injection point). Thus, we require the specification of an additional fifth constant of motion, $(P/n)_{\text{inj}}$; in the cold limit, $(P/n)_{\text{inj}} \rightarrow 0$ and $\mu \rightarrow m_p (\equiv \mu_c)$.

By using the conserved quantities, the equation of motion projected on the direction of the poloidal magnetic field, which is called the poloidal equation (and is often referred to as the relativistic Bernoulli equation), can be expressed by (e.g., [4])

$$(1 + u_p^2) = (E/\mu)^2 [(\alpha - 2M^2)f^2 - k], \quad (9)$$

where

$$\alpha \equiv g_{tt} + 2g_{t\phi}\Omega_F + g_{\phi\phi}\Omega_F^2, \quad (10)$$

$$k \equiv (g_{\phi\phi} + 2g_{t\phi}\tilde{L} + g_{tt}\tilde{L}^2)/\rho_w^2, \quad (11)$$

$$f \equiv -\frac{(g_{t\phi} + g_{\phi\phi}\Omega_F) + (g_{tt} + g_{t\phi}\Omega_F)\tilde{L}}{\rho_w(M^2 - \alpha)}, \quad (12)$$

and $\tilde{L} \equiv L/E$. The relativistic Alfvén Mach-number M is defined by

$$M^2 \equiv \frac{4\pi\mu n u_p^2}{B_p^2} = \frac{4\pi\mu\eta u_p}{B_p}. \quad (13)$$

The differential form of the poloidal equation (9) is expressed to the following form (see [4] for the concrete expression)

$$(\ln u_p)' = \frac{N(r, M^2; \tilde{L}, \Omega_F, \mu_{\text{inj}}; \Psi)}{D(r, M^2; \tilde{L}, \Omega_F, \mu_{\text{inj}}, \eta; \Psi)}, \quad (14)$$

where the prime $(\dots)'$ denotes $[(\partial_\theta \Psi)\partial_r - (\partial_r \Psi)\partial_\theta]/(\sqrt{-g}B_p)$ which is a derivative along a stream line. The denominator of the above equation can be reduced to

$$D \propto (u_p^2 - u_{\text{AW}}^2)^2 (u_p^2 - u_{\text{FM}}^2) (u_p^2 - u_{\text{SM}}^2), \quad (15)$$

where u_{AW} is the relativistic Alfvén wave speed, u_{FM} is the fast magnetosonic wave speed and u_{SM} is the slow magnetosonic wave speed. When $u_p = u_{\text{AW}}$, $u_p = u_{\text{FM}}$ or $u_p = u_{\text{SM}}$, the denominator becomes zero. Therefore, at these critical points, we must require $N = 0$ to obtain physical wind and accretion solutions which pass through these critical points smoothly.

3 Critical Conditions

For the physical solution of the accretion onto the black hole and wind/jet solution, we also should require the critical conditions at both the fast and slow magnetosonic points. In this section, we will discuss restrictions on the flow's field-aligned parameters; we see that the locations of the fast and slow magnetosonic points give the total energy and the particle number flux along the magnetic field lines. We use the indices “F” and “S” to denote the quantities evaluated at the fast and slow magnetosonic points, respectively, and use the index “X” to be united the quantities at these magnetosonic points.

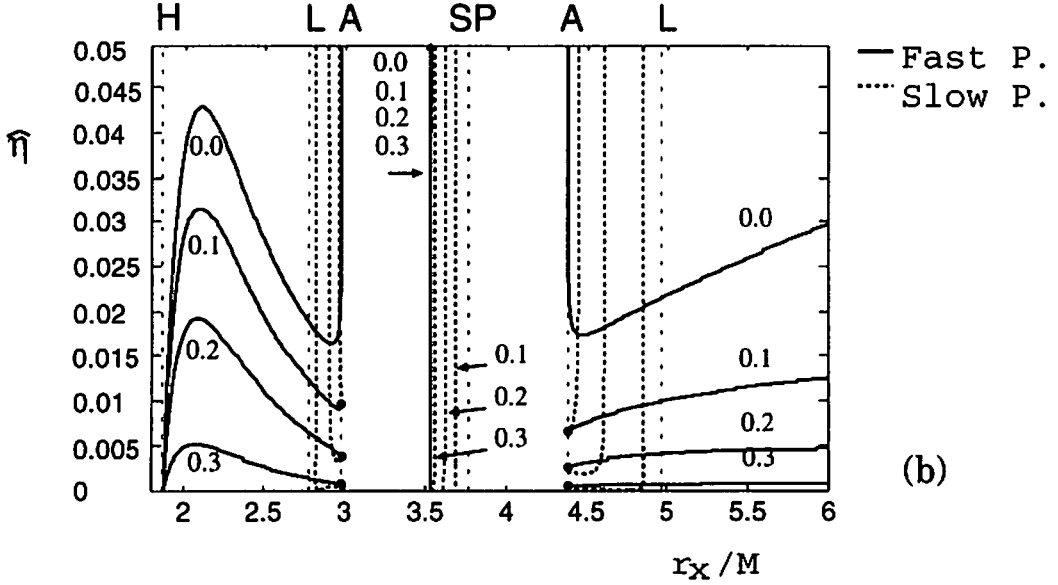
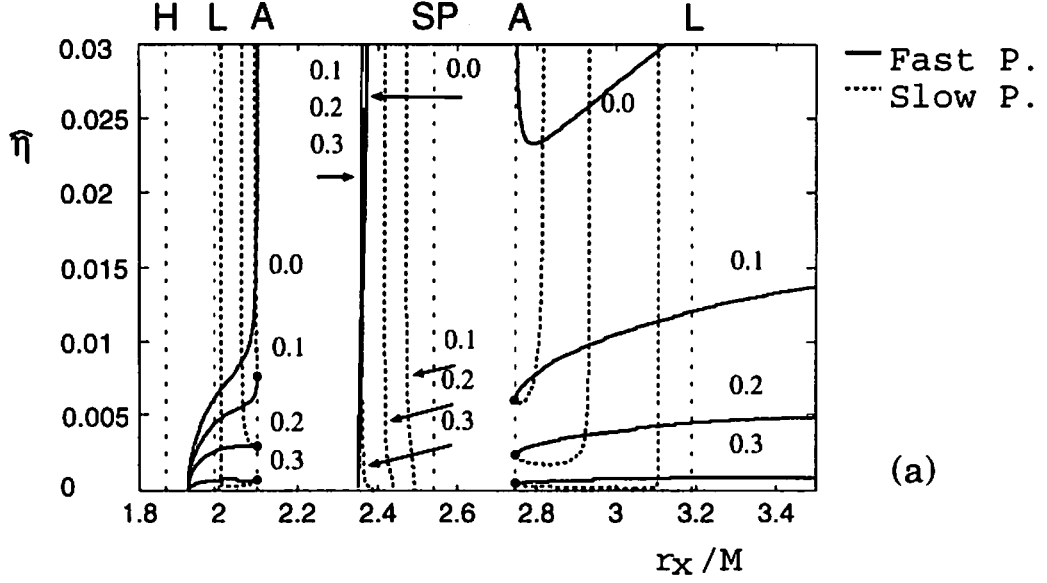


Figure 2: Relations of r_X to $\hat{\eta}$ for (a) $a = 0.5$, (b) $a = -0.5$ with several α_X -values (0.00, 0.10, 0.20), where $\Omega_F = 0.9\Omega_{\max}$, $r_A = 0.8$, $\delta = 0.0$. The solid-curves indicate $\hat{\eta} = \hat{\eta}(r_F)$, and the dotted-curves indicate $\hat{\eta} = \hat{\eta}(r_S)$.

From the conditions $D = 0$ at the fast or slow magnetosonic point and the definition of Mach-number (13), we obtain the particle flux through a flux tube η as a function of r_X with given parameters m , a , Ω_F , $\tilde{L}$, a_{sw}^X and Ψ

$$\eta = \left(\frac{B_p M^2}{4\pi\mu} \right)_X \left[\frac{\mathcal{R}(M^2 - \alpha) - M^2 \mathcal{A}^2}{\mathcal{R}C_{sw}^2(M^2 - \alpha) + M^2 \mathcal{A}^2} \right]^{1/2}, \quad (16)$$

where $\mathcal{R} \equiv e^2\alpha - 2e^2M^2 - kM^4$, $e \equiv 1 - \Omega_F\tilde{L}$, $\mathcal{A}^2 \equiv e^2 + \alpha k = f^2(M^2 - \alpha)^2$. The Mach-number $M_X^2 \equiv M^2(r_X)$ is a solution (or one of three solutions) of the cubic-equation which is derived from $N = 0$ with the poloidal equation (9). The relativistic sound velocity a_{sw} is given by

$$a_{sw}^2 \equiv \left(\frac{\partial \ln \mu}{\partial \ln n} \right)_{ad} = (\Gamma - 1) \frac{\mu - m_p}{\mu}, \quad (17)$$

and the sound four-velocity is given by $C_{sw}^2 = a_{sw}^2/(1 - a_{sw}^2)$.

Here, to discuss a trans-fast MHD flow in a black hole magnetosphere, we consider magnetic flux-tubes given by $\partial_r \Psi = 0$ and $\partial_\theta \Psi = B_{inj}(\Psi)r^{-\delta(\Psi)}$; that is, the poloidal magnetic field is denoted as

$$B_p(r; \Psi) = \frac{\partial_\theta \Psi}{\sqrt{\Delta\Sigma}} = \frac{B_{inj}(\Psi)}{\sqrt{\Delta\Sigma}} r^{-\delta(\Psi)}, \quad (18)$$

where $B_{inj}(\Psi)$ is the poloidal magnetic field at the plasma injection source, and $\delta(\Psi)$ is constant along the field lines; $\delta = 0$ means the radial configuration of magnetic field lines.

We define a non-dimensional parameter, $\hat{\eta} \equiv \mu_c \eta / B_{inj}$. Figures 2 show $\hat{\eta}$ against r_X ; we regard a , Ω_F , $\tilde{L}$, δ as known for a given Ψ , and a_{sw}^X (a_{sw}^F or a_{sw}^S) as a parameter. In the cold limit, we see three branches of $\hat{\eta} = \hat{\eta}(r_F)$ curves (solid curves of $a_{sw}^X = 0.0$), while we do not see any $\hat{\eta} = \hat{\eta}(r_S)$ curves. For hot MHD flows of $a_{sw}^X = 0.10, 0.20$, however, we can see both the $\hat{\eta} = \hat{\eta}(r_F)$ and $\hat{\eta} = \hat{\eta}(r_S)$ curves (dotted curves). We should note that $\hat{\eta} = \hat{\eta}(r_F)$ curves and $\hat{\eta} = \hat{\eta}(r_S)$ curves are connected at $r_F = r_S = r_A^{in/out}$ limit; that is, hot MHD flows have a finite value of η at the Alfvén point, while in the cold limit the value of η diverges at there.

In the region $r_H < r_F < r_A^{in}$, a $\hat{\eta} = \hat{\eta}(r_F)$ curve of $a_{sw}^X > 0$ has a maximum value ($\equiv \eta_{max}$), while, in the region $r_A^{in} < r_F < r_A^{out}$, it has no restriction. If $\eta < \eta_{max}$, the location of fast magnetosonic point is available between $r_H < r_F < r_A^{in}$ or $r_A^{in} < r_F < r_A^{out}$. Whether an accretion passed through the former point or the latter point depends on the flow's total-energy, which is determined at the plasma source. On the other hand, if $\eta > \eta_{max}$, the location of fast magnetosonic point is only available between $r_A^{in} < r_F < r_A^{out}$. We can also show that in the $\hat{\eta} \rightarrow \infty$ the fast magnetosonic point transposes to the sonic point.

4 Trans-Fast MHD Accretions

We will demonstrate accretion flows injected from the disk surface with slow velocity, which is at least smaller than the slow magnetosonic wave speed. The accretion flow must pass through the slow magnetosonic point, the Alfvén point, the fast magnetosonic point, in order; at these point, the five field-aligned constants of the accretion are restricted. The fast and slow critical points have X-type (physical) or O-type (unphysical) topology of solution, while the Alfvén point does not restrict the flow's total energy; hereafter, we will call the flow passing through both the X-type fast and X-type slow magnetosonic points as a “double-critical” solution.

To plot a double-critical solution, we need to determine the five field-aligned constant quantities: Ω_F , L , η , E , μ_{inj} . Although these constants should be given as boundary conditions at a plasma source, we can set the locations of r_H , r_F , r_L , r_A , r_S as parameters to fix the conserved quantities. First, to exist plasma sources in a black hole magnetosphere, we require that $\Omega_{min} < \Omega_F < \Omega_{max}$, where $\Omega_{min/max}$ is the minimum/maximum angular velocity of the magnetosphere to exist two light surfaces r_L^{in} and r_L^{out} . Second, $\tilde{L}$ is denoted by the Alfvén radius r_A , which locates between above two light surfaces [4]. Third, as we have seen in the previous section, when r_F and a_{sw}^F are given, η is calculated by equation (16); similarly, when r_S and a_{sw}^S are setted, η is also calculated. Of course, the η -value given at the fast magnetosonic

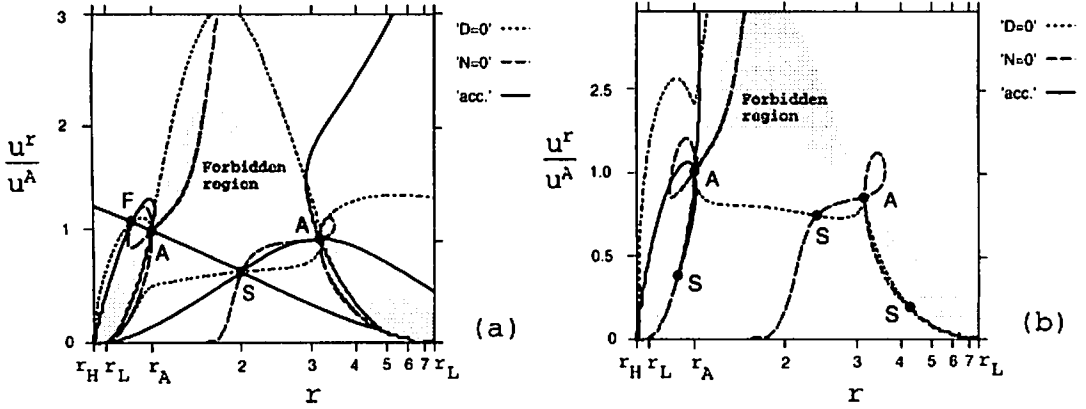


Figure 3: Examples of accretion solutions. (a) A Trans-fast MHD accretion passing through the inner Alfvén point, where $a/m = 0.1$, $\Omega_F = 0.5\Omega_{\max}$, $\tilde{L} = 0.319$, $\delta = 0.1$, $\Gamma = 3/4$, $a_{sw}^X = 0.25$ and $x_F = 0.21872$ ($E/\mu_c = 9.05$). The requirement of $\eta < \eta_{\max}$ is satisfied. (b) No trans-fast MHD accretion passing through the inner Alfvén point exists, where $x_S = 0.215$ ($E/\mu_c = 1.63$). This is a case of $\eta > \eta_{\max}$. The shaded regions are forbidden regions of the flow solutions. The dotted and broken curves denote the $D = 0$ and $N = 0$ curves, respectively; the crossings of these curves give the magnetosonic points.

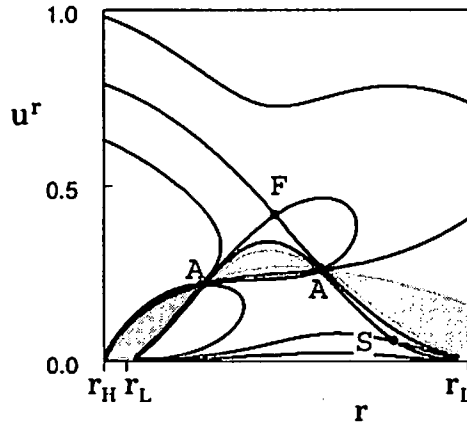


Figure 4: An accretion solution passing through the outer Alfvén point with $a/m = 0.1$, $\Omega_F = 0.6\Omega_{\max}$, $\tilde{L} = 0.39$ ($x_A^{\text{in}} = 2.48m$ and $x_A^{\text{out}} = 3.49m$), $\delta = 0.0$, $\Gamma = 3/4$, $a_s^X = 0.20c$, $x_F = 3.01m$. The value of η is $\hat{\eta} = 0.313$.

point and the η -value given at the slow magnetosonic point should be coincident for the double-critical solution. Finally, we can plot a contour map of E/μ_c on the r - u^r plane. The double-critical solution is given by a curve of $E = E_F = E_S$.

When the forbidden region is type-IA or IIIA (see [4]), where the inner and outer Alfvén points appear on the r - u^r plane, there are two branches of the double-critical MHD accretion solutions:

case (i) The flow passes through the slow magnetosonic point S^{cen} located between two Alfvén points and, after passing through the inner Alfvén point, A^{in} passes through the fast magnetosonic point F^{in} located between the inner Alfvén point and the event horizon; and then it falls into the black hole.

case (ii) The flow passes through the slow magnetosonic point S^{out} located between the outer light surface and the outer Alfvén point or S^{cen} located between two Alfvén points. Then, after passing through the outer Alfvén point A^{out} , the flow passes through the fast magnetosonic point F^{cen} located between two Alfvén points; and then it falls into the black hole.

Figure 3a shows an example of **case (i)**, which satisfies the requirement of $\eta < \eta_{\text{max}}$. The $D = 0$ and $N = 0$ curves are also plotted; the crossings of these curves give the fast (slow) magnetosonic points in the super (sub)-Alfvén region. (The super- and sub-Alfvén region are separated by the forbidden regions on the r - u^r plane.) Figure 3b shows a case of $\eta > \eta_{\text{max}}$; a slow magnetosonic point appears between r_H and r_A^{in} , and no trans-fast MHD accretion exists. An example of **case (ii)** is shown in Fig. 4. This solution corresponds to a hydrodynamical accretion in the hydro-dominated limit.

Similarly, for outflows, we see also two branches of double-critical solutions; that is,

case (iii) $S^{\text{cen}} \rightarrow A^{\text{out}} \rightarrow F^{\text{out}} \rightarrow \infty$

case (iv) $S^{\text{in}} \rightarrow A^{\text{in}} \rightarrow F^{\text{cen}} \rightarrow \infty$

where the slow magnetosonic point S^{in} located between the inner light surface and the inner Alfvén point, and the fast magnetosonic point F^{out} located outside the outer Alfvén point.

5 Concluding Remarks

We have considered hot MHD accretions along a flux tube connected from a plasma source to the event horizon. We have shown that there exist two physically different accretion regimes: *magnetic-like* MHD accretion and *hydro-like* MHD accretion; the hydro-like MHD accretion passes through the outer Alfvén point for sufficiently large η , while the magnetic-like MHD accretion passes through the inner Alfvén point for smaller η . When the value of η increases, the magnetic-like MHD accretion solution must transit to the hydro-like MHD accretion solution. (For larger η , the plasma cannot accrete onto the black hole stationary after passing through the inner Alfvén point.) The criterion for distinguishing between the two regimes is based on the location of the fast magnetosonic point, which changes discontinuously during the transition.

Further, we have found that, for the magnetic-like MHD accretion, the location of the X-type fast magnetosonic point is not unique for fixed intrinsic parameters of the accreting plasma. In Fig. 3b, between the range of $r_H < r_F < r_A^{\text{in}}$, we can see three crossings of the solid curve of $a_{\text{sw}}^F = 0.0$ (or 0.1) and a $\hat{\eta} = \text{constant}$ line; the first and third fast magnetosonic point are X-type critical points, while the second one is O-type. The number of these fast magnetosonic point depends on the spin of the black hole effectively. Transition between these two modes is also discontinuous.

Because to argue the details of the boundary conditions at the plasma source would take us beyond the scope of this paper, we have surveyed the dependence of the trans-fast MHD flows on a wide range of source parameters. To realize a physical accretion flow, the configuration of magnetic field lines would be changed by the thermal-pressure dominated flow. The self-consistent structure of the magnetic field is also remained. If a shock front is generated at somewhere between the plasma source and the event horizon, the trans-fast MHD accretion would be realized again with different wind parameter sets. These questions will be discussed in detail in a further paper.

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General Relativistic Magnetohydrodynamic Simulation of Jet Formation around Rotating Black Hole

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Abstract

Recent observations of Galactic Black Hole Candidates (BHCs) suggest that those that are superluminal jet sources have a more rapid black hole spin rate than otherwise normal BHCs. This conclusion provides observational support for models of relativistic jet formation that extract rotational energy of the central black hole. To investigate this mechanism, we have developed a new general relativistic magnetohydrodynamic code in Kerr geometry. Here we report on the first numerical simulation of the formation of a highly relativistic jet in a rapidly-rotating ($a = 1$) Kerr black hole magnetosphere. We find that the maximum velocity of the jet reaches $\geq 0.98c$ (Lorentz factor $> 5!$) and the terminal velocity of the jet is $0.9c$ (Lorentz factor $= 2$). On the other hand, for a non-rotating ($a = 0$) Schwarzschild black hole, the maximum jet velocity is less than $0.3c$ as far as we calculated with fall coronal condition. Based on these results, we propose a new model of relativistic jet formation from rotating black holes in our Galaxy and active galactic nuclei. This model may also be applicable to gamma-ray bursts.

1 Introduction

Radio jets ejected from radio loud active galactic nuclei (AGN) sometimes show proper motion with apparent velocity exceeding the speed of light c [1, 2]. The widely-accepted explanation for this phenomenon, called superluminal motion, is relativistic jet flow in a direction nearly along the observer's line-of-sight with a Lorentz factor greater than 2 [3]. Such relativistic motion is thought to originate from a region very close to the putative supermassive black hole thought to power each AGN [4, 5]. On the other hand, the great majority of AGN are radio quiet and do not produce powerful relativistic radio jets [5]. These two classes of active object, furthermore, (radio loud and quiet) are also found in the black hole candidates (BHCs) in our own Galaxy. Objects with superluminal jets, such as GRS 1915+105 and GRO J1655-40, belong to the radio loud class [6, 7]. Other objects such as Cyg X-1 and GS 1124-68 are relatively radio quiet and produce little or no jet.

What causes the difference between the two classes? Recent observations of the BHCs in our Galaxy suggest that the Galactic superluminal sources contain very rapidly rotating black holes (specific angular

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momentum of rotating black hole $a = 0.9 - 0.95$), while the black holes in Cyg X-1 and GS 1124-68 are spinning much less rapidly ($a = 0.3 - 0.5$) [8]. According to recent (nonrelativistic) studies of magnetically-driven jets by Kudoh and Shibata [9, 10], the terminal velocity of the formed jet is comparable to the rotational velocity of the disk at the foot of the jet and the mass outflow rate depends on the disk poloidal magnetic field strength. Further nonrelativistic simulations of jet formation from accretion disks themselves confirm these results [11, 12], except for the extremely large magnetic field case [13], which can produce very fast jets. The rotation velocity at the innermost stable orbit of the Schwarzschild black hole ($r = 3r_S$) is $0.5c$. (Here $r_S = 2GM_{BH}/c^2$ is the Schwarzschild radius, where G and M_{BH} are the gravitational constant and black hole mass, respectively.) In addition, recent studies indicate that the poloidal magnetic field strengths in disks around non-rotating black holes may be rather weak [14]. Therefore, a jet produced by MHD acceleration from an accretion disk around a non-rotating black hole should be sub-relativistic and very weak. In fact, numerical simulations of jet formation in a Schwarzschild metric show only sub-relativistic jet flow [15], except for the artificial condition of a hydrostatic corona in equilibrium [16].

Several mechanisms for relativistic jet formation from rotating black holes have been proposed [17, 18]. However, up until now no one has performed a self-consistent numerical simulation of the dynamic process of jet formation in a rotating black hole magnetosphere. To this end, therefore, we have developed a Kerr general relativistic magnetohydrodynamic (KGRMHD) code. In this paper we report briefly on what we believe is the first calculation of its kind — simulation of relativistic jet formation in a rotating black hole magnetosphere. When the rotation parameter of the rotating black hole $a = 1$, the maximum velocity of the formed jet reaches $\geq 0.98c$ (Lorentz factor ≥ 5), and the terminal velocity is $\sim 0.9c$ (Lorentz factor ~ 2). On the other hand, when a vanishes, only a very slow jet is formed.

2 Numerical Method

We use a 3 + 1 formalism of general relativistic conservation laws of particle number, momentum, and energy and Maxwell equations with infinite electric conductivity (see Appendix C in [15]). The Kerr metric, which provides the spacetime around a rotating black hole, is used in the calculation. When we use the Boyer-Lindquist coordinates, $x^0 = ct$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \phi$, the Kerr metric $g_{\mu\nu}$ is written as follows,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -h_0^2 (cdt)^2 + \sum_i h_i^2 (dx^i)^2 - 2h_3 \Omega_3 c dt dx^3. \quad (1)$$

When we modify the lapse function $\alpha = \sqrt{1 - r_S/r}$ in the Schwarzschild case by $\alpha = \sqrt{h_0^2 + \Omega_3^2}$ and add some terms about Ω_3 , we get a KGRMHD code relatively easily from that of Schwarzschild code [16]. We use the Zero Angular Momentum Observer (ZAMO) system of Thorne, Price, & Macdonald [19] for the 3-vector quantities, such as velocity, magnetic field, and so on. For scalars, we use the comoving frame to the fluid flow. The simulation is performed in the region $0.8r_S \leq r \leq 20r_S$, $0 \leq \theta \leq \pi/2$ with 210×70 meshes, assuming the axisymmetry with respect to the z -axis and the mirror symmetry with respect to the plane $z = 0$. The (simplified) radiative boundary condition is employed at $r = 0.8r_S$ and $r = 20r_S$. In the simulation, we use the simplified tortoise coordinates, $x = \log(r/r_H - 1)$, where r_H is the radius of the black hole horizon. Here $r_H = 0.5r_S$ when $a = 1$. To avoid the numerical oscillation, we use the simplified TVD method [20, 21, 22]. We check this KGRMHD code by Kepler motion around the rotating black hole.

3 Result

Figure 1 shows the initial and final states of the calculation of the relativistic jet formation around the Kerr black hole. These figures show the rest mass density (gray-scale), velocity (vector), and magnetic field (solid lines) in $0 \leq R \leq 10r_S$, $0 \leq z \leq 10r_S$. The black regions at the origin show the black hole horizon. The specific angular momentum of the black hole is $a = 1$ (maximally rotating black hole) and the radius is $r_H = 0.5r_S$. The initial state in the simulation consists of a hot corona and a cold accretion disk around a black hole (Fig. 1a). In the corona, plasma are assumed to be isothermal and nearly stationary falling with the specific enthalpy $h/\rho c^2 = 1 + \Gamma p/[(\Gamma - 1)\rho c^2] = 1.3$, where Γ is specific

heat ratio. Far from the hole, it becomes the stationary transonic solution exactly. The accretion disk is located at $|\cot\theta| \leq 0.125$, $r \geq r_D = 3r_S$ and the initial rotational velocity of the disk is assumed to be much smaller than that of circular orbit (5% of rotating velocity of circular orbit). The mass density of the disk is 300 times that of the corona. In addition, the magnetic field crosses the accretion disk almost perpendicularly. We use the vector potential A_ϕ of Wald solution to set the magnetic field, which provides the uniform magnetic field far from the hole [23]. However, we do not use A_t of Wald solution. We use ideal MHD condition $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$ to determine the electric field in spite of A_t . Here Alfvén velocity at the disk ($r = 3r_S$) is $v_A = 0.015c$.

Figure 1b shows the final stage of this simulation at $t = 63\tau_S$ where τ_S is defined as $\tau_S \equiv r_S/c$. The accretion disk falls into the black hole rapidly. The magnetic field becomes radial due to the dragging by the disk. The jet is formed almost along the magnetic field lines. Inside and outside of the jet, the magnetic field lines are bent strongly. The maximum velocity of the jet reaches to $\geq 0.98c$ (Lorentz factor $\gamma \geq 5$) at $R = 4r_S$, $z = 3.2r_S$ and its terminal velocity is $0.9c$ ($\gamma \geq 2$). We found the jet has a two-layered-shell structure with respect to density. The fast part of the jet has low density, while the high density part at the outer shell is slower. The high density region is connected to the high density region of the accretion disk. The magnetic tension of the bent magnetic field in the high density region may cause the acceleration of the high density plasma from the accretion disk. The disk fall into the black hole and touch it. This is important to form the relativistic jet form the fast rotating black hole. In fact, the jet is ejected from the upper region of the disk, as we can trace by the field lines.

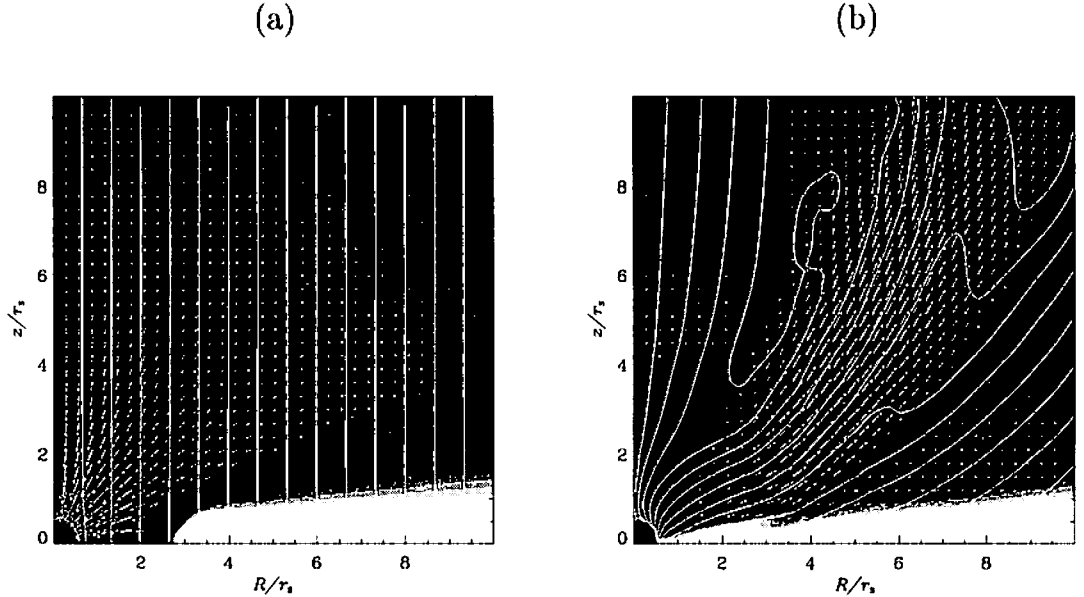


Fig. 1 A relativistic jet formation around the Kerr black hole (the rotation parameter $a = 1$). (a) Initial. (b) Final stage at $t = 63\tau_S$. Gray-scale shows the logarithm of mass density. At initial stage, the mass density of the accretion disk is three hundreds times of that of back ground corona. Vector indicates the velocity. The lines show the magnetic field. The relativistic jet are formed at the final stage. The maximum velocity of the jet is $\geq 0.98c$ (Lorentz factor ≥ 5) and the terminal velocity is $0.9c$ (Lorentz factor ~ 2).

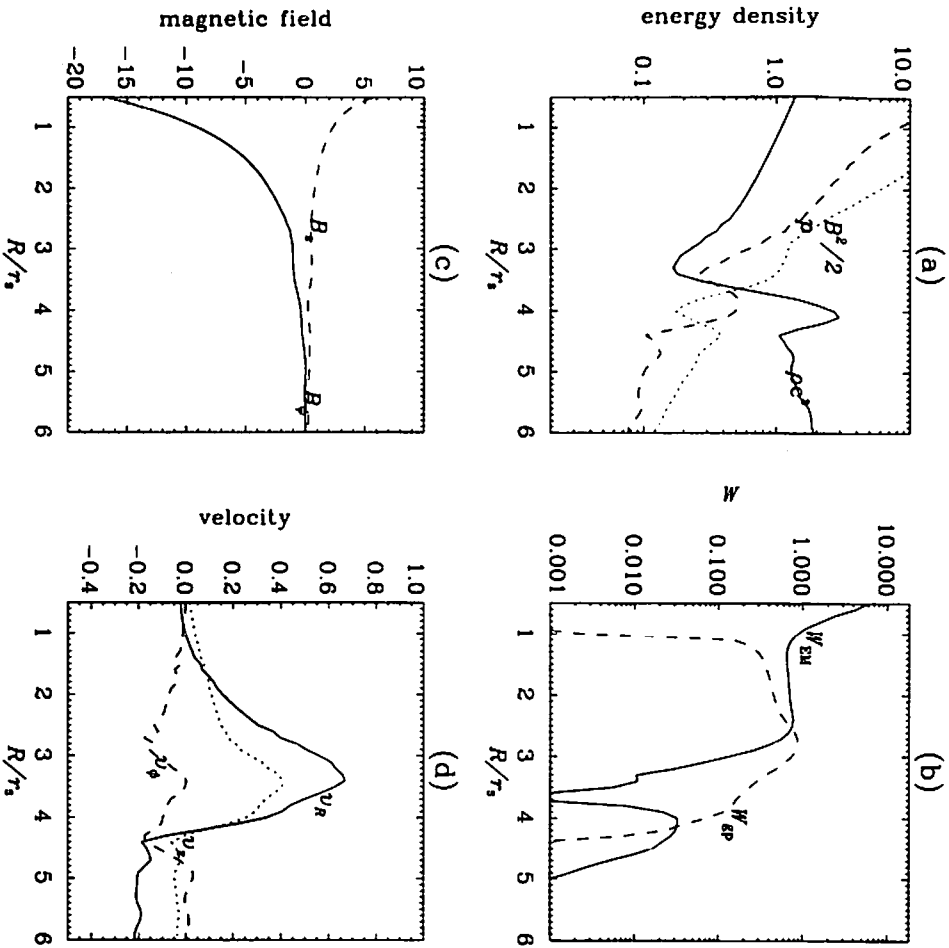


Fig. 2 Physical quantities at $z = 1r_s$, $t = 63r_s$. (a) Mass density ρ , magnetic pressure $B^2/2$, gas pressure p . (b) The power of the electromagnetic field (W_{EM}) and gas pressure (W_{gp}) to the acceleration of the jet. (c) Azimuthal (B_ϕ) and longitudinal components (B_z) of magnetic field. (c) Radial (v_R), azimuthal (v_ϕ), and longitudinal components (v_z) of velocity.

To clarify the acceleration mechanism of the relativistic jet, various physical quantities at $z = 1r_s$ are shown in Fig. 2. As shown in Fig. 1b, the foot of the jet is located at $R = 3r_s$, $z = 1r_s$. Figure 2a shows that the density around $R = 3r_s$ is low and this region corresponds to the foot of the fast jet. The high density region near $R = 4r_s$ corresponds to the slow, high density jet. Figure 2c shows the magnetic field, especially the poloidal component B_ϕ , becomes very large. Because initial magnetic field is $B_\phi = 0$, $B_z = 0.3\rho_0 c^2$, where ρ_0 is the unit mass density, the field increases more than 60 times of initial magnetic field. This amplification is caused by *metric-shear-driven MHD dynamo* by frame dragging effect of the rotating black hole [24]. When we assume azimuthal component of the plasma is zero, general relativistic Faraday law of induction and ideal MHD condition yield,

$$\frac{\partial B_\phi}{\partial t} = f_1 B_r + f_2 B_\theta, \quad (2)$$

where $f_1 = c(h_3/h_1)\partial(\Omega_3/h_3)/\partial r$, and $f_2 = c(h_3/h_2)\partial(\Omega_3/h_3)/\partial \theta$. It is noted that f_2 is one order smaller than f_1 when $a = 1$. If there is no disk, the magnetic tension causes the azimuthal motion of the plasma to decrease the dynamo. The heavy disk plays an important role to store the magnetic energy from the energy of the black hole rotation to produce a relativistic jet. The very strong magnetic pressure

$B_\phi^2/2\mu$ pushes the plasma against the strong gravity of the black hole (μ is the magnetic permeability of the vacuum). We estimate the power from the electro-magnetic field $W_{EM} = \mathbf{v} \cdot (\mathbf{E} + \mathbf{J} \times \mathbf{B})$ and the gas pressure $W_{gp} = -\mathbf{v} \cdot \nabla p$ (Fig. 2b). Near the black hole ($R \sim 0$), the power of the electromagnetic field is dominant due to the very strong magnetic pressure $B_\phi^2/2$. At $R \sim 3r_s$, the contributions of the electromagnetic field and gas pressure are comparable. However the high gas pressure is also caused by the very strong magnetic pressure. Figure 2d shows the azimuthal velocity is small. This indicates the jet acceleration is caused by the magnetic (gas) pressure rather than the magnetic tension (or centrifugal force). This magnetically driven mechanism is the same as that of Uchida & Shibata [25] and Shibata & Uchida [26].

Punsly & Coroniti [18] proposed the alternative magnetic force driven mechanism of the relativistic jet formation. They considered the magnetic tension causes the azimuthal velocity to produce the jet due to centrifugal force like the model of non-relativistic jet formation from the accretion disk [27]. Our model is complementary to their model and supported by the numerical calculation.

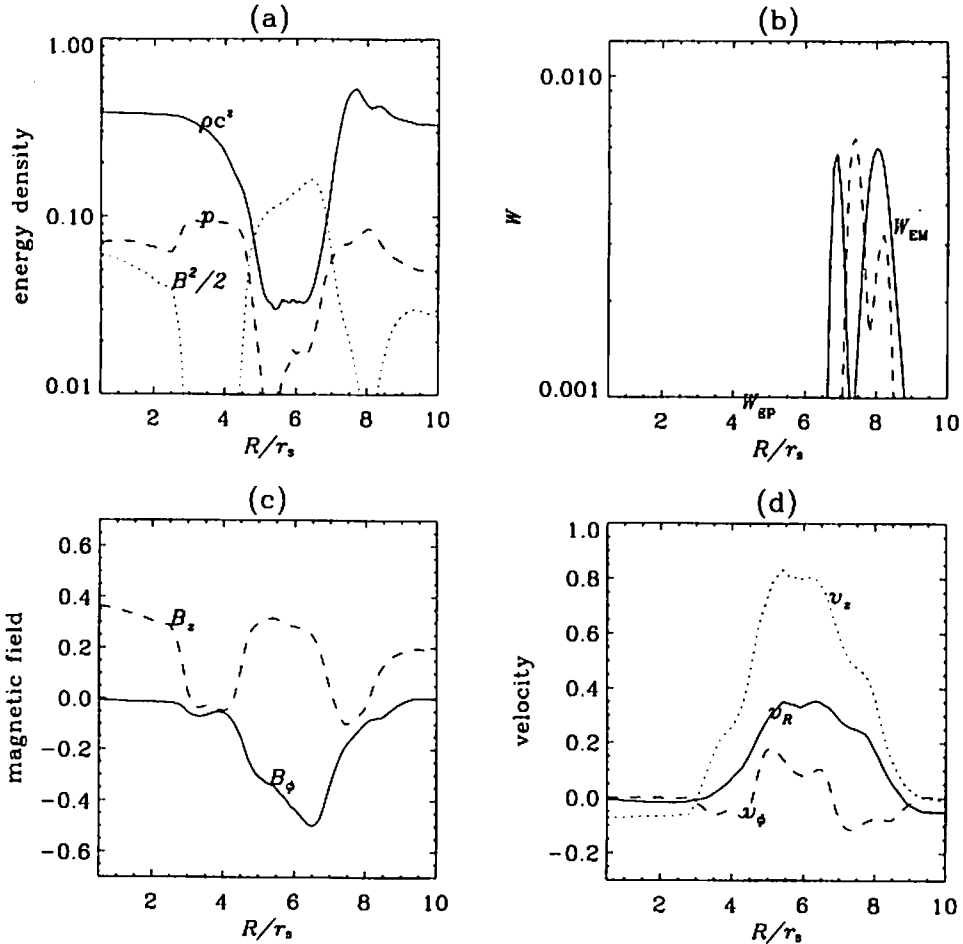


Fig. 3 Physical quantities at $z = 6r_s$, $t = 63\tau_s$. (a) Mass density ρ , magnetic pressure $B^2/2$, gas pressure p . (b) The power of the electromagnetic field (W_{EM}) and gas pressure (W_{gp}) to the acceleration of the jet. (c) Azimuthal (B_ϕ) and longitudinal components (B_z) of magnetic field. (d) Radial (v_R), azimuthal (v_ϕ), and longitudinal components (v_z) of velocity.

To see the structure of the jet, we plot several quantities across the jet at $z = 6r_s$. Figure 3d shows that the fast jet is located at $4r_s \leq R \leq 7r_s$. The maximum poloidal velocity of the fast, low density jet is $v_p = \sqrt{v_R^2 + v_z^2} = 0.9c$ (Lorentz factor $\gamma \sim 2.3$). The total velocity is $v_p = \sqrt{v_R^2 + v_z^2 + v_\phi^2} = 0.92c$ ($\gamma \sim 2.6$). The axial component of the jet velocity v_z is larger than its radial component v_R , while axial velocity v_z is smaller than radial one v_R at $z = 1r_s$ (see Fig. 2). This is because of the pinch effect of the jet. In the region, the mass density and pressure are low while the magnetic field is strong (Fig. 3a). The azimuthal component of the magnetic field B_ϕ is finite. This is the remnant of the magnetic pressure drive source near the black hole. The longitudinal component of the magnetic field B_z is also larger than that beside the jet. This may be caused by a magnetohydrodynamic pinch effect. It is noted that the azimuthal component of the velocity v_ϕ is also finite. It indicates the magnetic tension (or centrifugal force) also contributes the jet acceleration like Blandford-Payne type mechanism [27]. Anyway, in this region, the acceleration of the fast jet stops (Fig. 3b). On the other hand, in the outer layer region ($7r_s \leq R \leq 8r_s$), the acceleration power of both electromagnetic force and gas pressure is positive. This corresponds to the outer, slow, high density jet. This magnetic acceleration is caused by the N-shaped strongly bent magnetic field lines (see Fig. 1b).

4 Discussion

We also performed the simulations with similar initial condition except for rotation parameters of the black hole ($a = 0$). In that case, a jet is ejected but the maximum velocity is less than $0.3c$ and the terminal velocity is much smaller than the light speed. Therefore, the relativistic jet in the Kerr metric case is caused by the energy extraction from the rotational energy of the black hole. According to the numerical result, following model is proposed. When the black hole rotate very rapidly ($a = 1$), the frame of the spacetime is dragged rapidly toward the direction of the black hole rotation. This frame dragging causes the *metric-shear-driven MHD dynamo* (see eq. (2)). When the mass density of the accretion disk is large enough near the black hole horizon, the azimuthal component of the magnetic field increases linearly. Finally, the magnetic pressure $B_\phi^2/2\mu$ becomes stronger than the gravitational force of the black hole. The very strong magnetic pressure blows the surrounding plasma off. The expanding plasma is also pinched by the magnetic tension B_ϕ^2/μ . Then the flow becomes the jet. When no accretion disk, the azimuthal component of the magnetic field stops to increase because the magnetic tension prevents it. The rotation of the black hole is fast and the mass density of the accretion disk is large, the induced azimuthal magnetic field becomes strong enough to blow the plasma off very powerfully. Then the relativistic jet with the Lorentz factor more than 2 is formed. It is natural to consider the acceleration of the plasma by the magnetic tension. This mechanism is discussed by Punsly & Coroniti [18]. In fact, it contributes to the jet acceleration partially, but the power is small. Because while it needs the strong poloidal component of the magnetic field, metric-shear-driven MHD dynamo never cause it. They discussed their model as analogy of Blandford-Payne model [27]. In such point of view, our model corresponds to the Uchida-Shibata [25], Shibata-Uchida [26] models.

Recent X-ray observation of jet related black hole candidates (BHCs) in our Galaxy shows the relationship between the jet speed and the rotation of the central objects [8]. A quasi-periodic-oscillations (QPO) of the power spectrum is found in the Galactic superluminal source, GRO J1655-40 [28] and GRS 1915+105 [29]. Using the QPO, it is concluded that the both Galactic superluminal sources contain very rapidly rotating black holes ($a \sim 0.95$). Similarly, using QPO in Cyg X-1 [30] and in GS 1124-68 [31], it is concluded that the black holes in these two sources are spinning less rapidly ($a \sim 0.35$ and $a \sim 0.48$, respectively). Based on these results, Cui et al. [8] proposed that the difference between BHCs that are also superluminal jet sources and otherwise *normal* BHCs is the spin of the black hole. This observational result is just the same as our conclusion from the numerical results.

This model is also applicable to one of the most mysterious objects in the modern astrophysics, γ -ray burst, at least, one important phase of the burst related with *hypernova* [32]. The *hypernova* is the supernova with an extremely large explosion energy ($> 10^{52}$ erg) such as SN1998bw. The object is identified with the γ -ray burst GRB980425 [33], while a counterargument against the identification is presented [34]. Anyway, if we believe the identification, some of γ -ray bursts are originated from the unusual supernova. Our model of the magnetic extraction of the rotation energy of the black hole and

jet formation may be a key mechanism of producing the hypernova and induction of a γ -ray burst from the hypernova. First, the evolutionary process leading to the hypernova would be as follows [32]. The massive progenitor of initially $\sim 40M_{\odot}$ had a large angular momentum and a strong magnetic field, owing possibly to the spiraling-in of a companion star in a binary system. The collapse of the massive Fe core at the end of the evolution led to the formation of a rapidly rotating black hole. Then as shown in our simulations, the large rotational energy of the black hole was extracted with a strong magnetic field to induce a relativistic jet. The hypernova could induce a γ -ray burst in the following way: when the relativistic jet collides with circumstellar or interstellar matter, a relativistically expanding strong shock is formed. Non-thermal electrons that are produced at the shock front emits high-energy photons via synchrotron emission. The energy of these photons is given approximately by $160\text{keV}(\gamma/100)^4 n_1^{1/2}$ [35], where γ denotes the Lorentz factor of the expanding shell and n_1 is the density of the interstellar matter in cm^{-3} . Thus the event could be observed as a γ -ray burst if γ becomes as large as ~ 100 . The spherical symmetric models (like *fire-ball* model) may not produce large enough energies in γ -rays. However, an axi-symmetric explosion like jet could produce particularly high-speed material by a focused shock wave in a polar direction. The strong radio emission at early phases, which suggests the existence of such a relativistic jet, is consistent with the above model [33]. This model may be applicable to not only the γ -ray burst associated with hypernova but also other type γ -ray bursts.

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Event horizon far in the future

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Abstract

In our previous work, it was shown that the topology of an event horizon (EH) is determined by the past endpoint of the EH. A toroidal EH (coalescing EHs) is related to a two-dimensional (one-dimensional) crease set under the assumption that the EH is topologically sphere far in the future. In the present article, we discuss this assumption. We suggest that we can relate the topology of EHs to Weyl curvature on the future null infinity. For example, if the tail of gravitational wave was sufficiently long, the topology of EHs is not a single sphere far in the future.

1 Introduction

In the theoretical research of a black hole after Schwarzschild, the uniqueness theorem of the black hole is one of the most remarkable results. The topology of an EH (TOEH) is the most important assumption for the uniqueness theorem. In other words, the TOEH is the only remaining unsolved problem about the uniqueness theorem.

In our previous works, we investigated the TOEH by relating the change of the TOEH to the crease set of the EH[1]. In these works, the topology of the section of the EH by a timeslice is considered. Then, it is supposed that the TOEH is a single sphere far in the future.

Now, is the assumption that the TOEH is a single sphere far in the future really valid? Naively, it is likely valid under a physically appropriate condition, that is the asymptotic flatness. The reason is because the topology of the future null infinity in an asymptotically flat spacetime should be $R \times S^2$.

On the other hand, we in fact know that the existence of exact solutions like a Majumdar-Papapetrou solution whose EHs are always two spheres on any certain timeslicing. [In this solution, though the future null infinity is topologically $R \times S^2$, the topology of the section is about to change at the future end of the future null infinity from a sphere to two spheres.] Then we cannot consider that the EH always to be a single sphere. In the present article, we discuss the TOEH far in the future by investigating the relation between the structure of the future null infinity and the TOEH.

2 conjecture

First, we conjecture the following statement,

Conjecture 2.1 *Considering the following null surface and call it a modified horizon: A timeslicing $\{S_T = p|T(p) = T = \text{const.}\}$ of a spacetime induces a retarded time v on the future null infinity which diverges at the future end of the future null infinity. On the analogy of the definition of the EH $\sim \partial(J^-(\mathcal{I}^+))$, we define a series of modifications (depending on the timeslicing) of the EH as $H(v(T)) = \partial(J^-(\mathcal{I}^+ \cap S_T))$, since $v(T)$ determines a series of spatial two surfaces on the future null infinity for the timeslicing.*

Then for a sufficiently large T , there exist $T' > T$ such that the section of the EH by S_T is homeomorphic to $S_T \cap H(v(T'))$.

Supposing this conjecture, it turns out that the TOEH far in the future is known by the deformation of the future null infinity along its null generators with a sufficiently large retarded time at there. The deformation of the future null infinity along its null generators is described by the deviation of the

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congruence of the null generators. In the next section, considering a null tetrad formalism, one of whose null vectors are the tangent vector of the null generators, we investigate the deviation of the null generators with the Newman-Penrose equations.

3 deformation of the future null infinity

The deformation of the future null infinity is represented by the deviation of the null generators of the future null infinity. The tangent vector field of the null generators is given by

$$n^a = \bar{g}^{ab} \nabla_b \Omega, \quad (1)$$

where the conformal completion is define by $\bar{g}_{ab} = \Omega^2 g_{ab}$. Now we consider a null tetrad $(n^a, l^a, m^a, \bar{m}^a)$. The relation between the Ricci tensors of the physical metric g and The unphysical metric $\bar{g}$ is given by

$$\bar{R}_{ab} = R_{ab} + 2\Omega^{-1} \bar{\nabla}_a \bar{\nabla}_b \Omega + \bar{g}_{ab} \bar{g}^{cd} (\Omega^{-1} \bar{\nabla}_c \bar{\nabla}_d \Omega - 3\Omega^{-2} \bar{\nabla}_c \Omega \bar{\nabla}_d \Omega). \quad (2)$$

From now on, we expand the variables around the future null infinity with the order of $\Omega(\Omega(\mathcal{I}^+) = 0)$ as $X = \Sigma_n X^{(\alpha)} \Omega^\alpha [2]$. From Eq.1 and Eq.2, we know that an asymptotic flatness ($R_{ab} \sim O(\Omega^2)$) imposes a condition on the tangent vector n^a . The relation in an order Ω^{-2} tells that n^a is null at the null infinity where Ω vanishes. The equation gives relations between spin-coefficients ω_A in the higher orders,

$$F^{(\alpha)}(\omega_A^{(\beta)}, R_{ij}^{(\gamma)}) = 0. \quad (3)$$

On the other hand, the spin-coefficients are related to the curvature of a spacetime by the Newman-Penrose equations,

$$NP^{(\alpha)}(\omega_A^{(\beta)}, \partial_{\omega_A}^{(\gamma)}, \Psi^{(\delta)}, \Phi^{(\epsilon)}, \Lambda^{(n)}). \quad (4)$$

Though we should treat the equations in the sufficient order, for the present we give a discussion in the leading order. Even with the estimation in the leading order, we get the following preliminary result. Essential differential equations deduced from Eqs.4 are given by

$$\begin{aligned} \Delta \lambda^{(1)} &= 3\mu^{(0)} \lambda^{(1)} - \psi_4^{(0)}, \\ \Delta \sigma^{(0)} &= 2\mu^{(0)} \sigma^{(0)} - \lambda^{(1)}, \end{aligned} \quad (5)$$

where λ and σ are the shear of n^a and l^a , respectively, and μ is the expansion of n^a . Since $\psi_4^{(0)}$ corresponds to gravitational radiation, Eqs.5 means the gravitational radiation becomes a source of the shear of the null generators and the deformation of the future null infinity. By

$$M(v) = \exp \int^v -\mu^{(0)}, \quad (6)$$

their solution is given by

$$\begin{aligned} \lambda^{(1)} &= -M^{-3} \int \psi_4^{(0)} M^3 \\ \sigma^{(0)} &= M^{-2} \int M^{-1} \int \psi_4^{(0)} M^3. \end{aligned} \quad (7)$$

By the change of the gauge transformation $\Omega \rightarrow \Omega' = \omega \Omega$, the v -(the parameter of $n^a = \partial/\partial v$) dependence of $\mu^{(0)}$ changes. Therefore we can fix the v -dependence of $\mu^{(0)}$ for the following discussion. In order that the null generators reach the end of the future null infinity [in other words for the existence of the endpoint of the null generators], $M(v)$ must vanishes with $v \rightarrow \infty$. For example, choosing a gauge condition $\mu^{(0)} = 1/v$ and supposing that the tail of the gravitational radiation obeys a power law $\psi_4^{(0)} = v^{-\epsilon}$, the shear of n^a and l^a behave as following, near the future end of the future null infinity.

$$\lambda^{(1)} \sim v^{1-\epsilon}, \sigma^{(0)} \sim v^{2-\epsilon} \quad (8)$$

The above result means that the shears of an affine parametrized null tetrad $(\bar{n}^a, \bar{l}^a, m^a, \bar{m}^a)$ decay provided that ϵ is larger than two. Therefore a sufficiently long tail of the gravitational radiation will cause the divergence of the shear. Poincaré-Hopf theorem tells that this divergence of the shear means the topology change of the future null infinity from a sphere to two spheres. Thus from the conjecture 2.1, it is concluded that the Ehs are topologically two sphere far in the future.

Finally we consider the case of Majumdar-Papapetrou solution whose Ehs are always two spheres. The Weyl curvature Ψ_2 and Ψ_0 of this solution appears on the equator of the spatial S^2 -section of the future null infinity where the sphere is pinched, while only Ψ_2 appears in the north and the south poles. Since from the Newman-Penrose equation it is guessed that Ψ_0 and Ψ_2 become the sources of the shear and the expansion, respectively, an analysis similar to above will explain the situation in a higher order.

Furthermore, the studies about the tail of the gravitational radiation would determine the possibilities of such a TOEH far in the future.

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Shell Crossing in Spherically Symmetric Space-Time

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Abstract

We consider collisions between two spherical massive shells in vacuum space-time by assuming that the shells are transparent. The energy transfer and the change in the 3-momentum occur in the collision. Compared with the Newtonian situation, the relativistic effect always increases the energy transfer. Further while the 3-momentum conserves in the Newtonian situation, the 3-momentum necessarily decreases in the collision due to a relativistic effect. This effect is recognized to be a kind of Ricci focusing effect.

1 Overview

The dust fluid is often assumed to be a dominant matter field of the universe when we investigate various kinds of structure formation, such as the formation of galaxies, clusters of galaxies, and large-scale structures. Due to the pressure-less nature of the dust fluid, neighboring fluid elements will overlap with each other, or in other words, the so-called *shell crossing* will occur in the course of its time evolution from rather generic initial conditions. When the density of the dust becomes very large in some region (which we may refer to shell crossing), the resulting strong gravitational interaction may trigger a subsequent structure formation. Hence it is important in cosmological and astrophysical problems to understand the dynamics at the shell crossing and the subsequent time development of the system.

Shell crossing leads to an infinite mass density, and thus the treatment of the matter field as a dust fluid becomes inappropriate. However, we may realize the later evolution of this system by imposing a physical assumption on the dust fluid. One natural assumption is that the dust fluid represents a system composed of many small collisionless particles with vanishing velocity dispersion, and then a velocity dispersion will appear after the shell crossing. Since the analytic treatment of such a system is in general very difficult, an N -body simulation is often adopted to treat the complicated dynamics of these systems. However, an analytic, or at least semi-analytic treatment is desirable in order to determine the behavior in the vicinity of the shell crossing with sufficiently high precision to study its dynamics rigorously. In the framework of Newtonian gravity, an analytic treatment is possible if we restrict ourselves to a highly symmetric system. In systems with planar symmetry, collisionless particles can be replaced by massive planes parallel with each other, while they are replaced by concentric massive spherical shells in spherically symmetric cases. We can also replace the collisionless nature of the particles by treating each plane and each spherical shell as “transparent” at the collision. The equations of motion for a transparent massive plane and spherical shell are analytically integrable across the collision so that the N -body systems can also be treated analytically. With this procedure, Gouda and Nakamura[1] and Gouda[2] investigated shell crossing in the plane symmetric and spherically symmetric cases, respectively, in connection with structure formation in the universe.

Even if general relativity is adopted as the theory of gravity, the situation is similar, as long as we consider spherically symmetric systems. Although shell crossing leads to infinite curvature polynomials, i.e., a physical singularity, this singularity is weak in the sense that the first order derivative of the metric tensor is finite. Hence, the solution of the metric might be continued after the occurrence of this singularity under some physical conditions concerning the shell crossing.[3]

The equation of motion for a massive spherical shell composed of non-rotating particles, which is often called the dust shell, is analytically integrable using Israel’s formalism[4]. Even when the shell is

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composed of counter-rotating particles, the motion is obtained by merely integrating ordinary differential equations.[5] Hence, the spherically symmetric N -body system (N -shell system) can be treated analytically, or at least semi-analytically, in the framework of general relativity. Using this formalism, Sugiura et al. studied the behavior of the distance-redshift relation in the N -shell system. They found that the spherical N -shell model with an appropriate initial condition imitates the Einstein-de Sitter universe quite well even if the number of shells is only ten within the initial cosmological horizon.[6] This result implies that the N -shell model is a good tool to study the dynamics of a spherically symmetric system composed of dust. We should note, however, that in this example shell crossing, or equivalently collisions between the shells, does not occur.

There are already several works on collisions between spherical shells, but almost all works were concerned with null shells.[7] The fundamental nature of the collision between transparent massive shells whose trajectories are timelike, such as energy transfer and change in the 3-momentum, has not been made clear yet. Hence in this workshop, we discussed the collision between timelike spherical shells in vacuum space-time in detail and give an explicit expression of the energy transfer and the change in the 3-momentum due to their the collision.

Several years ago, Núñez, de Oliveira and Salim investigated the collision between transparent spherical shells, and their analysis included the case of massive shells.[8] However, the constraint equation they gave from the transparent condition is not sufficient to determine the motion of the shells after the collision. Recently two of the present authors proposed a procedure to obtain the motion of spherical massive shells after their collision under essentially the same condition as Núñez et al.[9]

The collision between two transparent massive shells is considered as a simple approximation of realistic shell crossings, and hence it is important for its own sake to investigate what happens in the collision between such shells in highly relativistic situations: this basic investigation might give some insight, for example, into the gravitational collapse of an N -body system composed of compact stars in the super massive black hole formation[10] and the formation of large-scale structure beyond the $100h^{-1}\text{Mpc}$ scale[11] which corresponds to the scale of the Hubble horizon at the radiation-matter equality.

Please see Ref.[12] for detailed calculations and results.

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Collision of timelike spherical shells

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Abstract

The spacetime representing a collision of two timelike shells is studied. It is constructed by 'cutting and pasting' four distinct Reissner-Nordstrom-de Sitter spacetimes. The junction condition is uniquely determined under the condition that two colliding shells are transparent. In the limit of the collision of two neutral null shells, the junction condition reduces to the Dray-'t Hooft-Redmount (DTR) relation. The collision near the horizon is also investigated and it is shown that the phenomena like mass inflation do not occur unless one of colliding shells is escaping from the horizon.

1 Introduction

More than thirty years ago, Israel [1] formulated the method for treating singular hypersurfaces and derived the equation of motion of the infinitely thin shell of dust in the spherically symmetric vacuum spacetime. Shell models are available to study the dynamics of bubbles in the early universe [3], the formations of galaxies and voids in more recent era [4, 5], the gravitational collapse, the super nova explosion, etc. On the other hand, since the dust fluid approximation is thought of good for describing our universe in macroscopic view, the dust fluid models, for example Tolman-Bondi spacetime, are also useful to model such inhomogeneous universes. The dust fluid universes, however, develop for general initial conditions the caustics where the energy density of dust diverges and we cannot calculate the later evolution of the caustics because they are the spacetime singularities. The divergence of the density may not occur in realistic systems because the pressure works in the high density region.

It is one of advantages of the shell models that they can treat such caustics. Since the shell models treat the matter as discrete set, the caustics may be described as successive collisions of the thin shells. Therefore, to study the collision of two massive shells as the elementary process of the caustics is important. The collision of massless shells in the spherically symmetric spacetime was studied by Dray and 't Hooft [6], and Redmount [7]. They constructed this model by 'cutting and pasting' four distinct Schwarzschild spacetimes and derived the junction condition known as the Dray-'t Hooft-Redmount (DTR) relation [6, 7, 8]. In the case of massless shells, since the world-surface of a shell is the null hypersurface, there are few freedom of the dynamics, while in the case of massive shells the problem is more complicated. Though Núñez, de Oliveira and Salim [9] investigated the collision of two massive shells and derived the constraint equation, their analysis is incomplete in the sense that their constraint equation does not uniquely determine the solution. Here, we treat the same problem to that of Núñez et al [9] and determine the junction condition explicitly [12].

In the following, Newton's constant of gravitation and the speed of light are taken to be unity.

2 Dynamics of a thin shell

We here briefly review [1] in this section. Let a timelike hypersurface Σ divide a spacetime (M, g) into two subsets V_1 and V_2 , and let Σ be the common boundary of V_1 and V_2 . The metric g is required to be continuous but its components of first derivative may have finite difference across Σ with respect to a suitable coordinate system, so the energy-momentum tensor will have ' δ -function-like' singularity there.

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Let h be the first fundamental form of the hypersurface Σ and K_a ($a = 1, 2$) be the second fundamental form with respect to the imbedding of Σ into V_a :

$$\begin{aligned} h &:= g - n \otimes n, \\ K_a &:= -1/2 h^+ [L_n h]_a, \quad (a = 1, 2), \end{aligned} \quad (1) \quad (2)$$

where n is the unit normal form to Σ directed from V_1 to V_2 , L_n the Lie derivative with respect to n , and h^+ the projection operator constructed from h , which maps the tensor field on the spacetime M to that on the submanifold Σ . The first fundamental form h is the induced metric on Σ , so this should be determined uniquely, while K_1 and K_2 may be different from each other.

When the hypersurface Σ is singular in the above sense, Σ may represent a thin shell, i.e. there is, additional matter on Σ . We here introduce the *surface energy-momentum tensor* S on Σ which represents the energy-momentum of matter associated to Σ

$$\begin{aligned} S &:= (1/8\pi)[\delta K - \text{trace}(\delta K)h], \\ \delta K &:= K_2 - K_1, \end{aligned} \quad (3)$$

where $\text{trace}(\cdot)$ denotes the trace with respect to h (possibly g).

On the other hand, the Gauss' and the Codazzi-Mainardi equations lead to

$$\begin{aligned} {}^h R + K_a(K_a) - \text{trace}(K_a)^2 &= -2G_a(n, n), \\ d \text{trace}(K) - ({}^h \nabla_{\partial/\partial x^\mu} K_a)(dx^\mu) &= h^+[G_a(n)], \end{aligned} \quad (4) \quad (5)$$

where ${}^h R$ and ${}^h \nabla$ denote the scalar curvature and the Riemannian connection of (Σ, h) , respectively, and $G_a := \text{Ric}_a - 1/2 R_a g$ the Einstein tensor evaluated in V_a . By substructing from Eq.(4)(Eq.(5)) for $a = 1$ to that for $a = 2$, we find the following relationships

$$\begin{aligned} (K_2 + K_1)(S) &= (1/4\pi)[G_1(n, n) - G_2(n, n)], \\ ({}^h \nabla_{\partial/\partial x^\mu} S)(dx^\mu) &= (1/8\pi)h^+[G_1(n) - G_2(n)]. \end{aligned} \quad (6) \quad (7)$$

These formulas (6) and (7) are used later to derive the law of conservation of the proper mass of the shell and the equation of motion of the shell.

Hereafter we assume that (M, g) is spherically symmetric, so Σ is invariant under this isometry. We also assume that the source-free coupled Einstein-Maxwell equations with a non-vanishing cosmological term are satisfied in V_1 and V_2 . Then a generalized Birkhoff's theorem says that each of V_1 and V_2 is isometric to one of Carter (m, q, Λ) solutions [10, 11] (as a special but important case, the collision in the Schwarzschild background was discussed in the paper [13]). The metric and the potential 1-form of the RNdS spacetime in the static coordinate system $\{t, r, \theta, \phi\}$ are given by

$$g = -f(r)dt \otimes dt + f(r)^{-1}dr \otimes dr + r^2(d\theta \otimes d\theta + \sin^2 \theta d\phi \otimes d\phi), \quad (8)$$

$$A = -(q/r)dt, \quad (9)$$

$$f(r) := 1 - 2m/r + q^2/r^2 - \Lambda r^2/3, \quad (10)$$

where m represents the gravitational mass, q the electric charge and Λ the cosmological constant.

We here introduce the coordinate system of V_a ($a = 1, 2$) by $\{t_a, r, \theta, \phi\}$ that is the static coordinate system of the RNdS spacetime with the gravitational mass m_a and the electric charge q_a , where each of r, θ and ϕ have common values on both sides of Σ , so omission of the subscripts does not lead to confusion.

In general, due to the spherical symmetry, the surface energy-momentum tensor takes the form of the perfect fluid: $S = (\sigma + P)u \otimes u + Ph$, where u represents the velocity vector of the fluid particle. Each world-line of particles is represented only by the coordinates t_a and r ($\theta, \phi = \text{const.}$) due to the spherical symmetry, i.e. u is represented by

$$\begin{aligned} u = d/d\tau &= (dt_1/d\tau)\partial_{t_1} + (dr/d\tau)\partial_r, \\ &= (dt_2/d\tau)\partial_{t_2} + (dr/d\tau)\partial_r, \end{aligned} \quad (11)$$

where τ is the proper time along the world-line of the particle. Let n be the unit spacelike normal form to Σ directed from V_1 to V_2 :

$$\begin{aligned} n &= -(dr/d\tau)dt_1 + (dt_1/d\tau)dr \\ &= -(dr/d\tau)dt_2 + (dt_2/d\tau)dr. \end{aligned} \quad (12)$$

All the non-vanishing components of K_a are

$$K_a(u, u) = (f_a(r)dt_a/d\tau)^{-1} (d^2r/d\tau^2 + f'_a(r)/2), \quad (13)$$

and

$$K_a(\partial_\theta, d\theta) = K_a(\partial_\phi, d\phi) = -(f_a(r)/r)dt_a/d\tau, \quad (14)$$

where $f_a := 1 - 2m_a/r + q_a^2/r^2 - \Lambda r^2/3$, ($a = 1, 2$). By evaluating the components of Eq.(3), we obtain

$$(f_2 dt_2/d\tau)^{-1} (d^2r/d\tau^2 + f'_2/2) - (f_1 dt_1/d\tau)^{-1} (d^2r/d\tau^2 + f'_1/2) = 4\pi(\sigma + 2P), \quad (15)$$

and

$$f_1 dt_1/d\tau - f_2 dt_2/d\tau = 4\pi\sigma r. \quad (16)$$

On the other hand, the r.h.s. of Eq.(7) vanishes, since $G_a(n) = -(q_a^2/r^4 + \Lambda)n$, which implies $h^+ \{G_a(n)\} = 0$. Then Eq.(7) leads to

$$d\mu/d\tau = -8\pi P r dr/d\tau, \quad (17)$$

where $\mu := 4\pi\sigma r^2$ is the proper mass of the shell. The proper mass μ is a function of r , when the equation of state: $P = P(\sigma)$ is given. Especially, in the case of dust, i.e. $P = 0$, Eq.(17) leads to the law of conservation of the proper mass of the shell: $\mu = \text{const.}$

By substituting Eqs.(13), (14) and $G_a(n, n) = -(q_a^2/r^4 + \Lambda)$ into Eq.(6) we obtain the following relationship

$$\begin{aligned} (f_2 dt_2/d\tau)^{-1} (d^2r/d\tau^2 + f'_2/2) + (f_1 dt_1/d\tau)^{-1} (d^2r/d\tau^2 + f'_1/2) \\ = (q_2^2 - q_1^2)/\mu r^2 + (2P/\sigma r)(f_1 dt_1/d\tau + f_2 dt_2/d\tau). \end{aligned} \quad (18)$$

From Eqs.(15), (16) and (18), we obtain

$$f_1 dt_1/d\tau = m_-/\mu + \mu/2r - q_+q_-/2\mu r, \quad (19)$$

$$f_2 dt_2/d\tau = m_-/\mu - \mu/2r - q_+q_-/2\mu r, \quad (20)$$

and the equation of motion of a shell:

$$\begin{aligned} d^2r/d\tau^2 &= (m_-q_+q_-/\mu^2 - m_+)/2r^2 - (\mu^2 - q_+^2)(\mu^2 - q_-^2)/4\mu^2 r^3 \\ &\quad + \Lambda r/3 + 2\pi P[\mu - (q_+q_- - 2m_-r)^2/\mu^3]/r, \end{aligned} \quad (21)$$

where $m_\pm := m_2 \pm m_1$ and $q_\pm := q_2 \pm q_1$. Furthermore, by substituting Eq.(19) or (20) into the normalization condition: $g(u, u) = -1$, we obtain the first integral of Eq.(21) or the law of conservation of energy:

$$\begin{aligned} (dr/d\tau)^2 &= -1 + m_-^2/\mu^2 + (m_+ - m_-q_+q_-/\mu^2)/r \\ &\quad + (\mu^2 - q_+^2)(\mu^2 - q_-^2)/4\mu^2 r^2 + \Lambda r^2/3. \end{aligned} \quad (22)$$

3 Colliding transparent shells

We consider here the situation in which two shells Σ and Σ^* with respective proper mass μ and μ^* collide with each other at p with radius r . Then, the world-surfaces of two shells will divide the spacetime into four regions V_a ($a = 1, 2, 3, 4$). Suppose that Σ divides V_1 and V_2 before the collision, V_4 and V_3 after the collision, and that Σ^* divides V_2 and V_3 before the collision, V_1 and V_4 after the collision (Fig.1).

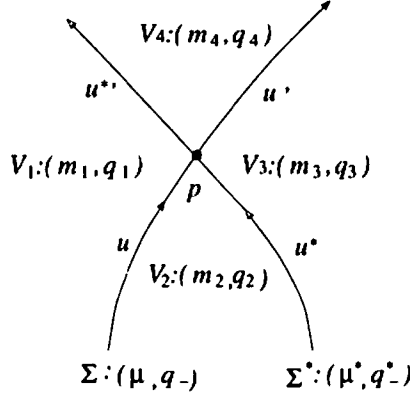


Figure 1: The configuration of Σ , Σ^* and V_a ($a = 1 \sim 4$).

We require that each charge of shells should not change and that each proper mass should be continuous throughout the collision. Each region V_a has the static coordinate system $\{t_a, r, \theta, \phi\}$ of the RNdS spacetime with parameters m_a , q_a and Λ .

We now consider the problem of determining m_4 , q_4 and each velocity of shells, when m_a , q_a ($a = 1, 2, 3$), Λ , μ and μ^* are given. Each charge of shells is simply the subtraction from q_a of outside region to that of inside one according to the Gauss' law. Let V_1 be inside region and V_3 outside one without loss of generality. Then Σ and Σ^* have the charges $q_- = q_2 - q_1$ and $q_+^* = q_3 - q_2$, respectively. Since they are required to conserve throughout the crossing, $q_4 = q_1 + q_+^* = q_1 - q_2 + q_3$. The gravitational mass m_4 of V_4 is not known until the velocity of a shell after the collision is determined.

Let Σ (Σ^*) have the velocity $u = d/d\tau$ ($u^* = d/d\tau^*$) before the collision and $u' = d/d\tau'$ ($u'' = d/d\tau''$) after the collision, the normal n (n^*) before the collision and n' (n'') after the collision, respectively, where both u and u^* are future pointing. Applying Eqs.(19), (20) and (22), we can determine each velocity and normal of shells before the collision up to two sign ambiguities. Respective components of velocity and normal are expressed as

$$dt_1/d\tau = (f_1 - f_2 + \nu^2)/2\nu f_1, \quad (23)$$

$$dt_2/d\tau = (f_1 - f_2 - \nu^2)/2\nu f_2, \quad (24)$$

$$dr/d\tau = \epsilon[\nu^4 - 2(f_1 + f_2)\nu^2 + (f_1 - f_2)^2]^{1/2}/2\nu, \quad (25)$$

$$dt_2/d\tau^* = (f_2 - f_3 + \nu^{*2})/2\nu^* f_2, \quad (26)$$

$$dt_3/d\tau^* = (f_2 - f_3 - \nu^{*2})/2\nu^* f_3, \quad (27)$$

$$dr/d\tau^* = \epsilon^*[\nu^{*4} - 2(f_3 + f_2)\nu^{*2} + (f_3 - f_2)^2]^{1/2}/2\nu^*, \quad (28)$$

where $\nu := \mu/r$, $\nu^* := \mu^*/r$ and $f_a := 1 - 2m_a/r + q_a^2/r^2 - \Lambda r^2/3$, ($a = 1, 2, 3$), and ϵ (ϵ^*) is the sign factor with respect to the velocity of the shell Σ (Σ^*), i.e. if $\epsilon = 1$ ($\epsilon^* = 1$) then Σ (Σ^*) is expanding or else contracting.

We now need the boundary condition to determine m_4 and each velocity of shells after the collision. We impose here the *condition of transparent shells*, by which we mean that each velocity of Σ and Σ^* is conserved throughout the collision, i.e. u and u^* are continuous to u' and u'' at p , respectively. (There isn't a notion of the continuity of vector fields across a singular hypersurface a priori, however, when a continuous metric is given as in the present situation, we may naturally introduce this notion by identifying the normal vectors on both sides of the singular hypersurface.) This condition is equivalent to that of Núñez et al [9]. Other conditions are of course possible, however this one is geometrically invariant and has a simple physical meaning, so would be among the most natural assumption. Furthermore, this uniquely determine m_4 , as we show in the following. To realize this condition, we evaluate u at p with respect to the coordinate system $\{t_3, r\}$ (coordinates θ and ϕ are ignored). We decompose u into the directions of u^* and n^* . Since the metric g on Σ^* have a form: $g = -u^* \otimes u^* + n^* \otimes n^*$, u can be expressed in the form $u = -g(u, u^*)u^* + g(u, n^*)n^*$ on Σ^* . Since both u^* and n^* can be expressed with respect

to the coordinate system $\{t_3, r\}$, we obtain the desired expression for u at p . As we have required, u' is continuous to u at p , so $u' = (dt_3/dr')\partial_{t_3} + (dr/dr')\partial_r$ becomes

$$\begin{aligned} u' = & \{f_2 \frac{dt_2}{d\tau} \frac{dt_2}{d\tau^*} \frac{dt_3}{d\tau^*} - f_2^{-1} \frac{dr}{d\tau} \frac{dr}{d\tau^*} \frac{dt_3}{d\tau^*} - f_3^{-1} [\frac{dt_2}{d\tau} \left(\frac{dr}{d\tau^*}\right)^2 - \frac{dr}{d\tau} \frac{dt_2}{d\tau^*} \frac{dr}{d\tau^*}]\} \partial_{t_3} \\ & + \{f_2 \frac{dt_2}{d\tau} \frac{dt_2}{d\tau^*} \frac{dr}{d\tau^*} - f_3 [\frac{dt_2}{d\tau} \frac{dt_3}{d\tau^*} \frac{dr}{d\tau^*} - \frac{dr}{d\tau} \frac{dt_2}{d\tau^*} \frac{dt_3}{d\tau^*}]\} - f_2^{-1} \frac{dr}{d\tau} \left(\frac{dr}{d\tau^*}\right)^2 \partial_r. \end{aligned} \quad (29)$$

Therefore, using Eqs.(23)~(28), we obtain following relationships

$$\begin{aligned} dt_3/dr' = & (4\nu f_2 f_3)^{-1} \{(\nu^2 - f_1 + f_2)(\nu^{*2} - f_2 - f_3) \\ & - \epsilon\epsilon^* [\nu^4 - 2(f_1 + f_2)\nu^2 + (f_1 - f_2)^2]^{1/2} \\ & \times [\nu^{*4} - 2(f_3 + f_2)\nu^{*2} + (f_3 - f_2)^2]^{1/2}\}, \end{aligned} \quad (30)$$

and

$$\begin{aligned} dr/dr' = & (\epsilon/2\nu)[-1/2 - f_3/f_2 + (1 + f_2 f_3)(f_2 - f_3)^2/4\nu^{*2} f_2^2 f_3 \\ & + (f_2 f_3 - 1)\nu^{*2}/4 f_2^2 f_3][\nu^4 - 2(f_1 + f_2)\nu^2 + (f_1 - f_2)^2]^{1/2} \\ & + (\epsilon^*/8 f_2 f_3 \nu \nu^{*2})(\nu^2 - f_1 + f_2)[(f_3 - f_2)^2 - (f_3 + f_2)\nu^{*2}] \\ & \times [\nu^{*4} - 2(f_3 + f_2)\nu^{*2} + (f_3 - f_2)^2]^{1/2}. \end{aligned} \quad (31)$$

Finally, applying Eqs.(20) and (30), we obtain an expression of m_4 after some tedious manipulation:

$$\begin{aligned} m_4 = & -(\nu^2 + \nu^{*2})r/4 + (f_2 - f_1 - f_3)r/4 \\ & - (\nu^2 - f_1)(\nu^{*2} - f_3)r/4 f_2 + r/2 + q_4^2/2r - \Lambda r^3/6 \\ & - (\epsilon\epsilon^* r/4 f_2)[\nu^4 - 2(f_1 + f_2)\nu^2 + (f_1 - f_2)^2]^{1/2} \\ & \times [\nu^{*4} - 2(f_3 + f_2)\nu^{*2} + (f_3 - f_2)^2]^{1/2}, \end{aligned} \quad (32)$$

or equivalently,

$$\begin{aligned} f_4 : = & 1 - 2m_4/r + q_4^2/r^2 - \Lambda r^2/3 \\ = & (\nu^2 + \nu^{*2})/2 + (f_1 + f_3 - f_2)/2 + (\nu^2 - f_1)(\nu^{*2} - f_3)/2 f_2 \\ & + (\epsilon\epsilon^*/2 f_2)[\nu^4 - 2(f_1 + f_2)\nu^2 + (f_1 - f_2)^2]^{1/2} \\ & \times [\nu^{*4} - 2(f_3 + f_2)\nu^{*2} + (f_3 - f_2)^2]^{1/2}, \end{aligned} \quad (33)$$

We can obtain u'' , or equivalently $dt_1/d\tau''$ and $dr/d\tau''$, in a similar manner. However, we have only to substitute as $t_1 \leftrightarrow t_3$, $f_1 \leftrightarrow f_3$, $\tau \leftrightarrow \tau^*$, $\nu \leftrightarrow \nu^*$ in Eqs.(30) and (31). Both m_4 and f_4 are of course invariant under such substitution. Eq.(33) is a generalization of DTR relation to the case of timelike shells, which is uniquely determined under the condition that two shells are transparent.

4 Neutral null shell limit

We consider here the neutral null shell limit and rederive the original DTR relation. It is pointed out by Núñez et al [9] that there are possibilities $f_1 + f_3 = f_2 + f_3$ or $f_4 = f_2$ holds instead of the DTR relation $f_1 f_3 = f_2 f_4$ in this limit. We show in the following that the former can occur only when one of shells violates the energy condition and that the latter is irrelevant.

The neutral null shell limit of, say Σ , is formally achieved by setting $P = 0$ and $\mu, q_- \rightarrow 0$. Let λ be the affine parameter of Σ , which can be constructed by the formal substitution: $\lambda = \tau/\mu$. Tangent null vector $d/d\lambda$ is future pointing only when the limit $\mu \rightarrow 0$ is taken with keeping $\mu > 0$.

By taking the limits of Eqs.(19), (20) and (22), we obtain the components of each velocity of two neutral null shells with respect to the coordinate system $\{t_2, r\}$:

$$\Sigma : dt_2/d\lambda = (f_1 - f_2)r/2f_2, \quad (34)$$

$$dr/d\lambda = \epsilon|f_1 - f_2|r/2, \quad (35)$$

$$\Sigma^* : dt_2/d\lambda^* = (f_2 - f_3)r/2f_2, \quad (36)$$

$$dr/d\lambda^* = \epsilon^*|f_2 - f_3|r/2, \quad (37)$$

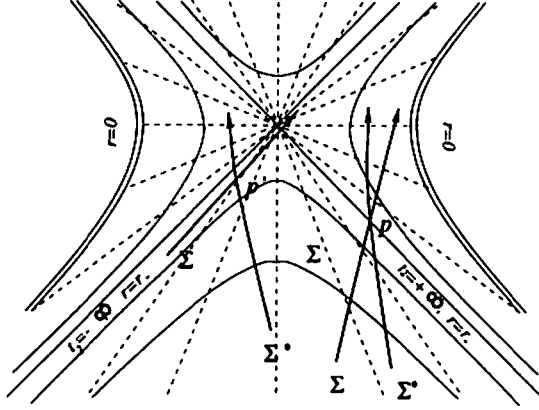


Figure 2: The Kruskal diagram of the RN spacetime with $m_2 > q_2$ is depicted. The solid lines denote hypersurfaces: $r = \text{const.}$, the double lines the singularities: $r = 0$ and the dashed lines hypersurfaces: $t_2 = \text{const.}$. The mass inflation does not occur at p , while it may occur at p' , where a falling shell and another escaping one collide.

Let the point of collision is located with respect to V_2 e.g. in the block that is characterized by the property: (P1) t_2 strictly increases in the future null directions, (P2) r strictly increases (decreases) in the out-going (in-going) null directions and (P3) f_2 is positive. Then, $dt_2/d\lambda, dt_2/d\lambda^* > 0$ hold by the property (P1), so $f_1 > f_2 > f_3$ is obtained from Eqs.(34) and (36) with the property (P3). Furthermore, the property (P2) leads $\epsilon = 1$ and $\epsilon^* = -1$. Similary, we can show that $f_1 < f_2, f_2 > f_3$ and $\epsilon = \epsilon^* = 1$ hold both in II and IV, $f_1 < f_2 < f_3, \epsilon = -1$ and $\epsilon^* = 1$ both in I' and III', and $f_1 > f_2, f_2 < f_3$ and $\epsilon = \epsilon^* = -1$ both in II' and IV'. Four cases listed above are exhaustive even if V_2 has degenerate horizons, negative or vanishing mass and/or Λ . In all cases, the following relationship holds:

$$\epsilon\epsilon^*|f_1 - f_2||f_3 - f_2| = (f_1 - f_2)(f_3 - f_2). \quad (38)$$

Then, Eq.(33) reduces to the well-known DTR relation [6, 7]:

$$f_2 f_4 = f_1 f_3. \quad (39)$$

Eq.(38) is violated only if one of $d/d\lambda$ and $d/d\lambda^*$ is past pointing, i.e. one of shells does not satisfy the energy condition. Then the following relationships instead of Eq.(38) and (39) holds:

$$\epsilon\epsilon^*|f_1 - f_2||f_3 - f_2| = -(f_1 - f_2)(f_3 - f_2), \quad (40)$$

and

$$f_2 + f_4 = f_1 + f_3. \quad (41)$$

Eq.(41) was derived by Núñez et al [9], though it was unclear when this equation holds.

5 Collisions near horizons

When two neutral null shells collide near the Cauchy horizon in the Reissner-Nordström (RN) spacetime, it is seen from DTR relation that the resulting mass: m_4 is very large, which is a simple model of the mass inflation [8].

We treat here the collision near the horizon of two timelike shells. For example, consider the collision slightly outside the inner black hole horizon of V_2 : $r = r_- := m_2 - (m_2^2 - q_2^2)^{1/2}$ in the RN spacetime with $m_2 > q_2$ (see Fig.2; this restriction is not essential and the following discussion may be applied to any horizon in general). Let the radius of the collision be $r = r_- + m_2\delta$, where δ is a small positive parameter. $f_2 = O(\delta)$ is negative, since the collision is assumed to occur outside the inner horizon. When

δ is sufficiently small, both $dr/d\tau$ and $dr/d\tau^*$ are positive, so $\epsilon = \epsilon^* = -1$. Assume that both shells fall into the same horizon, i.e. both $dt_2/d\tau$ and $dt_2/d\tau^*$ are positive (possibly both negative), then, from Eqs.(24) and (26), we obtain the following inequalities:

$$f_1 - \nu^2 < f_2, \quad (42)$$

$$f_3 - \nu^{*2} > f_2. \quad (43)$$

The condition: $f_2 < 0$ leads that the l.h.s. of Eq.(42) is negative:

$$f_1 - \nu^2 < 0. \quad (44)$$

While the l.h.s. of Eq.(43) may be negative, however, in the limit of $\delta \rightarrow +0$, this possibility is denied. To show this, note that $dt_2/d\tau^*$ has to diverge in this limit. If $f_3 - \nu^{*2} \leq -f_2$ is kept then $dt_2/d\tau^* \leq 1/\nu^*$ holds by Eq.(26), which leads contradiction, so we conclude that $f_3 - \nu^{*2} > -f_2$ holds in the limit of $\delta \rightarrow +0$, especially,

$$f_3 - \nu^{*2} > 0, \quad (45)$$

is obtained.

On the other hand, Eq.(33) becomes

$$f_4 = [(\nu^2 - f_1)(\nu^{*2} - f_3) + \epsilon\epsilon^*|\nu^2 - f_1||\nu^{*2} - f_3|]/2f_2 + O(1), \quad (46)$$

where we have singled out the only possible singular terms. However, these singular terms cancel each other for $\epsilon = \epsilon^* = -1$ under the conditions (44) and (45). Thus, f_4 remains finite in the limit of $\delta \rightarrow +0$. We obtain the similar result in the case of $\delta < 0$. Moreover, f_4 converges to the same value in both limits:

$$\lim_{\delta \rightarrow 0} f_4 = \frac{(f_1 f_3 - \nu^2 \nu^{*2})(f_1 + f_3 - \nu^2 - \nu^{*2})}{(f_1 - \nu^2)(f_3 - \nu^{*2})}. \quad (47)$$

This expression would be valid even if the horizon were outer black hole or cosmological horizons, degenerate or non-degenerate horizons. m_4 also converges to finite value, which suggests that phenomena like the mass inflation do not occur just in the presence of the massive flow falling into the horizon. In a physical viewpoint, the result is natural, since timelike flows of matter themselves do not suffer the infinite blue-shift unlike null flows or the gravitational waves.

Secondaly, we assume that one of shells does not fall into the inner black hole but tries to escape from that (one of $dt/d\tau$ and $dt/d\tau^*$ is negative), which means that this shell would fall into the other horizon in the limit: $\delta \rightarrow 0$. Then a similar argument leads

$$f_4 = (f_1 - \nu^2)(f_3 - \nu^{*2})/f_2 + O(1) \rightarrow -\infty, \quad (f_2 \rightarrow 0), \quad (48)$$

which means m_4 can in general take arbitrarily large positive values in this case. The collision of neutral null shells rather corresponds to this case; it is not sufficient to take the limit: $\nu, \nu^*, q_-, q_+ \rightarrow 0$ in Eq.(47) to obtain the neutral null shell limit, since the condition that $dt/d\tau, dt/d\tau^* > 0$ is no longer satisfied in the case of the collision of null shells.

Thus we obtain another conclusion that the mass inflation occurs when a falling shell and another escaping one collide near the horizon.

We here examine the relative velocity of colliding shells. Define the Lorentz factor between Σ and Σ^* by $\gamma := -g(u, u^*)$, which may be related to the relative velocity v by $\gamma = (1 - v^2)^{-1/2}$. When two falling shells collide near the horizon,

$$\gamma = \nu(f_3 - \nu^{*2})/2\nu^*(\nu^2 - f_1) + \nu^*(\nu^2 - f_1)/2\nu(f_3 - \nu^{*2}) + O(|\delta|), \quad (49)$$

while when a falling shell and another escaping shell collide,

$$\gamma = (\nu^2 - f_1)(f_3 - \nu^{*2})/2\nu\nu^* f_2 + O(1). \quad (50)$$

The former remains finite, while the latter in general diverges with $\delta \rightarrow 0$. This is another feature distinguishing two types of the collision near the horizon.

6 Conclusions

In this work we have constructed the spacetime representing the collision of two massive shells in the spherically symmetric background, under the condition that the colliding shells are transparent.

The junction condition Eq.(33) reduces to the DTR relation in the limit of the collision of two neutral null shells. Núñez et al [9] suggested that DTR relation has not to be satisfied but two other relations may hold, however as we have shown, one of their relation (Eq.(41)) is possible only when one of shells violates the energy condition and another relation is irrelevant.

We have investigated the collisions near horizons and have shown that phenomena like mass inflation does not occur when both colliding shells fall into the horizon. Physical reason of this result might be that the falling shells do not suffer infinite blue-shifts unlike the collision of null shells or gravitational waves [14, 15] corresponding to the finiteness of the Lorentz factor between the colliding shells, while the Lorentz factor between the falling shell and the escaping one is large.

The junction condition Eq.(33) has a form available to solve the Cauchy problem, so we can perform the N -body simulation applying the junction condition presented here.

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Spherical collapse of counterrotating particles and cosmic censorship

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Abstract

The collapse of counterrotating particles with spherical symmetry is investigated in the context of cosmic censorship. The stress-energy tensor is derived by summation of that of counterrotating collisionless particles. The metric functions are explicitly integrated using the elliptic integral. The solution contains three arbitrary functions which specifies initial distribution of mass $F(r)$, specific energy $E(r)$ and squared specific angular momentum $L^2(r)$. From generic C^∞ initial data, the continued collapse must result with shell-crossing singularities. Without shell-crossing singularities, the generic collapse must be bounced due to the angular momentum. This fact strongly suggests that the shell-focusing singularity in the collapse of an inhomogeneous dust ball is not generic and therefore not a genuine counterexample to the cosmic censorship. From nongeneric C^∞ initial data, that is, $L^2(r) = O(r^6)$, the collapse may result in a shell-focusing globally naked singularity. The naked singularity does satisfy the limiting focusing condition by Królak but not the strong curvature condition by Tipler.

1 introduction

In Newton gravity, the final fate of the gravitational collapse of a cloud of collisionless particles crucially depends on whether or not it is initially rotating and whether or not its velocity dispersion vanishes. Hence, also in general relativity, it is expected that rotation and velocity dispersion would play a crucial role. In general case, the collapse of a rotating cloud will proceed in a very aspherical manner. It is, however, known that spherical symmetry is very powerful simplification in investigating the space-time. Here we introduce *counterrotation* of collisionless particles which is consistent with spherical symmetry. A spherical cloud of counterrotating particles has been studied by several authors. The static case was first investigated by Einstein [1]. The static model is often referred to as an “Einstein cluster”. The dynamical model was investigated by Datta [2], Bondi [3] and Evans [4]. They obtained the partially differential equations for the metric functions and had qualitative understanding, but could not obtain the metric functions in the explicit integral form. Harada, Iguchi and Nakao [5] determined the metric functions in the explicit form and thereby examined how the occurrence of the shell-focusing naked singularity in the Lemaître-Tolman-Bondi solution [6] is altered and the implication to the cosmic censorship.

2 construction

In a spherically symmetric space-time, without loss of generality, the line element is written as

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}dr^2 + R^2(t,r)(d\theta^2 + \sin^2\theta d\phi^2). \quad (1)$$

We assume that each particle has negligibly small mass m and that collisions of particles are negligible. If we put a particle with a certain velocity and a certain inclination α ($0 \leq \alpha < 2\pi$) with respect to the equatorial plane $\theta = \pi/2$ at a certain point, it begins to move along a timelike geodesic in the plane

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inclined by the angle α . We choose the radial coordinate r as the one which follow the radial motion of the particle. The geodesic equation of the particle is integrable as

$$u^t = e^{-\nu} \sqrt{1 + \frac{L^2}{R^2}}, \quad (2)$$

$$u^r = 0, \quad (3)$$

$$(u^\theta)^2 + (u^\phi)^2 \sin^2 \theta = \frac{L^2}{R^4}, \quad (4)$$

$$u^\phi = \frac{M}{R^2 \sin^2 \theta}, \quad (5)$$

where $u^\mu \equiv dx^\mu/d\lambda$ is the 4-velocity of the particle and λ is the proper time of the particle. L and M are constants of motion, which are the magnitude and z -component of the specific angular momentum, i.e., the angular momentum per unit rest mass, of the particle, respectively. M satisfies

$$M = L \cos \alpha. \quad (6)$$

Moreover, from the comoving condition for the radial coordinate, we obtain the condition for the lapse function,

$$\nu' = \left(1 + \frac{L^2}{R^2}\right)^{-1} \frac{R'}{R}, \quad (7)$$

where the prime denotes the derivative with respect to r .

We put another particle at the same point with the identical phase of the identical orbital motion except for the inclination. Repeating this many times, we can finally obtain the homogeneous distribution with respect to the inclination with common orbital motion at the point. We also obtain the homogeneous distribution with respect to the inclination at another point of the same r . Repeating this many times, we can finally obtain a sufficiently homogeneous shell of collisionless particles at r with the common phase of the common orbital motion. We call this a thin shell. A thin shell is characterized by its energy density, specific energy and specific angular momentum. By constructing the thin shell, we recover spherical symmetry in the matter distribution. If we put together a lot of thin shells with smoothly distributed energy density, specific energy and specific angular momentum with respect to r , we can obtain a thick shell or a spherical cluster.

We regard this as a source of the spherically symmetric gravitational field which has been dealt with until now as if it was a given “background” space-time. It is clear that the local radial velocity dispersion of collisionless particles is not allowed though the coherent local radial motion is, of course, allowed. Clearly, an isotropic tangential velocity dispersion is introduced.

3 stress-energy tensor

Without loss of generality, we can concentrate on a point p on the equatorial plane in estimating the averaged stress-energy tensor. From Eqs. (4), (5) and (6), we can parametrize the tangential velocity by the inclination angle α as

$$u^\theta = \frac{L(r)}{R^2} \sin \alpha, \quad u^\phi = \frac{L(r)}{R^2} \cos \alpha, \quad (8)$$

for each particle comprising the thin shell at r . By construction, particles that pass through p are distributed homogeneously with respect to the inclination angle α . Therefore, if we average $u^\mu u_\nu$ with respect to the inclination angle α at p , we obtain, from Eqs. (2), (3) and (8),

$$\langle u^\mu u_\nu \rangle_\alpha \equiv \frac{1}{2\pi} \int_0^{2\pi} d\alpha u^\mu u_\nu = \begin{pmatrix} -\left(1 + \frac{L^2}{R^2}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \frac{L^2}{R^2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \frac{L^2}{R^2} \end{pmatrix}. \quad (9)$$

Here, we note that the inclination-averaged stress-energy tensor $T^{\mu\nu}$ is given as $T^{\mu\nu} = mn \langle u^\mu u^\nu \rangle$, where $n = n(t, r)$ is the proper number density of particles. Then we finally obtain T^μ_ν for the spherical cluster of counterrotating particles as

$$T^\mu_\nu = \begin{pmatrix} -\epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \Pi & 0 \\ 0 & 0 & 0 & \Pi \end{pmatrix}, \quad (10)$$

where ϵ and Π are the energy density and the tangential pressure given as

$$\epsilon = mn \left(1 + \frac{L^2}{R^2} \right), \quad \Pi = \frac{mn}{2} \frac{L^2}{R^2}, \quad (11)$$

respectively. The “equation of state” for the cluster is given as

$$\Pi = \frac{1}{2} \frac{L^2}{R^2 + L^2} \epsilon. \quad (12)$$

This stress-energy tensor is consistent with spherical symmetry of the space-time.

4 metric functions

4.1 comoving coordinates

The Einstein equation in the comoving coordinates (1) with only tangential pressure is given as follows. $\lambda(t, r)$, $\nu(r, t)$ and $R(r, t)$ satisfy the following coupled set of partially derivative equations:

$$e^{2\lambda} = \frac{R^2}{1+f}, \quad (13)$$

$$\nu' = -\frac{1}{h} \frac{\partial h}{\partial R} R', \quad (14)$$

$$\dot{R}^2 e^{-2\nu} = -1 + \frac{2F}{R} + \frac{1+f}{h^2}, \quad (15)$$

where the dot denotes the derivative with respect to t and the function $h(r, R)$ gives the “equation of state” as

$$\Pi = -\frac{R}{2h} h_{,R} \epsilon. \quad (16)$$

The energy density is given by

$$\epsilon = \frac{F'}{4\pi R^2 R'}. \quad (17)$$

The arbitrary functions $F(r)$ and $f(r) > -1$ are the conserved Misner-Sharp mass and the specific energy, respectively. The function $h(r, R) > 0$ has a meaning of the internal elastic energy per volume and the dust limit is given by $h = 1$. From regularity at the center, $f(0) = h^2(0, 0) - 1$, $R(0, t) = 0$, $|\nu(0, t)| < \infty$ and $F(r)/r^3 < \infty$ at $r \rightarrow 0$. Note that we can set $f(0) = h^2(0, 0) - 1 = 0$. The solution can be matched with the Schwarzschild space-time at an arbitrary radius $r = r_b$ if we identify the mass parameter M with $F(r_b)$. Assuming that R is initially a monotonically increasing function of r and rescaling the radial coordinate r , we can identify r with the circumferential radius R on the initial space-like hypersurface $t = 0$.

4.2 mass-area coordinates

In order to give an explicit solution for the metric functions, we introduce the mass-area coordinates following the procedure of Magli [7]. The line element is written as

$$ds^2 = -A(m, R)dm^2 - 2B(m, R)dmdR - C(m, R)dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (18)$$

The derivation of an exact solution with general tangential pressure is described in Magli [7]. From the equation of state for the cluster (12) and boundary condition $h(0,0) = 1$, the function $h(m,R)$ is given as

$$h^2 = 1 + \frac{L^2}{R^2}. \quad (19)$$

Since, in this model, $h(r,R)$ depends on r , the coupled partially differential equations in the comoving coordinates cannot be decoupled. Adopting the results obtained in Magli [7] to the present problem, we obtain the following metric functions of the explicit form:

$$A = H \left(1 - \frac{2m}{R} \right), \quad (20)$$

$$B = -E\sqrt{H} [E^2 - 2V(R;m)]^{-1/2}, \quad (21)$$

$$C = \left(1 + \frac{L^2}{R^2} \right) [E^2 - 2V(R;m)]^{-1}, \quad (22)$$

$$u = \pm \left(1 + \frac{L^2}{R^2} \right)^{-1/2} [E^2 - 2V(R;m)]^{1/2}, \quad (23)$$

$$\begin{aligned} \sqrt{H(m,R)} &= \frac{(R^0)_{,m}}{E} \left(1 + \frac{L^2}{(R^0)^2} \right) [E^2 - 2V(R^0;m)]^{-1/2} \\ &\quad + \int_{R^0}^R \frac{1}{E} \left[\left(1 + \frac{L^2}{x^2} \right)^2 + \frac{1}{2} \left(1 + \frac{L^2}{x^2} \right) x E_{,m} - \frac{E^2 L_{,m}^2}{2x} \right] \\ &\quad \times [E^2 - 2V(x;m)]^{-3/2} \frac{dx}{x}, \end{aligned} \quad (24)$$

$$\epsilon = \frac{1}{4\pi R^2 E \sqrt{H}} \left(1 + \frac{L^2}{R^2} \right) [E^2 - 2V(R;m)]^{-1/2}, \quad (25)$$

where $E^2(m) \equiv 1 + f(F^{-1}(m))$ and the effective potential $V(R;m)$ is defined as

$$V(R;m) \equiv \frac{1}{2} \left(1 - \frac{2m}{R} \right) \left(1 + \frac{L^2}{R^2} \right) = \frac{1}{2} - \frac{m}{R} + \frac{L^2}{2R^2} - \frac{mL^2}{R^3}. \quad (26)$$

The integral in the expression of $\sqrt{H}$ is reduced to

$$\int_{R^0}^R \frac{2(x^2 + L^2)^2 + x^3[(E^2)_{,m}(x^2 + L^2) - (L^2)_{,m}E^2]}{2E[(E^2 - 1)x^3 + 2mx^2 - L^2x + 2mL^2]\sqrt{(E^2 - 1)x^4 + 2mx^3 - L^2x^2 + 2mL^2x}} dx. \quad (27)$$

Therefore we find that the general solution is expressed by an elliptic integral.

If $L(m) = 0$, the solution is reduced to the LTB solution in the mass-area coordinates. If $E(m) = 1$ and $L(m) = 4m$, the integral is expressed by elementary functions as follows:

$$A = H \left(1 - \frac{2m}{R} \right), \quad (28)$$

$$B = -\frac{R}{|R - 4m|} \sqrt{\frac{RH}{2m}}, \quad (29)$$

$$C = \frac{R(R^2 + 16m^2)}{2m(R - 4m)^2}, \quad (30)$$

$$u = \pm |R - 4m| \sqrt{\frac{2m}{R(R^2 + 16m^2)}}, \quad (31)$$

$$\begin{aligned} \sqrt{H} &= \frac{(R^0)_{,m}((R^0)^2 + 16m^2)}{|R^0 - 4m|\sqrt{2mR^0}} + \text{sign}(R^0 - 4m) \\ &\quad \times \left[\left(\frac{(R - 8m)^2 + 80m^2}{3m(R - 4m)} \sqrt{\frac{R}{2m}} + 4\sqrt{2} \ln \frac{\sqrt{R} + 2\sqrt{m}}{|\sqrt{R} - 2\sqrt{m}|} \right) \right] \end{aligned}$$

$$- \left(\frac{(R^0 - 8m)^2 + 80m^2}{3m(R^0 - 4m)} \sqrt{\frac{R^0}{2m}} + 4\sqrt{2} \ln \frac{\sqrt{R^0} + 2\sqrt{m}}{|\sqrt{R^0} - 2\sqrt{m}|} \right), \quad (32)$$

$$\epsilon = \frac{R^2 + 16m^2}{4\pi R^2 |R - 4m| \sqrt{2mR\sqrt{H}}}, \quad (33)$$

5 causal structure

Since a shell-crossing singularity is widely believed not to be a true singularity, we suppose that no shell-crossing singularity occurs. Furthermore, no light ray can emanate from noncentral ($r > 0$) shell-focusing singularities. Therefore it turns out to be sufficient to consider a central ($r = 0$) shell-focusing singularity. Note that the divergence of the energy density means the scalar curvature singularity.

5.1 general case

The motion of each thin shell labeled by r is described by Eq. (15) or (23), which transforms as

$$\frac{1}{2} \left(1 + \frac{L^2}{R^2} \right) \left(\frac{dR}{d\tau} \right)^2 + \frac{1}{2} \left(1 - \frac{2F}{R} \right) \left(1 + \frac{L^2}{R^2} \right) = \frac{1}{2} (1 + f), \quad (34)$$

where τ is the proper time of the thin shell. The motion of a mass shell labeled by r is the same as the radial motion of a test particle with the specific angular momentum $L(r)$ and specific energy $(1 + f(r))$ at infinity in the Schwarzschild space-time of which mass parameter is $F(r)$. The factor $(1 + L^2/R^2)$ in the first term in the left hand side of Eq. (34) is the Lorentz factor which originates from the difference between λ , the proper time of the particle which is orbiting the center, and τ , that of the shell-comoving observer which moves only in the radial direction.

In convenience for further discussions, we adopt here the radial coordinate r which agrees with the circumferential radius R on the initial spacelike hypersurface $t = 0$. From this choice of the radial coordinate, the initial radius $R^0(m)$ is specified as $R^0(m) = F^{-1}(m)$. We assume that the metric functions and physical quantities are not only regular and but also C^∞ . From this assumption, the metric variables in the comoving coordinates can be expanded as

$$\nu(t, r) = \nu_0(t) + \nu_2(t)r^2 + \dots, \quad R(t, r) = R_1(t)r + R_3(t)r^3 + \dots, \quad (35)$$

and we set $\nu_0(t) = 0$ by using the rescaling freedom of the time coordinate. From the choice of r ,

$$R_1(0) = 1, \quad R_3(0) = \dots = 0. \quad (36)$$

Then, from Eqs. (7) and (34), the arbitrary functions $F(r)$, $f(r)$ and $L^2(r)$ should be expanded as

$$F(r) = F_3 r^3 + F_5 r^5 + \dots, \quad f(r) = f_2 r^2 + f_4 r^4 + \dots, \quad L^2(r) = L_4 r^4 + L_6 r^6 + \dots, \quad (37)$$

which imply C^∞ initial data. From Eq. (17), the energy density should be expanded as

$$\epsilon(t, r) = \epsilon_0(t) + \epsilon_2(t)r^2 + \dots, \quad (38)$$

and then

$$\epsilon(t, 0) = \epsilon_0(t) = \frac{3F_3}{4\pi R_1(t)^3}. \quad (39)$$

Observing the lowest order of Eq. (23) or (34), the time development of $R_1(t)$ is given by

$$\left(\frac{dR_1}{dt} \right)^2 = \frac{2F_3}{R_1} + f_2 - \frac{L_4}{R_1^2}. \quad (40)$$

From Eqs. (36) and (40), we obtain the condition for initial data

$$2F_3 + f_2 - L_4 \geq 0. \quad (41)$$

The equality holds only when $\dot{R}_1(0) = 0$.

First we consider the case of $L_4 > 0$. This condition corresponds to the generic C^∞ initial data within this model. Then, from Eq. (40), we find that $R_1(t)$ cannot vanish. Therefore no central singularity occurs for $L_4 > 0$. Next we examine the motion of the shell $r > 0$. From Eq. (23) or (34), allowed regions for given r are obtained by

$$V(R; r) \leq \frac{1}{2}(1 + f). \quad (42)$$

For sufficiently small r ,

$$V(R; r) \rightarrow -\infty \quad \text{as } R \rightarrow 0, \quad (43)$$

$$V(2F; r) = 0, \quad (44)$$

$$V(4F; r) \approx \frac{1}{64} \frac{L^2}{F^2} > \frac{1}{2}(1 + f) \quad (45)$$

$$V(R; r) \rightarrow \frac{1}{2} \quad \text{as } R \rightarrow \infty, \quad (46)$$

where $\approx$ means the equality up to the lowest order. Considering that $V(R; r) - (1 + f)/2$ for fixed r has at most three zeros and that $f(\tau) = O(\tau^2)$, the allowed regions are

$$0 \leq R \leq R_I, \quad R_{II} \leq R, \quad (\text{for } f \geq 0), \quad (47)$$

$$0 \leq R \leq R_I, \quad R_{II} \leq R \leq R_{III}, \quad (\text{for } f < 0), \quad (48)$$

where $R_I < R_{II} < R_{III}$ are positive zeros of $V(R; r) - (1 + f)/2$. The following inequality is satisfied

$$2F < R_I < 4F < R_{III}. \quad (49)$$

Since $F = O(\tau^3)$, $R(0, r) = r$ cannot be in the inner allowed region $0 \leq R \leq R_I$. This means that $R(t, r)$ must be in the outer allowed region at $t = 0$. Therefore we conclude that the region around $r = 0$, which was initially in a collapsing phase, necessarily experiences a bounce and begins to expand. The motion after that is an eternal expansion for $f \geq 0$ or oscillations for $f < 0$. This implies that, since $R > R_{II} > 4F > 2F$, the region around $r = 0$ is untrapped. We summarize this case by no central singularity, no apparent horizon and a bounce of the region around the center.

We proceed to the case of $L_4 = 0$. From Eq. (40), we find that the initially collapsing cloud inevitably form a central shell-focusing singularity after a finite proper time. In order to see whether this central singularity is naked or covered, we examine the motion of the region around the center. Here we define

$$D \equiv \lim_{r \rightarrow 0} \frac{L}{F} = \frac{L_0^{1/2}}{F_3}. \quad (50)$$

For $D > 4$, for sufficiently small $r > 0$,

$$V(R; r) \rightarrow -\infty \quad \text{as } R \rightarrow 0, \quad (51)$$

$$V(2F; r) = 0, \quad (52)$$

$$V\left(\frac{D^2}{4}F; r\right) \approx \frac{1}{2} + \frac{4(D^2 - 16)}{D^4} > \frac{1}{2}(1 + f). \quad (53)$$

$$V(R; r) \rightarrow \frac{1}{2} \quad \text{as } R \rightarrow \infty. \quad (54)$$

Then, the allowed regions are

$$0 \leq R \leq R_I, \quad R_{II} \leq R \quad (\text{for } f \geq 0), \quad (55)$$

$$0 \leq R \leq R_I, \quad R_{II} \leq R \leq R_{III} \quad (\text{for } f < 0), \quad (56)$$

where the following inequality is satisfied:

$$2F < R_I < \frac{D^2}{4}F < R_{II}. \quad (57)$$

Therefore we conclude that the region around $r = 0$, which was initially in a collapsing phase, necessarily experiences a bounce and begins to expand. The motion after that is an eternal expansion for $f \geq 0$ or oscillations for $f < 0$. This implies that, since $R > R_{II} > 4F > 2F$, the region around $r = 0$ is untrapped. Therefore the central singularity is naked. Since the space-time can be matched to the Schwarzschild space-time at an arbitrary radius $r = r_b$, we can construct the space-time with a globally naked singularity. The results do not depend on details of initial density profile.

Then we proceed to the case $0 \leq D < 4$. Consider the sign of

$$g(R; r) \equiv 2R(R^2 + L^2) \left[V(R; r) - \frac{1}{2}(1 + f) \right] \approx -fR^3 - 2FR^2 + D^2F^2R - 2D^2F^3, \quad (58)$$

for sufficiently small $r > 0$. For $f \geq 0$, it is easily found that $g(R; r)$ is always negative. For $f < 0$, it is easily found that $g(R; r)$ takes the relative maximum and minimum at $R = R_{max}$ and R_{min} ($R_{max} < R_{min}$) respectively and that the relative maximum is negative. Therefore the allowed region is

$$0 \leq R \quad (\text{for } f \geq 0), \quad (59)$$

$$0 \leq R \leq R_I \quad (\text{for } f < 0). \quad (60)$$

Therefore, the collapse continues to a covered singularity in the region around the center. An apparent horizon is formed in this case.

5.2 special case

Here we present the critical case in which $f(r) = 0$ and $L(r) = 4F(r)$. Since $L_4 = 0$, the central singularity occurs. The effective potential for a mass shell is given in this case as

$$V(R; r) = \frac{1}{2} - \frac{F(R - 4F)^2}{R^3}. \quad (61)$$

Then each initially collapsing shell of $r > 0$ approaches $R = 4F$ as

$$R - 4F \propto \exp\left(-\frac{\tau}{8F}\right). \quad (62)$$

In the region around the center $r = 0$, the collapse frozen to $R = 4F$ due to the angular momentum braking. Since $R > 4F > 2F$, the region around the center is untrapped eternally. Therefore the central singularity is naked and can be globally naked. We can prove that no shell crossing occurs and the coordinate system is valid at least in the region around the center by directly showing that $\sqrt{H} > 0$ in Eq. (32).

The central singularity is naked if and only if there exists a future-directed outgoing null geodesic which emanates from the singularity. The root equation which determines the tangent of the null geodesic from the singularity is obtained in Magli [7] as

$$y_0 = \frac{1}{2\beta} \lim_{m \rightarrow 0} \left[m^{3(1-\beta)/2} \sqrt{H} \sqrt{\left(-1 + \frac{1}{h^2}\right) m^{-(1-\beta)} + \frac{1}{y_0} \left(\frac{1}{h} - \sqrt{\frac{m^{1-\beta}}{y_0} - 1 + \frac{1}{h^2}}\right)} \right] \Bigg|_{R=2y_0m^\beta}, \quad (63)$$

with $\beta > 1/3$. Considering the explicit metric functions, i.e., Eq. (32), we obtain the positive finite root of the root equation with $\beta = 7/9$ as

$$y_0 = \left(\frac{24F_3^2 - F_5}{4\sqrt{2}F_3^{13/6}} \right)^{2/3}, \quad (64)$$

if $F_5 < 24F_3^2$, i.e., no shell-crossing singularity occurs. The above result implies that, along the future-directed outgoing radial null geodesic which emanates from the naked singularity with the definite tangent,

$$R \approx 2y_0m^{7/9} \approx 2y_0F_3^{7/9}r^{7/3}. \quad (65)$$

Therefore, unlike marginally bound dust collapse, the central shell-focusing singularity is naked even for initially homogeneous ball.

The procedure for determining the curvature strength of the naked central singularity in a spherically symmetric space-time with vanishing radial pressure is given in Harada, Nakao and Iguchi [8]. Solving the null geodesic equation, we obtain

$$m \propto \lambda^{9/7}, \quad (66)$$

where λ is the affine parameter. Then the Ricci tensor contracted by the tangent vector k^a is given by

$$R_{ab}k^ak^b \propto \lambda^{-12/7}. \quad (67)$$

Therefore the naked singularity satisfies the limiting focusing condition by Królak [9] but not the strong curvature condition by Tipler [10]. This means that the cross section of any congruence of null geodesics which hit the singularity is not crushed to zero but is contracted.

6 discussions

In the realistic collapse from generic C^∞ initial data, net rotation will induce very large deformation from spherical symmetry. Nevertheless, this model may display us pure effect of counterrotation or tangential velocity dispersion on the formation of shell-focusing naked singularity. From this model we learn a lesson that generic counterrotation or tangential velocity dispersion will rub off the shell-focusing naked singularity which forms without it. This strongly suggests that the shell-focusing naked singularity which may occur in the generic LTB solution cannot be a counterexample to the CCH because the effect of counterrotation will be necessarily present in generic collapse of a cloud of collisionless particles which obey the Einstein-Vlasov equation.

The important corollary in this paper is that the continued collapse of counterrotating particles must result with shell-crossing singularities. This strongly suggests that the extension of the space-time beyond the shell-crossing singularity is very important problem to overcome. It is expected that the extension of the space-time with a cluster of counterrotating particles beyond the shell-crossing singularity results in incorporating a radial velocity dispersion. Therefore we must deal with the radial velocity dispersion to obtain a picture of the formation of black holes and singularities from generic gravitational collapse of collisionless particles.

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Initial value problems in singular spacetimes

— Probing timelike singularities with finite energy waves —

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Abstract

To probe naked singularities in spacetime with waves rather than with particles we study the well-posedness of initial value problems for test scalar fields with finite energy so that the natural function space of initial data is the Sobolev space. In the case of static and conformally static spacetimes we examine the essential self-adjointness of the time translation operator in the wave equation defined in the Hilbert space. For some spacetimes the classical singularity becomes regular if probed with waves while stronger classical singularities remain singular. If the spacetime is regular when probed with waves we may say that the spacetime is “globally hyperbolic.”

1 Introduction

In general relativity a singular spacetime is defined by the geodesic incompleteness [1]. However, sometimes such a definition gives a very weak singularity which seems almost harmless from physical point of view. For example, a spacetime from which a single point is taken out is a singular spacetime because there is a geodesic curve which terminates at the point outside of the spacetime with a finite affine time. A stronger ‘physical singularity’ appears for example at the center of a spherically symmetric black hole, where the curvature scalar diverges and therefore the resultant infinite tidal force will tear off any physical object. The classification of singularities is yet under way but has not been completed [2].

The standard definition of a spacetime singularity is physically based on a probe with a classical point particle. In this article we shall discuss a wave probe to timelike singularities which was initiated by Wald [3] and later developed by Horowitz and Marolf [4]. The idea of the probe with waves rather than with classical particles is motivated by quantum field theory because everything should be described by quantum fields. The wave may propagate through the would-be singularity with a definite and unique way. For example, in the case of hydrogen atom the wave function is finite at the origin, which is a classical singularity. It is known that if the space is geodesically complete the Laplacian operator has a self-adjoint extension and the extension is unique so that the wave propagation is well defined. Converse is not always true. If the geodesic completeness is replaced by the well-posedness of initial value problems for test fields the concept of the global hyperbolicity should be drastically changed as Clarke [5] has advocated.

We shall be concerned with a classical wave propagation dictated by the Klein-Gordon equation in a curved spacetime with timelike singularities. To simplify the problem we shall assume that there is a (conformal) Killing vector throughout this article, which is the price we have to pay in the present work. Let us consider an $(n + 2)$ -dimensional static spacetime of the metric form

$$ds^2 = -V^2 dt^2 + h_{ij} dx^i dx^j, \quad (1)$$

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with a timelike Killing vector field $\xi^\mu = (\partial_t)^\mu$. For the initial value problem we introduce the following norm on a function space on each $t = \text{const.}$ hypersurface:

$$\|f\| := \left(\frac{q^2}{2} \int_{\Sigma} d\Sigma V^{-1} f^* f + \frac{1}{2} \int_{\Sigma} d\Sigma V h^{ij} D_i f^* D_j f \right)^{\frac{1}{2}}, \quad (2)$$

with q^2 being a positive constant and D_i denotes the covariant derivative with respect to the induced metric h_{ij} on Σ . Here $d\Sigma = d^{n+1}x \sqrt{h}$ is the natural volume element on Σ . We call the function space $\mathcal{H} = \{f \mid \|f\| < \infty\}$ the *Sobolev space* or H^1 . The Sobolev norm has been used in the standard proof of the well-posedness of initial value problems in a general globally hyperbolic spacetime [1].³ It is known that the norm is bounded above by the field energy so that the finiteness of the energy implies the finiteness of the norm [7]. Since we cannot afford to prepare an infinite energy field configuration as initial data, the function space is naturally limited by the condition:⁴

$$\|f\|^2 < \text{const.} \times E < \infty, \quad \text{where} \quad E := \int_{\Sigma} d\Sigma n^\mu \xi^\nu T_{\mu\nu}[f], \quad (3)$$

with n^μ being the unit normal to Σ and the energy momentum tensor is given by

$$T_{\mu\nu}[f] := \frac{1}{2} (\nabla_\mu f^* \nabla_\nu f + \nabla_\nu f^* \nabla_\mu f) - \frac{1}{2} g_{\mu\nu} (\nabla^\sigma f^* \nabla_\sigma f + m^2 f^* f). \quad (4)$$

The corresponding natural inner product in our Hilbert space is defined as

$$(f, g) := \frac{q^2}{2} \int_{\Sigma} d\Sigma V^{-1} f^* g + \frac{1}{2} \int_{\Sigma} d\Sigma V h^{ij} D_i f^* D_j g. \quad (5)$$

so that $\|f\|^2 = (f, f)$.

In static or conformally static spacetimes the wave equation, $\square\phi = 0$, becomes of the form,

$$\partial_t^2 \phi = -A\phi, \quad (6)$$

where $A := -VD^i V D_i$ is an operator on $\mathcal{H}$, whose self-adjoint extension is a time translation operator in the sense of Wald [3]. In this case the well-posedness of the initial value problem is translated into the essential self-adjointness of the operator A because of the spectral theorem [8]. The operator A is symmetric on $\mathcal{H}$ if the preliminary domain of A is suitably chosen, e.g., $C_0^\infty(\Sigma)$, a set of smooth functions with compact support on Σ . The symmetric operator A is said to be self-adjoint if the domain of the operator A and that of its conjugate A^* coincide, $\mathcal{D}(A) = \mathcal{D}(A^*)$. In this case the eigenvalue of the operator A is real and positive and therefore the time evolution of the field is well-defined [3]. We can obtain self-adjoint extensions of A by relaxing the boundary condition. If there are many possibilities of the self-adjoint extensions, we have to choose one of them by imposing a particular boundary condition, which is normally imposed by some physical requirement. In the case of naked singularities we do not have any criterion to choose the boundary condition. Therefore, if the self-adjoint extension is unique, there remains no ambiguity in the choice of the boundary conditions. The operator is called essentially self-adjoint if the self-adjoint extension is unique. In such a case we can say that the spacetime is “wave-regular.” Namely, we prepare a smooth and nice initial data at some spatial hypersurface by choosing the Sobolev space H^1 as the Hilbert space. The wave will propagate and eventually hit the timelike singularity and will be scattered off in some way. The point of the essential self-adjointness is that any unwanted singular modes which are not contained in the Hilbert space will not appear after scattering so

³In the standard formulation of well-posed initial value problems Sobolev norms containing higher order derivatives are chosen to define a topology on the space of initial data. Here we study especially the uniqueness of the solution in a non-globally hyperbolic spacetime with a timelike singularity. For this purpose it is sufficient to adopt H^1 as our Sobolev space, which is larger than H^m with norms containing $m(> 1)$ th order derivatives. Our results in this paper hold also for $H^m(> 1)$.

⁴The difference between ours and that of reference [4] is in the definition of the norm and therefore of the Hilbert space. In the case of quantum mechanics the natural Hilbert space is the linear function space with the square integrability, L^2 , because of the probabilistic interpretation of the wave function.

that the initial value problem is well-posed with no arbitrariness in the choice of the boundary conditions and the prediction is unique.

The essential self-adjointness can be tested by considering solutions of equations

$$(A^* \pm i)\psi = 0, \quad (7)$$

and showing that such solutions do not belong to our Hilbert space [8].

We would like to emphasize the relevance of the present work in quantum field theory in curved spacetime. It will be natural to expand a quantum field in terms of the normal modes which belong to the Sobolev space rather than L^2 . We assign the coefficients of the mode expansion as annihilation and creation operators. The quantum states are constructed by applying the creation operators to the vacuum state which is defined by the condition that the vacuum is annihilated by all the annihilation operators. Note that the constructed quantum states belong to L^2 class in the Fock space because of the probabilistic interpretation. This construction implies that if the initial value problem is well-posed the vacuum expectation value of the energy momentum tensor should be well-behaved near the would-be singularities so that the field energy is finite.

2 Wave probe in spherically symmetric static spacetimes

To illustrate the test of the essential self-adjointness of the operator A in Eq. (6) in curved spacetimes, we study the well-known spherically symmetric spacetimes.

In a general $(n + 2)$ -dimensional spherically symmetric static spacetime, the metric is given by

$$ds^2 = -V^2 dt^2 + V^{-2} dr^2 + R^2 d\Omega_n. \quad (8)$$

Here we assume that V^2 is a positive function of r for $0 < r < \infty$ and is singular at $r = 0$ so that the central singularity is timelike. Equations (7) reduce with the ansatz $\psi = f(r)Y(\Omega)$ to

$$f'' + \frac{(V^2 R^n)'}{V^2 R^n} f' - \frac{c}{V^2 R^2} f \mp i \frac{f}{V^4} = 0, \quad (9)$$

where the prime denotes the derivative with respect to r and c is the angular momentum quantum number. The norm of f is given by

$$\|f\|^2 = \frac{q^2}{2} \int d\mu dr R^n V^{-2} |f|^2 + \frac{1}{2} \int d\mu dr R^n V^2 |f'|^2, \quad (10)$$

where $d\mu$ is the volume element on the unit n -sphere ($d\Sigma = d\mu dr V^{-1} R^n$).

For the essential self-adjointness of A , the norms of the solutions of Eqs. (9) should be divergent for each c and each sign of the imaginary term. We can easily verify that the norm $\|f\|$ is divergent for $c > 0$ if it is for $c = 0$, so we will examine the essential self-adjointness for the $c = 0$ (S-wave) case.

• Negative mass Schwarzschild spacetime

The 4-dimensional negative mass Schwarzschild metric is given by

$$V^2 = 1 + \frac{2M}{r}, \quad r = R, \quad (M > 0). \quad (11)$$

Near the singularity, Eqs. (9) become

$$f'' + \frac{1}{r} f' = 0, \quad (12)$$

since the other terms are less singular or even regular at $r = 0$. Then the two independent solutions $f = \text{const.}$ and $g = \ln r$ are obtained. For the latter solution, the second term of the norm squared (10) behaves as

$$\sim \int_0 dr r^2 V^2 |g'|^2 \sim \ln r|_0 \rightarrow \infty. \quad (13)$$

Thus the operator A on this spacetime is essentially self-adjoint, hence the spacetime is wave-regular.

One might worry that our analysis does not involve the $\pm i$ part and only one of the two solutions for Eq. (12) is verified not to be in the Hilbert space. Here, we should remark that the solution f which well behaves near the singularity $r = 0$ is divergent at infinity $r = \infty$ so that the Sobolev norm is divergent. The point is that if one of the independent solutions fails to well behave near the singularity there are no ways for the other solution to meet the condition at infinity because there is no available other independent solution to superpose.

Note that if L^2 were chosen as the Hilbert space, the operator A would not be essentially self-adjoint in this case [4].

• Reissner-Nordström spacetime

For the over extreme Reissner-Nordström metric,

$$V^2 = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \quad R = r, \quad (Q^2 \geq M^2), \quad (14)$$

where Q denotes the electric (magnetic) charge. Near the timelike singularity at $r = 0$, Eqs. (9) for the S-wave ($c = 0$) become $f'' = 0$ ($V^2 R^2 \sim Q^2$) and f behaves as $f \sim r$ or a constant. Then, the norm squared (10) is finite. Thus, the classical singularity remains wave-singular. For $c > 0$ modes, the norms are divergent. This implies that only the S-wave can fall into the singularity.

Although we are motivated by the cosmic censorship, our analysis is basically local so that this wave probe can be also explored in cases that the timelike singularities are hidden behind the horizon by extending our analysis in a straightforward way.

To summarize the two examples above, the timelike singularity in the negative mass Schwarzschild spacetime is wave-regular while that of the over extreme Reissner-Nordström spacetime is not. The tendency that the over extreme Reissner-Nordström spacetime is more singular than the negative mass Schwarzschild spacetime sounds natural because the curvature is more divergent for the over extreme Reissner-Nordström spacetime. This reminds us of the well-known examples in quantum mechanics: the Coulomb potential and the r^{-2} potential problems in 3-dimensional space. The former is essentially self-adjoint and the latter is not if the r^{-2} potential is attractive and too strong [8].

In general, if the metric functions of the metric (8) behave as

$$R \sim r^p, \quad V^2 R^n \sim r^k, \quad (p > 0, k > 0), \quad (15)$$

near the singularity $r = 0$, Eqs. (9) become

$$f'' + \frac{k}{r} f' = 0, \quad (16)$$

with the ansatz $p > k - 1$, which holds for all our examples. Then, the solutions are $f = \text{const.}$ and $g = \ln r$ ($k = 1$), or r^{1-k} ($k \neq 1$). Since the norm squared (10) for the solution g is estimated as

$$\|g\|^2 \sim \int dr r^{-k+2(p-k+1)} + \int dr r^{-k} \sim \int dr r^{-k}, \quad (17)$$

the singularity turns out to be wave-regular in the case $k \geq 1$. The negative mass Schwarzschild case is $k = 1$ and the Reissner-Nordström case $k = 0$.

One might guess from the two examples above that the quasi-local mass given by

$$M_{\text{local}} = -\frac{R}{2} \{g^{\mu\nu} (\partial_\mu R) (\partial_\nu R) - 1\} \quad (18)$$

would be finite as $r \rightarrow 0$ in the wave-regular cases while it is infinite in wave-singular cases. However, this is not the case in other models as we shall see below.

• The Wyman solution

We shall consider some less known but hopefully more physical solutions of the Einstein equations coupled to matter fields. The Wyman solution is a static solution of the 4-dimensional Einstein equations coupled to a minimally coupled scalar field [9]. The metric is given by

$$V^2 = \left(\frac{\rho}{\rho + 2\eta} \right)^{m/\eta}, \quad R^2 = \rho^{1-m/\eta} (\rho + 2\eta)^{1+m/\eta}, \quad (19)$$

where $\eta = \sqrt{m^2 + \sigma^2}$ with a scalar charge σ , so $m/\eta < 1$. A curvature singularity is located at $\rho = 0$. Since $V^2 R^2 \sim \rho$, this is the case of the metric (15) with $k = 1$ and thus the spacetime is wave-regular.

- **Charged dilaton solution**

The 4-dimensional extremally charged dilaton solution is given by [4]

$$V^2 = \left(\frac{\rho}{\rho + r_+} \right)^{\frac{1}{1+a^2}}, \quad R^2 = (\rho + r_+)^{\frac{2}{1+a^2}} \rho^{\frac{2a^2}{1+a^2}}, \quad (r_+ = \text{const.} > 0), \quad (20)$$

with a^2 being a positive parameter in the model. In the case $a^2 > 1$, the central singularity at $\rho = 0$ becomes timelike, while it is null for $a^2 \leq 1$. For any $a^2 > 1$, $R^2 V^2 \sim \rho^2$ this is the case of the metric (15) with $k = 2$ and the singularity is wave-regular. The first term of the norm squared (17) diverges for $a^2 \leq 3$ so that we would reproduce the result in Ref. [4] if we chose the L^2 function space as our Hilbert space.

3 Generalizations

3.1 Probes with other fields

So far, the examples of timelike singularities have been probed with minimally coupled massless scalar fields. One can think of probing singularities with other fields such as spinor, vector, tensor fields, or metric perturbations. Our procedure of probing singularity is immediately generalized to each case by replacing the inner product (5), hence the norm, with that for the probe field in an appropriate way.

For example, when probing a timelike singularity with a massless scalar field coupled to the scalar curvature, we should adopt the inner product respecting the stress-tensor $T_{\mu\nu}^c$ for the field (see Eq. (3.190) in Ref. [10]). The field equation in the case is $(\square + \xi \mathcal{R})\phi = 0$, so the operator

$$A = -V D^i V D_i - \xi V^2 \mathcal{R} \quad (21)$$

should be examined for the essential self-adjointness, where ξ is a numerical factor and $\mathcal{R}$ the scalar curvature. As is well known, in the conformally coupled scalar field case, that is $\xi = (d-2)/4(d-1)$ for any spacetime dimension d , the field equation is invariant under the conformal transformations of the metric and the field, $g_{\mu\nu}(x) \rightarrow \bar{g}_{\mu\nu}(x) = C^2(x)g_{\mu\nu}(x)$, $\phi \rightarrow \bar{\phi} = C^{(2-d)/2}\phi$. Since $T_{\mu\nu}^c$ transforms as $\bar{T}_{\mu\nu}^c = C^{2-d}T_{\mu\nu}^c$, the corresponding inner product is conformally invariant. Thus, the calculation is as simple as that in the static case, when singularities in conformally static spacetimes are probed with conformally coupled scalar fields.

Generalizations to probes with spin 1/2 and 1 fields are similar. Explicit expressions of $T_{\mu\nu}$ for such fields are found, for example, in Ref. [10].

3.2 Conformally static spacetimes

In the previous section, we have examined the would-be naked singularities in static spacetimes. However, physically more interesting cases which can confront with the cosmic censorship are such dynamical spacetimes that naked singularities emerge after gravitational collapse. As the wave approach to fully dynamical cases is still far reaching for us at present, we will present a simple model to mimic such a dynamical problem on the basis of our study of the timelike singularity in conformally static spacetimes.

- **Self-similar case**

Dynamical problems such as the formation of naked singularities can be made tractable by assuming self-similarity [11]. Self-similar spacetimes also have attracted attention in connection with the critical behavior in gravitational collapse and have been studied in detail by many authors [12]. Probing timelike singularities especially in self-similar spacetimes therefore is an interesting issue concerning the cosmic censorship. A technical advantage of self-similar metrics is that they can be written in the conformally static form so that we can straightforwardly apply our procedure developed for the static spacetime in the previous section to self-similar spacetimes when probing with conformally coupled scalar fields.

For a massless scalar field, there is a non-static spherically symmetric solution discovered by Roberts [13]. As one of the models of naked singularities in self-similar spacetimes, we will analyze the timelike singularity in the Roberts solution, whose metric can be written in the conformally static form,

$$\begin{aligned} ds^2 &= e^{2\eta} d\bar{s}^2 = e^{2\eta} \{-d\eta^2 + dr^2 + R^2(r) d\Omega^2\}, \\ R^2(r) &:= \frac{1}{4} \{1 + p - (1 - p)e^{-2r}\} (e^{2r} - 1), \end{aligned} \quad (22)$$

where p is an integration constant and $(\partial_\eta)^\mu$ is the homothetic vector. For the value $0 < p < 1$, the curvature singularity located at $r = 0$ becomes timelike and the global structure is identical to that of the negative mass Schwarzschild spacetime. By using this solution the problem of self-similar scalar field collapse has been discussed [13, 14].

When probing the singularity with a conformally coupled scalar field, we can carry out the previous analysis with respect to the static metric $d\bar{s}^2$ instead of ds^2 . Since near $r = 0$, $R^2 \sim r$, this is the case of the metric (15) with $k = 1$. Thus, the norm squared (17) logarithmically diverges so that the singularity is wave-regular for the conformally coupled scalar field.

• Conformally flat spacetimes with emerging naked singularity

More physical situations of gravitational collapse require that the spacetime contains a regular initial spacelike hypersurface on which the collapsing matter has a compact support.

We can construct a spacetime which has a regular initial hypersurface Σ_{t_0} and a timelike singularity being formed in the future of Σ_{t_0} . For example, let us consider a conformally flat metric

$$ds^2 = C^2 (-dt^2 + dr^2 + r^2 d\Omega^2), \quad (23)$$

with a conformal factor which behaves near the center $r = 0$ as

$$C^2 \sim \begin{cases} r^p & (t \geq 0), \\ r^p + 1 & (t < 0), \end{cases} \quad (24)$$

where $p \geq 2$ and $C^2(r = 0, t) \rightarrow 0$ sufficiently smoothly as $t \rightarrow 0$. Then, the spacetime has a timelike singularity at the center $r = 0$ for $t \geq 0$ as depicted in Fig. 1. However, for sufficiently remote past, there are regular hypersurfaces Σ_t . Therefore we can take an initial regular hypersurface Σ_{t_0} at some $t_0 < 0$ and construct the Hilbert space on Σ_{t_0} .

Since all the hypersurfaces Σ_t are isomorphic to the initial hypersurface Σ_{t_0} up to a conformal factor, the Hilbert spaces of a conformally coupled scalar field on Σ_t are the same, even when Σ_t intersect the central singularity. Therefore the spacetime is wave-regular for the conformally coupled scalar field.

Timelike singularities of the type examined above turn out to be also wave-regular when probed with the Maxwell field since it is conformally invariant and Maxwell's equations are reduced to that of a massless Klein-Gordon field under a suitable gauge condition.

4 Summary and Discussion

We have studied the well-posedness of the initial value problem of the scalar wave equation in the case of static and conformally static spacetimes with timelike singularities choosing the Sobolev space as the natural Hilbert space. The physical idea behind the choice is that we can prepare an initial data only with finite energy. We have examined in detail the essential self-adjointness of the operator A in the wave equation defined in the Hilbert space in various models of spacetimes which contain timelike singularities in the conventional sense. In the spacetimes like the negative mass Schwarzschild spacetime the classical singularity becomes regular if probed with waves while more stronger classical singularities like the over extreme Reissner-Nordström spacetime remain singular. In the future work, we shall investigate what might occur in such a wave-singular spacetime in quantum field theory.

We have briefly touched upon the case that the timelike singularity emerges at some point in spacetime in a rather artificial model which is conformal to Minkowski spacetime. However, timelike singularities of general spacetimes without any (conformal) timelike Killing vector will be more interesting from the view

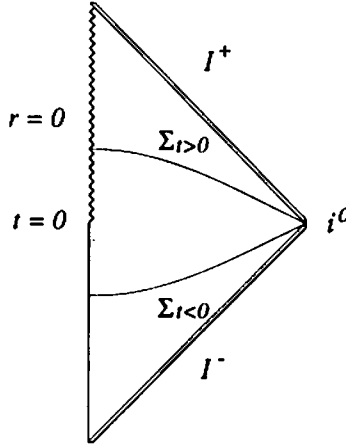


Figure 1: A spacetime with an emerging would-be naked singularity after $t = 0$.

point of the cosmic censorship especially in the case of gravitational collapse. In the case that “naked singularities” are found to be wave-regular, such a spacetime like the one illustrated by Fig. 1 can be said to be “globally hyperbolic” in the sense that the initial value problem is well-posed with the unique boundary value at the classical singularity. Clarke [5] gave a sufficient condition for the well-posedness of the initial value problem for test fields. However, it turns out that the curve integrability condition in Ref. [5] is not satisfied for all the wave-regular cases discussed in the present work. We suspect that the condition is too demanding for the well-posedness of the initial value problem and his theorem can be further sharpened.

We may carry out similar analyses to spinor, vector, and tensor fields. A natural question will be: Does the wave-regularity depend on what fields we probe? At the moment what we can say is that in principle yes, i.e., it depends and it is nothing wrong from physical point of view. If initial value problems are well-posed for *all fields*, “naked singularities” are harmless and nothing to be afraid of.

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