

**PROCEEDINGS OF THE SEVENTH WORKSHOP
ON
GENERAL RELATIVITY AND GRAVITATION**

Yukawa Institute for Theoretical Physics, Kyoto University,
October 27th - October 30th, 1997

edited by

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PREFACE

The Seventh Workshop on General Relativity and Gravitation was held in the week 27th October - 30th October, 1997 in the Yukawa Institute for Theoretical Physics at Kyoto University, Japan. A purpose of this workshop is to review the latest works in all fields relating to the gravity.

At present, there are many branches in issues on gravity and there are many researchers in Japan. Therefore, in a workshop restricted on ordinary scale, it seems to be impossible to cover all such fields even if it is important in order to get activities and to search for new directions by stimulating interaction there. This is just a further purpose of this series of workshops.

93 researchers participated and 52 speakers gave 14 invited talks and 41 short contributions. In reality, they came from various fields: the classical gravity (the space-time structure from a mathematical point of view), the observations of astrophysical and cosmological systems in which gravity plays an important role, and the theoretical research deeply related to that, the quantum field theory in curved space-times and the early stage of the universe in which the dynamics of quantum fields is highly influenced by gravity, the quantum nature of the gravitational interaction and the application for the very early stage of the universe. We believe that there were useful discussions and interactions between the participants.

We would like to thank Ms. Yokota, the secretary of the Department of Physics in Kyoto University, for her devoted transaction of various and tedious official works. We are also grateful to the graduate students of the nuclear astrophysics group in Kyoto University for their cooperation in the management of this workshop. If they did not help us, this workshop did not work. This workshop supported by Grant-in-Aid for Creative Basic Research (09NP0801) from the Japanese Ministry of Education, Science, Sports and Culture, which were the financial base of this workshop.

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Effects of Resistivity on Relativistic Trans-magnetosonic Flows

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Abstract

We study the effect of resistivity on the magnetized plasma accreting onto a black hole. To fall into the event horizon, the plasma injected from the plasma source with low velocity must pass through the Alfvén point and the fast and slow magnetosonic critical points. For the resistive magnetohydrodynamical (MHD) flow, we present the deviation from the ideal MHD case and discuss solution topologies around the magnetosonic points. It is found that nodal and spiral topologies are appeared on the flow solution.

1 Introduction

As a model of the central region of active galactic nuclei (AGNs), we consider a black hole magnetosphere which is composed of a super massive black hole, an accretion disk, and surrounding magnetic field, as illustrated in Figure 1. In the magnetosphere, an accretion onto the black hole would be generated from the vicinity of the inner edge of the accretion disk; and an outgoing wind/jet would be also arisen from the surface of the accretion disk. To discuss the properties of flows in the black hole magnetosphere, we treat the plasma flow by magnetohydrodynamics (MHD).

Accretion flow onto the black hole must be super-magnetosonic; and to explain a highly accelerated jet the outgoing flow should be also super-magnetosonic. We consider that such a MHD accretion is ejected from plasma sources (e.g., accretion disk and its corona) with a low velocity which is at least less than the slow magnetosonic wave speed, and accretes onto the black hole after passing through the slow magnetosonic point, the Alfvén point and the fast magnetosonic point smoothly (see Figure 2); at these points, the poloidal velocity of plasma equals the speeds of the slow magnetosonic wave, the Alfvén wave and the fast magnetosonic wave, respectively. This situation is analogous to a hydrodynamical flow which passes through a de Laval nozzle. (This nozzle has a minimum in its area and allows a transition from subsonic to supersonic.)

To determine the structure of a magnetosphere self-consistently, the so-called Bernoulli equation along magnetic field lines [1, 2], and the force-balance equation (e.g., [3, 4, 5]) have been discussed. The critical condition for trans-fast MHD flows in a black-hole magnetosphere was cleared by Takahashi[6].

We are interested in the solution topology near the magnetosonic point, where the solution is critical. For the ideal MHD case, the critical point described as X-type or O-type point. The O-type point is rejected as an accretion or a wind solution because there is no solution connecting the plasma source to the black hole or far distant region. The X-type point has two distinct solutions, one accelerating and one decelerating, that cross at the critical point. However, when the body forces acting on the flow depend explicitly on the flow velocity, as well as on the spatial coordinate, the other types of magnetosonic flows would be appeared. In fact, in the hydrodynamical cases, radiatively driven stellar winds, or deceleration of a supersonic wind by drag from interstellar or cometary neutrals or cosmic rays generates nodal/spiral type solutions(e.g., [7, 8, 9]). In the Newtonian resistive MHD case, Weber[10] discussed a trans-Alfvénic MHD flow. Here, we investigate the effect of *resistivity* on the relativistic MHD flow in the black hole magnetosphere.

2 Basic Equations

We assume that the magnetosphere is stationary and axisymmetric, and the plasma flow has finite conductivity and no viscosity. The background metric is denoted by the Boyer-Lindquist coordinate

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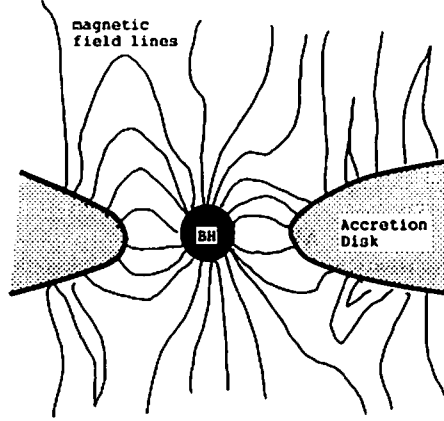


Figure 1: Black hole magnetosphere. An accretion disk rotates around a super massive black hole. The magnetized plasma is ejected from the disk surface, and accretes onto the black hole or goes away to far distant region.

(+ - - -) with $c = G = 1$ unit. The flow can be described by the following set of equations.

(i) Generalized Ohm's law : The electric current is the sum of the two terms corresponding respectively to the convection current and to the conduction current,

$$J^\mu = \varepsilon u^\mu + \sigma F^{\mu\nu} u_\nu, \quad (1)$$

where u^μ is the four-velocity of the fluid, $F^{\mu\nu}$ is the electromagnetic field tensor, ε is the proper density of electric charge and σ is the electrical conductivity of the fluid (see [11]).

(ii) Maxwell equations :

$$F_{;\nu}^{\mu\nu} = -4\pi J^\mu, \quad (2)$$

$$F_{[\mu\nu;\sigma]} = 0. \quad (3)$$

(iii) Baryon number conservation :

$$(nu^\mu)_{;\mu} = 0, \quad (4)$$

where n is the proper particle number density.

(iv) Energy and angular momentum current conservation : The existence of the Killing fields k and m , defined by the stationary and axisymmetry of the space-time, leads to two conserved quantities. By contraction with the energy momentum tensor $T^{\mu\nu}$ we get two currents; energy current, $P^\alpha = T_\beta^\alpha k^\beta$, and angular momentum current, $J^\alpha = -T_\beta^\alpha m^\beta$, which satisfy

$$P^\alpha_{;\alpha} = 0 = -J^\alpha_{;\alpha}, \quad (5)$$

where

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu} + \frac{1}{4\pi} \left(F^{\mu\delta} F_\delta^\nu + \frac{1}{4} g^{\mu\nu} F^2 \right), \quad (6)$$

and p is the pressure. The magnetic and electric field $\bar{B}$ and $\bar{E}$ seen by a comoving observer are defined by $\bar{B}_\alpha \equiv \frac{1}{2} \eta_{\alpha\beta\delta\sigma} u^\beta F^{\delta\sigma}$, where $\eta_{\alpha\beta\delta\sigma} \equiv \sqrt{-g} \varepsilon_{\alpha\beta\delta\sigma}$, and $\bar{E}_\alpha \equiv u^\beta F_{\alpha\beta}$ ($\bar{E}_\alpha = 0$ for ideal MHD case).

An axisymmetric stationary ideal MHD flow ($\sigma = \infty$) along a magnetic flux tube is specified by four constants of motion (e.g., Camenzind 1986a): the total energy $E(\Psi)$, total angular momentum $L(\Psi)$, angular velocity of the field line $\Omega_F(\Psi)$ and particle flux through a flux tube $\eta(\Psi)$. These conserved

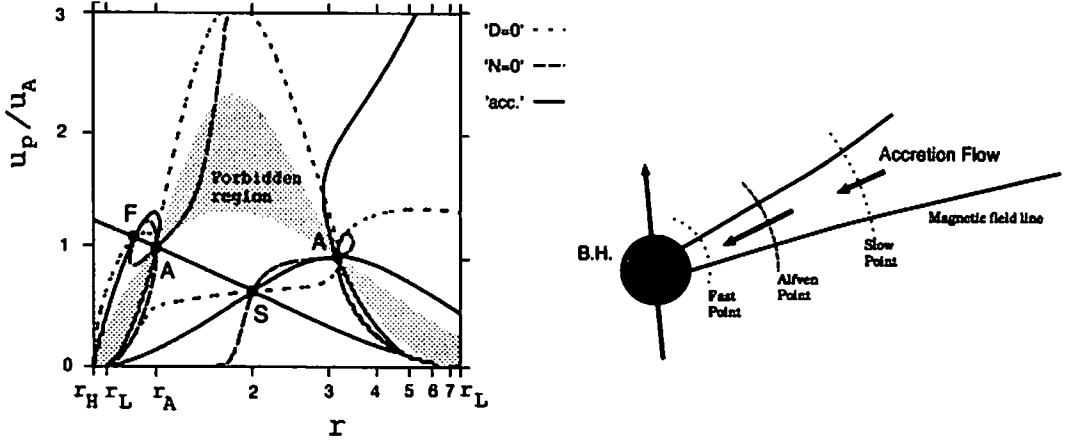


Figure 2: Ideal MHD accretion onto the black hole. The accreting plasma streams along magnetic field lines, and reaches to the event horizon ($r = r_H$) after passing the slow magnetosonic point (S), the Alfvén point (A), the fast magnetosonic point (F), in order.

quantities are given by

$$E = \mu u_t - (\Omega_F/4\pi\eta)B_\phi, \quad (7)$$

$$L = -\mu u_\phi - (1/4\pi\eta)B_\phi, \quad (8)$$

$$\Omega_F = -F_{tr}/F_{\phi r} = -F_{t\theta}/F_{\phi\theta}, \quad (9)$$

$$\eta = nu_p/B_p. \quad (10)$$

where μ is the specific enthalpy, B_ϕ is the toroidal magnetic field, u_p is the poloidal velocity defined by $u_p^2 \equiv -u_A u^A$ ($A = r, \theta$), B_p is the poloidal magnetic field seen by a lab-frame observer

$$B_p^2 \equiv -\frac{1}{\rho_w^2} [g^{rr}(\partial_r \Psi)^2 + g^{\theta\theta}(\partial_\theta \Psi)^2]. \quad (11)$$

With these conserved quantities, the MHD flow's differential equation can be written as following form (see [2])

$$(\ln u_p)' = \frac{N(r, u_p; \Omega_F, E, L)}{D(r, u_p; \Omega_F, E, L, \eta)}. \quad (12)$$

We can reduce the denominator as

$$D \propto (u_p^2 - u_{AW}^2)(u_p^2 - u_{FW}^2)(u_p^2 - u_{SW}^2), \quad (13)$$

where u_{AW} , u_{FW} , u_{SW} are the Alfvén wave speed, the fast magnetosonic wave speed, the slow magnetosonic wave speed, respectively. This denominator becomes zero, at the Alfvén point where $r = r_A$ and $u_p = (u_{AW})_A$, the fast magnetosonic point where $r = r_F$ and $u_p = (u_{FW})_F$, and the slow magnetosonic point where $r = r_S$ and $u_p = (u_{SW})_S$. Therefore, at these critical points, we must require $N = 0$ to obtain physical wind and accretion solutions which pass through these critical points smoothly. These critical conditions on the fast and slow magnetosonic point lead to the X-type or O-type topology of MHD flow solutions.

3 Deviation from Ideal MHD Flow Case

We extend the definition of Ω_F to the resistive case (Ω_F is indicative of the potential drop, but it is no longer a constant along the field lines). Equation (1) can be written

$$F_{\mu\nu}u^\nu = -(1/\sigma) [F_{\mu\nu}^\nu + \varepsilon u_\mu] \equiv -\mathcal{J}_\mu/\sigma, \quad (14)$$

or

$$F_{tr}u^r + F_{t\theta}u^\theta = -\mathcal{J}_t/\sigma, \quad (15)$$

$$F_{rt}u^t + F_{r\theta}u^\theta + F_{r\phi}u^\phi = -\mathcal{J}_r/\sigma, \quad (16)$$

$$F_{\theta t}u^t + F_{\theta r}u^r + F_{\theta\phi}u^\phi = -\mathcal{J}_\theta/\sigma, \quad (17)$$

$$F_{\phi r}u^r + F_{\phi\theta}u^\theta = -\mathcal{J}_\phi/\sigma. \quad (18)$$

Here, we introduce $\Omega_r \equiv F_{tr}/F_{r\phi}$ and $\Omega_\theta \equiv F_{t\theta}/F_{\theta\phi}$. From equations (15) and (18) with the Maxwell's equation (3), we have

$$u^A (\ln |\Omega_r|)_{,A} = -\frac{u^\theta}{F_{tr}} \left[\left(\frac{\mathcal{J}_t}{\sigma u^\theta} \right)_{,r} + \Omega_r \left(\frac{\mathcal{J}_\phi}{\sigma u^\theta} \right)_{,r} \right]. \quad (19)$$

Thus, we see that Ω_r is not conserved along the flow lines, and is coincident with Ω_F in the $\sigma = \infty$ limit. In the similar way, we can also obtain the following relation with respect to Ω_θ ,

$$u^A (\ln |\Omega_\theta|)_{,A} = -\frac{u^r}{F_{t\theta}} \left[\left(\frac{\mathcal{J}_t}{\sigma u^r} \right)_{,\theta} + \Omega_\theta \left(\frac{\mathcal{J}_\phi}{\sigma u^r} \right)_{,\theta} \right]. \quad (20)$$

Next, we introduce $\eta_r \equiv -nu^\theta \sqrt{-g}/F_{\phi r}$ and $\eta_\theta \equiv -nu^r \sqrt{-g}/F_{\theta\phi}$. From baryon-number conservation it follows that

$$u^\alpha_{;\alpha} = -u^\alpha [\ln(\sqrt{-g}n)]_{,\alpha}. \quad (21)$$

From equations (15), (18), (3) with the aid of the above equation, we have

$$u^A (\ln |\eta_r|)_{,A} = \frac{u^\theta}{F_{\phi r}} \left(\frac{\mathcal{J}_\phi}{\sigma u^\theta} \right)_{,r}. \quad (22)$$

In the similar way, we can also obtain the following relation

$$u^A (\ln |\eta_\theta|)_{,A} = \frac{u^r}{F_{\theta\phi}} \left(\frac{\mathcal{J}_\phi}{\sigma u^r} \right)_{,\theta}. \quad (23)$$

Thus, both η_r and η_θ are not conserved along each flow lines, and are coincident with η in the $\sigma = \infty$ limit.

Now, we will consider about the energy and angular momentum of the flow. From the conservation laws (4) and (5), we have

$$T^\lambda_{a;\lambda} = N^\lambda (\bar{\mu} u_a)_{,\lambda} - \frac{\mu}{4\pi} \bar{B}^\lambda (\bar{B}_a/\mu)_{,\lambda} + \xi_a = 0, \quad (24)$$

where $a = t$ or ϕ ; $\bar{\mu}$ is the generalized enthalpy, $\bar{\mu} = \mu + (\bar{B}^2 + \bar{E}^2)/(4\pi n)$; and ξ_a is a function of $\bar{B}^\mu$, $\bar{E}^\mu$, u^μ and becomes zero in the $\sigma \rightarrow \infty$ limit. Using the definitions of $\bar{B}^\mu$, η_r and η_θ , we see that

$$\bar{B}^r = -\left(\frac{u_t + \Omega_\theta u_\phi}{\eta_\theta} \right) N^r, \quad (25)$$

$$\bar{B}^\theta = -\left(\frac{u_t + \Omega_r u_\phi}{\eta_r} \right) N^\theta. \quad (26)$$

Because of $\Omega_r \neq \Omega_\theta$ and $\eta_r \neq \eta_\theta$, the stream lines of the flow are not identical with the magnetic field lines, while for ideal MHD case we get $B^A \propto N^A$.

We have already analyzed stationary and axisymmetric ideal MHD flows, so we will consider the deviation from ideal MHD flow and express Ω_r , Ω_θ , η_r and η_θ in the form of

$$\Omega_r = \Omega_F + \omega_r \quad (27)$$

$$\Omega_\theta = \Omega_F + \omega_\theta \quad (28)$$

$$\frac{1}{\eta_r} = \frac{1}{\eta} + \zeta_r \quad (29)$$

$$\frac{1}{\eta_\theta} = \frac{1}{\eta} + \zeta_\theta \quad (30)$$

where ω_r , ω_θ , ζ_r and ζ_θ are deviations from the stationary and axisymmetric ideal MHD flows, due to the effects of resistivity. Then, equations (25) and (26) are written as

$$\bar{B}^r = -(u_t + \Omega_F u_\phi) \frac{1}{\eta} N^r - N^r \left\{ \omega_\theta \frac{u_\phi}{\eta} + (u_t + \Omega_\theta u_\phi) \zeta_\theta \right\}, \quad (31)$$

$$\bar{B}^\theta = -(u_t + \Omega_F u_\phi) \frac{1}{\eta} N^\theta - N^\theta \left\{ \omega_r \frac{u_\phi}{\eta} + (u_t + \Omega_r u_\phi) \zeta_r \right\}. \quad (32)$$

Thus, equation (24) can be expressed as

$$N^A \left[\bar{\mu} u_a + \frac{u_t + \Omega_F u_\phi}{4\pi\eta} \bar{B}_a \right]_{,A} = \chi_a, \quad (33)$$

where $\bar{\mu} \equiv \mu - \frac{1}{4\pi\eta} (\bar{B}_t + \Omega_F \bar{B}_\phi)$; and χ_a is the deviation term including ω_r , ω_θ , ζ_r , ζ_θ , and it becomes zero in the $\sigma \rightarrow \infty$ limit.

In the resistive MHD flow case, the 'total energy' E and the 'angular momentum' L along the stream line can be defined by

$$\hat{E} \equiv \bar{\mu} u_t + \frac{u_t + \Omega_F u_\phi}{4\pi\eta} \bar{B}_t = \mu u_t - \frac{\Omega_F}{4\pi\eta} (u_t \bar{B}_\phi - u_\phi \bar{B}_t) \quad (34)$$

and

$$-\hat{L} \equiv \bar{\mu} u_\phi + \frac{u_t + \Omega_F u_\phi}{4\pi\eta} \bar{B}_\phi = \mu u_\phi + \frac{1}{4\pi\eta} (u_t \bar{B}_\phi - u_\phi \bar{B}_t). \quad (35)$$

These quantities are not conserved along the stream lines.

By using the definition of the magnetic field, we have

$$(u_t \bar{B}_\phi - u_\phi \bar{B}_t) = \frac{n}{\eta} [G_\phi u_t - G_t u_\phi] + \gamma, \quad (36)$$

where we define functions

$$G_t \equiv (g_{tt} + g_{t\phi} \Omega_A) \frac{\eta}{\eta_A} = (g_{tt} + g_{t\phi} \Omega_F) + \beta, \quad (37)$$

$$G_\phi \equiv (g_{t\phi} + g_{\phi\phi} \Omega_A) \frac{\eta}{\eta_A} = (g_{t\phi} + g_{\phi\phi} \Omega_F) + \alpha, \quad (38)$$

and

$$\alpha \equiv (g_{t\phi} + g_{\phi\phi} \Omega_A) \eta \zeta_A + g_{\phi\phi} \omega_A, \quad (39)$$

$$\beta \equiv (g_{tt} + g_{t\phi} \Omega_A) \eta \zeta_A + g_{t\phi} \omega_A, \quad (40)$$

$$\gamma \equiv \frac{n}{\eta_A} \left(\frac{J_A}{\sigma F_{\phi A}} \right) + \frac{\sqrt{-g}}{\sigma} (u^r J^\theta - u^\theta J^r). \quad (41)$$

From equations (34) and (35) with the above relation (36) and redefinitions $E \equiv \hat{E} + (\Omega_F/4\pi\eta)\gamma$ and $L \equiv \hat{L} + (1/4\pi\eta)\gamma$, we get the energy and angular momentum of the fluid part

$$\mu u_t = \frac{E(M^2 - G_t) + L \Omega_F G_t}{M^2 - \gamma_g}, \quad (42)$$

$$-\mu u_\phi = \frac{E G_\phi + L(M^2 - \Omega_F G_\phi)}{M^2 - \gamma_g}, \quad (43)$$

where $M^2 \equiv 4\pi\mu\eta^2/n$ is the Alfvén Mach number and $\gamma_g \equiv G_t + \Omega_F G_\phi$.

4 Poloidal MHD Flow Equation ($\sigma \neq \infty$)

From the relation $\mu^2(1 + u_p^2) = \mu^2(u' u_t + u^\phi u_\phi)$, we obtain the poloidal equation, which is the Bernoulli's equation for the relativistic MHD flow

$$\mu^2(1 + u_p^2) = \frac{-kM^4 - 2(e^2 + \mathcal{P})M^2 + (\gamma_g + \mathcal{R})e^2}{(M^2 - \gamma_g)^2}, \quad (44)$$

where

$$k \equiv (E^2 g_{\phi\phi} + 2ELg_{t\phi} + L^2 g_{tt})/\rho_w^2, \quad (45)$$

$$\mathcal{P} \equiv e[(Eg_{t\phi} + Lg_{tt})\alpha - (Eg_{\phi\phi} + Lg_{t\phi})\beta]/\rho_w^2, \quad (46)$$

$$\mathcal{R} \equiv \mathcal{A} - (g_{\phi\phi}\beta^2 - 2g_{t\phi}\alpha\beta + g_{tt}\alpha^2)/\rho_w^2, \quad (47)$$

$$\mathcal{A} \equiv \alpha\Omega_F + \beta, \quad (48)$$

where $\rho_w^2 \equiv g_{t\phi}^2 - g_{tt}g_{\phi\phi}$ and $e \equiv E - \Omega_F L$. At the Alfvén point, which is defined by $M^2 = \gamma_g$, we must require the critical condition

$$\left(\frac{L}{E}\right)_A = -\left(\frac{G_\phi}{G_t}\right)_A. \quad (49)$$

Next, by using the relations,

$$(M^2)' = M^2(1 - a_s^2)[(\ln u_p)' - (\ln B_p)'], \quad (50)$$

$$(\mu^2)' = -2\mu^2 a_s^2 [(\ln u_p)' - (\ln B_p)'], \quad (51)$$

we can obtain the differential form of the poloidal MHD flow equation in the following form

$$(\ln u_p)' = \frac{N}{D}, \quad (52)$$

where the prime denotes the derivative $\partial_r - (\partial_r \Psi / \partial_\theta \Psi) \partial_\theta$ along a magnetic field line. Hereafter, we assume the radial magnetic field geometry, and $E = E(r, u_p)$, $L = L(r, u_p)$, $\omega_r = \omega_r(r)$, $\omega_\theta = \omega_\theta(r)$, $\zeta_r = \zeta_r(r)$, $\zeta_\theta = \zeta_\theta(r)$. The denominator and numerator of the above equation are given by

$$D = \mu^2(u_p^2 - C_s^2)(M^2 - \gamma_g)^3 - (\gamma_g k + e^2 + \mathcal{P})M^4 + \mathcal{Q} + \mathcal{Z}, \quad (53)$$

$$N = [-\mu^2(1 + u_p^2)(M^2 - \gamma_g)^3 C_s^2 - (\gamma_g k + e^2 + \mathcal{P})M^4 + \mathcal{Q}](\ln B_p)' + \frac{M^2 - \gamma_g}{1 - a_s^2} \left[\left\{ \mu^2(1 + u_p^2)(M^2 - \gamma_g) + \frac{e^2}{2} \right\} \gamma_g' - \frac{1}{2} M^4 k' + \frac{1}{2} \mathcal{W} \right] \quad (54)$$

with definitions

$$\mathcal{Q} \equiv (\mathcal{R}e^2 - \mathcal{P}\gamma_g)M^2, \quad (55)$$

$$\mathcal{Z} \equiv - \left[(\dot{E}g_{\phi\phi} + \dot{L}g_{t\phi})(EM^2 - e\beta) + (\dot{E}g_{t\phi} + \dot{L}g_{tt})(LM^2 + e\alpha) \right] M^2/\rho_w^2 + (1/e)(\dot{E} - \Omega_F \dot{L}) [-M^2\mathcal{P} + e^2(\gamma_g - 2M^2 + \mathcal{R})], \quad (56)$$

$$\mathcal{W} \equiv (\gamma_g - 2M^2 + \mathcal{R})(e^2)' - 2M^2\mathcal{P}' + e^2\mathcal{R}', \quad (57)$$

where the expression $(\dot{}) \equiv u_p(\partial/\partial u_p)$ are introduced. The relativistic sound velocity a_s is given by

$$a_s^2 \equiv \left(\frac{\partial \ln \mu}{\partial \ln n} \right)_{ad} = (\Gamma - 1) \frac{\mu - mc^2}{\mu}, \quad (58)$$

and the sound four-velocity C_s is defined by $C_s^2 = a_s^2/(1 - a_s^2)$.

To determine the behavior of MHD wind solutions near the fast/slow magnetosonic point where $D = N = 0$, we must apply l'Hospital rule. Thus

$$(\ln u_p)'_{cr} = \frac{(N')_{cr}}{(D')_{cr}} = \text{finite}. \quad (59)$$

The denominator and numerator take the form of $D' = a(\ln u_p)' + b_D$ and $N' = b_N(\ln u_p)' + c$. Then equation (59) can be expressed in the form

$$a[(\ln u_p)'_{cr}]^2 + (b_D - b_N)[(\ln u_p)'_{cr}] - c = 0 , \quad (60)$$

where

$$a = 3\mu^2(u_p^2 - C_s^2)(M^2 - \gamma_g)^2 M^2(1 - a_s^2) + 2\mu^2(M^2 - \gamma_g)^3 C_s^2(1 + S) - Q(1 - a_s^2) + \dot{D} , \quad (61)$$

$$b_D = [-a - \dot{D} + 2\mu^2 u_p^2 (M^2 - \gamma_g)^3](\ln B_p)' - [3\mu^2(u_p^2 - C_s^2)(M^2 - \gamma_g)^2 + kM^4 + \mathcal{P}M^2]\gamma_g' - \gamma_g M^4 k' + \mathcal{X} , \quad (62)$$

$$\begin{aligned} \dot{D} = & -M^4 \left[\dot{E}(\dot{E}g_{\phi\phi} + \dot{L}g_{t\phi}) + \dot{L}(\dot{E}g_{t\phi} + \dot{L}g_{tt}) \right] / \rho_w^2 \\ & - 2M^2 \dot{E} \left[(\dot{E}g_{t\phi} + \dot{L}g_{tt})\alpha - (\dot{E}g_{\phi\phi} + \dot{L}g_{t\phi})\beta \right] / \rho_w^2 \\ & - \dot{E}(M^4 + \gamma_g M^2) [(Eg_{t\phi} + Lg_{tt})\alpha - (Eg_{\phi\phi} + Lg_{t\phi})\beta] / \rho_w^2 \\ & - e(M^4 + \gamma_g M^2) \left[(\dot{E}g_{t\phi} + \dot{L}g_{tt})\alpha - (\dot{E}g_{\phi\phi} + \dot{L}g_{t\phi})\beta \right] / \rho_w^2 \\ & - 2\gamma_g M^4 \left[E(\dot{E}g_{\phi\phi} + \dot{L}g_{t\phi}) + L(\dot{E}g_{t\phi} + \dot{L}g_{tt}) \right] / \rho_w^2 - 2e\dot{E}(M^4 - \mathcal{R}M^2) \end{aligned} \quad (63)$$

with definitions

$$S \equiv (1/2)[(\mu/m_p) - 1]^{-1} C_s^2 , \quad (64)$$

$$\mathcal{X} \equiv -M^4(e^2 + \mathcal{P})' + M^2(\mathcal{R}e^2)' - \gamma_g M^2 \mathcal{P}' , \quad (65)$$

and we also obtain

$$b_N = -b_D + (\dot{D} - \dot{Z})(\ln B_p)' + \mathcal{Y} , \quad (66)$$

$$\begin{aligned} c = & [2b_D + (a - \dot{D})(\ln B_p)'](\ln B_p)' - \frac{2}{1 - a_s^2} \mu^2 u_p^2 (M^2 - \gamma_g)^3 [(\ln B_p)']^2 \\ & + \frac{5}{1 - a_s^2} \mu^2 u_p^2 (M^2 - \gamma_g)^2 \gamma_g' (\ln B_p)' - \frac{1}{1 - a_s^2} \mu^2 (1 + u_p^2) (M^2 - \gamma_g) (\gamma_g')^2 \\ & + \frac{1}{1 - a_s^2} (M^2 - \gamma_g) \left(\frac{1}{2} \gamma_g + \mathcal{R} \right)' (e^2)' + N_w , \end{aligned} \quad (67)$$

$$\begin{aligned} \dot{Z} = & -M^2 \left[(\dot{E}g_{\phi\phi} + \dot{L}g_{t\phi}) \{ \dot{E}M^2 - 2\dot{E}\beta \} \right. \\ & \left. + (\dot{E}g_{t\phi} + \dot{L}g_{tt}) \{ \dot{L}M^2 + 2\dot{E}\alpha \} \right] / \rho_w^2 + \dot{E}^2 (\gamma_g - 2M^2 + \mathcal{R}) \end{aligned} \quad (68)$$

with definitions

$$\begin{aligned} N_w \equiv & -[\mu^2(1 + u_p^2)(M^2 - \gamma_g)^3 C_s^2 + (\gamma_g k + e^2 + \mathcal{P})M^4 - Q](\ln B_p)'' \\ & + \frac{M^2 - \gamma_g}{1 - a_s^2} \left\{ \left[\mu^2(1 + u_p^2)(M^2 - \gamma_g) + \frac{e^2}{2} \right] \gamma_g'' - \frac{1}{2} M^4 k'' + \frac{1}{2} \mathcal{W} \right\} , \end{aligned} \quad (69)$$

$$\mathcal{W} \equiv (\gamma_g - 2M^2 + \mathcal{R})(e^2)'' - 2M^2 \mathcal{P}'' + e^2 \mathcal{R}'' , \quad (70)$$

$$\begin{aligned} \mathcal{Y} \equiv & \frac{M^2 - \gamma_g}{1 - a_s^2} \left[\{ e\gamma_g' + (\gamma_g - 2M^2 + \mathcal{R})e' + e\mathcal{R}' \} \dot{E} \right. \\ & \left. - (1/2)M^4 k' + e(\gamma_g - 2M^2 + \mathcal{R})\dot{E}' - M^2 \dot{p}' \right] . \end{aligned} \quad (71)$$

All quantities appearing in a , b_D , b_N and c are evaluated at the fast or slow magnetosonic point.

Because the magnetosonic wave speeds have dependences of r , we rewrite equation (59) to the form of

$$a[(\ln u_p)'_{cr} - (\ln u_{MW})'_{cr}]^2 + \hat{b}[(\ln u_p)'_{cr} - (\ln u_{MW})'_{cr}] + \hat{c} = 0 , \quad (72)$$

where u_{MW} indicates u_{SW} or u_{FW} and $(\ln u_{MW})'_{cr} = -b_D/a$; and then we see that

$$\dot{b} = a(\ln u_{MW})'_{cr} - b_N = -(\dot{D} - \dot{Z})(\ln B_p)'_{cr} + \mathcal{Y} , \quad (73)$$

$$\dot{c} = -b_N(\ln u_{MW})'_{cr} - c . \quad (74)$$

The solution (72), which is the critical point slopes, are given by

$$(\ln u_p)'_{cr} = \frac{1}{2a} \left[-\dot{b} \pm (\dot{b}^2 + 4a\dot{c})^{1/2} \right] + (\ln u_{MW})'_{cr} . \quad (75)$$

In general, $\dot{E}$ and $\dot{L}$ are not zero in the resistive case; so that we see $\dot{b} \neq 0$. Therefore, it is possible that the two possible critical slopes have the same sign. This means that *nodal* topology appears on an accretion solution (e.g., [7, 8, 12]). A *spiral* topology can be also appeared on the solution. The O-type point disappeared from the solution of resistive MHD flows, while it can be realized on ideal MHD flow solution, where $\dot{b} = 0$.

5 Summary

We have determined the expression how the energy and angular momentum lost or gain through the magnetic flux tube for the resistive MHD flow in a black hole magnetosphere. We also find the possibility of nodal/spiral type flow solution. However, to discuss the accretion solution which streams from a plasma source to the event horizon, we need to know the current distribution, which will be obtained by using equation (2) under a suitable boundary conditions. Application of these results to specific accretion/jet sources is not considered here and will be reported in separate publications.

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Effect of Void Network on CMB Anisotropy

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Abstract

We study the effect of a void network on the CMB anisotropy in the Einstein-de Sitter background using Thompson & Vishniac's model. We estimate both the Rees-Sciama effect and the gravitational lensing effect. As well as the Sachs-Wolfe effect, the Rees-Sciama effect can be appreciable even for multipoles $l \lesssim 1000$ of the anisotropy spectrum, if primordial voids exist at recombination, which is realized in some inflationary models associated with a first-order phase transition. The gravitational lensing effect, on the other hand, slightly smoothes the primary anisotropy; quantitatively, our results for the void model are similar to the previous results for a CDM model. Both effects, together, would give some constraints on the configuration or origin of voids with high-resolution data of the CMB anisotropy up to $l \sim 1000$. Those effects, on the other hand, are negligible for $l \lesssim 500$, which does not obstruct the determination of the cosmological model from the anisotropy spectrum.

1 Introduction

The anisotropy of the cosmic microwave background (CMB) is an important probe of primordial fluctuations at recombination, which carries information on the cosmological parameters as well as the nature of dark matter [1]. CMB photons, however, are also affected gravitationally by nonlinear structures between recombination and the present epoch. In fact, a network of voids filling the entire universe has been suggested by redshift surveys such as the CfA2 and the SSRS2 [2]. Moreover, using such redshift surveys, El-Ad *et al.* [3] quantified voids in the galaxy distribution and confirmed the description of a void-filled universe: they showed that $\sim 50\%$ of the volume is filled with the voids and that the voids have a diameter of at least $40h^{-1}\text{Mpc}$ with an average underdensity of -0.9 . In this paper we investigate the effect of the void network on the CMB anisotropy in the Einstein-de Sitter background.

Rees and Sciama [4] (hereafter RS) showed that an evolving nonlinear structure perturbs the redshift of a photon passing through it by use of the "Swiss-cheese" model of overdensity. Later Thompson & Vishniac [5] (hereafter TV) estimated this RS effect in a void network model. First, they considered a spherical void in the Einstein-de Sitter background and derived an analytic expression of the redshift deviation $\delta T/T$ under the thin-shell approximation. Then, using this expression of $\delta T/T$, they calculated the variance of $\delta T/T$ for a universe filled with voids. As we will review in §2, the CMB anisotropy produced by the void network is of the order 10^{-6} if the present diameter of all voids is $60h^{-1}\text{Mpc}$ and if they have formed at $z < 10$; if the formation time is earlier, the anisotropy becomes larger. These results were supported by several authors [6, 7]. Although density perturbations are usually assumed to be linear at the last-scattering surface (LSS), nonlinear voids can exist there if voids are originated by primordial bubbles which are nucleated in a phase transition during inflation [8, 9]. Although the hypothesis of primordial voids is quite different from a conventional scenario, it may explain the present void-network structure and deserves further consideration. Therefore, more quantitative analysis of the RS effect for that case is important. One of the purposes of the present analysis is to calculate the power spectrum of the CMB anisotropy by extending TV's analysis.

The lensing effect of density perturbations, on the other hand, has also been investigated by several authors [10, 11, 12]. It has been concluded, as a whole, that the effect is appreciable on arcminute angular scales for some models while it is negligible on degree scales. In particular, Seljak [11] solved the shortcomings of the previous studies to include the nonlinear effects by modeling the power spectrum

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evolution in the nonlinear regime, and Martínez-González *et al.* [12] extended his method to study more general models; it was shown [11, 12] that for a CDM model the lensing effect changes the CMB angular power spectrum C_l considerably for $l \gtrsim 1000$. This power spectrum approach can be applied to general models as long as the power spectrum of density perturbations is known. Because the power spectrum of a void-network universe is not obtained, however, we shall estimate the lensing effect in a different way. That is, we estimate the correction of the primary anisotropy, using TV's formula of the scattering angle of a photon by a void. An advantage of our approach is that we make no approximation for nonlinearity nor relativistic effect.

In case nonlinear voids already exist at recombination — which is the case we are most interested in — we should also include the Sachs-Wolfe [13] (hereafter SW) effect of voids sitting on the LSS. Because that effect has already been investigated by Baccigalupi *et al.* [9], we just make some comments on it in the last section.

The plan of this paper is as follows. In §2, we briefly review TV's model and results, which we extend in the following sections. In §3, we calculate the temperature fluctuations caused by the RS effect of a void network. In §4, we investigate how the primordial fluctuations are modified by the gravitational lensing effect of a void network. §5 is devoted to summary and discussions.

2 Rees-Sciama Effect — Thompson & Vishniac's Model and Results —

Because our analysis is based on TV's model of a void-network universe and their analytical results, here we review them briefly. Consider a single spherical void in the Einstein-de Sitter background:

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2) \quad \text{with} \quad a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3}}, \quad (1)$$

where t_0 is the present time. The void itself is assumed to be empty. From momentum conservation and energy conservation, respectively, Maeda & Sato and Bertschinger [14] showed that the shell radius expands asymptotically as

$$r_v(t) \propto t^\beta \quad \text{with} \quad \beta \approx 0.13. \quad (2)$$

Figure 1 shows TV's model of a photon passing through a spherical void. With double Lorentz transformation of a momentum vector at the shell, we obtain the redshift deviation and the scattering angle of a photon:

$$\frac{\delta T}{T} = (HR)^3 \cos \psi \left(3\beta - \frac{2}{3} \cos^2 \psi\right), \quad \delta \alpha = (HR)^2 \sin(2\psi), \quad (3)$$

where H is the Hubble parameter, $R \equiv ar_v$ is the proper length of the shell radius, and ψ is defined in Fig. 1. All variables take values at the point where a CMB photon goes out of the void. The same results as (3) were also obtained by Martínez-González *et al.* in a different scheme using the “potential approximation” [7].

Next, TV estimated the CMB anisotropy produced by a void network. Their model consists of randomly distributed, equally sized, and non-overlapped voids, which formed at some time t_i simultaneously. Divide the universe into shells of the comoving thickness $2r_v(t_0)$, as depicted in Figure 2. For each shell, the probability of a ray intersecting a void is given by

$$P = \frac{3}{2} F_0 \left(\frac{t}{t_0}\right)^{2\beta}, \quad (4)$$

where F_0 is the fractional volume of space occupied by voids and normalized at the present. The variance of $\delta T/T$ for each shell is

$$\sigma_{\text{RS,shell}}^2 \equiv \left\langle \left| \frac{\delta T}{T} \right|^2 \right\rangle_{\text{shell}} - \left\langle \frac{\delta T}{T} \right\rangle_{\text{shell}}^2 \quad \text{with} \quad \langle \rangle_{\text{shell}} = P \int_0^{\frac{\pi}{2}} \sin \psi d\psi. \quad (5)$$

The net variance is

$$\sigma_{\text{RS}}^2 = \sum \sigma_{\text{RS,shell}}^2 \approx \sigma_{\text{RS}}^2 = \int_{t_i}^{t_0} \sigma_{\text{RS,shell}}^2 \frac{dt}{\Delta t} \quad \text{with} \quad \Delta t \equiv 2a(t)r_v(t_0). \quad (6)$$

After an algebraic calculation, we obtain

$$\sigma_{\text{RS}} = \frac{3}{2}(H_0 R_0)^{\frac{1}{2}} \left[\frac{3F_0}{5-24\beta} \left(\frac{2}{81} - \frac{8}{27}\beta + \beta^2 \right) \left\{ (1+z_f)^{\frac{1}{2}-12\beta} - 1 \right\} - \frac{4F_0^2}{5(1-6\beta)} \left(\beta - \frac{2}{15} \right)^2 \left\{ (1+z_f)^{\frac{1}{2}-15\beta} - 1 \right\} \right]^{\frac{1}{2}}. \quad (7)$$

TV mentioned that the second term in equation (7), which stems from the second term in equation (5), goes to zero for $\beta = 2/15 \approx 0.133$. Although we choose $\beta = 0.13$ in this paper, $\langle \delta T/T \rangle \approx 0$ is still satisfied.

In this model there are three parameters: the radius of voids R_0 , the volume fraction F_0 , and the formation time z_f . Keeping in mind the results of El-Ad *et al.* [3], we tentatively choose $R_0 = 40$ or $60 h^{-1} \text{Mpc}$ and $F_0 = 2/3$ in the following. Figure 3 shows a plot of equation (7). We see that the net variance depends strongly on the formation time of voids. As TV concluded, the RS effect cannot make a dominant contribution to the CMB anisotropy if nonlinear voids form at $z < 10$; on the other hand, if voids form before or just after recombination as some inflationary models predicted, the RS effect may not be negligible.

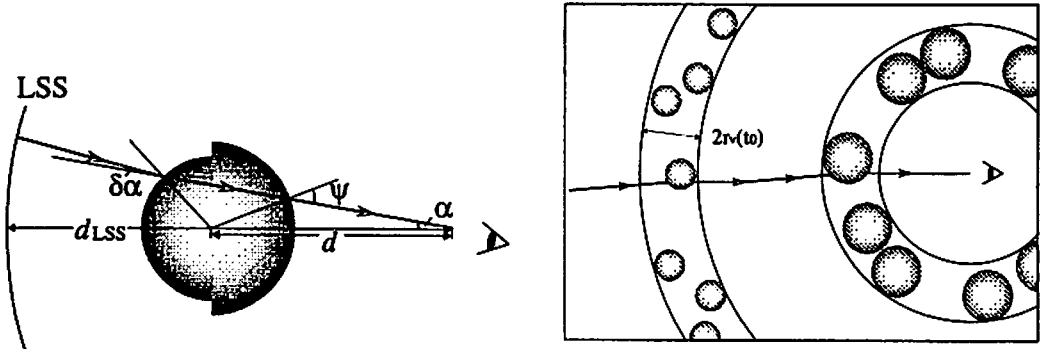


Fig. 1. Cross section of a void on the $\varphi = \pi/2$ plane. We define $\delta\alpha$ as the scattering angle of a photon, α as the angle formed between the direction of observation and the direction of the void's center, d as the comoving distance of the void's center, and d_{LSS} as the comoving distance of the LSS.

Fig. 2. Schematic sketch of a void-network universe (TV's model). Divide the universe into shells of comoving thickness $2r_v(t_0)$.

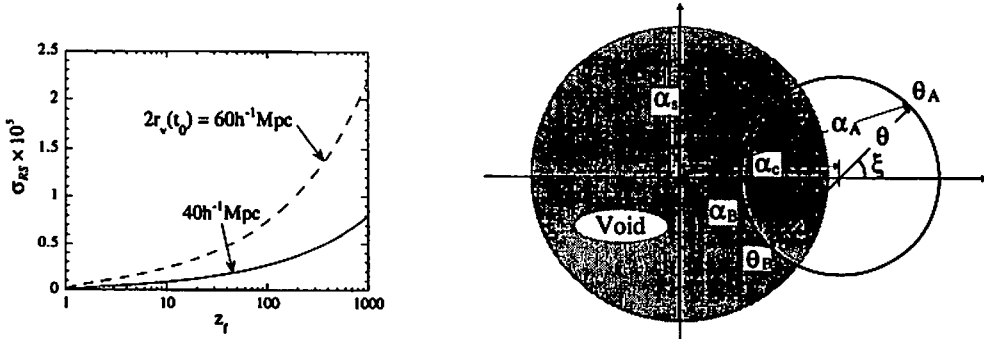


Fig. 3. RS effect: plot of eq. (7) (TV's results) for $2r_v(t_0) = 40 h^{-1} \text{Mpc}$ and $60 h^{-1} \text{Mpc}$. We set $\beta = 0.13$ and $F_0 = 2/3$.

Fig. 4. Projection of a void on the celestial sphere. Neglecting the curvature of the sphere in a local region around a void, we consider two light rays, which are labeled as θ_A and θ_B .

3 Rees-Sciama Effect — Calculation of C_l —

As we discussed in the previous section, for the case where the primordial voids exist from recombination, more quantitative analysis of the RS effect is important. We extend TV's analysis and calculate the angular correlation function and its multipole moment, which are defined by

$$C(\theta) \equiv \left\langle \frac{\delta T}{T}(\theta_A) \frac{\delta T}{T}(\theta_B) \right\rangle_\theta \equiv \sum_l \frac{(2l+1)C_l}{4\pi} P_l(\cos \theta), \quad (8)$$

where θ_A and θ_B are angular positions with the separation angle θ . In general, in order to calculate the correlation function with density configuration in a real space, simulation-like computation is needed. However, if we evaluate

$$D(\theta) \equiv \left\langle \left[\frac{\delta T}{T}(\theta_A) - \frac{\delta T}{T}(\theta_B) \right]^2 \right\rangle_\theta = 2C(0) - 2C(\theta). \quad (9)$$

for each shell, we can sum up the contributions from all shells by using the same relation as equation (6).

Furthermore, for each shell, by the assumption of random distribution of voids and by the relation $(\delta T/T) \approx 0$, the correlation function $C(\theta)$ for the case where two light rays A and B pass different voids becomes zero; this allows us to consider only one void in our calculation. Let us imagine a void projected on the celestial sphere, as depicted in Figure 4. The angles α_A , α_B , α_c , α_s , and ξ are defined by reference to Figure 4. We calculate $D(\theta)$ for each shell by neglecting the curvature of the sphere in a local region around a void. Defining the midpoint of θ_A and θ_B as

$$\theta_c \equiv \frac{\theta_A + \theta_B}{2} = (\alpha_c, 0), \quad (10)$$

the positions of θ_A and θ_B are written as

$$\theta_A = \left(\alpha_c + \frac{\theta}{2} \cos \xi, \frac{\theta}{2} \sin \xi \right), \quad \theta_B = \left(\alpha_c - \frac{\theta}{2} \cos \xi, -\frac{\theta}{2} \sin \xi \right). \quad (11)$$

Then we can calculate $D(\theta)$ for each shell with the expression

$$D(\theta)_{\text{shell}} = \frac{\alpha_m^2}{\alpha_s^2} P \frac{1}{\pi} \int_0^\pi d\xi \frac{2}{\alpha_m^2} \int_0^{\alpha_m} \alpha_c d\alpha_c \left[\frac{\delta T}{T}(\alpha_A) - \frac{\delta T}{T}(\alpha_B) \right]^2, \quad (12)$$

where the relation between ψ and α (α_A or α_B) is

$$\sin \psi = \frac{d}{r} \sin \alpha \quad \text{with} \quad d = 3t_0 \left[1 - \left(\frac{t}{t_0} \right)^{\frac{1}{3}} \right], \quad (13)$$

and the angular size α_s which corresponds to the void radius and the integral boundary α_m are defined as

$$\alpha_s \equiv \alpha|_{\psi=\frac{\pi}{2}} = \arcsin \left(\frac{r}{d} \right), \quad \alpha_m \equiv \alpha_s + \frac{\theta}{2}. \quad (14)$$

By numerical integration of equation (12), we obtain $D(\theta)$ for each shell. Then we sum up their values of all shells from t_f to t_0 . Figure 5 shows plots of $C(\theta)$ for $z_f = 5$ and for $z_f = 1000$. As we expected, the RS effect for $z_f = 5$ is negligibly small while it is not for $z_f = 1000$. Therefore, we plot the values of C_l only for the case of $z_f = 1000$. There is much difference in C_l between the cases of $R_0 = 40h^{-1}\text{Mpc}$ and of $R_0 = 60h^{-1}\text{Mpc}$. For the latter case the RS effect is large enough to be observed at the scale $l \approx 1000$ by the next Map and Planck experiments.

The dependence of the CMB fluctuations on the volume fraction of voids F_0 is easily understood: as equation (7) shows, $\delta T/T$ is proportional to $\sqrt{F_0}$, and C_l to F_0 , accordingly.

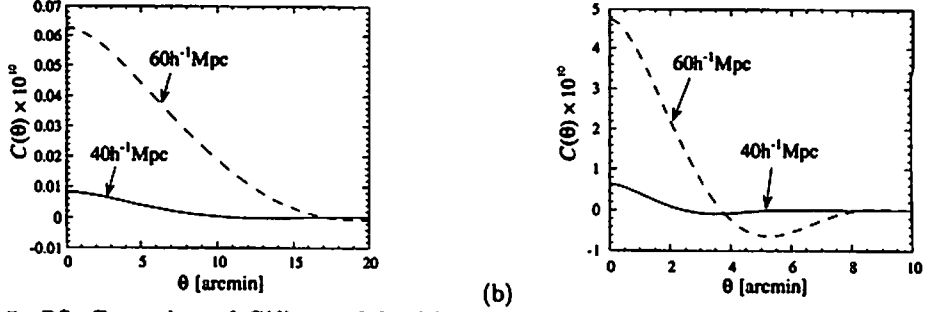


Fig. 5. RS effect: plots of $C(\theta)$ v.s. θ for (a) $z_f = 5$ and for (b) $z_f = 1000$. We set $\beta = 0.13$ and $F_0 = 2/3$.

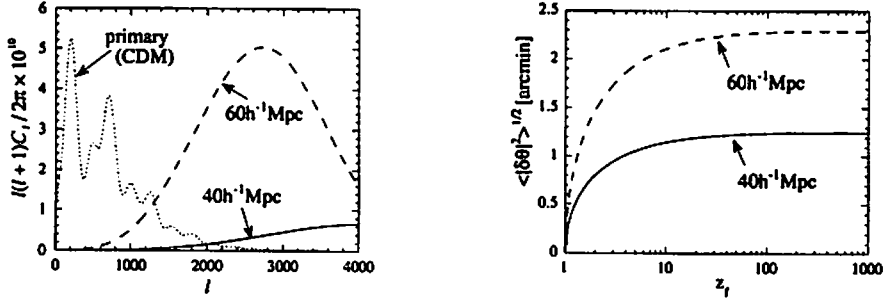


Fig. 6. RS effect: plot of the CMB angular power spectrum $l(l+1)C_l$ v.s. l for $C(\theta)$ in Fig. 5(b). For reference, we also plot the values of C_l estimated for a CDM model.

Fig. 7. Lensing effect: plot of $\sqrt{\langle |\delta\theta|^2 \rangle}$ v.s. z_f . We set $\beta = 0.13$, $F_0 = 2/3$, and $z_{\text{LSS}} = 1000$.

4 Gravitational Lensing Effect

We now turn to the lensing effect. First, we estimate the characteristic angular scale below which the lensing effect is appreciable, by calculating the angular excursion of a photon on the LSS, $\delta\theta$. The lens equation for a single void is given by

$$\delta\theta_{\text{one void}} = W\delta\alpha k \quad \text{with} \quad W \equiv \frac{d_{\text{LSS}} - d\cos\alpha}{d_{\text{LSS}}} \quad \text{and} \quad k \equiv \frac{\delta\theta}{|\delta\theta|}. \quad (15)$$

Replacing $\delta T/T$ with $W\delta\alpha$ in TV's calculation of σ_{RS} in §2, we find

$$\begin{aligned} \sqrt{\langle |\delta\theta|^2 \rangle} &= (H_0 R_0)^{\frac{1}{2}} \frac{\sqrt{F_0}}{\sqrt{z_{\text{LSS}} + 1} - 1} \left[\frac{1}{3(6\beta - 1)} - \frac{\sqrt{z_{\text{LSS}} + 1}}{9\beta - 1} + \frac{z_{\text{LSS}} + 1}{18\beta - 1} \right. \\ &\quad \left. - (z_f + 1)^{\frac{1}{2} - \theta\beta} \left\{ \frac{1}{3(6\beta - 1)} - \sqrt{\frac{z_{\text{LSS}} + 1}{z_f + 1}} \frac{1}{9\beta - 1} + \frac{z_{\text{LSS}} + 1}{z_f + 1} \frac{1}{18\beta - 1} \right\} \right]^{\frac{1}{2}}. \end{aligned} \quad (16)$$

We draw a plot of equation (16) in Figure 7, assuming $z_{\text{LSS}} = 1000$. The lensing effect also depends on the formation time of voids, though the dependence is not so strong as in the case of the RS effect. The typical angular scale of lensing is several arcminutes.

These angles, however, are not directly observable; we would rather calculate the dispersion of the relative angular separation θ , which is defined as

$$\sigma_{\text{gl}}(\theta) \equiv \frac{1}{\sqrt{2}} \langle |\delta\theta_A - \delta\theta_B|^2 \rangle^{\frac{1}{2}}. \quad (17)$$

σ_{g1} is, like the case of the RS effect, proportional to $\sqrt{F_0}$. The values of $\sigma_{g1}(\theta)$ for each shell and their sum are computed just like the computation of $D(\theta)$ in §3. Figure 8 shows a plot of $[\sigma_{g1}(\theta)/\theta]_{\theta=0.01 \text{ arcmin}}$ for each shell. Voids at $z \approx 50$ make the largest contribution to the CMB anisotropy: due to the factor W in equation (15), no effect arises from voids on the LSS, contrary to the case of the RS effect. To show that the dependence on the formation time z_f is not so strong in this case, we calculate $\sigma_{g1}(\theta)$ for several values of z_f , and some examples are presented in Figure 9. We note that the results are similar to the previous results for a CDM model [11, 12], though models and methods are quite different.

Once $\sigma_{g1}(\theta)$ is known, we can calculate the lensing effect on the CMB fluctuations. Here we adopt the approximate formula of Seljak [11] for the lensed correlation function:

$$\tilde{C}(\theta) = \frac{1}{2\pi} \int_0^\infty l dl e^{-\sigma_{g1}^2(\theta) l^2 / 2} C_l J_0(l\theta). \quad (18)$$

An example of the CMB anisotropy spectrum including lensing is presented in Figure 10a. As Seljak and Martínez-González *et al.* showed for a CDM model [11, 12], the lensing effect is to slightly smooth the main features appearing in the spectrum. To see the dependence on R_0 and z_f , we show the relative changes of the spectrum due to lensing for several cases in Figure 10(b), (c). In contrast with the case of the RS effect, the lensing effect has a weak dependence on the formation time.

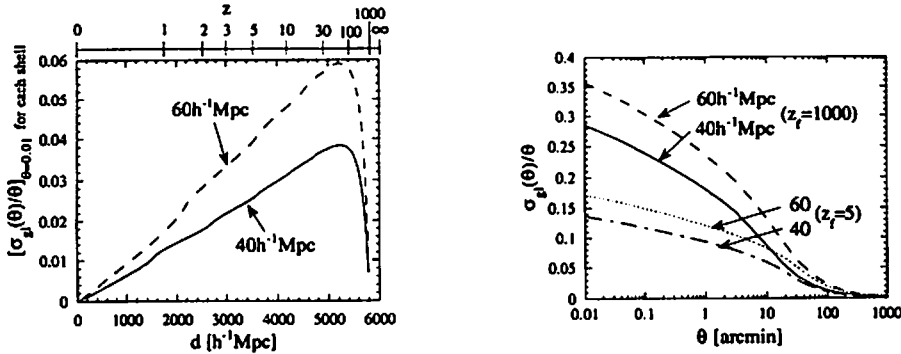
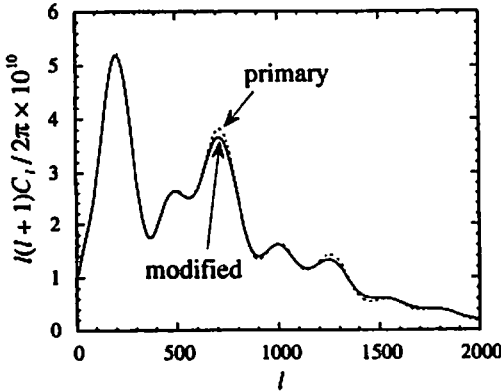


Fig. 8. Lensing effect: plot of $[\sigma_{g1}(\theta)/\theta]_{\theta=0.01 \text{ arcmin}}$ for each shell v.s. the position of shells ($h^{-1} \text{Mpc}$ and z). We set $\beta = 0.13$, $F_0 = 2/3$, and $z_{\text{LSS}} = 1000$.

Fig. 9. Lensing effect: plot of $\sigma_{g1}(\theta)/\theta$ v.s. θ . We set $\beta = 0.13$, $F_0 = 2/3$, and $z_{\text{LSS}} = 1000$.



(a)

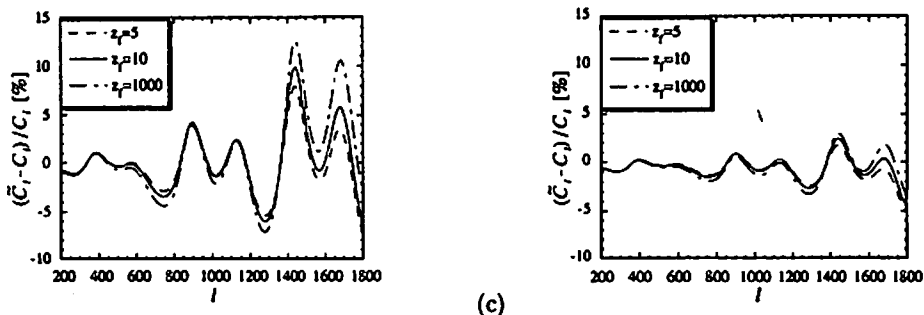


Fig. 10. Lensing effect. (a) shows plots of the CMB angular power spectrum $l(l+1)C_l$ v.s. l with lensing and without lensing for $2R_0 = 60h^{-1}\text{Mpc}$ and $z_t = 1000$. In (b) and in (c), respectively, the relative changes of the spectrum due to lensing for $2R_0 = 60h^{-1}\text{Mpc}$ and for $2R_0 = 40h^{-1}\text{Mpc}$ are presented. We set $\beta = 0.13$, $F_0 = 2/3$, and $z_{\text{LSS}} = 1000$.

5 Summary and Discussions

We have investigated the effect of the void network on the CMB anisotropy for TV's model, where many voids are distributed in the Einstein-de Sitter background. In particular, we have calculated the RS effect and the gravitational lensing effect separately.

As TV and other authors have already shown, the RS effect cannot make a dominant contribution to the CMB anisotropy in conventional scenarios like a CDM model, where nonlinear structure are formed at late time ($z_t < 10$). Hence, as to the RS effect, we have concentrated our attention on the case where primordial voids already exist at recombination, which is the case in some inflationary models. Calculating the angular power spectrum C_l , we have found that the RS effect can be appreciable in that case.

As we have mentioned in §1, however, the SW effect of voids lying on the LSS is also important and should be included. On the contrary, Baccigalupi *et al.* [9] claimed that the RS effect is negligible with respect to the SW effect, showing that the RS effect results from a partial cancellation of the redshift occurring during the void crossing, namely the SW effect, with the blueshift occurring at either shell crossing. We agree that, if we estimate the redshift distortion by a single void, the SW effect is much larger than the RS effect. What we would like to remark is that the number of voids between the LSS and us, which contribute to the RS effect, is much larger than the number of voids lying on the LSS. Hence, the total distortion of the redshift of photons by the RS effect can also be appreciable. In this sense, together with the analysis of the lensing effect, our work is complementary to that of Baccigalupi *et al.* for the probe of inflationary models associated with a first-order phase transition.

As for gravitational lensing, the effect of nonlinear voids has not been investigated so far. Our present work is the first trial of that investigation without any approximation for nonlinearity nor relativistic effect. We have shown how the primary anisotropy is smoothed out for some values of the void radius R_0 and of the formation time z_t . We have found that our results are similar to those for a CDM model [11, 12].

Our results as a whole suggest that, if the real universe is filled with nonlinear voids, they make some appreciable effects on the CMB anisotropy even for multipoles $l \lesssim 1000$. Those effects are expected to give some constraints on the configuration or origin of voids, particularly on the inflationary models in which primordial bubbles are nucleated, with the next generation of CMB satellites.

From another point of view, this multi-void model describes the extremely nonlinear universe that can be extrapolated from the distribution of luminous galaxies. Our results for $z_t = 1000$ provide the maximal contribution by nonlinear structure to the CMB anisotropy. We may, therefore, also conclude that, if one wants to determine the cosmological model from multipoles $l \lesssim 500$ of the anisotropy spectrum, the effect of nonlinear structure can be ignored.

Acknowledgments

We thank Paul Haines for reading the manuscript. Numerical Computation of this work was carried out at the Yukawa Institute Computer Facility. N. Sakai was supported by JSPS Research Fellowships for Young Scientists. This work was supported partially by the Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Science and Culture (No.9702603, No.09740334, and No.09440106).

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Cosmological Simulations of Gravitational Lensing

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Abstract

Gravitational weak lensing is analyzed by solving directly null-geodesic equations in the cosmological background. Statistical averages of optical quantities (the convergence, shear, and amplification) for ray bundles passing through cosmological inhomogeneities are derived in flat models (the S model with $(\Omega_0, \lambda_0) = (1.0, 0)$ and the L model with $(0.2, 0.8)$). The evolution of the spatial mass distribution is given by the N -body simulations in a tree-code. In the L model all particles are regarded as equivalent (compact) lens objects with a galactic size and in the S model particles consist of compact lens objects with a galactic size and clouds with weaker lens powers. The compact lens objects in the two models are assumed to have the same mass density. It is found as a result that the ratios of optical quantities in the S model to those in the L model increases with the separation angle (θ) of rays. If $\theta \sim 2$ arcsec and 360 arcsec, the ratios are ~ 1.5 and ~ 2.5 , respectively, for sources at $z = 1$.

1 Introduction

Since pioneering theoretical works on weak lensing by Gunn,[1] Blandford and Jaroszyński,[2] and Schneider and Weiss,[3] the deformation and amplification of distant sources due to the large-scale matter distribution have been studied successively,[4]- [11] together with local weak lensing caused by clusters of galaxies.[12] Recently their observational aspects also have been paid attention to and the determination of cosmological parameters through the quantitative analyses of weak lensing may be realistic in the near future.

The studies on cosmological gravitational lensing are classified into two types:

- (1) those due to the multi-lens-plane theory,[3][4] and
 - (2) those solving directly the null-geodesic equations for sources to observers' positions.[8][9][10]
- In the latter, multiple deflections can be automatically taken into account. Only for successive single deflections, both approaches must be consistent, if we follow light rays in high accuracy. Both approaches could include strong lensing, but the information on weak lensing was extracted by excluding singular cases or performing some smoothing. The inhomogeneous mass distributions have been derived hydrodynamically in the Newtonian and relativistic linear approximations, or using various N -body simulations.

In this work we treat ray bundles in the approach of second type, in flat background model with $(\Omega_0, \lambda_0) = (1.0, 0)$ and $(0.2, 0.8)$, and derive the optical quantities, such as the convergence and so on. The evolution of the mass distributions of inhomogeneities is given by the N -body simulation in the tree-code (which was provided by Suto) at the time interval $z = 0 - 5$, where the perturbations are in the CDM model with the power spectrum $n = 1$. [13] The difference from our previous works is that (1) we compared the weak lensing behaviors of various optical quantities in the standard model and low-density model, and derived the dependence of their quantities on the separation angle of ray bundles, (2) we adopted more realistic model universes, and (3) the calculation method of forces was improved.

In §2 the model universes and the equations for ray shooting are shown, and in §3 the statistical behaviors of optical quantities are derived. The details including figures and appendices are shown in a paper (Prog. Theor. Phys. 99, No. 1, 1998).

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2 Model universes and the ray shooting

The background model universes are assumed to be spatially flat, and the behavior of light rays passing through inhomogeneities in them is considered at the stage from an initial time t_1 (with redshift $z_1 = 5$) to the present time t_0 . In the Newtonian approximation, the line-element is expressed as

$$ds^2 = -(1 + 2\varphi/c^2)c^2 dt^2 + (1 - 2\varphi/c^2)a^2(t)(d\mathbf{x})^2. \quad (1)$$

The normalized scale factor $S \equiv a(t)/a(t_0)$ satisfies

$$\frac{dS}{d\tau} = \left(\frac{\Omega_0 + \lambda_0 S^3}{S} \right)^{1/2} \quad (2)$$

in the flat model, where $\tau \equiv H_0 t$. The gravitational potential φ is described by the Poisson equation

$$a^{-2} \frac{d^2 \varphi}{d\mathbf{x}^2} = 4\pi G \rho_B [\rho(\mathbf{x})/\rho_B - 1], \quad (3)$$

where $\rho_B (= \rho_{B0}/S^3)$ is the background density and

$$\rho_{B0} = \frac{3H_0^2 \Omega_0}{8\pi G} = 2.77 \times 10^{11} \Omega_0 h^2 M_\odot \text{ Mpc}^{-3}, \quad (4)$$

where $H_0 = 100 h \text{ Mpc}^{-1} \text{ km s}^{-1}$. In our treatment the inhomogeneities are spatially periodic in the sense that the physical situation at $\mathbf{x}$ is the same as that at $\mathbf{x} + l\mathbf{n}$, where the components of $\mathbf{n} (= (n^1, n^2, n^3))$ are integers. In an arbitrary periodic box with coordinate volume l^3 there are N particles with the same mass m . It is assumed that the force at an arbitrary point is the sum of forces from N particles in the box whose center is the point in question, and that the forces from outside the box can be neglected.

In this work we take two models (S model and L model) with $(\Omega_0, \lambda_0) = (1.0, 0)$ and $(0.2, 0.8)$, respectively. The present lengths of the boxes are

$$L_0 \equiv a(t_0)l = 32.5h^{-1}, 50h^{-1} \text{ Mpc}, \quad (5)$$

respectively. The particle number is 32^3 in both models, and so

$$m (= \rho_{B0} L_0^3 / N) = 2.90 \times 10^{11} h^{-1} M_\odot, 2.11 \times 10^{11} h^{-1} M_\odot, \quad (6)$$

respectively.

The distributions of particles in the two models were derived by the numerical N -body simulations using Suto's tree-code during the time interval $z = 0$ and $z = z_1$, where $z_1 = 5.0$ and 4.5 for models S and L, respectively. The details of his simulation are found in Suto's review paper.[13] Here all particles in model L are regarded as equivalent (compact) lens objects with a galactic size, which consist of galaxies and dark matter. In model S, particles in a small portion (20%) are regarded as compact lens objects with a galactic size and particles in the other portion (80%) are clouds with the weaker lens power. The latter particles have the same mass as that of the first ones, but a larger size. Their difference is represented using the softening radii (x_s). The compact lens objects were chosen randomly.

For the compact lens objects we assume the physical softening radius $a(t)x_s = 20h^{-1} \text{ kpc}$. In model S we consider three cases (a), (b) and (c) with respect to the radii of the clouds: $a(t)x_s = (100, 40, 20)h^{-1} \text{ kpc}$, for (a), (b) and (c), respectively. In Case (a) the clouds are very weak lenses and in Case (b) they are intermediate lenses which are not so weak as in Case (a) but weaker than compact lens objects. In both cases the clouds can contribute to cluster lenses as their components. The trivial case (c), in which the clouds have the same radius as compact lens objects, is added only for comparison.

Light propagation is described by solving the null geodesic equation with the null condition. Here let us use $T \equiv \frac{1}{2} \ln[a(t)/a(t_1)]$ as a time variable and $T_0 \equiv (T)_{t=t_0}$. Then we have $dS = 2 \exp[2(T - T_0)]dT$, so that

$$cdt = R c_R (\Omega_0 + \lambda_0 S^3)^{-1/2} e^{3T} dT, \quad (7)$$

where

$$R \equiv L_0 / [(1 + z_1)N^{1/3}] \quad (8)$$

and

$$c_R \equiv 2(c/H_0)/[R(1+z_1)^{3/2}]. \quad (9)$$

The line-element is

$$ds^2/R^2 = -c_R^2(\Omega_0 + \lambda_0 S^3)^{-1} e^{6T} (1 + \alpha\phi) dT^2 + (1 - \alpha\phi) e^{4T} dy^2, \quad (10)$$

where $y^0 = T$, $y_i = a(t_1)x_i/R$, $\varphi = \frac{Gm}{R}\phi$, and

$$\alpha \equiv \frac{2Gm}{c^2 R} = \frac{3}{\pi} \frac{\Omega_0}{(c_R)^2}. \quad (11)$$

The equations for light rays to be solved are

$$\frac{dy^i}{dT} = c_R e^T \tilde{K}^i, \quad (12)$$

$$\begin{aligned} \frac{d\tilde{K}^i}{dT} &= -3 \frac{\lambda_0 e^{6(T-T_0)}}{\Omega_0 + \lambda_0 e^{6(T-T_0)}} \tilde{K}^i + \alpha \frac{\partial \phi}{\partial T} \tilde{K}^i \\ &\quad - \gamma c_R^{-1} e^T \left[\frac{\partial \phi / \partial y^j}{\Omega_0 + \lambda_0 e^{6(T-T_0)}} - 2 \frac{\partial \phi}{\partial y^j} \tilde{K}^j \tilde{K}^i \right], \end{aligned} \quad (13)$$

$\gamma \equiv \alpha(c_R)^2$, and the null condition is

$$\sum_i (\tilde{K}^i)^2 = 1 + 2\alpha\phi. \quad (14)$$

The potential ϕ is given as a solution of the Poisson equation. For point sources with $\rho = m \sum_n \delta(aR(\mathbf{y} - \mathbf{y}_n))$ (n is the particle number), we have $\phi = \phi_1 + \phi_2$, where

$$\phi_1 = -e^{-2T} \sum_n \frac{1}{|\mathbf{y} - \mathbf{y}_n|}, \quad (15)$$

and ϕ_2 displays the contribution from the homogeneous background density. For the integration of the above equations, we calculate the potential at a finite number of points on the ray which are given at each time step (ΔT). Then particles near one of the points on the ray have a stronger influence upon the potential than particles far from any points. To avoid this unbalance in the calculation of the potential, we take an average of the potential ϕ_1 by integrating it analytically over the interval between one of the points and the next point. Moreover, to take into account the finite particle size as galaxies or clouds, we modify the above potential for point sources using the softening radii $y_s = a(t_1)x_s/R$. The modified potential is produced by replacing $(y - y_n)^2$ to $(y - y_s)^2 + (y_s)^2$ in the potential for point sources.

The initial values of $\tilde{K}^i$ are given so as to satisfy Eq. (14). The integration of Eqs. (12) and (13) with the modified potential was performed using the Adams method as in our previous papers. Use of the above averaged potential improved the accuracy for the integration of the null-geodesic equation.

3 Statistical behavior of optical quantities

We treat the deformation of ray bundles over the interval from $z = 0$ to $z = 5$ measured by an observer in a periodic box. For convenience we consider here the ray bundles reaching the observer in a regular form such that the rays are put in the same separation angle θ , and calculate the change in the angular positions of the rays increasing with redshift in the past direction. From this change we find the behavior of optical quantities.

Basic ray bundles consist of 5×5 rays which are put in the square form with the same separation angle $\theta = 2 - 360$ arcsec. Many bundles coming from all directions in the sky are considered. Here we take 1000 bundles coming from randomly chosen directions. In order to express angular positions of rays,

we use two orthogonal vectors $e_{(1)}^i$ and $e_{(2)}^i$ in the plane perpendicular to the first background ray vector $(\tilde{K}^i)_B$. These vectors satisfy

$$\sum_i (e_{(1)}^i)^2 = \sum_i (e_{(2)}^i)^2 = 1, \quad \sum_i e_{(1)}^i e_{(2)}^i = 0, \quad \sum_i e_{(1)}^i \tilde{K}_B^i = \sum_i e_{(2)}^i \tilde{K}_B^i = 0. \quad (16)$$

Then the angular coordinates $[X(m, n), Y(m, n)]$ of 25 rays relative to the first ray with $(m, n) = (1, 1)$ at any epoch are defined by

$$\begin{aligned} X(m, n) &= \sum_i [y^i(m, n) - y^i(1, 1)] e_{(1)}^i / y_B(1, 1) + X(1, 1), \\ Y(m, n) &= \sum_i [y^i(m, n) - y^i(1, 1)] e_{(2)}^i / y_B(1, 1) + Y(1, 1), \end{aligned} \quad (17)$$

where $y^i = y_B^i + \delta y^i$ and $y_B = [\sum_i (y_B^i)^2]^{1/2}$. Since all angular intervals of rays at the observer's point are the same ($= \theta$), differentiation of angular coordinates of the rays at any epoch with respect to those at observer's points is given by the following differences:

$$\begin{aligned} A_{11}(m, n) &= [X(m+1, n) - X(m, n)] / \theta, \\ A_{12}(m, n) &= [X(m, n+1) - X(m, n)] / \theta, \\ A_{21}(m, n) &= [Y(m+1, n) - Y(m, n)] / \theta, \\ A_{22}(m, n) &= [Y(m, n+1) - Y(m, n)] / \theta, \end{aligned} \quad (18)$$

where m and n run from 1 to 5. From the matrix $A_{ij}(m, n)$ we derive the optical quantities in the standard manner,[14] as the convergence ($\kappa(m, n)$), the shear ($\gamma_i(m, n)$, $i = 1, 2$), and the amplification ($\mu(m, n)$) defined by

$$\begin{aligned} \kappa(m, n) &= 1 - \text{tr}(m, n)/2, \quad \gamma_1(m, n) = [A_{22}(m, n) - A_{11}(m, n)]/2, \\ \gamma_2(m, n) &= -[A_{12}(m, n) + A_{21}(m, n)]/2, \\ \gamma^2 &\equiv (\gamma_1)^2 + (\gamma_2)^2 = [\text{tr}(m, n)]^2 - \det(A_{ij}(m, n)), \\ \mu(m, n) &= 1 / \det(A_{ij}(m, n)), \end{aligned} \quad (19)$$

where the trace is $\text{tr}(m, n) = A_{11}(m, n) + A_{22}(m, n)$. The average optical quantities in each bundle are defined as the averages of optical quantities for all rays in the bundle as follows:

$$\bar{\kappa} = [\sum_m \sum_n \kappa(m, n)] / 4^2, \quad \bar{\kappa}^2 = [\sum_m \sum_n (\kappa(m, n))^2] / 4^2, \quad (20)$$

and so on. Here it is to be noted that the above optical quantities at the separation angle θ are derived in the coarse-graining on this scale, because the contributions from smaller scales can be cancelled and smoothed-out in the averaging process.

As the result of our ray shooting we found the appearance of caustics in about 3 % of 1000 ray bundles, in which μ diverged for $z \leq 5$. For the present statistical analysis we excluded these caustic cases and considered only the cases of weak lensing in the sense of no caustics. The averaging for all non-caustic ray bundles is denoted using $\langle \rangle$ as $\langle \kappa^2 \rangle$, $\langle \gamma^2 \rangle$ and $\langle (\mu - 1)^2 \rangle$. Because $\kappa(m, n)$, $\gamma_i(m, n)$ and $\mu(m, n) - 1$ take positive and negative values with almost equal frequency, $\langle \kappa \rangle$, $\langle \gamma_i \rangle$ and $\langle \mu \rangle - 1$ are small.

From the behavior of $\langle \kappa^2 \rangle$ and $\langle \gamma^2 \rangle$ for various separation angles $\theta = 2 \text{ arcsec} - 360 \text{ arcsec}$ ($= 6 \text{ arcmin}$), it is found that between model L and S(b), which is the intermediate of the three cases of model S, $\langle \kappa^2 \rangle$ and $\langle \gamma^2 \rangle$ have nearly equal differences for $\theta = 2 - 360 \text{ arcsec}$, though the ratio of their values in model S to those in model L increases with the separation angle (θ) of rays. If $\theta \sim 2 \text{ arcsec}$ and 360 arcsec , the ratios are ~ 1.5 and ~ 2.5 , respectively, for sources at $z = 1$ for models S(b) and L. The differences among their values for S(a), S(b) and S(c) decrease with the increase of θ , and at $\theta = 360 \text{ arcsec}$ their differences almost vanish. This is because in model S, both compact lens objects and clouds work as similar lenses for $\theta = 360 \text{ arcsec}$ and compact lens objects with small radii work as dominant lenses for $\theta \sim 2$. At $\theta = 360 \text{ arcsec}$ the difference between $\langle \kappa^2 \rangle$ in model S and that in model L is about 6, which is consistent with the result of Bernardeau et al. (cf. their Figs. 3 and 4).

In Tables I, II, III and IV, we show for models L and S (a) $\sim$ (c) the numerical values of $\langle \kappa \rangle$, $\langle \kappa^2 \rangle^{1/2}$, $\langle \mu \rangle$, $\langle (\mu - 1)^2 \rangle^{1/2}$, $\langle \gamma_1 \rangle$ and $\langle \gamma^2 \rangle^{1/2}$ at epochs $z = 1, 2, \dots, 5$ for $\theta = 2$ arcsec. The optical quantities treated in our previous papers can be expressed using κ and γ . For example, $\beta [= \sqrt{G(\theta)}]$ in the previous ones is equal to $(\kappa^2 + \gamma^2)^{1/2}$. The value of β in this paper is smaller by a factor 10 than the corresponding ones in the previous paper.[9] This is because the mass of lens objects in this work is smaller by a factor 30 than the masses in the previous one.

The amplitude of lens fluctuations in source magnitudes is $\Delta m \equiv \frac{5}{2} \log[1 + \langle (\mu - 1)^2 \rangle^{1/2}] \simeq 1.09$ for $(\mu - 1)^2 \rangle^{1/2} > 1/2$. Its values are $0.03 - 0.1$ for $z = 1 - 5$, respectively, as can be seen from these tables.

Fig. I: Optical quantities in model L.

z	$\langle \kappa \rangle$	$\langle \kappa^2 \rangle^{1/2}$	$\langle \mu \rangle$	$\langle (\mu - 1)^2 \rangle^{1/2}$	$\langle \gamma_1 \rangle$	$\langle \gamma^2 \rangle^{1/2}$
1	-0.0004	0.0316	1.0038	0.0767	-0.0008	0.0316
2	-0.0025	0.0618	1.0129	0.1555	-0.0008	0.0648
3	-0.0047	0.0843	1.0247	0.2270	0.0009	0.0897
4	-0.0056	0.1011	1.0393	0.3031	0.0018	0.1077
5	-0.0061	0.1081	1.0463	0.3443	0.0021	0.1152

Fig. II: Optical quantities in model S(a).

z	$\langle \kappa \rangle$	$\langle \kappa^2 \rangle^{1/2}$	$\langle \mu \rangle$	$\langle (\mu - 1)^2 \rangle^{1/2}$	$\langle \gamma_1 \rangle$	$\langle \gamma^2 \rangle^{1/2}$
1	-0.0023	0.0435	1.0045	0.1055	0.0014	0.0478
2	-0.0055	0.0716	1.0151	0.1928	0.0021	0.0787
3	-0.0083	0.0892	1.0254	0.2702	0.0046	0.0968
4	-0.0104	0.1015	1.0347	0.3430	0.0067	0.1086
5	-0.0115	0.1105	1.0444	0.4180	0.0081	0.1172

Fig. III: Optical quantities in model S(b).

z	$\langle \kappa \rangle$	$\langle \kappa^2 \rangle^{1/2}$	$\langle \mu \rangle$	$\langle (\mu - 1)^2 \rangle^{1/2}$	$\langle \gamma_1 \rangle$	$\langle \gamma^2 \rangle^{1/2}$
1	-0.0023	0.0526	1.0103	0.1438	0.0018	0.0606
2	-0.0067	0.0847	1.0268	0.2520	0.0022	0.0990
3	-0.0101	0.1062	1.0443	0.3418	0.0048	0.1218
4	-0.0127	0.1215	1.0620	0.4551	0.0070	0.1366
5	-0.0139	0.1328	1.0779	0.5813	0.0085	0.1474

Fig. IV: Optical quantities in model S(c).

z	$\langle \kappa \rangle$	$\langle \kappa^2 \rangle^{1/2}$	$\langle \mu \rangle$	$\langle (\mu - 1)^2 \rangle^{1/2}$	$\langle \gamma_1 \rangle$	$\langle \gamma^2 \rangle^{1/2}$
1	-0.0027	0.0637	1.0183	0.2087	0.0023	0.0730
2	-0.0070	0.1018	1.0511	0.4011	0.0014	0.1194
3	-0.0103	0.1282	1.0903	0.6355	0.0036	0.1484
4	-0.0128	0.1474	1.1374	1.3270	0.0055	0.1678
5	-0.0141	0.1617	1.1895	2.0991	0.0067	0.1819

4 Conclusion

In this work we assumed that the low-density model L contains only compact lens objects and the Einstein-de Sitter model S contains compact lens objects and clouds with weaker lens power, and compared the roles of compact lens objects and the clouds (with larger radii). As the result it was found that the difference of optical quantities (the convergence and shear) between three cases of model S (S(a), S(b), and S(c)) decreases with the separation angle θ , the difference in models L and S(b) is almost constant at the interval $\theta = 2 - 360$ arcsec, though their ratios in model S to those in model L increases with the separation angle (θ) of rays. Their values at small angles depend sensitively on the radii of the clouds, which were derived at epochs of $z = 1 - 5$.

The mass density of compact lens objects (for $\Omega_0 = 0.2$) may be by ~ 10 larger than the observed mass density of galaxies. If compact lens objects are limited to observable galaxies, these optical quantities at small angles $\theta \approx 2$ arcsec will become smaller than those derived in the above, but the change in model S may be small, because the compact lenses occupy a small portion ($< 20\%$). At large angles $\theta > 360$ arcsec, the change is negligibly small.

Here we consider the comoving volume of the region through which a ray bundle with solid angle θ^2 pass at the interval $0 \leq z \leq z_i$. It is equal to the volume of a cone $V_R = \frac{1}{3}\theta^2[2cH_0(1 - 1/\sqrt{1+z_i})]^3$ in the flat model. Its ratio to the volume of the periodic box $V_B [= (32h^{-1}\text{Mpc})^3]$ is 1.2×10^{-3} for $z_i = 5$ and $\theta = 10$ arcsec. This small ratio means that the influence of the periodicity of the box on the the statistics of ray bundles is negligibly small.

Background models were assumed to be flat for simplicity. In the next work our analyses will be extended to an open model.

Acknowledgements

The author would like to thank Y. Suto for providing the results of N -body simulations which include the time evolution appropriate to the present studies. He owes also to the YITP computer system for the numerical analyses.

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Kamiokande and Super-Kamiokande

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Neutrino astrophysics has begun in 1987 when Kamiokande observed the neutrino burst from a supernova SN1987A. Kamiokande succeeded to observe solar neutrinos and demonstrated at the first time that the observed neutrinos are coming from the direction of the sun and confirmed the long standing solar neutrino problem. The atmospheric neutrino anomaly was first pointed out by the Kamiokande experiment in 1988 and is now thought to be an evidence of a neutrino oscillation. Super-Kamiokande, the bigger successor of Kamiokande, started its operation in April 1996. The accumulated data have already exceeded those of Kamiokande in its quality and quantity. New insights into neutrino masses will be obtained from the results of SK in very near future.

1 Introduction and Detector

The gigantic imaging water Cherenkov detectors located 1000 m underground (2,700m water equivalent), Kamiokande [1] and Super-Kamiokande [2], are the powerful tool to understand neutrino properties and study proton decay. The volume of Kamiokande (KM) is 4,500 tons (948 photo-multiplier tubes (PMTs) are arranged to watch inner volume of 2,140 tons), and that of Super-Kamiokande (SK) is 50,000 tons (11,146 PMTs are arranged for the inner volume of 32,000 tons). KM has started its operation in 1984, primarily to study nucleon decay [3] and later in 1987, KM has upgraded to be able to detect low energy events above 9.3 MeV to study solar neutrinos. In February, 1987, just after the upgrade, KM has observed the neutrino bursts from a supernova SN1987A [4] exploded in the Large Magellanic Cloud, 160k light years away. This is the first observation of astrophysical neutrinos other than solar neutrinos, and is the beginning of neutrino-astronomy. KM has succeeded to obtain a finite solar neutrino flux in 1987, soon after the neutrino burst detection. This observation provided the first evidence that the sun creates high energy neutrinos produced by the ${}^8B$ decay and confirmed the solar neutrino problem [5]. Unexpected anomaly of the atmospheric neutrinos, which is now thought to be an indication of a neutrino oscillation was first pointed out by the KM experiment in 1988 [6]. KM has stopped major physics run in February in 1995 and stopped its operation in summer 1997. The last stage of the experiment is solely devoted to a supernova watch. The physics achievements and the history of KM are summarized in references [7] [8].

Since the space is limited in this paper, we mostly discuss about the SK experiment in the following parts and physics discussion will also be limited to neutrino oscillation studies.

The schematic view of the SK detector is shown in figure 1. SK is not only bigger than KM in its size, but also has better energy, position and directional resolutions: 14%(19%), 26°(27°), 87cm(100cm) for 10 MeV electrons, respectively (the numbers shown in the parentheses are those for KM). The better resolutions are achieved by the increased number density of PMTs. The PMTs are arranged on the inner surface of the detector with the density of 2 PMTs / 1 m² (1 PMT / 1 m² for KM) which covers the 40% of the inner surface. This arrangement more than compensates the light loss (due to the attenuation) of the longer travel distance of the Cherenkov light than KM. Outside of the

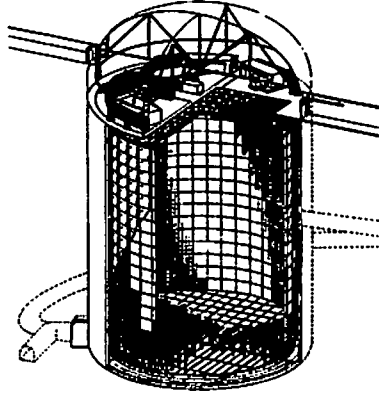


Figure 1: A schematic view of the 50,000 ton imaging water Cherenkov detector, Super-Kamiokande. The detector is a cylindrical shape, 39.3m in diameter and 42m in height. The inner volume of 32,000 tons, 16.9m in radius and 36.2m in height, is viewed by 11,146 photo-multiplier tubes with 50 cm diameter.

inner detector is surrounded by the outer detector of 2m thick active water slab viewed by 1,185 20 cm PMTs. This provides the shield against the external gamma rays and neutrons, and positively identifies incoming particles like through-going cosmic ray muons, particles exiting the detector and so on.

The energy of neutrinos, which we are interested in detecting in SK, ranges from a few MeV to more than 100 TeV. The interactions of neutrinos in the detector will be described later for solar and atmospheric neutrino cases.

The relativistic charged particles traversing faster than the light velocity in the water emit Cherenkov lights with an angle of

$$\cos\theta = \frac{1}{\beta n},$$

where $\theta=42^\circ$ for particles with $\beta=1$, and n is the refractive index of water. Those ring image Cherenkov lights are detected by the PMTs. The hit pattern, pulse height and timing information are used to reconstruct event vertex, direction, energy and particle species. The details of the event reconstruction can be found in one of those theses[9].

2 Operation and Calibration

Super-Kamiokande (SK) has started the operation on 1st of April in 1996 and been continuously taking data with the cumulative live-time close to 90%. Most of the time not for taking data are spent to calibrations, especially to those using electron LINAC [10].

Water is essential for the SK experiment. It serves as a target. But the light scattering and absorption affects on the energy resolution, vertex resolution and so on. For the first two months after the start-up, there was a rapid improvement of the water transparency, since we had just started to circulate the water a few days before the start-up. Therefore in the analysis presented here we have not used the data taken in April and May, 1996. There is a small change of the water transparency during the experiment. The water transparency, which is about 100m at 420nm now, is monitored and measured by several methods: muon-decay electrons, through-going muons and a direct measurement

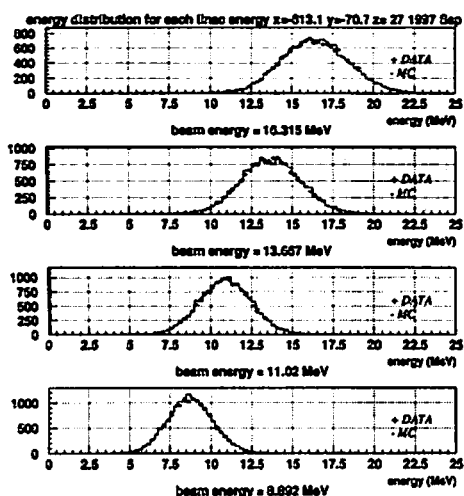


Figure 2: The energy distribution of LINAC data taken at the central height of the detector, but with about 8m off the center towards the side wall. The energies of the electrons are at 16.3, 13.7, 11.0 and 8.9 MeV from the top figure to the bottom figure. The results from the MC simulations are also shown by the solid lines. The data and the MC agree very well.

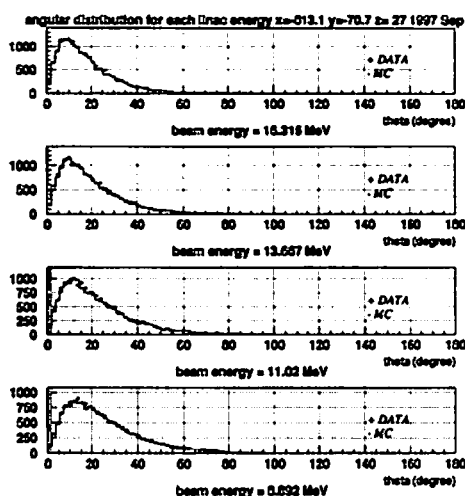


Figure 3: Same as figure 2, but for the angular distributions.

using a Dye laser and a CCD camera. A Change of the water transparency was taken into account when the event energy was reconstructed.

Another important element related to the water quality is Radon concentration in the water. The daughter of the Radon decay (^{214}Bi) emits β rays with the maximum energy of 3.26MeV and the resolution tail of this decay products is the major background for the solar neutrino measurement. The current concentration of Rn in the water is measured to be less than a few mBq/m³ which is already more than an order of magnitude lower than the KM level, but is still a source of the background. We are making extensive efforts to make further reductions of Rn in the water. Analysis threshold of 6.5 MeV currently set is mainly determined by this background level.

Trigger is made by the number of PMT hits within 200nsec time window. Data are being taken with two threshold levels—the low energy (LE) and the super-low energy (SLE) trigger. The threshold of the LE trigger is ~ 29 hits corresponding to about 5.7 MeV on an average. The SLE trigger is set at 24 hits (4.6 MeV on an average). The event vertex of those events only having a SLE trigger are quickly reconstructed and a crude fiducial cut are applied in the on-line host computer. This selection reduces the 100 Hz SLE trigger to about 5 Hz which is add on top of the 11 Hz of the LE trigger.

The energy of an event is determined differently for high energy (atmospheric neutrino) and low energy (solar neutrino) events. In the atmospheric neutrino analysis, the energy is determined based on the number of photo-electrons (9 p.e. approximately corresponds to 1 MeV) and in the solar

neutrino analysis the energy is determined by the number of hit PMTs since in such low energy most of the PMT hits are one photo electrons (5.6 hits approximately corresponds to 1 MeV).

The accuracy of the energy scale for high energy events is estimated to be $\pm 2.4\%$ by the various methods by using cosmic ray through-going muons, stopping muons, muon-decay electrons, invariant mass of π^0 's and so on. The energy resolution for electrons and muons are $2.5\%/\sqrt{E(\text{GeV})}+0.5\%$ and 3% , respectively.

In the solar neutrino analysis, the absolute energy of the events is calibrated by using the electron beam produced by the electron LINAC placed near the SK tank, which covers the energy range between 5 and 15 MeV, exactly matching to the energy spectrum of the solar neutrinos. The LINAC was also used to obtain the angular resolution and the vertex resolution. The beam energy of the LINAC was calibrated by the germanium detector which was calibrated by using the monochromatic electrons selected by a magnetic spectro-meter. The typical energy and angular distributions taken at various energies and at the injection position of about 8 m from the wall are shown in figures 2 and 3.

The uniformity throughout the entire fiducial volume and the directional dependence were checked by the LINAC data taken at different positions, γ -ray sources using $\text{Ni}(n,\gamma)\text{Ni}'$ reactions placed many different places in the detector and the spallation events which happen uniformly in the detector volume. After considering those effects, the energy scale error for the total flux measurement is estimated to be $\pm 0.85\%$. The details of the calibration will be found in ref[10], [18].

3 Physics with Super-Kamiokande

The primarily physics subject of Kamiokande (KM) is a search for nucleon decay; the name Kamiokande comes from Kamioka Nucleon Decay Experiment. After the observation of the neutrino burst from SN1987A and the success of the solar neutrino measurement, people would call the experiment as Kamioka Neutrino Detection Experiment.

The main purposes of Super-Kamiokande (SK) are, off course, to search for nucleon decay with an order of magnitude better sensitivity, to measure neutrinos from the sun and to study atmospheric neutrinos.

If neutrinos have finite masses and non-zero mixings among mass eigen-states, then neutrinos produced at the flavor eigen-states may oscillate among different states with the following probability (for two neutrino oscillation case),

$$P(\nu_\alpha \rightarrow \nu_\beta) = \sin^2 2\theta \sin^2 \left(1.27 \cdot \Delta m^2 \cdot L/E \right),$$

where E is the energy of neutrino and L is the distance between the source and the detector: L and E are determined by the experimental arrangement. If $\Delta m^2 \gg E/L$, then $P(\nu_\alpha \rightarrow \nu_\beta) = \frac{1}{2} \sin^2 2\theta$, and if $\Delta m^2 \ll E/L$, then experiments hardly see the oscillation effect and are able to set the upper limit by $\sin^2 2\theta \cdot \Delta m^2 \ll 0.8[E/L]\sqrt{P(\nu_\alpha \rightarrow \nu_\beta)}$. If $\Delta m^2 \sim E/L$, experiments are in a most optimum position. For example, for a configuration $E \sim 1\text{GeV}$ and $L \sim 10$ to $10,000$ km (in the case of the sub-GeV atmospheric neutrinos), $\Delta m^2 \sim 10^{-1} \sim 10^{-4} \text{eV}^2$ and for $E \sim 10$ MeV and $L \sim 150,000,000$ km (for 'Just So' solar neutrino oscillation [11]), $\Delta m^2 \sim 10^{-11} \sim 10^{-10} \text{eV}^2$.

When neutrinos pass through the media, the electron neutrinos obtain an additional potential, $\sqrt{2}G_F n_e$ through the charged current forward scattering amplitude. The equation of the neutrino

propagation for $\nu_e \rightarrow \nu_\mu$ case is,

$$i \frac{d}{dt} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} -\frac{\Delta m^2 \cos 2\theta}{4p} + \sqrt{2} G_F n_e & \frac{\Delta m^2 \sin 2\theta}{4p} \\ \frac{\Delta m^2 \sin 2\theta}{4p} & \frac{\Delta m^2 \sin 2\theta}{4p} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}.$$

The mixing angle in the matter becomes,

$$\tan 2\theta_m = \frac{\tan 2\theta_v}{1 - (2p\sqrt{2}G_F n_e)/\Delta m^2 \cos 2\theta_v}.$$

If $(2p\sqrt{2}G_F n_e)/(\Delta m^2 \cos 2\theta_v) = 1$ (resonance condition), then the mixing angle in the matter becomes maximal, even though the vacuum mixing angle is small. If the neutrino propagates adiabatically through the resonance region, electron neutrino converts into ν_μ 's. For 10 MeV neutrinos produced in the sun satisfy the resonance condition when pass through the sun ($0 < n_e/N_A < 100$) if

$$\Delta m^2 \leq 1.6 \times 10^{-4} \text{eV}^2.$$

Adiabatic condition for the 10 MeV neutrinos is,

$$\Delta m^2 \frac{\sin^2 2\theta}{\cos 2\theta} \geq 6.3 \times 10^{-8} \text{eV}^2.$$

These conditions determines the parameter region explaining the 'observed' conversion rate.

Searches for neutrino masses have been done not only by the neutrino oscillation searches, but also by other methods like direct mass searches and double beta decay experiments although the indication of the finite neutrino masses only come from oscillation experiments. The current upper limit on the neutrino mass from the experiments looking directly for the evidence of finite neutrino masses are,

$$m_{\nu_e} \leq 18.2 \text{MeV}/c^2 [12]$$

$$m_{\nu_\mu} \leq 170 \text{keV}/c^2 [13]$$

$$m_{\nu_\tau} \leq 3.5 \sim 5.6 \text{MeV}/c^2 [14]$$

The best limits for the Majorana neutrino masses of $0.48 \sim 1.5 \text{eV}^2$ [15] was obtained from the double beta decay experiment (We have included the uncertainty of the nuclear matrix element).

As you see above, neutrino oscillations are the only way to approach small masses (difference) well below 1eV: sensitive to the very wide Δm^2 range down to 10^{-12}eV^2 .

The neutrino oscillation studies in SK will be done not only with high statistics and better qualities but with different approaches: model independent and flux calculation independent analyses are the key issue for the SK experiment.

3.1 Atmospheric Neutrinos

Atmospheric neutrinos are produced in the upper atmosphere through the decay of mesons produced by interactions of primary cosmic rays on nuclei in the atmosphere. The neutrino energy ranges from below GeV to over TeV and the distance from the points where neutrinos are produced and the detector location ranges from 10 to 13,000km. Therefore the sensitivity to the mass difference is down to $\Delta m^2 \sim 10^{-4} \text{eV}^2$.

At the low energy limit where most of the muons produced in the atmosphere decay, the ratio of $(\nu_\mu + \bar{\nu}_\mu)/(\nu_e + \bar{\nu}_e)$ becomes ~ 2 . The more the energy increases, the more the ratio becomes larger.

The uncertainty of the atmospheric neutrino flux calculations is mainly come from the uncertainty of the primary cosmic ray flux, meson productions, and neutrino interactions, and is estimated to be $\sim 25\%$ (20% from the flux and 15% from the interaction uncertainties). In order to minimize those uncertainties in the neutrino oscillation studies, the double ratio $R=(\mu/e)_{data}/(\mu/e)_{MC}$ is usually taken.

Another important approach is to study the zenith angle distribution of the neutrino events which will be described later.

In SK the atmospheric neutrinos are studied by two classes of the events: the contained events and upward going muons. The upward going muons are produced by high energy muon neutrino interactions taken underneath of the detector and penetrating the detector entering from the bottom of the detector. We will not discuss the upward going muons in this paper.

3.1.1 Contained events

The contained events are those produced in the detector by neutrino interactions. Traditionally we divide the contained events into sub-GeV ($E_l < 1.33$ GeV) and multi-GeV ($E_l > 1.33$ GeV). The sub-GeV samples were used originally to search for proton decay.

The strategy to study atmospheric neutrinos is to identify neutrino flavor through the identification of the lepton flavor of single ring events (in the sub-GeV sample) and of most energetic particle (in the multi-GeV sample). In the multi-GeV sample we have also included so called partially contained events (PC)—events which have one track exiting from the inner detector. For example, 91.1% of the electron events come from ν_e charged current interactions and 95.9% of the muon events come from ν_μ charged current interactions in sub-GeV sample. For the multi-GeV sample, the correlation is much better. Therefore the particle identification is the key issue for the neutrino oscillation analysis.

The data between May-27 in 1996 and Oct-16 in 1997 (414.2 effective live days) are used for the present analysis. The contained events are first selected by applying simple computer algorithm to the raw data ($\sim 10^6$ events/day): (1) no significant outer detector activity, (2) the inner total charge is > 200 photo-electrons. We have also applied further cuts to remove stopping muons, through going muons and electrons from stopping muon decays. This reduction step yields ~ 30 events/day for the fully contained event (FC) selection and ~ 20 events for partially contained event (PC) selection. Then the events were scanned by two independent physicists and the remaining background events were rejected. We finally obtained ~ 8.0 fully contained events/day and ~ 0.5 partially contained events/day after applying the 2m cut from the inner PMT surface (22.5kton fiducial volume) and the minimum energy cuts of 30 MeV (fully contained) and 3,000 photo-electrons (partially contained).

The identification of electron and muons are done by using the information of the diffuseness of the Cherenkov rings. The probability is defined by

$$P_{e(\mu)}^{(1)} = \prod_{\theta_i < 1.5\theta_c} \frac{1}{\sqrt{2\pi}\sigma_i} \exp \left[-\frac{1}{2} \left(\frac{p.e.^i - p.e.^i_{e(\mu)}}{\sigma_i} \right)^2 \right],$$

where σ_i is the resolution of PMT including also an effect of the gain spread. We have also similar probability function for the angular distribution, $P_{e(\mu)}^{(2)}$. Then the log-likelihood function is determined by

$$\mathcal{L} = \log(P_\mu^1 \cdot P_\mu^2) - \log(P_e^1 \cdot P_e^2).$$

The likelihood distribution of the particle identification is shown in figure 4 and you can clearly see the good separation between the μ -like and e -like events. The mis-identification probability is estimated

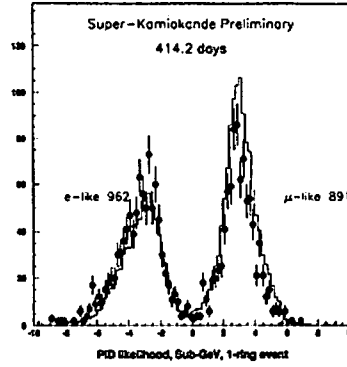


Figure 4: The distribution of the likelihood to identify electrons and muons. The separation between μ -like and e-like events are clearly seen. The probability of the mis-identification is less than 1%

Table 1: The summary of the contained events caused by the atmospheric neutrino interactions in Super-Kamiokande for 414.2 effective days. Also listed is the predictions by the Monte Carlo Calculation.

	Sub-GeV		Multi-GeV	
	DATA	MC(Honda)	DATA	MC(Honda)
1R	1853	2001.3	394	411.6
e-like	962	796.1	218	182.7
μ -like	891	1205.2	176	229.0
2R	594	657.2	153	143.7
$\geq 3R$	155	190.1	151	156.3
PC			200	244.8

to be less than 1 %. The validity of the particle identification was verified by the test done at KEK using a 1kt water Cherenkov detector by the electron and muon beam with known momentum [16].

The results after the final data selections are summarized in table 1. All the events are used to make the double ratio of the muons and electrons:

$$\frac{(\mu/e)_{DATA}}{(\mu/e)_{MC}} = 0.635^{+0.029}_{-0.028} \pm 0.009 \pm 0.048(sub - GeV),$$

$$\frac{(\mu/e)_{DATA}}{(\mu/e)_{MC}} = 0.665^{+0.059}_{-0.054} \pm 0.020 \pm 0.078(multi - GeV(FC)),$$

$$\frac{(\mu/e)_{DATA}}{(\mu/e)_{MC}} = 0.664^{+0.069}_{-0.062} \pm 0.022 \pm 0.097(multi - GeV(FC + PC)).$$

The first, second and third errors come from statistics, MC statistics and systematics, respectively. The double ratios, $(\mu/e)_{DATA}/(\mu/e)_{MC}$, as a function of the energy for sub-GeV and multi-GeV sample are shown in figures 5 and 6. The monotonous suppression of $\sim 60\%$ is seen over entire energy region.

The zenith angle distribution, which is of most importance to understand about neutrino oscillations, is similar to the E/L plot since the zenith angle, θ_z , approximately determines the distance(L)

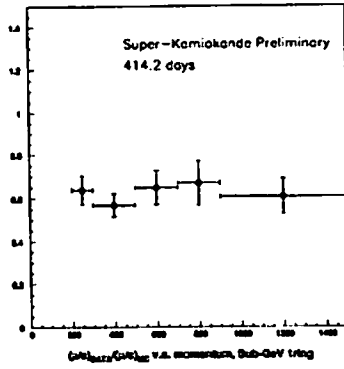


Figure 5: The double ratios, $(\mu/e)_{DATA}/(\mu/e)_{MC}$, for the sub-GeV sample as a function of energy for 414.2 days of data.

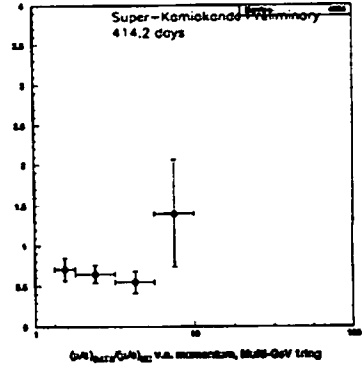


Figure 6: Same as figure 5, but for the multi-GeV data sample.

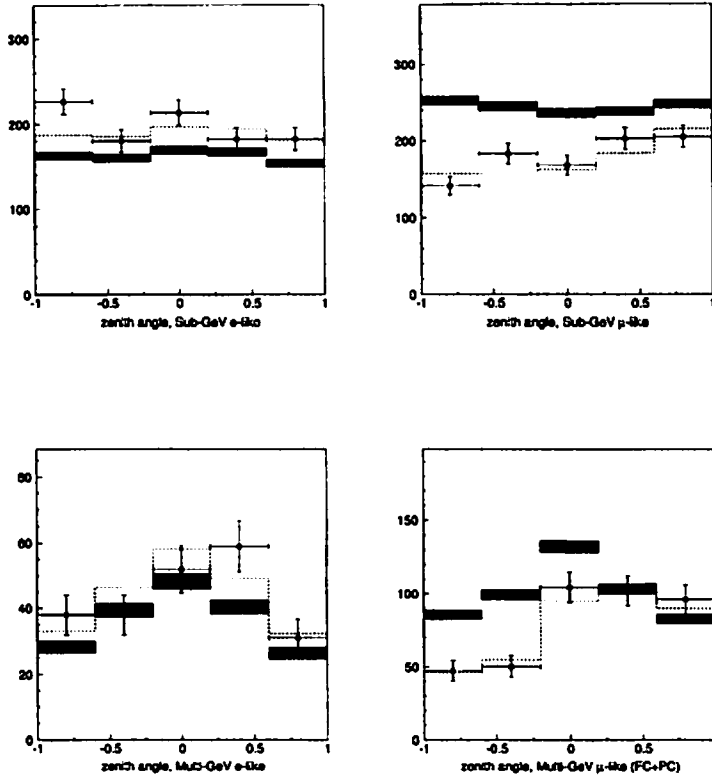


Figure 7: The zenith angle distribution of the e-like and μ -like events for the sub-GeV and multi-GeV data samples. The solid bars show the prediction from the MC calculation (no oscillation) being symmetric for all the distributions. In the high energy muon sample, the zenith angle distortion is prominent. The predicted distributions for the neutrino oscillation, $\nu_\mu \rightarrow \nu_\tau$ ($\Delta m^2 = 5 \times 10^{-3} \text{eV}^2$, $\sin^2 2\theta = 1.0$) are also shown by the dotted lines. For e-like data, the difference between no oscillation (solid bars) and the oscillation (dotted) comes from the flux re-normalization due to the absolute flux uncertainty of 25%.

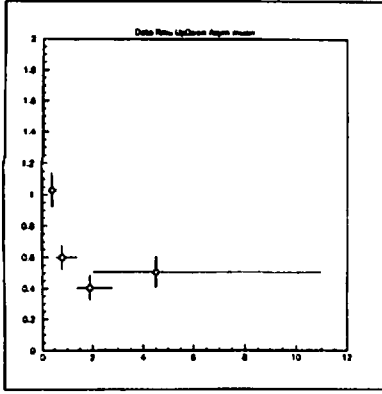


Figure 8: The up/down ratio of the upward going ($-1 < \cos\theta_z < -0.2$) and downward going ($0.2 < \cos\theta_z < 1.0$) events for the μ -like events. The horizontal axis shows the 'neutrino' energy estimated with the help of the MC calculations. The lowest data less than 500 MeV should always be symmetric (1 in this figure) since there is poor angular correlation between leptons and neutrinos in this energy region.

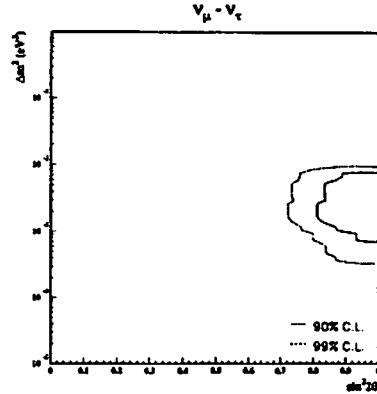


Figure 9: An Allowed region assuming $\nu_\mu \rightarrow \nu_\tau$ neutrino oscillation.

between the detector and the place where neutrinos are produced. If you see the distortion in the distribution, you can immediately get crude guide on the value for the oscillation parameter. Another important characteristics is that the zenith angle distribution should be symmetric.

This nature of the symmetry does not depend upon any flux calculations except for a small difference (less than 3% for the events below 1 GeV: negligible above 1 GeV) of the geo-magnetic cut-off. Therefore, if any up/down asymmetry is found, it indicates immediately that some unknown situation must exist. In this sense the data stands by itself.

In figure 7, we have shown the number of e-like and μ -like events as a function of zenith angle for the sub-GeV and multi-GeV data sets. The predictions shown by the solid bars are symmetric for all of the data samples although the predictions have 25% systematic uncertainties which are not shown in figure 7. The electron samples show more or less symmetric distributions, but in the muon sample it is evidential that more neutrinos come from top of the detector and less from the bottom of the detector. The multi-GeV muon data has most prominent characteristics.

In order to magnify the problem, we have shown in figure 8, the ratio of the upward going ($-1 < \cos\theta_z < -0.2$) and downward going ($0.2 < \cos\theta_z < 1.0$) events for the μ -like events. All the data except one less than 500 MeV are $\sim 50\%$. The lowest data less than 500 MeV should always be symmetric since there is poor angular correlation between leptons and neutrinos in this energy region. Therefore the data below 500 MeV can be used as a controlled sample and should be 1 in figure 8.

By using those information on the energy and zenith angle distribution, we can get an confidence contour for the neutrino oscillation parameters (for $\nu_\mu \rightarrow \nu_\tau$ case) as shown in figure 9. The expected zenith angle distributions for $\Delta m^2 = 5 \times 10^{-3} \text{eV}^2$, $\sin^2 2\theta = 1$ are shown in figure reffig:zenith by the

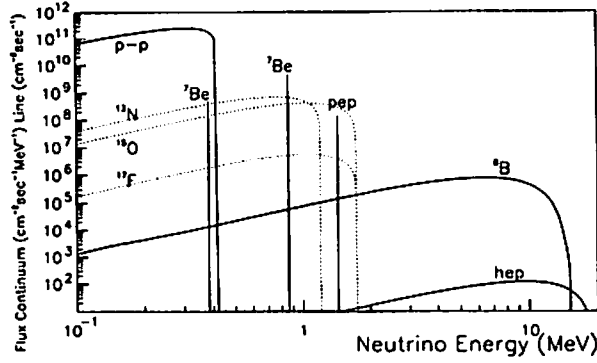


Figure 10: The solar neutrino spectra. KM and SK can only measure the ^8B -neutrinos and the Chlorine experiment can measure both the ^8B and ^7Be -neutrinos. The gallium experiments have lowest energy threshold and are able to observe neutrinos from the pp-fusion process which produces most of the solar neutrinos.

dotted lines which quite agree with the data.

It is interesting that a part of the allowed region can be accessible by planned long baseline neutrino oscillation experiments.

3.2 Solar Neutrinos

The solar neutrinos are produced in the central part of the sun through the various nuclear reactions [17]. The dominant processes producing neutrinos are those in the pp-chain:

$$p + p \rightarrow d + e^+ + \nu_e (E_\nu \leq 0.420 \text{ MeV}),$$

$$p + e^- + p \rightarrow d + \nu_e (E_\nu = 1.442 \text{ MeV}),$$

$$^3\text{He} + p \rightarrow 4\text{He} + \nu_e (E_\nu \leq 18.77 \text{ MeV}),$$

$$^7\text{Be} + e^- \rightarrow ^7\text{Li} + \nu_e (E_\nu = 0.862 \text{ MeV} (89.7\%), 0.384 \text{ MeV} (10.3\%)),$$

$$^8\text{B} \rightarrow ^8\text{Be}^* + e^+ + \nu_e (E_\nu \leq 15 \text{ MeV}),$$

The solar neutrino spectra predicted by a standard solar model is shown in figure 10.

The present energy threshold of SK is 6.5 MeV, and we therefore can only detect ^8B neutrinos, maximum energy of which is ~ 15 MeV.

The main interaction in this energy region is the neutrino electron elastic scattering, since the neutrino nucleus scattering is suppressed by the Pauli principle. The cross sections of the $\nu + e$ interactions are well calculated by the standard theory of the electro-weak interactions:

$$\frac{d\sigma}{dE_e} = \frac{2G_F^2 m_e}{\pi} \left[c_L^2 + c_R^2 \left(1 - \frac{E'_e}{E_\nu} \right)^2 + c_L c_R \frac{m_e}{E_\nu} \right],$$

where $c_L = 1/2 + \sin^2\theta_W$ and $c_R = \sin^2\theta_W$ for $\nu_e e$ scattering, and $c_L = -1/2 + \sin^2\theta_W$ and $c_R = \sin^2\theta_W$ for $\nu_\mu e$ scattering. The $\nu_e e$ interaction is mediated by both charged and neutral currents, and the $\nu_\mu e$ interaction is mediated only by the neutral current. With $\sin^2\theta_W = 0.225$, the total cross sections are

$$\sigma(\nu_e + e \rightarrow \nu_e + e) = 0.920 \times 10^{-43} (E_\nu / 10 \text{ MeV}) \text{ cm}^2,$$

$$\sigma(\nu_\mu + e \rightarrow \nu_\mu + e) = 0.157 \times 10^{-43} (E_\nu/10 \text{ MeV}) \text{ cm}^2.$$

The $\nu_\mu e$ neutral current interaction must be taken into account when the neutrino oscillation of the $\nu_e \rightarrow \nu_\mu$ mode is considered.

Although the cross sections are small, those $\nu + e$ interactions have the following good characteristics. The direction of the recoil electron is constrained to be very forward: $E_e \theta^2 \leq 2m_e$ —18deg for 10 MeV recoil electrons. Therefore the directions of the neutrinos are well kept in the direction of the recoil electrons. The recoil electron energy reflects the incident neutrino energy, therefore the shape of the neutrino energy spectrum can be studied by the recoil electron energy.

For some scenarios, like the spin-flavor precession, ν_e may convert into $\bar{\nu}_e$. These $\bar{\nu}_e$'s are detected through the $\bar{\nu}_e + p \rightarrow e^+ + n$ interactions. The cross section of this interaction is

$$\sigma(\bar{\nu}_e + p \rightarrow e^+ + n) = \frac{G_F^2 E_\nu^2}{\pi} |\cos^2 \theta_c|^2 \left[1 + 3 \left(\frac{g_A}{g_V} \right) \right] = 9.23 \times 10^{-42} \left(\frac{E_\nu}{10 \text{ MeV}} \right)^2 \text{ cm}^2.$$

The cross section in water is about 20 times larger than that of the $\nu_e e$ scattering. The incident neutrino energy is exactly obtained by $E_\nu = E_e + 1.3 \text{ MeV}$, however the directional information would be lost. The angular distribution of the interaction is almost flat:

$$d\sigma/d\theta \sim 1 - 0.103 \cdot \cos\theta.$$

The purpose of the solar neutrino study in SK is not only to confirm the flux deficits, but also to obtain an conclusive evidence of neutrino oscillation independent from the solar model calculations.

The distortion of the recoil electron energy shape, if found, is a strong evidence for the MSW effect or the Just So neutrino oscillations. The daytime and the night-time flux difference is the evidence of the regeneration of neutrinos through the earth: the MSW effect on the earth's matter. Other time variations like seasonal and semi-annual are also the key issue and a correlation with the solar activity would be an evidence of a large neutrino magnetic moment.

In order to study those phenomena in detail, the experiment must acquire very high statistical data and reduce systematic errors. The energy determination is of most crucial and difficult among them. The energy is basically determined by number of hit PMTs corrected for the light absorption through the water and PMT geometry and two photon hit and is calibrated by using the electron beam produced by the electron LINAC as was already explained.

The data presented here is based on 374.2 days of data taken between May 31, 1996 and Oct 20, 1997. The expected solar neutrino event rate is 37 events per day for SSM_{BP95} while the number of event triggered per day is about one million. Most serious background is cosmic ray muons ($\sim 3 \text{ Hz}$) and subsequent β and γ rays originating from the decay of the spallation products. Those muons can be removed easily by using the information of the visible energy in the inner volume and those in the outer detector. The spallation products were removed by using the distance and time correlation between the spallation products and the parent muons. There are significant amount of background entering from out side of the detector: mostly γ rays. Those background were removed by applying the fiducial volume cut (2m from the surface of the inner PMTs). After selecting the data between 6.5 MeV and 20 MeV, we obtained 196 events/day as a final sample.

The $\cos\theta_{\text{sun}}$ distribution of the final sample is shown in figure 11 and the two dimensional plot of those events, in a coordinate system where the sun always sits at the center, is shown in figure 12: this figure may be called as a Neutrino Heliograph. From figure 11 we obtained the number of solar neutrinos by using a maximum likelihood method. We assumed the shape of the ^8B neutrino spectrum

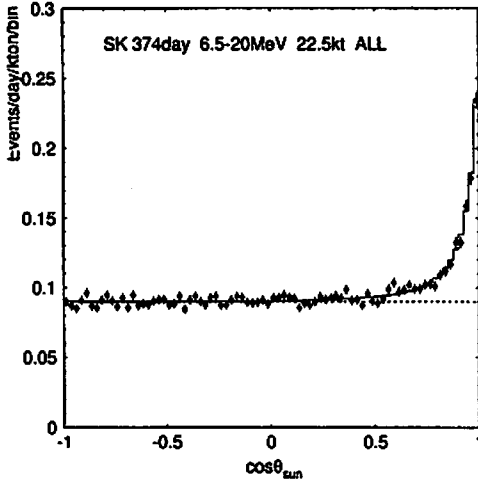


Figure 11: The $\cos\theta_{sun}$ distribution. The forward peak caused by the solar neutrino interaction is prominent.

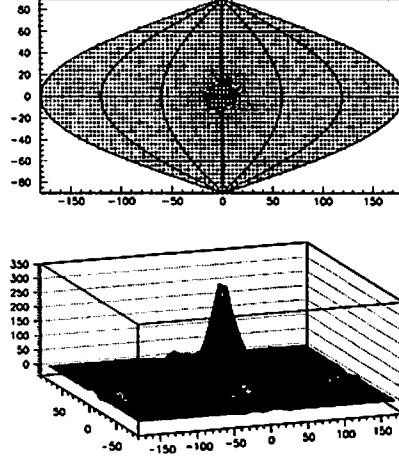


Figure 12: The image of the sun measured by the solar neutrino:Neutrino Heliograph.

of reference [19] and for the likelihood calculation the data in the $\cos\theta_{sun}$ distributions are further divided into energy bins. The number of events thus obtained is 4,951.8 events in 374.2 days between 6.5 and 20 MeV in 22.5kton fiducial volume.

The solar ^8B neutrino flux is:

$$\phi_{^8B} = 2.37^{+0.06}_{-0.05}(stat.)^{+0.09}_{-0.07}(syst.) \times 10^6/cm^2/s,$$

Note that KM observed 597 events in 2079 days: the measured flux is $\phi_{^8B} = 2.80 \pm 0.19(stat.) \pm 0.33(syst.) \times 10^6/cm^2/s$. The result from SK agrees that from KM within experimental errors.

The ratio to the prediction the standard solar model of BP95 is,

$$\frac{DATA}{SSM_{BP95}} = 0.358^{+0.009}_{-0.008}(stat.)^{+0.014}_{-0.010}(syst.).$$

The daytime and night-time flux difference has a solar model independent information on the large angle MSW solutions. The day and night flux ratio obtained by SK is,

$$\frac{(Day - Night)}{(Day + Night)} = -0.31 \pm 0.024(stat.) \pm 0.014(syst.).$$

No significant overall flux difference is found. The effect of the regeneration through the earth depends upon the path length and the density profile of the earth where the solar neutrinos pass through. We therefore divided the night-data ($\cos\theta_z < 0.0$: the nadir is the z-coordinate) into five bins ($\Delta\cos\theta_z = 0.2$). We have shown in figure 13 the fluxes of day time and night-time divided into 5 bins. The typical day/night fluxes expected from a large angle solution ($\Delta m^2 = 2.82 \times 10^{-5} eV^2$, $\sin^2 2\theta = 0.66$) and a small angle solution ($\Delta m^2 = 6.31 \times 10^{-6} eV^2$, $\sin^2 2\theta = 9.12 \times 10^{-3}$) of MSW effect are also plotted in figure 13.

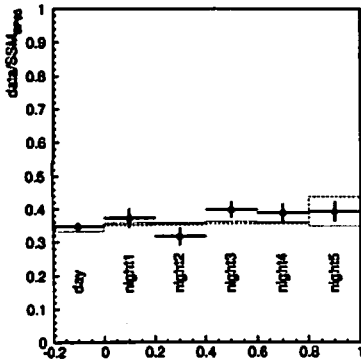


Figure 13: The day and night-time fluxes. The night data are divided into 5 bins with $\Delta\cos\theta = 0.2$. The expected day-night fluxes for $\Delta m^2 = 2.82 \times 10^{-5} eV^2$, $\sin^2 2\theta = 0.66$ (dashed line) and $\Delta m^2 = 6.31 \times 10^{-6} eV^2$, $\sin^2 2\theta = 9.12 \times 10^{-3}$ (dotted line) are shown.

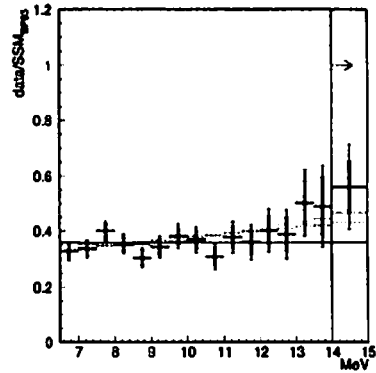


Figure 14: The spectrum of the recoil electrons. The thick parts show the statistical errors and the extensions with the thin bars show the combined error of statistics and systematics. Also shown are a typical small angle solution ($\Delta m^2 = 6.31 \times 10^{-6} eV^2$ and $\sin^2 2\theta = 9.12 \times 10^{-3}$ (dotted line)) and a typical just so solution ($\Delta m^2 = 7.08 \times 10^{-11} eV^2$ and $\sin^2 2\theta = 0.83$ (dashed line)).

By using the data we can study neutrino oscillation hypothesis. The confidence contour obtained is shown in figure 15 at 95% C.L. It is quite interesting that even with this 'small' data sample we can draw an excluded region where half of the large angle allowed regions obtained by Hata et al. [20] are excluded.

The recoil electron energy spectrum is a key issue to untangle the solution of the solar neutrino problem, since the neutrino oscillation may have a strong energy dependence. It is sensitive especially to the so called small angle solutions and the Just So vacuum oscillations. In figure 14, the spectrum of the recoil electrons is shown. The current data indicate that the distribution is consistent to be flat, but have better fitting for a small angle or just so solutions (namely non-flat distribution), although statistically not significant. In figure 14, we also plotted the expected energy distortion for a typical small angle ($\Delta m^2 = 6.31 \times 10^{-6} eV^2$ and $\sin^2 2\theta = 9.12 \times 10^{-3}$) and a just so solution ($\Delta m^2 = 7.08 \times 10^{-11} eV^2$ and $\sin^2 2\theta = 0.83$).

From the energy spectrum, we can get the confidence contour for the neutrino oscillation parameters. For the MSW regions, we have obtained excluded region with 95% C.L. which is shown as regions surrounded by the solid curves in figure 15. For the just so region we have obtained excluded region at 99% confidence level, but we get allowed region at 95% C.L. as shown in figure 16. However we should not take this too seriously at this time not only because the statistics is not sufficient and the indication is very marginal but also because the SK experiment will soon increase the statistics very rapidly at least for a next couple of years. We should wait another year or so before the experiment can make a definite conclusion.

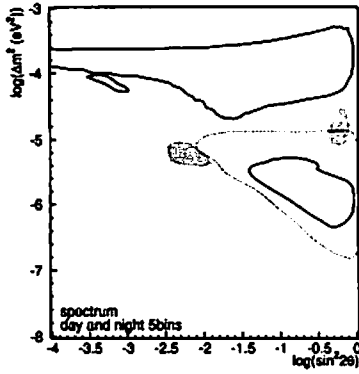


Figure 15: The confidence contour for the MSW effect by the SK data. The day/night flux difference and the electron energy distribution are used. No absolute flux information was used. The solid line shows the 95% excluded region by the energy spectrum and the dotted line shows the 95% excluded region by day/night data. Shaded regions are those obtained by Hata et al.

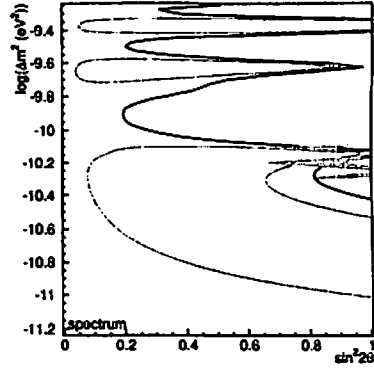


Figure 16: The confidence contour for the Just So oscillation obtained by the electron energy distribution. The solid line shows the region excluded by 99% C.L. and the dotted line shows the allowed region at 95% C.L.

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Microlens Diagnostics of Central Region of Active Galactic Nuclei

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Abstract

The innermost regions of quasars can be resolved by a gravitational-lens ‘telescope’ on scales down to a few AU (or $\sim \mu\text{as}$). This scale is hardly revealed by today’s observational instruments directly.

Microlensing light curves depend on structures and/or sizes of the lensed source (an accretion disk). Inversely, we consider to constrain the size or the structure of an accretion disk that believed to place in central region of active galactic nuclei.

On the basis of this idea, we performed numerical simulations of microlensing of two types of accretion disk and conclude that we can get more information about central region of active galactic nuclei by microlensing event.

1 Introduction

There has been long discussion regarding the central structure of active galactic nuclei (AGN). Generally, it is believed that there is a supermassive black hole and a surrounding accretion disk. However, it is quite difficult to obtain direct information about the center of AGN, the accretion disk, because the disk size is far too small to resolve. As for Q2237+0305, for example, the angular diameter distance is $\sim 4 \times 10^{27}$ cm. If a black hole placed at the center has a mass of $\sim 10^8 M_\odot$, the disk size ($\sim 1000 r_g$, r_g is Schwarzschild radius) will be $\sim 3 \times 10^{16}$ cm. Hence, apparent disk size is $\sim 10^{-11}$ radian $\sim 1 \mu\text{as}$.

Instead, microlensing can be used as a ‘gravitational telescope’ to resolve the disk structure[?][?], since a typical Einstein-ring radius[?] of a stellar mass lens is very small; e.g., $\theta_E \sim 10^{-11}$ radian for Q2237+0305. With such a gravitational telescope we can obtain information about the disk emergent spectra as a function of the distance from the central black hole. When a point source object is lensed, its luminosity is amplified without any color change. But, in some actual situations, we cannot treat a source as a point and the effect of a finite source size should be taken into account. Many authors examined such finite size effects and they conclude that different size and/or structure of source shows

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different microlensing light curve [?][?][?][?]. However, they only considered simple and unrealistic source.

The purpose of this work is to obtain some criteria for distinguishing disk physics based on more realistic disk models. In section 2 we describe our methods of simulation. In section 3 our results are displayed. Section 4 is devoted to discussion.

2 Method of the Simulation

2.1 Accretion Disk Models

First, we describe the adopted accretion disk models. We consider two representative types of disks:

1. An optically thick disk:

The first case we consider is the geometrically-thin and optically-thick standard disk[?]. Since we are concerned with disk structure on scales ranging from a few tens to a few thousands of r_g , we ignore relativistic effects, such as the gravitational redshift by the black hole, for simplicity. The temperature distribution of the optically thick standard disk is then given by

$$T(r) = 2.2 \times 10^8 \left(\frac{\dot{M}}{10^{28} g s^{-1}} \right)^{1/4} \left(\frac{M}{10^8 M_\odot} \right)^{1/4} \left(\frac{r}{10^{14} cm} \right)^{-3/4} \left[1 - \left(\frac{r_{in}}{r} \right)^{1/2} \right]^{1/4} K, \quad (1)$$

where $\dot{M}$ is the mass accretion rate and M is the black hole mass. The inner edge of the disk, r_{in} , is set to be $r_{in} = 3r_g$, the radius of the marginally stable last circular orbit around a non-rotating black hole.

Since the disk is optically thick, we can assume blackbody radiation and calculate spectrum.

2. An optically thin disk

The second model is the optically thin version of the advection dominated accretion flow (ADAF). We use spectrum from one of this model[?] and derive luminosity. Here, included are synchrotron, bremsstrahlung, and inverse Compton scattering of soft photons created by the former two processes [?]. Optical flux is mainly due to Comptonized synchrotron photons.

2.2 Microlensing Events

Since we now consider microlensing effects on an extended source, we must integrate the magnification effects over the entire source. We thus, need to calculate the angular separation u between a part of the source in question and the lens center, from which the amplification factor, $A(u)$, can be found. For this amplification factor, we consider two cases of microlensing event and use two types of the factor, i.e., microlensing event by a single lens and by many lenses at the same time (caustics). Former case, we substitute exact formula[?],

$$A(u) = \frac{u^2 + 2}{u(u^2 + 4)^{1/2}}, \quad (2)$$

and latter case, we use analytical approximate formula[?],

$$A(u) = \begin{cases} u^{-1/2} + A_0 & (\text{for } u > 0) \\ A_0 & (\text{for } u \leq 0), \end{cases} \quad (3)$$

as $A(u)$, where A_0 represents a constant amplification factor due to the effect of another caustics and the initial effect of the considering caustic. This is because that exact values of latter case are too complicated and the formula we used is known to represent exact values well.

In the present study, we first divide an accretion disk plane into azimuthal (φ) and radial (r) elements, calculate observed (i.e, microlensed) flux from each elements, and summing up over the entire disk plane from the inner boundary (r_{in}) to the outer boundary (r_{out}), we obtain the total observed flux at frequency ν , or the spectrum of the microlensed accretion disk.

For calculations of microlensing light curves, we move lens position(s) with appropriate transverse velocity, and re-calculate u of each elements and amplification factor iteratively.

3 Results of the Calculations

3.1 Model Parameters

In this simulation, we consider Q2237+0305[?] (quasar) as a source. This object is macrolensed by a foreground galaxy, and the source is split into four (or five) images. Now, we suppose that one of the macrolensed image is further microlensed by a star in the foreground galaxy causing the macrolensing. Actually, several microlensing events were observed in this image[?]. The redshift parameters are[?]

$$z_{os} = 1.675, z_{ol} = 0.039, \text{ and } z_{ls} = 1.575. \quad (4)$$

As for Hubble's constant, we adopt the approximate value[?], $H_0 \sim 60 \text{ km s}^{-1} \text{ Mpc}^{-1}$, and we consider the Einstein-de Sitter universe.

For simplicity, we assume that the accretion disk is face-on; i.e. $i = 0$, to the observer. Actually, based upon a grand unified paradigm of AGNs, quasars are not far from face-on views. We also assume the mass of a lensing object to be $M_{\text{lens}} = 1.0 M_{\odot}$.

3.2 Light Curves

In figure 1², we plot the radial distribution of emitted photons with several wavelengths. This figure help us to understand what is expected in observation when microlensing events have occurred.

The upper panel (standard accretion disk) of figure 1 shows rather gradual change of each flux over a wide spatial range. This reflects the fact that viscous heating and radiative cooling are balanced in the standard disk. Since the potential energy only gradually changes with the radius, so is the viscous heating rate and is the radiative cooling rate. Moreover, the lower a photon frequency is (or the longer the wavelength is), the wider becomes the parts of the disk which generate photons.

In the optically-thin accretion disk, on the other hand, the optical-UV flux is totally dominated by that from the inner region ($r \leq 10r_g$). The lower panel (ADAF) of figure 1 clearly shows that almost 100% of optical flux originates from the region inside $10r_g$, and that the contribution from the outer parts is smaller by several orders. This is possible in ADAF, since radiative cooling is no longer directly related to the shape of the gravitational potential well (which has a rather smooth profile). The reason why optical flux is produced in a rather restricted region inside $10r_g$ is due to enhanced synchrotron emission in the vicinity of the black hole, where density and pressure (and thus magnetic pressure) are likely to be at maximum within the flow. Nearly irrespective of the u value, as a consequence, it is always the innermost region (within $10 r_g$) that dominates the entire disk spectrum. The amplification of the total emission just reflects that of the emission from the innermost parts. Therefore, the overall flux will be amplified by lensing without large spectral changes.

In X-ray ranges, in contrast, radiation originate from wider disk ranges; the contribution from the outer part at $100 - 1000r_g$ is not negligible at frequency of $f \sim 10^{18}$ (a few keV). Figure 1 also shows that X-rays are produced over a wider range. This is because of a high electron temperature, $T_e \sim 10^9 K$ maintained even at $r \sim 100r_g$ [?]. We can thus predict that small frequency dependence will be seen in X-ray ranges in this model.

Such quite different spectral behavior give rise to different light curves of different disk models.

Figure 2³ shows the microlensing light curves of optically thick (upper panel) and thin (lower panel) disks for the case that the event caused by a single lens. Here, we use a value of $0.01 \times \theta_E$ (θ_E : Einstein ring radius) as an impact parameter (i.e., the shortest distance between lens object and center of the accretion disk). The light curve of the optically thin disk is still 'achromatic (i.e., there is no color change)' because of its flat temperature distribution, whereas the light curve of the optically thick disk shows strong frequency dependence (i.e., this leads 'chromatic' feature).

²The radial distribution of the emergent flux with several wavelength for the case of a standard-type disk (upper) and ADAF (lower), respectively. The adopted bands are U-band (3650 Å, by the solid line), V-band (5500 Å, by the dotted line), I-band (8800 Å, by the dashed line), K-band (22000 Å, by the long dashed line), and X-ray (5 keV, by the dot-dashed line).

³The microlensed light curve of standard disk (upper) and for ADAF (lower) for a single lens case. The different type of lines correspond to the same as in figure 1

Figure 1 is again useful to understand this chromaticity and achromaticity. Since flux distribution is similar among different optical fluxes in the case of optically thin disks, each optical flux is amplified in a similar way and thus the color does not change; i.e. achromatic feature appears. X-ray emitting region is wider than that of optical flux, small chromaticity may appear when we compare optical and X-ray light variations.

In an optically thick disk, conversely, low-energy photons are created in a wider spatial range, compared with high-energy photons. If the microlensing event occurs, therefore, low-energy emission will be the first to be amplified, followed by the significant amplification of high-energy emission. Note that each optical flux has the same level at the beginning (at -120 d) in the upper panel of figure 2, whereas a higher-energy optical flux is greater than that of a lower-energy flux in the intrinsic spectra in the absence of lensing. This indicates that the amplification of a lower-energy flux has already been appreciable at -120 d. This results in the color change; i.e. chromatic feature is produced.

Next, we simulate the case of caustic induced event (i.e., the event caused by many lens objects at the same time). This resultant light curves are shown in figure 3⁴. Same as the case of single lens, but strongly, there is chromatic feature in light curves of standard accretion disk, although light curves of optically thin ADAF is still achromatic. In this case, difference between two disk models is more apparent than the case of single lens. This is because different formula of amplification factor of these two cases. Especially, in the case of caustic, caused by sharp change of the amplification factor when source comes into (or goes out from) caustic, such interesting feature is strengthened.

4 Discussion

We have demonstrated how spectral behavior and microlensing light curves change with time, depending on the disk structure. We considered two representative disk models: the standard-type disk emitting predominantly optical-UV photons, and the optically thin, advection-dominated flow (ADAF) producing a wide range of photons from radio to X- γ rays. Using the microlensing events, we can distinguish the different structure of AGN accretion disks and map them in details.

At the peak of the microlensing event, the inner part of the accretion disk is strongly magnified (especially, when the impact parameter is small). Broad band photometry will be able to detect the color changes, thereby revealing the structure of AGN accretion disks. Even if there is an accretion disk that we have never studied, we will find unexpected light curves, and will be inspired to construct new accretion disk model. Fortunately, such observations do not require good time resolution, as long as a lens is located far from the observer.

Finally, recent high-resolution observations, including those from *HST*, have revealed circumnuclear starburst regions in AGNs[?]. If starburst region exists around an accretion disk, microlensing event of AGN will make an interesting light curve; large-amplitude, short timescale variation (on a few months) caused by the microlensing of the accretion disk will be superposed on more gradual light variation (over a year) caused by the microlensing of starburst region. Thus, long-term monitoring may provide a possible tool to elucidate the extension of circumnuclear starbursts.

We submit these works to journal and they will be published [?][?].

Acknowledgements

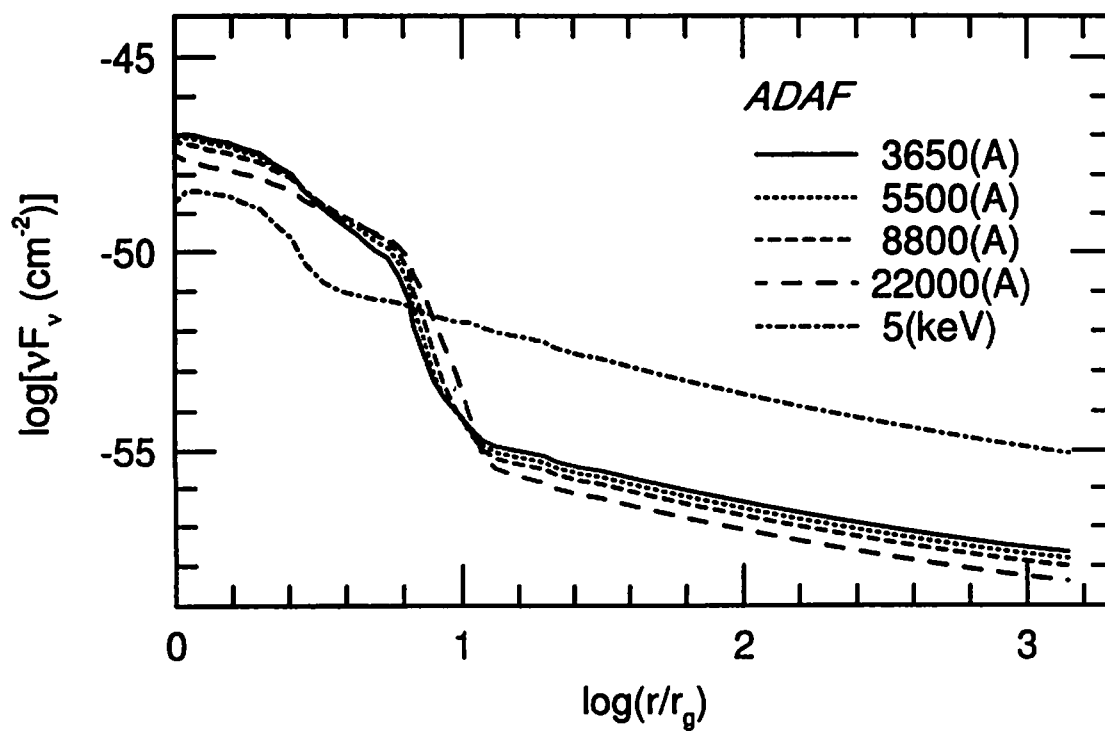
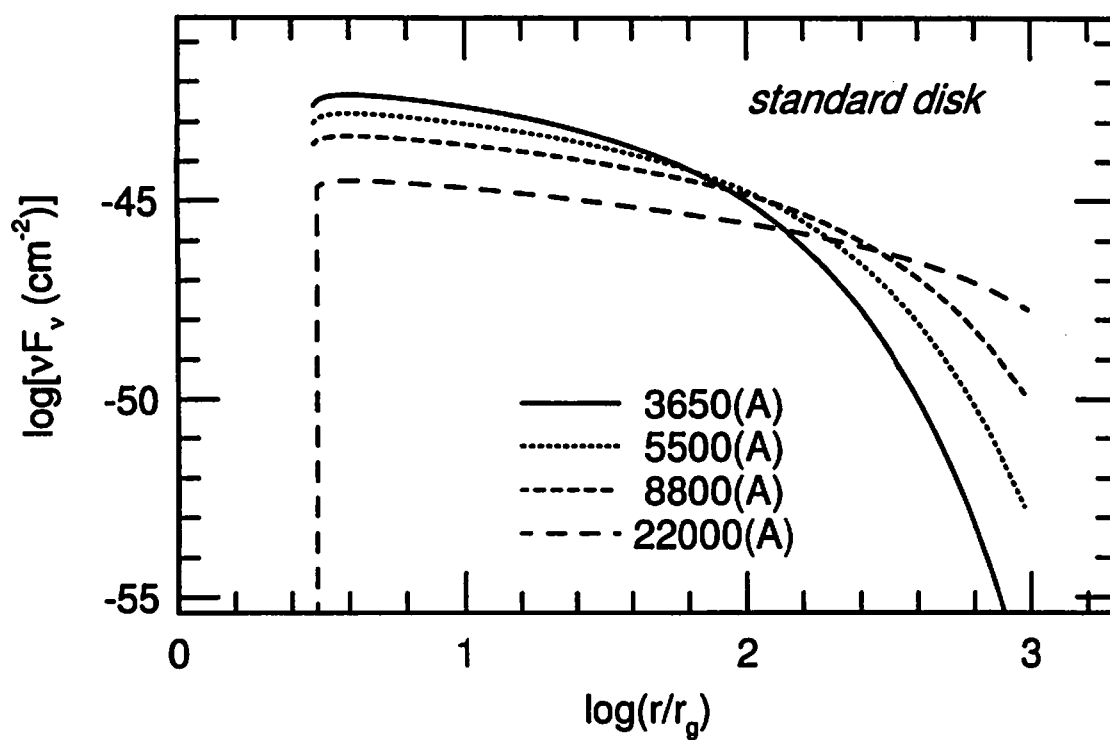
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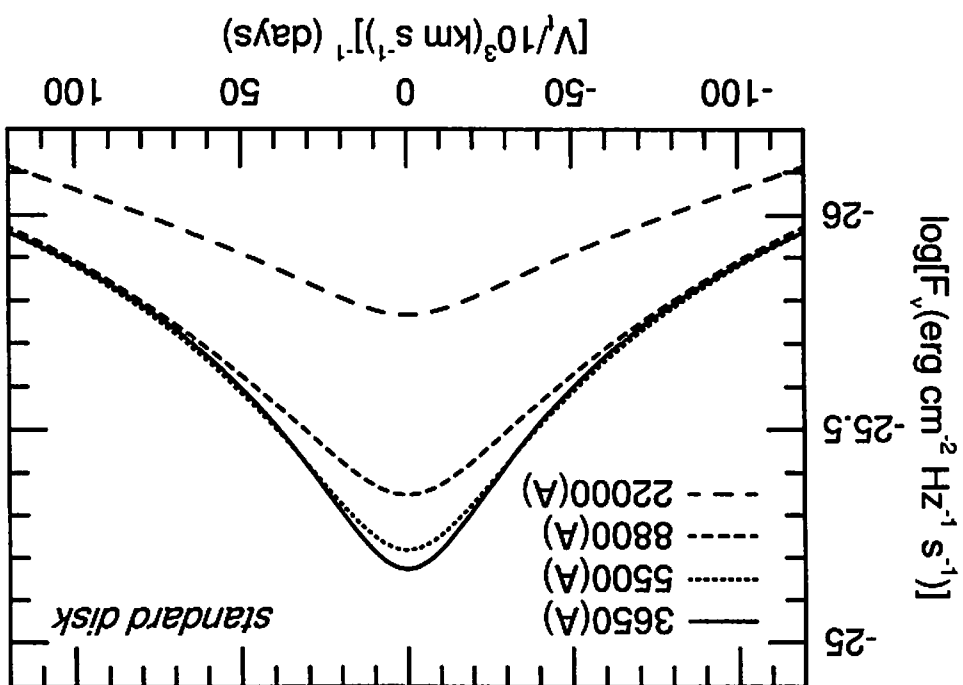
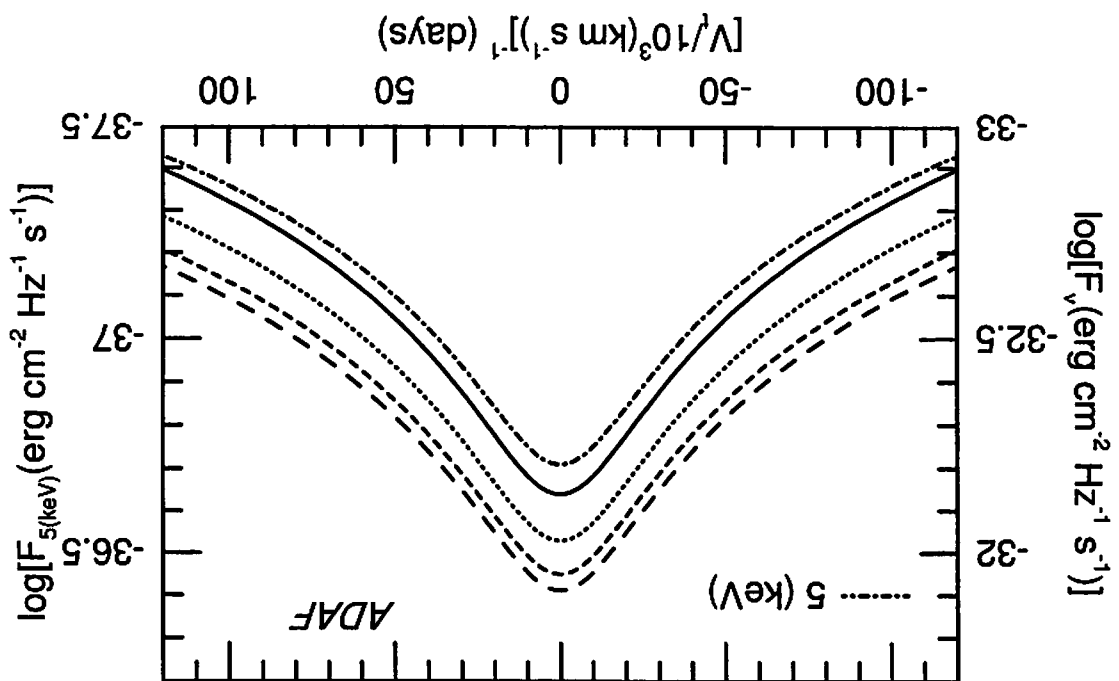
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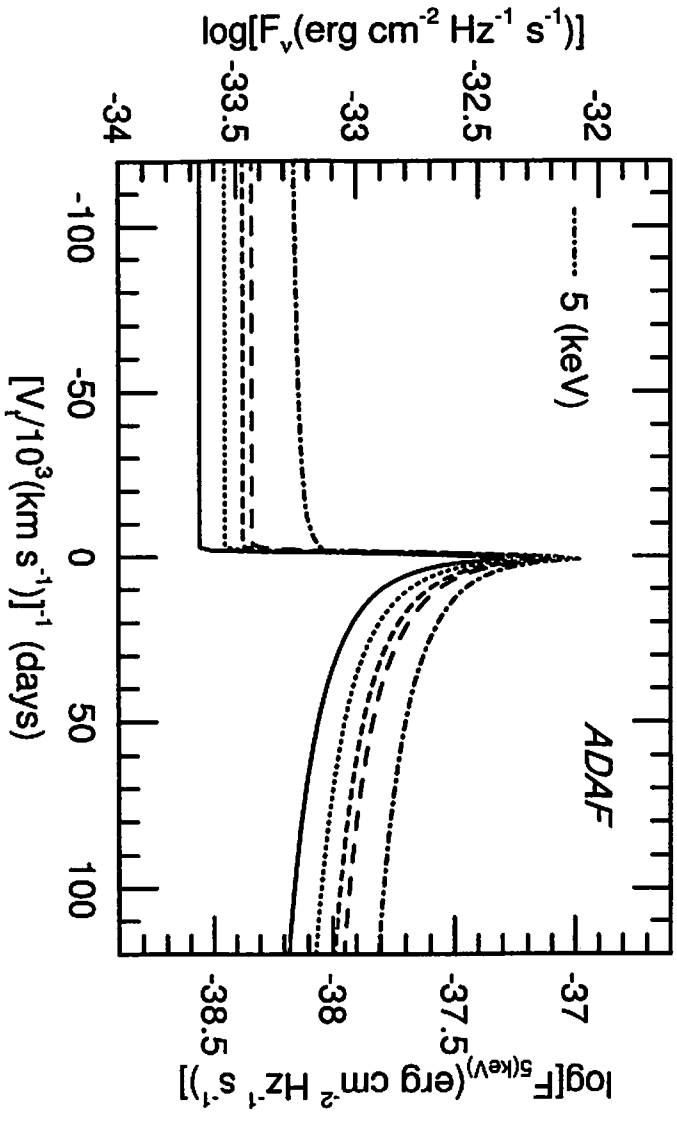
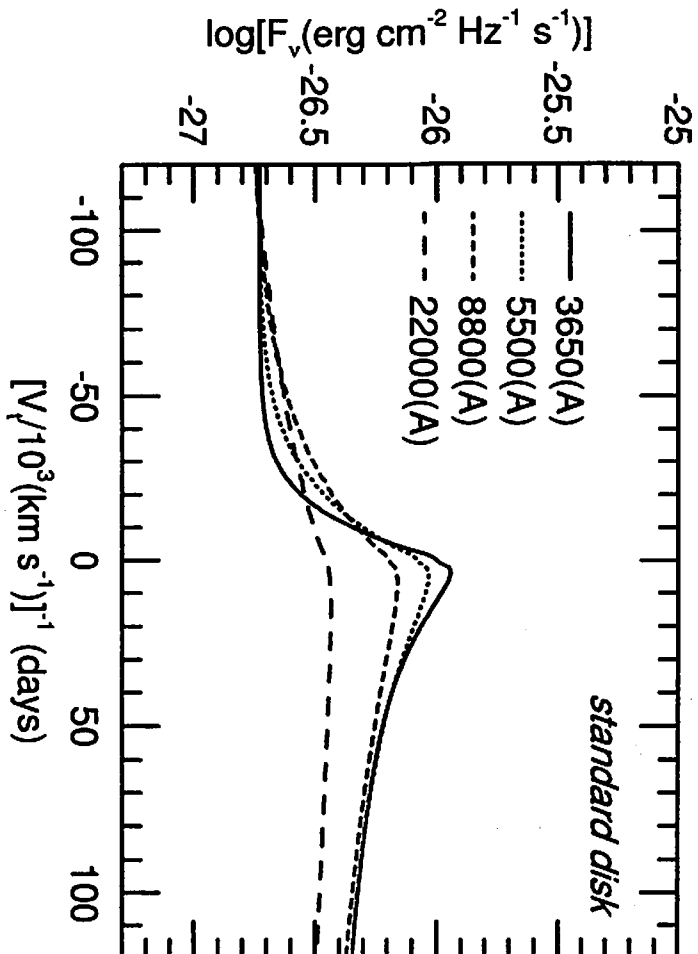
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⁴ Same as figure 2, but for caustic case

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Gravitational lensing of gravitational waves from inspiraling binaries by a point mass lens

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Gravitational lensing of gravitational waves from inspiraling binaries is discussed in the context of advanced laser-interferometer detectors, taking correct account of the diffraction effect. Convolving the spectrum of the lensed waveform with that of the detector noise, we calculate how much the signal-to-noise ratio is magnified by gravitational lensing as a function of the mass and position of the lens. When the lens is much lighter than $\sim 10^2 M_\odot$, the diffraction is so effective that the waveflux is not magnified appreciably. We predict that lensed waveforms are distinguishable from unlensed ones in that the signal-to-noise ratio shows an oscillatory behavior as the frequency sweeps up.

The gravitational waves emitted during the inspiral stage before the final coalescence of two neutron stars (NS-NS binary) is the most promising source for the laser-interferometer detectors under construction such as the LIGO [1]. To this end, much effort has been made to calculate accurately the waveform template [2] which is matched-filtered with the observed signal, and also to estimate the event rate [3]. Thus it is important to consider every possible source of noise that alters the results of these theoretical calculations. One such kind of noise may be the gravitational focusing of gravitational waves [4], the subject we shall investigate in this paper.

Previously, Wang et al. [5] considered the microlensing of gravitational waves from inspiraling binaries by (hypothetical) stellar mass lenses distributed over the universe (since the probability of lensing by galaxies is negligible), and concluded that, due to the lensing magnification, there may be a significant number of those inspiral events which would be too distant to be detected had it not been for the lensing. They simply calculated the lensing magnification using geometric optics limit in the same way as for the optical light, and took the maximum magnification to be infinite on the caustics. As shown by several authors [6], however, if the wavelength λ is longer than the Schwarzschild radius of the lens mass M , then the wave effect (diffraction) becomes important so the maximum magnification should be small. Since the wavelength is typically $\sim 10^2$ – 10^4 km in the observable frequency range of the planned detectors, the waveflux cannot be magnified significantly for lens masses lighter than $\sim 10^2 M_\odot$, as is the case for the microlensing considered in Ref. [5].

This paper provides the first proper treatment of gravitational lensing of gravitational waves from inspiraling binaries in the context of advanced laser-interferometer detectors, taking correct account of the diffraction effect. We show that the lensed events should be much rarer than estimated in Ref. [5] for the LIGO type detectors, but that it is possible to distinguish lensed waveforms from unlensed ones once the former are actually detected. We assume the $\Omega = 1$ and $A = 0$ cosmology and use the

units $c = G = 1$.

To give a rough account of why the ratio M/λ determines the significance of diffraction, consider monochromatic waves with wavelength λ passing through a double-slit with the slit width of order the Einstein radius $\xi_s \sim (MD)^{1/2}$ where D is the distance from us to the lens. Interference of two waves coming from the slits would produce a oscillating pattern in the intensity distribution on the screen located at the observer. The width ℓ of the central peak of the distribution is given by $\ell \sim \lambda D / \xi_s \sim (D/M)^{1/2} \lambda$. Then the maximum magnification of the waveflux is on the order of $\sim \xi_s / \ell \sim M/\lambda$. Thus large magnification can occur only when the wavelength is much shorter than the Schwarzschild radius of the lens.

More quantitatively, we perform the phase integral according to the Huygens principle in the wave optics theory to obtain the wave amplitude at the observer. Consider a point source of radiation whose position vector on the source plane is $\vec{\eta}$, and a Schwarzschild lens of mass M located at redshift of z_L . The arrival time at the observer for a ray passing through $\vec{\xi}$ on the lens plane is written as [7]

$$\tau(\vec{x}, \vec{y}) = 4M(1+z_L) \left[\frac{1}{2} |\vec{x} - \vec{y}|^2 - \ln x - \phi_m(y) \right], \quad (1)$$

where $\vec{x} := \vec{\xi}/\xi_s$ and $\vec{y} := \vec{\eta}/\eta_s$ are dimensionless variables normalized by the Einstein radius, and the origins of $\vec{x}$ and $\vec{y}$ planes are on the optic axis which connects the observer and lens. The first and second terms of equation (1) represent the geometrical and gravitational time-delays, respectively, and we choose $\phi_m(y)$ so that the minimum value of τ on the $\vec{x}$ -plane is zero. For a Fourier component of waves with the frequency f at the observer, the wave amplitude (apart from the time-dependent $e^{2\pi i f t}$ factor) is given by the Kirchhoff integral on the lens plane as [7]

$$\psi(\omega, y) = \frac{\omega}{i\pi} \int d^2x \exp[2\pi i f \tau(\vec{x}, \vec{y})] \quad (2)$$

$$= \exp\left\{\frac{1}{2}\pi\omega + i\omega[\ln\omega - \phi_m(y)]\right\} \\ \times \Gamma(1 - i\omega) F(i\omega, 1; i\omega y^2), \quad (3)$$

where $\hat{\omega} := 4\pi f \hat{M}$ measures the ratio of the Schwarzschild radius of the lens to the wavelength, $\hat{M} := M(1+z_L)$ is the redshifted lens mass, and F is the confluent hypergeometric function. The prefactor of equation (2) is chosen such that $\psi = 1$ without lensing. Then the lensing magnification $\bar{\mu} = |\psi|^2$ for the Fourier component is

$$\bar{\mu}(\hat{\omega}, y) = \bar{\mu}_m(\hat{\omega}) |F(i\hat{\omega}, 1; i\hat{\omega}y^2)|^2, \quad (4)$$

where $\bar{\mu}_m(\hat{\omega}) := 2\pi\hat{\omega}/(1-e^{-2\pi\hat{\omega}})$ is the maximum magnification which occurs when the source is just behind the lens $y = 0$. Thus one again verifies the fact that $\bar{\mu}_m \gg 1$ only when $\hat{\omega} \gg 1$ as shown above. It is interesting to note that equation (3) can also be derived by solving the propagation equation $\square\psi = 0$ with the metric of linearized gravity, which yields the Schrödinger equation in a Coulomb scattering potential [8].

The assumption of a point source is highly justified for radiation from a binary so it is unnecessary to average equation (3) over y , since the source size is smaller than the wavelength because of its quadrupolar character. In fact, the change in the time delay $\tau(\vec{x}, \vec{y})$ due to the source size η_s ($< \lambda$) is $\sim \tau\eta_s/\eta_z \sim (M/D)^{1/2}\eta_s \ll f^{-1}$, implying that the phase in equation (2) does not move significantly over the source size. Also it is evident that, even when the lens is not a point mass but have finite size, the above formulae are valid if the lens size is much smaller than the Einstein radius ξ_E (typically greater than 100AU for cosmological lensing).

Before examining the lensing effect on gravitational waves, let us briefly summarize the notation of some quantities which characterize the laser-interferometer detectors. Performing the matched filtering between the incoming waveform $h(t)$ to the detector and the trial waveform $h_s(t)$ in templates, the signal-to-noise ratio (SN) is given by $\rho[h, h_s] = \langle h|h_s \rangle / \langle h_s|h_s \rangle^{1/2}$ [9], where the inner product is defined as

$$\begin{aligned} \langle a|b \rangle &:= 4 \int_0^\infty df \frac{\text{Re}[\bar{a}^*(f) \bar{b}(f)]}{S_n(f)}, \\ \bar{a}(f) &:= \int_{-\infty}^\infty dt a(t) e^{-2\pi i f t}, \end{aligned} \quad (5)$$

and $S_n(f)$ is the power spectrum of noise in the detector. For the advanced LIGO detector, we use [10]

$$f S_n(f) = A x/g(x), \quad g(x) := \frac{x^4 H(x - x_{\min})}{1 + 2x^4(1 + x^2)}, \quad (6)$$

where $x := f/f_k$, $f_k = 70\text{Hz}$, $A = 2.1 \times 10^{-46}$, $x_{\min} = 1/7$ and $H(x)$ is the step function. The maximum SN is achieved when the trial waveform coincide with the incoming waveform: $\rho_m[h] = \rho[h, h] = 2[\int_0^\infty df |\bar{h}(f)|^2 / S_n(f)]^{1/2}$. For waves from a binary at redshift of z in circular orbit, the lowest order (Newtonian) quadrupole formula gives the expression for the maximum SN as [9,10]

$$\rho_m[h] = \frac{(5/6)^{1/2} \Theta \hat{M}}{4\pi^{2/3} d_L(z)} \left[\int_0^{f_m} df \frac{(\hat{M}f)^{-1/3}}{f S_n(f)} \right]^{1/2}, \quad (7)$$

where $d_L(z) = 2cH_0^{-1}(1+z)[1 - (1+z)^{-1/2}]$ is the luminosity distance from us to the source redshift z , $\hat{M} = \mathcal{M}(1+z)$ is the redshifted chirp mass ($\mathcal{M} = [(M_1 M_2)^3 / (M_1 + M_2)]^{1/5}$ for the binary masses M_1 and M_2), Θ is the orientation function depending on the angular position and inclination of the source ($0 \leq \Theta \leq 4$), and f_m is the maximum frequency at the innermost stable circular orbit. The post-Newtonian calculation of point mass limit shows that $f_m = 1.42\text{kHz}(1.4M_\odot/M_1)/(1+z)$ for equal-mass binaries $M_1 = M_2$ [11].

When gravitational waves are lensed by an intervening point mass lens, each Fourier component of the observed (lensed) waveform $\bar{h}_L$ is the product of the unlensed waveform $\bar{h}$ and the wave function ψ given in equation (3); $\bar{h}_L = \bar{h}\psi$. Consequently the maximum SN² is magnified due to gravitational lensing by the factor (cf. eq.4):

$$\mu(m, y) := \frac{\rho_m^2[h_L]}{\rho_m^2[h]} = \frac{\int_0^{x_m} dx x^{-7/3} g(x) \bar{\mu}(mx, y)}{\int_0^{x_m} dx x^{-7/3} g(x)}, \quad (8)$$

where $m := 4\pi f_k \hat{M}$. This is plotted in Figure 1(a,b) versus the source position y and the redshifted lens mass $\hat{M}$. Also plotted is the magnification factor in geometric optics limit, $\mu_g(y) = (y^2 + 2)/[y(y^2 + 4)^{1/2}]$ [7], which is independent of the lens mass. The results are very insensitive to the upper limit of the integrals $x_m := f_m/f_k$ when $x_m \gtrsim 10$, and in all the figures we set $x_m = 15$ (corresponding to a binary of $M_1 \sim M_2 \sim 1.4M_\odot$ at redshift $z \sim 0.35$ [11]). The figure shows the tendency that geometric optics becomes good as the lens mass increases and would be valid for lensing by galactic masses. In case of microlensing by stellar masses, however, the diffraction is so effective that the lensing cross section ($\propto y^2$) for large magnification is negligibly small. Therefore we conclude that the number of high-redshift, highly magnified inspiral events detectable by the advanced LIGO is over-estimated in Ref. [5] who uses geometric optics for the magnification and considers microlensing for the lensing probability.

The most important information in the matched filtering is the phase of the wave signal [10]. Suppose that we detect a lensed signal and we do not have lensed waveforms in templates. Then we cannot get the possible maximum value of SN since the phase difference between the lensed and unlensed waveforms (i.e., the phase of ψ in eq.3) changes with time as the frequency sweeps up. If our templates include unlensed waveforms only, SN would decrease typically by the following factor

$$\frac{\rho[h_L, h]}{\rho_m[h_L]} = \frac{\int_0^{x_m} dx x^{-7/3} g(x) \text{Re}[\psi(mx, y)]}{\mu^{1/2} \int_0^{x_m} dx x^{-7/3} g(x)}, \quad (9)$$

which is plotted in Figure 1(c,d). Note that in geometric optics the phase shift due to lensing remain constant

and so SN does not decrease. Certainly it is desirable to be equipped with the lensed waveform in our templates to achieve the maximum SN for every signal. Moreover it may be possible to determine the lens mass from the matched-filtering of the lensed signal with such templates if this SN decrease is sufficiently large. However the figure indicates that the SN decrease due to diffraction is insignificant except for some rare cases of close encounter of sources with very large lens mass, occurring with extremely low probability (discussion on the probability is given below).

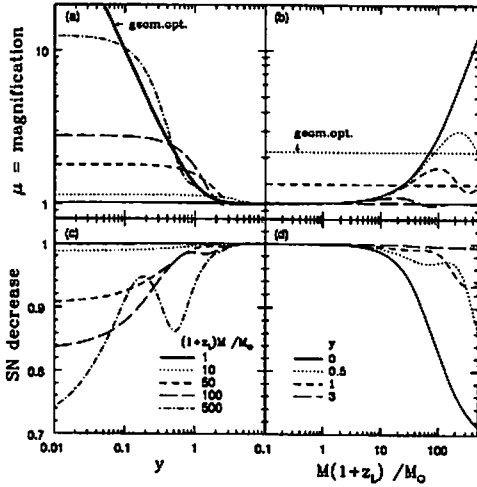


FIG. 1. (a,b): The lensing magnification factor of the signal-to-noise ratio squared (eq.8). (c,d): The decreasing factor in the signal-to-noise ratio (eq.9) when the lensed waveform h_L is matched filtered with the unlensed waveform h . The abscissae in (a,c) are the source position y in units of the Einstein radius and in (b,d) the redshifted lens mass $\tilde{M} = M(1+z_L)$. The magnification factor in the geometric optics limit is plotted by thick curve in (a) and by horizontal lines in (b).

Next we calculate the maximum redshift z_m of sources reached by the single advanced LIGO detector with SN greater than a certain threshold ρ_0 , in the presence of the lensing magnification. Solving $\rho_m[h_L] = \rho_0$ with $\Theta = 4$ and $y = 0$ (maximum magnification) with respect to z yields [12]

$$z_m = (\alpha + \beta + \alpha^2/\beta)^6 - 1, \quad (10)$$

where

$$\alpha := \frac{(5/6)^{1/2} H_0 \mathcal{M}}{6\pi^{2/3} \rho_0} \left[\int_0^{f_m} \frac{df}{f} \frac{(\mathcal{M}f)^{-1/3} \tilde{\mu}_m(\tilde{\omega})}{f S_n(f)} \right]^{1/2} \quad (11)$$

and $\beta := [\alpha^3 + \frac{1}{2} + (\alpha^3 + \frac{1}{4})^{1/2}]^{1/3}$. Figure 2 plots the maximum redshift z_m versus the redshifted lens mass

$\tilde{M}$ for various values of $h(\rho_0/5)^{-1}(\mathcal{M}/1.2M_\odot)^{5/6}$ where $h := H_0/(100\text{km s}^{-1}\text{Mpc}^{-1})$. Note that $\tilde{M} \rightarrow 0$ and $\tilde{M} \rightarrow \infty$ correspond to no lensing limit and lensing in geometric optics limit, respectively. The figure shows that the maximum detectable redshift hardly increases due to microlensing by stellar mass lenses. Thus we conclude that the high-redshift tail in the redshift distribution of inspiral events due to the lensing magnification (Fig. 3 of Ref. [5]) should not exist (but much lower tail due to galaxy lensing may exist).

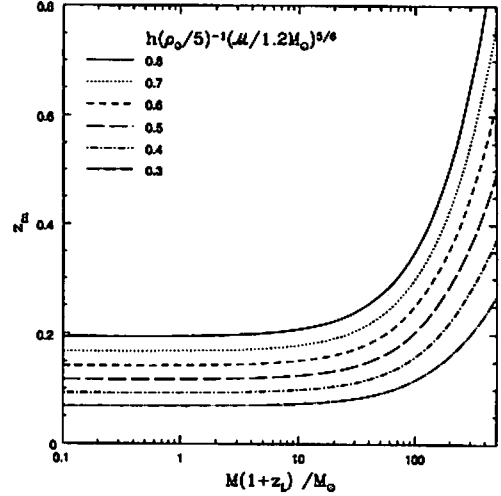


FIG. 2. The maximum redshift reached by the advanced LIGO detector taking the lensing magnification into account (eq.10) versus the redshifted lens mass $\tilde{M}$ for some values of $h(\rho_0/5)^{-1}(\mathcal{M}/1.2M_\odot)^{5/6}$. Equation (6) was used for the noise spectrum.

Finally we estimate the lensing probability and the lensed event rate very roughly. Following Ref. [5] we consider hypothetical point mass lenses distributed over the universe, since the probability of galaxy lensing ($\propto z^3$) is an order of magnitude smaller. The optical depth of lensing for source redshift z with impact parameter smaller than y is $\tau = \frac{1}{4}\Omega_L y^2 z^2$ for small z [13], where Ω_L is the density parameter contributed from lensing objects. It is estimated in Ref. [5] that unlensed events with SN > 5 have the rate $\dot{N} \sim 150\text{--}200\text{yr}^{-1}$ for $h \sim 0.5\text{--}0.8$ and the average redshift $\bar{z} \sim 0.2$. So the lensed event rate is of order $\dot{N}_L \sim \tau \dot{N} \sim (0.16\text{--}0.2)y^2(\Omega_L/0.1)(z/0.2)^2\text{yr}^{-1}$. Thus we expect that a lensed event with $y = 1$ occurs about once in every 5–8 years if $\Omega_L = 0.1$. These crude arguments may slightly under-estimate the number of small y events since we have neglected entirely the increase of events due to the lensing magnification. More accurate calculation of the lensing probability requires detailed information on the mass function of lenses and is not pre-

sented here.

Although our results are negative in that the event rate would not significantly increase even with the lensing magnification, in closing let us predict a possible observable effect from diffraction. For unlensed waveforms in the Newtonian formula (eq.7), plots of the maximum SN ρ_m versus the sweeping-up frequency f_m have universal shape irrespective of individual binaries. In fact this f_m - ρ_m relation is observable if we filter the observed signal with the function $w(t) = 2f_m j_0(2\pi f_m t)$, though the error bar is likely to be very large. On the other hand, lensed waveforms should yield deviation from the universal curve because the magnification factor (eq.3) depends on the frequency. We plot in Figure 3 the frequency distribution of SN, $(d\rho_m^2/d\ln f_m)^{1/2}$, versus f_m for some values of the source position y and the lens mass $\tilde{M}$. Detection of large deviations from the unlensed universal curve, in particular the oscillatory behavior like those in Figure 3, is suspected as a signature of gravitational lensing. Though the frequency of the detection of such events is too low to discuss any statistical properties of lensing objects, a single discovery of one such phenomenon – diffraction of gravitational waves – itself is physically very interesting.

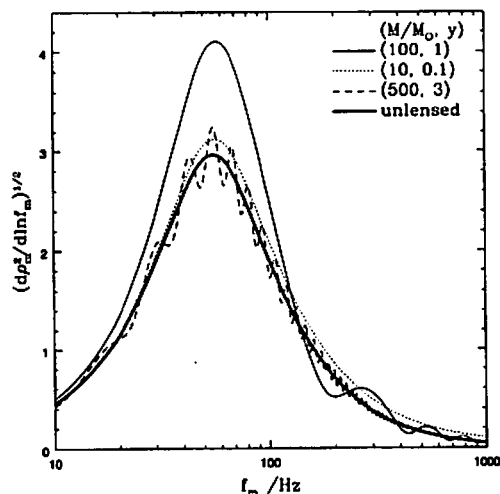


FIG. 3. Frequency distribution of the signal-to-noise ratio for some values of the source position y in units of the Einstein radius and the redshifted lens mass $\tilde{M}$, with $h = 0.7$, $\Theta = 1.6$, $\mathcal{M} = 1.2M_\odot$ and $z = 0.1$. Shown in thick curve is the unlensed “universal” distribution.

The author thank N. Sugiyama, M. Shibata and M. Sasaki for helpful discussions, the anonymous referees for valuable comments, and J. Yokoyama for the hospitality of Yukawa institute for theoretical physics where the substantial portion of this work was developed. He

gratefully acknowledges support from a JSPS fellowship. This work is supported in part by grants-in-aid by the Ministry of Education, Science, Sports and Culture of Japan (4125).

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Timelike Infinity

- Introduction and Basics -

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Abstract

By extending the Ashtekar and Romano's definition[1] of spacelike infinity to the timelike direction, a new definition of asymptotic flatness at timelike infinity for isolated systems with a source is proposed. The order of the asymptotic flatness is also defined, which tells how fast the energy-momentum falls off at timelike infinity. It leads to a picture of the hierarchy in the asymptotic structure. It is found that the fundamental asymptotic structure consists of a unit spacelike hyperboloid timelike infinity, not a point timelike infinity as in the previous works. Hence, the treatment is free from the direction dependent differentiability found in them.

1 Introduction

If a gravitational collapse takes place in asymptotically flat spacetime, we expect naively the following things to happen. The energy of fields and particles in the exterior will radiate away to infinity or fall into the black hole and the spacetime eventually becomes vacuum. As this process is irreversible, it will die away and the spacetime accordingly becomes stationary. Thus, we expect the result product of the collapse to be a Kerr black hole because the uniqueness theorem states that a vacuum, stationary and asymptotically flat spacetime is uniquely the Kerr spacetime [2]. Here, we see that following things are important to be checked to have the naive picture more concrete; whether the spacetime actually becomes vacuum or not; and whether vacuum state actually induces stationary state or not. One of our purpose of this paper is to investigate the later in *asymptotic* way by looking at timelike infinity, the very late regime of the spacetime. This approach may be more reasonable because the *exact* vacuum or stationary state do not exist in general cases. In this approach, we will find out specifically how the rate of a spacetime becoming stationary is related with the rate of the spacetime becoming vacuum. This helps us to fill the gap between the uniqueness theorem and the naive picture of the real gravitational collapse.

The understanding of timelike infinity might be also helpful to the investigation of the binary neutron star coalescence. The study of timelike infinity gives us concrete definitions on 'quasi-stationary' state and accordingly the moments of the spacetime. This may help the analysis of gravitational wave from the binary neutron stars at the quasi-stationary state found before they coalesce.[3]

There are a number of works done on timelike infinity. Cutler provides a definition of timelike infinity with everywhere-smooth unphysical metric [4]. However, as mentioned in the literature, the high differentiability of the metric demands the spacetime to have vanishing Bondi mass at timelike infinity. In other words, the timelike infinity is defined only for spacetimes with no bounded source such as a black hole. All the 'mass' is to be carried away to null infinity in such spacetimes. On the other hand, Penrose's timelike infinity differs in that it is defined to consider the spacetimes with a bounded source [5] by extending the Ashtekar and Hansen's work [12] on spacelike infinity. However, the differentiability at timelike infinity is substantially reduced in exchange: the unphysical metric has only direction dependent differentiability at timelike infinity. This low differentiability is not something that can be avoided because the originally distributed information of the physical spacetime with a source is compressed down to a point at timelike infinity in the treatment. As a result, the conditions that appear in the argument are always written in the language of 'awkward' [1] differentiability. This makes it somewhat troublesome to grasp the physical meaning of the conditions.

Ashtekar and Romano has formulated an elegant treatment of spacelike infinity, in which the corresponding 'awkward' differentiability at spacelike infinity is avoided by leaving the spacelike infinity as a 3-manifold in coordinate independent manner. Their work is new in that spacelike infinity appears

as a 3-manifold boundary of unphysical spacetimes and that the information of the unphysical spacetime is given not by a conformally completed unphysical metric but by a normal vector and an intrinsic metric of time slices foliating the unphysical spacetime. The ‘awkward’ differentiability is avoided and the conditions in the argument is easy to comprehend: it is given on the energy-momentum tensor in straightforward manner.

We apply Ashtekar and Romano’s method to the timelike direction and investigate timelike infinity. Our investigation is basically direct translation of their work. However, it is different in a number of important points. First, special care is paid for a bounded source that is still present in the very late regime of a spacetime, in contrast with the spacelike direction. As a result, it is found that there is a translation symmetry only in timelike direction globally. This is a reasonable result because we expect the spacelike traslation symmetry be broken by the presence of a bounded source at timelike infinity. The care is also found to be important in that the function that is present only if there is a source at timelike infinity is ill-defined without the care. This is also something that is not found at the spacelike infinity. Second, the misuse of the energy-momentum condition in their work is corrected. The correction is important in that it gives a picture of a hierarchy in the asymptotic structure, a point overlooked in their work. The weakest condition needed to have an asymptotic flat metric at timelike infinity and that needed to define the four-momentum or the angular-momentum are different from each other. This is because we need to restrict the asymptotic structure further to have the platform to define the conserved quantities. In other words, there is a hierarchy in the asymptotic structure and each quantities at timelike infinity belong to different levels in the hierarchy. Third, it is clarified where the information on the four-momentum is contained. It is contained in the function associated with the lapse function as thier coefficient of the mode $\ell = 0, 1$. This is a consistent picture with a four-momentum understood as ‘monopole’ and ‘dipole’ moments. Forth, a concept of asymptotic Killing field is introduced. It gives a picture of the order of asymptotic symmetry, which was difficult to grasp in the previous work. As a result, it helps to understand the hierarchy of the asymptotic structure further: the stronger the energy-momentum condition is, the stronger the spacetime has asymptotic symmetry. It is a concept that is very important to see, for example, how fast the spacetime becomes stationary.

The rest of the paper is organized as follows. In sec. 2, we investigate the Minkowski spacetime to find out how to complete general spacetimes to have timelike infinity with desirable features. In sec. 3, following the completion of Minkowski spacetime, we propose a definition of asymptotic flatness at future timelike infinity for general spacetimes with a source. In sec. 4, we examine the zero-th order asymptotic structure of the spacetimes that possess the defined asymptotic flatness at timelike infinity. An asymptotic symmetry is proposed to preseve such structure in sec. ?? . Then, in sec. ?? , asymptotic Killing fields are defined to find out with what rate the spacetime becomes stationary compared with the rate of spacetimes becoming vacuum. Its relation with the asymptotic symmetry introduced in the previous section is also discussed. In sec. ?? , we define conserved quantities with the results of the symmetry. The appendices give some examples of spacetimes with the timelike infinity defined in this paper and the confusing one which does not possess it. In this paper, we follow the notation of Wald[13].

2 Minkowski SpaceTime

In this section, we complete Minkowski spacetime $(\hat{\mathcal{M}}, \hat{\eta}_{ab})$ to find the suitable way of completion for general spacetimes, in which timelike infinity is defined not as a point but as a 3-manifold.

In a standard static spherical coordinate (t, r, θ, ϕ) , the metric of Minkowski spcaetime is given by

$$\hat{\eta}_{ab} = -(dt)_a(dt)_b + (dr)_a(dr)_b + r^2(d\Omega_2)_{ab},$$

where $(d\Omega_2)_{ab} := (d\theta)_a(d\theta)_b + \sin^2\theta(d\phi)_a(d\phi)_b$. Since our attention is now on the future timelike direction, we introduce the standard hyperbolic coordinate $(\rho, \chi, \theta, \phi)$ to the *interior* of the future light cone placed at the origin: $t = \rho \cosh \chi$ and $r = \rho \sinh \chi$, where $\chi = [0, \infty)$ and $\rho = (0, \infty)$. In this coordinate, the timelike infinity is given by the limit $\rho \rightarrow \infty$ and the metric by

$$\hat{\eta}_{ab} = -(d\rho)_a(d\rho)_b + \rho^2 h_{ab},$$

where $h_{ab} := (d\chi)_a(d\chi)_b + \sinh^2\chi(d\Omega_2)_{ab}$ is a unit spacelike hyperboloid metric. To consider the timelike infinity, introduce a new coordinate $\Omega := \rho^{-1}$. Then, we obtain a singular metric at the timelike infinity,

$\Omega \rightarrow 0$:

$$\hat{\eta}_{ab} = -\Omega^{-4}(d\Omega)_a(d\Omega)_b + \Omega^{-2}h_{ab}.$$

If we were to complete it in conformal way, we would take a conformal factor of Ω^4 to have a non-singular completed metric:

$$\eta_{ab} := \Omega^4 \hat{\eta}_{ab} = -(d\Omega)_a(d\Omega)_b + \Omega^2 h_{ab}.$$

However, this inevidently results in the timelike infinity compressed to a point because of the Ω^2 factor of h_{ab} . This suggests that if the spacetime is completed in different powers of Ω for the directions normal to and tangent to Ω -const. surface, the ‘point compression’ may be avoided and 3-manifold timelike infinity is obtained. Thus, define

$$n^a := \Omega^{-4} \hat{\eta}^{ab} (d\Omega)_b = -(\partial\Omega)^a$$

for the normal direction to the Ω -const. surfaces, and

$$q_{ab} := \Omega^2 \left(\hat{\eta}_{ab} + \Omega^{-4} F^{-1} (d\Omega)_a (d\Omega)_b \right) = h_{ab}, \quad \text{where } F := -\mathcal{L}_n \Omega$$

for the tangent direction to the Ω -constant surfaces. Now, the information of the gravitational field is fully given by the pair (n^a, q_{ab}) , where in the conformal completion it is given by the conformally completed unphysical metric $\eta_{ab} := \Omega^4 \hat{\eta}_{ab}$. The method is new in that timelike infinity is investigated with the completed pair (n^a, q_{ab}) not with the completed unphysical metric. In view of this pair (n^a, q_{ab}) , the timelike infinity, $\Omega = 0$, is a 3-manifold that has its intrinsic metric h_{ab} .

3 Definitions

In this section, following the investigation of Minkowski spacetime in the previous section, we define asymptotic flatness to order n at 3-manifold timelike infinity.

DEFINITION 1: A spacetime $(\hat{\mathcal{M}}, \hat{g}_{ab})$ is said to possess an *asymptote at timelike infinity to order n* (ATI- n) for $n = 0, 1, 2, \dots$, if there exists a manifold $\mathcal{M}$ with boundary $\mathcal{H}$, a smooth function Ω defined on $\mathcal{M}$, and an imbedding of an open subset of $\hat{\mathcal{M}}$, $\hat{\mathcal{F}}$, to $\mathcal{M} - \mathcal{H}$ satisfying the following conditions:

- (1) $\Omega \neq 0$ and $\nabla_a \Omega \neq 0$, where $\hat{=}$ denotes the equality evaluated on the boundary $\mathcal{H}$;
- (2) $n^a := \Omega^{-4} \hat{g}^{ab} \nabla_b \Omega$ and $q_{ab} := \Omega^2 (\hat{g}_{ab} + \Omega^{-4} F^{-1} \nabla_a \Omega \nabla_b \Omega)$ admit smooth limits to $\mathcal{H}$ with q_{ab} having signature $(+++)$ on $\mathcal{H}$, where $F := -\mathcal{L}_n \Omega = -n^a \nabla_a \Omega$ is a function that admits a smooth limit to $\mathcal{H}$ by the definition of n^a and $\nabla_a \Omega$; and
- (3) $\lim_{\rightarrow \mathcal{H}} \Omega^{-(2+n)} \hat{T}_{ab} = 0$, where $\hat{T}_{ab}$ is the physical energy-momentum tensor.

DEFINITION 2: A spacetime $(\hat{\mathcal{M}}, \hat{g}_{ab})$ is said to be *asymptotically flat at future timelike infinity with a source to order n* (AFFTIS- n) if it possesses an asymptote at timelike infinity to order n and its boundary $\mathcal{H}$ has topology $S^2 \times R$.

Some remarks on the notations: we write ATI- ∞ or AFFTIS- ∞ for the *exactly* vacuum spacetimes since they are ATI or AFFTIS to any order; and we attach a hat ($\hat{}$) to physical functions and tensors and denote functions and tensors that admit a smooth limit to $\mathcal{H}$ without hats.

Remark that the conditions for the definition of ATI- n spacetimes do not contain any ‘awkward’ differentiability. The condition (1) can be understood as the definition of the timelike infinity. The condition (2) gives the basis for the asymptote to be considered, following the investigation of Minkowski spacetime. And finally, the condition (3) constrains the spacetime so that it is Minkowskian-like. It is easy to grasp the condition’s physical meaning. It demands how fast the energy-momentum tensor should fall off in order to get the corresponding results. It will be shown in the next section that n must be non-negative to be asymptotically flat. The condition might be supported by the Price’s theorem[6] related to the late time behaviour of the gravitational collapse. According to the theorem, scalar field ϕ with mode ℓ decays asymptotically as $\phi \propto t^{-(2\ell+3)}$ or $\propto t^{-(2\ell+2)}$ resulting in $\hat{T}_{ab} \propto t^{-6}$ or $\hat{T}_{ab} \propto t^{-4}$ even for the slowest decay. The difference depends on the initial arrangement of the scalar fields.

Note that the condition (3) in DEFINITION 1 is given differently from that of Ashtekar and Romano: they demand that the energy-momentum tensor fall off faster than Ω^1 , $\lim_{\rightarrow \mathcal{H}} \Omega^{-1} \hat{T}_{ab} = 0$. Using the

condition, they discard such term as $q_a^m q_b^n \Omega^{-1} \hat{R}_{mn}$ in an equation evaluated on $\mathcal{H}$. However, this means that the fall-off of tensors are viewed in the components with respect to the *unphysical* basis in $\mathcal{M}$. Recalling that the $\hat{T}_{ab}$ is a *physical* metric and the condition is *physical*, it should be viewed in the components with respect to the *physical* basis in $\hat{\mathcal{F}} \subset \hat{\mathcal{M}}$ not with respect to the unphysical basis. Thus, in this paper, we use a physical unit tetrad consisting of a unit normal vector to Ω -const. surface, $\hat{n}^a$, and a triad on Ω -const. surface, $\{(\hat{e}_i)^a\}_{i=1,2,3}$ to get the components to view the fall-off of physical tensors. They satisfy

$$\hat{g}^{ab} = -\hat{n}^a \hat{n}^b + (\hat{e}_i)^a (\hat{e}_i)^b \quad (1)$$

$$\hat{n}^a = \frac{n^b}{\sqrt{-\hat{g}_{ab} n^a n^b}} = \Omega^2 F^{-1/2} n^a. \quad (2)$$

$$(\hat{e}_i)^a = \Omega (e_i)^a \quad (3)$$

where $\{(e_i)^a\}_{i=1,2,3}$ is a smooth triad of q_{ab} : the inverse of q_{ab} on Ω -const. surface is related to it by $q^{ab} =: (e_i)^a (e_i)^b$. (Eq.(3) will be derived below.) The tetrad is also used to see the fall of the physical tensor $\mathcal{L}_{\hat{g}} \hat{g}_{ab}$ in sec. VI. With this correction, it is found that the weakest energy-momentum condition needed to obtain each result is different from one another. In other words, there is a hierarchy in the asymptotic structure: the stronger energy-momentum condition constrains the higher order of asymptotic structure at timelike infinity. As a result, we need to have n dependence in the energy-momentum condition and to define the order of the asymptotic flatness.

Note that asymptotical flatness is defined only for a certain sub-region of $\hat{\mathcal{M}}$, $\hat{\mathcal{F}}$, not for the whole spacetime. This is to consider the spacetime with a bounded source. If there is a bounded source, we expect not the whole spacetime but only the region outside the source to be asymptotically flat. Hence, by restricting to the sub-region of the spacetime, the asymptotical flatness can be considered also for the spacetime with a source. However, this results in $\hat{\mathcal{F}}$ with topology R^4 minus $R \times S^3$, if the spacetime $\hat{\mathcal{M}}$ has topology of R^4 and the world line of the source, that is to be excluded from $\hat{\mathcal{F}}$, can be surrounded by $R \times S^3$. Then, the timelike infinity misses a point which corresponds to the future of the region excluded: the resulting timelike infinity would be of topology $S^2 \times R$. This motivates DEFINITION 2. It is important to note that the boundary of $\hat{\mathcal{F}}$ can be chosen to be null in this treatment in contrast to the Penrose's treatment in which the boundary can be only timelike from its construction of asymptotic structure at timelike infinity. This allows future investigation on the relation of the event horizon and timelike infinity. This point be further discussed at the end of the paper. In the appendix A, explicit procedures of deciding the region $\hat{\mathcal{F}}$ and completion to get n^a , F and q_{ab} for AFFTIS spacetimes are shown.

For the convenience, we define some useful tensors and show the relation among them. From the definition, with smooth functions and tensors, the physical metric $\hat{g}_{ab}$ is given by

$$\hat{g}_{ab} = -\Omega^{-4} F^{-1} \nabla_a \Omega \nabla_b \Omega + \Omega^{-2} q_{ab}$$

and its inverse $\hat{g}^{ab}$ by

$$\hat{g}^{ab} = -\Omega^4 F^{-1} n^a n^b + \Omega^2 q^{ab}.$$

Comparing the above with eq.(1), we obtain eq.(3). A smooth projection operator on the Ω -const. surface may be defined with smooth tensors as follows:

$$\begin{aligned} q^a_b &:= q^{ac} q_{cb} \\ &= \delta^a_b + F^{-1} n^a \nabla_b \Omega. \end{aligned} \quad (4)$$

Since the physical extrinsic curvature of the Ω -const. surface does not admit a smooth limit to $\mathcal{H}$,

$$\begin{aligned} \hat{K}_{ab} &:= \frac{1}{2} \mathcal{L}_{\hat{n}} \hat{q}_{ab} \\ &= \Omega^{-1} F^{1/2} q_{ab} + \frac{1}{2} F^{-1/2} \mathcal{L}_n q_{ab}, \end{aligned}$$

we define a smooth tensor K_{ab} by

$$\begin{aligned} K_{ab} &:= \Omega \hat{K}_{ab} \\ &= F^{1/2} q_{ab} + \frac{1}{2} \Omega F^{-1/2} \mathcal{L}_n q_{ab}. \end{aligned} \quad (5)$$

Hereafter, we raise and lower the indices of tensors without hats on Ω -const. surface with the smooth metric q_{ab} . For example, $K^a_b := q^{ac} K_{cb}$. The derivative operator associated with q_{ab} and $\hat{q}_{ab}$, D_a and $\hat{D}_a$ respectively, do not differ because q_{ab} and $\hat{q}_{ab}$ are different only by the factor of Ω . Hence, their Riemann tensors and the Ricci tensors do not differ either:

$${}^3\hat{R}_{abc}{}^d = {}^3R_{abc}{}^d \quad \text{and} \quad {}^3\hat{R}_{ab} := {}^3R_{ab}$$

where $(\hat{D}_a \hat{D}_b - \hat{D}_b \hat{D}_a)\omega_c =: {}^3\hat{R}_{abc}{}^d \omega_d$, ${}^3\hat{R}_{ab} := \hat{R}_{acb}{}^c$, $(D_a D_b - D_b D_a)\omega_c =: {}^3R_{abc}{}^d \omega_d$ and ${}^3R_{ab} := R_{acb}{}^c$

4 The zero-th order Asymptotic Structure

In this section, we explore the asymptotic structure of a spacetime that possesses asymptote at timelike infinity to order zero using the Einstein equation and define the zero-th order asymptotic structure.

To begin with, we analyze the *momentum constraint* equation:

$$\Omega^{-2}(\hat{e}_t)^a \hat{n}^b \hat{T}_{ab} = [D_c K_a^c - D_a K](e_t)^a.$$

For ATI-0 spacetimes, the left-hand side vanishes on timelike infinity by the energy-momentum condition:

$$D_c K_a^c - D_a K \doteq 0.$$

Substituting eq.(3.5) we find

$$F \doteq \text{const.}$$

Here, note that there is conformal freedom on F . Think of another completion, $(\mathcal{M}, \Omega')$, related to the original $(\mathcal{M}, \Omega)$ by $\Omega' = \alpha \Omega$ where α is a smooth function and is constant on $\mathcal{H}$. Then, F in this completion $(\mathcal{M}, \Omega')$, is related to the original by

$$F' := -\mathcal{L}_{n'} \Omega' \doteq \alpha^{-2} F.$$

That is, there is always a corresponding completion that gives $F \doteq 1$ for any completion: there is no loss of generality in demanding the completions to satisfy $F \doteq 1$. Such completions are said to be the *unit hyperboloid completions* and such Ω is said to *induce the unit hyperboloid completions* for the reason that will be clarified below and only consider such completions hereon. Then, in the unit hyperboloid completion, F is expanded around $\mathcal{H}$ as

$$F = 1 + \sum_{\ell=1}^{\infty} {}^{(\ell)}F \cdot \Omega^\ell, \quad (6)$$

where $\{{}^{(\ell)}F\}$ is a set of smooth functions that do not depend on Ω , and the extrinsic curvature K_{ab} is given by

$$K_{ab} = q_{ab} + \frac{1}{2}\Omega({}^{(1)}F q_{ab} + \mathcal{L}_n q_{ab}) + O(\Omega^2). \quad (7)$$

Next, we analyze the space-space components of the Ricci tensor given by

$$\begin{aligned} \Omega^{-2}(\hat{e}_t)^a (\hat{e}_t)^b \hat{R}_{ab} = [{}^3R_{ab} + K K_{ab} &- F^{1/2} K_{ab} - 2K_{ac} K_b^c \\ &+ \Omega F^{-1/2} \mathcal{L}_n K_{ab} + F^{-1/2} D_a D_b F^{1/2}](e_t)^a (e_t)^b. \end{aligned}$$

For ATI-0 spacetimes, the left-hand side vanishes again on timelike infinity by the energy-momentum condition:

$$0 \doteq [{}^3R_{ab} + K K_{ab} + F^{1/2} K_{ab} - 2K_{ac} K_b^c + \Omega F^{-1/2} \mathcal{L}_n K_{ab} + F^{-1/2} D_a D_b F^{1/2}](e_t)^a (e_t)^b.$$

Using eqs. (6) and (7), we find it reduce to

$${}^3R_{ab} \doteq -2q_{ab}$$

on timelike infinity, or pulling back this equation to the submanifold $\mathcal{H}$,

$$\underline{{}^3R}_{ab} \doteq -2\underline{g}_{ab}, \quad (8)$$

where the underbars denote that they are tensors and functions pulled-back to $\mathcal{H}$. The above equation says that the three-dimensional manifold $(\mathcal{H}, \underline{g}_{ab})$ has negative-constant curvature:

$$\underline{{}^3R}_{abcd} \doteq -2\underline{g}_{a[c}\underline{g}_{d]b}. \quad (9)$$

That is, the timelike infinity is isometric to a unit spacelike hyperboloid in Minkowski spacetime, $\underline{g}_{ab} = h_{ab}$. The isometric hyperboloid is *unit* because the *unit* hyperboloid completion is chosen. This is where the name comes from.

The time-time component of the energy-momentum condition of the ATI-0 spacetime gives nothing new. Hence, we have explored all the asymptotic structure of the ATI-0 spacetime, namely, Ω that induces the unit hyperboloid completion and the negative constant curvature of timelike infinity. The energy-momentum condition of ATI-0 spacetime cannot constrain the asymptotic structure any further. This structure is common to all the ATI-0 spacetimes and it motivates the following definition.

DEFINITION 3: The *zero-th order asymptotic structure* of ATI-0 spacetimes is

- (1) Ω inducing the unit hyperboloid completion, and
- (2) the intrinsic metric on the unit spacelike hyperboloid $\mathcal{H}$, h_{ab} .

The zero-th order asymptotic structure corresponds to the universal structure of AM spacetimes in [1]. However, it does not contain the equivalence class of Ω in which two completions are considered equivalent if they differ from second order in Ω and the difference is constant on timelike infinity:

$$\lim_{\rightarrow \mathcal{H}} \frac{\Omega' - \Omega}{\Omega^2} = \text{const.}$$

This is because the structure is of the next order, as it is also stated in [1] and seen in sec. ??.

Reminding its topology is restricted, the corresponding structure for AFTIS-0 spacetimes would be defined as follows.

DEFINITION 4: The *zero-th order asymptotic structure* of AFTIS-0 spacetimes is

- (1) Ω inducing the unit hyperboloid completion Ω ,
- (2) the intrinsic metric on the unit spacelike hyperboloid $\mathcal{H}$, h_{ab} and
- (3) the topology of timelike infinity, $S^2 \times R$.

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Timelike Infinity II

– Timelike Infinity and Asymptotic Symmetry –

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Abstract

An asymptotic symmetry group is proposed to preserve the fundamental structure and its structure is found to be similar to that of the Poincaré group just as the BMS group is at null infinity. It is also shown that such spacetimes with the energy-momentum tensor falling at the rate $\sim t^{-3}$ is asymptotically stationary with the rate $\sim t^{-2}$ by defining asymptotic Killing fields, which give a new point of view on the asymptotic symmetry. With the investigated symmetry, conserved quantities are defined. It is found that the four-momentum of spacetimes can be defined only with an order higher energy-momentum condition, showing that it is as an asymptotic object of an order higher. In similar fashion, the angular momentum is defined with two orders higher condition.

1 Asymptotic Symmetry

In this section, we define the asymptotic symmetry groups that preserves the zero-th order structure of ATI-0 and AFFTIS-0 spacetimes, introduced in the previous section. Then, we investigate their structures. The basic ideas are already given in [1] but here we reorganize and represent it.

The generator of the symmetry can be defined as follows to preserve the zero-th order asymptotic structure of ATI-0 and AFFTIS-0 spacetimes. (Note that as thier asymptotic structures differs only by the global nature, thier *infinitesimal* symmetry group is the same.)

DEFINITION 5: The *infinitesimal ‘Ti’ symmetry* $\mathcal{L}_G$ is a set of an asymptotic symmetry generator, ξ^a , which is a vector field of class C^∞ and satisfies following conditions:

(1) the function

$$\alpha := \lim_{\rightarrow \mathcal{H}} \frac{\mathcal{L}_\xi \Omega}{\Omega^2} \quad (1)$$

is smooth; and

(2) ξ^a is tangential to $\mathcal{H}$ and its pull-back is a Killing field of h_{ab} on $\mathcal{H}$:

$$q^a_c \xi^c \doteq \xi^a \quad \text{and} \quad \mathcal{L}_\xi h_{ab} \doteq 0. \quad (2)$$

(‘Ti’ stands for timelike infinity; it rhymes with ‘Scri’($\mathcal{I}^\pm$) as the Ashtekar’s ‘Spi’(i^0) does.)

The conditions (1) and (2) are imposed so that the generator of the symmetry preserves the zero-th order structure (1) and (2), correspondingly. First, for the completion to remain unit hyperboloid completion, the transformed completion can differ from the original completion by no more than order two in Ω . Second, for the intrinsic metric to remain unit spacelike hyperboloid metric, its action should be closed on $\mathcal{H}$, or it should be tangential to $\mathcal{H}$, and pulled back to $\mathcal{H}$ be a Killing field.

Let us investigate the structure of the infinitesimal ‘Ti’ group. First of all, note that the action of the generator is totally characterized by the pair $(\underline{\alpha}, \xi^a)$. Hence, we define equivalence class of ξ as follows.

$$\xi_1^a \stackrel{\text{eqv}}{\sim} \xi_2^a \stackrel{\text{def}}{\iff} (\underline{\alpha}_1, \xi_1^a) \doteq (\underline{\alpha}_2, \xi_2^a). \quad (3)$$

where $\underline{\alpha}_n$ is $\underline{\alpha}$ generated by ξ_n^a . We investigate the structure of $\mathcal{L}_G$ using the pairs.

Because its element is a vector field of class C^∞ , $\mathcal{L}_G$ is a Lie algebra and the bracket of the pairs can be written as

$$[(\underline{\alpha}, \xi^a), (\underline{\beta}, \eta^a)] \doteq (\mathcal{L}_\xi \underline{\beta} - \mathcal{L}_\eta \underline{\alpha}, [\xi, \eta]^a). \quad (4)$$

To see its structure further, we introduce a subgroup of $\mathcal{L}_G$.

DEFINITION 6: An *infinitesimal Ti supertranslation* $\mathcal{L}_S$ is a subset of $\mathcal{L}_G$ whose element $(\underline{\alpha}, \xi)$ satisfies

$$\xi^a \doteq 0. \quad (5)$$

(It is well defined as ξ^a and thus the above equation is completion-independent.)

Its name comes from the fact that it contains an infinitesimal Ti *translation* group defined as follows.

DEFINITION 7: An *infinitesimal Ti translation* $\mathcal{L}_T$ is a subset of $\mathcal{L}_S$ whose element $(\underline{\alpha}, 0)$ satisfies

$$\underline{D}_a \underline{D}_b \underline{\alpha} - \underline{\alpha} h_{ab} \doteq 0 \quad (6)$$

where $\underline{D}_a$ is the derivative operator associated with h_{ab} on $\mathcal{H}$.

(The group is well defined for the following reason. As ξ^a vanish on $\mathcal{H}$, it is in the form $\xi^a = \Omega^{(1)} \xi^a + O(\Omega^2)$. Thus, α is completion-independent: in another completion Ω' that differs from the original Ω by $\Omega' = \Omega + \omega \Omega^2 + O(\Omega^3)$, $\alpha' = \alpha + \Omega \mathcal{L}_{(1)} \omega$ or $\underline{\alpha}' \doteq \underline{\alpha}$.)

There are four independent functions that satisfy eq. (6):

$$\underline{\alpha}_+ \doteq \cosh \chi, \quad \underline{\alpha}_x \doteq \sinh \chi \sin \theta \cos \phi, \quad \underline{\alpha}_y \doteq \sinh \chi \sin \theta \sin \phi \quad \text{and} \quad \underline{\alpha}_z \doteq \sinh \chi \cos \theta \quad (7)$$

where (χ, θ, ϕ) is the standard hyperbolic coordinate on $\mathcal{H}$. One way to obtain the four functions is to write eq.(6) in the components with the coordinate and to solve the resulting six partial differential equations. We will find out that this group is indeed associated with the asymptotic translation symmetry and the four-momentum of the AFFTIS spacetime in Sec. 2 and 3.

Because $\mathcal{L}_S$ is an Abelian subalgebra of $\mathcal{L}_G$,

$$[(\underline{\alpha}, 0), (\underline{\beta}, 0)] \doteq (0, 0) \in \mathcal{L}_S,$$

and its commutator with an element of $\mathcal{L}_G$ is its element,

$$[(\underline{\alpha}, 0), (\underline{\beta}, \eta^a)] \doteq (-\mathcal{L}_{\underline{\eta}} \underline{\alpha}, 0) \in \mathcal{L}_S,$$

$\mathcal{L}_S$ is an ideal of $\mathcal{L}_G$. That is, the quotient $\mathcal{L}_G/\mathcal{L}_S$ is a Lie algebra, consisting of cosets $(\{\underline{\alpha}\}, \xi^a)$ where $\{\underline{\alpha}\}$ is the set of all the smooth functions on $\mathcal{H}$. Hence, each element of the quotient is characterized only by its pullback to $\mathcal{H}$. In another words, it is isomorphic to the group of the Killing field of a hyperboloid, or the Lie algebra of the Lorentz group $\mathcal{L}_L$:

$$\mathcal{L}_G/\mathcal{L}_S \cong \mathcal{L}_L. \quad (8)$$

As every element of $\mathcal{L}_G$ can be written as the sum of elements of $\mathcal{L}_S$ and $\mathcal{L}_L$,

$$(\underline{\alpha}, \xi^a) = (\underline{\alpha}, 0) + (0, \xi^a), \quad (9)$$

we can further deduce that

$$\mathcal{L}_G \cong \mathcal{L}_S \oplus \mathcal{L}_L \quad (10)$$

where $\oplus$ denote the semi-direct product.

Next, we think of the ‘finite’ asymptotic symmetry group. To consider finite transformation, the global nature of the structure to be preserved needs to be well defined. That is, the ‘finite’ symmetry group should be defined to preserve the zero-th order asymptotic structure of AFFTIS-0 spacetimes

DEFINITION 8: The ‘Ti’ group $\mathcal{G}$ is a group of a diffeomorphism generated by an element of $\mathcal{L}_G$ on geodesically complete $\mathcal{H}$ with topology $S^2 \times R$.

$\mathcal{H}$ is demanded to be geodesically complete to have complete integrated curves of the generator field for the diffeomorphism. However, a spacelike 3-manifold with negative constant curvature with topology of $S^2 \times R$ must be a spacelike hyperboloid with a *point missing on it*. If we choose the missing point as a origin for the Lorentz transformation, the integrated curves of the generator of the Lorentz boosts are

incomplete: only the rotation around the missing point is allowed. Hence, the structure of the ‘ T ’ group is

$$\mathcal{G} \cong \mathcal{S} \otimes \mathcal{R} \quad (11)$$

where $\mathcal{R}$ is the rotation group. It is the semi-direct product of the infinite dimensional additive group of smooth functions on the unit spacelike hyperboloid and the rotation group. The additive group is infinite dimensional in contrast with the Poincaré group’s four dimensional additive group, the translation group. It is because from the ‘zero-th order asymptotic structure’s point’ of view, the translation is no different from the changes in completions, which is possible in ‘infinite-dimensional way’. (The translation can be selected out as the particular change in completions. See the definition of $\mathcal{L}_T$.) It is interesting that this structure is very similar to the BMS group at null infinity which is a semi-direct product of the infinite dimensional additive group of conformally weighted functions on the 2-sphere and the Lorentz group [9, 10, 11].

2 asymptotic killing fields

In this section, we investigate the symmetry of the ATI-0 and AFFTIS-0 spacetimes from different point of view by introducing the concept of asymptotic Killing fields. The meaning of $\mathcal{L}_T$ introduced in the previous section is also clarified.

The asymptotic Killing fields may be defined as follows.

DEFINITION 9: A spacetime $(\mathcal{M}, \hat{g}_{ab})$ is said to admit an *asymptotic Killing field* $\hat{\xi}^a$ to order n in Ω if the following condition is satisfied:

$$\lim_{\Omega \rightarrow 0} \Omega^{-n} \mathcal{L}_{\hat{\xi}} \hat{g}_{ab} = 0. \quad (12)$$

This newly-defined local asymptotic symmetry is related to the infinitesimal ‘ T ’ group defined in the previous section as follows.

PROPOSITION 1: $\hat{\xi}^a$ is an element of the infinitesimal ‘ T ’ group if and only if $\hat{\xi}^a$ is an asymptotic Killing field admitted to order 0 by an ATI-0 spacetime.

Proof of If: With smooth tensors, the components of the metric with respect to a physical unit tetrad are given by

$$\begin{aligned} \hat{n}^a \hat{n}^b \mathcal{L}_{\hat{\xi}} \hat{g}_{ab} &= 4\Omega^{-1} \mathcal{L}_{\hat{\xi}} \Omega - 2F^{-1} \mathcal{L}_{[\hat{\xi}, m]} \Omega - F^{-1} \mathcal{L}_{\hat{\xi}} F \\ (\hat{e}_i)^a \hat{n}^b \mathcal{L}_{\hat{\xi}} \hat{g}_{ab} &= \left[\Omega^{-1} F^{-1/2} D_m (\mathcal{L}_{\hat{\xi}} \Omega) - \Omega q_{ma} F^{-1/2} \mathcal{L}_{\hat{\xi}} n^a \right] (e_i)^m \\ (\hat{e}_i)^a (\hat{e}_j)^b \mathcal{L}_{\hat{\xi}} \hat{g}_{ab} &= \left[-2\Omega^{-1} (\mathcal{L}_{\hat{\xi}} \Omega) q_{mn} + q_m^a q_n^b \mathcal{L}_{\hat{\xi}} q_{ab} \right] (e_i)^m (e_j)^n. \end{aligned} \quad (13)$$

The fact that a vector field $\hat{\xi}^a$ is an asymptotic Killing field to order 0 demands all the components to be of the order 1 in Ω , $\lim_{\Omega \rightarrow 0} \Omega^0 \mathcal{L}_{\hat{\xi}} \hat{g}_{ab} = 0$. Hence, the time-time component of $\mathcal{L}_{\hat{\xi}} \hat{g}_{ab}$ has neither terms of the order Ω^{-1} nor that of the order Ω^0 :

$$\mathcal{L}_{\hat{\xi}} \Omega = \beta \Omega^2 + O(\Omega^3)$$

where β is a smooth function. That is, $\alpha := \mathcal{L}_{\hat{\xi}} \Omega / \Omega^2$ is a smooth function. Using this α and reminding F is constant on $\mathcal{H}$ for ATI-0 spacetimes, eq. (??), the above identities (13) are given by

$$\begin{aligned} \hat{n}^a \hat{n}^b \mathcal{L}_{\hat{\xi}} \hat{g}_{ab} &= 0 + \Omega \mathcal{L}_{\hat{\xi}}^{(1)} F + O(\Omega^2) \\ (\hat{e}_i)^a \hat{n}^b \mathcal{L}_{\hat{\xi}} \hat{g}_{ab} &= \left[0 + \Omega (D_m \alpha - q_{ma} \mathcal{L}_{\hat{\xi}} n^a) + O(\Omega^2) \right] (e_i)^m \\ (\hat{e}_i)^a (\hat{e}_j)^b \mathcal{L}_{\hat{\xi}} \hat{g}_{ab} &= \left[q_m^a q_n^b \mathcal{L}_{\hat{\xi}} q_{ab} - 2\Omega \alpha q_{mn} \right] (e_i)^m (e_j)^n. \end{aligned} \quad (14)$$

For the same reason, the space-space component of $\mathcal{L}_{\hat{\xi}} \hat{g}_{ab}$ is demanded to be of order 1 in Ω :

$$q_m^a q_n^b \mathcal{L}_{\hat{\xi}} q_{ab} = 0.$$

Pulling back this equation to $\mathcal{H}$, we obtain

$$\mathcal{L}_{\hat{\xi}} h_{ab} \doteq 0.$$

That is, $\hat{\xi}^a$ is a Killing field of h_{ab} on $\mathcal{H}$.

Hence, $\hat{\xi}^a$ is an element of the infinitesimal 'Ti' group if $\hat{\xi}^a$ is an asymptotic Killing field to order 0 because it induces a smooth function α and its pull back to $\mathcal{H}$ is a Killing field of h_{ab} . $\square$

(Note that the second of eqs.(14) shows that a equivalence class of completions, $\{\Omega\}$, introduced for the universal structure in[1] is indeed of the next order in Ω because the preservation of the equivalence class demands that $D_m \alpha - q_{ma} \mathcal{L}_{\hat{\xi}} n^a$, appearing in the next order, vanish. This is why we have excluded it from the zero-th order asymptotic structure.)

Proof of Only If: If $\hat{\xi}^a$ is an element of infinitesimal 'Ti' group, from its definition, $\alpha := \mathcal{L}_{\hat{\xi}} \Omega / \Omega^2$ is a smooth function, and thus eq. (14) is satisfied. Further, pulled back to $\mathcal{H}$, the element of $\mathcal{L}_{\mathcal{G}}$ is a Killing field of h_{ab} . Hence, the first term of the space-space component of $\mathcal{L}_{\hat{\xi}} \hat{g}_{ab}$ in eq. (14) vanishes. This results in

$$\lim_{\Omega \rightarrow 0} \mathcal{L}_{\hat{\xi}} \hat{g}_{ab} = 0.$$

That is, an asymptotic Killing field $\hat{\xi}^a$ to order 0 is admitted if it is an element of $\mathcal{L}_{\mathcal{G}}$. $\square$

Note that the proposition states an 'exact' Killing field, $\hat{\xi}^a$ with $\mathcal{L}_{\hat{\xi}} \hat{g}_{ab} = 0$, is also an element of the infinitesimal 'Ti' group.

Next, we investigate the asymptotic translational Killing fields.

PROPOSITION 2: AT1-0 spacetimes admit four independent translational asymptotic Killing fields,

$$\begin{aligned} \hat{\xi}_t^a &= \Omega [\sinh \chi (e_1)^a] + \Omega^2 F^{-1/2} \cosh \chi n^a \\ \hat{\xi}_x^a &= \Omega [\cosh \chi \sin \theta \cos \phi (e_1)^a + \cos \theta \cos \phi (e_2)^a - \sin \phi (e_3)^a] + \Omega^2 F^{-1/2} \sinh \chi n^a \\ \hat{\xi}_y^a &= \Omega [\cosh \chi \sin \theta \sin \phi (e_1)^a + \cos \theta \sin \phi (e_2)^a + \sin \phi (e_3)^a] + \Omega^2 F^{-1/2} \sinh \chi n^a \\ \hat{\xi}_z^a &= \Omega [\cosh \chi \cos \theta (e_1)^a - \sin \theta (e_2)^a] + \Omega^2 F^{-1/2} \sinh \chi n^a, \end{aligned} \quad (15)$$

to order one in Ω , $\lim_{\Omega \rightarrow 0} \Omega^{-1} \mathcal{L}_{\hat{\xi}} \hat{g}_{ab} = 0$, where $e_1 = \partial_\chi$, $e_2 = (\sinh^{-1} \chi) \partial_\theta$ and $e_3 = (\sinh^{-1} \chi \sin^{-1} \theta) \partial_\phi$. They are constant with respect to the physical metric and orthogonal to each other:

$$\hat{g}_{ab} (\hat{\xi}_{x^\mu})^a (\hat{\xi}_{x^\nu})^b = \mp \delta_{\mu,\nu} \quad \text{where } - \text{ for } \mu = 0 \text{ and } + \text{ for } \mu = 1, 2 \text{ or } 3 \\ \text{and } \{x^\mu\}_{\mu=0,1,2,3} = \{t, x, y, z\}.$$

Proof: Since generators for translation are constant vectors, they vanish on $\mathcal{H}$, which can be also seen from (15):

$$\hat{\xi}^a = \sum_{\ell=1}^{\infty} {}^{(\ell)} \xi^a \Omega^\ell. \quad (16)$$

Then, if $\alpha := \mathcal{L}_{\hat{\xi}} \Omega / \Omega^2$ is a smooth function, we have

$$\begin{aligned} \hat{n}^a \hat{n}^b \mathcal{L}_{\hat{\xi}} \hat{g}_{ab} &= 0 + O(\Omega^2) \\ (e_1)^a \hat{n}^b \mathcal{L}_{\hat{\xi}} \hat{g}_{ab} &= \left[\Omega (D_m \alpha - q_{ma} {}^{(1)} \xi^a) + O(\Omega^2) \right] (e_1)^m \\ (e_1)^a (e_2)^b \mathcal{L}_{\hat{\xi}} \hat{g}_{ab} &= \left[\Omega (q_m^a q_n^b \mathcal{L}_{(1)} q_{ab} - \alpha q_{mn}) + O(\Omega^2) \right] (e_1)^m (e_2)^n. \end{aligned} \quad (17)$$

That is, if equations

$$\begin{aligned} D_m \alpha - q_{ma} {}^{(1)} \xi^a &\doteq 0 \\ q_m^a q_n^b \mathcal{L}_{(1)} q_{ab} - \alpha q_{mn} &\doteq 0 \end{aligned}$$

are satisfied, the vector field, ξ^a , is admitted as an asymptotic Killing field to order one, given that α is a smooth function. By pulling back the equations to $\mathcal{H}$, the above conditions can be restated as follows on $\mathcal{H}$:

$$\begin{aligned} {}^{(1)}\xi^a &\doteq \underline{D}^a \alpha. \\ \mathcal{L}_{(1)}\xi h_{ab} - \alpha h_{ab} &\doteq 0 \end{aligned} \quad (18)$$

or

$$\underline{D}_a \underline{D}_b \alpha - \alpha h_{ab} \doteq 0 \quad (19)$$

$${}^{(1)}\xi^a \doteq \underline{D}^a \alpha. \quad (20)$$

However, (19) is the same as (6) and we already know that $\underline{\alpha}_t$, $\underline{\alpha}_x$, $\underline{\alpha}_y$ and $\underline{\alpha}_z$ given in eq.(7) are the four independent smooth functions that satisfy the equation. Then, eq.(20) tells us that ${}^{(1)}\xi^a$ satisfying the condition can be obtained from these α :

$$\begin{aligned} {}^{(1)}\xi_t^a &\doteq \sinh \chi (\underline{e}_1)^a \\ {}^{(1)}\xi_x^a &\doteq \cosh \chi \sin \theta \cos \phi (\underline{e}_1)^a + \cos \theta \cos \phi (\underline{e}_2)^a - \sin \phi (\underline{e}_3)^a \\ {}^{(1)}\xi_y^a &\doteq \cosh \chi \sin \theta \sin \phi (\underline{e}_1)^a + \cos \theta \sin \phi (\underline{e}_2)^a + \cos \phi (\underline{e}_3)^a \\ {}^{(1)}\xi_z^a &\doteq \cosh \chi \cos \theta (\underline{e}_1)^a - \sin \theta (\underline{e}_2)^a. \end{aligned} \quad (21)$$

Because the vector fields eqs.(15) have their pulled-back $O(\Omega^1)$ terms same as those in eqs.(21), we see that four independent translational asymptotic Killing fields to order one in Ω are admitted. $\square$.

Note that eqs.(18)-(20) tell us that the function α generated by an element of $\mathcal{L}_{\mathcal{T}}$ provides a conformal Killing field on $\mathcal{H}$, $\underline{D}^a \alpha$, that represent the asymptotic translational symmetry of Λ T1-0 spacetimes. We will use this conformal Killing field to define a conserved quantity in the next section. As a corollary to the PROPOSITION 2, we can show how the infinitesimal 'Ti' translational group $\mathcal{L}_{\mathcal{T}}$ is associated with the above translational asymptotic Killing fields.

COROLLARY: The translational asymptotic Killing fields to order one, eqs.(15), are the elements of the infinitesimal 'Ti' translational group $\mathcal{L}_{\mathcal{T}}$.

Proof: Admitted as asymptotic Killing fields to order one, the translational asymptotic Killing fields are elements of the infinitesimal 'Ti' group from PROPOSITION 1. In addition, they vanish on $\mathcal{H}$ and generate smooth functions satisfying eq.(6). They satisfy all the conditions of $\mathcal{L}_{\mathcal{T}}$. $\square$

Finally, we consider finite translation symmetry in AFFTIS-0 spacetimes. It is shown that there is only stationary symmetry in the global sense, no spacelike translation symmetries due to the missing point on $\mathcal{H}$.

PROPOSITION 3: An AFFTIS-0 spacetime is globally asymptotic stationary to order one, in the sense that the integral curves of stationary asymptotic Killing field to order one are complete in $\mathcal{N}$, the neighbourhood of $\mathcal{H}$ in $\mathcal{F}$, where $\mathcal{F}$ is imbedded $\mathcal{F}$ in $\mathcal{M}$. However, in contrast, it does not possess global asymptotic translation symmetry in spacelike direction.

Proof: From the definition, an AFFTIS-0 spacetime misses a point on the hyperboloid $\mathcal{H}$ and the neighbourhood of $\mathcal{H}$ misses the region that corresponds to the past of the point, or the region that is not in $\mathcal{F}$ and thus out of $\mathcal{N}$. To label this region, first, choose the standard hyperbolic coordinate $\{X^i\}_{i=1,2,3} = \{\chi, \varrho, \varphi\}$ on $\mathcal{H}$ with the origin $\chi = 0$ labelling the missing point. Then, extend the coordinate to the neighbourhood of $\mathcal{H}$ by requiring $\mathcal{L}_n X^i = 0$. Now, in this coordinate, some neighbourhood of $\chi = 0$ on Ω -const. surface for $0 < \Omega < \varepsilon \ll 1$ is the region not in $\mathcal{N}$. Thus, the integrated curve that extends to $\chi = 0$ is incomplete in $\mathcal{N}$.

The timelike translational asymptotic Killing field to order one, the first of eqs.(15), behaves as

$$\xi_t^a \rightarrow \Omega^2 F^{-1/2} n^a \quad \text{for} \quad \chi \rightarrow 0.$$

on a Ω -const. surface for $0 < \Omega < \varepsilon \ll 1$. It is orthogonal to the $\Omega = \text{const.}$ surface and does not head towards the neighbourhood of $\chi = 0$. Hence, their integrated curves are complete in $\mathcal{N}$ and, in this sense, the global stationary symmetry to order one is admitted in AFFTIS-0 spacetime.

On the other hand, the rest of the spacelike asymptotic Killing fields are tangential to the $\Omega = \text{const.}$ surface in the neighbourhood of $\chi = 0$,

$$\begin{aligned}\hat{\xi}_x^a &\rightarrow \Omega [\sin \theta \cos \phi (e_1)^a + \cos \theta \cos \phi (e_2)^a - \sin \phi (e_3)^a] \\ \hat{\xi}_y^a &\rightarrow \Omega [\sin \theta \sin \phi (e_1)^a + \cos \theta \sin \phi (e_2)^a + \cos \phi (e_3)^a] \quad \text{as } \chi \rightarrow 0, \\ \hat{\xi}_z^a &\rightarrow \Omega [\cos \theta (e_1)^a - \sin \theta (e_2)^a]\end{aligned}$$

and do not vanish. This can be seen by introducing a new coordinate $(\bar{t}, \bar{x}, \bar{y}, \bar{z})$ defined as

$$\begin{aligned}\bar{t} &= \cosh \chi, \quad \bar{x} = \sinh \chi \sin \theta \cos \phi, \quad \bar{y} = \sinh \chi \sin \theta \sin \phi \quad \text{and} \quad \bar{z} = \sinh \chi \cos \theta \\ \text{where } \bar{t}^2 &- (\bar{x}^2 + \bar{y}^2 + \bar{z}^2) = 1\end{aligned}$$

in order to avoid the bad behaviour of the coordinate (θ, ϕ) at $\chi = 0$. In this coordinate, we can see that they are non-vanishing around $\chi = 0$:

$$\begin{aligned}\hat{\xi}_x^a &\rightarrow \Omega (\partial_{\bar{x}})^a \\ \hat{\xi}_y^a &\rightarrow \Omega (\partial_{\bar{y}})^a \quad \text{as } \chi \rightarrow 0, \\ \hat{\xi}_z^a &\rightarrow \Omega (\partial_{\bar{z}})^a\end{aligned}$$

where $(\partial_{\bar{x}})^a$, $(\partial_{\bar{y}})^a$ and $(\partial_{\bar{z}})^a$ are smooth vectors from the construction. To summarize, the spacelike Killing field is tangential to the Ω -const. surface for $0 < \Omega < \varepsilon \ll 1$ and non-vanishing around $\chi = 0$. That is, their integrated curves extend to the region around $\chi = 0$ and thus are incomplete in $\mathcal{N}$. That is, spacelike translation are not admitted in the global sense. $\square$

We conclude the section with a remark on the hierarchy of the asymptotic structure. Note that eqs.(14) shows that the $O(\Omega^1)$ terms are indeed of the next order asymptotic structure. It contains a term such as ${}^{(1)}F$ of the next order. However, eqs.(17) contain the terms of the next order asymptotic structure in $O(\Omega^2)$. (For example, the $O(\Omega^2)$ term of the time-time component is $\Omega^2[D^a({}^{(1)}F D_a \alpha + {}^{(1)}F \alpha - 2n^a \nabla_a \alpha)]$.) This is because $\hat{\xi}^a$ is in general of the order 1 and thus, written in terms of ${}^{(1)}\xi^a$, the order of Ω in the equation is shifted an order higher. This is why asymptotic translational Killing field to order one is admitted even in ATI-0 spacetimes. Note also that terms of the first order such as ${}^{(1)}F$ is not constrained in any way for general ATI-0 spacetimes. That is, the $O(\Omega^2)$ terms in eqs.(17) have no constraint in general and thus ATI-0 spacetimes do not admit a translational asymptotic Killing field to order two in general.

3 conserved quantities and the first and second order asymptotic structure

In this section, we find that it is inconsistent to define a conserved quantity for *general* AFFTIS-0 spacetimes. Thus, we define the four-momentum for AFFTIS-1 spacetimes and angular momentum for AFFTIS-2 spacetimes.

Since ATI-0 spacetimes possesses an infinitesimal translational symmetry group, $\mathcal{L}_T$, it should be possible to define a conserved four-momentum respecting the symmetry, as it is the case for Minkowski spacetime. Now, we investigate how we can define it.

First, we need something that gives information on the gravitational field. We know that the information is to be found in the traceless part of the gravitational field, the Weyl tensor. To investigate the Weyl tensor, it is helpful to decompose it to its electric part, $\hat{E}_{ab} := \hat{C}_{amb\ell} \hat{n}^m \hat{n}^\ell$, and its magnetic part, $\hat{B}_{ab} := {}^* \hat{C}_{amb\ell} \hat{n}^m \hat{n}^\ell$, with respect to the Ω -const. surface. With smooth tensors, they are given by

$$\begin{aligned}\hat{E}_{ab} &= F^{1/2} K_{ab} + K_{am} K_b^m + q_a^m q_b^\ell \hat{R}_{m\ell} + F^{-1/2} D_a D_b F^{1/2} - \frac{1}{2} (q_a^m q_b^\ell + q_{ab} q^{mn}) \hat{L}_{m\ell} \\ \hat{B}_{ab} &= \epsilon^{\ell m}{}_a D_\ell K_{mb} + \frac{1}{2} \Omega^{-2} \epsilon_{abc} (e_1)^\epsilon (\hat{e}_1)^m \hat{n}^\ell \hat{R}_{m\ell}.\end{aligned}$$

For AT1-0 spacetimes, they vanish on $\mathcal{H}$:

$$\hat{E}_{ab} = \Omega^{(1)} E_{ab} + O(\Omega^2) \quad \text{and} \quad \hat{B}_{ab} = \Omega^{(1)} B_{ab} + O(\Omega^2)$$

where

$$\begin{aligned} {}^{(1)}E_{ab} &:= \frac{1}{2} \left\{ D_a D_b {}^{(1)}F - {}^{(1)}F q_{ab} - (e_1)_a (e_1)_b \left[\Omega^{-3} ((\hat{e}_1)^m (\hat{e}_1)^\ell + \delta_{1,j} (\hat{e}_\kappa)^m (\hat{e}_\kappa)^\ell) \hat{L}_{m\ell} \right] \right\} \\ {}^{(1)}B_{ab} &:= \frac{1}{2} \left\{ \epsilon^{\ell m}{}_a D_\ell {}^{(1)}K_{mb} + \epsilon_{abc} (e_1)^c \left[\Omega^{-3} (\hat{e}_1)^m \hat{n}^\ell \hat{R}_{m\ell} \right] \right\} \end{aligned}$$

with $\hat{L}_{ab} := \hat{R}_{ab} - \frac{1}{6} \hat{g}_{ab} \hat{R}$ and ${}^{(1)}K_{ab} := {}^{(1)}F q_{ab} + \mathcal{L}_n q_{ab}$. Hence, the $O(\Omega^1)$ terms are the possible candidate that give us information on the gravitational field. Note here that ${}^{(1)}E_{ab}$ and ${}^{(1)}B_{ab}$ depend on the nature of the particular AT1-0 spacetime through the smooth functions in brackets [] in the above equations. This can be understood as the definition of AT1-0 spacetimes constrain only up to the zero-th order asymptotic structure and thereby does not constrain the heigher asymptotic structure such as ${}^{(1)}E_{ab}$ and ${}^{(1)}B_{ab}$. Hence, we cannot talk on the general features of the first order asymptotic structure in AT1-0 spacetimes. For instance, it is not possible to state whether ${}^{(1)}E_{ab}$ is divergence free or not in general AT1-0 spacetimes:

$$\underline{D}^b {}^{(1)}E_{ab} \doteq -\frac{1}{2} (\underline{e}_1)_a (\underline{e}_1)_b \underline{D}^b \left\{ \lim_{\mathcal{H}} \left[\Omega^{-3} ((\hat{e}_1)^m (\hat{e}_1)^\ell + \delta_{1,j} (\hat{e}_\kappa)^m (\hat{e}_\kappa)^\ell) \hat{L}_{m\ell} \right] \right\}$$

where eq.(??) is used in the evaluation. It depends on the function in the brackets []. However, it will be found out later that ${}^{(1)}E_{ab}$ must be divergence-free in order to define a conserved four-momentum from it. Thus, we should consider AT1-1 spacetimes for the conserved quantity. The stronger fall-off condition on the energy-momentum tensor, $\lim_{\mathcal{H}} \hat{T}_{ab}$ is supported by the Price's theorem[6]. In those spacetimes, the terms in the brackets [] vanish on $\mathcal{H}$,

$${}^{(1)}E_{ab} := \frac{1}{2} \left(D_a D_b {}^{(1)}F - {}^{(1)}F q_{ab} \right) \quad (22)$$

$${}^{(1)}B_{ab} := \frac{1}{2} \left(\epsilon^{\ell m}{}_a D_\ell {}^{(1)}K_{mb} \right), \quad (23)$$

and we have divergence-free ${}^{(1)}E_{ab}$:

$$\underline{D}^b {}^{(1)}E_{ab} \doteq 0. \quad (24)$$

Moreover, ${}^{(1)}E_{ab}$ obeys the field equation:

$$\underline{D}_{[a} {}^{(1)}E_{b]c} \doteq 0, \quad (25)$$

where eq.(??) is again used in the evaluation. We can show that ${}^{(1)}B_{ab}$ also obeys a field equation and is divergence-free. Contracting the Bianchi identity $\nabla_{[a} \hat{R}_{b]c}{}^{de} = 0$ with $\hat{n}^a \hat{\epsilon}^{bc}{}_m \hat{\epsilon}^{den}$ yields

$$\begin{aligned} \epsilon^{ae}{}_m D_a \hat{B}_{cn} &- \Omega F^{-1/2} n^a \nabla_a \hat{E}_{mn} + \epsilon^{ac}{}_m \epsilon^{de}{}_n K_{ad} \hat{E}_{ce} \\ &+ \epsilon^{bc}{}_m (\hat{B}_n)_c F^{-1} D_b F + \frac{1}{2} \Omega F^{-1/2} \left(n^{[a} q^{c]e} q_{mn} - n^{[a} q^{c]e}{}_n q^e{}_m \right) \nabla_a \hat{L}_{ce} = 0. \end{aligned}$$

Decomposing $\hat{L}_{ab}$ into its components,

$$\hat{L}_{ab} = (\hat{L}_{mn} \hat{n}^m \hat{n}^n) \hat{n}_a \hat{n}_b + 2(\hat{L}_{mn} (\hat{e}_1)^m \hat{n}^n) (\hat{n}_a (\hat{e}_1)_b) + (\hat{L}_{mn} (\hat{e}_1)^m (\hat{e}_1)^n) (\hat{e}_1)_a (\hat{e}_1)_b,$$

in AT1- n spacetimes, we find

$$\epsilon^{ae}{}_m D_a {}^{(1)}B_{cn} = O(\Omega^1) + O(\Omega^n) \quad (26)$$

where $O(\Omega^n)$ contains such terms as $\frac{1}{2} \Omega^{-3} K^a{}_a (\hat{L}_{mn} \hat{n}^m \hat{n}^n)$. Thus, we have

$$\underline{D}_{[a} {}^{(1)}B_{b]c} \doteq 0 \quad \text{and} \quad \underline{D}^b {}^{(1)}B_{ab} \doteq 0 \quad (27)$$

on $\mathcal{H}$ in ATI-1 spacetimes. Further, the constraint on the energy-momentum tensor in ATI-1 spacetimes gives a differential equation of ${}^{(1)}F$, which is also an asymptotic object of the first order. The fact that ${}^{(1)}E_{ab}$ is traceless by its construction gives

$$\underline{D}^a \underline{D}_a {}^{(1)}F - 3 {}^{(1)}F \doteq 0.$$

The general solution of the differential equation is given by

$${}^{(1)}F(\chi, \theta, \phi) = \sum_{\substack{l \geq 0 \\ |m| \leq l}} a_{lm} {}^{(1)}F_{lm}(\chi, \theta, \phi) \quad (28)$$

where

$${}^{(1)}F_{lm}(\chi, \theta, \phi) := Y_{lm}(\theta, \phi) P_{\frac{l}{2}}^{l+\frac{1}{2}}(\cosh \chi) \frac{1}{\sqrt{\sinh \chi}}. \quad (29)$$

Y_{lm} and P_m^l are, respectively, the spherical harmonic and the associated Legendre function defined by

$$P_\nu^\mu(z) := \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1} \right)^{\mu/2} F\left(-\nu, \nu+1, 1-\mu; \frac{1-z}{2}\right) \quad \text{for } z > 1$$

where $F(\alpha, \beta, \gamma; z)$ is the hypergeometric functions. For the later convinience, we choose the coefficients of the spherical hramonics as:

$$Y_{lm}(\theta, \phi) = \begin{cases} \sqrt{2\pi} P_l^m(\cos \theta) \cdot \cos m\phi & m \geq 0 \\ \sqrt{2\pi} P_l^m(\cos \theta) \cdot \sin m\phi & m < 0. \end{cases}$$

Explicitly calculated, we find

$$\begin{aligned} {}^{(1)}E_{00} &= 2 \left(2 \sinh \chi + \frac{1}{\sinh \chi} \right) \\ {}^{(1)}E_{11} &= 2 \left(2 \cosh \chi - \frac{\cosh \chi}{\sinh^2 \chi} \right) \sin \theta \cos \phi \end{aligned} \quad (30)$$

These functions indeed appear when the spacetimes are completed and the function F is calculated. See eqs.(??) and (??). Note that they are smooth only on $\mathcal{H}$ with a point $\chi = 0$ excluded. Otherwise, it is ill-defined.

On the other hand, the fact that ${}^{(1)}B_{ab}$ is traceless does not give any non-trivial equation. This is because the discarded term of ${}^{(1)}B_{ab}$ containing the Ricci tensor is also traceless. Or in another words, the Einstein equation for AFFTIS 1 spacetimes is not used in ‘non-tivial’ way when the term is discarded for ${}^{(1)}B_{ab}$ in contrast with the ${}^{(1)}E_{ab}$ ’s case.

Now that we have the tensor field, ${}^{(1)}E_{ab}$, to give us the information on the gravitaional field and the conformal Killing vector on $\mathcal{H}$, $\underline{D}^a \underline{\alpha}$, that represents a transrational symmetry, we are ready to define the conserved quantity at timelike infinity $\mathcal{H}$.

DEFINITION 10: The four-momentum of AFFTIS-1 spacetimes, $P(\underline{D}^a \underline{\alpha})$, is a linear mapping of a conformal Killing field on $\mathcal{H}$, $\underline{D}^a \underline{\alpha}$, to the reals defined by

$$P(\underline{D}^a \underline{\alpha}) : \underline{D}^a \underline{\alpha} \mapsto \mathbb{R} \quad \stackrel{\text{def}}{=} \quad P(\underline{D}^a \underline{\alpha}) := -\frac{1}{8\pi} \int_{\mathcal{S}} \epsilon_{ab} {}^{m(1)}E_{mn} \underline{D}^n \underline{\alpha} \quad (31)$$

where $\mathcal{S}$ is a cross section of $\mathcal{H}$ and $\underline{\alpha}$ is a smooth funtion generated by an element of $\mathcal{L}_{\mathcal{T}}$.

Note that the four-momentum is defined for AFFTIS spacetimes not for ATI spacetimes. This is because we need timelike infinity which admits a 2-sphere cross section on which we can define the integration.

The four-momentum is conserved because its value does not depend on the choice of the cross section on which it is evaluated. This can be seen by noting that the 2-form integrand is closed: the 3-form $\underline{D}_{[c} \epsilon_{ab]} {}^{m(1)}E_{mn} {}^{(1)}\xi^n$ is proportional to

$$\begin{aligned} \epsilon^{ab} \underline{D}_c \left(\epsilon_{ab} {}^{m(1)}E_{mn} \underline{D}^n \underline{\alpha} \right) &\doteq (\underline{D}^m {}^{(1)}E_{mn}) \underline{D}^n \underline{\alpha} + {}^{(1)}E_{mn} \left(\frac{1}{2} \alpha h^{mn} \right) \\ &\doteq 0 \end{aligned}$$

in which the fact that $\underline{D}^a \underline{\alpha}$ is a conformal Killing field is used to get the second term on the right hand side in the first line, and the facts that ${}^{(1)}\underline{E}_{ab}$ is divergence-free and that ${}^{(1)}\underline{E}_{ab}$ is traceless on the second equality. Note that the integration is guaranteed to be conserved by the facts that $\underline{D}^a \underline{\alpha}$ represents an infinitesimal translational symmetry and thus is a conformal Killing field and that the gravitational field ${}^{(1)}\underline{E}_{ab}$ is divergence-free in AFFTIS-1 spacetimes.

There are four 'independent' components of $P(\underline{D}^a \underline{\alpha})$ corresponding to the four linearly independent vector fields, $\{\underline{D}^a \underline{\alpha}_t, \underline{D}^a \underline{\alpha}_x, \underline{D}^a \underline{\alpha}_y, \underline{D}^a \underline{\alpha}_z\}$. Here, we evaluate each component of the four-momentum and see how the information on the four-momentum is contained in the gravitational field. To evaluate the integration, choose the $\chi = \text{const.}$ surface as $\mathcal{S}$. Then, for the timelike translation, we obtain

$$P(\underline{D}^a \underline{\alpha}_t) = \frac{1}{16\pi} \int_{\chi=\text{const.}} \left(\frac{\partial^2}{\partial \chi^2} {}^{(1)}\underline{F} - {}^{(1)}\underline{F} \right) \sinh \chi \sqrt{h} d\theta d\phi,$$

where $h := \det(h_{ij}) = \sinh^2 \chi \sin^2 \theta$. Substituting the general solution for ${}^{(1)}\underline{F}$, eq.(28), we find that other than the $\ell = 0$ term vanishes:

$$P(\underline{D}^a \underline{\alpha}_t) = a_{00}, \quad (32)$$

where the orthogonality of spherical harmonics

$$\int Y_{\ell m} d(\cos \theta) d\phi = 4\sqrt{2}\pi^{\frac{1}{2}} \delta_{\ell,0} \delta_{m,0},$$

is used. The spacelike momentum can be evaluated in the same way, resulting in

$$\begin{aligned} P(\underline{D}^a \underline{\alpha}_x) &= a_{11} \\ P(\underline{D}^a \underline{\alpha}_y) &= a_{1-1} \\ P(\underline{D}^a \underline{\alpha}_z) &= a_{10}. \end{aligned} \quad (33)$$

That is, the information on the four-momentum is contained in the $\ell = 0, 1$ terms of ${}^{(1)}\underline{F}$ as their coefficients. Calculating ${}^{(1)}\underline{F}$ explicitly, it is found that the Kerr and the Schwarzschild spacetimes have the mass parameter m for their a_{00} , eqs.(??) and (??), and that the Schwarzschild spacetimes boosted by constant velocity v along the x -axis have $-mv$ for its a_{11} , eq.(??). In this respect, it might be more reasonable to call ${}^{(1)}\underline{F}$ the *four-momentum aspect*, rather than 'the mass aspect' as suggested by Ashtekar and Romano in [1].

Note here that if the spacetime contains a bounded source and has a mass, a_{00} is non-zero. That is, the function $\underline{F}$ contains a term such as ${}^{(1)}\underline{F}_{00}$, eq.(30), which is not smooth at $\chi = 0$ in contrast with the fact that the corresponding function for spacelike infinity is everywhere smooth. However, we do not need to worry over this because the point is excluded from $\mathcal{H}$ for AFFTIS- n spacetimes. Here, we see again that it is important and natural to consider $\mathcal{H}$ with a point missing, in case of considering timelike infinity.

As already mentioned, the conservation of the four-momentum defined above is guaranteed by the facts that the tensor ${}^{(1)}\underline{E}_{ab}$ is divergence-free and traceless and that $\underline{D}^a \underline{\alpha}$ is a conformal Killing vector field on $\mathcal{H}$. Remark that ${}^{(1)}\underline{B}_{ab}$ is also a divergence-free and traceless tensor field and that an element of $\mathcal{L}_L$, ξ^a , is also a conformal Killing field on $\mathcal{H}$. They give us three more conserved quantities. However, they identically vanish since the 2-form integrand constructed from them are exact and the cross section on which it is evaluated has no boundary[1].

In Minkowski spacetime, there is also a conserved quantity associated with the symmetry under Lorentz transformation, known as the angular momentum. As we have the symmetry in the T_i group, we would expect it is also possible to define the angular momentum in AFFTIS spacetimes. However, to define it the origin of the angular momentum must be specified, which can be done by demanding ${}^{(1)}\underline{B}_{ab}$ vanish on $\mathcal{H}$ and further restricting the gauge to those that keep this condition[1]. Then, in ATI-2 spacetimes it can be shown that ${}^{(2)}\underline{B}_{ab}$ is divergence free and the angular momentum can be defined as the integration of the 2-form $\underline{\epsilon}_{ab}{}^m{}^{(2)}\underline{B}_{mn}\xi^n$, where ξ^n is a Killing field of h_{ab} on $\mathcal{H}$, in much the same way as the four-momentum. Note that the angular momentum is a conserved quantity of the second order asymptotic structure. It can be also seen from eq.(??) as the angular momentum parameter a is contained in ${}^{(2)}\underline{F}$.

The above results may be summarized as follows. The four-momentum can be defined only for AFTIS-1 spacetimes, that is, the spacetimes satisfying the stronger energy condition that constrain the asymptotic structure to higher order so that the spacetimes have a universal first order asymptotic structure as a platform for the four-momentum to be defined. For such spacetimes, ${}^{(1)}F$ is constrained and it contains the information of the four-momentum as its coefficient. The four-momentum may be said to be a conserved quantity of the first order. The discussion on the construction of a conserved quantity with ${}^{(1)}B_{ab}$ or $\xi^a \in \mathcal{L}_L$ shows that the four-momentum is the only conserved quantity in the first order asymptotic structure since there are no more candidate for the conserved quantity to be defined after we have elucidated all the possible candidate for gravitational fields and symmetry in Ti group at this order. This is somewhat expected result as the four-momentum is the only 'monopole and dipole moments' of spacetimes and the first order asymptotic structure corresponds to monopole and dipole moments of a spacetime. The angular momentum can be defined only for AFTIS-2 spacetimes and it is a conserved quantity of the second order.

4 summary and discussions

In this paper, we have defined asymptotic flatness at timelike infinity and its order by extending the new completion method introduced by Ashtekar and Romano[1] to the timelike direction. In the treatment, timelike infinity is represented as a 3-manifold unit spacelike hyperboloid and thus there is no 'awkward' differentiability, which is unavoidable in the previous treatment of timelike infinity represented as a point. To accommodate spacetimes with bounded sources, our asymptotic flatness at the timelike infinity is defined for a subregion of the whole spacetime, from which the region with the bounded source is excluded. This results in 3-manifold timelike infinity with a point missing on it, which corresponds to the future of the excluded region. The condition of the asymptotic flatness is simple. It is given as a fall-off condition of the energy-momentum tensor. We found that the stronger the fall-off condition is, the more the asymptotic structure is constrained and the platform is prepared for asymptotic objects such as asymptotic Killing fields, the four-momentum and the angular-momentum. First, we explored the zero-th order asymptotic structure of spacetimes with such asymptotic flatness at timelike infinity. It is found to consist of unit hyperboloid completion and the negative constant curvature of timelike infinity. The infinitesimal Ti group is defined to preserve this structure and found to be the semidirect product of the infinite dimensional additive group of smooth functions on the 3-manifold timelike infinity and the generator of the Lorentz transformation. The corresponding finite symmetry group lacks Lorentz group due to the missing point on timelike infinity. Next, a concept of asymptotic Killing fields is introduced to see the asymptotic symmetry from different point of view. Its relation with the Ti group is checked and translational asymptotic Killing fields are proved to be admitted to order zero. The discussion also shows that without stronger energy-momentum condition, there is no asymptotic symmetry to higher order in general. It shows the hierarchy of the asymptotic symmetry, which was difficult to grasp with the definition of the Ti group only. It is also proved that in global sense there is no asymptotic translation symmetry in spacelike direction but in timelike direction. A consistent picture with the idea that the presence of a bounded source would break the symmetry in the spacelike direction. With the symmetry, conserved quantities are defined. It is found that stronger energy-momentum condition is needed so that there is enough basis asymptotic structure for them to be defined. The investigation shows that the four-momentum is a first order asymptotic structure and the angular momentum is a second order asymptotic structure. It is also shown that the four-momentum is the only conserved quantity for a first order asymptotic structure.

Some problems remain for future works. Although we stepped into the first order asymptotic structure to define conserved quantity, it has not been explored in full. The Einstein equation of the first order remains to be investigated. Only after the investigation, we can see whether the translational asymptotic Killing field to order 1 is admitted and whether the condition ${}^{(1)}B_{ab} \doteq 0$ needed to define the angular momentum is naturally satisfied in ATI-1 spacetimes. Our proposal strategy is to use a cosmological perturbation method since the foliation of the interior of light cone in Minkowski spacetime with a hyperboloid coordinate can be regarded as a Milne universe. Next, the meaning of the defined four-momentum should be cleared. It should be the final value of the Bondi mass but this is something to be

proved.

We remark that in the present work there is no reference to the timelike curves converging to timelike infinity on which the previous works' analyses[4][5] are based. This fact makes it possible to take the region $\hat{\mathcal{F}}$ as just the outside of the event horizon, resulting in the boundary of $\mathcal{F}$ being the event horizon. In another words, there is a possibility to consider the event horizon in connection with timelike infinity. This may answer to the question how timelike infinity is placed between the null infinity and the event horizon in multi black hole spacetimes, such as a Majumdar-Papapetrou spacetime[7, 8]. It is questionable because the topology of the event horizon is of multi components of $S^2 \times R$ where the topology of null infinity is of a single component of $S^2 \times R$.

Acknowledgment

We would like to thank Katsuhiko Sato, Akio Hosoya and Yasushi Suto for their encouragements. We also thank Keita Ikumi for his fruitful discussion with us.

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Chronology Protection and Non-Naked Singularity

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Abstract

We test the chronology protection conjecture in classical general relativity by investigating finitely vicious space-times. First we present singularity theorems in finitely vicious space-times by imposing some restrictions on the chronology violating sets. In the theorems we can refer to the location of an occurring singularity and do not assume any asymptotic conditions such as the existence of null infinities. Further introducing the concept of a *non-naked singularity*, we show that a restricted class of chronology violations cannot arise if all occurring singularities are the non-naked singularities. Our results suggest that the causal feature of the occurring singularities is the key to prevent the appearance of causality violation.

1 Introduction

Whether or not there are causality violating regions in the universe is an interesting problem in physics. Hawking has proposed the chronology protection conjecture which states that the laws of physics prevent closed timelike curves from appearing [1]. He investigated the instability of a Cauchy horizon caused by causality violation and showed that the renormalized stress-energy tensor always diverges at the Cauchy horizon by using quantum field theory. The instabilities of such Cauchy horizons have also been shown in this approach by several authors [2]. On the other hand, Li et al. [3] obtained the opposite result that the Cauchy horizon is stable against the vacuum fluctuations under some conditions. There also remains an unsettled problem that the back reaction of quantum effects on the geometry is difficult to take into account. At present, there seems no consensus on the quantum instability of the Cauchy horizon caused by causality violation.

In the framework of classical general relativity, only a few attempts have so far been made at verifying the chronology protection conjecture. Hawking [1] considered a compactly generated Cauchy horizon that all the past-directed null geodesic generators of the Cauchy horizon are totally imprisoned in a compact set. He showed that such a Cauchy horizon cannot occur under the weak energy condition. Tipler provided a partial answer to the conjecture by showing that the creation of a closed timelike curve leads to singularity formation under the stronger energy condition than the usual energy conditions [4].

In the context of testing the conjecture, one would like to examine the case that causality violation could arise in a finite region of space from a regular initial data by physical process such as gravitational collapse. In such a case space-time may be *finitely vicious*, whose notion was first introduced by Tipler [4]. Space-times containing time machines are examples of finitely vicious space-times and many models of such space-times have been proposed. Morris et al. [5] suggested a model of a time machine in a wormhole space-time by accelerating one mouth of the wormhole with respect to the other mouth. Deutsch and Politzer [6] constructed a simple model of a time machine (the Deutsch-Politzer time machine) by identifications of spacelike line segments in two-dimensional Minkowski space-time. It is interesting that

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the null geodesic generators of the boundary of causality violating set in this model are not imprisoned in any compact sets, while those of a compactly generated Cauchy horizon are.

Most recently, Krasnikov [7] proposed a simple model of time machine by modifying the Deutsch-Politzer time machine. Surprisingly, there are no singularities associated with the chronology violation in this model even though the weak energy condition is satisfied. This model demonstrates that the weak energy condition does not necessarily prevent causality violations of the Deutsch-Politzer time machine type from occurring in classical general relativity; the chronology protection conjecture is violated even if this energy condition holds.

In this paper, we test the chronology protection conjecture by investigating finitely vicious space-times under some reasonable conditions, especially, the strong energy condition. To begin with we define a class of finitely vicious space-times whose causal structures are similar to that of a space-time with the Deutsch-Politzer time machine. Next we show that there exists an inextendible and incomplete causal geodesic in the chronology violating region under physically reasonable conditions. To prove this, we do not assume the existence of any asymptotic regions. Introducing the notion of a *non-naked singularity*, we finally show that some classes of chronology violations which will be specified by a suitable boundary condition for the chronology violating set cannot arise if all occurring singularities are the non-naked singularities.

2 A geometry of chronology violation

We shall consider space-times in which causality violating sets (chronology violating sets) are formed in gravitational collapse or “manufactured” as time machines by future technology. In such a space-time causality violation should begin in a finite region of space in the future of a partial Cauchy surface S . It is expected that associated with the causality violation a Cauchy horizon $H^+(S)$ with compact spatial section develops in the future of S . Such space-times are classified in a *finitely vicious space-time* by Tipler [4].

There are some works on constructions of time machines [5, 6, 7] in finitely vicious space-times with non-trivial topology. The four-dimensional Deutsch-Politzer time machine [6], constructed below, is one of the simplest models of such time machines. Consider two-dimensional Minkowski space-time I with metric

$$ds^2 = -dt^2 + dx^2, \quad (1)$$

and two horizontal line segments $l_{\pm} := \{t, x | t = \pm a, |x| < b\}$ on I . Remove the points $(|t| = a, |x| = b)$ from I and make cuts along $l_{\pm}$. Identifying the upper banks of l_+ and l_- with the lower banks of l_- and l_+ , respectively, one can construct a two-dimensional space-time I' containing the two-dimensional Deutsch-Politzer time machine. Then one can obtain the four-dimensional Deutsch-Politzer time machine as a product space-time $M = I' \times Q$, where Q is a closed compact two-dimensional space, like S^2 or T^2 . The causal structure of this space-time is schematically depicted in Fig. 1.

The chronology violating-region, the interior of the hexagon in Fig. 1, is described as $I^+(q) \cap I^-(q)$, $q \in M$. The boundary of the chronology violating-region consists of portions of $\dot{I}^+(q)$ and $\dot{I}^-(q)$. As easily seen in Fig. 1, the space-time is finitely vicious because there is a hypersurface S whose intersection with $\dot{I}^+(q) \cap \dot{I}^-(q)$ is compact.

From the above observations, as a simple and reasonable class of finitely vicious space-times, we focus our attention on a finitely vicious space-time (M, g) with a chronology violating set V satisfying the following three conditions;

Conditions

1. there is a locally spacelike three-dimensional hypersurface S (with no edge) such that $K := V \cap S$ has a compact closure $\bar{K}$ and the boundary $\bar{K}$ is equal to $\dot{I}^+(q) \cap \dot{I}^-(q)$,
2. every closed causal curve in $\bar{V}$ is a noncontractible curve which intersects $\bar{K}$,
3. $R_{ab}K^aK^b \geq 0$ for every non-spacelike vector K^a and every causal geodesic with a tangent vector K^a contains a point at which $K_{[a}R_{b]c}d_cK_{f]}K^cK^d \neq 0$.

Condition 1 is a boundary condition for the chronology violating set. Condition 2 is imposed for the simplicity of arguments since we consider closed causal curves associated with non-trivial topology such

as the topology in the Deutsch-Politzer time machines. Condition 2 ensures that the causality condition holds on a covering manifold $\hat{M}$ that will be considered in the following sections. Condition 3 means that the strong energy condition and the generic condition are satisfied.

3 Singularity theorems

In this section, we will investigate finitely vicious space-times satisfying conditions 1, 2, 3 and present theorems which state that there is a singularity in the causality violating-region. Before showing this, we shall explicitly define the statement “there is an incomplete causal geodesic curve (singularity) in a sub-region N in a space-time” as follows.

Definition 1 Consider a submanifold N of a space-time (M, g) . If there is an affinely parameterized causal geodesic $\gamma : [0, a) \rightarrow M$ which is inextendible in M and incomplete at the value a such that for each value $t \in [0, a)$ the point $\gamma(t)$ exists in N , then we can say that N is causally geodesically incomplete.

Now we shall show that $\bar{V}$ is causally geodesically incomplete without imposing any singularity conditions (Theorem 2) to begin with. On the basis of Theorem 2, we next present a theorem (Theorem 9) which states that V is causally geodesically incomplete under the strong curvature singularity condition. To prove the theorems, for the space-time (M, g) under consideration, we often use the covering manifold $\hat{M}$ defined as the set of all pairs $(p, [\lambda])$ where $[\lambda]$ is an equivalence class of curves from a locally spacelike hypersurface (with no edge) S to $p \in M$ homotopic modulo S and p (see Ref. [8], p. 205). Let $\pi : \hat{M} \rightarrow M$ be a projection. Then, for any point p of M , there exists an open neighborhood U of p such that $\pi^{-1}(U)$ is a collection of open sets of $\hat{M}$, each mapped diffeomorphically onto U by π . For chronology violating set V , a set $\hat{V}$ is defined as the set of all points $\hat{q} \in \hat{M}$ such that $\pi(\hat{q}) \in V$. Similarly, for a subset N in M , $\hat{N}$ denotes hereafter the corresponding subsets $\pi^{-1}(N)$ in $\hat{M}$, unless otherwise explicitly stated.

3.1 Singularity Theorem I

We present the following theorem.

Theorem 2 If a finitely vicious space-time (M, g) with a chronology violating set V satisfies conditions 1, 2, 3, then $\bar{V}$ is causally geodesically incomplete.

To prove this theorem, we give the following proposition and lemmas under conditions 1, 2, 3, provided that $\bar{V}$ is causally geodesically complete.

Proposition 3 There is no causal curve which leaves V and then returns to V .

proof. Suppose that there was a causal curve γ which left V and reentered V , i.e., γ had endpoints $p, r \in V$ and contained a point of $M - V$ between p and r . Let us consider only the case that γ was past-directed from p to r without loss of generality. Then there would be a point $s \in \gamma$ on $\bar{V}$ such that $p > s > r$. Since V can be described as $I^+(p) \cap I^-(p)$, s was in $\dot{I}^+(p)$ or $\dot{I}^-(p)$. Consider the case that $s \in \dot{I}^-(p)$. Because $p \in V$, there would be a closed timelike curve γ_1 through p . Since the causal curve $\gamma_1 + \gamma$ is not a null geodesic, one could obtain a past-directed timelike curve γ_2 from p to s by deformation of $\gamma_1 + \gamma$ following Prop. 4.5.10 of Ref. [8]. This contradicts $s \in \dot{I}^-(p)$. Therefore $s \in \dot{I}^+(p)$. Because $r \in I^+(p)$ [recall that $r \in V = I^+(p) \cap I^-(p)$], there would be a past-directed causal curve γ_3 from r to p . Then there was a past-directed causal curve γ_4 from s to p which consists of a segment of γ from s to r and γ_3 . Then, from Prop. 4.5.10 of Ref. [8], one could obtain a past-directed timelike curve which connects s with p by deformation of $\gamma_4 + \gamma_1$. This is impossible because $s \in \dot{I}^+(p)$. $\square$

Lemma 4 The strong causality condition holds in $\bar{V}$.

proof. The proof is similar to that of Prop. 6.4.6 of Ref. [8]. $\square$

Lemma 5 Let $\hat{K}$ be one of the connected components of $\pi^{-1}(\bar{K})$ and consider a future (past) Cauchy horizon $H^+(\hat{K})$ [resp. $H^-(\hat{K})$]. If $H^+(\hat{K}) \cap \hat{V}$ [resp. $H^-(\hat{K}) \cap \hat{V}$] is non-empty, then each past-(future-)directed null geodesic generator through a point of $H^+(\hat{K}) \cap \hat{V}$ [resp. $H^-(\hat{K}) \cap \hat{V}$] has no past (future) endpoint and remains in $\bar{V}$.

proof. This can be proved by using Prop. 3. $\square$

Lemma 6 *There exists a future-inextendible timelike curve $\hat{\gamma}$ in $\text{int}(D^+(\hat{K}))$ and $J^-(\hat{\gamma}) \cap J^+(\hat{K})$ is in $\hat{V}$.*

proof. This proof is very similar to the proof of Lemma 8.2.1 of Ref. [8]. $\square$

Lemma 7 *There is a future and past-inextendible timelike curve in $\text{int}[D(E^-(\hat{T}))] \cap \hat{V}$.*

proof. We have only to show that there is a past-inextendible timelike curve $\hat{\lambda}$ in $\text{int}[D^-(E^-(\hat{T}))]$ because of Lemma 6.

There is a closed timelike curve λ through a point of $\text{int}(\pi(\hat{T}))$ because $\text{int}(\pi(\hat{T})) \subset V$. This means that there exists a past-inextendible timelike curve $\hat{\lambda} = \pi^{-1}(\lambda)$ in $\hat{V}$ from a point of $\text{int}(\hat{T})$ because $\hat{\lambda}$ is inextendible. Let us consider a Cauchy development $D^-(J^-(\hat{T}))$.

(a) Assume that $\hat{\lambda}$ does not intersect $H^-(J^-(\hat{T}))$. Each point of $\hat{\lambda}$ is in $\text{int}[D^-(J^-(\hat{T}))]$ and hence every future-inextendible causal curve from $\hat{\lambda}$ intersects $J^-(\hat{T})$ and also $I^+(J^-(\hat{T}))$. Then $[J^+(\hat{\lambda}) \cap J^-(\hat{T})] \subset E^-(\hat{T})$ as in the proof of the previous lemma and hence $\hat{\lambda}$ is a desired curve.

(b) Assume that $\hat{\lambda}$ intersects $H^-(J^-(\hat{T}))$. Denote the intersection point and the future-inextendible null geodesic generator of $H^-(J^-(\hat{T}))$ through the point by $\hat{p}$ and $\hat{\xi}_p$, respectively. $\hat{\xi}_p$ is future-inextendible by Prop. 6.5.2 of Ref. [8]. Let us consider a set $\hat{\Sigma} := [I^+(\hat{p}) \cap J^-(\hat{T})]$. By Lemma 8.2.1 of Ref. [8], each null geodesic generator of $\hat{\Sigma}$ is contained in $E^-(\hat{T}) \cap \hat{V}$. If $\hat{\Sigma}$ was non-compact, there would be a past-inextendible null geodesic generator $\hat{\eta}$ of $E^-(\hat{T}) \cap \hat{\Sigma}$ in $\hat{V}$. This is impossible because the null geodesic generator $\hat{\eta}$ has a past endpoint by condition 3, otherwise one could get a past incomplete null geodesic $\hat{\eta}$ in $\hat{V}$ or a future incomplete null geodesic generator of $J^-(\hat{\gamma}, \hat{M})$ in $\hat{V}$ since $J^-(\hat{\gamma}, \hat{M}) \cap J^+(\hat{\gamma}, \hat{M}) \subset \hat{V}$. Therefore $\hat{\Sigma}$ is compact.

(b-I) Assume that $\hat{\xi}_p$ does not intersect $E^-(\hat{T}) \cap \hat{\Sigma}$. It can be shown that $\hat{\xi}_p$ remains in $\hat{V}$ by the same argument in Lemma 5. Then $H^-(\hat{\Sigma})$ is non-compact by the facts that $\hat{\xi}_p \subset (\hat{V} \cap H^-(\hat{\Sigma}))$ and the strong causality condition holds in $\hat{V}$ as shown in the proof of Lemma 6. Then one can get a past-inextendible timelike curve $\hat{\lambda}$ in $\hat{V} \cap \text{int}[D^-(E^-(\hat{T}))]$ because of the compactness of $\hat{\Sigma}$ as in Lemma 6.

(b-II) Assume that $\hat{\xi}_p$ intersects $E^-(\hat{T}) \cap \hat{\Sigma}$. $\hat{\xi}_p$ is a null geodesic generator of $J^-(\hat{T}) - E^-(\hat{T})$ because $\hat{\xi}_p$ is future-inextendible (remind that the future-directed null geodesic generators of $J^-(\hat{T}) - E^-(\hat{T})$ are inextendible and also $\hat{\xi}_p$ and $J^-(\hat{T})$ are achronal). Denote an intersection point of $\hat{\xi}_p \cap [E^-(\hat{T}) \cap \hat{\Sigma}]$ by $\hat{q}$ ($\in \hat{V}$) and take an arbitrary small neighborhood $\hat{U}_q$ of $\hat{q}$ such that $\overline{\hat{U}_q} \subset \hat{V}$. Since $\hat{q} \in [J^-(\hat{T}) \cap H^-(J^-(\hat{T}))]$, there is a future-directed causal curve $\hat{\mu}_0$ from a point $\hat{q}_0$ ($\in \hat{V}$) of $J^-(\hat{T}) \cap [\hat{U}_q - D^-(J^-(\hat{T}))]$ to a point of $\hat{T} \cap \hat{V}$ by a simple argument. Then from Prop. 3, it follows that $\hat{\mu}_0$ is contained in $\hat{V}$. Consider a collection of local causality neighborhoods $\hat{U}_\alpha$ covering $\hat{M}$ as a locally finite atlas and its subcollection $\{\hat{U}_i\}$ which covers $H^-(E^-(\hat{T}))$. As discussed in Lemma 8.2.1 of Ref. [8], $\hat{\mu}_0$ intersects $H^-(E^-(\hat{T}))$ at a point $\hat{r}_1 \in \hat{V}$. Then one can take a neighborhood $\hat{U}_1$ of $\hat{r}_1$ such that $\overline{\hat{U}_1}$ also is in $\hat{V}$. Let $\hat{p}_1$ be a point of $J^-(\hat{T}) \cap [\hat{U}_1 - D^-(J^-(\hat{T}))]$. There is a future-inextendible causal curve $\hat{\lambda}_1$ from $\hat{p}_1$ which does not intersect either $J^-(\hat{T})$ or $D^-(E^-(\hat{T}))$. Let $\hat{q}_1$ be a point on $\hat{\lambda}_1$ not in $\overline{\hat{U}_1}$. Since $\hat{q}_1 \in J^-(\hat{T})$ and $\hat{p}_1 \in \hat{V}$, there is a future-directed causal curve $\hat{\mu}_1$ in $\hat{V}$ which connects $\hat{q}_1$ with a point of $\hat{T} \cap \hat{V}$. By Lemma 4 this curve intersects $H^-(E^-(\hat{T}))$ at a point $\hat{r}_2 \in \hat{V}$ not in $\overline{\hat{U}_1}$. One can take a neighborhood $\hat{U}_2$ of $\hat{r}_2$ such that $\overline{\hat{U}_2}$ also is in $\hat{V}$. Thus one obtain an infinite sequence of points $\hat{r}_n \in \hat{V}$ such that any subsequence of $\{\hat{r}_n\}$ does not converge to any point in $\hat{M}$ by the construction of $\{\hat{r}_n\}$.

Let us consider a future-directed null geodesic generator $\hat{\alpha}_n \in H^-(E^-(\hat{T}))$ from $\hat{r}_n$ ($\hat{\alpha}_m \neq \hat{\alpha}_{m'}$ if $m \neq m'$). There are now two different situations to be taken into account;

(b-II-i) Every $\hat{\alpha}_m$ does not have a future endpoint $\hat{s}_m$. There is a future-inextendible null geodesic generator $\hat{\alpha}_{m_0}$ of $H^-(E^-(\hat{T})) \cap \hat{V}$. Then one can obtain a past-inextendible timelike curve $\hat{\lambda}$ in $\hat{V} \cap \text{int}[D^-(E^-(\hat{T}))]$ by the same way that we obtained it in the case (b-I).

(b-II-ii) Every $\hat{\alpha}_m$ has a future endpoint $\hat{s}_m$. One can get an infinite sequence of points $\hat{s}_m \in [\text{edge}(E^-(\hat{T})) \cap \hat{V}]$ because $\hat{\alpha}_m \subset \hat{V}$ as $\hat{\mu}_n \subset \hat{V}$. Let $\hat{\beta}_m$ be a null geodesic generator of $E^-(\hat{T}) \cap \hat{V}$ whose

past endpoint is $\hat{s}_m$. Take a sequence of future endpoints $\hat{i}_m (\in (\hat{T} \cap \hat{V}))$ of $\hat{\beta}_m$. From the compactness of $\hat{T}$ one can take a subsequence $\{\hat{i}_l\}$ of $\{\hat{i}_m\}$ which converges to a point $\hat{i}$ of $\hat{T} \cap \hat{V}$. By Lemma 6.2.1 of Ref. [8] there exists a limit curve $\hat{\beta}$ through a point $\hat{i}$ which is also a null geodesic generator of $E^-(\hat{T})$. If $\{\hat{s}_l\}$ did not converge, $\hat{\beta}$ would be past incomplete or there was a future incomplete null geodesic generator of $J^-(\hat{\gamma})$ in $\hat{V}$. Thus, $\hat{\beta}$ has a point $\hat{s}$ in $\text{edge}(E^-(\hat{T}))$ which is a limit point of $\{\hat{s}_l\}$. Because each null geodesic generator $\hat{\beta}_l$ is contained in $\hat{V}$, the segment of the limit curve $\hat{\beta}$ from $\hat{s}$ to $\hat{i}$ is contained in $\hat{V}$. Consider an infinite sequence of null geodesics $\hat{\alpha}'_l$ such that each $\hat{\alpha}'_l (\supset \hat{\alpha}_l)$ is past-inextendible in $\hat{M}$ and passes through two points $\hat{s}_l$ and $\hat{r}_l$. By Lemma 6.2.1 of Ref. [8] through $\hat{s}$ there exists a limit null geodesic curve $\hat{\alpha}'$ which is past-inextendible. Since each $\hat{\alpha}_l$ is a null geodesic generator of $H^-(E^-(\hat{T}))$, a portion of $\hat{\alpha}'$ is a null geodesic generator $\hat{\alpha}$ of $H^-(E^-(\hat{T}))$ from $\hat{s}$. Suppose that $\hat{\alpha}$ had a past endpoint in $H^-(E^-(\hat{T}))$. Then one could take a point $\hat{a}$ on $\hat{\alpha} \cap [\hat{M} - H^-(E^-(\hat{T}))]$ and the neighborhood $\hat{U}_a$ of $\hat{a}$ such that $\hat{U}_a \subset [\hat{M} - H^-(E^-(\hat{T}))]$. Since $\hat{a}$ was a limit point of $\{\hat{\alpha}'_l\}$, one could take a point $\hat{a}_l (< \hat{r}_l)$ in $\hat{\alpha}'_l \cap \hat{U}_a$ for l large enough. Then a subsequence of $\{\hat{r}_l\}$ would also converge to a point $\hat{b} (\hat{s} > \hat{b} > \hat{a})$ of $\hat{\alpha}$. This contradicts the fact that any subsequence of $\{\hat{r}_l\}$ has no limit point as shown above. Therefore $\hat{\alpha} (= \hat{\alpha}')$ is past-inextendible in $H^-(E^-(\hat{T}))$. Because $\hat{\alpha}_l \subset \hat{V}$, $\hat{\alpha}$ is contained in $\hat{V}$. Obviously there is a past-inextendible timelike curve $\hat{\lambda}$ from $\text{int}(\hat{T})$ in $I^+(\hat{\alpha}) \cap \text{int}[D^-(E^-(\hat{T}))]$. Then it can be also shown that $\hat{\lambda} \subset \hat{V}$ from Prop. 3 as in the case of $\hat{\mu}_0 \subset \hat{V}$. Thus, the proof of the case (b-II) is complete. $\square$

Proof of theorem 2. By Lemma 7, there is a future and past-inextendible timelike curve $\hat{\rho}$ in $\text{int}[D(E^-(\hat{T}))] \cap \hat{V}$. Then, following the proof of the theorem of Hawking and Penrose (see Ref. [8], p. 269), applied to $\hat{\rho}$, one can complete the proof of Theorem 2.

Take infinite sequences of points $\hat{a}_n, \hat{b}_n$ on $\hat{\rho}$ such that;

- (I) $\hat{a}_{n+1} \in I^-(\hat{a}_n), (\hat{b}_{n+1} \in I^+(\hat{b}_n))$,
- (II) no compact segment of $\hat{\rho}$ contains more than a finite number of the $\hat{a}_n (\hat{b}_n)$,
- (III) $\hat{b}_1 \in I^+(\hat{a}_1)$.

Then there would be a timelike geodesic $\hat{\mu}_n$ of maximum length between $\hat{a}_n$ and $\hat{b}_n$ since $\hat{\rho}$ was contained in the globally hyperbolic set $\text{int}[D(E^-(\hat{T}))] \cap \hat{V}$. Because $\hat{T}$ is compact, there would be an inextendible causal geodesic $\hat{\mu}$ in $J^-(\hat{\rho}) \cap J^+(\hat{\rho}) (\subset \hat{V})$. If $\hat{\mu}$ was future and past complete in $\hat{V}$, there would be conjugate points $\hat{x}$ and $\hat{y}$ on $\hat{\mu}$ with $\hat{y} \in I^+(\hat{x})$. This is a contradiction. $\square$

3.2 Singularity Theorem II

Hereafter, for simplicity, we assume that the strong causality condition is satisfied in $\hat{V}$ (condition 2') because this assumption is reasonable for our considering space-time satisfying condition 2.

We have seen that $\bar{V}$ is causally geodesically incomplete; singularities are formed in $\bar{V}$. If a space-time singularity is caused by a physically realistic process such as gravitational collapse, the curvatures become unboundedly large near the singularity and consequently all objects falling into it are crushed to be zero volume. This is conjectured by Tipler et al. [10] and independently by Królak [11]. They introduced the notion of a *strong curvature singularity* [9, 16]. As a situation that the strong curvature singularity condition holds, we assume that the space-time is *maximal* [12]. In Ref. [12] it is shown that there is no incomplete null geodesic generating an achronal set in a maximal space-time under some physical conditions. From this result we immediately have the following proposition.

Proposition 8 *Let (M, g) be a maximal space-time on which the weak energy condition holds and N a three-dimensional compact acausal set in (M, g) . If there is a point p of $H^+(N) - N$ such that $J^-(p) \cap J^+(N)$ contains a future-inextendible timelike curve λ , then the null geodesic generator of an achronal set $[J^-(p) \cap J^-(\lambda)] \cap J^+(N)$ is future complete.*

Thus, we can present the following theorem by combining Corollary 12 and Lemma 13 below with Theorem 2.

Theorem 9 *If a finitely vicious space-time (M, g) with a chronology violating set V is maximal and satisfies conditions 1, 2', 3, then V is causally geodesically incomplete.*

Proposition 8 leads to the following Corollary.

Corollary 10 *There is no future incomplete null geodesic generator of $\dot{V} \cap J^-(\hat{\gamma}) \cap J^+(\hat{K})$ in a maximal space-time $(\bar{M}, \hat{g})$, where $\hat{\gamma}$ is the future inextendible timelike curve obtained in Lemma 6.*

Lemma 11 *There is a past incomplete null geodesic generator of $J^+(\hat{\beta}) \cap \hat{V}$ if $\hat{\beta}$ is past incomplete, where $\hat{\beta}$ is the null geodesic generator of $E^-(\hat{T})$ in Lemma 7.*

proof. This proof is essentially an extension of Prop. 3 and Prop. 8. $\square$

proof of theorem 11. By Theorem 2, the assumption that $\bar{V}$ was causally geodesically complete is incorrect. Suppose that V was causally geodesically complete but $\bar{V}$ was not. Then, recalling the proof of Theorem 2, one could see that the null geodesic generator of $\dot{J}^-(\hat{\gamma})$ in Lemma 7 was future incomplete or $\hat{\beta}$ in Lemma 7 was past incomplete in $\hat{V}$. By Corollary 10 the former case is impossible. By Lemma 11 the latter case contradicts the supposition that V was causally geodesically complete. $\square$

4 Chronology protection conjecture and causal feature of singularity

In the previous section we showed that there exists an incomplete causal geodesic in V under the some suitable conditions and the assumption that a finitely vicious space-time is maximal.

In this section we shall test the chronology protection conjecture by examining space-times considered in the previous sections, assuming further that the occurring singularities are physically realistic singularities. Here, we shall say that chronology is protected if $V = \emptyset$.

We begin by inspecting exact solutions of the Einstein equation. One of the most well-known solutions containing both singularities and chronology violating sets is the maximally extended Kerr solution, which represents a rotating black hole. (See Ref. [8], for its metric and the maximal extension of the solution.) A conformal diagram of this solution is illustrated in Fig. 2. Chronology violating set is inside the inner horizons and the space-time is finitely vicious since a hypersurface slicing $\{S(\tau)\}$ can be taken so that the intersection of an $S(\tau)$ and the Cauchy horizon $H^+(S)$ for a partial Cauchy surface $S = S(0)$ is compact. As depicted in Fig. 2, this space-time has timelike singularities in the chronology violating set. It can be seen that the singularity is a real, scalar curvature singularity as the curvature scalar polynomial $R_{abcd}R^{abcd}$ diverges there.

From the observation above, one may expect that even in more generic space-time an appearance of chronology violation of the finitely vicious type is accompanied by the formation of timelike (naked) singularities. In other words, from the standpoint of the chronology protection, one may conjecture that if all singularities are spacelike, chronology violation of the type cannot occur. To see that this conjecture is indeed true, we shall introduce the notion of a *non-naked singularity*, which is a generalization of a spacelike singularity observed, for example, in Schwarzschild space-time.

Definition 12 *We shall say that a future (past) incomplete causal geodesic γ terminates at a non-naked singularity if there is a point p on γ such that every future (past)-directed causal curve from p terminates at scalar curvature singularities.*

The non-naked singularity condition conforms to the belief that the formation of naked (timelike) singularities seems unlikely in a generic space-time in connection with the cosmic censorship conjecture [13]. So this condition may be regarded as a seemingly reasonable condition for physically realistic singularities. It should be noted that this condition specifies the causal feature of a singularity in question.

Now, under this condition, we shall give a partial answer to the chronology protection conjecture by establishing the following theorem.

Theorem 13 *If every incomplete causal geodesic terminates at non-naked singularities, there is no maximal space-time (M, g) which contains chronology violating set V satisfying conditions 1, 2', 3.*

proof. Suppose that there were V which satisfied conditions 1, 2', 3. By Theorem 9 there is an incomplete causal geodesic γ in V . Without loss of generality, one can consider γ as a future-directed causal geodesic. Because all the occurring singularities are of the non-naked type, γ has a point p in the definition 12. On the other hand, there would be a closed timelike curve λ through p since p was in the chronology violating set V . This contradicts the fact that all the future-directed causal curves from p terminate at scalar curvature singularities. $\square$.

The non-naked singularity condition in Theorem 13 implies that a version of the cosmic censorship conjecture [13] is correct. Though this conjecture was proved in some seemingly physical cases [16], whether or not this conjecture is correct in a more general context is still an open question. One may say that at least our result suggests that *the basic problems behind chronology protection strongly depend on the causal feature of occurring singularities.*

Acknowledgments

We are grateful to I. Racz for useful discussion. We would also like to thank A. Hosoya and H. Ishihara for their continuous encouragement. The work is supported in part by the Japan Society for Promotion of Science (K.M.).

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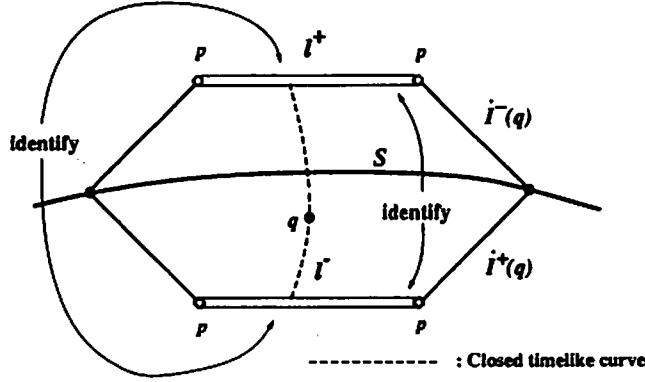


Figure 1: The four-dimensional Deutsch-Politzer time machine M is shown. The hexagon is a chronology violating-region. S is a locally spacelike hypersurface. Each point represents a closed spacelike two-surface Q . The points p , which correspond to the points $(|t| = a, |x| = b)$ in I , are removed.

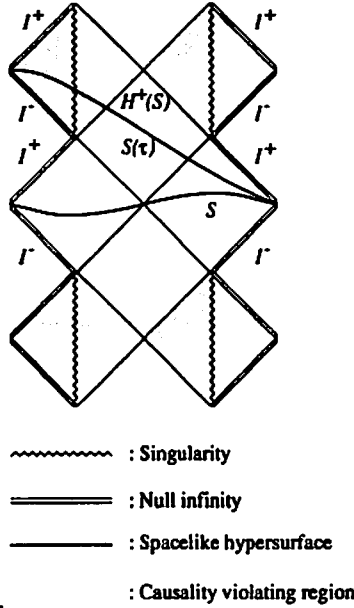


Figure 2: A conformal diagram of a maximally extended Kerr solution along the symmetry axis for non-extreme case. Singularities exist inside the causality violating-regions (the gray regions). Precisely, the zig-zag line in this figure describes the axis passing through, without intersecting, the ring of singularity rather than the singularity itself, since the only causal geodesics reaching the ring singularity are those in the equatorial plane. $S(\tau)$ is a locally spacelike hypersurface and $S(0)$ is a partial Cauchy surface. $H^+(S)$ is a future Cauchy horizon for S .

Generalization of the Lorentzian splitting theorem

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Abstract

We present a new proof of the cosmic no hair conjecture by using global analysis for differential geometry. We also prove the “Lorentzian” warped products splitting theorem, which plays important roles in the proof of this conjecture. It is worth noticing that we do not assume any spacetime symmetry (e.g. homogeneity or the existence of Killing fields).

The *cosmic no hair conjecture* states that the only stationary single-horizon solution to vacuum Einstein's equations with a positive cosmological constant Λ is de Sitter spacetime [17, 7]. This is an important and unsolved problem in the context of the generalization of the black hole uniqueness theorem and of the evidence of naturalness for inflation [18].

As for the uniqueness theorem, it is well known that Einstein manifolds in three dimension have constant sectional curvature [4]. In four dimension, however, it has been proved by Boucher [5] and Boucher, Gibbons and Horowitz [7] that the only stationary solution to vacuum Einstein's equations with a negative cosmological constant is anti-de Sitter spacetime. Unfortunately, their method cannot be directly extended to the case of a positive cosmological constant since their asymptotic structures are different.

The inflation scenario is widely believed that it took place in the early stage of our universe where Λ behaves itself as false vacuum energy. A problem is whether or not de Sitter-like expansion is natural consequence in a spacetime having Λ . Also, in most studies on the inflation, the Robertson-Walker metric is assumed initially by hand. Such models cannot answer the question why the universe is homogeneous and isotropic [22, 18]. Therefore it is necessary to investigate Einstein's equations without assuming any spacetime symmetries. The cosmic no hair conjecture is useful to answer these problems. That is, the cosmic no hair conjecture, if completely true, support that the inflation is very natural and the spacetime region becomes homogeneous and isotropic.

Many authors, using different methods each other, have shown that de Sitter spacetime is stable against small perturbations (cf. [2, 6, 21, 18]). Wald [22] has shown that all Bianchi type (except type IX) spacetime with Λ and with matter fields satisfying the dominant and strong energy conditions always approaches de Sitter spacetime locally within one Hubble expansion time $\sqrt{3/\Lambda}$. Note that the dominant and strong energy conditions are, however, unreasonable for the inflation universe.

If we assume the existence of spacetime symmetries, it is possible to derive uniqueness theorems for solutions to vacuum Einstein's equations with Λ . Actually, Boucher [5] has shown that the only regular, conformally stationary and asymptotically de Sitter solution is de Sitter spacetime. Moreover, Morrow-Jones and Witt [19] have shown that the only stable, locally static solution is locally de Sitter spacetime, if the spacetime has no black hole. These analyses strongly depend on the existence of Killing or conformal Killing field, that is, symmetries of the spacetime. Thus it is a problem whether or not such symmetries are reasonable.

In this paper we present a new proof for the conjecture by using *global analysis of differential geometry*. In particular, *comparison theorems* [10, 13] and *maximum principles* [1, 12] are very useful tools to determine the global structure of spacetime manifolds. One of the most important results derived from these theorems is the *Lorentzian splitting theorem* [11, 14, 20, 15], which states the spacetime being static if the spacetime is timelike geodesically complete (or globally hyperbolic), satisfying the strong energy condition and, furthermore, having a timelike line. That is, this theorem indicates that such a spacetime has a Killing field. We generalize this theorem under weakening the strong energy condition. Indeed, the strong energy condition is not satisfied in the inflation universe. The cosmological constant Λ behaves like negative energy density. For the Riemannian manifold such generalizations have been made by several

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authors [13, 8]. It is called the *warped products splitting theorem*. From results of this theorem we have learned that there exists a conformal Killing field which is not a Killing field. It is well known that a manifold having such a symmetry is locally isometric to a constant curvature manifold if it is an Einstein manifold [16].

Our main purpose is to prove the Lorentzian version of the warped products splitting theorem and to prove the cosmic no hair conjecture by using this theorem. We would like to emphasize that symmetries or global structure are not assumed in our theorem.

Our notation and fundamental definitions are as those of Beem, Ehrlich and Easley [3] unless otherwise mentioned.

Let (M, g) be a *spacetime*, i.e. a n -dimensional, connected and time-oriented Lorentzian manifold where $n \geq 4$. The vacuum Einstein's equation with Λ is $Ric = \frac{2}{n-2}\Lambda g$ and $g(T, T) = -1$ for every timelike unit vector T . Hereafter, we put $\Lambda = \frac{1}{2}(n-1)(n-2)$. By *Lorentzian warped products* we mean that the spacetime splits as $L \times_f N$ where L is a Lorentzian, N is a Riemannian manifold and f is a function on L [3]. If $\dim L = 1$ N is a $(n-1)$ -dimensional spacelike hypersurface. Given a smooth spacelike hypersurface N there exists the future pointing unit normal vector n . *Mean curvature* of N is defined by trace of the second fundamental form $-g(\nabla_X Y, n)$ where X, Y are smooth vector fields tangent to N . Let $\gamma : (-\infty, \infty) \rightarrow M$ be a *timelike line* ($\gamma : [0, \infty) \rightarrow M$ be a *future timelike ray*) which realizes the distance between any two of its points. For $p \in I(\gamma)$ and $s \in (-\infty, \infty)$ such that $p \in I^-(\gamma(s))$, let $\alpha_{p,s}$ be a maximal geodesic connecting p and $\gamma(s)$. If there exists a sequence $\{s_k\}_{k=1}^\infty \subset (-\infty, \infty)$ such that $s_k \rightarrow \infty$, $p \in I^-(\gamma(s_k))$ and $\alpha'_{p,s_k}(0) \rightarrow v \in T_p M$, then the maximal geodesic α_p starting at p in the direction v is called an *asymptote to γ at p* . Let $K \subset TM$ be a compact set of future pointing timelike unit vectors. We call b_γ^+ the *(future) Lorentzian Busemann function* of a future timelike maximal geodesic ray $\gamma_\eta := \exp(r\eta)$ for $r > 0$ and $\eta \in K$ where $b_\gamma^+ := \lim_{r \rightarrow \infty} b_{\gamma,r}^+$ and $b_{\gamma,r}^+ := r - d(x, \gamma_\eta(r))$ for $x \in I^-(\gamma_\eta(r))$. The *Lorentzian horosphere* S_γ^+ is the zero level set of b_γ^+ . Note that Lorentzian horospheres are Lorentzian distance spheres centered at infinity. Similarly, the *(past) Lorentzian Busemann function* is defined by $b_\gamma^- := \lim_{r \rightarrow -\infty} b_{\gamma,r}^-$ and $b_{\gamma,r}^-(x) := -r - d(\gamma_\eta(r), x)$.

First, we review two important theorems which are a comparison theorem and a maximum principle. The Laplacian comparison theorem is what as follows (cf. [13, 10]).

Theorem 1 (Comparison Theorem) Let $\gamma_\eta(r) : (-\infty, 0] \rightarrow M$ be a *past ray* and $x \in I^+(\gamma_\eta(r))$. The mean curvature $H(d(\gamma_\eta(r), x))$ of any smooth part (which is spacelike) of a Lorentzian distance sphere of radius $d(\gamma_\eta(r), x)$ centered at $\gamma_\eta(r)$ in M satisfying $Ric(T, T) \geq -(n-1)$ for every timelike unit vector T is bounded from above by $H(d(\gamma_\eta(r), x)) = \Delta b_{\gamma,r}^-(x) \leq (n-1) \coth(d(\gamma_\eta(r), x))$.

Remarks 1 In our convention, positivity of mean curvature of a Lorentzian distance sphere means that the Lorentzian distance ball is convex in the outward direction. $\square$

Proof of Theorem 1. Note that $\nabla d(\gamma_\eta(r), x)$ is a past directed unit normal vector along the Lorentzian distance sphere. From the Riccati equation [3] we have

$$-\frac{\partial}{\partial f}(\Delta f) + Ric\left(\frac{\partial}{\partial f}, \frac{\partial}{\partial f}\right) + |\text{Hess}(f)|^2 = 0, \quad (1)$$

where $f = d(\gamma_\eta(r), x)$. Using the Cauchy-Schwarz inequality, $|\text{Hess}(f)|^2 \geq (\Delta f)^2/(n-1)$, and $Ric(T, T) \geq -(n-1)$ we get

$$-\frac{\partial}{\partial f}(\Delta f) - (n-1) + \frac{(\Delta f)^2}{n-1} \leq 0. \quad (2)$$

Put $\psi := (n-1)/\Delta f$. Then $\psi' - \psi^2 + 1 \leq 0$ where $'$ is derivative with respect to f . By integrating this inequality we conclude $\Delta d(\gamma_\eta(r), x) \geq -(n-1) \coth(d(\gamma_\eta(r), x))$. Note that $\Delta d(\gamma_\eta(r), x) \rightarrow -\infty$ as $d(\gamma_\eta(r), x) \rightarrow 0$. Thus we have $\Delta b_{\gamma,r}^-(x) \leq (n-1) \coth(d(\gamma_\eta(r), x))$. The equality $H(d(\gamma_\eta(r), x)) = \Delta b_{\gamma,r}^-(x)$ is shown by direct calculation. $\square$

Using *support hypersurfaces* [23], the geometric maximum principle has been formulated and proved by Andersson, Galloway and Howard [1].

Theorem 2 (Lorentzian Geometric Maximum Principle) Let S_1 and S_2 be C^0 spacelike hypersurfaces in a spacetime (M, g) which meet at a point q , such that S_1 is locally to the future of S_2 near q . Assume for some constant H_0 : (1) S_1 has mean curvature $\leq H_0$ in the sense of support hypersurfaces. (2) S_2 has mean curvature $\geq H_0$ in the sense of support hypersurfaces with one-sided Hessian bounds. Then $S_1 = S_2$ near q , i.e., there is a neighborhood U_q of q such that $S_1 \cap U_q = S_2 \cap U_q$. Moreover, $S_1 \cap U_q = S_2 \cap U_q$ is a smooth spacelike hypersurface with mean curvature H_0 . $\square$

Remarks 2 This theorem is a generalization of one given by Eschenburg. In Eschenburg's theorem (Theorem 1b of [12]) one of two spacelike hypersurfaces (S_1 and S_2) is C^2 . $\square$

To use the Lorentzian geometric maximum principle we need support hypersurfaces whose second fundamental form has lower (or upper) bound. The Lorentzian horosphere introduced above have such support hypersurfaces. Indeed, the next lemma follows from a standard argument of the Lorentzian distance sphere (cf. [9, 13, 12, 1]).

Lemma 1 Consider γ_η and S_γ^+ as above. For any $x \in S_\gamma^+ \cap U_{\gamma_\eta(0)}$ where $U_{\gamma_\eta(0)}$ is neighborhood of $\gamma_\eta(0)$ let α_x be the asymptote to γ_η at x . For any sequence $r_j \rightarrow \infty$ put Lorentzian distance spheres $S_{\alpha, r_j}^+ := \{x \in U_{\gamma_\eta(0)} | b_{\alpha, r_j}^+ = 0\}$. Then the following properties hold: (a) $x \in S_{\alpha, r_j}^+$ for every j ; (b) η is normal to S_{α, r_j}^+ at x ; (c) S_{α, r_j}^+ is locally to the future of S_{α, r_i}^+ for $i < j$; (d) S_{α, r_j}^+ is a spacelike hypersurface near x ; (e) S_γ^+ is locally to the future of S_{α, r_j}^+ ; (f) there is a locally uniform lower bound on the second fundamental form of S_{α, r_j}^+ for every j . $\square$

Remarks 3 Properties (a) and (b) are evident. Properties (c) and (e) are shown by the reverse triangle inequality and the reverse Lipschitz inequality, $b_\gamma^+(q) \geq b_\gamma^+(p) + d(p, q)$ ($p \ll q$), respectively. Property (d) directly follows from property (c). Since x is outside the cut locus of $\alpha_x(r_j)$ Property (f) holds. Note that, for small r_1 , a lower curvature bound near x gives us this result [12]. If we consider S_γ^- , there is a upper bound on the second fundamental form of S_γ^- . $\square$

Next lemma is directly derived from Theorem 1 and Lemma 1 (cf. [13, 1]).

Lemma 2 Let (M, g) be a globally hyperbolic spacetime satisfying $\text{Ric}(T, T) \geq -(n-1)$ for every timelike unit vector T and containing a timelike line γ_η . Then there exists a nice neighborhood $U_{\gamma_\eta(0)}$ of $\gamma_\eta(0)$ [11] so that the Lorentzian horosphere $S_\gamma^- := \{x \in U_{\gamma_\eta(0)} | b_\gamma^-(x) = 0\}$ is a C^0 spacelike hypersurface in $U_{\gamma_\eta(0)}$ and S_γ^- has mean curvature $\leq n-1$ in the sense of future support hypersurfaces.

Proof. Note that the set of initial vectors $\{\alpha'_x(0) | x \in U_{\gamma_\eta(0)}\}$ is contained in a compact subset K [11, 14]. Let $x \in S_\gamma^-$ and let $\eta = \alpha'_x(0)$ and $|\alpha'_x(0)| = 1$ where an asymptote α_x at x is a past co-ray of γ_η . For $r \in (-\infty, r_0]$ ($r_0 \leq 0$) let $S_{\alpha, r}^-$ be defined by Lemma 1. Then, as $\exp(r\eta) = \alpha_x(r)$ is a past ray, this implies that $\alpha_x(r)$ maximizes distance on $[r_0 - \delta, 0]$ for some $\delta > 0$. Also the segment $\alpha_x|_{[r, 0]}$ maximizes the distance between $x = \alpha_x(0)$ and $\exp(r\eta) = \alpha_x(r)$ and this segment is free of cut points. This implies the past Lorentzian distance sphere $S_{\alpha, r}^-$ is smooth near the point x [15]. By Theorem 1 the mean curvature $H_x^{S_{\alpha, r}^-}$ of $S_{\alpha, r}^-$ at x satisfies $H_x^{S_{\alpha, r}^-} \leq (n-1) \coth(-r)$. Thus for any $\epsilon > 0$ if we choose $r \in (-\infty, r_0]$ so that $\coth(-r) < 1 + \epsilon$ then $S_{\alpha, r}^-$ will be future support hypersurface for S_γ^- at x with mean curvature $< (n-1)(1 + \epsilon)$ since $S_{\alpha, r}^-$ is in the causal future S_γ^- follows from Lemma 1. Thus we have that the Lorentzian horosphere S_γ^- has mean curvature $\leq n-1$ in the sense of future support hypersurfaces. The proof of Lemma 2 is complete. $\square$

Now we show our main theorem which is the Lorentzian warped products splitting theorem.

Theorem 3 Let (M, g) be a globally hyperbolic spacetime satisfying $\text{Ric}(T, T) \geq -(n-1)$ for every timelike unit vector T and containing a timelike line γ_η . If every acausal hypersurface containing $\gamma_\eta(0)$ has mean curvature $\geq n-1$ in the sense of past support hypersurfaces, then there exists a complete Riemannian manifold (N, h) such that (M, g) is isometric to a Lorentzian warped products $\mathbb{R} \times N$ with metric

$$g = -dt^2 + \exp(2t)h. \quad (3)$$

Remarks 4 The condition which every acausal hypersurface containing $\gamma_\eta(0)$ has mean curvature $\geq n-1$ in the sense of past support hypersurfaces is a physical assumption. This means that the universe is initially expanding since, roughly speaking, the mean curvature of the spacelike hypersurface is regarded as the “local Hubble constant” for an observer on the spacelike hypersurface. $\square$

Proof of theorem 3. If $S_\gamma^\pm := \{x \in U_{\gamma_\eta(0)} | b_\gamma^\pm(x) = 0\}$ then both $S_\gamma^\pm$ are C^0 spacelike hypersurfaces in $U_{\gamma_\eta(0)}$. And, for any point in $U_{\gamma_\eta(0)}$, S_γ^+ has mean curvature $\geq n-1$ in the sense of past support hypersurfaces with one-sided Hessian bounds from the assumption and Lemma 1. Furthermore, S_γ^- has mean curvature $\leq n-1$ in the sense of future support hypersurfaces from Lemma 2. By the reverse triangle inequality

$$b_\gamma^+ + b_\gamma^- \geq 0. \quad (4)$$

Thus, by the Lorentzian geometric maximum principle (Theorem 2), $b_\gamma^+ = b_\gamma^- = 0$ on $\Sigma := S_\gamma^+ = S_\gamma^-$ which is a smooth spacelike hypersurface with mean curvature

$$H_\Sigma = (\Delta b_\gamma^-) = n-1. \quad (5)$$

Because of, for $x \in S_\gamma^-$, $b_\gamma^-(x) = 0$ and hence $b_\gamma^+(x) = b_\gamma^+(x) + b_\gamma^-(x) \geq 0$ by equation (4) and, furthermore, $\{x | b_\gamma^+(x) \geq 0\}$ is in the causal future of S_γ^+ .

It is easy to see that α_x is orthogonal to Σ where α_x is asymptote to γ_η start at $x \in \Sigma$. Σ is smooth and the support hypersurfaces are orthogonal to α_x at x . Therefore Σ has the same tangent spaces as any of its support hypersurfaces at a contact point. Furthermore, we can see that at points $x \in \Sigma$ the asymptote to γ_η from x is unique and is given by $\alpha_x(t) = \exp(t\alpha'_x)$ for $0 \leq t < \infty$. From the Lipschitz inequality and the equality $b_\gamma^+(\alpha_x(t)) = b_\gamma^+(x) + t$ for $0 \leq t < \infty$, we have that the ray α_x maximizes the distance between any of its points $\alpha_x(t) = \exp(t\alpha'_x)$ and Σ for $0 \leq t < \infty$. This means that for $x \in \Sigma$ every α_x is a Σ -ray [3, 15]. Note that Σ has no focal points along α_x . Similar results are also satisfied for $-\infty < t \leq 0$. Thus the map $\Phi : \Sigma \times (-\infty, \infty) \rightarrow M$ by $\Phi(x, t) = \exp(t\alpha'_x(0))$ is non-singular where $\alpha'_x(0)$ is normal to Σ .

Next we will show that, for $x_1, x_2 \in U_{\gamma_\eta(0)}$, a asymptote α_{x_1} to γ_η has no intersection with another asymptote α_{x_2} to γ_η , that is, the map $\Phi|_{\Sigma \times (-\infty, \infty)}$ is injective. It suffices to show that $\Phi(x_1, s) \neq \Phi(x_2, t)$ for $0 \leq s, t < \infty$ if $x_1 \neq x_2$. Assume $\Phi(x_1, s) = \Phi(x_2, t)$ with $s, t > 0$. Since the second argument of Φ is the distance from Σ to its point, we have $s = t$. Now suppose $x_1 \neq x_2$ but $\Phi(x_1, s) = \Phi(x_2, s)$. Then choose $s_1 \in (s, \infty)$ and let β be the broken geodesic from x_1 to $\alpha_{x_2}(s_1)$ formed by a segment of α_{x_1} from x_1 to $\alpha_{x_1}(s) = \alpha_{x_2}(s)$ and a segment of α_{x_2} from $\alpha_{x_1}(s) = \alpha_{x_2}(s)$ to $\alpha_{x_2}(s_1)$. The length of β is s_1 . Also we can find a curve β_1 near β that has length greater than s_1 by a standard argument of smoothing the corner. But this contradicts that α_{x_2} realizes the distance between Σ and its point $\alpha_{x_2}(s_1)$. For $-\infty < t \leq 0$ we have the same result. Thus $\Phi|_{\Sigma \times (-\infty, \infty)}$ is injective.

Now since $\nabla b_\gamma^- = \partial/\partial t$ which is the future directed unit tangent vector field to the normal geodesics of Σ , Raychaudhuri equation [3] is

$$-\nabla b_\gamma^-(\Delta b_\gamma^-) = Ric(\nabla b_\gamma^-, \nabla b_\gamma^-) + |\text{Hess} b_\gamma^-|^2. \quad (6)$$

From equations (5), (6), the Cauchy-Schwarz inequality and $Ric(\nabla b_\gamma^-, \nabla b_\gamma^-) \geq -(n-1)$ we have

$$\text{Hess} b_\gamma^-(X, Y) = g(X, Y), \quad (7)$$

for tangent vectors X, Y to Σ . Note that Σ is a totally umbilic hypersurface. It follows that $g = -dt^2 + \exp(2t)h|_\Sigma$.

Recall that the map $\Phi|_{\Sigma \times (-\infty, \infty)}$ is non-singular and injective. Therefore the set $M_\Sigma := \{\exp(t\alpha'_x(0)) | x \in \Sigma, -\infty < t < \infty\}$ has a Lorentzian warped products metric as equation (3).

This Lorentzian local warped products splitting theorem may be globalized by precisely the same arguments as used by Eschenburg [11]. Finally, by Theorem 3.66 of Ref. [3], (N, h) is a complete Riemannian manifold since (M, g) is globally hyperbolic. Thus the proof of Theorem 3 is complete. $\square$

Now we get the *cosmic no hair theorem* as follows.

Theorem 4 *Let (M, g) be a globally hyperbolic spacetime satisfying $\text{Ric}(T, T) = -\frac{2}{n-2}\Lambda$ for every timelike unit vector T where $\Lambda = \frac{1}{2}(n-1)(n-2)$ and containing a timelike line γ_η . If every acausal hypersurface containing $\gamma_\eta(0)$ has mean curvature $\geq n-1$ in the sense of past support hypersurfaces, then the spacetime is locally isometric to de Sitter spacetime.*

Proof. From Theorem 3 this spacetime splits as equation (3). Therefore there exists obviously a conformal Killing field which is not a Killing field on M . Also, the spacetime is an Einstein manifold. Consequently, the spacetime is locally isometric to de Sitter spacetime by the theorem of Ref. [16]. This completes the proof of Theorem 4. $\square$

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[23] For a constant H_0 a C^0 spacelike hypersurface S has mean curvature $\leq H_0$ ($\geq H_0$) in the sense of future (past) support hypersurfaces iff for all $q \in N$ and $\epsilon > 0$ there is a C^2 future (past) support hypersurface $F_{q,\epsilon}(P_{q,\epsilon})$ to N at q and the mean curvature of $F_{q,\epsilon}(P_{q,\epsilon})$ at q satisfies $H_q^{F_{q,\epsilon}} \leq H_0 + \epsilon$. ($H_q^{P_{q,\epsilon}} \geq H_0 - \epsilon$.) Here, we say that F is a future support hypersurface for S at $x \in S$ iff $x \in F$ and F is locally to the future of S near x . A past support hypersurface is time dual. Furthermore, S has mean curvature $\leq (\geq) H_0$ in the sense of support hypersurfaces with one-sided Hessian bounds iff for all compact sets $K \subseteq S$ there is a compact set $\hat{K} \subset TM$ and a constant $C_K > 0$ such that for all $q \in K$ and $\epsilon > 0$ there is a C^2 future (past) support hypersurface $F_{q,\epsilon}(P_{q,\epsilon})$ to S so that: (1) the future pointing unit normal $n^{F_{q,\epsilon}}(q)$ ($n^{P_{q,\epsilon}}(q)$) to $F_{q,\epsilon}(P_{q,\epsilon})$ at q is in $\hat{K}$, (2) at the point q the mean curvature $H_q^{F_{q,\epsilon}}$ ($H_q^{P_{q,\epsilon}}$) and second fundamental form $h^{F_{q,\epsilon}}$ ($h^{P_{q,\epsilon}}$) of $F_{q,\epsilon}(P_{q,\epsilon})$ satisfy $H_q^{F_{q,\epsilon}} \leq H_0 + \epsilon$ ($H_q^{P_{q,\epsilon}} \geq H_0 - \epsilon$), and $h_q^{F_{q,\epsilon}} \leq C_K g|_{F_{q,\epsilon}}$ ($h_q^{P_{q,\epsilon}} \geq -C_K g|_{P_{q,\epsilon}}$).

Fate of a Geon

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Abstract

An attempt to investigate the fate of a geon is presented. First, the theory of the geon is reviewed briefly. New solutions are also found numerically. The linear perturbation equations around the geon solution are derived. The stability analysis is leaved for the future work.

1 Introduction

A geon is a classical configuration trapped by own gravitational potential. The electro-magnetic geon was found by Wheeler in 1955 [1] as a non-singular solution of the Einstein equations. Recently, the case of gravitational geon has been discussed by Cooperstock et. al. [2] and Anderson and Brill. [3] As for the gravitational case, the geon solution depends on the choice of the variables. However, the qualitative behavior of the solutions is similar to that of the electro-magnetic case. In the case of the scalar field, the final equations are also the same as the electro-magnetic case. Thus, the geon solution exists irrespective of the type of the classical field. On the other hand, it is known that these solutions are unstable. [4] Hence, it is interesting to know the fate of the geon. As the trapped region is near the Schwarzschild horizon, it is reasonable to imagine the geon will collapse and form the black hole. Of course, the geon might be erased. This situation remind us the critical phenomena in black hole formation. So far, this kind of phenomena is investigated in the context of the spherical symmetric system. It should be stressed that the geon system is not spherically symmetric. To settle the above issue, we will look for the time dependent solution that include the geon solution as a static solution. For simplicity, we will consider the case of the scalar field here without losing the generality.

The organization of this paper is as follows. In sec.2, the geon theory is reviewed. And, new solutions are found numerically. In sec.3, the dynamics of the geon is studied. The final section is devoted to the conclusion.

2 The Static Geon

To study the geon, we consider as follows. First, the classical waves are trapped by own gravitational potential, and go around the the great circle of spherical surface. These waves are consisted by counter-clockwise wave and clockwise wave. And the waves make the standing wave (see Fig.1(left)). For simplicity, we will think the spherical symmetric solution (see Fig.1(right)). It is expected that the geon solution can be found as in Fig.2. So we consider the static and spherical symmetric background:

$$ds^2 = -e^{\nu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1)$$

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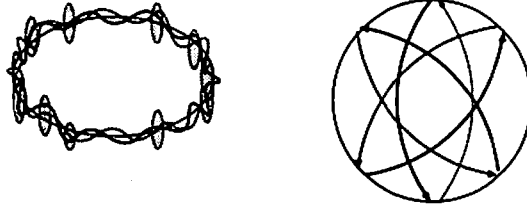


Figure 1: Geon

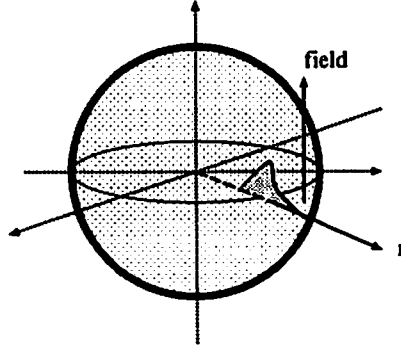


Figure 2: Geon

And we consider the scalar field in the above background. So we can separate the form of scalar field to time, radial, θ and φ part. As for θ and φ parts, we can use the spherical harmonic Y_{lm} . Then, the scalar field ϕ can be represented as follows

$$\phi = \frac{\bar{f}(r)}{r} Y_{lm} \sin \omega t \quad (2)$$

The variation of the action with respect to the scalar field ϕ gives the wave equation. First, we substitute the form of ϕ into the wave equation and introduce the tortoise coordinate $dr^* \equiv e^{\frac{\lambda-\nu}{2}} dr$ and new variables given by

$$\rho = \omega r, \quad e^{-\lambda} \equiv 1 - \frac{2L(\rho)}{\rho}, \quad e^{\lambda+\nu} \equiv Q(\rho)^2, \quad e^\nu = \left[1 - \frac{2L(\rho)}{\rho} \right] Q(\rho)^2, \quad d\rho^* \equiv Q^{-1} \left[1 - \frac{2L(\rho)}{\rho} \right]^{-1} d\rho. \quad (3)$$

And, the normalization $f \equiv \left(\frac{GN}{2} \right)^{\frac{1}{2}} \omega \bar{f}$ simplifies the Einstein equations. The above transformation of variables yields the wave equation as follows:

$$\frac{d^2 f}{d\rho^{*2}} + \left(1 - \left(\frac{Ql}{\rho} \right)^2 \left(1 - \frac{2L}{\rho} \right) \right) f - \frac{2Q}{\rho^3} \left(1 - \frac{2L}{\rho} \right) \left(2QL - \frac{Q^2 l^2}{\rho^2} \right) f = 0 \quad (4)$$

The energy-momentum tensor is given by the vibration of the action. Now, let us consider the trapped field on the spherical surface. We have to eliminate the time dependence and make the energy-momentum tensor spherically symmetric like the background space-time. First, we eliminate the time dependence of the energy-momentum tensor. For the periodicity of the field, we can give the time-averaged energy-momentum tensor by integrating it for a period of the wave. Next, we shall rotate the standing wave like the image in Fig.3. Here is the section including the standing wave around the great circle.

We perform the angular averaging:

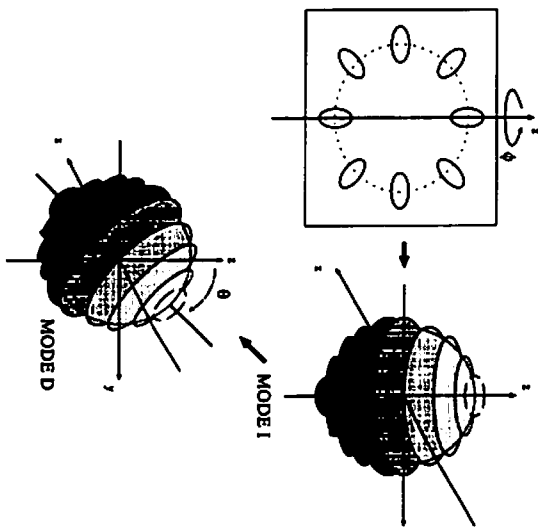


Figure 3:

(MODE I)

$$< X >_I$$

First, we take the axis going through the center of the great circle which includes the standing wave and rotate the waves around this axis. By this operation, we can eliminate the dependence of the rotational direction.

(MODE D)

$$< X >_D$$

In the next step, we rotate this axis. Thus, we can eliminate the dependence of the another rotational direction. So we have the spherical symmetry for the energy-momentum tensor.

Now, we can choose the axis as z and the rotational direction as θ , φ as in Fig.3. Consequently, the averaging procedure becomes as follows

$$< X >_D = \frac{N}{4\pi} \int < X >_I \sin \theta d\theta. \quad (5)$$

Here, N denotes the number of active mode and the normalization is given by $\int Y_{lm}^* Y_{lm} \sin \theta d\theta = \delta_{ll} \delta_{mm}$. After this averaging operation, we use the variables same as in case of the wave equation. Then, the Einstein equation becomes

$$\frac{dL}{d\rho^2} = \frac{1}{2Q} \left[\left(\frac{df}{d\rho^2} \right)^2 - \frac{2Q}{\rho} \left(1 - \frac{2L}{\rho} \right) f \frac{df}{d\rho^2} + \frac{Q^2}{\rho^2} \left(1 - \frac{2L}{\rho} \right)^2 + \left(\frac{Q'}{\rho} \right)^2 \left(1 - \frac{2L}{\rho} \right) f^2 \right] \quad (6)$$

$$\frac{dQ}{d\rho^2} = \frac{1}{\rho} \left(1 - \frac{2L}{\rho} \right)^{-1} \left[\left(\frac{df}{d\rho^2} \right)^2 - \frac{2Q}{\rho} \left(1 - \frac{2L}{\rho} \right) f \frac{df}{d\rho^2} + \left(1 + \frac{Q^2}{\rho^2} \left(1 - \frac{2L}{\rho} \right)^2 \right) f^2 \right]. \quad (7)$$

The equation of $\frac{dL}{d\rho^2}$ determines the distribution of the mass for the radial distance ρ^2 .

We will consider the limit of large l . In this limit, the relevant quantities can be expanded in the inverse powers of $l^{\frac{1}{2}}$. By considering the consistency, the expansion of these variables are given by

$$\rho = l^* + l^{\frac{1}{2}} r_0(x) + \dots \quad (8)$$

$$L = l \lambda_0(x) + l^{\frac{3}{2}} \lambda_1(x) + \dots \quad (9)$$

$$Q = \frac{1}{k(x)} + l^{-\frac{1}{2}}q_1 + \dots \quad (10)$$

$$f = l^{\frac{1}{2}}\phi + \phi_1 + \dots \quad (11)$$

$$x \equiv (\rho^* - l)l^{-\frac{1}{2}} \quad (12)$$

$$\frac{d}{d\rho^*} = l^{-\frac{1}{2}} \frac{d}{dx} \quad (13)$$

So we substitute this expansion into the equations and see the leading order of l . We can define the new variable j as follows

$$\left[1 - \left(\frac{l^*Q}{\rho}\right)^2 \left(1 - \frac{2L}{\rho}\right)\right] \equiv j(x)k(x)l^{-\frac{3}{2}} + \dots \quad (14)$$

Then, the wave and Einstein equations become the following three equations

$$\frac{d^2\phi}{dx^2} + j(x)k(x)\phi(x) = 0 \quad (15)$$

$$\frac{dk}{dx} = -\phi(x)^2 \quad (16)$$

$$\frac{dj}{dx} = 3 - \frac{1}{k(x)^2} \left[1 - \left(\frac{d\phi}{dx}\right)^2\right] \quad (17)$$

These equations for ϕ , j and k form the closed system and the problem becomes the eigenvalue problem. Moreover, in the case of the electro-magnetic or gravitational wave, if the appropriate variables are taken, the final equations are the same as these three equations. We have solved the above equations numerically. These graphs represent the field ϕ as the function of the radial distance.

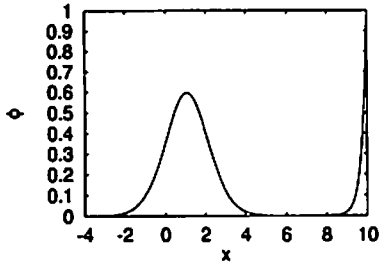


Figure 4: $\phi_0 = 0.669428385$

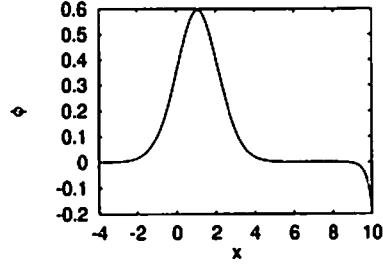


Figure 5: $\phi_0 = 0.66942838$

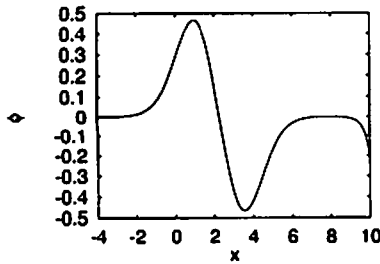


Figure 6: $\phi_0 = 0.8119509383$

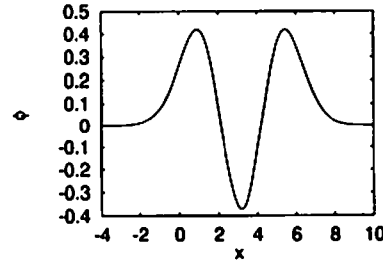


Figure 7: $\phi_0 = 0.7413322907$

3 The Dynamical Geon

To investigate the time dependent case, we take spherically symmetric and time dependent background spacetime. The metric is given by

$$ds^2 = -e^{\nu(r,t)} dt^2 + e^{\lambda(r,t)} dr^2 + r^2 d\Omega^2 = h(r,t)(-dt^{*2} + dr^{*2}) + r^2 d\Omega^2 \quad (18)$$

The * stuck coordinates are conformally flat coordinates. The form of scalar field is written as

$$\Phi = \frac{f(r,t)}{r} Y_{l,m} \sin(\omega t + \delta(r,t)) . \quad (19)$$

It should be noted that $f(r,t)$ depends on spatial and time coordinates and the phase $\delta(r,t)$ is also introduced.

In the static case, the tortoise coordinate was useful. In the present case, the correspondent is the conformal coordinate system. By using the following notation

$$t^* \equiv x^1, \quad r^* \equiv x^2, \quad t \equiv x^1 \equiv \alpha(\bar{x}^1, \bar{x}^2), \quad r \equiv x^1 \equiv \beta(\bar{x}^1, \bar{x}^2), \quad (20)$$

the metric can be written as

$$\begin{aligned} ds^2 &= -e^{-\nu} dx_1^2 + e^{-\lambda} dx_2^2 = h^{-1}(-d\bar{x}_1^2 + d\bar{x}_2^2) \\ &= \frac{1}{h}(\alpha_{,2}^2 - \alpha_{,1}^2) dx_1^2 + \frac{1}{h}(\beta_{,2}^2 - \beta_{,1}^2) dx_2^2 + \frac{2}{h}(\alpha_{,2}\beta_{,2} - \alpha_{,1}\beta_{,1}) dx_1 dx_2 . \end{aligned} \quad (21)$$

The relations between α and β are given by

$$\beta_{,1} = e^{(\frac{\nu-\lambda}{2})} \alpha_{,2}, \quad \beta_{,2} = e^{(\frac{\nu-\lambda}{2})} \alpha_{,1} \quad (22)$$

By substituting the the above relations between α and β into the integrability condition $(\beta_{,1})_{,2} = (\beta_{,2})_{,1}$, we get the equation for α

$$(e^{(\frac{\nu-\lambda}{2})} \alpha_{,2})_{,2} - (e^{(\frac{\nu-\lambda}{2})} \alpha_{,1})_{,1} = 0 \quad (23)$$

Furthermore, the inverse transformation becomes

$$\begin{pmatrix} \frac{dt^*}{dt} & \frac{dr^*}{dt} \\ \frac{dt^*}{dr} & \frac{dr^*}{dr} \end{pmatrix} = \begin{pmatrix} \frac{1}{h} e^{\nu} \alpha_{,1} & -\frac{1}{h} e^{\nu} \alpha_{,2} \\ -\frac{1}{h} e^{\frac{\nu+\lambda}{2}} \alpha_{,2} & \frac{1}{h} e^{\frac{\nu+\lambda}{2}} \alpha_{,1} \end{pmatrix} . \quad (24)$$

In the non-static case, the method for deriving the wave and Einstein equations are basically same as that of the static case. But we do not know the explicit form of coordinate transformation, therefore we change the equations by using the relation of coordinate transformation.

We expand the variables in the inverse powers of $l^{\frac{1}{2}}$ here. However, now we must expand the time t and transform t to a new variable τ . Moreover, we must expand the new variables δ and h by considering the consistency of the equations as

$$h = h_1(x, \tau) + l^{-\frac{1}{2}} h_2(x, \tau) \quad (25)$$

$$\delta = l^{\frac{1}{2}} \delta_1 + \dots \quad (26)$$

$$t = l + l^{\frac{1}{2}} y_0(x, \tau) + \dots \quad (27)$$

$$\frac{d}{dt^*} = l^{-\frac{1}{2}} \frac{d}{d\tau} \quad (28)$$

The coefficients of the expansion are functions of x and τ in this case.

We expand the sin part of the wave equation and see the leading order $\mathcal{O}(l^0)$, $\mathcal{O}(l^{-\frac{1}{2}})$, $\mathcal{O}(l^{\frac{1}{2}})$ of this equation:

$$\mathcal{O}(l^{\frac{1}{2}}) : h_1 = \left(\frac{\partial y_0}{\partial \tau} + \frac{\partial \delta}{\partial \tau} \right)^2 + \left(\frac{\partial y_0}{\partial x} + \frac{\partial \delta}{\partial x} \right)^2 \quad (29)$$

$$\mathcal{O}(l^0) : h_2 = 0 \quad (30)$$

And we can define the new variable j as follows

$$\left[\left(\frac{\partial t}{\partial t^*} + \frac{\partial \delta}{\partial t^*} \right)^2 - \left(\frac{\partial t}{\partial r^*} + \frac{\partial \delta}{\partial r^*} \right)^2 - \frac{h}{r^2} l(l+1) \right] \equiv l^{-\frac{3}{2}} j(x, \tau) k(x, \tau) + \dots \quad (31)$$

So, we get the equation for ϕ .

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial \tau^2} + j k \phi = 0 \quad (32)$$

The leading order of cos part gives the equation for δ :

$$2 \frac{\partial \phi}{\partial \tau} \left(\frac{\partial y_0}{\partial \tau} + \frac{\partial \delta_1}{\partial \tau} \right) + 2 \frac{\partial \phi}{\partial x} \left(\frac{\partial y_0}{\partial x} + \frac{\partial \delta_1}{\partial x} \right) + 2 \phi \left(\frac{\partial^2 y_0}{\partial x^2} + \frac{\partial^2 \delta_1}{\partial x^2} - \frac{\partial^2 y_0}{\partial \tau^2} - \frac{\partial^2 \delta_1}{\partial \tau^2} \right) = 0 \quad (33)$$

On the other hand, by using the relation for h_1 and expanding the coordinate transformation law, we get the following equations:

$$\frac{1 - 2\lambda_0}{k^2} \left(\frac{\partial y_0}{\partial \tau} \right)^2 = \left(\frac{\partial y_0}{\partial \tau} + \frac{\partial \delta_1}{\partial \tau} \right)^2 - \left(\frac{\partial y_0}{\partial x} + \frac{\partial \delta_1}{\partial x} \right)^2 \quad (34)$$

$$\left(\frac{\partial y_0}{\partial x} \frac{\partial}{\partial x} - \frac{\partial y_0}{\partial \tau} \frac{\partial}{\partial \tau} \right) \frac{1 - 2\lambda_0}{k} + \frac{1 - 2\lambda_0}{k} \left(\frac{\partial^2 y_0}{\partial x^2} - \frac{\partial^2 y_0}{\partial \tau^2} \right) = 0 \quad (35)$$

As for the Einstein equation, we use these expansion and take their leading order. Then, we introduce the operator $\lambda \equiv \frac{\partial y_0}{\partial \tau} \frac{\partial}{\partial x} - \frac{\partial y_0}{\partial x} \frac{\partial}{\partial \tau}$ that corresponds to $\frac{\partial}{\partial \tau}$.

Thus, the final equations are given by

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial \tau^2} + j k \phi = 0 \quad (36)$$

$$\Lambda k = -\frac{k^2}{1 - 2\lambda_0} \phi^2 \left(\left(\frac{\partial y_0}{\partial \tau} \right)^2 - \left(\frac{\partial y_0}{\partial x} \right)^2 \right)^{-1} \left[\left(\left(\frac{\partial y_0}{\partial \tau} \right)^2 + \left(\frac{\partial y_0}{\partial x} \right)^2 \right) (\tilde{\Omega}_1^2 + \tilde{\Omega}_2^2) + 12 \frac{\partial y_0}{\partial \tau} \frac{\partial y_0}{\partial x} \Omega_1 \Omega_2 \right] \quad (37)$$

$$\begin{aligned} \Lambda j = & \frac{P}{k} + \frac{2}{k^4} (1 - 2\lambda_0)(1 - 3\lambda_0) - \frac{1}{k^2} \left[(\alpha_{,1}^2 + \alpha_{,2}^2) \left(\left(\frac{\partial \phi}{\partial \tau} \right)^2 + \left(\frac{\partial \phi}{\partial x} \right)^2 \right) + 12 \alpha_{,1} \alpha_{,2} \frac{\partial \phi}{\partial \tau} \frac{\partial \phi}{\partial x} \right] \\ & - j \phi^2 \left[1 - \frac{k^2}{1 - 2\lambda_0} (\alpha_{,1}^2 - \alpha_{,2}^2) \left((\alpha_{,1}^2 + \alpha_{,2}^2) (\Omega_1^2 + \Omega_2^2) + 4 \alpha_{,1} \alpha_{,2} \Omega_1 \Omega_2 \right) \right] \end{aligned} \quad (38)$$

The last equation includes the higher order term represented by P . For simplicity, we have defined the variables:

$$\Omega_1 \equiv \frac{\partial t}{\partial t^*} + \frac{\partial \delta}{\partial t^*}, \quad \Omega_2 \equiv \frac{\partial t}{\partial r^*} + \frac{\partial \delta}{\partial r^*}, \quad \tilde{\Omega}_1 \equiv \frac{\partial y_0}{\partial \tau} + \frac{\partial \delta_1}{\partial \tau}; \quad \tilde{\Omega}_2 \equiv \frac{\partial y_0}{\partial x} + \frac{\partial \delta_1}{\partial x} \quad (39)$$

It is easy to see that the above equations are closed, and these equations are reduced to the original geon equations when we take the static limit. Therefore, these equations are considered as the generalization of the geon equations.

It is complicated to analyze the full equations of motion. However, what we need is to know the behavior near the geon configuration. For that purpose, we need to linearize the equations of motion which we have obtained above. If we take the terms which is necessary when we do the linearization, we

obtain the following equations:

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial \tau^2} + jk\phi = 0 \quad (40)$$

$$\frac{\partial k}{\partial x} = -\phi^2 \left(\frac{\partial y_0}{\partial \tau} \right) \quad (41)$$

$$\frac{\partial j}{\partial x} = \left(\frac{\partial y_0}{\partial \tau} \right)^{-1} \frac{2}{k^2} (1 - 2\lambda_0)(1 - 3\lambda_0) - \frac{1}{k^2} \left(\frac{\partial y_0}{\partial \tau} \right) \left(\frac{\partial \phi}{\partial x} \right)^2 - \frac{j}{k} \phi^2 \left(\frac{\partial y_0}{\partial \tau} \right)^{-1} \left(1 - \left(\frac{\partial y_0}{\partial \tau} \right)^2 \right) \quad (42)$$

$$\frac{1 - 2\lambda_0}{k^2} \left(\frac{\partial y_0}{\partial \tau} \right) = \frac{\partial y_0}{\partial \tau} + 2 \frac{\partial \delta_1}{\partial \tau} \quad (43)$$

$$\left(\frac{\partial y_0}{\partial x} \frac{\partial}{\partial x} - \frac{\partial y_0}{\partial \tau} \frac{\partial}{\partial \tau} \right) \frac{1 - 2\lambda_0}{k} + \frac{1 - 2\lambda_0}{k} \left(\frac{\partial^2 y_0}{\partial \tau^2} - \frac{\partial^2 y_0}{\partial x^2} \right) = 0 \quad (44)$$

It is easy to linearize the above equations. Hence, the remaining task is to solve the eigen value problem. We will report the results soon.

4 Conclusion

In the first part of this paper, we reviewed the geon theory and presented the new kind of solutions of the geon equations. It is not so trivial to have many eigen functions for the non-linear eigen value problem. From the numerical experience, we guess there are infinite number of geons with the same mass. The physical implication of this result is not apparent. In the second part, we formulated the stability analysis of the geon system. The final equations are rather simple. We will publish the results elsewhere.

Although we have not yet obtained the final results, we regard our study important from the following points of view. It is a possible analytic method for treating the non-spherically symmetric system. It might be related to the critical phenomena in black hole formation. It might give some insight to the no-hair conjecture or hoop conjecture.

Acknowledgements

We would like to thank Masaaki Sakagami for helpful comments.

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Neutron Stars in Scalar-Tensor Theories of Gravity and Catastrophe Theory

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Abstract

We investigate neutron stars in scalar-tensor theories. We examine their secular stability against spherically symmetric perturbations by use of a turning point method. For some choices of the coupling function contained in the theories, the number of the stable equilibrium solutions changes and the realized equilibrium may change discontinuously as the asymptotic value of the scalar field or total baryon number is changed continuously. The behaviour of the stable equilibrium solutions is explained by fold and cusp catastrophes. Whether the cusp catastrophe appears or not depends on the choices of the coupling function. These types of the catastrophes are structurally stable. Recently discovered spontaneous scalarization, which is non-perturbative strong-field phenomenon due to the presence of the gravitational scalar field, is well described in terms of the cusp catastrophe.

1 Introduction

The scalar-tensor theories [1, 2] are viable theories of gravity for some choices of the coupling function which is contained in the theories. Predictions of these theories in a strong field may be drastically different from those of general relativity. Recently, Damour and Esposito-Farèse [3] discovered one example of such phenomena. They showed that, for some choices of the coupling function, the configuration of a massive neutron star deviates significantly from that in general relativity, even if the post-Newtonian limit of the theory is extremely close to or even agrees with that of general relativity. This deviation in a strong field may be easily tested from binary-pulsar timing observations, if it exists, because of the extra energy loss by scalar gravitational radiation [3]. The deviation from general relativity can be no longer dealt with as a perturbative effect from general relativity. Damour and Esposito-Farèse referred this non-perturbative strong field effect as “spontaneous scalarization” on the analogy of the spontaneous magnetization of the ferromagnets.

We use the units in which $c = 1$. The Greek indices run over 0 to 3. We follow the MTW [5]’s sign conventions for curvature quantities.

2 Basic Equations

Here we consider a class of scalar-tensor theories in which gravity is mediated by not only a metric tensor but also a massless scalar field. The action is given by [2]

$$I = \frac{1}{16\pi G_*} \int \sqrt{-g_*} (R_* - 2g_*^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu}) d^4x + I_m[\Psi_m, A^2(\varphi)g_{*\mu\nu}], \quad (1)$$

where $g_{*\mu\nu}$ is the ‘Einstein’ frame metric tensor, Ψ_m denotes matter fields collectively and G_* is some dimensionful constant. In this Einstein frame, the Einstein-Hilbert term is isolated from other sectors. The ‘Brans-Dicke’ frame metric tensor $\tilde{g}_{\mu\nu}$ is related to the Einstein frame metric tensor by the following conformal transformation:

$$\tilde{g}_{\mu\nu} = A^2(\varphi)g_{*\mu\nu}. \quad (2)$$

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In the Einstein frame, the field equations are given by

$$G_{*\mu\nu} = 8\pi G_* T_{*\mu\nu} + 2 \left(\varphi_{*,\mu} \varphi_{*,\nu} - \frac{1}{2} g_{*\mu\nu} g_*^{\alpha\beta} \varphi_{*,\alpha} \varphi_{*,\beta} \right), \quad (3)$$

$$\square_* \varphi = -4\pi G_* \alpha(\varphi) T_*, \quad (4)$$

while the equations of motion for matter are

$$\nabla_{*\nu} T_{*\mu}^{\nu} = \alpha(\varphi) T_* \nabla_{*\mu} \varphi, \quad (5)$$

where the energy-momentum tensor of the matter, $T_*^{\mu\nu}$, is defined and related to the physical energy-momentum tensor $\tilde{T}^{\mu\nu}$ as

$$T_*^{\mu\nu} \equiv \frac{2}{\sqrt{-g_*}} \frac{\delta I_m[\Psi_m, A^2(\varphi) g_{*\mu\nu}]}{\delta g_{*\mu\nu}} = A^6(\varphi) \tilde{T}^{\mu\nu}. \quad (6)$$

$G_{*\mu\nu}$ and $\square_*$ are the Einstein tensor and d'Alembertian of $g_{*\mu\nu}$, respectively. T_* and $\alpha(\varphi)$ are defined as

$$T_* \equiv T_{*\mu}^{\mu} \equiv T_*^{\mu\nu} g_{*\mu\nu}, \quad (7)$$

$$\alpha(\varphi) \equiv \frac{d \ln A(\varphi)}{d\varphi}. \quad (8)$$

The parameters in the parametrized post-Newtonian framework are given by [1, 2]

$$1 - \gamma_{Edd} = \frac{2\alpha_0^2}{1 + \alpha_0^2}, \quad \beta_{Edd} - 1 = \frac{\beta_0 \alpha_0^2}{2(1 + \alpha_0^2)^2}, \quad (9)$$

$$\xi = \alpha_1 = \alpha_2 = \alpha_3 = 0, \quad (10)$$

where β_{Edd} and γ_{Edd} are the so-called Eddington parameters. We have defined

$$\alpha_0 \equiv \alpha(\varphi_0), \quad \beta_0 \equiv \frac{d\alpha}{d\varphi}(\varphi_0), \quad (11)$$

and φ_0 is the value of the scalar field φ in the spatial asymptotic region. We assume that the cosmological evolution of the scalar field is sufficiently slow in comparison with the characteristic time scale of the local gravitational process of the isolated object considered here. From this assumption φ_0 is regarded as the cosmological value of the scalar field. Then, the solar-system experimental constraints are $\gamma_{Edd} = 0.9996 \pm 0.0017$ [6], and $4\beta_{Edd} - \gamma_{Edd} - 3 = -0.0007 \pm 0.0010$ [7].

For complete equations for the structure of a relativistic star in the scalar-tensor theory, see [3]. We restrict ourselves to the static and spherically symmetric case. The matter is described as a perfect fluid, i.e.,

$$\tilde{T}_{\mu\nu} = (\bar{\rho} + \bar{p}) \tilde{u}_\mu \tilde{u}_\nu + \bar{p} \tilde{g}_{\mu\nu}. \quad (12)$$

We use the polytropic and isentropic equations of state: $\bar{\rho} = \bar{n} m_b + K n_0 m_b / (\Gamma - 1) (\bar{n}/n_0)^\Gamma$, $\bar{p} = K n_0 m_b (\bar{n}/n_0)^\Gamma$, $m_b = 1.66 \times 10^{-24} \text{g}$, $n_0 = 0.1 \text{fm}^{-3}$, where $\bar{n}$ is the baryon number density in the Brans-Dicke frame. We then take the parameters $\Gamma = 2.34$ and $K = 0.0195$ (EOS II of [3]).

Then, from the asymptotic properties at spatial infinity for a static and isolated system

$$g_{*\mu\nu} = \eta_{\mu\nu} + \frac{2G_* m}{r} \delta_{\mu\nu} + O\left(\frac{G_*^2}{r^2}\right), \quad (13)$$

$$\varphi = \varphi_0 + \frac{G_* \omega}{r} + O\left(\frac{G_*^2}{r^2}\right), \quad (14)$$

where $\eta_{\mu\nu}$ is the Minkowskian metric. We call m ADM energy and ω scalar charge [2]. We use the unit in which $G_* = 1$.

3 Stability Criteria

In the scalar-tensor theory, control parameters of the static, spherically symmetric and isolated neutron star are not only the baryon number N but also the 'external field', that is, the asymptotic value of the scalar field φ_0 . For static systems, the partial derivative of m in terms of φ_0 with N constant is given by [2]

$$\left(\frac{\partial m}{\partial \varphi_0}\right)_N = -\omega. \quad (15)$$

The energy injection of the system by increasing baryons is described as

$$\int_0^{r_*} 4\pi \tilde{u}_0 \delta \tilde{\rho} A^3 r^2 \left(1 - \frac{2\mu}{r}\right)^{-1/2} dr = \int_0^{r_*} 4\pi e^{\nu/2} \tilde{\mu} \delta \tilde{n} A^4 r^2 \left(1 - \frac{2\mu}{r}\right)^{-1/2} dr, \quad (16)$$

where $\tilde{\mu} \equiv d\tilde{\rho}/d\tilde{n}$ is the chemical potential. The first law of thermodynamics in an adiabatic process is

$$d\left(\frac{\tilde{\rho}}{\tilde{n}}\right) = -\tilde{p}d\left(\frac{1}{\tilde{n}}\right). \quad (17)$$

From the equations for the structure of the star and (17), we find that the quantity, $Ae^{\nu/2}\tilde{\mu}$, is constant all over the star. Therefore this quantity can be estimated by its value at the stellar surface. Using this fact, the expression of the energy injection Eq. (16) is rewritten as

$$A_s e^{\nu_s/2} \tilde{\mu}_s \int_0^{r_*} 4\pi A^3 r^2 \left(1 - \frac{2\mu}{r}\right)^{-1/2} \delta \tilde{n} dr = A_s e^{\nu_s/2} \tilde{\mu}_s \delta N, \quad (18)$$

where the suffix 's' indicates that the quantity is evaluated at the stellar surface $r = r_*$. Therefore, the effective chemical potential, μ_{eff} , is given by

$$\mu_{eff} \equiv \left(\frac{\partial m}{\partial N}\right)_{\varphi_0} = A_s e^{\nu_s/2} \tilde{\mu}_s. \quad (19)$$

From the above discussions the variation of m for static systems in a quasi-static process is written in the following form:

$$\delta m = -\omega \delta \varphi_0 + \mu_{eff} \delta N, \quad (20)$$

where by "quasi-static process" we mean successive changes among the infinitesimally nearby equilibria.

Suppose that the isolated neutron star is perturbed slightly with φ_0 and N constant for some reason other than incident waves, while the spherical symmetry is preserved. Then the outgoing waves can carry out some positive energy to infinity and the system cannot keep its original state if there exists an energetically favorable configuration which is infinitesimally deformed from the system with the same φ_0 and N . Therefore, an equilibrium solution X is secularly stable against spherically symmetric (infinitesimal) perturbations if and only if there is no momentarily static and spherically symmetric configuration Y which is arbitrarily close to X with the same φ_0 and N but strictly smaller m .

In order to examine the stability of the equilibrium solution, we follow the turning point method [8, 9]. In the present problem, m is a potential function, since the equilibrium is a stationary point of m ($\delta m = 0$) and the stable equilibrium is a minimal point of m ($\delta m = 0$ and $\delta^2 m > 0$). The asymptotic value of the scalar field φ_0 and baryon number N form a 2-dimensional control space. The equilibria are uniquely parametrized by two parameters, i.e., the central value of the scalar field, φ_c , and central total baryonic density, $\tilde{\rho}_c$.

We adopt the following stability criteria [9]:

(i) The stability of $X(\varphi_c, \tilde{\rho}_c)$ can change typically only at a "turning point". Here the "turning point" $(\varphi_c^0, \tilde{\rho}_c^0)$ is a point where there exists a nontrivial vector $(\delta\varphi_c, \delta\tilde{\rho}_c)$ such that

$$\delta\varphi_0 = \left(\frac{\partial\varphi_0}{\partial\varphi_c}\right)_{\tilde{\rho}_c} \delta\varphi_c + \left(\frac{\partial\varphi_0}{\partial\tilde{\rho}_c}\right)_{\varphi_c} \delta\tilde{\rho}_c = 0, \quad (21)$$

$$\delta N = \left(\frac{\partial N}{\partial\varphi_c}\right)_{\tilde{\rho}_c} \delta\varphi_c + \left(\frac{\partial N}{\partial\tilde{\rho}_c}\right)_{\varphi_c} \delta\tilde{\rho}_c = 0. \quad (22)$$

From Eqs. (21) and (22),

$$\frac{\partial(\varphi_0, N)}{\partial(\varphi_c, \bar{\rho}_c)} = 0 \quad (23)$$

at the turning point. Therefore the change of stability can be detected as envelopes of a family of curves $\bar{\rho}_c = \text{const.}$ in the (φ_0, N) plane. Of course, this is also true for a family of curves $\varphi_c = \text{const.}$

(ii) In order to specify an unstable branch at the turning point, we draw the sequence of equilibria in the (φ_0, ω) plane, maintaining N constant. Then, as one proceeds along the curve in a counterclockwise direction, a branch beyond the turning point is unstable. This is also the case with the curve $(N, -\mu_{eff})$ with φ_0 constant. This is the direct consequence of Theorem I of [9].

Here we describe the meaning of the criterion (i) in the context of the catastrophe theory. We regard the ADM energy m as a function of three variables, φ_0 , N and ω . We take ω as a state variable. We consider an equilibrium space

$$M_m = \left\{ (\varphi_0, N, \omega) \mid \left(\frac{\partial m}{\partial \omega} \right)_{\varphi_0, N} = 0 \right\}, \quad (24)$$

and a control space

$$\mathbf{R}^2 = \{(\varphi_0, N)\}. \quad (25)$$

We define a catastrophe map

$$\begin{aligned} \chi_m : M_m &\longrightarrow \mathbf{R}^2 \\ (\varphi_0, N, \omega) &\longmapsto (\varphi_0, N). \end{aligned} \quad (26)$$

A point $P \in M_m$ is called a singular point of χ_m if the Jacobian of χ_m vanishes at P . A point $Q \in \mathbf{R}^2$ is called a singular value if there is at least one singular point in $\chi_m^{-1}(Q)$. A bifurcation set $B_m \subset \mathbf{R}^2$ is a set of singular values. At a singular point $P \in M_m$, the normal vector of the tangent space of M_m , which is

$$\left(\left(\frac{\partial^2 m}{\partial \varphi_0 \partial \omega} \right)_{N, \omega}, \left(\frac{\partial^2 m}{\partial N \partial \omega} \right)_{\omega, \varphi_0}, \left(\frac{\partial^2 m}{\partial \omega^2} \right)_{\varphi_0, N} \right), \quad (27)$$

is parallel to the $\varphi_0 N$ -plane. Therefore, the set of singular points, $\Sigma_m \subset M_m$, satisfies,

$$\Sigma_m = \left\{ (\varphi_0, N, \omega) \mid \left(\frac{\partial m}{\partial \omega} \right)_{\varphi_0, N} = \left(\frac{\partial^2 m}{\partial \omega^2} \right)_{\varphi_0, N} = 0 \right\}, \quad (28)$$

and the bifurcation set $B_m \subset \mathbf{R}^2$ satisfies

$$B_m = \left\{ (\varphi_0, N) \mid \left(\frac{\partial m}{\partial \omega} \right)_{\varphi_0, N} = \left(\frac{\partial^2 m}{\partial \omega^2} \right)_{\varphi_0, N} = 0 \right\}. \quad (29)$$

The envelopes of the family of the curves $\bar{\rho}_c = \text{const.}$ in the (φ_0, N) plane form a bifurcation set B_m of the catastrophe map χ_m because the Jacobian of χ_m vanishes at points on the envelopes. The criterion (i) says that a sequence of the equilibria can change its stability only at the points of the bifurcation set.

From the criteria (i) and (ii), we examine stability of equilibria of neutron stars in the scalar-tensor theory. From the turning point method alone, however, we can not say that an equilibrium solution is *stable*. Therefore the stability of an equilibrium must be investigated by perturbation study *once for all*. For this purpose, we examine the case in which $\alpha(\varphi_0) = 0$ and m/r_s is sufficiently small ($\bar{\rho}_c$ is sufficiently small). In this case, there is an equilibrium solution that is identical to that in general relativity. This solution in which the central density is sufficiently small is stable for the case $\alpha(\varphi_0) = 0$ if $\Gamma > \Gamma_c \simeq 4/3$ [4].

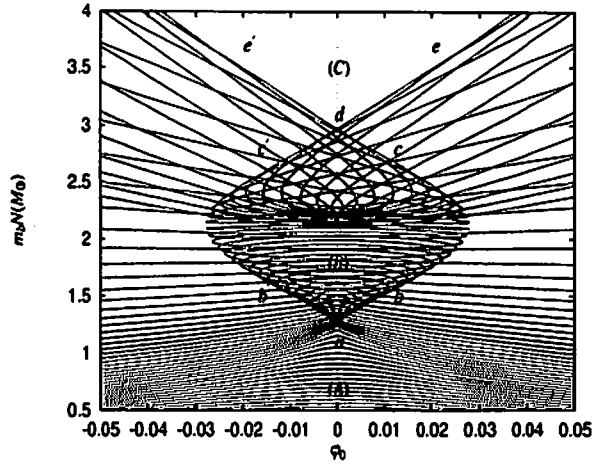


Figure 1: A family of curves of $\bar{\rho}_c = \text{const.}$ in the (φ_0, N) plane for $\beta = -4$ case. The ordinate is $m_b N$ in place of the baryon number N . In the region (A), only one stable equilibrium exists. In the region (B), two distinct stable equilibria exist. In the region (C), no stable solution exists. The point a is a cusp point. At the point a , the cusp catastrophe occurs, while the fold catastrophe occurs at a point on the envelopes except for a .

4 Results

Hereafter we restrict our attention to the coupling function of the quadratic form:

$$A(\varphi) = \exp\left(\frac{1}{2}\beta\varphi^2\right). \quad (30)$$

We concentrate our attention to the case $\beta \lesssim -4.35$. We present the results of the case $\beta = -6$.

Figure 1 shows $\bar{\rho}_c = \text{const.}$ curves in the (φ_0, N) plane. On the envelope of the family of curves, $e'dcbab'c'de$, the sequence of equilibria changes its stability. Although there are other envelopes in the region (B), they have nothing to do with the change of the number of stable equilibria. Figure 2 shows curves (φ_0, ω) with N constant. From the criteria (i) and (ii), the number of stable equilibria is as follows: In the region (A), only one stable equilibrium exists. In the region (B), two distinct stable equilibria exist. Surprisingly, these stable equilibria are different even for $\varphi_0 = 0$ from their counterparts in general relativity. For $\varphi_0 = 0$, the unstable solution agrees with the stable solution in general relativity. One of the two stable equilibria disappears on the envelope $dcbab'c'd$. In the region (C), no stable equilibrium exists. The point a is a bifurcation point for $\varphi_0 = 0$. This is seen in Fig. 3 that displays the curves $(\bar{\rho}_c, m)$ and $(\bar{\rho}_c, N)$ for $\varphi_0 = 0$, where the solid lines denote stable branches and the dotted lines denote unstable branches. In this figure, two stable branches are degenerate because, for $\varphi_0 = 0$, two stable equilibria are identical except for the sign of the scalar field. The equilibrium of the bifurcated stable branches is more compact for smaller mass but less compact for larger mass than the general relativistic sequence. Figure 4 shows the equilibrium space M_m near the point a . This type of the catastrophe at the point a is classified to the cusp catastrophe in which the control space is 2-dimensional. The map χ_m is a cusp catastrophe map. This catastrophe is elementary and structurally stable, which suggests that this structure is stable against adding small higher order terms to the exponent of the coupling function (30). The point a is called a cusp point.

We restrict our attention to the cusp point $a(0, N_a)$. ($m_b N_a \simeq 1.24 M_\odot$ for $\beta = -6$.) The potential function, m , is written around the cusp point a as

$$m = \frac{C}{4}\omega^4 - \frac{D(N - N_a)}{2}\omega^2 - \varphi_0\omega + m_a, \quad (31)$$

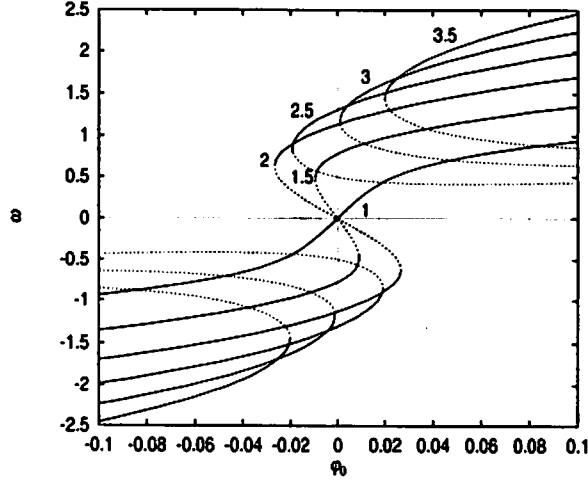


Figure 2: Curves (φ_0, ω) with N constant for $\beta = -4$ case. The number attached to each curve is $m_b N$ in the solar mass unit. The solid lines denote stable branches and the dotted lines denote unstable branches.

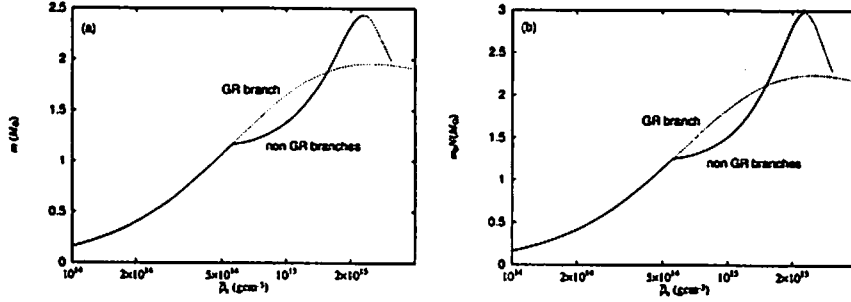


Figure 3: (a) $(\bar{\rho}_c, m)$ and (b) $(\bar{\rho}_c, N)$ curves with $\varphi_0 = 0$ for $\beta = -6$. The solid lines denote stable branches, while the dotted lines denote unstable branches. The two distinct bifurcated branches are degenerate because they have identical ADM energies and baryon numbers but scalar fields of the opposite sign. The number of stable equilibria changes as 1, 2, 0 as N increases.

where C and D are some positive constants. The reason why the coefficient of $\varphi_0 \omega$ is determined is that Eq. (15) holds. This form of expansion agrees with the usual Landau ansatz for the second-order phase transition, which has been used to explain the spontaneous scalarization by Damour and Esposito-Farèse [3]. The scalar charge is given by roots of the following cubic equation:

$$\left(\frac{\partial m}{\partial \omega} \right)_{\varphi_0, N} = C\omega^3 - D(N - N_a)\omega - \varphi_0 = 0. \quad (32)$$

From Eq. (32), near the cusp point a , the scalar charge is given by

$$\omega = 0, \quad (33)$$

for $N < N_a$ with $\varphi_0 = 0$. This is a stable branch. For $N > N_a$ with $\varphi_0 = 0$,

$$\omega = \begin{cases} \pm \left(\frac{D}{C} \right)^{1/2} (N - N_a)^{1/2} & \text{for the stable branches,} \\ 0 & \text{for the unstable branch.} \end{cases} \quad (34)$$

At the point a the stable equilibrium changes *continuously*, but its derivative with respect to N is *discontinuous*. If Eq. (31) involved a cubic term in ω , the number of the roots of the equation $\partial m / \partial \omega = 0$

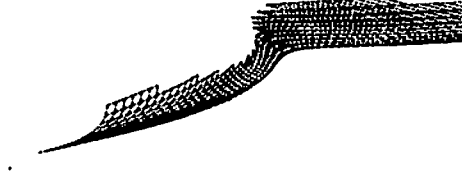


Figure 4: Equilibrium space M_m in the (φ_0, N, ω) space around the cusp point a for $\beta = -6$ case. This structure of the equilibrium space is classified to the cusp catastrophe.

did change at the point a . But this catastrophe was classified to the fold type and therefore not the case for the point a because of the shape of the bifurcation set seen in Fig. 1. That is why Eq. (31) does not contain the cubic term. We also note that, for the case of 2-dimensional control space, the structurally stable catastrophe is classified to either the fold or cusp type by Thom's theorem. Therefore, at the point a , the second-order phase transition occurs. From Eqs. (32-34), with $\varphi_0 = 0$ near the point a , it is derived that the scalar susceptibility χ_φ is given by

$$\chi_\varphi = \begin{cases} D^{-1}(N_a - N)^{-1} & \text{for } N < N_a, \\ \frac{1}{2}D^{-1}(N - N_a)^{-1} & \text{for } N > N_a. \end{cases} \quad (35)$$

Near the point a in the region (A) in Fig. 1, only one real root of the cubic equation (32) corresponds to the stable equilibrium, while, in the region (B), the smallest and largest roots of three real roots correspond to the stable equilibria and the intermediate root corresponds to the unstable one. If $\varphi_0 > 0$, the largest root corresponds to the globally stable equilibrium. If $\varphi_0 < 0$, the smallest root corresponds to the globally stable equilibrium. If $\varphi_0 = 0$, the two stable equilibria have identical ADM energies. The bifurcation set B_m , which is the envelope $b'ab$, is given by

$$\left(\frac{\partial m}{\partial \omega}\right)_{\varphi_0, N} = C\omega^3 - D(N - N_a)\omega - \varphi_0 = 0, \quad (36)$$

$$\left(\frac{\partial^2 m}{\partial \omega^2}\right)_{\varphi_0, N} = 3C\omega^2 - D(N - N_a) = 0, \quad (37)$$

i.e.,

$$\varphi_0 = \pm \left(\frac{4D^3}{27C}\right)^{1/2} (N - N_a)^{3/2}, \quad (38)$$

near the point a . The cusp catastrophe has been named after this shape. On the envelope $dcbab'c'd$ except for the points a and d , one of the two distinct stable equilibria, the locally but not globally stable one, disappears, and hence the first-order phase transition occurs if the system obeys a perfect delay convention. On the envelopes ede' , the stable equilibrium disappears. The catastrophic feature on the envelopes except for the point a is the fold catastrophe. The point d is not a cusp point but an intersection of two folds.

Here we present the physical interpretation on why the spontaneous scalarization occurs. In spite of the absence of the potential in the Lagrangian, the scalar field φ obtains an effective potential term $W(\varphi)$

which satisfies

$$\frac{\partial W}{\partial \varphi} = -4\pi G_* \alpha(\varphi) T_* \quad (39)$$

because of the coupling with the matter. Note that T_* depends on φ . Then, if we consider $A(\varphi)$ of the form (30),

$$\frac{\partial V}{\partial \varphi} = -4\pi G_* \beta \varphi T_*, \quad (40)$$

and if $T_* = A^4(-\tilde{\rho} + 3\tilde{p})$ is negative, $\varphi = 0$ is an unstable stationary point of the effective potential, if $\beta < 0$. On the other hand the term from the spatial derivative in the Eq. (4) has a contribution to stabilize the solution. By these two competing effects, the stability of the trivial configuration $\varphi = 0$ against the spontaneous scalarization is governed. For detailed analysis on the stability of the trivial configuration, see [4].

If the spontaneous scalarization occurs, the effective gravitational constant, which is $A^2(\varphi) = \exp(\beta\varphi^2)$ in the sense of the inverse of the Brans-Dicke scalar field, becomes considerably smaller than unity. Thereby the gravitation becomes weaker and considerably larger mass than in general relativity can be supported by the lower matter pressure than in general relativity.

5 Summary

The behaviour of equilibria of neutron stars in the scalar-tensor theories of gravitation shows a catastrophic feature, which is characterized by the discontinuous change of the system. When we consider the function $A(\varphi)$ of the form, $A(\varphi) = \exp(\frac{1}{2}\beta\varphi^2)$, the catastrophe types are classified to fold and cusp catastrophes. The appearance of the cusp catastrophe depends on whether $\beta \gtrless -4.35$ or $\beta \lesseqgtr -4.35$. From the fact that those types of the catastrophes are structurally stable, it is expected that they would be seen in a wide class of the coupling functions.

Here we comment on a continuous change of the asymptotic value of the scalar field φ_0 . If we identify φ_0 with the cosmological value of the scalar field, the evolution of φ_0 can be described by the equation of motion (4) in the Friedmann-Robertson-Walker universe. Through the cosmological evolution of the scalar field φ_0 , the neutron stars may collapse and radiate the scalar gravitational wave.

We also comment on a continuous change of N , which may be a result of a mass accretion onto the neutron star. If the baryon number of the neutron star exceeds the maximum value, the neutron star collapses and the scalar gravitational waves are radiated and this is a candidate of the source of the scalar gravitational waves.

See [10] for further details.

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An Analytic Approach to Charged Dilatonic Boson Stars

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Abstract

We consider a classical solution for a charged dilatonic boson star. We concentrate on the nearly critical case, in which the long range forces (gravity, electric and dilatonic forces) between two particles are almost cancelled. The critical behaviors of the mass and the radius of the boson star is studied in the vicinity of the critical case.

1 Introduction

The study of charged boson stars is thoroughly reviewed by Jetzer[1]. The charged boson star is constituted of the particles with the electric charge e and the mass m , which is represented by a complex scalar field ϕ . Then we consider the following action:

$$S = \int d^4x \frac{\sqrt{-g}}{16\pi} \left[\frac{1}{G} R - F^2 - |D_\mu \phi|^2 - m^2 |\phi|^2 - \frac{\lambda}{2} |\phi|^4 \right], \quad (1)$$

where R is the scalar curvature, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength, A_μ is the $U(1)$ gauge field. The covariant derivative is defined as $D_\mu = \nabla_\mu + ieA_\mu$. From this action, we can find the classical solution for charged boson stars[1].

Jetzer found[1] that the mass and the radius of the spherical charged boson star show critical behaviors in the limit of $Gm^2 - e^2 \rightarrow 0$. In this limit of couplings, the gravitational and electric forces are cancelled between each charged boson pair. Jetzer illustrated the fact through numerical study of solutions in his review[1], thus the detailed aspects of the critical phenomenon is unclear, at least for a primary reader of [1].

Here we study an approximated solution for a charged boson star, in the vicinity of the critical case, including a mutual dilatonic force. By using an analytic expression of the approximation, we examine the critical behaviours of the physical quantities of a charged dilatonic boson star, such as its mass and radius.

2 Dilaton-coupled charged boson stars

We start with the following action:

$$S = \int d^4x \frac{\sqrt{-g}}{16\pi} \left[\frac{1}{G} \{ R - 2(\nabla_\mu \phi)^2 \} - e^{-2a\phi} F^2 - e^{-2b\phi} |D_\mu \phi|^2 - m^2 e^{-2c\phi} |\phi|^2 - \frac{\lambda}{2} e^{-2d\phi} |\phi|^4 \right], \quad (2)$$

where ϕ is the dilaton field and a, b, c and d are dimensionless couplings. λ is a scalar four point coupling. The field equations are obtained by taking the variations of the fields in the action.

We take the spacetime metric as

$$ds^2 = - \left(1 - \frac{2GM(r)}{r} \right) e^{-2\delta} dt^2 + \left(1 - \frac{2GM(r)}{r} \right)^{-1} dr^2 + r^2 d\Omega_2^2. \quad (3)$$

We also take the ansatz for the fields:

$$\phi = \varphi(r) e^{i\omega t}, \quad eA_\mu = m(A(r) - \omega)\delta_{\mu 0}, \quad \phi = \phi(r). \quad (4)$$

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These assumption leads to the spherical solution. We also use the new definitions for couplings:

$$q^2 \equiv \frac{e^2}{Gm^2}, \quad \Lambda \equiv \frac{\lambda}{Gm^2}, \quad (5)$$

and replace the variables to the dimensionless ones:

$$mr \rightarrow r, \quad GmM \rightarrow M, \quad \sqrt{G}\varphi \rightarrow \phi. \quad (6)$$

The field equations to be solved are then written in the form:

$$\begin{aligned} \varphi'' + \left(\frac{2}{r} - \delta' + \frac{2}{r} \frac{M - rM'}{r - 2M} - 2b\phi' \right) \varphi' \\ + \left[\frac{e^{2\delta}}{1 - \frac{2M}{r}} A^2 - e^{-2(c-b)\phi} - e^{-2(d-b)\phi} \Lambda \varphi^2 \right] \frac{1}{1 - \frac{2M}{r}} \varphi = 0 \end{aligned} \quad (7)$$

$$A'' + \left(\frac{2}{r} + \delta' - 2a\phi' \right) A' - \frac{1}{2} q^2 \frac{e^{-2(b-a)\phi}}{1 - \frac{2M}{r}} \varphi^2 A = 0 \quad (8)$$

$$\begin{aligned} \phi'' + \left(\frac{2}{r} - \delta' + \frac{2}{r} \frac{M - rM'}{r - 2M} \right) \phi' - a \frac{e^{2\delta - 2a\phi}}{1 - \frac{2M}{r}} \frac{A'^2}{q^2} \\ + \frac{b}{2} \left[- \frac{e^{2\delta - 2b\phi}}{\left(1 - \frac{2M}{r}\right)^2} A^2 \varphi^2 + \varphi'^2 \right] + \frac{c}{2} \frac{e^{-2c\phi}}{1 - \frac{2M}{r}} \varphi^2 + \frac{d}{4} \Lambda \frac{e^{-2d\phi}}{1 - \frac{2M}{r}} \varphi^4 = 0 \end{aligned} \quad (9)$$

$$\begin{aligned} \frac{2}{r^2} M' = \left(1 - \frac{2M}{r} \right) \phi^2 + e^{2\delta - 2a\phi} \frac{A'^2}{q^2} \\ + \frac{1}{2} e^{-2b\phi} \left[\frac{e^{2\delta}}{1 - \frac{2M}{r}} A^2 \varphi^2 + \left(1 - \frac{2M}{r} \right) \varphi'^2 \right] + \frac{1}{2} e^{-2c\phi} \varphi^2 + \frac{\Lambda}{4} e^{-2d\phi} \varphi^4 \end{aligned} \quad (10)$$

$$-\frac{1}{r} \delta' = \phi'^2 + \frac{1}{2} e^{-2b\phi} \left[\frac{e^{2\delta}}{\left(1 - \frac{2M}{r}\right)^2} A^2 \varphi^2 + \varphi'^2 \right], \quad (11)$$

where ' denotes the derivative with respect to dimensionless r .

The boundary condition of the field variables are:

$$\begin{aligned} \varphi'(0) &= 0, \quad A'(0) = 0, \quad \phi'(0) = 0, \\ M(0) &= 0, \quad \delta(0) = 0, \quad \phi(0) = 0, \\ \varphi(\infty) &= 0, \quad A(\infty) = \text{const.} \end{aligned} \quad (12)$$

3 The solution for the linear order approximation

Here we consider the simplest case that the self coupling of the charged scalar field is infinitely large, i.e., $\Lambda \rightarrow \infty$. This situation has been considered for both cases for the neutral and charged boson stars (see [1]).

We translate the variables as

$$\varphi_* = \varphi \sqrt{\Lambda}, \quad r_* = \frac{r}{\sqrt{\Lambda}}, \quad M_* = \frac{M}{\sqrt{\Lambda}}. \quad (13)$$

In the limit of $\Lambda \rightarrow \infty$, the field equations (7-11) are reduced to the followings:

$$\varphi_*^2 = \frac{e^{2\delta + 2(d-b)\phi}}{1 - \frac{2M_*}{r_*}} A^2 - e^{-2(c-d)\phi} \quad \text{if } \varphi_*^2 > 0, \text{ otherwise } \varphi_*^2 = 0 \quad (14)$$

$$A'' + \left(\frac{2}{r_*} + \delta' - 2a\phi' \right) A' - \frac{1}{2} q^2 \frac{e^{-2(b-a)\phi}}{1 - \frac{2M_*}{r_*}} \varphi_*^2 A = 0 \quad (15)$$

$$\begin{aligned} \phi'' + \left(\frac{2}{r_*} - \delta' + \frac{2}{r_*} \frac{M_* - r_* M_*'}{r_* - 2M_*} \right) \phi' - a \frac{e^{2\delta-2a\phi}}{1 - \frac{2M_*}{r_*}} \frac{A'^2}{q^2} \\ - \frac{b}{2} \frac{e^{2\delta-2b\phi}}{\left(1 - \frac{2M_*}{r_*}\right)^2} A^2 \varphi_*^2 + \frac{c}{2} \frac{e^{-2c\phi}}{1 - \frac{2M_*}{r_*}} \varphi_*^2 + \frac{d}{4} \frac{e^{-2d\phi}}{1 - \frac{2M_*}{r_*}} \varphi_*^4 = 0 \end{aligned} \quad (16)$$

$$\frac{2}{r_*^2} M_*' = \left(1 - \frac{2M_*}{r_*} \right) \phi^2 + e^{2\delta-2a\phi} \frac{A'^2}{q^2} + \frac{1}{2} \frac{e^{2\delta-2b\phi}}{1 - \frac{2M_*}{r_*}} A^2 \varphi_*^2 + \frac{1}{2} e^{-2c\phi} \varphi_*^2 + \frac{1}{4} e^{-2d\phi} \varphi_*^4 \quad (17)$$

$$-\frac{1}{r_*} \delta' = \phi'^2 + \frac{1}{2} \frac{e^{2\delta-2b\phi}}{\left(1 - \frac{2M_*}{r_*}\right)^2} A^2 \varphi_*^2, \quad (18)$$

where ' denotes the derivative with respect to r_* . Notice that φ_* has been solved in terms of other field variables.

Now we consider the nearly critical case. In this case, the long range forces are mutually almost cancelled, then the density of the scalar field become very low, i.e. $\varphi_*(0)$ is very small. Thus one can find that $A_0 \equiv A(0)$ is nearly equal to unity. Then we define a small parameter $\epsilon \equiv A_0^2 - 1$, and expand all field variables in the small parameter ϵ :

$$A(r_*) = A_0(1 + \epsilon \alpha_1(r_*) + \epsilon^2 \alpha_2(r_*) + \dots), \quad (19)$$

$$\phi(r_*) = \epsilon \phi_1(r_*) + \epsilon^2 \phi_2(r_*) + \dots, \quad (20)$$

$$M_*(r_*) = \epsilon M_{*1}(r_*) + \epsilon^2 M_{*2}(r_*) + \dots, \quad (21)$$

$$\delta(r_*) = \epsilon \delta_1(r_*) + \epsilon^2 \delta_2(r_*) + \dots. \quad (22)$$

We can obtain the sets of differential equations order by order in the power of ϵ .

3.1 linear order in ϵ

In the first order of ϵ , we get a remarkably simple set of linear differential equations. That is,

$$\beta_1'' + \frac{2}{r_*} \beta_1' = [q^2 - (b-c)^2] \left(\frac{1}{2} + \beta_1 + \frac{M_{*1}}{r_*} + \delta_1 \right), \quad (23)$$

$$\frac{1}{r_*^2} M_{*1}' = \frac{1}{2} + \beta_1 + \frac{M_{*1}}{r_*} + \delta_1, \quad (24)$$

$$\frac{1}{r_*} \delta_1' = -\left(\frac{1}{2} + \beta_1 + \frac{M_{*1}}{r_*} + \delta_1 \right), \quad (25)$$

where $\beta_1 = \alpha_1 - (b-c)\phi_1$. At the same time, we find that $\phi_1 = ((b-c)/q^2)\alpha_1$.

The solutions that satisfy the boundary conditions inside a boson star are obtained by simple manipulations and found to be

$$\beta_1 = \frac{q^2 - (b-c)^2}{2(1 + (b-c)^2 - q^2)} \left(1 - \frac{\sin \sqrt{1 + (b-c)^2 - q^2} r_*}{\sqrt{1 + (b-c)^2 - q^2} r_*} \right), \quad (26)$$

$$M_{*1} = \frac{1}{2(1 + (b-c)^2 - q^2)} \left(\frac{\sin \sqrt{1 + (b-c)^2 - q^2} r_*}{\sqrt{1 + (b-c)^2 - q^2}} - r_* \cos \sqrt{1 + (b-c)^2 - q^2} r_* \right), \quad (27)$$

$$\delta_1 = \frac{1}{2(1 + (b-c)^2 - q^2)} \left(\cos \sqrt{1 + (b-c)^2 - q^2} r_* - 1 \right). \quad (28)$$

Therefore in the same order one can get

$$e^{-2d\phi} \varphi_*^2 = (A_0^2 - 1) \frac{\sin \sqrt{1 + (b-c)^2 - q^2} r_*}{\sqrt{1 + (b-c)^2 - q^2} r_*} + O(\epsilon^2). \quad (29)$$

From the solution, we see that the critical relation in the couplings is $1 + (b-c)^2 - q^2 \rightarrow 0$, since the radius or the typical scale of the boson star must approach infinity in the critical configuration. This is

a simple but important result from our analytic approach. We expect that the radius of the boson star behaves as $\approx 1/\sqrt{1+(b-c)^2-q^2}$ when $1+(b-c)^2-q^2 \rightarrow 0$.

We must note that the solution for φ is obtained also in a non-relativistic model which is described by the Hamiltonian

$$H = \int d^3x \frac{1}{16\pi G} \left[2(\nabla_\mu \phi)^2 - \sqrt{m^2(1+(b-c)^2) - e^2/G} |\varphi|^2 \phi + \frac{|\nabla_\mu \varphi|^2}{2m} + \frac{\lambda}{8m^2} |\varphi|^4 \right]. \quad (30)$$

Our approximation of the first order in ϵ we have considered here just corresponds to the non-relativistic model, but moreover, we have also obtained the lowest order solution in the metric.

3.2 beyond the first order

If we are interested in the specific mass of the boson star, we must examine the higher order in ϵ , because the mass of the boson star actually realized is given by the extremum of the value of $M(\infty)$ when the boundary condition at $r=0$ is varied[1].

We can find the differential equations of the order of ϵ^2 , which are turn out to be

$$\beta_2'' + \frac{2}{r_*} \beta_2' = [q^2 - (b-c)^2](\beta_2 + \frac{M_{*2}}{r_*} + \delta_2) + f_\beta \quad (31)$$

$$\frac{1}{r_*^2} M_{*2}' = (\beta_2 + \frac{M_{*2}}{r_*} + \delta_2) + f_M \quad (32)$$

$$\frac{1}{r_*} \delta_2' = -(\beta_2 + \frac{M_{*2}}{r_*} + \delta_2) + f_\delta, \quad (33)$$

and

$$\beta_2 = \alpha_2 - (b-c)\phi_2, \quad (34)$$

where

$$f_\alpha = 3 \frac{M_{*1}}{r_*} \frac{M_{*1}'}{r_*^2} + 2a(b-c) \frac{1}{r_*^2} \left(\frac{M_{*1}}{r_*} \right)^2 + \alpha_1 \frac{M_{*1}'}{r_*^2} + 2(a-b)\phi_1 \frac{M_{*1}'}{r_*^2} + X \quad (35)$$

$$f_\phi = 3(b-c) \frac{M_{*1}}{r_*} \frac{M_{*1}'}{r_*^2} - 2c(b-c)\phi_1 \frac{M_{*1}'}{r_*^2} + (2b-d) \left(\frac{M_{*1}'}{r_*^2} \right)^2 + (aq^2 - 2(b-c)) \frac{1}{r_*^2} \left(\frac{M_{*1}}{r_*} \right)^2 + X \quad (36)$$

$$f_\beta = q^2 f_\alpha - (b-c) f_\phi \quad (37)$$

$$f_M = \frac{3}{2} \left(\frac{M_{*1}'}{r_*^2} \right)^2 - 2c\phi_1 \frac{M_{*1}'}{r_*^2} + \frac{1}{2} (q^2 + (b-c)^2) \frac{1}{r_*^2} \left(\frac{M_{*1}}{r_*} \right)^2 + X \quad (38)$$

$$f_\delta = 2 \left(\frac{M_{*1}'}{r_*^2} \right)^2 + 2 \frac{M_{*1}}{r_*} \frac{M_{*1}'}{r_*^2} - 2c\phi_1 \frac{M_{*1}'}{r_*^2} + (b-c)^2 \frac{1}{r_*^2} \left(\frac{M_{*1}}{r_*} \right)^2 + X \quad (39)$$

with

$$X = \frac{\alpha_1^2}{2} + \delta_1 + 2 \frac{M_{*1}}{r_*^2} + (d-b)\phi_1 - (c-d)^2 \phi_1^2 + (\delta_1 + (d-b)\phi_1)^2 + \alpha_1(1 + 2\delta_1 + 2(d-b)\phi_1) + \frac{M_{*1}}{r_*} (1 + 2\alpha_1 + 2\delta_1 + 2(d-b)\phi_1). \quad (40)$$

ϕ_2 is solved as

$$\phi_2 = \frac{b-c}{q^2} \alpha_2 - \frac{b-c}{r} \left[\int_0^r dR \int_0^R dR' R' \{ f_\alpha(k) - \frac{1}{b-c} f_\phi(R') \} \right]. \quad (41)$$

The rest of solutions can be solved analytically, but does not seem to be simple. Note that the critical behavior as $q^2 - (b - c)^2 \rightarrow 1$ still appears when the solution is obtained by the series expansion in the powers of ϵ .

Figure 1 shows the solutions for $M(r_*)$, in the case of $a = b = c = d = 0$.

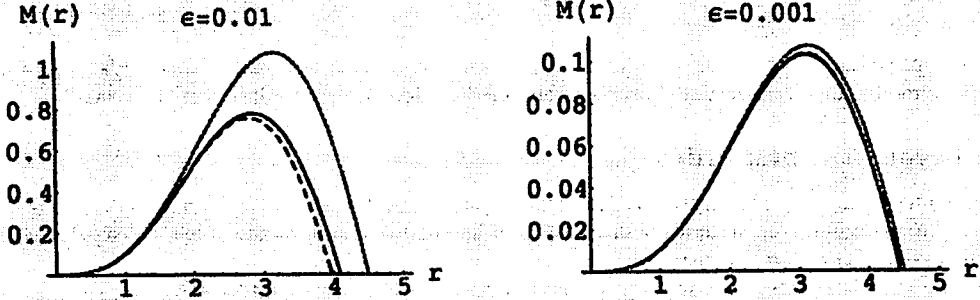


Figure 1: The plots of the solutions for $M(r_*)$. The frequently-broken line shows the solution in the order of ϵ . The broken line shows the solution up to the order of ϵ^2 . The bold line shows the numerical solution. Actually, $M(r_*)$ gives the mass function only inside the boson star. In this figure, for simplicity, we have not omitted the physically meaningless region.

Although we do not show the detailed calculation here, we can show that the critical behavior of mass of a boson star is $M \approx 1/\sqrt{1 - q^2}$, when $q^2 \rightarrow 1$, in this case.

4 Summary and prospect

Here we studied the critical behavior of charged dilatonic boson stars in the limit of the cancellation of the long range forces. We treated the model analytically and examined the case of the infinite limit of the self coupling. The analytic approach is possible for the no self coupling case, but the analysis will be more complicated.

The approximation in the linear order presented above revealed a very beautiful form of the solution, which coincide with the one in the non-relativistic model. From this solution, we concluded that the critical relation in the couplings is $Gm^2(1 + (b - c)^2) - \epsilon^2 \rightarrow 0$. This is also expected by a non-relativistic analysis.

Higher order equations are very complicated and the validity of the critical relation on the couplings seems to be nontrivial. At the second order in ϵ , the relation and critical behavior is unchanged when no dilaton force exists. The critical behavior may be changed due to the other couplings a and d , from this order and beyond. Since the numerical investigation has not sufficiently been done yet, the critical behavior of boson stars in the dilatonic case must be studied further.

Finally, we must notice that the analytic approximation is useful for studying the wave field in the boson star background (e.g. [2]).

Note Added

A very simple but viable analytical approximation for the solution considered above has been found. Such as,

$$\alpha \approx \frac{q^2}{2(A_0^2 - q^2)} \left(1 - \frac{\sin \sqrt{A_0^2 - q^2} r_*}{\sqrt{A_0^2 - q^2} r_*} \right), \quad (42)$$

when $b = c = 0$.

The derivation and validity of this form will be reported[3].

Acknowledgements

I would like to thank K. Shiraishi.

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Topological Appearance of Event Horizon

-What Is the Topology of the Event Horizon That We Can See? -

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Abstract

The topology of the event horizon (TOEH) is usually believed to be a sphere. Nevertheless, some numerical simulations of gravitational collapse with a toroidal event horizon or the collision of event horizons are reported. Considering the indistinguishability of the event horizon (EH), we see that such non-trivial TOEHs are caused by the set of endpoints (the crease set) of the EH. The two-dimensional (one-dimensional) crease set is related to the toroidal EH (the coalescence of the EH). Furthermore, examining the stability of the structure of the endpoints, it becomes clear that the spherical TOEH is unstable under linear perturbation. On the other hand, a discussion based on catastrophe theory reveals that the TOEH with handles is stable and generic. Also, the relation between the TOEH and the hoop conjecture is discussed. It is shown that the Kastor-Traschen solution is regarded as a good example of the hoop conjecture by the discussion of its TOEH. We further conjecture that a non-trivial TOEH can be smoothed out by rough observation in its mass scale.

1 Introduction

The existence of an event horizon is one of the most essential concepts in general relativity. As general relativity is the theory of spacetime dynamics, it informs us about the causal structure of a spacetime; there are several kinds of horizons which are the limits of our communication or influence. In studying the fate of stars, the event horizon (EH) is the most important among them as the surface of a black hole.

To reveal the appearance of a black hole, many authors have been studying the geometrical properties of the EH. In particular, the area theorem of an EH[1] is a noteworthy result, and its importance in relationship to the properties of black hole entropy is being increasingly recognized.

Nevertheless, this is not the whole of the EH. It has both more abstract and more detailed properties. For example, we might be interested in concrete figures of the EH, e.g., the ellipticity of the EH or the multipole moments of its pulsation. Now, what we wish to emphasize is that consideration of the EH from the viewpoint of geometrical concepts is always based upon more abstract concepts. So, we must concentrate on the abstract concept of the geometry of the EH.

The (spatial) topology of an EH is the most fundamental concept when one investigates the various physical properties of the EH, and it is sometimes believed to be trivial. For example, one may assume that the topology of each component of the EH is a sphere for the uniqueness theorem of a black hole. On the other hand, it is natural for the topology of the event horizon (TOEH) to be a sphere, in an astrophysical sense since it is just the surface of a black hole.

So, are the possibilities of a non-spherical TOEH denied? There is a numerical work reporting the existence of an EH with non-spherical topology.[2] After all, attempts to prove the TOEH must be a sphere in general, are far from perfect. In this situation, it might be more interesting to reveal the mechanism of the TOEH than to prove the spherical nature of the TOEH. In the present article we consider this problem.

In a physically realistic gravitational collapse, it is believed that a spacetime is quasi-stationary far in the future. Then, it is natural to assume that the TOEH should be a sphere for a single asymptotic region, in such a quasi-stationary phase.[1][7] Supposing that the TOEH is a sphere far in the future,

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the problem of the TOEH turns out that of a topology change in a three-dimensional null surface, i.e., from a two-surface with an arbitrary topology to a single sphere. From this viewpoint we adapt the well-known theories of the topology change of a spacetime to this case. Our investigation will show the relation between the TOEH and the endpoints (the crease set) of the EH.

Understanding the mechanism of a non-trivial TOEH, the question as to what topology is probable in general arises. To get a final answer to this we should determine the whole dynamics of spacetimes. Though numerical simulations might achieve this, we will examine the stability or generality of the TOEH to stress something with regard to this question. In a background of the Oppenheimer-Snyder spacetime, the stability of the TOEH under linear perturbation is studied. Furthermore, we discuss the structural stability of the TOEH in the context of catastrophe theory. Moreover, we obtain some information about the EH of the Kastor-Traschen (KT) solution numerically. Since the KT solution describes the merging of some spherical EHs, we analyze this solution as an example of a topologically changing EH. According to our theories of the TOEH, it is shown that the solution can be regarded as a good example of the hoop conjecture of an EH. We also state a conjecture concerning the scale dependence of the TOEH assuming a “hoop theorem”.

In the next section, we briefly summarize previous works done by other authors regarding the TOEH. Though in §3 the theorems of topology change are prepared, and it is applied to the EH[3] in §4, we do not spare sufficient space for these issues. Refs.[3] and [5] give the whole investigation about these. Section 5 gives discussion of linear perturbation in a spherically symmetric spacetime.[4] In §6, the structural stability of the TOEH is investigated,[4] based on catastrophe theory. We study the relation between an example of the exact solution with the change of the TOEH (the KT solution) and the hoop conjecture, and give a conjecture concerned with it[6] in §7. The final section is devoted to summary and discussion.

2 Background

The existence of an EH is the most interesting concept of the causal structure of a spacetime. Many authors have studied the EH. Mathematically, the EH is defined as the boundary of the causal past of the future null infinity.[1] Since the natural asymptotic structure of a spacetime is supposed to be asymptotically flat, where the topology of the future null infinity is $S^2 \times R$, we naively think that the (spatial) topology of the EH will always be S^2 .

Earlier studies of exact black hole solutions, e.g., the Schwarzschild solution, Kerr solution, etc., found spherical EHs. The first work generally dealing with the TOEH is due to Hawking in 1972.[1] In his work, it is proved in a stationary spacetime that each component of any (smooth) EH must have a spherical topology. Its extension to non-stationary spacetimes was first attempted by Gannon.[8] With the physically reasonable conditions of the asymptotic flatness and the dominant energy condition, it is proved that the topology of a smooth EH must be either a sphere or a torus. In 1993, a new lead to investigate the TOEH, the “topological censorship” theorem, was proven by Friedmann, Schleich and Witt.[9] Assuming asymptotic flatness, global hyperbolicity, and a suitable energy condition, that theorem stipulates that any two causal curves extending from the past to the future null infinity are homotopy equivalent to each other in the sense of a continuous map preserving their causality. This suggests that the region outside black holes is simply connected. From this, Browdy and Galloway concluded that if no new null generators enter the horizon at later times, the topology of an EH must eventually be spherical.[11] This theorem was strengthened by Jacobson and Venkataramani, limiting the time for which a torus EH can persist.

For all these works restricting the possibilities of the TOEH, it is assumed that the EH is differentiable or there is no endpoint of the EH implying the indifferentiability of the EH in a given region. Nevertheless there is no a priori reason for analyticity. Considering the entire structure of an EH, it cannot be differentiable in general. For example, even in the case of a spherically symmetric spacetime (Oppenheimer-Snyder spacetime) the EH is not differentiable where the EH is formed. For stationary black holes, Chruściel and Wald pointed out that the toroidal topology of black holes—as well as all other non-spherical topologies—can be excluded as a simple consequence of the “topological censorship” theorem, when a suitable energy condition is imposed, without assuming the differentiability conditions on the EH implicitly assumed in Ref. [1]. On the other hand, there is no attempt at dealing with indifferentiable dynamical EHs like the above exemplified formation of the EH. If the EH is not smooth, we

cannot say that such an EH should generally be a sphere.

In fact, the existence of an EH whose topology is not a single S^2 is reported in the numerical simulations of gravitational collapse. Shapiro, and Teukolsky et al.[2] numerically discovered a toroidal EH in the collapse of toroidal matter, and Seidel et al. numerically showed the coalescence of two spherical EHs.[12] This is because, as it will be shown in the present paper, an EH is not differentiable at the endpoints of the null geodesic generating the EH. In the present paper, we are mainly concerned with such an indifferenciability at the endpoints. We are not concerned with indifferenciability not caused by the endpoints (for example, indifferenciability related to the pathological structure of the null infinity[18]).

3 The topology change of a spacetime

Many works have been concerned with the topology change of a spacetime. Some of these are useful to discuss the topology of an EH which is a three-dimensional null surface imbedded into a four-dimensional spacetime. Refs. [3] and [5] show them in detail.

3.1 The Poincaré-Hopf theorem

In particular, the Poincaré-Hopf theorem given by Sorkin[14] is useful.

3.2 Geroch's theorem

Geroch's theorem is for Lorentzian manifolds. Here we rearrange it for null surfaces.

4 The topology of event horizon

Now, we apply the topology change theories given in the previous section to EHs. Let (M, g) be a four-dimensional C^∞ spacetime whose topology is R^4 . In the remainder of this article, the spacetime (M, g) is assumed to be strongly causal, and also the weak cosmic censorship is assumed. Furthermore, for simplicity, the topology of the EH (TOEH²) is assumed to be a smooth S^2 far in the future and the EH is not an eternal one (in other words, the EH begins somewhere in the spacetime, and it is open to the infinity in the future direction with a smooth S^2 section). We expect that those assumptions could be valid if we consider only one regular ($\sim R \times S^2$) asymptotic region, namely the future null infinity $\mathcal{J}^+$, to define the EH, and the formation of a black hole. The following investigation, however, is easily extended to the case of different final TOEHs far in the future.

From these assumptions, we conclude that the TOEH is related to the crease set[3][5][16] (the set composed of the endpoints), as depicted in Fig. 1, i.e., the coalescence of EHs is related to the one-dimensional crease set and a torus EH is related to the two-dimensional crease set.

5 The linear perturbation of an event horizon with a spherical topology

The purpose of this section is to investigate the stability of the TOEH which is always a sphere under linear perturbation. From our investigation in Ref.[3] such an EH has only one zero-dimensional crease set (see Fig. 1(a)). Then, we investigate whether this zero-dimensional crease set is stable under linear perturbation. Now, it should be noted the 'stability' does not imply that the perturbation does not blow up but that the TOEH is not changed by the perturbation.

As the background spacetime, a spherically symmetric spacetime is appropriate. If the spherically symmetric spacetime has a non-eternal EH, it has only one endpoint at the origin, and the TOEH is always a sphere. In this case, it is possible to study the linear perturbation in an established framework[23]

Now we consider the Oppenheimer-Snyder spacetime as a familiar example of the EH with a spherical topology. Its line element is given by

²The TOEH is the topology of the spatial section of the EH. Of course, it depends on a timelicing.

• interior:

$$ds^2 = a(\eta)^2 (-d\eta^2 + d\chi^2 + \sin^2 \chi d\Omega^2), \quad 0 \leq \chi \leq \chi_0 \quad (1)$$

$$a(\eta) = \frac{1}{2} a_m (1 + \cos \eta), \quad (2)$$

• exterior:

$$ds^2 = -(1 - 2m/R) dt^2 + \frac{dR^2}{1 - 2m/R} + R^2 d\Omega^2, \quad R_B(t) \leq R, \quad (3)$$

$$= \left(\frac{32m^3}{R} \right) e^{-R/2m} (-dV^2 + dU^2) + R^2 d\Omega^2, \quad (4)$$

where V and U are Kruskal-Szekeres coordinates. When these geometries are continued at $\chi = \chi_0$, the parameters of the exterior region are related to a_m and χ_0 as

$$m = \frac{1}{2} a_m \sin^3 \chi_0, \quad (5)$$

$$R_B = \frac{a_m \sin \chi_0}{2} (1 + \cos \eta). \quad (6)$$

In the background spacetime the equations of null geodesics are easily solved and integrated. The background values of an outgoing null geodesic γ in the direction θ_0, ϕ_0 and from the origin at $\eta = \eta_0$ are

$$l_0^a = \left(\frac{\partial}{\partial \eta} \right)^a + \left(\frac{\partial}{\partial \chi} \right)^a \quad (7)$$

$$= \left(\frac{\partial}{\partial V} \right)^a + \left(\frac{\partial}{\partial U} \right)^a \quad (8)$$

$$\gamma(\eta_0, \theta_0, \phi_0) : \begin{cases} \chi = \eta - \eta_0, \\ U - U_0(\chi = \chi_0, \eta = \chi_0 + \eta_0) = V - V_0(\chi = \chi_0, \eta = \chi_0 + \eta_0) \end{cases} \quad (9)$$

$$\theta = \theta_0, \quad \phi = \phi_0, \quad (10)$$

$$\eta_{crit} = \pi - 3\chi_0, \quad (11)$$

where l_0 is an outgoing null vector field and η_{crit} is the supremum of the time η when a light ray emitted from the origin can reach to the future null infinity $\mathcal{J}^+$. The crease set of the EH is a point at the origin with $\eta = \eta_{crit}$.

We treat the freedom of linear perturbation with a spherical harmonics Y_{LM} expansion, and they are decomposed into odd parity $[(-1)^{L+1}]$ modes and even parity $[(-1)^L]$ modes. Since they are decoupled in the spherically symmetric background, we discuss the stability of the TOEH under each mode of the perturbation with a parity, L , and M . First we develop the properties of null geodesics in a perturbed spacetime. The equations of null geodesics are given by

$$0 = g_{ab} l^a l^b \quad (12)$$

$$= (g_{0ab} + h_{ab}) (l_0^a + \delta l^a) (l_0^b + \delta l^b) \quad (13)$$

$$= h_{ab} l_0^a l_0^b + 2g_{0ab} \delta l^a l_0^b, \quad (14)$$

and

$$l^a \nabla_a l^b = \alpha l^b \quad (15)$$

$$l^a \partial_a l^b + \Gamma_{ac}^b l^a l^c = \alpha l^b \rightarrow \quad (16)$$

$$l_0^a \partial_a \delta l^b + \delta l^a \partial_a l_0^b + 2\Gamma_{0ac}^b l_0^a \delta l^c + \delta \Gamma_{ac}^b l_0^a l_0^c = \delta \alpha l_0^b + \alpha_0 \delta l^b, \quad (17)$$

where g_0, Γ_0 is given by (1), (3) and l_0 (7), (8). the quantity $\delta \alpha$ corresponds to the parametrization of l , and it is set so that $\delta l^\eta + \delta l^\chi (\delta l^V + \delta l^U)$ vanishes. The deformation δx^a of the light path γ fixing its end

on the same position of the future null infinity $\mathcal{J}^+$ is integrated backward along the background light path γ from the future null infinity to a point in the interior region as

$$\delta\alpha = \delta\Gamma_{ab}^{(V)} l_0^a l_0^b + \delta\Gamma_{ab}^{(U)} l_0^a l_0^b \quad (18)$$

$$\delta\eta = -\delta\chi = \int_{\chi=\chi_0}^{\chi} d\chi[\gamma] \frac{h_{\eta\eta} + 2h_{\eta\chi} + h_{\chi\chi}}{4a^2} + \delta\eta(\chi = \chi_0), \quad (19)$$

$$\delta\eta(\chi = \chi_0) = \frac{\partial\eta}{\partial V}\delta V_0 + \frac{\partial\eta}{\partial U}\delta U_0, \quad (20)$$

$$\delta V_0 = -\delta U_0 = \int_{U=\infty}^{U_0} dU[\gamma] \frac{(h_{VV} + 2h_{VU} + h_{UU})Re^{R/2m}}{128m^3}, \quad (21)$$

since $\delta V_0 = -\delta U_0$ implies $\delta\eta(\chi_0) = -\delta\chi(\chi_0)$.

5.1 Even parity mode

The metric perturbation of the even parity mode is given by

$$h_{ab} = \begin{pmatrix} \eta, V & \chi, U & \theta & \phi \\ \bar{h}_{AB} Y_{LM} & \bar{h}_A Y_{LM, \alpha} \\ Sym & r^2(K\gamma_{\alpha\beta} Y_{LM} + GY_{LM; \alpha\beta}) \end{pmatrix}, \quad (22)$$

where r is a circumference radius, and $\gamma_{\alpha\beta}$ is the metric of the unit sphere.[23] For the even parity mode, the angular distribution of the $\delta\eta$ and $\delta\chi$ (19) is given by the spherical harmonics Y_{LM} . So, it is helpful to discuss the symmetry of each Y_{LM} .

Since Y_{00} is a spherically symmetric function it causes no change of the crease set, unless the perturbation is unstable and destroys the entire EH. The even parity modes with $L = 1$, $M = \pm 1$, can change into the mode of Y_{10} through a certain rotation, and we only consider $M = 0$ for the $L = 1$ mode. By a Y_{10} perturbation, the wave front of light around the origin is shifted along the z -axis. Then, we only need to determine perturbed light paths starting from the origin and moving toward the north and the south. From Eq. (22), we see $\delta\chi(\gamma(\eta = \eta_0, \theta = 0)) = -\delta\eta(\gamma(\eta = \eta_0, \theta = 0)) = -\delta\chi(\gamma(\eta = \eta_0, \theta = \pi)) = \delta\eta(\gamma(\eta = \eta_0, \theta = \pi))$. Furthermore, $\delta\theta$ and $\delta\phi$ for these light paths vanish due to axial-symmetry. These imply the intersection of $\gamma(\eta = \eta_{crit}, \theta = 0)$ and $\gamma(\eta = \eta_{crit}, \theta = \pi)$ does not undergo a change of its time η but, rather, its position χ by $2\delta\chi$ along the z -axis. Since there is no peak of Y_{10} between $\theta = 0$ and $\theta = \pi$, all the other $\gamma(\eta = \eta_{crit}, \theta, \phi)$ should also be shifted so as to pass the same position of $2\delta\chi$ on the z -axis at $\eta = \eta_{crit}$. Therefore the original zero-dimensional crease set is only shifted in the z -direction by $2\delta\chi$. There is no change of the TOEH.

The even parity mode with $L = 2$ possesses reflection symmetries about three orthogonal planes. For small perturbation, these modes change the wave front of light from spherical to ellipsoidal. By an appropriate rotation, the principal axes of the ellipsoidal wave front become the x -, y - and z -axis. Then, it is sufficient to determine light paths along these axes. By the symmetry, $\delta\theta$ and $\delta\phi$ vanish for these light paths. Since $\delta\eta = -\delta\chi$ implies that the change of η_{crit} is given by $2\delta\eta(\gamma(\eta_{crit}))$, the $\delta\eta_{crit}$ of the light paths on the principal axes by each even parity Y_{2M} mode are given by

$$\begin{aligned} \delta\eta_{crit}(L = 2, M = 0) &= \sqrt{\frac{5}{\pi}}H, -\frac{1}{2}\sqrt{\frac{5}{\pi}}H, -\frac{1}{2}\sqrt{\frac{5}{\pi}}H, \\ \delta\eta_{crit}(L = 2, M = 1) &= \frac{3}{2}\sqrt{\frac{5}{6\pi}}H, 0, -\frac{3}{2}\sqrt{\frac{5}{6\pi}}H, \\ \delta\eta_{crit}(L = 2, M = 1) &= \frac{3}{2}\sqrt{\frac{5}{6\pi}}H, 0, -\frac{3}{2}\sqrt{\frac{5}{6\pi}}H, \end{aligned}$$

where H (the factor not depending on Y_{LM}) is given by Eq. (19). From these results, we see the shape of the crease set around the origin. Light paths from the latest direction (maximal $\delta\eta_{crit}$) form an endpoint at the origin (for example, see Fig. 2). On the other hand, light paths on the other axes cross a light path from another direction not passing through the origin, at a position different from the origin, so

that their intersections provide the dimensions of the crease set to their directions. Thus, the case of $L = 2, M = \pm 1, 2$ provides a two-dimensional crease set. Contrastingly the crease set with $L = 2, M = 0$ depends on the signature of H . If H is negative (positive), the crease set is one (two)-dimensional (see Fig. 2). Since H is generally not equal to zero, the TOEH is not stable under the perturbation with $L = 2$.

By the mode with $L > 2$, the wave front will experience more complicated deformation. By such a deformation, the crease set will undergo branching and become highly complicated, as stated in the remark of Theorem??. For these modes, $\delta\theta$ and $\delta\phi$ will not be excluded from the discussion. A detailed investigation, however, would show the change of the structure of the crease set occurs even with non-vanishing $\delta\theta, \delta\phi$.

5.2 Odd parity mode and higher order contribution

The metric perturbation of the odd parity mode is given by

$$h_{ab} = \begin{pmatrix} \eta, V & \chi, U & \theta & \phi \\ 0 & 0 & \bar{h}_A S_\alpha & \\ 0 & 0 & & \\ Sym & & \bar{h}_{S(\alpha;\beta)} \end{pmatrix}, \quad (23)$$

where the S_α are the transverse vector harmonics on the unit sphere.[23] From (19) and (23), it is clear that the odd parity mode does not affect $\delta\eta, \delta\chi$ at linear order. On the other hand, though $\delta\theta$ and $\delta\phi$ exist, they do not affect the structure of the crease set, because without $\delta\eta$ and $\delta\chi$, all the perturbed outgoing light paths whose original past endpoint in background is the origin at $\eta = \eta_0$ start at the origin at the same time η_0 . They still have only one endpoint at the origin with $\eta = \eta_{crit}$.

For the modes not changing the structure of the crease set, it would be necessary to investigate contributions from higher order evaluation. The higher order contributions are contained in the back-reaction of the changes of the light path to the equation of null geodesics. Nevertheless, it is also necessary to include a second order metric-perturbation. It will cause difficulty in further investigations. At second order, there should be mode coupling between different parities of L and M . This fact implies generally that the structure of the crease set is unstable at higher order. Even if this is the case, however, there are differences in the sensitivity of the crease set among the modes. The TOEH is insensitive to the odd parity mode and $L = 1$ even parity mode.

6 The structural stability of the topology of the event horizon

In the previous section, it was shown that the spherical TOEH is unstable under linear perturbation. Since there is no appropriate example of a spacetime, however, with non-spherical topology, similar analysis is impossible for other TOEHs. In this section we discuss the structural stability of the crease set of the EH in the context of catastrophe theory. As discussed in §2, this corresponds to the stability of the TOEH. First, we investigate this structural stability in a (2+1)-dimensional spacetime.

6.1 In (2+1)-dimensional spacetime

The plan of analysis is following. First, we consider the appropriate wave front of light in a flat spacetime. According to geometrical optics, the wave front produces backward caustics and the endpoints of a null surface related to the wave front. In the context of catastrophe theory, Thom's Theorem states that the structures of such caustics are classified[24] if they are structurally stable. Thus we analyze the structure of the caustics and judge whether the crease set is classified by Thom's Theorem. Here, we consider the structural stability as corresponding to stability under the small change of the shape of the wave front and the local geometry around the endpoints. The shape of the wave front reflects the global structure of the spacetime between $\mathcal{J}^+$ and the wave front. Furthermore, the stability is that of only the local structure of the caustics. Therefore, to discuss stability in a flat spacetime is valid as long as we deal with the structure of a small neighborhood.

For simplicity, we consider only an elliptical wave front,

$$E_2 : \left(\frac{x-x_0}{a} \right)^2 + \left(\frac{y}{b} \right)^2 = 1 \quad (24)$$

$$x_0 = -\frac{a^2 - b^2}{a}, \quad a \geq b. \quad (25)$$

Then, the square of the distance between (x, y) and (X, Y) is given by

$$f_{XY}(\mathbf{x}) = (X - x)^2 + (Y - y)^2 \quad (26)$$

$$= \left(X - \left(a \sqrt{1 - (y/b)^2} + \frac{b^2 - a^2}{a} \right) \right)^2 + (Y - y)^2, \quad (27)$$

where (X, Y) is an arbitrary point and $x - x_0$ is positive. As known from geometrical optics, in a flat spacetime, a light path through (X, Y) is given by the stationary points of $f_{XY}(\mathbf{x}, y)$:

$$\frac{\partial f_{XY}(\mathbf{x})}{\partial y} = 0 \Rightarrow Y = A(y)X + B(y) \quad (28)$$

$$A(y) = - \left(\frac{\partial x(y)}{\partial y} \right)_{E_2}, \quad B(y) = x \left(\frac{\partial x(y)}{\partial y} \right)_{E_2} + y, \quad (29)$$

where $(\partial/\partial)_{E_2}$ represents partial differentiation with the constraint E_2 . The light paths are drawn in Fig. 3. From this figure, we see that they form a caustic at the origin, and the set of endpoints of the null surface concerning the wave front is a one-dimensional set, an interval on the x -axis $[2x_0, 0]$.

To see the structure of the caustic, we derive the Taylor series of $f_{XY}(\mathbf{x})$ around the origin,

$$\widetilde{f_{XY}}(\mathbf{x}) = \left(\frac{b^4}{a^2} - \frac{2b^2X}{a} + X^2 + Y^2 \right) - 2Yy + \frac{aXy^2}{b^2} \quad (30)$$

$$+ \left(\frac{a^2}{4b^4} - \frac{1}{4b^2} + \frac{aX}{4b^4} \right) y^4 + O(y^5). \quad (31)$$

Hence, the light paths form a cusp (a type A_3 catastrophe) at $(X, Y) = (0, 0)$ because $\widetilde{f} \sim y^4$. From Thom's Theorem, we know it is structurally stable except for at $a = b$. Of course, $a = b$ corresponds the circular wave front and the zero-dimensional crease set.

6.2 In (3+1)-dimensional spacetime

For (3+1)-spacetime, the investigation above can be carried out in a similar manner, though the situation becomes a little complex. In this case, there are three possibilities of the crease set of the EH, even after sufficient simplification. As shown in Fig. 4, the endpoint forms a point, line, or surface. As in the previous subsection, we consider the ellipsoidal wave front,

$$E_3 : \left(\frac{x}{a} \right)^2 + \left(\frac{y}{b} \right)^2 + \left(\frac{z-z_0}{c} \right)^2 = 1 \quad (32)$$

$$z_0 = -\frac{c^2 - a^2}{c}, \quad 0 < a \leq b \leq c. \quad (33)$$

For a $z - z_0 > 0$ branch, the square of the distance between (x, y, z) and an arbitrary point (X, Y, Z) is given by

$$f_{XYZ}(\mathbf{x}) = (X - x)^2 + (Y - y)^2 + (Z - z)^2 \quad (34)$$

$$= (X - x)^2 + (Y - y)^2 + \left(Z - \left(c \sqrt{1 - (x/a)^2 - (y/b)^2} + z_0 \right) \right)^2. \quad (35)$$

The light path through (X, Y, Z) is given by

$$\frac{\partial f_{XYZ}(x)}{\partial x} = 0, \quad \frac{\partial f_{XYZ}(x)}{\partial y} = 0 \quad (36)$$

$$\Rightarrow X = A(x, y)Z + B(x, y), \quad \text{and} \quad Y = C(x, y)Z + D(x, y), \quad (37)$$

$$A = -\left(\frac{\partial z(x, y)}{\partial x}\right)_{E_3}, \quad B = z\left(\frac{\partial z(x, y)}{\partial x}\right)_{E_3} + x, \quad (38)$$

$$C = -\left(\frac{\partial z(x, y)}{\partial y}\right)_{E_3}, \quad D = z\left(\frac{\partial z(x, y)}{\partial y}\right)_{E_3} + y. \quad (39)$$

From Fig. 5, exhibiting the light paths, it is seen that a caustic is formed around the origin. Only when a, b and c are equal the crease set becomes zero-dimensional (at the origin). The relations $a = b \neq c$ imply that endpoints form a one-dimensional set which is an interval on the z -axis, $[2z_0, 0]$. Otherwise, the crease set is two-dimensional (Fig. 4).

The Taylor series of the potential f at the origin is given by

$$\widetilde{f_{X=0}}(x) = \frac{a^4}{c^2} + \frac{-a^2 + c^2}{4a^4}x^4 + \frac{-a^2 + c^2}{8a^6}x^6 + \frac{b^2 - a^2}{b^2}y^2 + \frac{-a^2 + c^2}{2a^2b^2}x^2y^2 \quad (40)$$

$$+ \frac{3(-a^2 + c^2)}{8a^4b^2}x^4y^2 + \frac{-a^2 + c^2}{4b^4}y^4 + \frac{3(-a^2 + c^2)}{8a^2b^4}x^2y^4 + \frac{-a^2 + c^2}{8b^6}y^6 \quad (41)$$

$$+ O(x^7). \quad (42)$$

The structure of the caustic is controlled by the leading term of $\widetilde{f}$ about x, y . When $a < b \leq c$, $\widetilde{f} \sim \alpha x^4 + \beta y^2 + \gamma$ produces a cusp (type A_3). Then, the two-dimensional crease set is structurally stable. On the other hand, $\widetilde{f}$ becomes $\alpha(y^4 + 2x^2y^2 + x^4) + \gamma$ with $a = b \neq c$. This case corresponds to the line crease set and it is not structurally stable. Incidentally, if $a < b < c$, there is also another cusp at $(0, 0, -(b^2 - a^2)/c)$ (as seen by carefully studying Fig. 5). The Taylor expansion of f around this cusp is given by

$$\widetilde{f_{X=X_c}}(x) = \frac{b^4}{c^2} + \frac{-b^2 + c^2}{4b^4}y^4 + \frac{-b^2 + c^2}{8b^6}y^6 + \frac{a^2 - b^2}{a^2}x^2 + \frac{-b^2 + c^2}{2a^2b^2}x^2y^2 \quad (43)$$

$$+ \frac{3(-b^2 + c^2)}{8a^2b^4}x^2y^4 + \frac{-b^2 + c^2}{4a^4}x^4 + \frac{3(-b^2 + c^2)}{8a^4b^2}x^4y^2 + \frac{-b^2 + c^2}{8a^6}x^6 \quad (44)$$

$$+ O(x^7). \quad (45)$$

This implies that the cusp is also stable as long as $a \neq b$ and $b \neq c$. With $b \rightarrow a$, this cusp approaches the cusp at the origin, and they degenerate into an unstable structure. Contrastingly, when b is equal to c , this cusp disappears at the center of the ellipsoid. The example given in Ref.[2] corresponds to this case. Of course, the zero-dimensional crease set ($a = b = c$) is not structurally stable.

7 An example and conjecture

Figure 1, and the discussion in the section 4 and 5 suggest that the TOEH is affected by the distortion of the black hole. Then, qualitative discussion is possible in regard to the relation between the TOEH and collapsing matter.

7.1 The hoop conjecture and KT-solution

First, we discuss the hoop conjecture in the sense of the TOEH, analyzing the EH of the Kastor-Traschen (KT) solution.[6] The KT solution is the only known exact solution describing the coalescence of black holes. Since its EH is an eternal one, it does not satisfy the assumption of §4. Nevertheless, it can be stated that the crease set is acausal and connected (not compact) also in the case considered here (which is the coalescence of two black holes with identical mass parameters). This fact leads us to relate the KT solution to the hoop conjecture.

On the grounds of physical considerations, one expects the formation of a black hole to occur when the matter or inhomogeneity of spacetime itself (non-linear gravitational waves) is concentrated into a sufficiently “small” region. This might be the basis of the so-called *hoop conjecture* (HC) proposed by Thorne[26] which states that *black holes with horizons form when and only when a mass M gets compacted into a region whose circumference in every direction is $C \leq 4\pi M$* . We point out here, as mentioned in §4 that the coalescence of a black hole can be regarded as the formation of a black hole which is expected to be highly distorted. Hence, if we adopt a timeslicing in which the two stars *instantaneously* form a single, highly distorted black hole, the length scale of the black hole on this hypersurface is expected to be of the order of the Schwarzschild radius determined by its gravitational mass. Thus, the discussion in this subsection shows that the KT spacetime provides a good example, at the intuitive level, for this conjecture, at least as formulated for event horizons.

Since there is a positive cosmological constant, Λ , in the KT spacetime, this spacetime is not asymptotically flat but, rather, asymptotically de Sitter. Although the hoop conjecture has been proposed for an asymptotically flat spacetime, if the mass scale of black holes is much smaller than the cosmological horizon scale, $\sqrt{3/\Lambda}$, this conjecture should hold also in an asymptotically de Sitter spacetime. Furthermore it has been suggested that the hoop conjecture holds even for the case of gravitational collapse with a large mass scale in such an asymptotically de Sitter spacetime.[27]

Now, we focus on the KT spacetime of two black holes with identical mass parameters. The line element and the electromagnetic potential one-form in the contracting cosmological chart are given in the form

$$ds^2 = -U^{-2}d\tau^2 + U^2(dx^2 + dy^2 + dz^2), \quad (46)$$

$$A = -U^{-1}d\tau, \quad (47)$$

where

$$U = -H\tau + \frac{m}{r_+} + \frac{m}{r_-}, \quad (48)$$

$$r_{\pm} = \sqrt{x^2 + y^2 + (z \mp a)^2}, \quad (49)$$

and H , m and a are constant. The constant H is related to the cosmological constant via $H = \sqrt{\Lambda/3}$. The above line element represents two point sources, each with mass parameter m and charge $|e| = m$, located at $\mathbf{x} = (0, 0, \pm a)$ in the background Euclidean 3-space. The AD mass, M_{tot} , of this system is given by $2m$, which must be less than the critical value $M_{crit} = 1/4H$, so that the spacetime describes the merging of two black holes. Then its EH is located in the range $\tau < 0$. With regard to the region of the spacetime which is covered by the above cosmological chart, see Refs.[28] and [29].

We numerically integrated equations for the null geodesics backward in time from a sufficiently late time $\tau = -\epsilon$, where ϵ is a sufficiently small positive parameter. In the contracting cosmological chart (47), the marginally trapped surface enclosing both black holes goes to infinity, $r \equiv \sqrt{x^2 + y^2 + z^2} \rightarrow +\infty$, in the limit $\epsilon \rightarrow 0$. [29] Hence, at $\tau = -\epsilon$, the marginally trapped surface enclosing both holes is sufficiently spherical, since the higher multipole moments are much smaller than the monopole term in the line element (47) in the limit $r \rightarrow +\infty$. The vicinity of the marginally trapped surface is well approximated by the one-black-hole solution, i.e., the Reissner-Nordström-de Sitter (RNdS) spacetime. Moreover, for this reason, the EH is very close to the marginally trapped surface enclosing both holes at $\tau = -\epsilon$. Therefore we integrate the equations for the past directed ingoing null geodesics normal to the marginally trapped sphere at $\tau = -\epsilon$, which is easily derived as

$$r = \frac{(1 - \sqrt{1 - 8Hm})/2H - 2m}{H\epsilon}. \quad (50)$$

Here the *ingoing* direction is determined in the usual sense with respect to the radial coordinate r . The above expression agrees with the location of the event horizon in the RNdS solution with mass parameter $2m$. [30] If we choose sufficiently small ϵ , the deviation of the spacelike two-sphere defined by Eq.(50) from the EH should be smaller than the error due to the numerical integration.

Figure ?? in Ref.[31] depicts the time evolution of the EH in the contracting cosmological chart (47), which shows the usual picture of a two-black-hole collision. Adopting the normalization $H = 1$, Fig. ??

in Ref.[31] corresponds to the case with parameters $m = 0.1$ and $\alpha = 0.1$. Then the AD mass of this system is $M_{tot} = 0.8M_{crit}$. As can be seen from this figure, the crease set of the EH is a curve. In order to investigate whether or not the crease set is acausal, we have calculated the quantity

$$N \equiv g_{\mu\nu} m^\mu m^\nu, \quad (51)$$

where $g_{\mu\nu}$ is the metric tensor which gives the line element (47) and m^μ is the tangent vector of the crease set. With this analysis it is confirmed that the crease set is a spacelike one-dimensional curve.

In Fig. ?? of Ref.[31] we give the numerical result for the proper length of the crease set, which is related to a spindle EH, as a function of τ . This figure shows that the length of the crease set converges on πM_{tot} from below, while the crease set is non-compact. This implies that there exists a distorted hoop surrounding the crease set whose circumference never exceeds $4\pi M_{tot}$.

Although the KT solution considered here describes an eternal black hole spacetime, we regard this spacetime as the far future of two stars composed of charged fluid. In the “past” of this picture, the physical distance on the cosmological chart (47) between each star is sufficiently large. Thus these stars should collapse and be enclosed by their marginally trapped surfaces, which do not enclose both stars but, rather, only one of the two. Let us consider such a case. Since the crease set is a spacelike curve, we can always adopt a timeslicing which includes a spacelike hypersurface, as shown in Fig. ??; the shape of a section of the event horizon for this spacelike hypersurface is dumbbell-like. Our result that the length of the crease set of the KT solution converges on $2\pi M_{tot}$ suggests that there should not be the case in which the circumference of the EH produced from two massive charged stars is longer than order $4\pi M_{tot}$ by such an appropriate choice of the timeslicing, even if the initial separation in the usual picture is much larger than order M_{tot} . We should note that there exist marginally trapped surfaces which enclose each hole (star) and will be apparent horizons, inside that EH.[29] This implies that the circumference of a black hole with (apparent) horizons is just bound by $4\pi M_{tot}$ in general. In this sense, the hoop conjecture seems to hold even in such a spacetime including a cosmological constant. Here it should be noted that the result presented here is only a tentative one. More detailed and confirmed results will appear in our forthcoming paper.[6]

7.2 Speculation: Scale conjecture of the TOEH

One may expect to give some restriction on the TOEH introducing certain conditions on the matter field. The present results in the section 4,5,6 imply, however, that this is probably hopeless. It seems that the symmetry of the spacetime affects the structure of the crease set of the EH and the TOEH, and it is easily disturbed by perturbation. If one is not concerned with the scale of the topological structure of the EH, the TOEH can generally become complicated. On the other hand, the discussion is the previous subsection is based on the expected relation between the scale of the TOEH and the deformation of the black hole, where the scale dependent structure of the topology would be defined in a certain manner, for example Ref. [32]. Here, taking the standpoint that we believe the existence of something like a “hoop theorem” in a wide class of gravitational collapses, we also conjecture that the TOEH cannot be non-spherical in the sense of a smoothed topology (if it can be well defined) in its mass scale without a naked singularity because, if we observe the non-spherical TOEH with a determination in its mass scale, the size of the crease set should be larger than the mass scale. The hoop theorem, however, will not admit such a deformed collapse that the crease set has the length of the mass scale, without the naked singularity.

Conjecture 7.1 (The hoop theorem of the TOEH) *Smoothing the topological structure of an EH by its mass scale, the EH whose (spatial) topology is not a single sphere cannot be formed without a naked singularity.*

Here, we must note that there is the possibility of a wide crease set without the corresponding configuration of matter, since the structure of the EH can also be determined by the asymptotic structure of a spacetime. Such a possibility would have to be eliminated by a certain asymptotic condition. Assuming the hoop theorem,[34][33] it might not be so difficult to construct the theorem conjectured here.

8 Summary and discussion

We have studied the topology of the EH (TOEH), partially considering the indifferentiability of the EH. We have found that the coalescence of EHs is related to the one-dimensional crease set and a torus EH is related to the two-dimensional crease set. Furthermore these changes of the TOEH can be removed by an appropriate timeslicing, since the crease set of an EH is a connected acausal set. We see that the TOEH depends strongly on the timeslicing. The dimension of the crease set, however, plays an important role for the TOEH and, of course, is invariant under the change of the timeslicing. Then, a question arises, which dimension is generally possible for the crease set.

Following these results, we have investigated the stability of the topology of the EH (TOEH). First the stability of a spherical topology was investigated under linear perturbation in a spherically symmetric background. To linear order, the $L = 2$ even parity mode changes the structure of the crease set and the TOEH, and the odd parity mode and $L = 1$ mode do not. At higher order, however, mode coupling between the modes with different L or parity cause the instability of the TOEH even for these modes not changing the TOEH at the linear order. For $L > 2$ even parity modes, more detailed investigation is required. In any case, we have seen that the trivial TOEH is generally unstable under linear perturbation.

In this discussion of the linear perturbation, we consider the Oppenheimer-Snyder spacetime as an example of an always spherical EH. Nevertheless, the result would be same for other non-eternal EHs with spherical symmetry, since we have never used a concrete geometry of the spacetime, other than the spherical symmetry.

How can we interpret the fact that the TOEH is insensitive to some modes of the perturbation? In a sense, when we give an odd parity or $L = 1$ perturbation to the spherical EH, the change of the TOEH at higher order would not be detectable (though it is not trivial how one can observe it). This follows from the fact that, while a local geometry around an observer is perturbed with the same strength as the given perturbation, the TOEH is not so perturbed. The change of the TOEH at higher order would be prevented from being observed.

Next, using simple analysis from catastrophe theory, the structural stability of the crease set was studied in a more general situation. Assuming an ellipsoidal wave front, the stability of zero-, one-, and two-dimensional crease sets were investigated. We see that the two-dimensional crease set is stable, and the one- and zero-dimensional crease sets are not. Therefore, TOEHs with handles (torus, double torus, ...) are generic.

Though in the present article we find only the crease set with a cusp catastrophe, as discussed in Ref. [25], there is the possibility of additional types, the ‘swallowtail’, the ‘pyramid’, and so on. They should form other structurally stable crease sets. The TOEH in these cases them will be studied in a forthcoming work.

As an example of such a change of the TOEH, the KT solution is examined in its causal structure. Determining the EH of the KT solution describing the collision of two black holes with identical mass numerically, we see that the length of its crease set converges on a finite value πM_{tot} from below, while the crease set is non-compact. As mentioned in the §4, the crease set is a connected acausal set, and we can relate it to the hoop conjecture, because, since such an EH can be regarded as the formation of a distorted black hole in a certain timeslicing. Then the crease set with its length being $2\pi M_{tot}$ implies the existence of a hoop whose circumference is $4\pi M_{tot}$. In this sense, the bounds for the length of the crease set in the KT solution support the hoop conjecture.

A conjecture about the scale of the TOEH was given. Assuming the “hoop theorem” we expect that only topological structure (e.g. handles) smaller than its mass scale is admitted without naked singularity. After the definition of a scale-dependent topological structure, this conjecture could be proved in the situation where a restricted hoop theorem holds.

Finally we are reminded of an essential question. How can we see the topology of the EH? Some of the previous works, for example “topological censorship”, [9] stresses that it is impossible. On the contrary, we expect phenomena depending strongly on the existence of the EH as the boundary condition of fields, for instance the quasi-normal mode of gravitational waves [21] or Hawking radiation, [22] reflects the TOEH. For example, with regard to Hawking radiation, we would like to construct a toy model for the change of the TOEH, something like the Rindler spacetime for the Schwarzschild spacetime. This is our future problem.

Acknowledgements

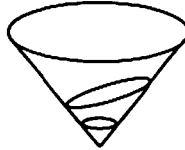
We would like to thank G. Galilei and I. Newton.

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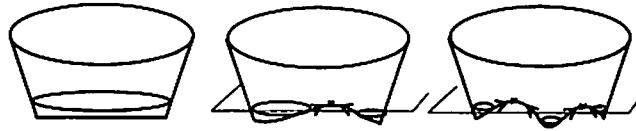
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zero-dimensional crease set



one-dimensional crease set



two-dimensional crease set

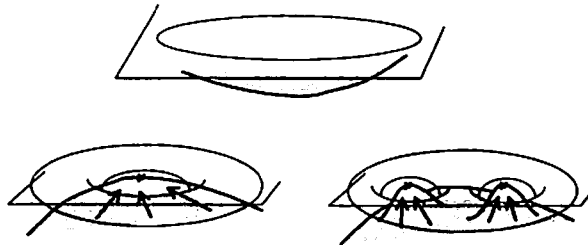


Figure 1: EHs with the zero-, one- and two-dimensional crease sets are shown. We see that the one-dimensional crease set becomes a coalescence of an arbitrary number of spherical EHs. For the two-dimensional crease set, only sections of the EH and the crease set are drawn. It can become an EH with an arbitrary number of handles. It is also possible to change the EH into a trivial creation of a spherical EH.

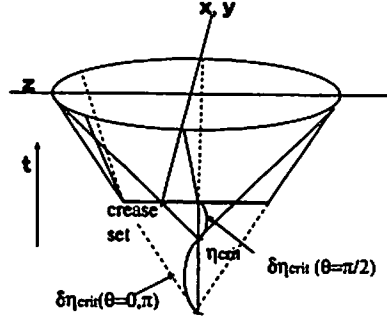


Figure 2: The latest light paths with maximal $\delta\eta_{crit}$ (x, y direction in this figure) form an endpoint at the origin with $\eta = \eta_{crit} + \delta\eta_{crit}(\theta = \pi/2)$. On the other hand, a light path on the other axis (z -axis in this figure) crosses light paths from other directions and forms an endpoint there. Thus the crease set comes to possess a dimension in this (z -)direction.

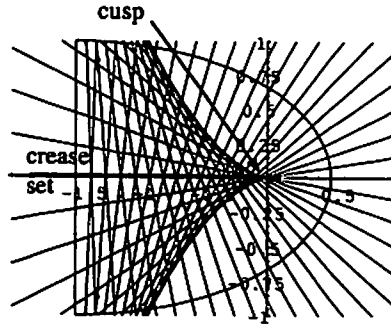


Figure 3: The light paths for the elliptic wave front with $a = 2, b = 1$. There are crossing points of the light paths which are the endpoints of a null surface corresponding to the wave front, on the x -axis, $[2x_0, 0]$. A cusp is formed at the origin.

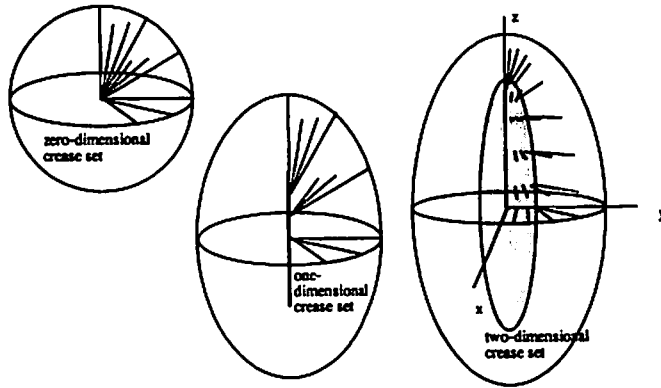


Figure 4: The crease set becomes zero-dimensional for the spherical wave front. In the prolate-spheroidal wave front, a one-dimensional crease set appears. Otherwise, the ellipsoidal wave front produces a two-dimensional crease set.

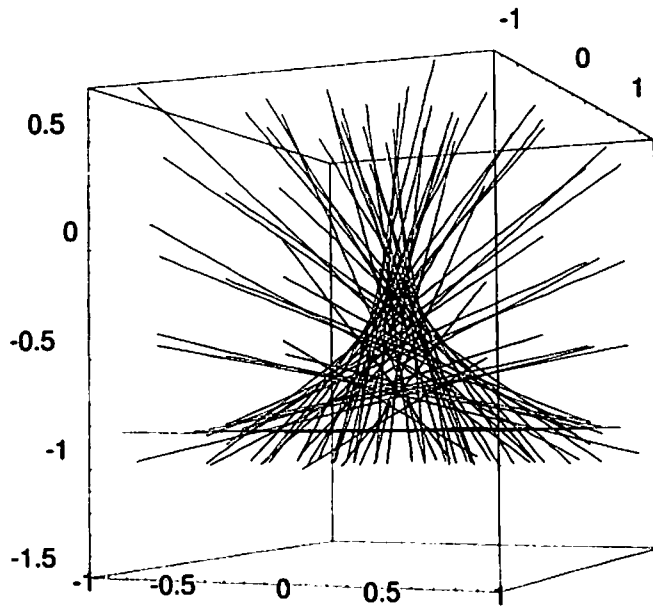


Figure 5: The light paths for the ellipsoidal wave front with $a = 1$, $b = 1.3$, $c = 1.5$. A cusp is formed at the origin. Studying this figure carefully, one sees that another cusp exists at $(0, 0, (b^2 - a^2)/c)$.

The Circumference of Black Holes in Kastor-Traschen Space-Time

- Implication for the Hoop Conjecture -

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The Kastor-Traschen solution is at present the only analytic solution which describes the collision of an arbitrary number of black holes. In this article, we investigate the dynamics of the event horizon of this space-time in the case of the collision of two black holes, using the ray-tracing method. We show the nature of the topology change of the event horizon and further show that the circumference of a section of the event horizon defined by an appropriate spacelike hypersurface has the length of the same order to the mass scale even if the separation of the black-holes is large enough. This fact is significant to understand general gravitational collapse without spherical symmetry and also for the hoop conjecture.

PACS numbers:

The Kastor-Traschen (KT) space-time is a family of solutions to the Einstein-Maxwell equations with a positive cosmological constant, Λ [1]. This space-time represents a multi-black-hole cosmos and each black hole has a $U(1)$ charge whose absolute value is equal to its mass parameter. The remarkable property of this solution is to describe the collisions of black holes. Hence in order to understand the dynamical process of the collision between compact objects and the following formations of black holes without spherical symmetry, it is important to make both the local and global structures of this space-time clear.

After this space-time was discovered, Brill et al. [2] and Brill [3] investigated, in part, the occurrence of the event horizon of this space-time and its global structure. The dynamics of marginally trapped surfaces in this space-time has been investigated by Nakao, Shiromizu and Hayward for the case of two black holes with an identical mass parameter [4]. These researches revealed that the KT space-time really describes the merging process of black holes if the Abbott-Deser (AD) mass [5] of

the system is smaller than or equal to the critical value, $M_{cr} \equiv \sqrt{3/16\Lambda}$.

In this article, we numerically investigate the location of the event horizon of the KT space-time in the case of two black holes with an identical mass parameter by the ray-tracing method [6,7]. By this investigation, the evolution of the topology of the event horizon in this space-time is clarified.

Recently, the numerical researches have shown that the section of the event horizon by a spacelike hypersurface might have a non-spherical topology [6,8]. In connection with the topological censorship about the event horizon [9,10], there were several theoretical studies of the topology of the event horizon [11–13]. Shapiro, Teukolsky and Winicour have shown that the numerical result given by Hughes et al. [6] does not conflict with the topological censorship theorem, using a simple flat space-time example [12].

Later one of the present authors revealed several important features of the topology of the event horizon [14]. This reference discussed the set composed of endpoints

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of null geodesic generators of an event horizon, which in this article will be called the *seam* of the event horizon. For an event horizon with trouser topology such as the one considered here, this reflects the well known tailor's theorem that trousers must have a seam. A remarkable fact pointed out in Ref. [14] is that the seam is spacelike and connected, with the assumption of a non-eternal black hole. Similarly, one can easily comprehend that the spacelike nature holds even for an eternal black hole like the KT space-time. This means that one can choose a spacelike hypersurface which crosses the seam arbitrarily often. This leads to the further important consequence that the number of black holes, or more exactly speaking, the number of connected components of a section of an event horizon by a spacelike hypersurface, depends on the choice of the time slicing. In other words, the number of black holes is a gauge-dependent notion, in the case of general non-spherically symmetric space-times [15]. Thus, for example, the statement of the *two-black-hole collision* is not gauge invariant. We can imagine that two massive stars which are very distant from each other collapse into "two black holes" and later they coalesce with each other to form one final black hole. However, if we adopt certain time slicings, the picture of collapse is drastically altered: two massive stars collapse into a single highly deformed black hole (a spindle black hole). Furthermore, by adopting another kind of time slicing, where the seam repeatedly crosses the spacelike hypersurface, it is possible to regard this space-time as, for example, that of a fifteen-black-hole collision.

On the grounds of physical consideration, one expects the formation of a black hole to occur when the matter or inhomogeneity of the space-time itself (non-linear gravitational waves) is concentrated into a sufficiently "small" region. This might be the base of the so-called *hoop conjecture* (HC) proposed by Thorne [16] which states *black holes with horizons form when and only when a mass M gets compacted into a region whose circumference in every direction is $C \lesssim 4\pi M$* [17]. Hence, if we adopt the time slicing on which the two stars form *instantaneously* a single highly deformed black hole, the length scale of the black hole on this hypersurface is expected to be the order of the Schwarzschild radius determined by its gravitational mass.

Since there is a positive cosmological constant, Λ , in the KT space-time, this space-time is not asymptotically flat but is asymptotically de Sitter. Although the hoop conjecture has been proposed for an asymptotically flat space-time, if the mass scale of black holes is much smaller than the cosmological horizon scale, $\sqrt{3/\Lambda}$, this conjecture should hold also in an asymptotically de Sitter space-time. Furthermore it has been suggested that the hoop conjecture holds even for the case of gravitational collapse with a large mass scale in such an asymptotically de Sitter space-time [18].

In this article, we follow the notation of Misner, Thorne and Wheeler [17] and adopt the geometrized units.

As mentioned, in this article, we focus on the KT

space-time of two black holes with an identical mass parameter. The line element and the electromagnetic potential one-form in the contracting cosmological chart are given in the form

$$ds^2 = -U^{-2}d\tau^2 + U^2(dx^2 + dy^2 + dz^2), \quad (1)$$

$$A = -U^{-1}d\tau, \quad (2)$$

where

$$U = -H\tau + \frac{m}{r_+} + \frac{m}{r_-}, \quad (3)$$

$$r_{\pm} = \sqrt{x^2 + y^2 + (z \mp a)^2}, \quad (4)$$

and H , m and a are constant. The constant H is related to the cosmological constant via $H = \sqrt{\Lambda/3}$. The above line element represents two point sources, each with mass parameter m and charge $|e| = m$, located at $x = (0, 0, \pm a)$ in the background Euclidean 3-space E^3 . The AD mass, M_{tot} , of this system is given by $2m$ which must be less than the critical value $M_{\text{crit}} = 1/4H$, so that the space-time describes the merging of two black holes. Then the event horizon is located in the range $\tau < 0$. Concerning what region of the space-time is covered by the above cosmological chart, see Refs. [2] and [4].

Here we should comment on the procedure to find the event horizon. The definition of the event horizon given by Hawking is the boundary of the causal past of future null infinity in the case of an asymptotically flat space-time [19]. For an asymptotically de Sitter space-time, the same definition is applicable by virtue of the existence of well defined future null infinity [20]. An important consequence from the definition of the event horizon is that both in the asymptotically flat space-time and the asymptotically de Sitter one, the trapped regions are located inside event horizons. The method of Hughes et al. [6] is to solve large numbers of future directed null geodesics starting from each space-time point to various directions and then to investigate whether those null geodesics go into the trapped region or not. The starting points from which the null geodesics enter into the trapped region are placed inside the event horizon and the others are outside of it. On the other hand, Anninos et al. [7] adopt a different method to simplify the numerical calculation drastically. The latter method uses the fact that the event horizon will be the attractor of null hypersurfaces generated by past directed ingoing null geodesics near the event horizon, and hence the equations for large numbers of null geodesics going to various directions need not be integrated. Here we essentially follow the latter method in the sense that the event horizon is determined backward in time. Nevertheless, we do not treat the equation for the null hypersurface generated by past directed ingoing null geodesics but directly solve the equations for the null geodesic generators themselves. In order to determine whether the null geodesic generators approach an attractor which will be the event horizon, we investigate the area of a section of a null hypersurface by the spacelike hypersurface labeled by τ , since the

coordinate values of the null geodesic trajectories do not have any invariant meaning. More detailed analysis will be reported in our forthcoming work [21].

We numerically integrate equations for the null geodesics backward in time from a sufficiently late time $\tau = -\epsilon$, where ϵ is a sufficiently small positive parameter. In the contracting cosmological chart, (1), the marginally trapped surface enclosing both black holes goes to infinity, $r \equiv \sqrt{x^2 + y^2 + z^2} \rightarrow +\infty$, in the limit of $\epsilon \rightarrow 0$ [4]. Hence, at $\tau = -\epsilon$, the marginally trapped surface enclosing both holes is sufficiently spherical since the higher multipole moments are much smaller than the monopole term in the line element (1) in the limit of $r \rightarrow +\infty$. The vicinity of the marginally trapped surface is well approximated by one-black-hole solution, i.e., the Reissner-Nordström-de Sitter (RNdS) space-time [22]. Moreover, by this reason, the event horizon is very close to the marginally trapped surface enclosing both holes at $\tau = -\epsilon$. Therefore we integrate the equations for the past directed ingoing null geodesics normal to the marginally trapped sphere at $\tau = -\epsilon$, which is easily derived as

$$r = \frac{(1 - \sqrt{1 - 8Hm})/2H - 2m}{H\epsilon}. \quad (5)$$

Here *ingoing* direction is determined by the usual sense with respect to the radial coordinate r . The above expression agrees with the location of the event horizon in the RNdS solution with a mass parameter $2m$ [23]. If we choose sufficiently small ϵ , the deviation of the spacelike two-sphere defined by Eq.(5) from the event horizon should be smaller than the error due to the numerical integration.

Figure 1 depicts the time evolution of the event horizon in the contracting cosmological chart, (1), which shows the usual picture of a two black-hole collision. Adopting a normalization $H = 1$, Fig.1 corresponds to the case with parameters $m = 0.1$ and $a = 0.1$. Then the AD mass of this system is $M_{tot} = 0.8M_{crit}$. As can be seen from this figure, the seam (set of past endpoints of the null geodesic generators) of the event horizon is a curve. In order to investigate whether the seam is spacelike or not, we have calculated the following quantity,

$$N = g_{\mu\nu} m^\mu m^\nu, \quad (6)$$

where $m^\mu = dx^\mu/ds$ is the tangent vector of the seam, and it has been confirmed that N is always positive, i.e. the seam is the spacelike curve, which is expected from Ref. [14]. Then, we measure the following quantity,

$$L(\tau') = \int_{\tau > \tau'} N^{1/2} ds. \quad (7)$$

This is the proper length of the $\tau > \tau'$ part of the seam, which is related to a spindle event horizon. The result is shown in Fig.2. This figure shows that the length of the seam converges to a finite value of order M_{tot} . In addition, the result in the case $a = 100$, the other parameters

same to Fig.2, is shown in Fig.3. In this case also, the proper length of the seam converges like above case. It is contrary to intuition that this property of convergence is kept how far two black-holes were separated.

These results may be interpreted as follows. Roughly speaking, the KT solution describes comoving black holes in the contracting chart of de Sitter space-time. This means that in the past, in the sense τ has large enough negative value, the physical distance between each black-holes is large as far as $\tau = \text{constant}$ hypersurface is concerned, so one can not find a hoop of length smaller than $4\pi M_{tot}$ enclosing two black holes in this hypersurface. On the other hand, since the seam is a spacelike curve, we can always adopt the time slicing which includes a spacelike hypersurface as shown in Fig.5; the shape of a section of the event horizon by that spacelike hypersurface is dumbbell-like one. Now if the hypersurface is deformed as Fig.5, one may then find such a hoop according to above result, while in the deformed spacelike hypersurface, the black hole consists of one connected component, which is consistent with HC.

HC is usually associated to the gravitational collapse. The researches for the initial data [24–26] and of the dynamics of spindle gravitational collapses of collisionless matter [27] suggest that the HC catches the important physical feature of the gravitational collapse. While our result suggests that HC may be applied also to the case of the black-hole collision. If so, one may have the knowledge whether black holes collide or not from the local geometry.

I found a mistake in the calculations reported at the meeting. It was reported that the ‘hoop’ of the black hole in the Kastor-Traschen space-time has infinite length, so the hoop conjecture may not be correct. However, it was found out by subsequent check of the calculations that the calculations reported were unreliable after the length of the hoop had begun to diverge. The results with more careful treatments are reported here.

ACKNOWLEDGMENTS

We are grateful H. Sato for his encouraging discussion. This work was partly supported by the Grant-in-Aid for ????(No.????) from the Ministry of Education, Science, Sports and Culture.

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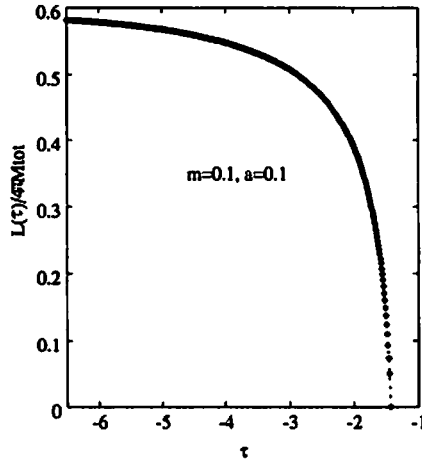


FIG. 1. The shape of the EH of the KT space-time with parameters $m = 0.1, a = 0.1$. A closed curve represents the section of the EH by the $\tau = \text{constant}$ hypersurface.

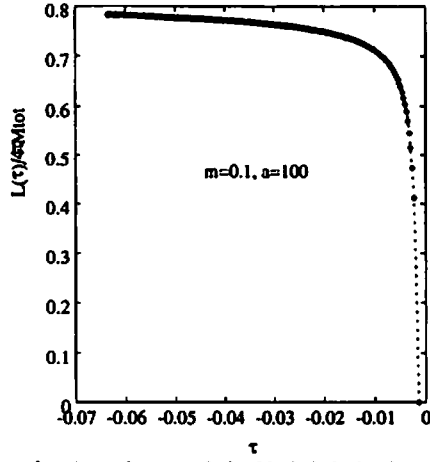


FIG. 2. $L(\tau)$ which is analogous to the circumference of the black hole is shown as a function of τ with parameters $m = 0.1, a = 0.1$.

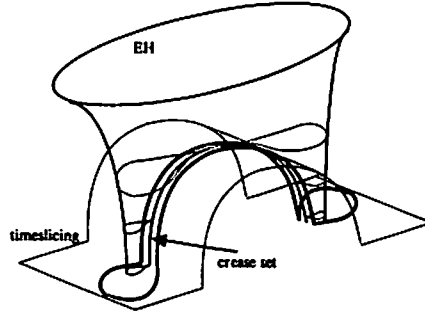


FIG. 3. Same to Fig.3 but with a parameter $a = 100$.

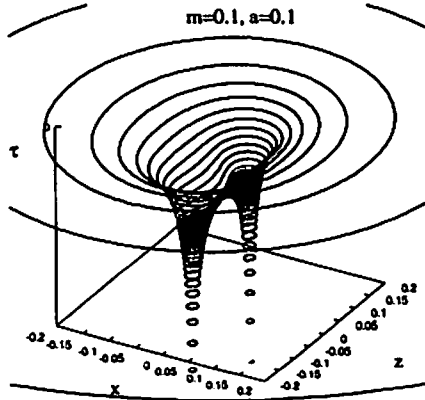


FIG. 4. Since the crease set is a spacelike curve, it may be possible to define the circumference of the black hole as the length of the crease set by taking a time slicing like this figure.

Cosmic string coupled to gravitational waves

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Abstract

The interaction between a straight infinitesimally thin string and gravitational waves is considered. To investigate the interaction between them, we propose the string bundle shell which can be regarded as one of regularized cosmic string. We define the deformation variables of string and solve the scattering problem of gravitational wave by straight strings. We show that the scattering data is no phase shift nor resonance, nevertheless the string is deformed by the gravitational waves.

1 Introduction

Domain walls and cosmic strings are topological defects associated with the symmetry breaking of unified theories in the early universe. They are also idealized by infinitesimally thin Nambu-Goto wall or string and their motions are determined by the minimization of their world volume or world area. By using these infinitesimally thin wall or string, one can avoid introducing internal structure. If one neglects its gravity, the Nambu-Goto action has oscillatory solutions and the oscillations will become to be a source of gravitational waves[1]. However, in the case of domain walls coupled to gravitational field, the recent investigations[2] using special models show that there is no dynamical freedom of free oscillations. This result shows that the picture in the treatment neglecting gravity is incorrect; the motion of test wall and the one of a self-gravitating wall are quite different. How about cosmic strings?

In general relativity, there is a natural mathematical frame work only for three dimensional singular hypersurface in four dimensional spacetime[3], which is used in Ref.[2]. However, as emphasized in Ref.[4], there exists no simple general prescription for obtaining the physical characteristics of an arbitrary line source, nevertheless there are some examples which describe a self-gravitating cosmic string[5]. One of well-known examples of a self-gravitating straight infinitesimally thin cosmic string is the conical Minkowski spacetime with a finite deficit angle[6]. To clarify the interaction between a self-gravitating cosmic string and gravitational waves, one might think that the perturbation with the above background spacetime is useful tool. However, this background might not be appropriate to investigate the interaction between string and gravitational waves. Because we must impose the boundary condition on perturbations at the singular line which describes the string and the boundary condition at this singular line is not clear.

Thus, to discuss the interaction between a cosmic string and gravitational waves, it is necessary to propose a procedure of regularization of cosmic string spacetime which preserves the nature of cosmic strings. In this talk, we propose a regularized model as discussed below to describe a self-gravitating straight string and its perturbative motion and we discuss the interaction between the string model and gravitational waves. Our main idea to construct background self-gravitating string and its perturbative motion is based on a cylindrical shell with anisotropic tensions, which can be regarded as *string bundle*. To construct string spacetime, we take a limit in which the radius of the shell vanishes. Although more detailed discussion about the relation between our model and cosmic string are necessary, we only show the preliminary results here. Complete our discussion will be seen elsewhere.

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2 Cylindrical Shell and Straight Cosmic String

Here, we consider the cylindrical shell with anisotropic tensions in a vacuum cylindrically symmetric spacetime with two Killing vectors $z^\mu = (\partial_z)^\mu$ and $\phi^\mu = (\partial_\phi)^\mu$. We denote the world hypersheet of this shell by Σ . The shell separate the spacetime into two vacuum regions $\mathcal{M}_\pm$ with boundary $\partial\mathcal{M}_\pm$. By the identification $\partial\mathcal{M}_- = \Sigma = \partial\mathcal{M}_+$, whole spacetime consist of $\mathcal{M}_- \cup \Sigma \cup \mathcal{M}_+$. Since these vacuum regions $\mathcal{M}_+$ and $\mathcal{M}_-$ have cylindrical symmetry, it is natural to assume that both $\mathcal{M}_\pm$ are described by the Weyl canonical form[7],

$$ds^2 = e^{2(\gamma-\psi)}(-dt^2 + d\rho^2) + e^{2\psi}dz^2 + \rho^2 e^{-2\psi}d\phi^2, \quad (1)$$

where ψ and γ are functions of t and ρ . From this metric (1), the Einstein equations can be reduced to the following three equations,

$$\partial_t^2 \psi - \frac{1}{\rho} \partial_\rho (\rho \partial_\rho \psi) = 0, \quad \partial_\rho \gamma = \rho ((\partial_t \psi)^2 + (\partial_\rho \psi)^2), \quad \partial_t \gamma = 2\rho (\partial_t \psi)(\partial_\rho \psi). \quad (2)$$

The metric on $\mathcal{M}_\pm$ must satisfy with these equations. Here, we note that ψ corresponds to the cylindrical plus modes of gravitational waves.

Next, we derive the junction condition at the shell with anisotropic tension. Since we consider a cylindrical shell, the intrinsic metric on Σ is described by

$$ds^2|_\Sigma = q_{ij}dx^i dx^j = -d\tau^2 + e^{2\psi_0}dz^2 + \rho_0^2 e^{-2\psi_0}d\phi^2, \quad (3)$$

where τ is the proper time function of Σ , $\psi_0 = \psi_0(\tau)$ is the boundary value of $\psi(\rho, t)$, $\rho_0 = \rho_0(\tau)$ is the radius of the shell and $x^i = (\tau, z, \phi)$. The four dimensional extension $q_{\mu\nu}$ of three metric q_{ij} is the projection operator from the tangent space on $\mathcal{M}$ to that on Σ . On the other hand, junction formula also gives the condition for the extrinsic curvature $K_{j\pm}^i = -q^{i\mu} q_j^\nu \nabla_\mu n_{\nu\pm}$ of $\partial\mathcal{M}_\pm$ in $\mathcal{M}_\pm$ where $n_{\mu\pm}$ are unit normal to $\partial\mathcal{M}_\pm$, respectively, and the identification $\partial\mathcal{M}_- = \Sigma = \partial\mathcal{M}_+$ is done so that $n_{\mu+} = n_{\mu-} = n_\mu$, where n^μ directed from $\mathcal{M}_-$ to $\mathcal{M}_+$. In terms of this extrinsic curvature, the junction condition[3] is reduced to

$$[K_j^i] = 8\pi G \left(S_j^i - \frac{1}{2} \delta_j^i S_l^l \right), \quad (4)$$

where

$$[A] = A_+ - A_-, \quad (5)$$

for arbitrary functions $A_\pm$ on $\partial\mathcal{M}_\pm$. Here, S_j^i is the surface energy momentum tensor of the shell. Since we consider a cylindrical shell with anisotropic tensions, the four dimensional extension of the surface energy momentum tensor of this shell is given by

$$S_{\mu\nu} = \sigma(\tau)u_\mu u_\nu - \zeta(\tau)z_\mu z_\nu - \xi(\tau)\phi_\mu \phi_\nu, \quad (6)$$

where u_μ is the four velocity of Σ .

The non-vanishing components of the four-velocity of u^μ and the normal vector $n_\pm^\mu$ in terms of the Weyl canonical coordinate are given by

$$u_\pm^i = \frac{dt_\pm}{d\tau} = \sqrt{\left(\frac{d\rho_0}{d\tau}\right)^2 + e^{-2(\gamma_\pm - \psi_\pm)}}, \quad u^\rho = \frac{d\rho_0}{d\tau}, \quad n_\pm^t = \epsilon_\pm \frac{d\rho_0}{d\tau}, \quad n_\pm^\rho = \epsilon_\pm u_\pm^t, \quad (7)$$

where $\gamma_\pm = \gamma_\pm(\tau)$ are boundary values of the metric function $\gamma(t, \rho)$ in (1) on $\partial\mathcal{M}_\pm$, $\epsilon_\pm = \text{sgn}(n_\pm^\mu \partial_\mu r)$ and we used the orthonormal conditions $n_\pm^\mu n_{\mu\pm} = -u_\pm^\mu u_{\mu\pm} = 1$ and $u^\mu n_{\mu\pm} = 0$. In terms of the coordinate system $\{\tau, z, \phi\}$, the components of the extrinsic curvature $K_{j\pm}^i$ are given by

$$K_{\tau\pm}^\tau = \frac{1}{(D_\perp \rho)_\pm} \left(\frac{d^2 \rho_0}{d\tau^2} - (D_\parallel \psi) \frac{d\rho_0}{d\tau} + \rho_0 ((D_\parallel \psi)^2 + (D_\perp \psi)_\pm^2) \right) - (D_\perp \psi)_\pm, \quad (8)$$

$$K_{z\pm}^z = (D_\perp \psi)_\pm, \quad (9)$$

$$K_{\phi\pm}^\phi = \frac{(D_\perp \rho)_\pm}{\rho_0} - (D_\perp \psi)_\pm, \quad (10)$$

where $D_{\parallel}A = u^a \partial_a A$, $(D_{\perp}A)_{\pm} = n_{\pm}^a \partial_a A$ for an arbitrary scalar function A in the neighborhood of Σ .

First, from (9), we note that the z - z component of (4) with the energy momentum tensor (6) yields the condition

$$- [D_{\perp} \psi] = 4\pi G(\sigma - \zeta + \xi), \quad (11)$$

where $(D_{\perp}A)_{\pm} = n_{\pm}^a \partial_a A$ for an arbitrary scalar function on $\partial\mathcal{M}_{\pm}$. For a domain wall which is created by the symmetry breaking of scalar field, the equation of state of S_j^i is given by $\sigma = \zeta = \xi$ and the right hand side of (11) does not vanish. Because of this condition, a domain wall with cylindrical symmetry emits gravitational radiation of ψ [7]. On the other hand, if (6) satisfies the equation of state $\zeta = \sigma + \xi$, there is no excitation of ψ . In particular, the case $\xi = 0$ gives the surface energy momentum tensor

$$S_{\mu\nu} = \sigma(\tau)(u_{\mu}u_{\nu} - z_{\mu}z_{\nu}). \quad (12)$$

This energy momentum tensor (12) corresponds to the one which is reduced from the two dimensional Nambu-Goto action of string in four dimensional spacetime,

$$S_{\mu\nu} = \lambda\delta(\rho)(u_{\mu}u_{\nu} - z_{\mu}z_{\nu}), \quad (13)$$

where λ is constant and corresponds to the energy density of the string and $\delta(\rho)$ is the delta function. Hence, if we take the limit in which the radius of our shell to be zero, we may regard our cylindrical shell with anisotropic tension as a straight Nambu-Goto string. In this limit, $\sigma(\tau)$ must diverge, since $\sigma(\tau)$ must correspond to $\lambda\delta(\rho)$. Thus, we can obtain a self-gravitating cosmic string spacetime by taking the limit in which the radius of the shell with the surface energy momentum tensor (12) vanishes. Since (12) leads no excitation of any modes of ψ , we may choose $\psi = 0$ on the whole spacetime $\mathcal{M}$.

Next, we consider the other components of the junction condition (4) and the continuity equation of the surface energy momentum tensor (12). From (8), the τ - τ component of (4) and $\psi = 0$ yield $d^2\rho_0/d\tau^2 = 0$, which shows that the shell moves with constant velocity. Hereafter, we only consider the case with $d\rho_0/d\tau = 0$. In this case, from (10), the ϕ - ϕ component of the junction condition (4) and $\psi = 0$ yield

$$e^{-\gamma_+} = 1 - \alpha := 1 - 8\pi G\rho_0\sigma. \quad (14)$$

Here, we choose $\epsilon_{\pm} = 1$ and impose the regularity on the axis $\rho = 0$ in $\mathcal{M}_-$. Furthermore, the continuity equation of the surface energy momentum tensor (12) yields $d(\rho_0\sigma)/d\tau = 0$. This means that α defined in (14) is constant. Furthermore, it can be easily seen that α is the deficit angle in $\mathcal{M}_+$. Since there is no excitation of ψ , Einstein equations (2) with the regularity at $\rho = 0$ in $\mathcal{M}_-$ shows that $\mathcal{M}_-$ becomes to be Minkowski spacetime

$$ds^2|_{\mathcal{M}_-} = -dt^2 + d\rho^2 + dz^2 + \rho^2 d\phi^2, \quad \text{for } 0 < \rho < \rho_0, \quad (15)$$

and $\mathcal{M}_+$ becomes to be Minkowski spacetime with the deficit angle α

$$ds^2|_{\mathcal{M}_+} = -dt^2 + \frac{d\rho^2}{(1-\alpha)^2} + dz^2 + \rho^2 d\phi^2, \quad \text{for } \rho_0 < \rho. \quad (16)$$

Thus, we have constructed the whole spacetime in which there is a shell with anisotropic tensors (12). The surface energy momentum tensor (12) has the same form as (13) which is reduced from the Nambu-Goto action. This shell can be regarded as the string bundle produced by the sets of straight cosmic string. Furthermore, we can obtain the self-gravitating cosmic string spacetime by taking limit $\rho_0 \rightarrow 0$ with fixed value α . We must also note that $\sigma = \alpha/(8\pi G\rho_0)$ diverges in this limit, as we expected. This divergence of σ corresponds to the divergence in $\lambda\delta(\rho)$ of (13).

We use this string bundle spacetime as the background to investigate the interaction between a straight string and gravitational waves and we consider the scattering problem of gravitational wave by this string bundle and we obtain the scattering data with finite radius ρ_0 . The scattering data for an infinitesimally thin straight string can be obtained by taking the limit $\rho_0 \rightarrow 0$ with fixed α from that for our string bundle.

3 Gauge Invariant Perturbation

In this section, we consider the gauge invariant perturbation theory with the background (16) ($\mathcal{M}_+$). The perturbation on $\mathcal{M}_-$ can be easily obtained from the results with the background $\mathcal{M}_+$ by choosing $\alpha = 0$. We use the gauge invariant perturbation formalism developed by Gerlach and Sengupta[8]. Their formalism is the perturbation theory with spherical background. Although our background (16) does not have spherical symmetry, we will see the perturbation theory can be developed by the manner which is completely parallel to their formalism. Let $h_{\mu\nu}$ be the perturbative metric

$$g_{\mu\nu} = {}^{(0)}g_{\mu\nu} + h_{\mu\nu}, \quad (17)$$

where ${}^{(0)}g_{\mu\nu}$ is given by (16). The background geometry ($\mathcal{M}, {}^{(0)}g$) can be decomposed into two part, i.e., $\mathcal{M}_+ = \mathcal{M}_1 \times \mathcal{M}_2$. The topology of $\mathcal{M}_1$ is $R^1 \times S^1$ and $\mathcal{M}_2$ is R^2 . And the metric (16) is also written by

$$ds^2 = \gamma_{ab} dy^a dy^b + \eta_{pq} dz^p dz^q, \quad (18)$$

where

$$\gamma_{ab} dy^a dy^b = \frac{d\rho^2}{(1-\alpha)^2} + \rho^2 d\phi^2, \quad \eta_{pq} dz^p dz^q = -dt^2 + dz^2. \quad (19)$$

The metric γ_{ab} is that on $\mathcal{M}_1$ and η_{pq} is the two dimensional Lorentzian metric on $\mathcal{M}_2$. The indices $a, \dots, d$ represents (ρ, ϕ) and $p, \dots, s$ represents (t, z) . Let D_a , ∂_p and ∇_μ be the covariant derivatives associated with metrics γ_{ab} , η_{pq} and ${}^{(0)}g_{\mu\nu}$, respectively.

Under the infinitesimal gauge-transformation $\delta_G x^\mu := \bar{x}^\mu - x^\mu = \xi^\mu$, the perturbative metric $h_{\mu\nu}$ transforms as

$$\delta_G h_{\mu\nu} = -\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu. \quad (20)$$

The perturbation variables h_{ab} , h_{ap} , h_{pq} , ξ_a and ξ_p can be expanded by the harmonics on $\mathcal{M}_2$ as follows,

$$h_{ab} := \int d^2 k f_{ab} S, \quad h_{ap} := \int d^2 k \left\{ f_a^{(o1)} V_p^{(o1)} + f_a^{(e1)} V_p^{(e1)} \right\}, \quad (21)$$

$$h_{pq} := \int d^2 k \left\{ f_{(o2)} T_{pq}^{(o2)} + f_{(e0)} T_{pq}^{(e0)} + f_{(e2)} T_{pq}^{(e2)} \right\}, \quad (22)$$

$$\xi_a := \int d^2 k \zeta_a S, \quad \xi_p := \int d^2 k \left\{ \zeta_{(o)} V_p^{(o)} + \zeta_{(e)} V_p^{(e)} \right\}. \quad (23)$$

where

$$S := e^{-i\omega t + i k_z z}, \quad V_p^{(o1)} := \epsilon_{pq} \partial^q S, \quad V_p^{(e1)} := \partial_p S, \quad (24)$$

$$T_{pq}^{(e0)} := \frac{1}{2} \eta_{pq} S, \quad T_{pq}^{(e2)} := (\partial_p \partial_q - \frac{1}{2} \eta_{pq} \partial^r \partial_r) S, \quad T_{pq}^{(o2)} := -\epsilon_{r(p} \partial_{q)} \partial^r S, \quad (25)$$

and ϵ_{pq} is the two dimensional totally antisymmetric tensor on $\mathcal{M}_2$. In these mode decomposition (21)-(23), $\int d^2 k = \int d\omega dk_z$ and the expansion coefficients are tensors on $(\mathcal{M}_1, \gamma)$. The symbol o and e refer to the odd and the even modes, respectively. By inspecting the gauge transformation of the coefficients we can easily find gauge-invariant combinations.

First, for the odd modes, (20) yields

$$\delta_G f_{(o1)}^a = -D^a \zeta_{(o)}, \quad \delta_G f_{(o2)} = -2\zeta_{(o)}. \quad (26)$$

From these transformation law, one can easily find two independent gauge-invariant quantities,

$$F_a := f_{a(o1)} - \frac{1}{2} D_a f_{(o2)}. \quad (27)$$

Next, for the even modes, (20) yields

$$\delta_G f_{ab} = -D_a \zeta_b - D_b \zeta_a, \quad \delta_G f_{a(e1)} = -D_a \zeta_{(e)} - \zeta_a, \quad \delta_G f_{(e0)} = -2k^2 \zeta_{(e)}, \quad \delta_G f_{(e2)} = -2\zeta_{(e)}, \quad (28)$$

where $k^2 = \omega^2 - k_z^2$ is the eigen values of the differential operator $\eta^{pq}\partial_p\partial_q$. Here, we define the variable X^a as

$$X^a := f_{(e1)}^a - \frac{1}{2}D^a f_{(e2)}. \quad (29)$$

This variables X^a transforms as $\delta_G X^a = -\zeta^a$. Hence, we can easily find the gauge-invariant combinations,

$$F_{ab} := f_{ab} - D_a X_b - D_b X_a, \quad F := f_{(e0)} - k^2 f_{(e2)}. \quad (30)$$

Thus, we have all gauge invariant quantities.

4 Perturbation Equations

On the background spacetime (16), the perturbed vacuum Einstein equations for odd modes are given by

$$(\Delta + k^2)F^a = 0, \quad D_a F^a = 0, \quad (31)$$

where $\Delta := D^c D_c$. The second equation of (31) implies that F_a is expressed in terms of a scalar variable $\Phi_{(o)}$ as

$$F^a = \epsilon^{ab} D_b \Phi_{(o)}, \quad (32)$$

where $\epsilon^{ab} = \gamma^{ac}\gamma^{bd}\epsilon_{cd}$ is the two dimensional totally antisymmetric tensor on $\mathcal{M}_1$. In terms of $\Phi_{(o)}$, the first equation of (31) is written as

$$\epsilon^{ab} D_b [(\Delta + k^2)\Phi_{(o)}] = 0. \quad (33)$$

This implies that the term inside the square bracket is a constant. When $k^2 \neq 0$, by using the freedom of adding an arbitrary constant in the definition of $\Phi_{(o)}$, the perturbed Einstein equation can be reduced to the following single equation for $\Phi_{(o)}$:

$$(\Delta + k^2)\Phi_{(o)} = 0, \quad (34)$$

On the other hand, for even modes, perturbative vacuum Einstein equations are reduced to

$$F_c^c = 0, \quad D^c F_{ac} - \frac{1}{2}D_a F = 0, \quad (\Delta + k^2)F^{ab} = 0, \quad (\Delta + k^2)F = 0. \quad (35)$$

From the first and the second equation of (35), we can express F_{ab} and F in terms of a scalar master variable $\Phi_{(e)}$ as

$$F_{ab} = D_a D_b \Phi_{(e)} - \frac{1}{2}\Delta \Phi_{(e)}, \quad F = \Delta \Phi_{(e)}. \quad (36)$$

Substituting these expressions into the remaining equations of (35), we obtain the equation for $\Phi_{(e)}$:

$$D_a D_b (\Delta + k^2)\Phi_{(e)} = 0. \quad (37)$$

This equation can be easily integrated as follows:

$$(\Delta + k^2)\Phi_{(e)} = \mathcal{F}(\rho, \phi), \quad (38)$$

where $\mathcal{F}(\rho, \phi)$ is an arbitrary function of ρ and ϕ which is satisfied with $D_a D_b \mathcal{F} = 0$. $\mathcal{F}$ is also satisfied with $\Delta \mathcal{F} = 0$. As in the odd modes, when $k^2 \neq 0$, this freedom of function $\mathcal{F}$ can be chosen to be zero without loss of generality by utilizing the freedom of adding the appropriate function to $\Phi_{(e)}$. Hence the perturbed Einstein equations for even modes are reduced to the same equation as that for the odd modes:

$$(\Delta + k^2)\Phi_{(e)} = 0. \quad (39)$$

5 Global Solutions

Here, we construct global solutions to the perturbed Einstein equation in $\mathcal{M}$ by matching the solutions in $\mathcal{M}_-$ and $\mathcal{M}_+$ along our string bundle Σ . The perturbed Einstein equations must satisfy the perturbed junction condition

$$[\delta q_{ij}] = 0, \quad [\delta K_j^i] = 8\pi G(\delta S_k^i - \frac{1}{2}\delta_j^i \delta S_l^l). \quad (40)$$

The perturbative extrinsic curvature

$$\delta K_j^i = \delta(-q_\mu^i q_j^\nu \nabla_\mu n_\nu) \quad (41)$$

on $\partial\mathcal{M}_+$ can be represented in terms of the variables F , F^a , F^{ab} and X^a as follows:

$$\delta K_\epsilon^\epsilon = \int d^2k \left[D_\parallel F_{\perp\parallel} - \frac{1}{2}K_\epsilon^\epsilon F_{\perp\perp} - \frac{1}{2}D_\perp F_{\parallel\parallel} + (D_\perp K_\epsilon^\epsilon)X_\perp - D_\parallel^2 N \right] S, \quad (42)$$

$$\delta K_p^a = \int d^2k \left[\left\{ -D_\parallel N + K_\epsilon^\epsilon X_\parallel + \frac{1}{2}F_{\perp\perp} \right\} \tau^a V_p^{(\epsilon 1)} + \frac{1}{2}\epsilon^{cd} D_c F_d \tau^a V_p^{(\sigma 1)} \right], \quad (43)$$

$$\delta K_q^p = \int d^2k \left[F_\perp T_{(\sigma 2)q}^p + \left\{ -k^2 N - \frac{1}{2}D_\perp F \right\} T_{(\epsilon 0)q}^p - N T_{(\epsilon 2)q}^p \right], \quad (44)$$

where $K_\epsilon^\epsilon = -(1 - \alpha)/\rho_0$. Furthermore, we introduce the vector field $\tau_a = \rho(d\phi)_a$ and denote the inner product of arbitrary vector fields A_a on $\mathcal{M}_2$ with the basis n^a , τ^a by $A_\perp = A_a n^a$, $A_\parallel = A_a \tau^a$, respectively. In (42)-(44), N is the gauge invariant variable which corresponds to the components δn_p of the perturbative normal vector and defined by the following equations

$$N := \eta_{(\epsilon)} - X_\perp, \quad \delta n_p := \int d^2k \eta_{(\epsilon)} V_p^{(\epsilon 1)}. \quad (45)$$

The perturbative extrinsic curvature on $\partial\mathcal{M}_-$ can be obtained by choosing $\alpha = 0$ from the results for that on $\partial\mathcal{M}_+$.

Next, we consider the perturbation of the surface energy momentum tensor δS_j^i . This perturbation is directly related to the matter fields which construct the string. This introduction of the perturbative energy momentum tensor is necessary to make our treatment to be consistent. In terms of the mode decomposition of δS_j^i , we only consider the components of δS_j^i ,

$$\delta S_\delta^a = 0, \quad \delta S_p^a = \int d^2k s_{(\epsilon 1)}^a V_{(\epsilon 1)p}, \quad \delta S_q^p = \int d^2k s_{(\epsilon 0)} T_{(\epsilon 0)q}^p, \quad (46)$$

where $s_{(\epsilon 0)}$ is the energy density perturbation of the cosmic string (the perturbation of σ) and $s_{(\epsilon 1)}^a$ corresponds to the velocity perturbation to the direction of τ^a .

5.1 Odd mode junction

First, we consider global solution of the odd modes. The junction conditions for odd modes are given by

$$[F_\parallel] = 0, \quad [F_\perp] = 0, \quad [\epsilon^{ab} D_a F_b] = 0. \quad (47)$$

These conditions are reduced to the conditions:

$$[\Phi_{(\sigma)}] = 0, \quad [D_\perp \Phi_{(\sigma)}] = 0, \quad (48)$$

where we use (34). On the other hand, the solution of equation (34) are given by

$$\Phi_{(\sigma)+} = \sum_m e^{im\phi} \left[A_+ H_{\nu_+}^{(1)}(\beta_+ \rho) + B_+ H_{\nu_+}^{(2)}(\beta_+ \rho) \right], \quad \Phi_{(\sigma)-} = \sum_m e^{im\phi} A_- J_{\nu_-}(\beta_- \rho), \quad (49)$$

where $\Phi_{(e)\pm}$ are the solutions on $\mathcal{M}_\pm$, respectively, $H_{\nu_+}^{(1)}(\beta_+\rho)$ and $H_{\nu_+}^{(2)}(\beta_+\rho)$ are the Hankel function of first and second class, respectively, and A_+ and B_+ are the amplitude of outgoing and ingoing wave, $J_{\nu_-}(\beta_-\rho)$ is Bessel function and

$$\nu_+ = \frac{m}{1-\alpha}, \quad \beta_+ = \frac{k}{1-\alpha}, \quad \nu_- = m, \quad \beta_- = k. \quad (50)$$

Note that we have imposed the regularity at the axis $\rho = 0$ on $\Phi_{(e)-}$. From (48) and (49), we obtain

$$A_+ = SB_+ := \frac{J_{\nu_++1}(\beta_-\rho_0)H_{\nu_+}^{(2)}(\beta_+\rho_0) - J_{\nu_-}(\beta_-\rho_0)H_{\nu_++1}^{(2)}(\beta_+\rho_0)}{J_{\nu_-}(\beta_-\rho_0)H_{\nu_++1}^{(1)}(\beta_+\rho_0) - J_{\nu_++1}(\beta_-\rho_0)H_{\nu_+}^{(1)}(\beta_+\rho_0)}B_+. \quad (51)$$

From this relation between A_+ and B_+ , we can easily obtain the asymptotic behavior of the scattering matrix S such as $S \rightarrow 1$ when $\rho_0 \rightarrow 0$. This scattering matrix in the limit $\rho_0 \rightarrow 0$ shows that the current of gravitational waves is conserved in this scattering and there is no phase shift nor resonance.

5.2 Even mode junction

Next, we consider global solution for the even modes. Together with the junction conditions for the intrinsic and the extrinsic curvatures, the conditions for even mode are given by

$$[F] = 0, \quad [X_\parallel] = 0, \quad [F_\parallel] = 2[K_\epsilon^\epsilon X_\perp], \quad (52)$$

$$[N] = 0, \quad [D_\perp F] = 0, \quad [\tfrac{1}{2}F_\perp + K_\epsilon^\epsilon X_\parallel] = 8\pi G s_{(e1)}, \quad (53)$$

$$[D_\parallel F_{\perp\perp} - \tfrac{1}{2}D_\perp F_\parallel - \tfrac{1}{2}K_\epsilon^\epsilon F_{\perp\perp} + (D_\perp K_\epsilon^\epsilon)X_\perp] = -4\pi G s_{(e0)}. \quad (54)$$

To evaluate these conditions, it is convenient to fix the gauge freedom. First, we can fix the gauge so that $s_{(e1)} = 0$ by using gauge freedom $\delta_G s_{(e1)} = -\sigma\zeta^a$. In this gauge, the velocity perturbation is represented by the variable $X_\parallel$ which corresponds to the gauge invariant velocity perturbation. Furthermore, using the gauge freedom $\delta_G(X_\perp)_\pm = -(\zeta_\perp)_\pm$, we fix the gauge so that $(X_\perp)_\pm = -N$. By this gauge choice, $X_\perp$ represents deformation of the string bundle to the normal direction to the background bundle. These two variables (or equivalently, the vector field X^a) have the important physical meaning also in the limit of the string bundle to an infinitesimally thin string $\rho_0 \rightarrow 0$, i.e., the string motion can be represented by these variables.

First, we consider the scattering data. By using the even mode master equation (39) and the relation between the gauge invariant variables and the even mode master variable (36), we can rewrite the junction condition (52) and the second condition of (53) as

$$[\Phi_{(e)}] = 0, \quad [D_\perp \Phi_{(e)}] = 0. \quad (55)$$

These junction conditions are the same as (48). Furthermore, the master equation (39) for even modes has the same form as that for odd modes (34). Then, we obtain the same result as (51). Again, we have obtained the scattering matrix for an infinitesimally thin string which shows that the current of gravitational waves is conserved and there is no phase shift nor resonance.

Now, the above result of the scattering matrix give arise to a question, i.e., the string does move or not? To see this, we must evaluate the deformation variable $X_\perp$ and the velocity variable $X_\parallel$. Using the relation between the master variable and gauge invariant variables (36) and the master equation (39), we can reduce the last condition of (52) and the last one of (53) to

$$X_\perp = -\frac{1}{2}D_\perp \Phi_{(e)}, \quad X_\parallel = -\frac{1}{2}D_\parallel \Phi_{(e)}. \quad (56)$$

Since $X_\perp$ and $X_\parallel$ are continuous, we evaluate these variables on $\partial\mathcal{M}_-$. From the solution $\Phi_{(e)-}$, we can easily see that only $m = 1$ mode contributes to $X_\perp$ and $X_\parallel$ in the limit to string ($\rho_0 \rightarrow 0$). In this limit, we obtain

$$X_\perp^S := \lim_{\rho_0 \rightarrow 0} X_\perp = -\frac{kA_{-,m=1}}{4}e^{i\phi}, \quad X_\parallel^S := \lim_{\rho_0 \rightarrow 0} X_\parallel = -\frac{kA_{-,m=1}}{4}e^{i(\phi+\pi/2)}. \quad (57)$$

Then we can obtain the deformation vector of the string

$$X_S^a := \text{Re} \left[\int d^2k \left(-\frac{k A_{-,m=1}}{4} \right) S(e^{i\phi} n^a + e^{i(\phi+\pi/2)} \tau^a) \right] \quad (58)$$

$$= \int d^2k X S(n^a \cos(\phi + \phi_0) - \tau^a \sin(\phi + \phi_0)), \quad (59)$$

where $X = k|A_{-,m=1}|/2$ and $\phi_0 = \arg(A_{-,m=1})$ and X and ϕ_0 may depend on k . The norm of this string deformation vector X_S^a represents the amplitude of deformation and $n^a \cos(\phi + \phi_0) - \tau^a \sin(\phi + \phi_0)$ represent the direction of string deformation. Then, we conclude that the string will be deformed by gravitational wave scattering.

On the other hand, (54) can be reduced to

$$\delta\alpha := 8\pi G s_{(e0)} \rho_0 = -\frac{\alpha k^2}{2} \Phi_{(e)}, \quad (60)$$

where $\delta\alpha$ is the perturbative deficit angle. In the limit $\rho_0 \rightarrow 0$, this $\delta\alpha$ is given by

$$\lim_{\rho_0 \rightarrow 0} \delta\alpha = -\frac{\alpha k^2}{2} A_{-,m=0} \quad (61)$$

We must note that the only $m = 0$ mode contributes to $\delta\alpha$. However, the $m = 0$ mode does not contribute to the motion of string X_S^a . On the other hand, the $m = 1$ mode, which contributes to the motion of the string, does not contribute to the perturbative deficit angle. Hence, we may say that the $m = 0$ mode is energy density perturbation mode of the string and the $m = 1$ mode is the deformation mode of the string. When one consider the motion of the cosmic string, the $m = 1$ mode is only relevant mode.

6 Discussion

In summary, we proposed the model to investigate the interaction between a straight infinitesimally thin string and gravitational waves. Our model proposed here is the string model which is described by an appropriate limit of the thin shell of matter field with specific equation of state. This thin shell is regarded as string bundle and its zero radius limit with fixing the deficit angle represents a straight thin string. Further, we considered the gravitational wave scattering by this string. As a result, we showed that there is no phase shift nor resonance pole while there are many resonance poles when the radius of this bundle is finite. On the other hand, we defined the perturbative deformation variables $X_\perp$ and $X_\parallel$ in terms of the metric perturbation. In the above string limit, only $m = 1$ mode of gravitational waves contributes to finite perturbative deformation. Then we conclude that the string deformations are represented in terms of the metric variables and the string is deformed by the gravitational wave scattering.

The difference between the scattering data of the thin string and that of the string bundle shows that the fact that the dynamical degrees of freedom are described by the gravitational waves has nothing to do with no resonance. Furthermore, the existence of resonance pole in the string bundle scattering suggests that it crucially depends on the internal structure of string whether there exist resonance poles or not.

We must also note that the scattering matrix $S = 1$ for the infinitesimally thin string does not mean there is no gravitational lensing effect due to this string. Our mode functions (49) in $\mathcal{M}_+$ already include the effect of the deficit angle α . So, if we consider the wave packet scattering of gravitational waves by using our mode functions, we will obtain the gravitational lensing effect of gravitational wave.

Finally, all above results are based on the assumption that we can obtain the perturbed motion of the Nambu-Goto string by taking zero radius limit with the fixed deficit angle of the perturbed string bundle. So we must confirm that the above assumption is correct. To do this, we must derive the equations which the deformation vector of string X_S^a satisfies. If this equation corresponds to that derived from the Nambu-Goto action, our string can be regarded as the Nambu-Goto string. Although, to derive this equation, the delicate argument are necessary[9], we will be able to obtain in near future. Then, we will show whether our string corresponds to the Nambu-Goto string or not.

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Uniformly Accelerating Black Holes and the Gravitational-Wave Tails

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Abstract

We revisit the vacuum C-metric describing a uniformly accelerating Schwarzschild black hole to study the late-time behavior of gravitational waves. We analyze the radiation field by using the mass function defined in the Bondi-Metzner-Sachs coordinate system, and it is confirmed that the power-law tail predicted by linear perturbation theory really develops at future null infinity in this fully nonlinear axisymmetric dynamics, quite different from gravitational collapse and black hole collision. A special property of the tail formation is briefly discussed.

1 Introduction

In the framework of general relativity many works have been devoted to investigating the dynamical evolution of gravitational waves and any other wave fields on a curved spacetime. Then it has been shown that some *universal* behaviors appear at very late times as a result of the black hole formation in gravitational collapse and the collision of two black holes: The wave fields fall off with time according to specific shapes which are called the quasi-normal ringing and the power-law tail. These late-time behaviors have been extensively studied in the context of linearized (nonspherical) perturbations in spherically symmetric fixed backgrounds. In particular, the existence of the quasi-normal modes has been well-established even in fully nonlinear dynamical systems by virtue of many numerical calculations. The quasi-normal modes are characterized by exponential time dependence $\exp(i\omega t)$ with complex eigenfrequencies ω . Therefore, the power-law tails should eventually dominate the exponentially damped quasi-normal modes at very late times. The perturbation theory clearly shows that if a Schwarzschild black hole forms in the background, the evolving test-fields can develop the power-law tails at future null and timelike infinities as well as on the horizon [1]. The tail behavior seems to be more universal than the quasi-normal decay, because the former is insensitive to the spherical background geometry. For example, even if one considers perturbations in a Reissner-Nordström background, the exponent of the power law remains to be independent of the black hole charge Q . (Of course, the quasi-normal frequency ω depends on Q .) Further, the power-law tail can form in general at future null and timelike infinities even when no horizon is present in the background.

The linearized perturbation theory may be sufficient for understanding the tail phenomenon of weak fields at future null and timelike infinities. Nevertheless, it is important to verify that the power-law tail really develops as a result of the evolution of a fully nonlinear system. However, it is very difficult to observe accurately by numerical simulations such a wave decay at very late times and at very distant regions, and until now the existence of the wave tail in a nonlinear evolution has been numerically confirmed only in spherically symmetric models of a collapsing scalar field [2,3]. Our purpose in this paper is therefore to find the tail behavior in a fully nonlinear axisymmetric system.

Our approach is not to pursue numerical simulations of axisymmetric gravitational collapse and black hole collision. We would like to turn our attention to a simpler situation generating gravitational waves, i.e., a uniformly accelerating black hole. It should be emphasized that the uniform acceleration of a black hole is a dynamical process quite different from gravitational collapse and black hole collision, and we can regard it as a good example to check the universality of the power-law tail. Further we have the analytical model of a uniformly accelerating Schwarzschild black hole as an exact solution of the axisymmetric vacuum Einstein equations, which is known as the vacuum C-metric [4]. It becomes

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an easy task to investigate the late-time behavior of gravitational waves emitted from the accelerating hole. We will treat the radiation field only at future null infinity, by transforming the G-metric to a Bondi-Metzner-Sachs form [5]. The power-law decay in the retarded time u will be clearly shown to occur at future null infinity in complete accordance with the prediction by linear perturbation theory applied to gravitational collapse and black hole collision. This paper is a short report of the full version [6] which contains further details of the results given in the following.

2 Power-Law Tails

Let us begin with a brief review of the tail formation at future null infinity in Schwarzschild backgrounds [1]. We consider a massless scalar field ϕ as axisymmetric perturbations, which may be expanded in the form

$$\phi = \sum_l \psi_l(u, v) P_l(\cos \theta) / r, \quad (1)$$

where u and v are the usual retarded and advanced null coordinates in Schwarzschild geometry. (We use throughout this paper units such that $c = G = 1$.) The general solution of the Klein-Gordon equation for each l pole field ψ_l

$$-4\partial_u \partial_v \psi_l = V_l(r) \psi_l \quad (2)$$

contains two arbitrary functions $E(u)$ and $F(v)$ which represent the outgoing and ingoing fields and are determined through some boundary condition for r .

Collapsing objects may produce a primary outgoing radiation of ψ in the limited region $u_0 < u < u_1$. Then we require the boundary condition that $E(u_1) \sim -\mu$ if there exists an initial static l pole field $\psi_l \simeq \mu/r^l$ (which should completely radiate away within the duration $u_0 < u < u_1$), and $E(u_1) = 0$ for no initial static field. Even in this situation the wave field does not abruptly decay in the region $u > u_1$, because the backscattering of the primary flux becomes important in the region $u_0 < u < u_1$ owing to the potential term V_l in the Klein-Gordon equation. In fact the analytic continuation of the data on $u = u_1$ leads to the ingoing field with the form $F(v) \sim v^{-s}$, where $s = 1$ if the static field is present and $s = 2$ otherwise. This dependence of $F(v)$ constrains the solution ψ_l propagating to the distant flat region $r \gg M$ (i.e., the functional form of $E(u)$ in the limit $u \rightarrow \infty$), and we obtain the late-time behavior of the power-law tail

$$\psi_l \sim u^{-s-l} \quad (3)$$

at future null infinity.

Though this example of the power-law tail is restricted to the evolution of a scalar field, the exponent given by $s + l$ remains valid for any other fields at least within the framework of linear perturbation theory. We will derive the same power-law behavior of gravitational waves from an accelerating black hole described by the vacuum G-metric in fully nonlinear dynamics.

3 The Vacuum G-metric

The line element of the vacuum G-metric written in the accelerating frame with the coordinates σ, ξ, p and ϕ can have the static form

$$ds^2 = H d\sigma^2 - H^{-1} d\xi^2 - 2A\xi^2 H^{-1} d\xi dp - (G^{-1} + A^2 \xi^2 H^{-1}) \xi^2 dp^2 - \xi^2 G d\phi^2, \quad (4)$$

where

$$H = 1 - 2m\xi^{-1} + 6Amp + A\xi G_p - A^2 \xi^2 G, \quad (5)$$

and

$$G = 1 - p^2 - 2Amp^3, \quad G_p \equiv dG/dp = -2p - 6Amp^2. \quad (6)$$

The positive constants m and A represent the Schwarzschild mass parameter and the acceleration parameter, respectively. If $Am < 1/\sqrt{27}$, the Killing horizons corresponding to the Rindler and Schwarzschild ones exist on the two null surfaces satisfying $H = 0$, and the acceleration occurs along the axis given by

the roots of $G = 0$ [7]. We denote the one positive and two negative roots by $p_+ > 0$ and $0 > p_- > p_0$, respectively. The forward and backward directions of acceleration correspond to $p = p_-$ and $p = p_+$, respectively [9]. The physical range of p is limited to $p_- \leq p \leq p_+$ where $G \geq 0$.

To observe the radiation field at future null infinity, it is useful to transform the metric to a Bondi-Metzner-Sachs (BMS) form written by

$$ds^2 = V du^2 + 2e^{2\beta} du dr + 2U du d\theta - r^2(e^{2\gamma} d\theta^2 + e^{-2\gamma} \sin^2 \theta d\varphi^2). \quad (7)$$

For the limit of large r the metric functions have the form of power-series in powers of r^{-1} such that

$$V = 1 - 2M(u, \theta)r^{-1} + O(r^{-2}), \quad \gamma = N(u, \theta)r^{-1} + O(r^{-3}), \quad (8)$$

where the so-called news function N determines the mass function M via the equation

$$\partial_u M = -(\partial_u N)^2 + \frac{1}{2\sin\theta} \partial_\theta \left[\frac{1}{\sin\theta} \partial_\theta (\sin^2 \theta \partial_u N) \right]. \quad (9)$$

We will discuss the behavior of the radiation field in terms of a time evolution of the mass function M . Then our main task is to pick out the news function N from the C-metric.

The radiation field of the C-metric was first studied by Kinnersley and Walker [4] and further explored by Farhoosh and Zimmerman [8] without giving any mention about the tail phenomenon. To reveal the late-time behavior, we introduce a rescaling parameter a into the coordinate transformation defined by

$$T = (aH^{1/2}/A) \sinh(A\sigma/a), \quad Z = (aH^{1/2}/A) \cosh(A\sigma/a) \quad (10)$$

and

$$R = a\xi G^{1/2}, \quad \varphi = \phi/a, \quad (11)$$

where the period of the angular coordinate φ instead of ϕ is assumed to be 2π . For $m = 0$ we can see that the C-metric reduces to the flat metric, and the new coordinates T, Z, R and φ become the inertial cylindrical coordinates under the choice $a = 1$. For $m \neq 0$, of course, such a coordinate transformation leads to a slightly complicated form of the line element. However, we are interested in the asymptotic form of the metric at a distant region such that both T and $\rho \equiv \sqrt{R^2 + Z^2}$ infinitely increase by keeping the difference $w \equiv T - \rho$ finite,

$$ds^2 = dT^2 - dZ^2 - dR^2 - R^2 d\varphi^2 + ds_{(c)}^2, \quad (12)$$

where we have the leading correction $ds_{(c)}^2$ added to the flat metric as follows,

$$ds_{(c)}^2 = [1 - (a^2 W)^{-1}] [(Z^2 - T^2)^{-1} (T dT - Z dZ)^2 + dR^2] \quad (13)$$

and the function $W(p)$ is given by

$$W \equiv (G_p^2 - 2GG_{pp})/4 = 1 + 6Amp - 2Amp^3 - 3A^2 m^2 p^4. \quad (14)$$

Though this asymptotic form of the metric still does not satisfy the BMS radiation gauge, the transformation to the BMS coordinates r, u and θ in the limit $r \rightarrow \infty$ is easily found to be

$$\rho = r + \rho_0(u, \theta) + O(r^{-1}), \quad w = w_0(u, \theta) + O(r^{-1}), \quad (15)$$

and for $\tan \eta \equiv Z/R$,

$$\eta = \theta + \eta_0(u, \theta)r^{-1} + O(r^{-2}), \quad (16)$$

which lead to the relation between the original coordinate p and the BMS coordinates,

$$\int_0^p \frac{dp}{G^{3/2}} = \frac{aA(u - u_0(\theta))}{\sin \theta}, \quad (17)$$

where u_0 is an arbitrary function of θ . Since the news function $N(u, \theta)$ is determined by the equation

$$N = -(\rho_0 + \cot \theta \eta_0), \quad (18)$$

the time derivative can be rewritten into the form

$$\partial_u N = \frac{a^2 W - 1}{2\sin^2 \theta}, \quad (19)$$

from which we will discuss the evolutionary behavior of the mass function $M(u, \theta)$ by giving a suitable choice of a and u_0 .

4 The Late-Time Behavior

It is known that the condition for local flatness on the symmetry axis requires γ to satisfy the boundary condition such that $\gamma/\sin^2\theta$ becomes regular in the limit $\sin\theta \rightarrow 0$. This is essential to a choice of a , because (19) implies that the time-dependent field $\partial_u N$ breaks this boundary condition, unless $a^2 W \rightarrow 1$ on the symmetry axis. Note that G must vanish at $\sin\theta \rightarrow 0$ according to the relation (17). In particular, at late times $u - u_0 > 0$, the integral in (17) must be positive, and we obtain $p_+ - p \propto \sin^2\theta$ in the limit $\sin\theta \rightarrow 0$. Then, if we denote the value of W at $p = p_+$ by W_+ , the flatness condition at late times leads to the choice

$$a = 1/\sqrt{W_+}. \quad (20)$$

Further we obtain the equality $dW/dp = 6AmG$, which gives $a^2 W - 1 \propto (p_+ - p)^2 \propto (\sin\theta)^4$ in the limit $p \rightarrow p_+$. Therefore, such a choice of a is sufficient for the flatness condition of $\partial_u N$ on the symmetry axis.

Now let us evaluate (17) at early times $u < u_0$. Then the value of p is found to be equal to $p_- < 0$ (instead of $p_+ > 0$) in the limit $\sin\theta \rightarrow 0$. By virtue of this difference of p the choice of a given by (20) cannot satisfy, at early times, the condition for local flatness on the symmetry axis, and the time derivative $\partial_u N$ becomes infinite in the limit $\sin\theta \rightarrow 0$ according to

$$\partial_u N = \frac{(W_-/W_+) - 1}{2\sin^2\theta}, \quad (21)$$

if we denote the value of W at $p = p_-$ by W_- . This is similar to the vacuum axisymmetric metric describing a static balance of two Schwarzschild black holes, for which we set the coordinate system that only the segment of the axis between two black holes is not locally flat, and the other portions of the axis remain regular. (The singular segment may be interpreted as the existence of some strut to support two holes against their mutual attractive force.) For the vacuum C-metric discussed here the physical interpretation of the breakdown of local flatness on the symmetry axis will be the existence of a driving force necessary for uniform acceleration of a black hole. Now the flatness condition becomes a time-dependent problem. The black hole moving along the symmetry axis asymptotically approaches to future null infinity at a time $u = u_0$. Therefore, to reveal dynamical behaviors of the gravitational field, it is plausible to assume the coordinate system that the flatness condition is satisfied at late times $u > u_0$, and the line singularity representing a driving force as a source of strong gravitational radiation is limited to the duration $u < u_0$ of black hole acceleration.

Our claim in this paper is that the primary radiation (highly collimated in the direction $\sin\theta \rightarrow 0$) due to the line singularity and the accelerating black hole should die off sharply after $u = u_0$. At late times no source of any static gravitational fields is present, and the tail behavior will develop as all the field remaining in the inner region radiates away to future null infinity. In linear perturbation theory the radiation field is expanded in multipole modes to obtain the exponent of power-law tails. For the fully axisymmetric gravitational field we must pick out some component of the metric playing a role of the radiation field which allows a direct comparison with the prediction of linear perturbation theory. In the BMS coordinate system discussed here our proposal is to use the mass function M as such a radiation field. To show this, under a suitable choice of the function $u_0(\theta)$, we rewrite (17) into the expansion in powers of $f \equiv 2\sin^2\theta/aA^2u^2 \ll 1$ with constant coefficients b_n ($b_1 = 1$) as follows,

$$p_+ - p = \sum_{n=1}^{\infty} b_n f^n, \quad (22)$$

which is valid at very late times $Au \gg 1$. If we use (22) to the leading order, the late-time behavior of $\partial_u M$ is given by

$$\partial_u M \simeq \frac{3ma^3}{A^3u^4} P_2(\cos\theta). \quad (23)$$

We can verify that the quadrupole angular dependence appears in (23) as a dominant radiation field at very late times, and the field amplitude defined by the mass function M falls off as u^{-3} at future null infinity. It is also easy to evaluate the late-time decay for the higher l pole fields, by using the expansion

(22) to the higher order, and we arrive at the final result that only the multipole fields with even l exist, and the leading behavior of the amplitude for each l pole field in M falls off as u^{-l-1} .

We can also consider the decay law of the Bondi mass M_B defined by

$$M_B(u) = \frac{1}{2} \int_0^\pi M(u, \theta) \sin \theta d\theta. \quad (24)$$

Though at early times $u < u_0$ the Bondi mass becomes infinite owing to the existence of a line singularity, at late times $u > u_0$ we can recover the mass function well-behaved as a function of θ , and the decay rate $\partial_u M_B$ is found to be proportional to u^{-8} . The Bondi mass will completely vanish in the limit $u \rightarrow \infty$ as all the radiation field dies off, which has not been explicitly checked in the present work.

We have shown the existence of the power-law tail of gravitational waves from a uniformly accelerating black hole, which is a result of fully nonlinear axisymmetric dynamics. The power law u^{-l-1} of the l pole field obtained here is corresponding to the case that some initial static field exists, if compared with linear perturbation analysis. For a uniformly accelerating black hole such static multipole fields which behave as r^{-l} will be due to a deformed Schwarzschild field present at early times $u < u_0$. When the black hole as the source of static multipole fields moves far away with the primary collimated radiation, these multipole fields begin to radiate away to future null infinity, and the flat spacetime is finally left behind.

It should be noted that the radiation field at future null infinity is symmetric under the transformation $\theta \leftrightarrow \pi - \theta$. This absence of odd l pole fields in the wave tails will be related to the global structure of the vacuum C-metric, because the maximal extension allows the existence of a pair of oppositely accelerating black holes in a similar manner with the Kruskal extension of the Schwarzschild metric [4]. Nevertheless, it is sure that no direct interaction between two black holes can work. (One hole is moving in the region beyond the Rindler horizon of another hole.) Then we would like to conclude that the tail formation found here for even l pole fields is not essentially due to such a specific symmetry of the global structure of the C-metric, though some generic features of the late-time behavior of gravitational waves generated from a more realistic accelerating black hole may be missed in this simple analytical model.

Acknowledgements

The author wishes to thank Y. Oshiro for valuable discussions.

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Gravitational Waves from Coalescing Black Hole MACHO Binaries

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ABSTRACT

If MACHOs are black holes of mass $\sim 0.5M_{\odot}$, they should be formed in the early universe when the temperature is $\sim \text{GeV}$. We estimate that there exist $\sim 5 \times 10^8$ black hole binaries in the halo up to ($\sim 50\text{kpc}$) whose coalescing time is comparable to the age of the universe so that the event rate will be $\sim 5 \times 10^{-2}$ events/year/galaxy. Although detailed numerical simulations are needed for more accurate determination of the event rate, this suggests that we can expect a few events/year within 15Mpc. The gravitational waves from such coalescing black hole MACHOs can be detected by LIGO, VIRGO, TAMA and GEO. Therefore the existence of black hole MACHOs can be checked in near future by gravitational waves.

Subject headings: gravitation: microlensing — gravitational waves: coalescing binaries
— cosmology: dark matter

1. Introduction

The analysis of the first 2.1 years of photometry of 8.5 million stars in Large Magellanic Cloud (LMC) by MACHO collaboration (Alcock *et al.* 1996a) suggests that $0.62^{+0.3}_{-0.2}$ of the halo consists of MACHOs of mass of $0.5^{+0.3}_{-0.2} M_{\odot}$ in the standard spherical flat rotation halo model. The preliminary analysis of the four year data suggests the existence of at least four additional microlensing events with $t_{dur} \sim 90$ days in the direction of LMC (Pratt 1997). The estimated mass of MACHOs is just the mass of red dwarfs. However the contribution of the halo red dwarfs to MACHO events should be small since the observed density of the halo red dwarfs is too low (Bahcall *et al.* 1994, Graff & Freese 1996a, Graff & Freese 1996b). As for white dwarf MACHOs, the mass fraction of white dwarfs in the halo should be less than 10% since too many white dwarfs with bright progenitors with the Salpeter mass function are in conflict with the number counts of distant galaxies (Charlot & Silk 1995). If the IMF which has a sharp peak around $2M_{\odot}$ is adopted, the fraction can be 25% or so (Adams & Laughlin 1996). The existence of such a population of halo white dwarfs may be or may not be consistent with the observed luminosity function (Gould 1997, Lidman 1997, Freese 1997). In any case the future observation of the high velocity white dwarfs in our solar neighborhood (Lidman 1997) will prove whether white dwarf MACHOs exist or not.

If the number of high velocity white dwarfs is proved to be large enough, the stellar formation theory should explain why the IMF is sharply peaked at $\sim 2M_{\odot}$. If it is not, there arises a large possibility that MACHOs are absolutely new objects such as black holes of mass $\sim 0.5M_{\odot}$ which should be formed in the early universe, or boson stars with the mass of the boson $\sim 10^{-10}$ eV. Of course it is still possible that the clump of MACHOs exists only toward LMC (Nakamura, Kan-ya & Nishi 1996), MACHOs are brown dwarfs in the rotating halo (Spiro 1997) or MACHOs are stars in the thick disk (Turner 1997).

In this *Letter* we consider black hole MACHO (BH-MACHO) case. Since there are a huge number (at least $\sim 4 \times 10^{11}$) of black holes in the halo it is natural to expect that some of them are binaries. We estimate the fraction and the distribution function with respect to the semimajor axis and the eccentricity of binary BHMACHOs. The main purpose of the estimate is the following two points. Firstly if the semi-

major axis of a BHMACHO binary in a circular orbit is $\sim 10^{12}$ cm the binary coalesces in the time scale of the age of the universe so that the gravitational waves in the last three minutes (Cutler *et al.* 1993) can be detected by LIGO, VIRGO, TAMA and GEO (Barish 1996, Brillet 1996, Tsubono 1996, Hough 1996). Secondly in the microlensing events one event toward LMC is due to a binary with separation $\gtrsim 2 \times 10^{14}$ cm (Bennett *et al.* 1996). Although this may be a binary in the Galactic disk or in LMC, it is important to ask the possibility of it being a BHMACHO binary. In §2 we estimate the fraction and the distribution function with respect to the semimajor axis and the eccentricity of the binary BHMACHOs. In §3 we argue the detectability of the gravitational waves from the binary BHMACHOs. §4 will be devoted to discussions.

2. Formation of Solar Mass Size Black Hole MACHO Binary

Since it is impossible to make a black hole of mass $\sim 0.5M_{\odot}$ as a product of the stellar evolution, we have to consider the formation of solar mass size black holes in the very early universe due to large metric perturbations or some unknown mechanism in various possible phase transitions (Yokoyama 1995, Jedamzik 1997). Our stand point here, however, is not to study the detailed model but to know what is the outcome of the formation of BHMACHOs. In short we assume the existence of solar mass black holes in the early universe without asking its theoretical origin.

The density parameter of BHMACHOs, Ω_{BHM} , should be comparable to Ω_b (or Ω_{CDM}) to explain the number of the observed MACHO events. In order to simplify the discussion in this *Letter*, we assume that BHMACHOs are the only dark matter and dominate the matter energy density, i.e., $\Omega = \Omega_{BHM}$, although it is possible to consider other dark matter components than BHMACHOs. To determine the mean separation of the BHMACHOs, it is convenient to consider it at the time of matter-radiation equality, $t = t_{eq}$. At this time, both energy densities of the radiation and BHMACHOs are given by

$$\rho_{eq} = 1.4 \times 10^{-15} (\Omega h^2)^4 \text{ g/cm}^3, \quad (1)$$

where h is the Hubble parameter in units of 100 km/s/Mpc. Then the mean separation of BHMACHOs with mass, M_{BH} , at this time is given by

$$\bar{x} = \left(\frac{M_{BH}}{\rho_{eq}} \right)^{\frac{1}{3}}$$

$$= 1.1 \times 10^{16} \left(\frac{M_{BH}}{M_{\odot}} \right)^{\frac{1}{2}} (\Omega h^2)^{-\frac{1}{2}} \text{cm}. \quad (2)$$

We set the scale factor R unity at $t = t_{eq}$. Thus $\bar{x}$ can be also regarded as the comoving mean separation. Here we also note that the Hubble horizon scale at $t = t_{eq}$ becomes

$$\begin{aligned} L_{eq} &\sim \sqrt{\frac{3c^2}{8\pi G \rho_{eq}}} \\ &= 1.1 \times 10^{21} (\Omega h^2)^{-2} \text{cm}. \end{aligned} \quad (3)$$

During the radiation dominated era, the total energy inside the horizon changes as R^2 . Since the Jeans mass in this era is essentially the same as the horizon mass, black holes are formed only at the time when the horizon scale is equal to the Schwarzschild radius of a BH-MACHO. Thus the scale factor at the formation epoch becomes

$$R_f = \sqrt{\frac{GM_{BH}}{c^2 L_{eq}}} = 1.2 \times 10^{-8} \left(\frac{M_{BH}}{M_{\odot}} \right)^{\frac{1}{2}} (\Omega h^2). \quad (4)$$

The age of the universe and the temperature T_f at $R = R_f$ are $\sim 10^{-8}$ sec and $\sim \text{Gev}$, respectively. It is interesting to note that T_f depends neither on Ω nor on the present temperature of cosmic microwave background radiation.

Now we ask what the fraction of the binary BH-MACHOs is. Consider a pair of black holes with mass M_{BH} and the initial comoving separation $x < \bar{x}$. At the beginning, the comoving separation does not change. The energy density due to a pair of black holes is given by

$$\rho_{BH} \equiv \frac{\rho_{eq} \bar{x}^3}{x^3} \frac{1}{R^3}, \quad (5)$$

while that due to the radiation is given by

$$\rho_r = \frac{\rho_{eq}}{R^4}, \quad (6)$$

Hence, the energy density of BH-MACHOs dominates that of the radiation in the neighborhood of this BH-MACHO binary for

$$R > R_m \equiv \left(\frac{x}{\bar{x}} \right)^3. \quad (7)$$

This means that the binary decouples from the cosmic expansion when $R = R_m$ so that it becomes a

bound system. One can confirm this fact by comparing the kinetic energy due to the relative Hubble velocity of two black holes with their gravitational binding energy when $R = R_m$. Here we note that the background universe is still radiation dominated at this stage.

If the motion of the two black holes is not disturbed, the binary system cannot obtain any angular momentum so that they coalesce to be a single black hole in the free fall time scale. However, in reality, there exists tidal force from neighboring black holes so that the binary obtains the angular momentum and is kept from colliding with each other unless x is exceptionally small. We refer to the semimajor axis and the semiminor axis of the binary as a and b , respectively. Simply, a is estimated as

$$a = x R_m = \frac{x^4}{\bar{x}^3}, \quad (8)$$

Suppose that the comoving separation of the nearest neighboring black hole from the center of mass of the binary is y . Then b will be evaluated by (tidal force) $\times$ (free fall time)² as

$$b = \frac{GM_{BH} x R_m}{(y R_m)^3} \frac{(x R_m)^3}{GM_{BH}} = \left(\frac{x}{y} \right)^3 a. \quad (9)$$

Hence, the eccentricity e is given by

$$e = \sqrt{1 - \left(\frac{x}{y} \right)^6}. \quad (10)$$

Now we assume that x and y have the uniform probability distribution in the range $x < y < \bar{x}$. Then the probability distribution of the binary parameters, a and e , is calculated as

$$f(a, e) da de = \frac{18x^2 y^2}{\bar{x}^6} dx dy, \quad (11)$$

$$= \frac{3ea^{\frac{1}{2}}}{2\bar{x}^{\frac{1}{2}}(1-e^2)^{\frac{1}{2}}} da de. \quad (12)$$

From the condition that $y < \bar{x}$, the maximum value of the eccentricity for a fixed a is given by $e_{max} = \sqrt{1 - (a/\bar{x})^{\frac{1}{2}}}$. Integrating $f(a, e)$ with respect to e , we have the distribution of the semimajor axis $f_a(a)$ as

$$f_a(a) da = \frac{3}{2} \left[\left(\frac{a}{\bar{x}} \right)^{\frac{1}{2}} - \left(\frac{a}{\bar{x}} \right)^{\frac{1}{2}} \right] \frac{da}{a} \quad (13)$$

From Eq. (13), it is found that the fraction of the binary BH-MACHOs with $a \sim 2 \times 10^{14} \text{cm}$ is $\sim 8\%$

and $\sim 0.9\%$ for $\Omega h^2 = 1$ and 0.1 , respectively. The estimated fraction of the ~ 10 AU size BHMACHO binary is slightly smaller than the observed rate of the binary event (i.e. one binary event in 8 observed MACHOs).

3. Gravitational Waves from Coalescing BHMACHO Binary

We consider here the short period BHMACHO binaries. Their coalescing time due to the emission of gravitational waves is approximately given by (Peters and Mathews 1963, Peters 1964)

$$t = t_0 \left(\frac{a}{a_0} \right)^4 (1 - e^2)^{\frac{3}{2}}, \quad (14)$$

where $t_0 = 10^{10}$ year and

$$a_0 = 2 \times 10^{11} \left(\frac{M_{BH}}{M_\odot} \right)^{\frac{1}{2}} \text{ cm}, \quad (15)$$

is the semimajor axis of the circular orbit BHMACHO binary which coalesces in t_0 . Integrating Eq. (12) for a fixed t with the aid of Eq. (14), we obtain the probability distribution of the coalescing time $f_t(t)$ as

$$f_t(t)dt = \frac{3}{29} \left[\left(\frac{t}{t_{max}} \right)^{\frac{3}{2}} - \left(\frac{t}{t_{max}} \right)^{\frac{3}{2}} \right] \frac{dt}{t}, \quad (16)$$

where

$$t_{max} = t_0 \left(\frac{\bar{x}}{a_0} \right)^4. \quad (17)$$

If the halo of our galaxy consists of BHMACHOs of mass $\sim 0.5M_\odot$, $\sim 10^{12}$ BHMACHOs exist up to LMC. The number of coalescing binary BHMACHOs with $t \sim t_0$ becomes $\sim 5 \times 10^8$ for $\Omega h^2 = 0.1$ so that the event rates of coalescing binary becomes $\sim 5 \times 10^{-2}$ events/year/galaxy. If, however, BHMACHOs extend up to the half way to M31, the number of coalescing binary BHMACHOs with $t \sim t_0$ can be $\sim 3 \times 10^9$ and the event rate becomes ~ 0.3 events/year/galaxy, which is much larger than the best estimate of the event rate of coalescing neutron stars based on the number of the identified binary neutron stars (Phinney 1991, Narayan, Piran & Shemi 1991) by ~ 5 orders of magnitude.

The detectability of these waves by interferometers is most easily discussed in terms of the waves' "characteristic amplitude" h_c (Eq. (46b) of Thorne 1987,

with a well-known factor 2 correction $-\sqrt{2}$ originating in Eq. (29) of that paper, and $\sqrt{2}$ in the ft of Eq. (42), which should have been $\int f dt$):

$$h_c = 4 \times 10^{-21} \left(\frac{M_{chirp}}{M_\odot} \right)^{5/6} \left(\frac{\nu}{100 \text{ Hz}} \right)^{-1/6} \left(\frac{r}{20 \text{ Mpc}} \right)^{-1}. \quad (18)$$

Here $M_{chirp} = (M_1 M_2)^{3/5} / (M_1 + M_2)^{1/5}$ is the "chirp mass" of the binary whose components have individual masses M_1 and M_2 . This h_c is to be compared with an interferometer's "sensitivity to bursts" $h_{SB} = 11[fS_h(f)]^{1/2}$, where $S_h(f)$ is the spectral density of the interferometer's strain noise (cf. Eq. (111) of Thorne 1987, where h_{SB} is denoted $h_{3/yr}$). This $h_{SB}(f) \equiv h_{3/yr}(f)$ is plotted in various publications, e.g. Abramovici et. al. (1989) and Thorne (1995). It has a minimum (optimal sensitivity) at a frequency $f \simeq 100$ Hz. For the first LIGO and VIRGO interferometers, which are expected to be operational in 2001, that minimum is $h_{SBmin} \simeq 3 \times 10^{-21}$. The GEO600 and TAMA interferometers, with their somewhat shorter armlengths, will have h_{SBmin} a little worse than this in 2001. LIGO/VIRGO should be able to detect coalescing binaries, with high confidence, out to the distance for which $h_c = h_{SBmin}$ at the optimal frequency $f \simeq 100$ Hz. Inserting $M_1 = M_2 \simeq 0.5M_\odot$ for BHMACHOs into the above equations, we see that the first LIGO/VIRGO interferometers in 2001 should be able to see BHMACHO coalescences out to about 15Mpc distance, i.e., out to the VIRGO cluster, where our estimates ($\sim 1/100$ years in each galaxy like our own) suggest an event rate of several per year.

LIGO R&D for the first interferometers is now nearing completion and is beginning to be redirected toward interferometer enhancements, for which the sensitivity goal is a factor 10 improvement, to $h_{SBmin} \simeq 3 \times 10^{-22}$ (Barish et. al. 1996). In the mid 2000s, with these enhancements in place, LIGO should be able to see BHMACHO coalescences out to about 150Mpc, which would give a few events per year even if the event rate is 1000 times smaller than our estimates, $\sim 1/10^5$ yrs in our own galaxy.

4. Discussion

We have estimated the fraction and the distribution function of the binary BHMACHOs in the order of magnitude argument in this Letter. It is possible to perform N-body numerical simulations to determine the distribution function of the binary BHMA-

CHOs. However the dynamic range of a is very large ($10^5 \text{ cm} < a < 10^{16} \text{ cm}$) so that we need large scale numerical simulations using the biggest supercomputers. This is a challenging numerical problem which should be done urgently.

It is well known that coalescing binary neutron stars are one of the most important sources of gravitational waves. The event rate is estimated as 10^{-6} events/year/galaxy (Narayan, Piran & Shemi 1991, Phinney 1991). The event rate of the coalescing BHMACHO binary is comparable to or greater than the upper limit of that of coalescing binary neutron stars (Phinney 1991). The coalescing binary neutron stars are also considered as possible sources of the gamma-ray bursts (Mezardos 1995). If so the detection of gravitational waves should be accompanied by a gamma-ray burst. The time delay between the gravitational waves and the gamma-ray is not known well. If we consider the fire ball model (Mezardos 1995) the delay should be ~ 1 sec. That is, the gamma-ray appears ~ 1 sec after the amplitude of the gravitational waves becomes zero. However in the coalescence of binary BHMACHOs the emission of gamma-ray is not expected. Therefore we may be able to distinguish coalescing binary BHMACHOs from coalescing binary neutron stars.

If gamma-ray is not emitted by coalescing binary neutron stars, we may use the mass to distinguish them. The mass of a neutron star is expected to be $\sim 1.4M_{\odot}$. This is supported observationally as well as theoretically since the mass of the iron core before the collapse is $\sim 1.4M_{\odot}$. Therefore if the mass of a BHMACHO is smaller than $1M_{\odot}$ it is possible to identify them. However if the mass of a BHMACHO is $\sim 1M_{\odot}$, coalescing BHMACHO binaries and coalescing neutron star binaries are indistinguishable from the detected gravitational waves. In principle, however, black holes absorb the gravitational waves so that the evolution of the binary in the last three minutes (Cutler et al. 1993) is different from that of a neutron star binary of the same mass. The difference arises in the 2.5th post-Newtonian order for the Kerr black hole case (Tagoshi 1997) and 4th order for the Schwarzschild black hole case (Poisson & Sasaki 1995). This will be a challenging problem of the data analysis of the gravitational waves.

It is known that the Silk damping of density perturbations on small scales causes distortions of the CMB spectrum by dumping the acoustic energy into CMB (Daly 1991, Barrow & Coles 1991, Hu, Scott & Silk

1994). Thus it is important to examine if the large primordial density perturbations that are responsible for the BHMACHO formation are compatible with the observed upper limit of CMB spectral distortions (Mather et al. 1994).

For a rms amplitude of density perturbations $\delta(l)$ on a comoving scale l , the fraction of the universe that turns into black holes is given by (Carr 1975)

$$f(l) \approx \delta(l) \exp\left(-\frac{1}{18\delta(l)^2}\right), \quad (19)$$

at a radiation-dominated stage. In the present case, $f(l)$ is equal to R_f given by Eq. (4) and l is equal to the comoving scale l_{BH} of the density perturbations that give rise to BHMACHOs. Recalling that we fixed the scale factor R unity at the time of the matter-radiation equality, the present comoving scale corresponding to l_{BH} is given by

$$(1+z_{eq})l_{BH} = \frac{(1+z_{eq})GM_{BH}}{R_f c^2} = 5 \times 10^{17} \left(\frac{M_{BH}}{M_{\odot}}\right)^{\frac{1}{2}} \text{ cm}, \quad (20)$$

where z_{eq} is the redshift at the equal time. Since $R_f \sim 10^{-8}$, we need $\delta(l_{BH})^2 \sim 4 \times 10^{-3}$.

On the other hand, Hu, Scott and Silk (1994) showed that the observational limit constrains the rms amplitude of density perturbations at the horizon entry as

$$\delta(l)^2 \lesssim 10^{-4}, \quad \text{for } l > l_D, \quad (21)$$

where l_D is the comoving scale of the Silk damping scale at the double Compton thermalization time which is at present given by

$$(1+z_{eq})l_D = 3 \times 10^{20} \text{ cm}. \quad (22)$$

Hence the scale of our interest is approximately 3 orders of magnitude smaller than the scale to which the observational constraint applies.

In particular, if we assume the primordial density perturbation spectrum to have a power-law shape, $P(k) \propto k^n$, on scales greater than l_{BH} , the above result implies that $n \gtrsim 1.6$. Interestingly, this value is consistent with the one suggested by the observed CMB anisotropies (Bennett et al. 1994) and it is marginally consistent with the COBE normalization (Hu, Scott & Silk 1994). Thus it seems that a scenario with the primordial spectrum having a peak at $k = k_{BH} = 1/l_{BH}$ and having the power-law index

$n \sim 1.6$ for $k < k_{BH}$ can explain not only the large scale structure of the universe but also the MACHOs. To examine whether there exists a cosmological model that realizes such a scenario is out of the scope of the present *Letter*, but it is worth noting that a variant of the so-called hybrid inflation model seems to have a good chance of giving a successful scenario (Garcia-Bellido, Linde & Wands 1996).

This work was supported by a Japan-Us Co-operating program on gravitational waves from coalescing binary compact objects. The main part of the paper was completed when three of the authors (TN, MS and TT) were staying at Caltech. This work was also supported by the Grant-in-Aid for Basic Research of the Ministry of Education, Culture, and Sports No.08NP0801.

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Black Hole Binary Formation in the Expanding Universe -Three Body Problem Approximation -

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Abstract

We study the black hole MACHO binary formation through the three body interaction in the early universe at $t \sim 10^{-5}$ s. The probability distribution functions of eccentricity and semimajor axis of binaries as well as of the coalescence time are obtained assuming that the black holes are randomly formed in space. We confirm that the previous order-of-magnitude estimate for the binary parameters is valid up to the numerical factor. We found that the coalescence rate of the black hole MACHO binaries is $\sim 5 \times 10^{-2} \times 2^{\pm 1}$ events/year/galaxy taking several possible factors which may affect this estimate into consideration. This suggests that the event rate of coalescing binary black holes will be at least several events per year within 15 Mpc. The first LIGO/VIRGO interferometers in 2001 will be able to verify whether the MACHOs are black holes or not.

1 Introduction

The analysis of the first 2.1 years of photometry of 8.5 million stars in the Large Magellanic Cloud (LMC) by the MACHO collaboration [2] suggests that the fraction $0.62^{+0.3}_{-0.2}$ of the halo consists of MACHOs (MASSive Compact Halo Objects) of mass $0.5^{+0.3}_{-0.2} M_{\odot}$ in the standard spherical flat rotation halo model. The preliminary analysis of four years of data suggests the existence of at least four additional microlensing events with $t_{\text{dur}} \sim 90$ days in the direction of the LMC [3]. At present, we do not know what MACHOs are. This is especially because there are strong degeneracies in any microlensing measurements; the mass, velocity and distance of a lens object. The inferred mass is just the mass of red dwarfs. However, a tight bound is obtained from the observations [4, 5, 6]. The red dwarfs contribute at most a few percent to the mass of the halo.

The brown dwarfs are also restricted by the *Hubble Space Telescope* search. Extrapolating the mass function, the contribution of the brown dwarfs to the mass of the halo is less than a few percent [6]. However, the possibility of brown dwarfs cannot be rejected if the mass function has a peak at the brown dwarfs since they are so dim. The possibility that the MACHOs are neutron stars is ruled out by the observational constraints on the metal and helium abundance [7].

As for white dwarfs, assuming the Salpeter initial mass function (IMF) with an upper and a lower mass cutoff, the mass fraction of the white dwarfs in the halo should be less than 10 % from the number count of the high z galaxies [8]. The observation of the chemical yield does not favor that the MACHOs are white dwarfs [9]. However if the IMF has a peak around $\sim 2M_{\odot}$, MACHOs can be white dwarfs [10]. Future observations of high-velocity white dwarfs in our solar neighborhood will make clear whether white dwarf MACHOs exist or not. Of course, it is still possible that an overdense clump of MACHOs exist toward the LMC [11].

If the number of the high-velocity white dwarfs turns out to be large enough to explain the MACHOs, then star formation theory should explain why the IMF has a peak at $\sim 2M_{\odot}$. If it is not, we must consider other possibilities such that the MACHOs are primordial black holes or boson stars. Even if the MACHOs are black holes, however, it seems difficult to verify observationally whether the MACHOs are black holes or not. In fact, radiation from gas accreting to a black hole MACHO (BH MACHO) is too dim to be observed unless the velocity of BH MACHO is exceptionally small [12].

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This work is based on the collaboration with K.Ioka, T.Tanaka, and T.Nakamura [1].

Recently, however, Nakamura et al. [13] proposed the detectability of the gravitational wave from coalescence of BHMACHO binaries which are formed through the three body interaction in the early universe at $t \sim 10^{-5}$ s. The event rate of coalescing binary BHMACHOs was estimated as $\sim 5 \times 10^{-2}$ events/y/galaxy, which suggests that we can expect several events per year within 15 Mpc. If this estimate is true, not only we are able to confirm whether the MACHOs are black hole or not, but also we have a plenty of sources of gravitational wave. However in Ref.[13] they made only order-of-magnitude arguments and there are uncertainties in the estimate of the event rate. Especially, the estimate of the semimajor and the semiminor axis of the binary formed through the three body interaction is not based on accurate numerical calculations. In this paper we investigate up to what extent the order-of-magnitude argument in Ref.[13] is valid by calculating numerically the three body problem in the expanding universe.

We use units of $G = c = 1$.

2 Review of BHMACHO Binaries Formation Scenario

We briefly review the BHMACHO binaries formation scenario in Ref.[13] to introduce our notations.

For simplicity, we assume that black holes dominate the dark matter, i.e. $\Omega = \Omega_{BHM}$. The extension to the case that black holes do not dominate the dark matter is not difficult. Primordial black holes are formed when the horizon scale is equal to the Schwarzschild radius of a black hole. As in the previous paper [13], we do not argue detailed formation mechanisms of primordial black holes [15] but argue how to confirm the existence of BHMACHOs. The scale factor at the time of the formation is

$$R_f = \sqrt{M_{BH}/H_{eq}^{-1}} = 1.1 \times 10^{-8} \left(\frac{M_{BH}}{M_\odot} \right)^{\frac{1}{2}} (\Omega h^2), \quad (1)$$

where H_{eq} is the Hubble parameter at the time of radiation-matter equality and $H_{eq}^{-1} = \sqrt{3/8\pi\rho_{eq}} = 1.2 \times 10^{21} (\Omega h^2)^{-2}$ cm. The scale factor is normalized such that $R = 1$ at the time of radiation-matter equality.

The mean separation of black holes $\bar{x}$ with mass M_{BH} at the time of radiation-matter equality is

$$\begin{aligned} \bar{x} &= (M_{BH}/\rho_{eq})^{1/3} \\ &= 1.2 \times 10^{16} (M_{BH}/M_\odot)^{1/3} (\Omega h^2)^{-4/3} \text{ cm}. \end{aligned} \quad (2)$$

As a foundation for computing the distribution function $f(a, e)$ for BHMACHO binaries, in [13] it was assumed that (i) the BHMACHOs are created with a distribution of comoving separations x that is *uniform* over the range from an initial physical separation equal to the black hole size to a maximum separation $x = \bar{x}$ and that (ii) the BHMACHOs' peculiar motions are negligible compared to the Hubble flow. Obviously, it may be better to assume that BHMACHOs are distributed *randomly* rather than uniformly. The effect of the initial peculiar velocity is considered in [1].

Consider a pair of black holes with mass M_{BH} and a comoving separation $x < \bar{x}$. These holes' masses produce a mean energy density over a sphere with the radius of the size of their separation as $\bar{\rho}_{BH} \equiv \rho_{eq} \bar{x}^3 / (x^3 R^3)$. $\bar{\rho}_{BH}$ becomes larger than the radiation energy density $\rho_r = \rho_{eq} / R^4$ for

$$R > R_m \equiv \left(\frac{x}{\bar{x}} \right)^3. \quad (3)$$

The binary decouples from the cosmic expansion and becomes a bound system when $R = R_m$. The tidal force from neighboring black holes gives the binary enough angular momentum to keep the holes from colliding with each other unless x is exceptionally small.

The semimajor axis a may be estimated

$$a = \alpha x R_m = \alpha \frac{x^4}{\bar{x}^3}, \quad (4)$$

where α is a constant of order $O(1)$. We denote by y the comoving separation of the nearest neighboring black hole from the center of mass of the binary. Then the semiminor axis b may be estimated as (tidal

force) $\times$ (free fall time)² and is given by

$$b = \alpha\beta \frac{M_{BH} x R_m}{(y R_m)^3} \frac{(x R_m)^3}{M_{BH}} = \beta \left(\frac{x}{y}\right)^3 a, \quad (5)$$

where β is a constant of order $O(1)$. Hence, the binary's eccentricity e is given by

$$e = \sqrt{1 - \beta^2 \left(\frac{x}{y}\right)^6}. \quad (6)$$

In [13], $\alpha = \beta = 1$ is assumed. However, more accurate estimate of them as well as the probability function is necessary to estimate the fraction of binaries and the event rate of coalescing binaries, which is the prime subject of this paper.

3 Multi-particle System in the Expanding Universe

Although our main interest is in the three-body problem, we formulate the problem as generally as possible.

3.1 Basic Equations

We treat the motion of a black hole as that of a test particle within the Newtonian approximation [16]. We first assume that the line element is given by

$$ds^2 = -(1 + 2\phi)dt^2 + (1 - 2\phi)R(t)^2 d\mathbf{x}^2, \quad (7)$$

where ϕ is the Newtonian potential defined by

$$R^{-2}\Delta\phi = 4\pi\rho_{BH}, \quad (8)$$

with ρ_{BH} being the energy density of black holes. The action of the particle is given by

$$\int ds = \int \sqrt{1 + 2\phi - R^2 \dot{\mathbf{x}}^2} dt \simeq \int \left(1 - \frac{1}{2}R^2 \dot{\mathbf{x}}^2 + \phi\right) dt. \quad (9)$$

Then the equation of motion is derived as

$$(R^2 \dot{\mathbf{x}})' = -\nabla\phi. \quad (10)$$

For the potential, $\phi = -\sum_{j \neq i} \frac{m_j}{R|\mathbf{x}_i - \mathbf{x}_j|}$, of multi-particle system, the above equation is expressed as

$$(R^2 \dot{\mathbf{x}}_i)' = -\frac{1}{R} \sum_{j \neq i} \frac{m_j(\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^3}. \quad (11)$$

If we introduce $\mathbf{z}_i \equiv \mathbf{x}_i/\bar{x}$ and the scale factor R as the time variable, then Eq.(11) can be written as

$$\begin{aligned} \mathbf{z}_i'' + \frac{1}{R}\mathbf{z}_i' &= -\frac{M_{BH}}{\bar{x}^3 H^2 R^5} \sum_{j \neq i} \frac{(\mathbf{z}_i - \mathbf{z}_j)}{|\mathbf{z}_i - \mathbf{z}_j|^3} \\ &= -\frac{3}{8\pi R} \sum_{j \neq i} \frac{(\mathbf{z}_i - \mathbf{z}_j)}{|\mathbf{z}_i - \mathbf{z}_j|^3}, \end{aligned} \quad (12)$$

where a prime denotes the derivative with respect to R and we have used Eq.(2) and assumed the equal-mass black holes.

3.2 Validity of Newtonian Approximation

The cosmological Newtonian approximation is valid if (1) $|\phi| \ll 1$ and (2) the scale of inhomogeneity ℓ satisfies the relation $\ell \ll H^{-1}$ [16]. Since the minimum separation between the binary black holes is $a(1 - e)$, the condition (1) is satisfied if

$$M_{BH} \ll a(1 - e). \quad (13)$$

Then in terms of the comoving scale we have

$$y/x \ll 5.8 \times 10 (x/\bar{x})^{2/3} (M_{BH}/M_\odot)^{-1/3} (\Omega h^2)^{-2/3}, \quad (14)$$

where we use Eq.(4), Eq.(6) for $\alpha = \beta = 1$ and $(1 - e) \simeq (1 - e^2)/2$. The condition (2) is written as

$$Rx \ll H_{eq}^{-1} R^2. \quad (15)$$

Then we have

$$R \gg 1.0 \times 10^{-5} (x/\bar{x}) (M_{BH}/M_\odot)^{1/3} (\Omega h^2)^{2/3}. \quad (16)$$

Therefore we have to choose the initial scale factor so that the condition (16) is satisfied.

Note here that Eq. (12) does not depend on M_{BH} . This means that there is a scaling law, i.e. Eq.(12) is invariant under the transformation defined by

$$x \rightarrow \lambda x, R \rightarrow \lambda^3 R, \quad (17)$$

where λ is a constant.

4 Three Body Problem and Formation of Binary Black Holes

We solve Eq.(12) numerically for three particles using the fifth-order Runge-Kutta method with adaptive step size control[17]. Considering the conditions for the validity of the Newtonian approximation derived in the previous section, we set the initial conditions as $x/\bar{x} = \eta$ ($0.1 < \eta < 1$) and $\dot{x} = 0$ at $R = 10^{-3} \eta (M_{BH}/0.5 M_\odot)^{1/3}$ so that Eq.(16) is satisfied. Using the scaling law Eq.(17), we see that these initial conditions are the same as $x/\bar{x} = \lambda \eta$ and $\dot{x} = 0$ at $R = 10^{-3} (\lambda \eta) (M_{BH}/0.5 \lambda^{-6} M_\odot)^{1/3}$. Therefore we can obtain the results for different M_{BH} from a single numerical result. We then numerically estimate α and β for x and y in the range, $0.1 < x/\bar{x} < 1$ and $2 < y/x < 7$. The total number of the parameters we examined is 100 for each direction of the third body. In this section we show the main results for simplicity. The dependence of α and β on the initial direction of the third body is discussed in [1] in more details. The dependence of the results on the initial conditions is found to be small [1].

In Fig.1, the positions of the second body (thick curve) and the third body (dotted curve) relative to the first body are shown for (a) $x/\bar{x} = 0.3, y/x = 2.0$ and (b) $x/\bar{x} = 0.3, y/x = 4.0$. θ is chosen as $\pi/4$, where θ denotes the angle between the x -direction and the direction of the third body. The coordinate is normalized by $\bar{x}$. We see that the binary is formed through the three body interaction while the third body goes away. To see the accuracy of numerical calculations, we checked the time reversal of the problem. That is, we have re-started the numerical integration from the final time backward to the initial time, and have found that the differences from the true values of coordinates and velocities are very small: the relative error in the coordinate position, $|\mathbf{z}_{init} - \mathbf{z}(\text{time reversed})_{init}|/|\mathbf{z}_{init}|$, is less than 10^{-7} and the “velocity” component, $\mathbf{z}'_i(\text{time reversed})$, deviates from zero by at most 10^{-5} .

Fig.2 shows a as a function of initial separation $x/\bar{x}$. The filled triangles are numerical data. The solid line is the approximate equation, $a/\bar{x} = (x/\bar{x})^4$. It is found that a is well fitted by the following function as

$$\frac{a}{\bar{x}} \simeq 0.41 \left(\frac{x}{\bar{x}} \right)^{3.9} \quad (18)$$

irrespective of the direction of the third body we have examined ($\theta = \pi/6, \pi/4, \pi/3$).

Fig.3 shows b/a as a function of x/y for (a) $\theta = \pi/6$, (b) $\pi/4$, and (c) $\pi/3$. The filled triangles are numerical data. The solid line is the approximate equation, $b/a = (x/y)^3$. We see that the numerical

results are parallel to the approximate estimate in the previous paper [13]. We performed the least square fitting of the numerical results assuming that $b/a = \beta(x/y)^n$. The results are given as

$$\begin{aligned}\frac{b}{a} &= 0.74 \left(\frac{x}{y}\right)^{3.2} \quad (\theta = \pi/6) \\ \frac{b}{a} &= 0.77 \left(\frac{x}{y}\right)^{3.1} \quad (\theta = \pi/4) \\ \frac{b}{a} &= 0.62 \left(\frac{x}{y}\right)^{3.1} \quad (\theta = \pi/3)\end{aligned}\tag{19}$$

We see θ -dependence of β . However as for the power index n , it is almost constant so that we will not discuss the small deviation of n from 3 from now on.

The important conclusion is that we have verified that the power dependence is in good agreement with the previous analytic order-of-magnitude estimate of Eq.(4) and Eq.(5) and that numerical coefficients α and β are actually about unity. In the next section we will assume that α and β are constants. For simplicity we will adopt $\alpha = 0.4$ and $\beta = 0.8$.

5 Probability Distribution Function for Binaries

5.1 Distribution Function and Binary Fraction

If we assume that black holes are distributed randomly, then the probability distribution function $P(x, y)$ for BHMACHO binaries is

$$P(x, y)dx dy = \frac{9x^2y^2}{\bar{x}^6} e^{-y^3/\bar{x}^3} dx dy,\tag{20}$$

where $0 < x < y < \infty$ so that $\int_0^\infty \int_x^\infty P(x, y) dy dx = 1$. By changing the variables x and y in Eq.(20), to a and e with Eq.(4) and Eq.(6), we obtain the probability distribution function of eccentricity and semimajor axis of binaries,

$$f(a, e) da de = \frac{3}{4} \frac{\beta}{(\alpha \bar{x})^{3/2}} \frac{a^{1/2} e}{(1 - e^2)^{3/2}} \exp \left[-\frac{\beta}{(1 - e^2)^{1/2}} \left(\frac{a}{\alpha \bar{x}} \right)^{3/4} \right] da de,\tag{21}$$

where $\sqrt{1 - \beta^2} < e < 1$, $0 < a < \infty$ so that $\int_0^\infty \int_{\sqrt{1 - \beta^2}}^1 f(a, e) da de = 1$. For $0 < e < \sqrt{1 - \beta^2}$, $f(a, e) = 0$, i.e. no such binary is formed.

Integrating $f(a, e)$ with respect to e in the range $\sqrt{1 - \beta^2} < e < 1$, we obtain the distribution function of the semimajor axis

$$f_a(a) da = \frac{3}{4} \left(\frac{a}{\alpha \bar{x}} \right)^{3/4} \exp \left[-\left(\frac{a}{\alpha \bar{x}} \right)^{3/4} \right] \frac{da}{a}.\tag{22}$$

From Eq. (22), it is found that if $\alpha = 1$, the fraction of BHMACHOs that are in binaries with $a \sim 2 \times 10^{14} \text{cm}$ and $M_{BH} = 0.5 M_\odot$ is $\sim 4\%$ and $\sim 0.4\%$ for $\Omega h^2 = 1$ and 0.1 , respectively. On the other hand, if we adopt our numerical estimate of α and β that was given in the previous section, for example $\alpha = 0.4, \beta = 0.8$, the fraction becomes $\sim 7\%$ and $\sim 0.8\%$ for $\Omega h^2 = 1$ and 0.1 , respectively. This estimated fraction of ~ 10 AU size BHMACHO binaries is comparable to the observed rate of binary MACHO events (one binary event in eight observed MACHOs) [18] although the small number statistics prevents us from stating something definite.

5.2 Gravitational Waves from Coalescing BHMACHO Binaries

We consider here short period BHMACHO binaries. Their coalescence times due to the emission of gravitational waves are approximately given by [19]

$$t = t_0 \left(\frac{a}{a_0} \right)^4 (1 - e^2)^{\frac{1}{2}},\tag{23}$$

where $t_0 = 10^{10}$ year and

$$a_0 = 2.0 \times 10^{11} \left(\frac{M_{BH}}{M_\odot} \right)^{\frac{1}{3}} \text{ cm} \quad (24)$$

is the semimajor axis of a binary with circular orbit which coalesces in t_0 . Note that Eq.(23) is an approximation for $e \sim 1$ in [19], however, it is also a good approximation even for $e \sim 0$. Eq. (23) can be written in terms of x and y with Eq. (4) and Eq. (5).

$$t = \bar{t} \left(\frac{x}{\bar{x}} \right)^{37} \left(\frac{y}{\bar{y}} \right)^{-21}, \quad (25)$$

$$\bar{t} = \beta^7 \left(\frac{\alpha \bar{x}}{a_0} \right)^4 t_0. \quad (26)$$

Integrating Eq. (20) for a given t with the aid of Eq. (25), we obtain the probability distribution function for the coalescence time $f_t(t)$. We take the range of the integration as $0 < x < \bar{x}$, $x < y < \infty$. (This range turns out to be $(t/\bar{t})^{1/16} \bar{x} < y < (t/\bar{t})^{-1/21} \bar{x}$ for a given t .) The first condition is necessary for the binary formation. Then,

$$\begin{aligned} f_t(t)dt &= \frac{3}{37} \left(\frac{t}{\bar{t}} \right)^{3/37} \left[\Gamma \left(\frac{58}{37}, \left(\frac{t}{\bar{t}} \right)^{3/16} \right) - \Gamma \left(\frac{58}{37}, \left(\frac{t}{\bar{t}} \right)^{-1/7} \right) \right] \frac{dt}{t} \\ &\approx \frac{3}{37} \left(\frac{t}{\bar{t}} \right)^{3/37} \Gamma \left(\frac{58}{37} \right) \frac{dt}{t}, \end{aligned} \quad (27)$$

where $\Gamma(x, a) = \int_a^\infty s^{x-1} e^{-s} ds$. The second equality is valid when we consider $t \sim t_0$ because $t_0/\bar{t} \sim 2 \times 10^{-23} \ll 1$ for $\Omega h^2 = 0.1$, $M_{BH} = 0.5 M_\odot$, $\alpha = 0.4$, and $\beta = 0.8$.

If the halo of our galaxy consists of BHMACHOs of mass $\sim 0.5 M_\odot$, about 10^{12} BHMACHOs exist out to the LMC. The number of coalescing binary BHMACHOs with $t \sim t_0$ then becomes $\sim 1 \times 10^9$ for $\Omega h^2 = 0.1$, $\alpha = 0.4$ and $\beta = 0.8$ so that the event rate of coalescing binaries becomes $\sim 1 \times 10^{-1}$ events/year/galaxy. This rate is slightly larger than the estimate of [13]. If we consider various effects as discussed in [1], we find that the event rate does not decrease significantly and is estimated as $5 \times 10^{-2} \times 2^{\pm 1}$ events/year/galaxy including the uncertainty. However, if the BHMACHOs extend up to half way to M31, the number of coalescing binary BHMACHOs with $t \sim t_0$ can be $\sim 6 \times 10^9$ and the event rate becomes $\sim 6 \times 10^{-1}$ events/year/galaxy. Both of these estimates are much larger than the best estimate of the event rate of coalescing neutron stars based on the statistics of binary pulsar searches in our Galaxy, $\sim 1 \times 10^{-5}$ events/year/galaxy [20]. Because the first LIGO/VIRGO interferometers in 2001 should be able to see BHMACHO coalescence out to about 15 Mpc distance, i.e., out to the Virgo Cluster [13], the event rate will be several per year even if we pessimistically estimate it ($\sim 1/100$ events/year/galaxy in each galaxy like our own).

In deriving the probability distribution function for the coalescence time Eq.(27), we have neglected various effects, such as the angle dependence of β , 3-body collision, the effect of the fourth body, the effect of the mean fluctuation field, the initial condition dependence and the radiation drag. We consider these effects in [1].

6 Summary

In this paper we have discussed the black hole binary formation through the three body interaction in the expanding universe. We have confirmed that the order-of-magnitude argument in [13] is valid up to the numerical factor. Several effects have been considered in [1]. The effect of the 3-body collision and the mean fluctuation field may reduce the event rate of the coalescing BHMACHO binaries about half. On the contrary the angle dependence of the tidal force may increase the event rate about twice [1]. The results do not crucially depend on the initial peculiar velocity of BHMACHOs and the initial scale factor when the BHMACHOs begin to interact [1]. The radiation drag does not affect the motion of BHMACHOs. After all, the probability distribution function for the coalescence time $f_t(t)$ in Eq.(27) is a good estimation. The event rate will be $5 \times 10^{-2} \times 2^{\pm 1}$ including the uncertainty from the various

effects. This suggests that we can at least expect a few events per year within 15 Mpc. This event rate of coalescing BHMACHO binaries is comparable to or greater than the upper limit of that of coalescing binary neutron stars [20]. The gravitational wave from such coalescence should be able to be detected by LIGO/VIRGO/TAMA/GEO network.

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FIGURE CAPTION

- Fig.1. The positions of the second body (thick curve) and the third body (dotted curve) relative to the first body for (a) $x/\bar{x} = 0.3, y/x = 2.0$ and (b) $x/\bar{x} = 0.3, y/x = 4.0$. $\theta = \pi/4$. The coordinate is normalized by $\bar{x}$.
- Fig.2. The semimajor axis a as a function of initial separation $x/\bar{x}$. The filled triangles are numerical data. The solid line is the approximate equation, $a/\bar{x} = (x/\bar{x})^4$.
- Fig.3. $b(\text{semiminor axis})/a(\text{semimajor axis})$ as a function of x/y for (a) $\theta = \pi/6$, (b) $\pi/4$, and (c) $\pi/3$. The filled triangles are numerical data. The solid line is the approximate equation, $b/a = (x/y)^3$.

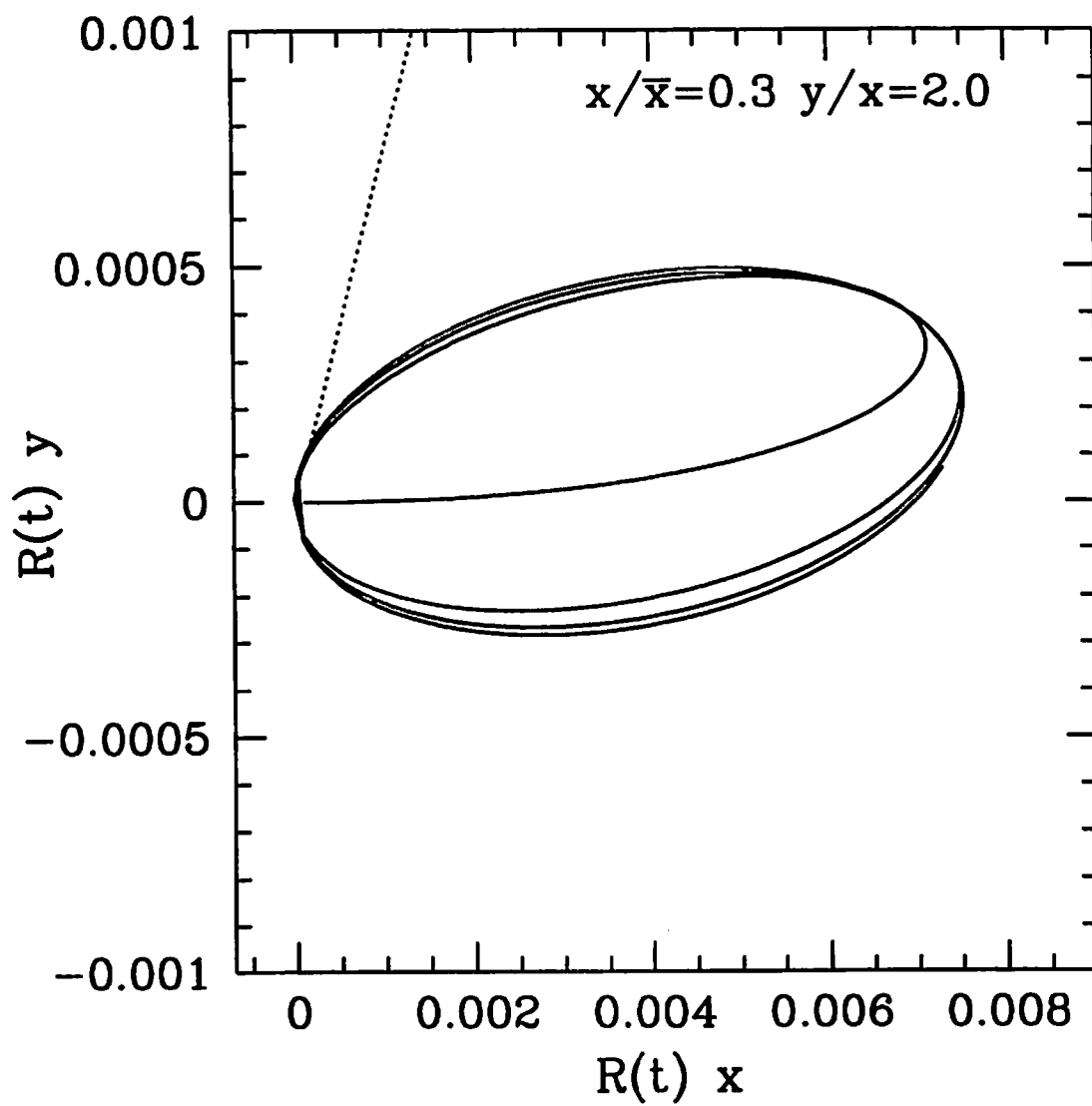


Fig.1(a)

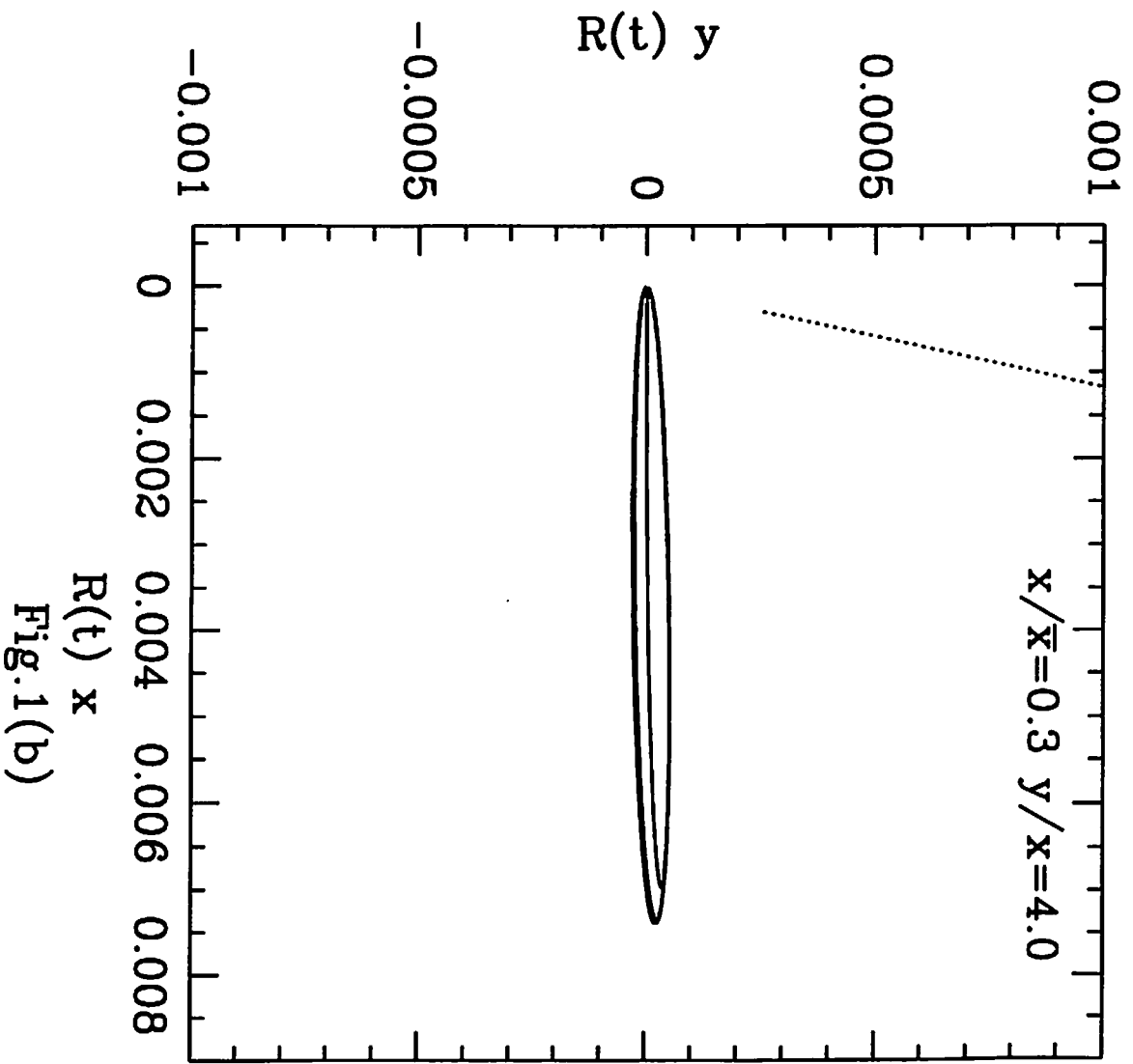


Fig.1(b)

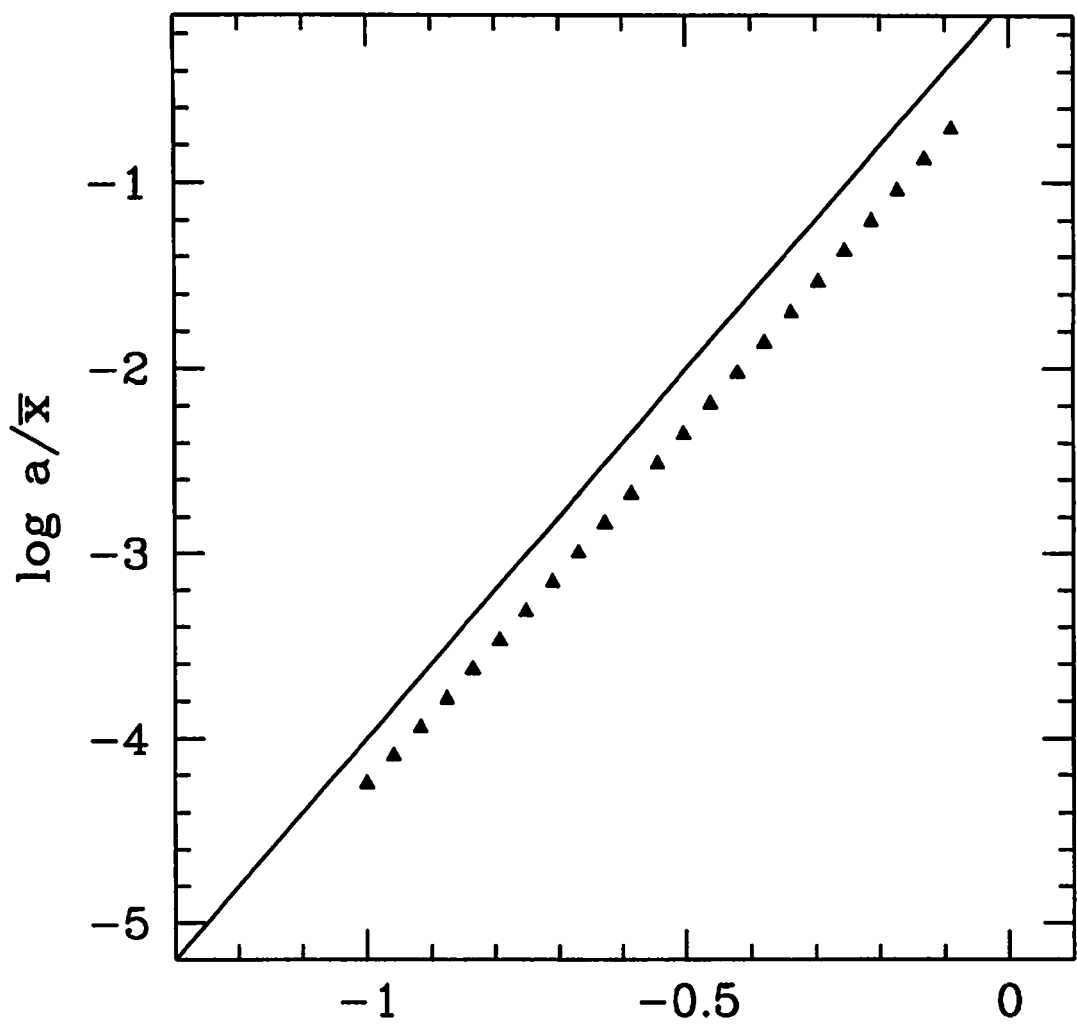


Fig.2

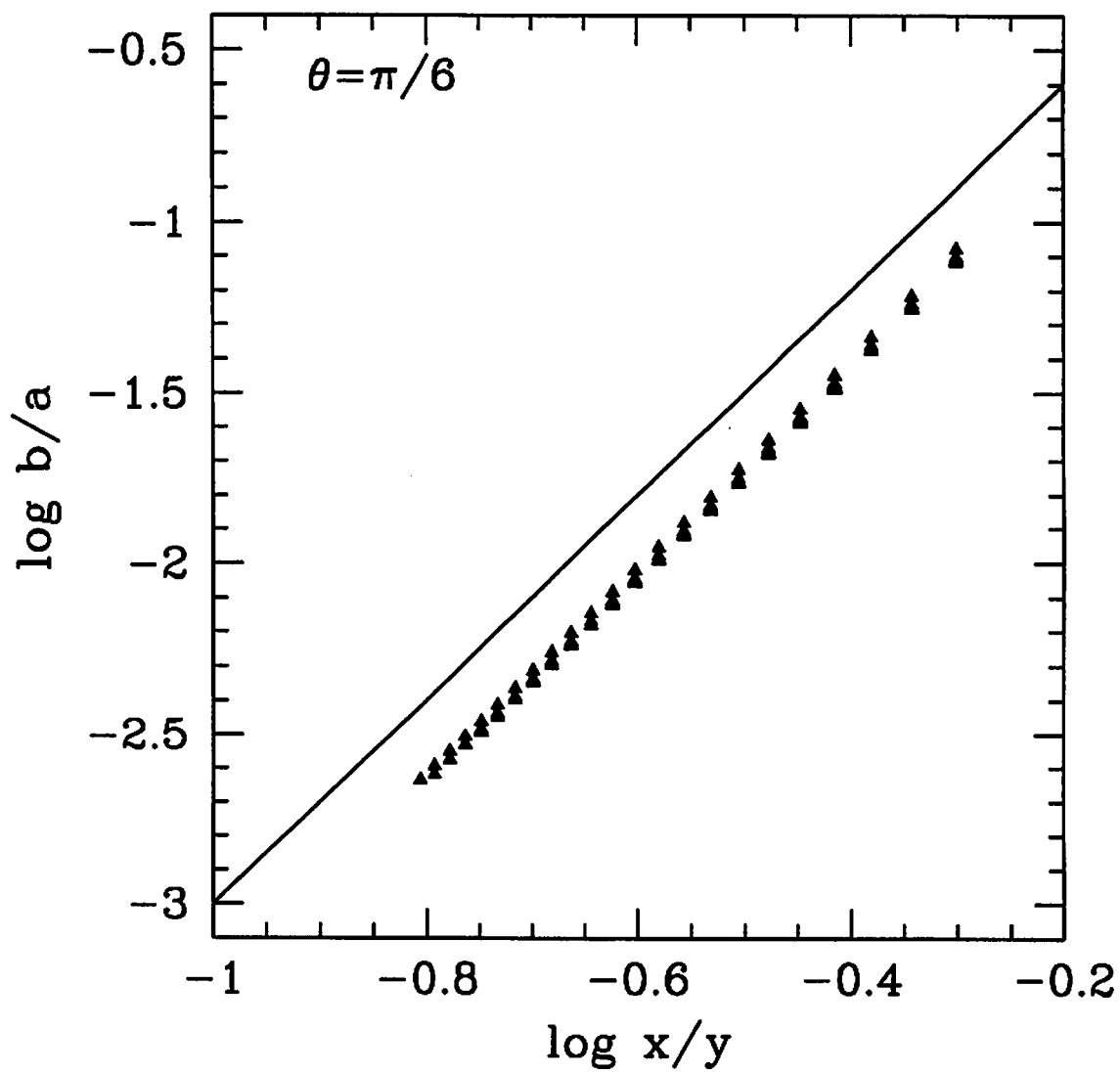


Fig.3(a)

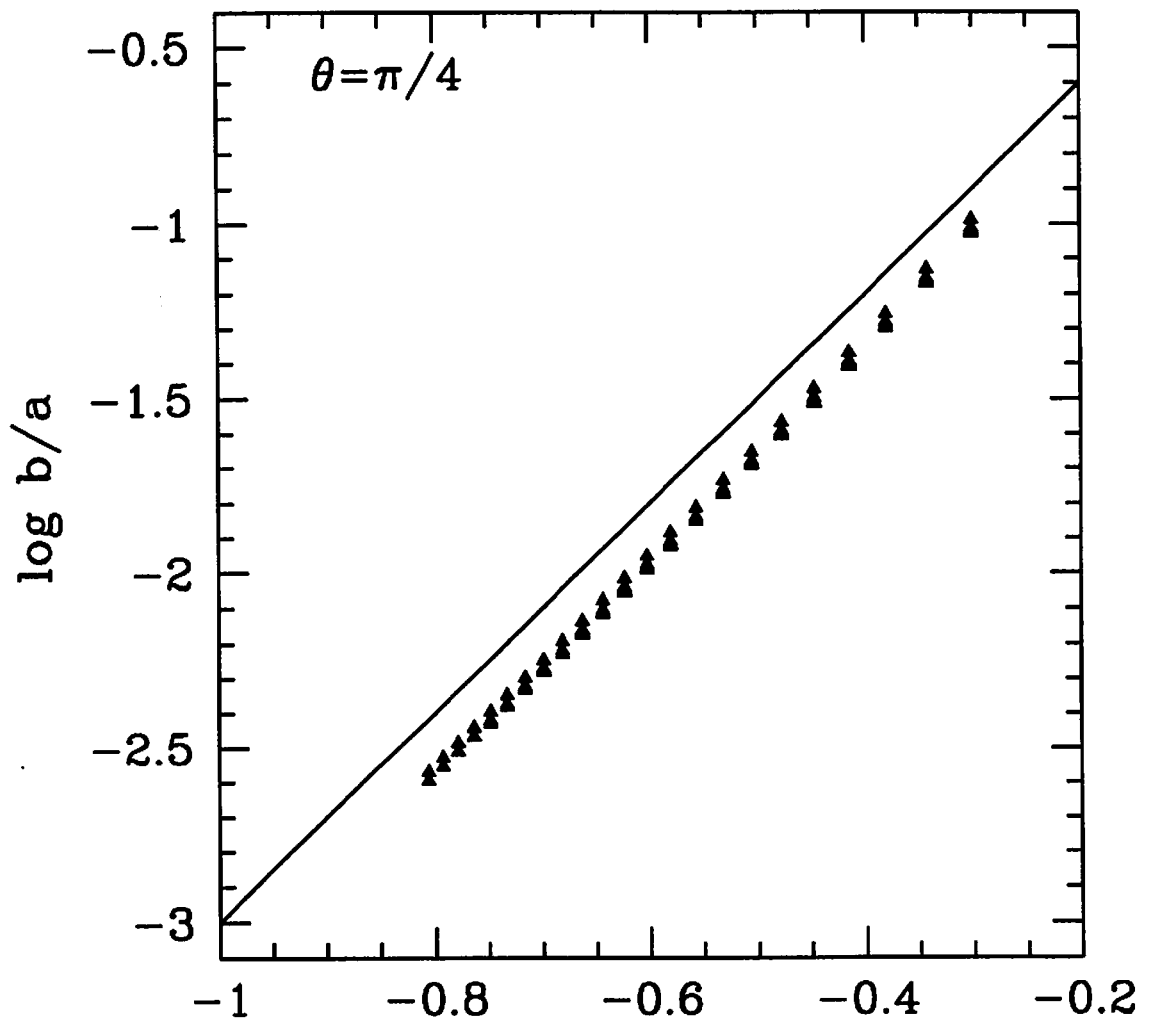
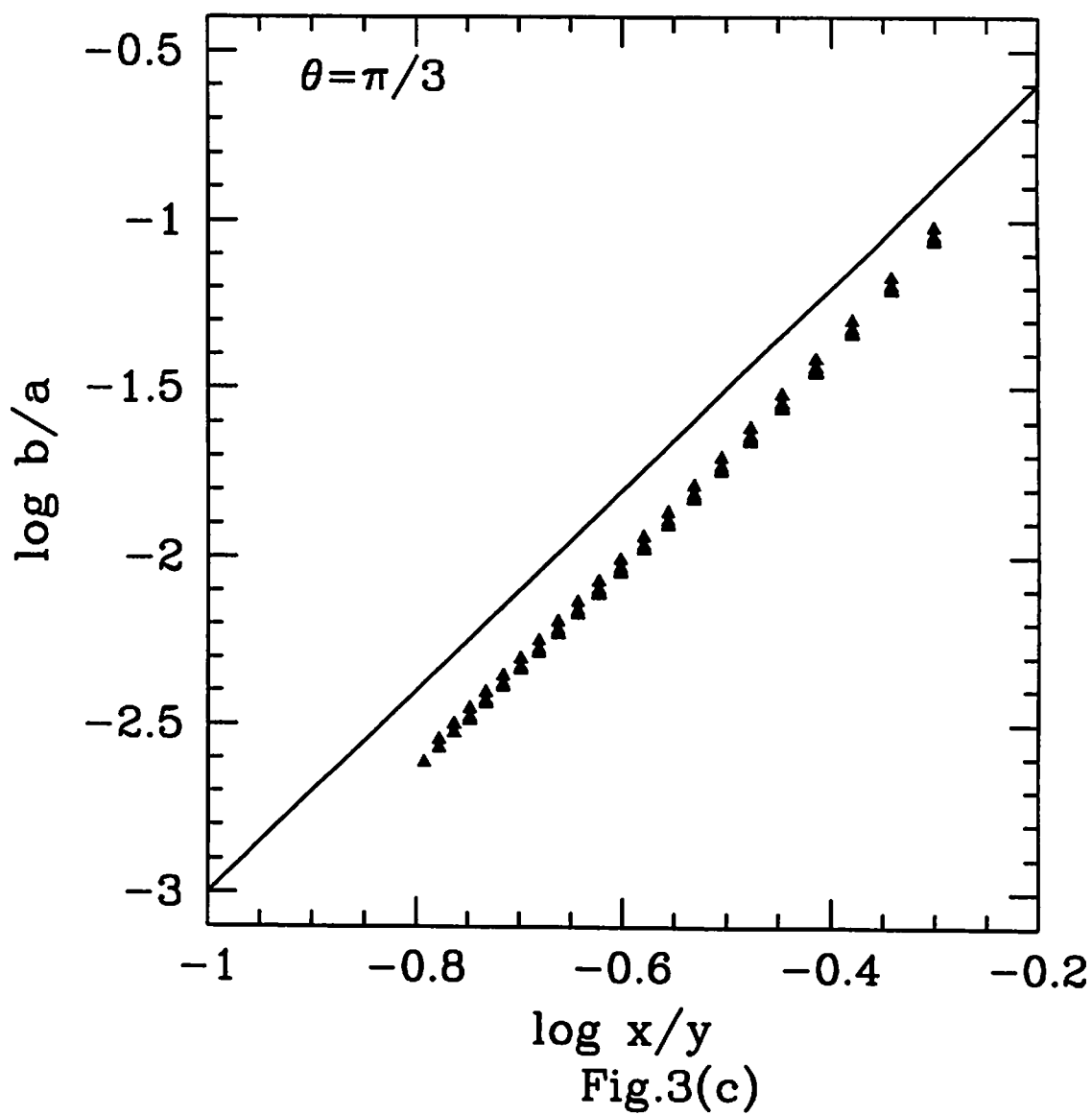


Fig.3(b)



Regularization of Radiation Reaction Force

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Abstract

In this talk, we propose a new method to extract the singular part from the linear metric perturbation induced by a point particle moving on a Schwarzschild/Kerr black hole. The only assumption we made here is the analyticity of the resulting metric perturbation. The method is expected to be useful in investigating the gravitational radiation reaction to the motion of the particle, its spin, and so on.

Owing to the recent progress of the technology, it becomes possible to detect gravitational waves and several projects of constructing laser-interferometric gravitational wave detectors are going on[1].

One of the most promising sources of gravitational waves is a coalescing compact binary system. The event rate is estimated to be 3 events per year, and, because of this reason, most of gravitational wave detectors are designed for gravitational waves from binary coalescence[1]. On the other hand, gravitational wave detectors are more sensitive to the frequency modulation of the gravitational waves rather than to the waveform modulation[2]. By the famous ‘quadrupole formula’, the frequency of the gravitational waves are directly related to the distribution of matter. Therefore, the problem is to consider the orbital evolution of the binary, taking into account the effect of radiation reaction.

Linear perturbation study is thought to be one of effective methods to investigate the orbital evolution of coalescing compact binary systems[3]. In linear perturbation study, one assumes that one compact object of a binary system makes a dominant contribution to the gravity and the gravitational field is approximated to be a Schwarzschild/Kerr black hole, and considers that the other object can be treated as a point particle which induces linear metric perturbation on the background black hole geometry.

In the previous works of perturbation study[3], one derives the asymptotic behavior of the metric perturbation induced by a point particle. Then the gravitational radiation loss of the energy and the angular momentum of the particle motion is derived by balance equations.

Though the use of balance equations are quite effective, they are not sufficient to investigate the radiation reaction. One example is the radiation reaction to a Carter constant. The orbit of a particle moving around a Kerr black hole is characterized by three parameter, i.e. energy, angular momentum and Carter constant. Thus the radiation reaction to the orbit cannot be computed until one finds the reaction to a Carter constant.

In this talk, we discuss a method to derive the explicit components of the radiation reaction force from the metric perturbation.

The local behavior of the metric perturbation around the position of the particle was investigated in our previous papers[4]. (We call this a ‘local’ analysis of the metric perturbation.) As one may easily infer, we found that the metric perturbation diverges at the particle, however, the metric perturbation is found to be divided into two part, i.e. the instantaneous part and the tail part. The instantaneous part owes to the divergent behavior of the perturbation around the particle and it is determined by the background metric and its derivative at the position of the particle. The tail part of the metric perturbation is regular at the particle and it is determined by the global structure of the background geometry and the past history of the particle motion.

Our previous paper[4] derived the equation of motion with the effect of gravitational radiation reaction. We found that the gravitational radiation reaction is totally due to the tail part of the metric perturbation, however, the explicit components of the tail part was unevaluated.

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In this talk, we propose a method to compute the explicit components of the tail part of the metric perturbation. The difficulty is that the tail part seems impossible to derive by a simple extension of a local analysis since the tail part depends on the global geometry of the background spacetime.

There are well-known methods to derive the metric perturbation of a Schwarzschild/Kerr black hole, i.e., Zerilli-Regge-Wheeler formalism[5] in the case of a Schwarzschild black hole, the method of Chrzanowski[6] in the case of a Kerr black hole. (We call this a 'global' analysis of the metric perturbation.) We consider the derivation of the tail part by a combination of the global analysis and the local analysis. The metric perturbation is obtained by the global analysis in the form of a Fourier-Harmonic expansion, in a certain gauge condition, and the instantaneous part is obtained by the local analysis in the harmonic gauge condition, and we consider the extraction of the instantaneous part from the metric perturbation obtained by a global analysis.

We suppose that the particle is placed at a point z^α . The metric perturbation and the instantaneous part of it diverge at z^α , therefore, we consider deriving them at a point other than z^α . After the derivation of the tail part, we consider taking the coincidence limit $x^\alpha \rightarrow z^\alpha$ to derive the regularized metric at the particle. At this stage, one may have two strategies. (1) One first derives the instantaneous part in the form of a Fourier-Harmonic expansion, then extracts each mode function of the instantaneous part from the corresponding mode function of the metric perturbation. (2) One first takes the summation of the Fourier-Harmonic expansion of the metric perturbation, then extracts the instantaneous part to derive the tail part.

The advantage of the strategy (1) is that the mode-by-mode extraction can be done after taking the coincidence limit $x^\alpha \rightarrow z^\alpha$ since each mode function of the tail part and the metric perturbation, hence each mode function of the instantaneous part is finite. However, there is a problem in this method. Since the instantaneous part of the metric perturbation is known just around the particle, it is not possible to derive the Fourier-Harmonic expansion of this part. Therefore we consider the strategy (2).

The problem of the strategy (2) is that it is difficult to sum a Fourier-Harmonic expansion of the metric perturbation before taking the coincidence limit $x^\alpha \rightarrow z^\alpha$. Our idea to overcome this difficulty is to use a Taylor expansion with respect to the radius (i.e. $1/x^r$ or $x^r - r_h$) where we adopt Boyer-Lindquist coordinates $\{t, r, \theta, \phi\}$ and r_h is the radius of the (outer) horizon. The summation of the Fourier-Harmonic expansion becomes much easier if one takes for each power of the radius with the post-newtonian approximation of the motion of the particle. We extract the instantaneous part from the metric perturbation in the form of a Taylor expansion, and then takes the coincidence limit $x^\alpha \rightarrow z^\alpha$.

It is important to note that the asymptotic behavior of the metric perturbation in the Fourier-Harmonic expansion at the horizon and at the spatial infinity is $\sim \exp[\pm i\omega(x^t + x^{r*})]$ and $\sim \exp[\pm i\omega(x^t - x^{r*})]$ respectively, thus, different for each mode function. (Here we use $x^{r*} = x^r + 2M \log[x^r - r_h]$, in which we denote the mass of the black hole by M .) We set $x^t - z^t = -(x^{r*} - z^{r*})$, $x^\theta = z^\theta$, $x^\phi = z^\phi$ for the Taylor expansion at the region $z^r > x^r > r_h$, and $x^t - z^t = x^{r*} - z^{r*}$, $x^\theta = z^\theta$, $x^\phi = z^\phi$ for the Taylor expansion at the region $\infty > x^r > z^r$, so that all the mode functions would have the same convergent asymptotic behavior, (Here we use $z^{r*} = z^r + 2M \log[z^r - r_h]$.) Then the coincidence limit $x^\alpha \rightarrow z^\alpha$ corresponds to the limit $x^r \rightarrow z^r$.

In the following, we summarized a procedure of the derivation of the radiation reaction force to the particle at z^α . A similar procedure is applicable to the derivation of the radiation reaction to the spin, and so on.

1) Bare Force in the Local Analysis:

Here we make use of a local analysis in the previous section, except that we calculate the explicit components in a coordinate dependent manner. We introduce a small parameter ϵ in order to see the singular behavior of the 'bare' force $F_{(bare)\alpha}(\epsilon, z)$ which is a reaction force derived by the full metric perturbation and its derivative at the position $x^\alpha(\epsilon, z)$,

$$F_{(bare)\alpha}(\epsilon, z) = -\delta\Gamma_{\alpha,\beta\gamma}(\epsilon, z)u^\beta(\epsilon, z)u^\gamma(\epsilon, z), \quad (1)$$

$$\delta\Gamma_{\alpha,\beta\gamma}(\epsilon, z) = \frac{1}{2}(\hbar_{\alpha\beta;\gamma}(x) + \hbar_{\gamma\alpha;\beta}(x) - \hbar_{\beta\gamma;\alpha}(x)), \quad (2)$$

$$\lim_{\epsilon \rightarrow 0} u^\alpha(\epsilon, z) = u^\alpha(z). \quad (3)$$

where $u^\alpha(z)$ is a tangential vector of the orbit at z^α . I define $x^\alpha(\epsilon, z)$ as,

$$x^t = z^t \pm (x^r \cdot x^r = z^r + \epsilon - z^{r*}), \quad (4)$$

$$x^r = z^r + \epsilon, \quad x^\theta = z^\theta, \quad x^\phi = z^\phi, \quad (5)$$

where the upper(lower) signature in $x^t(\epsilon, z)$ is taken for positive(negative) ϵ .

2) Laurent Expansion of the Bare Force:

As in the metric perturbation, the 'bare' force is divided into the instantaneous force and the tail force. We perform the Laurent expansion of each contribution regarding ϵ to see the singular behavior of the 'bare' force.

$$F_{(bare)\alpha}(\epsilon, z) = F_{(inst)\alpha}(\epsilon, z) + F_{(tail)\alpha}(\epsilon, z), \quad (6)$$

$$F_{(inst)\alpha}(\epsilon, z) = \frac{1}{\epsilon^2} f_{(inst)\alpha}^{(2)}(z) + \frac{1}{\epsilon} f_{(inst)\alpha}^{(1)}(z) + f_{(inst)\alpha}^{(0)}(z) + O(\epsilon), \quad (7)$$

$$F_{(tail)\alpha}(\epsilon, z) = f_{(tail)\alpha}^{(0)}(z) + O(\epsilon). \quad (8)$$

Here $O(\epsilon)$ represent terms that vanish in the coincidence limit $\epsilon \rightarrow 0$. $f_{(inst)\alpha}^{(2)}$, $f_{(inst)\alpha}^{(1)}$, $f_{(inst)\alpha}^{(0)}$ are possible to compute in an explicit manner and the problem becomes the evaluation of $f_{(tail)\alpha}^{(0)}$ that is non-trivial in the local analysis. Since one takes the coincidence limit $\epsilon \rightarrow 0$ after the extraction of the instantaneous force, it is convenient to re-define the instantaneous force as,

$$F_{(inst')\alpha}(\epsilon, z) = \frac{1}{\epsilon^2} f_{(inst)\alpha}^{(2)}(z) + \frac{1}{\epsilon} f_{(inst)\alpha}^{(1)}(z) + f_{(inst)\alpha}^{(0)}(z). \quad (9)$$

3) Taylor Expansion of the Instantaneous Force:

We consider a Taylor expansion of the instantaneous force $F_{(inst')\alpha}(\epsilon, z)$ regarding the radius $x^r = \epsilon + z^r$. Since the instantaneous force diverges at $\epsilon = 0$, the expansion is different for the sign of ϵ . For positive ϵ , it is expanded with $1/x^r$ that is valid for $\infty > x^r > z^r$. From (9), one obtains

$$F_{(inst')\alpha}(x^r - z^r, z) = \sum_{n=0}^{\infty} \frac{1}{x^r{}^n} F_{(inst+)\alpha}^{(n)}(z), \quad (10)$$

$$F_{(inst+)\alpha}^{(0)} = f_{(inst)\alpha}^{(0)}, \quad (11)$$

$$F_{(inst+)\alpha}^{(n)} = z^r{}^{(n-1)} f_{(inst)\alpha}^{(1)} + (n-1) z^r{}^{(n-2)} f_{(inst)\alpha}^{(2)}. \quad (12)$$

For negative ϵ , it is expanded with $x^r - r_h$ that is valid for $z^r > x^r > r_h$. From (9), one obtains

$$F_{(inst')\alpha}(x^r - z^r, z) = \sum_{n=0}^{\infty} (x^r - r_h)^n F_{(inst-)\alpha}^{(n)}(z), \quad (13)$$

$$F_{(inst-)\alpha}^{(0)} = f_{(inst)\alpha}^{(0)} - \frac{1}{z^r - r_h} f_{(inst)\alpha}^{(1)} + \frac{1}{(z^r - r_h)^2} f_{(inst)\alpha}^{(2)}, \quad (14)$$

$$F_{(inst-)\alpha}^{(n)} = -(z^r - r_h)^{n-1} f_{(inst)\alpha}^{(1)} + (n+1)(z^r - r_h)^{n-2} f_{(inst)\alpha}^{(2)}. \quad (15)$$

(In the following, we denote the Taylor expansion at $\infty > x^r > z^r$ ($z^r > x^r > r_h$) by $+(-)$ in $F^{(n)}(z)$.)

4) Bare Force in the Global Analysis:

Using the Zerilli-Regge-Wheeler formalism[5] or the method of Chrzanowski[6], the derivation of the metric perturbation is reduced to the derivation of ordinary differential radial equations of the mode functions of a Fourier-Harmonic expansion, which are called Zerilli-Regge-Wheeler functions or Teukolsky functions. A quite efficient method to solve these equations are known[7, 8], in which the mode functions are expanded with Bessel functions, hypergeometric functions, and so on. These functions are known in the form of a Taylor expansion, therefore, one can obtain the Taylor expansion of the bare force $F_{(bare)\alpha}(x^r - z^r, z)$ regarding $1/x^r$ for $\infty > x^r > z^r$, or regarding $x^r - r_h$ for $z^r > x^r > r_h$.

$$F_{(bare)\alpha}(x^r - z^r, z) = \begin{cases} \sum_{lm\omega} \sum_{n=0}^{\infty} \frac{1}{(x^r)^n} F_{(bare+)\alpha}^{(n)}(z) \\ \sum_{lm\omega} \sum_{n=0}^{\infty} (x^r - r_h)^n F_{(bare-)\alpha}^{(n)}(z) \end{cases} \quad (16)$$

5) Mode summation of the Bare Force:

We take the summation of the Fourier-Harmonic series of the bare force given in the global analysis for each power of the Taylor expansion. Here the summation becomes easier when taking into account the post-Newtonian expansion $v \ll 1$, as one has done in the previous method of perturbation study[3],

$$\mu x^r \approx \mu z^r \approx O(v^3), \quad \omega x^r \approx \omega z^r \approx O(v). \quad (17)$$

One then obtain,

$$F_{(bare)\alpha}(x^r - z^r, z) = \begin{cases} \sum_{n=0}^{\infty} \frac{1}{(x^r)^n} F_{(bare+)\alpha}^{(n)}(z) \\ \sum_{n=0}^{\infty} (x^r - r_h)^n F_{(bare-)\alpha}^{(n)}(z) \end{cases} \quad (18)$$

5) Regularized Force:

We extract the instantaneous force (10), (13) from the bare force (18) so as to obtain the tail force $F_{(tail)\alpha}(x^r - z^r, z)$ in the form of a Taylor expansion.

$$F_{(tail)\alpha}(x^r - z^r, z) = \begin{cases} \sum_{n=0}^{\infty} \frac{1}{(x^r)^n} F_{(tail+)\alpha}^{(n)}(z) \\ \sum_{n=0}^{\infty} (x^r - r_h)^n F_{(tail-)\alpha}^{(n)}(z) \end{cases} \quad (19)$$

$$F_{(tail\pm)\alpha}^{(n)} = F_{(bare\pm)\alpha}^{(n)} - F_{(inst\pm)\alpha}^{(n)} \quad (20)$$

Since the tail force is regular at $x^r = z^r$ by construction, one may take the coincidence limit $x^r = z^r$ of (19) so as to obtain the regularized force.

The crucial assumption we put here is that the tail force $F_{(tail)\alpha}(x^r - z^r, z)$ is analytical at $\infty > x^r > r_h$. Because of this assumption, the Taylor expansion derived at $\infty > x^r > z^r$ ($z^r > x^r > r_h$) becomes valid at $\infty > x^r > r_h$, and one can properly set $x^r = z^r$ in (19) to obtain the regularized force.

Until now, we have ignored the gauge problem in deriving the radiation reaction force. The local analysis done in our previous paper adopts the harmonic gauge condition, while the Fourier-Harmonic expansion of the metric perturbation adopts a different gauge². We assume that the gauge transformation between these two gauges is finite at the position of the particle. With this assumption, the finite gauge ambiguity is left in the radiation reaction force. However it is well known that gauge dependence vanishes when one considers the time-averaged reaction, which is the case when considering the adiabatic evolution of the orbit. Therefore we consider that the gauge ambiguity does not make trouble in constructing theoretical templates of gravitational waves.

Using the method developed here, the computation of the radiation reaction to a Carter constant and the spin evolution is in progress[9].

² In a Schwarzschild geometry, the Zerilli-Regge-Wheeler formalism[5] takes the so called Regge-Wheeler gauge condition. In a Kerr geometry, a radiation gauge is taken[6].

Acknowledgements

We thank all the members of the group in theoretical gravitational-wave physics in Japan and all the members of the theoretical astrophysics group in Osaka University for fruitful conversations. We also thanks Professor M. Sasaki, Professor H. Sato, Professor S. Ikeuchi, Professor F. Takahara and Professor A. Hosoya for their continuous encouragements. This work was supported in part by Monbusho Grant-in-Aid for Scientific Research No.5427.

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Gravitational radiation from corotating binary neutron stars

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Abstract

We analytically study gravitational radiation from corotating binary neutron stars composed of incompressible, homogeneous fluid in circular orbits. The energy loss rate is derived up to the first post-Newtonian (1PN) order beyond the quadrupole approximation including effects of the finite size of each star of binary. It is found that the leading term of finite size effects in the 1PN order is only $O(GM_*/c^2 a_*)$ smaller than that in the Newtonian order, where $GM_*/c^2 a_*$ means the ratio of the gravitational radius to the mean radius of each star of binary, and the 1PN term acts to decrease the Newtonian finite size effect in gravitational radiation.

1 Introduction

Around the beginning of the next century, the laser interferometers for detection of gravitational waves, such as LIGO[1], VIRGO[2], GEO600[3] and TAMA300[4] will be in operation. The most important targets for these detectors are coalescing binary neutron stars (BNS's) because in a period of their inspiraling phase they will emit gravitational waves of frequencies in the sensitive range of these detectors, i.e., from 10Hz to 1000Hz. We have to prepare accurate theoretical templates in order to extract informations of BNS's such as each mass, spin and so on using matched filtering technique[5].

In their early inspiraling phase, the hydrodynamical effect of each star of binary is less effective and the point particle approximation works well. In the point particle approximation, the energy loss rate is calculated up to 2.5 post-Newtonian (PN) order for arbitrary mass binaries[6]. The other approach for the inspiraling phase is the black hole perturbation method. In this formalism, the energy loss rate is calculated up to 4PN order in a Kerr black hole case[7], and 5.5PN order in a Schwarzschild black hole case[8].

In the late inspiraling phase, however, the hydrodynamical effects of each star of binary become important, and we have to study the evolution of the binary taking into account them[9][10][11][12]. The importance of the study of the late inspiraling phase comes from the existence of the innermost stable circular orbit (ISCO). If we are able to know the location of the ISCO from the signal of gravitational waves, we can see a strong general relativistic phenomenon as well as we may get the information of the equation of state of neutron stars, i.e., the relation between the mass and the radius[13][14]. To know the precise location of the ISCO, it is desirable to construct the theoretical template of gravitational waves in the late inspiraling phase as in the early one. Up to the Newtonian order, we have already understood gravitational radiation from binary systems of finite size stars[15][16][17], but apparently the Newtonian treatment is not appropriate because BNS's are general relativistic objects.

To investigate general relativistic effects on gravitational waves in the late inspiraling phase, in this paper, we analytically calculate the energy loss rate of the 1PN order including the finite size effects. In the calculation, we assume that BNS's are composed of the incompressible and homogeneous fluid and then obtain an ellipsoidal equilibrium configuration for each star. Also, we suppose the corotating circular orbit in which each star of binary uniformly rotates around the center of mass of the system. Although the assumption of corotation is not appropriate for the realistic one which will be almost irrotational[18][17], the energy loss rate we calculate in this paper will be useful to see contribution of the finite size effects.

This paper is organized as follows. In Sec. 2, we derive the mass quadrupole moments which are necessary for calculation of gravitational waves by using Blanchet-Damour formalism[19]. In Sec. 3, we

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give the the energy loss rate gathering the time derivatives of the mass quadrupole moments which we derive in Sec. 2. Section 4 is devoted to summary and discussion.

Throughout this paper, we use the unit $G = 1$ and c denotes the speed of light. Latin indices $i, j, k, \dots$ take 1 to 3, and δ_{ij} denotes the Kronecker's delta. We use I_{ij} and $\mathcal{I}_{ij}$ as the quadrupole moment and its trace free part,

$$I_{ij} \equiv \int d^3x \rho x_i x_j = \frac{M}{5} a_i^2 \delta_{ij}, \quad \mathcal{I}_{ij} = I_{ij} - \frac{1}{3} \delta_{ij} \sum_{k=1}^3 I_{kk}, \quad (1)$$

where M and a_i denote the Newtonian mass and the principal axes of the star.

2 Formulation

In this paper, we calculate the energy loss rate of gravitational radiation from corotating binary stars. Each star of binary is assumed to be composed of the incompressible and homogeneous fluid because all the calculations can be done analytically. We pay particular attention to BNS's of equal masses in circular orbits whose coordinate separation is R . We assume that the center of mass of the binary system locates at the origin of the *inertial* coordinate system (X_1, X_2, X_3) , and the center of mass of a star (star 1) locates at the origin of the *comoving* coordinate system (x_1, x_2, x_3) . In the comoving coordinate system, the other star (star 2) locates at $(x_1, x_2, x_3) = (-R, 0, 0)$. The transformation from the coordinate system (X_1, X_2, X_3) to (x_1, x_2, x_3) is given by

$$\begin{aligned} X_1 &= \left(x_1 + \frac{R}{2}\right) \cos \Omega t - x_2 \sin \Omega t, \\ X_2 &= \left(x_1 + \frac{R}{2}\right) \sin \Omega t + x_2 \cos \Omega t, \\ X_3 &= x_3. \end{aligned} \quad (2)$$

According to Blanchet and Damour[19], in the 1PN order, the energy loss rate is written as

$$\frac{dE}{dt} = -\frac{1}{5c^5} \left[M_{ij}^{(3)} M_{ij}^{(3)} + \frac{1}{c^2} \left(\frac{5}{189} M_{ijk}^{(4)} M_{ijk}^{(4)} + \frac{16}{9} S_{ij}^{(3)} S_{ij}^{(3)} \right) \right], \quad (3)$$

where $M_{ij} \dots$ and S_{ij} are Blanchet-Damour's mass multipole moments and the current quadrupole moment, and the superscript (n) where $n = 2, 3, 4$ denotes the time derivative d^n/dt^n . In the subsequent subsections, we derive M_{ij} which are necessary for calculation of gravitational waves. Note that in the case of the identical star binary the total M_{ijk} and S_{ij} vanish, and they have no contributions to the energy loss rate[20].

2.1 Mass quadrupole moment

First, we calculate M_{ij} up to the 1PN order. M_{ij} is the even term of the inertial coordinates X_1, X_2 , and X_3 , and we consider the equal mass binary. Then, the contribution from star 1 is the same as that from star 2. M_{ij} is expressed as

$$\begin{aligned} M_{ij}(t) &= \int d^3x \sigma \hat{X}_{ij} + \frac{1}{14c^2} \frac{d^2}{dt^2} \left(\int d^3x \rho \hat{X}_{ij} X^2 \right) - \frac{20}{21c^2} \frac{d}{dt} \left(\int d^3x \rho v_k \hat{X}_{ijk} \right), \\ &= \int d^3x \rho \hat{X}_{ij} + \frac{1}{c^2} \left[\int d^3x \rho \left(2v^2 + 2U + \frac{3P}{\rho} \right) \hat{X}_{ij} + \frac{1}{14} \frac{d^2}{dt^2} \left(\int d^3x \rho \hat{X}_{ij} X^2 \right) \right. \\ &\quad \left. - \frac{20}{21} \frac{d}{dt} \left(\int d^3x \rho v_k \hat{X}_{ijk} \right) \right], \end{aligned} \quad (4)$$

where

$$\sigma = \rho \left[1 + \frac{1}{c^2} \left(2v^2 + 2U + \frac{3P}{\rho} \right) \right], \quad (5)$$

$$\hat{X}_{ij} = X_i X_j - \frac{1}{3} \delta_{ij} X^2, \quad (6)$$

$$\hat{X}_{ijk} = X_i X_j X_k - \frac{1}{5} X^2 (\delta_{ij} X_k + \delta_{jk} X_i + \delta_{ki} X_j), \quad (7)$$

$$X^2 = X_1^2 + X_2^2 + X_3^2, \quad (8)$$

$$v^i = (-\Omega X_2, \Omega X_1, 0). \quad (9)$$

In the following calculation, we integrate only over star 1 because the total mass quadrupole moment is twice as large as that of star 1. Moreover, as shown in Ref. [10], the deformation from the Newtonian ellipsoid becomes higher order effect of R^{-1} . Therefore, we integrate over the ellipsoid simply.

2.1.1 Newtonian order

The quadrupole moment of the Newtonian order appears in the first term of Eq. (4) and expressed as

$$D_{ij} \equiv \int d^3 X \rho \left(X_i X_j - \frac{1}{3} \delta_{ij} X^2 \right). \quad (10)$$

Using Eq. (2), we get the components as

$$D_{11} = \hat{D} \cos^2 \Omega t + \text{const.}, \quad D_{22} = -\hat{D} \sin^2 \Omega t + \text{const.}, \quad D_{12} = \hat{D} \sin \Omega t \cos \Omega t, \quad (11)$$

where $\hat{D} = I_{11} - I_{22} + R^2 M/4$. In these equations, we use $M \equiv \int \rho d^3 x$ as the Newtonian mass. Since D_{33} does not depend on time and the other terms vanish, we do not write them here. Note that in these components, only the contribution from star 1 is included.

2.1.2 1PN order

Next, we calculate the 1PN order terms. For the 1PN order terms in Eq. (4), we define

$$\begin{aligned} V_{ij} &\equiv 2 \int d^3 X \rho v^2 \hat{X}_{ij}, \quad U_{ij}^{1 \rightarrow 1} \equiv 2 \int d^3 X \rho U^{1 \rightarrow 1} \hat{X}_{ij}, \quad U_{ij}^{2 \rightarrow 1} \equiv 2 \int d^3 X \rho U^{2 \rightarrow 1} \hat{X}_{ij}, \\ P_{ij} &\equiv 3 \int d^3 X \rho \frac{P}{\rho} \hat{X}_{ij}, \quad Q_{ij} \equiv \frac{1}{14} \frac{d^2}{dt^2} \left(\int d^3 X \rho \hat{X}_{ij} X^2 \right), \quad R_{ij} \equiv -\frac{20}{21} \frac{d}{dt} \left(\int d^3 X \rho v_k \hat{X}_{ijk} \right), \end{aligned} \quad (12)$$

because it is convenient to calculate the terms separately. In these equations, we use

$$U^{1 \rightarrow 1} = \pi \rho \left(A_0 - \sum_i A_i x_i^2 \right), \quad (13)$$

$$\begin{aligned} U^{2 \rightarrow 1} &= \frac{M}{R} \left[1 - \frac{x_1}{R} + \frac{2x_1^2 - x_2^2 - x_3^2}{2R^2} + \frac{-2x_1^3 + 3x_1(x_2^2 + x_3^2)}{2R^3} + O(R^{-4}) \right] \\ &\quad + \frac{3I_{11}}{2R^3} \left(1 - \frac{3x_1}{R} + O(R^{-2}) \right) + O(R^{-5}), \end{aligned} \quad (14)$$

$$P = P_0 \left(1 - \sum_i \frac{x_i^2}{a_i^2} \right), \quad (15)$$

$$P_0 = \frac{\rho}{3} \left[\pi \rho A_0 - \frac{\Omega_N^2}{2} (a_1^2 + a_2^2) - \frac{M}{2R^3} (2a_1^2 - a_2^2 - a_3^2) \right] + O(R^{-5}), \quad (16)$$

where $U^{1 \rightarrow 1}$, $U^{2 \rightarrow 1}$ and P denote the Newtonian potential generated by star 1 itself, the Newtonian potential generated by star 2, and the pressure, respectively[10]. $A_{ij} \dots$ are index symbols introduced by Chandrasekhar[21] and $A_0 = \sum_i A_i a_i^2$ is calculated from[21]

$$A_0 = a_1 a_2 a_3 \int_0^\infty \frac{du}{\sqrt{(a_1^2 + u)(a_2^2 + u)(a_3^2 + u)}}. \quad (17)$$

When we derive the 1PN order terms, we need to calculate only some combinations: The 1PN order terms in $M_{ij}^{(3)}M_{ij}^{(3)}$ are given by

$$\begin{aligned}
& D_{ij}^{(3)} \left(V_{ij}^{(3)} + U_{ij}^{1 \rightarrow 1(3)} + U_{ij}^{2 \rightarrow 1(3)} + P_{ij}^{(3)} + Q_{ij}^{(3)} + R_{ij}^{(3)} \right) \\
&= D_{11}^{(3)} \left[V_{11}^{(3)} - V_{22}^{(3)} + U_{11}^{1 \rightarrow 1(3)} - U_{22}^{1 \rightarrow 1(3)} + U_{11}^{2 \rightarrow 1(3)} - U_{22}^{2 \rightarrow 1(3)} + P_{11}^{(3)} - P_{22}^{(3)} \right. \\
&\quad \left. + Q_{11}^{(3)} - Q_{22}^{(3)} + R_{11}^{(3)} - R_{22}^{(3)} \right] \\
&\quad + 2D_{12}^{(3)} \left[V_{12}^{(3)} + U_{12}^{1 \rightarrow 1(3)} + U_{12}^{2 \rightarrow 1(3)} + P_{12}^{(3)} + Q_{12}^{(3)} + R_{12}^{(3)} \right], \tag{18}
\end{aligned}$$

where we use the relation $D_{22}^{(i)} = -D_{11}^{(i)}$. Then, we need to calculate only two combinations of the components expressed as $[(1, 1) - (2, 2)]$ and $[(1, 2)]$. We separately show the terms of the 1PN order in Eq. (18) in the following.

$$V_{11} - V_{22} = 2\Omega^2 \hat{V} \cos 2\Omega t, \quad V_{12} = \Omega^2 \hat{V} \sin 2\Omega t, \tag{19}$$

$$U_{11}^{1 \rightarrow 1} - U_{22}^{1 \rightarrow 1} = 2\hat{U}^{1 \rightarrow 1} \cos 2\Omega t, \quad U_{12}^{1 \rightarrow 1} = \hat{U}^{1 \rightarrow 1} \sin 2\Omega t, \tag{20}$$

$$U_{11}^{2 \rightarrow 1} - U_{22}^{2 \rightarrow 1} = 2\hat{U}^{2 \rightarrow 1} \cos 2\Omega t + O(R^{-3}), \quad U_{12}^{2 \rightarrow 1} = \hat{U}^{2 \rightarrow 1} \sin 2\Omega t + O(R^{-3}), \tag{21}$$

$$P_{11} - P_{22} = 3\hat{P} \cos 2\Omega t, \quad P_{12} = \frac{3}{2}\hat{P} \sin 2\Omega t, \tag{22}$$

$$Q_{11} - Q_{22} = -\frac{2}{7}\Omega^2 \hat{Q} \cos 2\Omega t, \quad Q_{12} = -\frac{1}{7}\Omega^2 \hat{Q} \sin 2\Omega t, \tag{23}$$

$$R_{11} - R_{22} = -\frac{16}{21}\Omega^2 \hat{Q} \cos 2\Omega t, \quad R_{12} = -\frac{8}{21}\Omega^2 \hat{Q} \sin 2\Omega t, \tag{24}$$

where

$$\hat{V} = \frac{R^4 M}{16} + \frac{3R^2}{2} I_{11} + I_{1111} - I_{2222}, \tag{25}$$

$$\hat{U}^{1 \rightarrow 1} = \pi \rho \left[A_0 \left(I_{11} - I_{22} + \frac{R^2 M}{4} \right) - \sum_i A_i \left(I_{11ii} - I_{22ii} + \frac{R^2}{4} I_{ii} \right) \right], \tag{26}$$

$$\hat{U}^{2 \rightarrow 1} = \frac{M}{R} \left(\frac{R^2 M}{4} - I_{22} + \frac{3}{4} I_{11} \right), \tag{27}$$

$$\hat{P} = \frac{P_0}{\rho} \left[I_{11} - I_{22} + \frac{R^2 M}{4} - \sum_i \frac{1}{a_i^2} \left(I_{11ii} - I_{22ii} + \frac{R^2}{4} I_{ii} \right) \right], \tag{28}$$

$$\hat{Q} = \frac{R^4 M}{16} + \frac{R^2}{4} (6I_{11} + I_{33}) + I_{1111} - I_{2222} + I_{1133} - I_{2233}, \tag{29}$$

and

$$\begin{aligned}
I_{ijkl} &\equiv \int d^3x \rho x_i x_j x_k x_l, \\
&= \frac{M}{35} \times \begin{cases} 3a_i^4 & (i = j = k = l) \\ a_i^2 a_k^2 & (i = j \neq k = l). \end{cases} \tag{30}
\end{aligned}$$

3 The energy loss rate

Our purpose is to know the finite size effects of the 1PN order in gravitational radiation. In the following, we calculate the energy loss rate up to $O(R^{-3})$ in the 1PN order beyond the Newtonian order of the quadrupole formula.

Taking into account the fact stated in section 2, the energy loss rate is expressed as

$$\frac{dE}{dt} = -\frac{1}{5c^5} \left[(2D_{ij}^{(3)})(2D_{ij}^{(3)}) + \frac{2}{c^2} (2D_{ij}^{(3)}) \left\{ 2V_{ij}^{(3)} + 2U_{ij}^{1 \rightarrow 1(3)} + 2U_{ij}^{2 \rightarrow 1(3)} + 2P_{ij}^{(3)} + 2Q_{ij}^{(3)} + 2R_{ij}^{(3)} \right\} \right]. \tag{31}$$

Collecting the results we derived in section 2, we can get the energy loss rate as

$$\begin{aligned} \frac{dE}{dt} = & -\frac{64M^5}{5c^5R^5} \left[1 + \frac{1}{5R^2} \{ 8(a_1^2 - a_2^2) + 9(2a_1^2 - a_2^2 - a_3^2) \} + O(R^{-4}) \right. \\ & + \frac{1}{c^2} \left[10\pi\rho A_0 - \frac{151M}{84R} + \frac{4\pi\rho A_0}{5R^2} \left\{ \frac{412}{21}(a_1^2 - a_2^2) + \frac{111}{5}(2a_1^2 - a_2^2 - a_3^2) \right\} \right. \\ & \left. \left. - \frac{M}{210R^3} (1658a_1^2 - 1103a_2^2 - 16a_3^2) + O(R^{-4}) \right] \right], \end{aligned} \quad (32)$$

where we use the angular velocity of the 1PN order[10]

$$\begin{aligned} \Omega^2 = & \frac{2M}{R^3} \left[1 + \frac{1}{c^2} \left\{ 2\pi\rho A_0 - \frac{9M}{4R} - \frac{M}{10R^3} (28a_1^2 - 14a_2^2 - 9a_3^2) + O(R^{-4}) \right\} \right] \\ & + \frac{18I_{11}}{R^5} \left(1 + \frac{28}{15c^2} \pi\rho A_0 + O(R^{-2}) \right). \end{aligned} \quad (33)$$

To express Eq. (32), we used M , but it does not conserve through the sequence of the binary. Moreover, we simply used R as the orbital separation, but in the 1PN case, the center of mass deviates from that in the Newtonian one and as a result, the orbital separation also deviates from R . We need a conserved mass and an appropriate definition of the center of mass. However, if we rewrite Eq. (32) using the conserved *proper* mass

$$\begin{aligned} M_* = & \int d^3x \rho \left[1 + \frac{1}{c^2} \left(\frac{v^2}{2} + 3U \right) \right] \\ = & M \left[1 + \frac{1}{c^2} \left\{ \frac{12\pi\rho A_0}{5} + \frac{13M}{4R} + \frac{M}{20R^3} (34a_1^2 - 11a_2^2 - 15a_3^2) + O(R^{-5}) \right\} \right], \end{aligned} \quad (34)$$

the energy loss rate seems to depend on the internal structure of the star of binary even if we take the limits $a_i/R \rightarrow 0$. This is simply because M_* is not the gravitational mass. Instead of M_* , we need an appropriate *gravitational* mass.

If we use the parameterized post-Newtonian (PPN) mass[22]

$$\begin{aligned} M_{\text{PPN}} = & \int d^3x \rho \left[1 + \frac{1}{c^2} \left(\frac{v^2}{2} + 3U - \frac{1}{2}U_{\text{self}} + \frac{v_{\text{self}}^2}{2} \right) \right] \\ = & M \left[1 + \frac{1}{c^2} \left\{ 2\pi\rho A_0 + \frac{13M}{4R} + \frac{M}{20R^3} (38a_1^2 - 7a_2^2 - 15a_3^2) + O(R^{-5}) \right\} \right], \end{aligned} \quad (35)$$

with the center of mass of each star defined as

$$x_{\text{PPN}}^i = \frac{1}{M_{\text{PPN}}} \int d^3x \rho x^i \left[1 + \frac{1}{c^2} \left(\frac{v^2}{2} + 3U - \frac{1}{2}U_{\text{self}} + \frac{v_{\text{self}}^2}{2} \right) \right] \quad (36)$$

and the orbital separation as

$$R_{\text{PPN}} = R \left[1 + \frac{1}{c^2} \left\{ -\frac{4Ma_1^2}{5R^3} + O(R^{-5}) \right\} \right], \quad (37)$$

the leading internal structure dependent term in the 1PN order is renormalized. Substituting the PPN mass and R_{PPN} into Eq. (32), we rewrite the energy loss rate as

$$\begin{aligned} \frac{dE}{dt} = & -\frac{64M_{\text{PPN}}^5}{5c^5R_{\text{PPN}}^5} \left[1 + \frac{1}{5R_{\text{PPN}}^2} \{ 8(a_{1\text{PPN}}^2 - a_{2\text{PPN}}^2) + 9(2a_{1\text{PPN}}^2 - a_{2\text{PPN}}^2 - a_{3\text{PPN}}^2) \} + O(R_{\text{PPN}}^{-4}) \right. \\ & + \frac{1}{c^2} \left[-\frac{379M_{\text{PPN}}}{21R_{\text{PPN}}} - \frac{2\pi\rho A_{0\text{PPN}}}{15R_{\text{PPN}}^2} \left\{ \frac{128}{7}(a_{1\text{PPN}}^2 - a_{2\text{PPN}}^2) + \frac{99}{5}(2a_{1\text{PPN}}^2 - a_{2\text{PPN}}^2 - a_{3\text{PPN}}^2) \right\} \right. \\ & \left. \left. - \frac{M_{\text{PPN}}}{210R_{\text{PPN}}^3} (24604a_{1\text{PPN}}^2 - 14620a_{2\text{PPN}}^2 - 7765a_{3\text{PPN}}^2) + O(R_{\text{PPN}}^{-4}) \right] \right], \end{aligned} \quad (38)$$

where we use the relation

$$a_i^2 = a_{iPPN}^2 \left[1 - \frac{1}{c^2} \left\{ \frac{4\pi\rho A_{0PPN}}{3} + \frac{13M_{PPN}}{6R} + \frac{M_{PPN}}{30R^3} (38a_{1PPN}^2 - 7a_{2PPN}^2 - 15a_{3PPN}^2) \right\} \right]. \quad (39)$$

Then, we recover the energy loss rate given for the point equal mass binary case[6] when we take the limits $a_i/R \rightarrow 0$. (However, M_{PPN} appears to be the imperfect definition of gravitational mass as shown below.)

From Eq. (38), we find that the leading finite size effect of 1PN order is only of $O(\rho A_0)(\sim O(M/a_i))$ smaller than the Newtonian finite size term. The next order terms are of $O(M/R)$ smaller than the Newtonian term. Here, we note that the effect of the spin-orbit (SO) coupling appears in the same order as the latter terms for the case of the corotating binary case. Including the SO coupling term, the energy loss rate of the point particle binary of equal mass is written as[23]

$$\frac{dE}{dt} = -\frac{64M_{PPN}^5}{5c^5 R_{PPN}^5} \left[1 - \frac{1}{c^2} \left\{ \frac{379M_{PPN}}{21R_{PPN}} + \frac{74M_{PPN}}{15R_{PPN}^3} (a_{1PPN}^2 + a_{2PPN}^2) \right\} \right]. \quad (40)$$

We can see from this equation that the latter finite size terms of the 1PN order in Eq. (38) is not explained only by the SO coupling term. In order to examine whether this fact is really true or not, we rewrite dE/dt using the angular velocity which is the invariant value of the coordinate.

Using the angular velocity, the orbital separation is written as

$$R = \left(\frac{2M_{PPN}}{\Omega^2} \right)^{1/3} \left[1 + \frac{1}{5} \left(\frac{\Omega^2}{2M_{PPN}} \right)^{2/3} (2a_{1PPN}^2 - a_{2PPN}^2 - a_{3PPN}^2) + \frac{1}{c^2} \left\{ -\frac{11M_{PPN}}{6} \left(\frac{\Omega^2}{2M_{PPN}} \right)^{1/3} - \frac{22\pi\rho A_{0PPN}}{75} \left(\frac{\Omega^2}{2M_{PPN}} \right)^{2/3} (2a_{1PPN}^2 - a_{2PPN}^2 - a_{3PPN}^2) - \frac{M_{PPN}}{30} \left(\frac{\Omega^2}{2M_{PPN}} \right) (2a_{1PPN}^2 + 5a_{2PPN}^2 + 6a_{3PPN}^2) \right\} \right]. \quad (41)$$

Substituting this expression into the equation of the energy loss rate, we have

$$\begin{aligned} \frac{dE}{dt} = & -\frac{2}{5c^5} (2M_{PPN}\Omega)^{10/3} \left[1 + \frac{(2M_{PPN}\Omega)^{4/3}}{5M_{PPN}^2} \left\{ 2(a_{1PPN}^2 - a_{2PPN}^2) + 2a_{1PPN}^2 - a_{2PPN}^2 - a_{3PPN}^2 \right\} \right. \\ & + \frac{1}{c^2} \left[-\frac{373}{84} (2M_{PPN}\Omega)^{2/3} \right. \\ & - \frac{2\pi\rho A_{0PPN}}{525M_{PPN}^2} (2M_{PPN}\Omega)^{4/3} \left\{ 160(a_{1PPN}^2 - a_{2PPN}^2) + 77(2a_{1PPN}^2 - a_{2PPN}^2 - a_{3PPN}^2) \right\} \\ & \left. \left. - \frac{(2M_{PPN}\Omega)^2}{840M_{PPN}^2} (2602a_{1PPN}^2 - 1697a_{2PPN}^2 - 443a_{3PPN}^2) \right\} \right], \end{aligned} \quad (42)$$

On the other hand, in the point particle case, the energy loss rate is written as

$$\frac{dE}{dt} = -\frac{2}{5c^5} (2M_{PPN}\Omega)^{10/3} \left[1 + \frac{1}{c^2} \left\{ -\frac{373}{84} (2M_{PPN}\Omega)^{2/3} - \frac{(2M_{PPN}\Omega)^2}{5M_{PPN}^2} (a_{1PPN}^2 + a_{2PPN}^2) \right\} \right]. \quad (43)$$

Comparing the last term of Eq. (42) with that of Eq. (43), we find that they do not agree. A part of the reason seems due to the fact that the 1PN quadrupole dependent term really appears in the last terms of Eq. (42). However, we cannot still explain the disagreement completely because there remains terms except for the SO coupling one even if we take a spherical limit $a_{1PPN} = a_{2PPN} = a_{3PPN}$. Then, what is the reason why the terms remain? We guess that this is due to the imperfection of the definition of gravitational mass. We adopt the PPN mass as gravitational mass of each star to write the equation of the energy loss rate, but the PPN mass can renormalize only the self-gravity and spin into mass. In the context of this paper, we need a conserved gravitational mass which renormalizes the quadrupole term. As far as we know, however, no one has proposed such a mass. Nevertheless, M_{PPN} is not expected to be different from "true gravitational mass" M_{true} so much because

$$M_{true} = M_{PPN} \left[1 + O\left(\frac{M a_i^2}{c^2 R^3} \right) \right]. \quad (44)$$

Hence, we regard M_{PPN} as an approximate gravitational mass in the following.

Note that we can also calculate the angular momentum loss rate as the energy one (see Ref. [20]).

4 summary and discussion

In this paper, we have analytically calculated the energy loss rate. The conclusions are as follows.

- (i) The leading finite size term of the 1PN order in dE/dt reduces $(dE/dt)_{N-\text{finite}}$ by $\sim 40\%$ which is $2C_s$ times as large as that of the Newtonian order where $C_s \equiv M_*/(c^2 a_*)$.
- (ii) The reason of the weak convergence of the finite size terms is that both of them are in the same order when we see Eq. (42) in the viewpoint of the series of Ω . However, the 1PN term is of $O(C_s)$ higher than the Newtonian one in the viewpoint of the PN approximation. Therefore, the 1PN term is only by a factor of $O(C_s)$ smaller than the Newtonian one.
- (iii) The leading finite size term of the 1PN order has the opposite sign of that of the Newtonian order. This is because the 1PN order terms make the self-gravity of the star stronger, then it is difficult to deform the star. Therefore, the finite size effects on the energy loss rate decrease.

Next, we discuss the definition of gravitational mass of each star. In this paper, we use the PPN mass as the gravitational mass in order to compare the equations of the energy loss rate with those of the point particle case. However, the PPN mass is appropriate only for the point particle case (i.e., as long as we ignore the tidal forces on each star of binary)[22]. In the context of present paper, we should use a gravitational mass which conserves and appropriately renormalizes the effect of the finite size of stars even if the tidal force exists. Unfortunately, no one has proposed such a mass as far as we know. To define such gravitational mass will be an unresolved problem.

Finally, we point out a problem with regard to the back reaction by gravitational radiation as follows. In this paper, we have calculated the total gravitational wave luminosity. However, it is not trivial what fraction of it contributes to the back reaction to Ω because the structure of each star (i.e., a_i in the incompressible case) is also affected due to the radiation reaction. This situation is in contrast to that for the point particle case where we only need to consider the radiation reaction to Ω . To know the change rate of Ω and a_i in the 1PN order, we need to solve the hydrodynamic equation in the 1PN order including the 3.5PN radiation reaction terms. Here, we note that from the observational point of view, we need not the total luminosity, but the back reaction of Ω . Hence, to resolve this problem seems one of the important issues in order to study the late time evolution of BNS's.

Acknowledgements

We thank H. Asada, L. Blanchet and K. Nakao for useful discussions and also K.T. would like to thank T. Nakamura and H. Sato for helpful comments and continuous encouragement. This work was in part supported by a Grant-in-Aid of Ministry of Education, Culture, Science and Sports (Nos. 08237210, 09740336 and 08NP0801).

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Gravitational waves from a spinning particle orbiting around a Kerr black hole

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Abstract

I study gravitational waves from a spinning particle moving on an equatorial plane orbiting around a Kerr black hole (BH) using a BH perturbation approach. I calculate the energy flux, spectrum, and waveform of gravitational waves to see the effects of the particle spin s . First, I consider the effect of the orbital eccentricity from a particle spin on the gravitational wave. The particle spin which is parallel to the BH spin enhances the energy flux by more than a factor of ten than that which is anti-parallel to the BH spin. Secondly, the characteristic frequency of the gravitational wave is useful for restricting a set of parameters from observations. This frequency which corresponds to the synchrotron radiation at the periastron radius, largely depends on the particle spin.

1 Introduction

A number of worldwide projects for detecting gravitational radiation directly using km-scale laser interferometers such as LIGO [1], VIRGO, GEO600, and TAMA are progressing. Among many possible sources of gravitational waves, one expected source is a coalescing binary, composed of neutron star (NS)-NS, NS-BH, or BH-BH. These direct detections of gravitational waves will enable us to see the strong gravitational phenomena such as a collision of compact stars or BH's.

In astrophysics, rotation of a system plays quite an important role. The angular momentum or spin may completely change the evolution of the system. As for a spin effect, there are so far several papers discussing its effect on gravitational radiation using the post-Newtonian expansion method. Kidder, Will, and Wiseman [2] calculated the spin effects on the energy flux of a gravitational wave. They clarify the spin effects in a change of orbital frequency and orbital radius in the binary evolution. Will and Wiseman [3] derive the gravitational waveform and the energy flux emitted from the binary NS or BH under the second post-Newtonian approximation and calculate a spin correction in the waveform and energy loss rate.

Analysis using a BH perturbation approach, which is based on the linearized Einstein equation, is another way to calculate gravitational waves from the inspiraling of BH-BH binaries. The advantage of this method is that it enables us to treat full GR effects of background BH and it may apply to an arbitrary motion of a particle. Sasaki and Nakamura [4] reformulated the Teukolsky equation and made it possible to calculate gravitational waves numerically. They also calculated gravitational waves from a particle falling along the spin axis of a Kerr BH using their technique. Subsequently, Nakamura, Oohara, and Kojima [5] investigated for a wide variety of parameters of a particle angular momentum and BH spin, and clarify the effects of them on gravitational waves emitted for the falling, plunging, and scattering motion.

In general, a compact binary system such as PSR1913+16 has an orbital eccentricity. Moreover, the final stage of a binary coalescence must diverge from the circular orbit. There are two papers which treat a highly deformed bound orbit motion of a test particle using a BH perturbation approach. Tanaka, Shibata, Sasaki, Tagoshi, and Nakamura [6] consider the bound orbit on an equatorial plane of Schwarzschild BH, Shibata [7] extended it into a Kerr BH. They show the difference between numerical calculation and post-Newtonian expansion is maximally 30% for the Schwarzschild BH case, while the factor of 10 for the Kerr BH case.

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In this paper, I extend these works to a spinning particle and figure out the effect of a particle spin. I discuss two points to see the effects of a particle spin. One is the effect of orbital eccentricity from a particle spin. The previous papers conclude that the orbital eccentricity largely affects the gravitational waves, I calculate this with a particle spin. The other is the characteristic frequency of gravitational wave in this system. Since the characteristic frequency of the plunging orbit is a quasinormal mode frequency which depends on the mass and the rotation of BH, I calculate this characteristic frequency in this system.

This paper is organized as follows. In Sec. 2, I derive explicit form of the equation of motion (EOM) for a spinning particle moving on the equatorial plane of a Kerr BH. I show a BH perturbation approach in Sec. 3. Then, I present numerical results in Sec. 4. Sec. 5 is devoted to summary. Throughout this paper, I use the units of $c = G = 1$ and the metric signature $(-, +, +, +)$.

2 The motion of a spinning particle

The EOM for a spinning particle were first derived by Papapetrou [8] (so that I call the EOM Papapetrou's EOM), and was then reformulated by Dixon[9], who discussed a generic treatment for the EOM of an extended body. Papapetrou's EOM are given as

$$\begin{aligned}\frac{D}{d\tau}p^\mu &= -\frac{1}{2}R^\mu{}_{\nu\rho\sigma}(z)v^\nu S^{\rho\sigma}, \\ \frac{D}{d\tau}S^{\mu\nu} &= p^\mu v^\nu - p^\nu v^\mu,\end{aligned}\tag{1}$$

where $z^\mu(\tau)$, $v^\mu(\tau) \equiv dz^\mu/d\tau$, $p^\mu(\tau)$, $S^{\mu\nu}(\tau)$ and $R^\mu{}_{\nu\rho\sigma}$ denote the world line, 4-velocity, momentum, spin angular momentum tensor of a spinning particle and Riemann tensor of Kerr spacetime, respectively, with τ being an affine parameter of the orbit. The quantity $\mu^2 \equiv -p_\mu p^\mu$ is conserved and then μ is regarded as the mass of the particle. I also introduce a normalized momentum $u^\mu = p^\mu/\mu$ such that $u^\mu u_\mu = -1$. The notation $D/d\tau$ denotes the covariant derivative along the particle trajectory.

In order to close the system, I need a set of supplementary conditions, i.e., a relation between v^μ and u^μ (or p^μ), which fixes the center of mass of a spinning particle, as[9]

$$S^{\mu\nu}p_\nu = 0.\tag{2}$$

The magnitude of spin is also conserved, so that I set

$$S^{\mu\nu}S_{\mu\nu} = 2\mu^2 s^2,\tag{3}$$

where a constant s is the specific spin angular momentum of the spinning particle. I fix an orbital affine parameter τ by setting

$$u^\mu v_\mu = -1.\tag{4}$$

Using Eqs. (1), (2), and (4), the relation between u^μ and v^μ becomes

$$v^\mu - u^\mu = \frac{S^{\mu\nu}R_{\nu\gamma\sigma\lambda}u^\gamma S^{\sigma\lambda}}{2[\mu^2 + \frac{1}{4}R_{\alpha\beta\delta\epsilon}S^{\alpha\beta}S^{\delta\epsilon}]}.\tag{5}$$

Thus, the system of basic equations is closed.

In the following, I restrict our analysis to the case where a spinning particle moves on the equatorial plane of a Kerr BH ($\theta = \pi/2$). From the EOM, I find that the spin direction of the spinning particle is always perpendicular to the equatorial plane. Then, Papapetrou's EOM are integrable. The explicit form of Papapetrou's EOM becomes

$$\Sigma_s \Lambda_s \frac{dt}{d\tau} = a \left(1 + \frac{3Ms^2}{r\Sigma_s} \right) \left\{ \bar{J}_z - (a+s)\bar{E} \right\} + \frac{r^2 + a^2}{\Delta} P_s,\tag{6}$$

$$\Sigma_s \Lambda_s \frac{dr}{d\tau} = \pm \sqrt{R_s},\tag{7}$$

$$\Sigma_s \Lambda_s \frac{d\varphi}{d\tau} = \left(1 + \frac{3Ms^2}{r\Sigma_s} \right) \left\{ \bar{J}_z - (a+s)\bar{E} \right\} + \frac{a}{\Delta} P_s,\tag{8}$$

where

$$\Sigma_s = r^2 \left(1 - \frac{Ms^2}{r^3} \right), \quad (9)$$

$$\Lambda_s = \left(1 - \frac{3Ms^2r\{-(a+s)\tilde{E} + \tilde{J}_z\}^2}{\Sigma_s^3} \right), \quad (10)$$

$$R_s = P_s^2 - \Delta \left[\frac{\Sigma_s^2}{r^2} + \left\{ -(a+s)\tilde{E} + \tilde{J}_z \right\}^2 \right], \quad (11)$$

$$P_s = \left\{ (r^2 + a^2) + \frac{as}{r}(r + M) \right\} \tilde{E} - \left(a + \frac{Ms}{r} \right) \tilde{J}_z, \quad (12)$$

and I normalized the energy and total angular momentum of a spinning particle as $\tilde{E} \equiv E/\mu$, and $\tilde{J}_z \equiv J_z/\mu$, respectively.

However, there still exists a problem solving the EOM of a spinning particle moving on an equatorial plane [10]. For some choice of a , s , and J_z (in particular for large $|s|$), the 4-velocity becomes null at a radius r_n and turns to spacelike beyond r_n . It is not clear why such a pathological feature appears. However, the cause is not clear, I simply exclude sets of parameters in which the timelike condition is violated at a radius $r \leq r_n$.

Using the EOM, the timelike condition, i.e., $v^\mu v_\mu < 0$, of a spinning particle is reduced to the inequality

$$3 \frac{(\tilde{J}_z - \tilde{E}(a+s))^2}{(Ms^2)^{2/3}} < \frac{(1-X)^4}{(2+X)X^{5/3}}, \quad (13)$$

with $X \equiv Ms^2/r^3$. This condition should be satisfied for any value of $r > r_p$ for the plunging trajectory always to be timelike, where r_p is the periastron radius from the Kerr BH. I show such a condition in Figure 1 and find that the allowed region is constrained for a large spin ($s/M \sim 1$).

As a source term of Einstein equation, I have to know the energy momentum tensor of a spinning particle $T^{\mu\nu}$, which is given as [11]

$$T^{\mu\nu}(x) = \frac{1}{\sqrt{-g}} \int d\tau \left[p^{(\mu}(x, \tau) v^{\nu)}(x, \tau) \delta^{(4)}(x - z(\tau)) - \nabla_\lambda \left[S^{\lambda(\mu}(x, \tau) v^{\nu)}(x, \tau) \delta^{(4)}(x - z(\tau)) \right] \right], \quad (14)$$

where $v^\alpha(x, \tau)$, $p^\alpha(x, \tau)$, and $S^{\alpha\beta}(x, \tau)$ are fields extended from $v^\mu(\tau)$, $p^\mu(\tau)$, and $S^{\mu\nu}(\tau)$ defined only on the particle trajectory. These extended fields are defined only near the particle trajectory, $x^\mu = z^\mu(\tau)$.

3 Gravitational waves from a spinning particle

I begin with a spinning particle motion in the Kerr background spacetime and analyze the gravitational waves emitted by a orbiting spinning particle. In order to calculate metric perturbations of a Kerr spacetime, I adopt the Sasaki-Nakamura formalism [4], in which the Regge-Wheeler equation is generalized by a transformation from the Teukolsky equation. The radial wave equation becomes

$$\left[\frac{d^2}{dr^{*2}} - F(r) \frac{d}{dr^*} - U(r) \right] X_{lm\omega}(r) = S_{lm\omega}(r), \quad (15)$$

where the tortoise coordinate r^* is defined as

$$dr^* = \frac{r^2 + a^2}{\Delta} dr. \quad (16)$$

with $\Delta = r^2 - 2Mr + a^2$. The explicit form for potential functions $F(r)$ and $U(r)$ and the method to calculate the source term $S_{lm\omega}(r)$ are given in [5]. And the method to construct the wave function $X_{lm\omega}(r)$ of Eq. (15) is also written in [5].

For observation, I only have to know the asymptotic behavior of the wave function $X_{lm\omega}(r)$ at infinity, which is

$$X_{lm\omega}^{(\infty)}(r) = A_{lm\omega}^{(\infty)} e^{i\omega r^*}, \quad (17)$$

where

$$A_{lm\omega}^{(\infty)} = \frac{c_0}{2i\omega A_{lm\omega}^{in}} \int_{-\infty}^{\infty} \frac{S_{lm\omega}(r)}{\gamma} X_{lm\omega}^{in(0)}(r) dr^*, \quad (18)$$

and the quantities c_0 , γ , and $A_{lm\omega}^{in}$ are also written in [5].

When I consider the case that the motion of the spinning particle is a bound orbit, I have to calculate an infinite time integral in principle. However, in this present case, the orbit is quasi-periodic:

$$\begin{aligned} r(t + \Delta t) &= r(t), \\ \varphi(t + \Delta t) &= \varphi(t), \end{aligned}$$

where Δt is the orbital period of a spinning particle. Thus, from the contribution to the source term, the frequency of the gravitational wave becomes discrete [6] and it can be depicted as

$$\omega_n = \frac{m\Delta\varphi + 2n\pi}{\Delta t} \quad n = 0, \pm 1, \pm 2, \dots, \quad (19)$$

where $\Delta\varphi$ denote the rotation angle period of a spinning particle, respectively.

Then the waveform and the time-averaged energy flux at infinity are

$$h_+ - ih_\times = \frac{16\pi}{r(\Delta t)} \sum_{n=-\infty}^{\infty} \sum_{l=2}^{\infty} \sum_{m=-l}^l \frac{A_{lm\omega_n}^{(\infty)}}{c_0} \frac{-2S_{lm}^{a\omega}(\theta)}{\sqrt{2\pi}} e^{i(\omega_n(r^* - t) + m\varphi)}, \quad (20)$$

$$\left\langle \frac{dE}{dt} \right\rangle = \sum_{n=-\infty}^{\infty} \sum_{l=2}^{\infty} \sum_{m=-l}^l \left(\frac{dE}{dt} \right)_{lm\omega_n}, \quad (21)$$

where I defined the spectrum as

$$\left(\frac{dE}{dt} \right)_{lm\omega_n} \equiv \frac{16\pi}{(\Delta t)^2} \omega_n^2 \left| \frac{A_{lm\omega_n}^{(\infty)}}{c_0} \right|^2, \quad (22)$$

and $S_{lm}^{a\omega}(\theta)$ is a spin-weighted spheroidal function given in [5]

4 Numerical Results

I show my numerical results in Figure 3 and Figure 4. I only calculate the $l = 2$ mode of the gravitational wave, this is the most dominant mode. Although the result of a detailed analysis must be different from this result, I think that the behavior is similar to that when all l modes are taken into account.

First, I consider the behavior of the energy flux in terms of a particle spin and a total angular momentum. There are two remarkable points. One is that, when I fix the total angular momentum and the energy of the spinning particle, the energy flux increases as the particle spin becomes parallel to the total angular momentum. The reason is as follows. When the particle spin is parallel to the BH spin, the particle spin suppresses the barrier of the effective potential (see Figure 2) and the particle is allowed to move towards the BH. Thus, the energy flux becomes larger as the particle moves into the strong gravitational field region. The other is that, when I fix the spin angular momentum and the energy of the spinning particle, the energy flux is suppressed as the total angular momentum of the spinning particle

becomes larger. The reason is the same as the previous one. When the total angular momentum of the spinning particle becomes larger, the centrifugal force on the particle from the BH becomes larger. Thus, the allowed region for the particle motion goes away from the BH.

Next, I discuss effect of the orbital eccentricity of the particle motion which affects the gravitational wave. From the EOM, when I fix the energy of the spinning particle, the orbital eccentricity becomes larger as the spin angular momentum (total angular momentum) becomes parallel to the total angular momentum (smaller) (see Figure 2). When I fix the energy of the particle, the energy flux with various parameters of particle spin and total angular momentum are on the single curved line (see Figure 3). From Figure 3, the orbital eccentricity is one of the most dominant contributions to the gravitational wave.

Finally, I show the gravitational waveform and the spectrum of some sets of parameters in Figure 4. In this figure, there are two remarkable points in terms of the particle spin effect. One is that, I can determine orbital period from the gravitational waveform by measuring the period of the gravitational waveform. Roughly speaking, the orbital period corresponds to the orbital eccentricity of the motion. The other is that, when the spinning particle comes near the periastron radius, the waveform bursts. This means that most of the gravitational wave in this system emits at the point of the periastron radius. The frequency for which the spectrum takes the maximum value also corresponds to the frequency ω_{circular} of the circular orbit at the periastron radius,

$$\omega_{\text{circular}} = m \left(\frac{d\varphi}{dt} \right)_{r \sim r_p}. \quad (23)$$

I show the frequency ω_{circular} using the EOM in Figure 6. The frequency ω_{circular} largely depends on the particle spin, it is possible to restrict some sets of parameters.

5 Summary

I study the gravitational wave from a spinning particle moving on an equatorial plane around a Kerr BH using the BH perturbation approach.

First, I discuss the particle spin effect on the orbital eccentricity. The particle spin (total angular momentum) enhances (suppresses) the orbital eccentricity of the particle motion. And the energy flux in this bound orbit system largely depends on the orbital eccentricity. For instance, it enhances the energy flux about a factor of 10 when the particle spin is opposite to each other. Therefore, I cannot neglect the particle spin effect, which affects the orbital eccentricity.

Secondly, observing the gravitational wave or spectrum, it is possible to restrict some sets of parameters from the frequency where most of the gravitational wave are emitted. For the plunging motion, the characteristic frequency of the gravitational wave is quasinormal mode frequency, while for the bound orbit, the characteristic frequency is the synchrotron radiation from the periastron radius. This result is suggested from that most of the gravitational wave is emitted at the strong gravitational field region.

To discuss the details of gravitational waves in the bound orbit system, the detailed analysis which takes care of all l modes is in progress.

Acknowledgements

I would like to thank Kei-ichi Maeda and Masaru Shibata for suggesting this project to me. I also thank Paul Haines for critical reading of my manuscript. This work is partly supported by JSPS Research Fellowship for Young Scientists, No. 5689.

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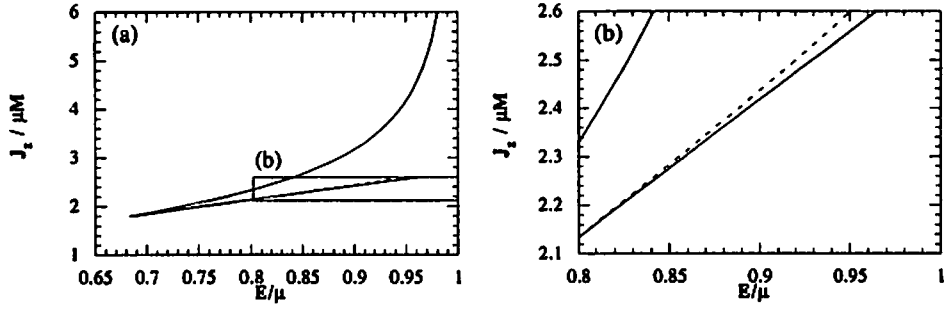


Figure 1: The allowed region for bound orbits of the spinning particle for $a/M = s/M = 0.9$. The upper solid line corresponds to the the parameter of the stable circular orbit, while the lower solid line corresponds to the limitation of the bound orbit. The regions enclosed by solid line are allowed for bound orbits, while the region below the dashed line are the violation of timelike condition.

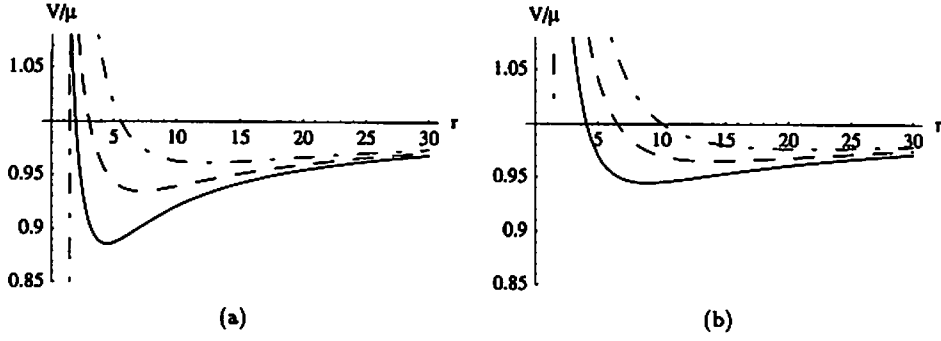


Figure 2: The effective potential of the spinning particle moving on an equatorial plane ((a) $J_z/\mu M = 3$ and (b) $J_z/\mu M = 4$). Solid line, dashed line, and dash-dotted line denote the cases of $s/M = 0.9, 0$, and -0.9 , respectively.

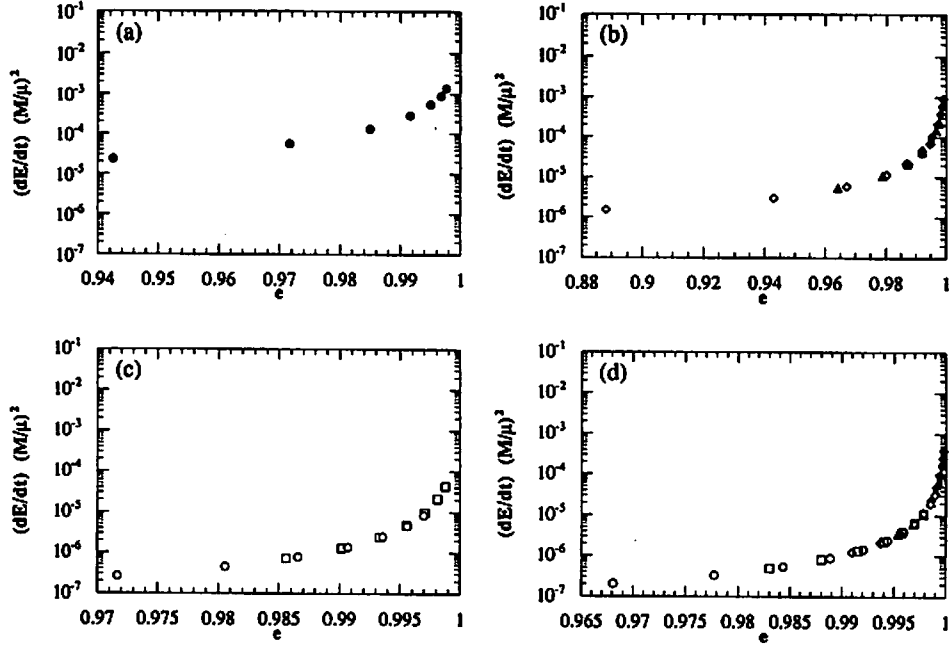


Figure 3: The energy flux of gravitational waves in terms of the orbital eccentricity e ((a) $E/\mu = 0.97$, $a/M = 0.9$, (b) $E/\mu = 0.97$, $a/M = 0.9$, (c) $E/\mu = 0.99$, $a/M = 0$, and (d) $E/\mu = 0.97$, $a/M = 0.9$). I calculate 7 cases of the particle spin for each J_z , while $s/M = -0.9, -0.6, -0.3, 0, 0.3, 0.6$, and 0.9 . Circle (filled), square (filled), diamond (filled), triangle (filled), diamond (open), square (open), and circle (open) denote the cases of $J_z/\mu M = 2.0, 2.5, 3.0, 3.5, 4.0, 4.5$, and 5.0 , respectively. I defined the orbital eccentricity as $e = (1 - (r_p/r_a)^2)^{1/2}$, which is the definition for an ellipsoidal orbit, while r_a is the apastron radius from the Kerr BH. In every case, when the particle spin becomes parallel to the total angular momentum, the orbital eccentricity increases.

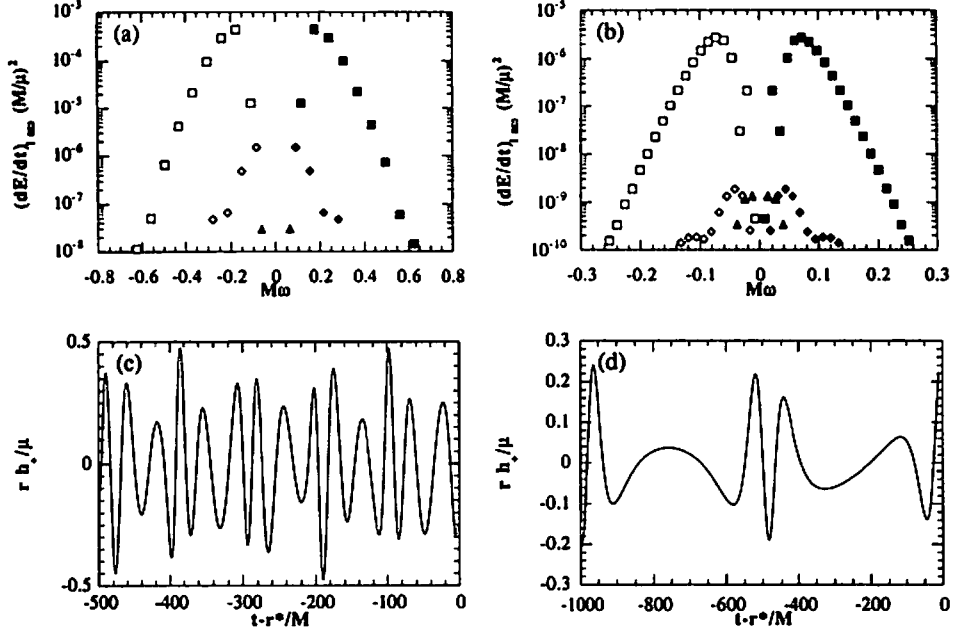


Figure 4: The spectrum ((a),(b)) and the waveform ((c),(d)) of gravitational waves (((a),(c)) $a/M = 0.9$, $s/M = 0.9$, $E/\mu = 0.89$, $J_z/\mu M = 3$, ((b),(d)) $a/M = 0.9$, $s/M = -0.9$, $E/\mu = 0.97$, $J_z/\mu M = 3$). Square (filled), diamond (filled), triangle (filled), diamond (open), and square (open) denote the cases of $m = 2, 1, 0, -1$, and -2 , respectively.

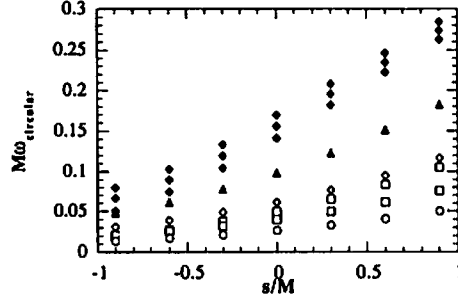


Figure 5: The relation between the particle spin and the frequency of synchrotron radiation at the periastron radius. Circle (filled), square (filled), diamond (filled), triangle (filled), diamond (open), square (open), and circle (open) denote the cases of $J_z/\mu M = 2.0, 2.5, 3.0, 3.5, 4.0, 4.5$, and 5.0 , respectively. The parameters are the same as Figure 3 (a), (b), (d). When the energy of the particle increases fixing the total angular momentum, the frequency also increases.

3D Numerical Relativity

— Coalescence of Binary Neutron Stars —

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Abstract

Emission of strong gravitational waves is expected in the last three milliseconds of coalescence of binary neutron stars or black holes. In this phase, analytic or semi-analytic studies are not so powerful since full nonlinear and dynamical treatment of matter and space-times are needed. General relativistic effects are very important when two stars finally merge into a single black hole. Moreover the event is completely three-dimensional so that large-scale numerical calculations are required. Numerical relativity is the unique method for studying such nonlinear phase of coalescence. We show some results of 3D, fully relativistic simulations.

1 Introduction

In order to compare the theory with observations, equations in the theory must be solved in some ways. Usually some simplification or some approximation must be made to solve the equations. For example, a system is assumed to be spherically or axially symmetric, or a certain quantity to be so small that functions may be expanded as a power series. In some realistic models, however, such assumptions are not applicable, or even with these simplifications it is hard to solve the equations.

Computational physics can provide a new mode of scientific investigation. In this new mode, we replace the spacetime continuum with a finite lattice and/or continuum matter with finite elements and simulate a natural phenomena using numerical methods. Computational physics can be, then, an intermediary between theory and experiment(observation).

In general relativity, one of the best ways of studying dynamical spacetimes is the analysis of gravitational radiation. The most promising source of gravitational radiation is coalescence of binary neutron stars or black holes. In *the last three minutes* of coalescence, the post-Newtonian expansion will give good approximation since v_{rot}/c is as small as 0.1. Thus an analytic or semi-analytic study will make good predictions for observations of gravitational radiation and analysis of observed data will make it possible to determine each mass, the distance and so on. In the final phase, or *the last three milliseconds*, of the coalescence, however, there is no such small parameters. Numerical relativity, that is, computational physics in general relativity, is one of methods for studying such nonlinear phase of coalescing binary neutron stars or black holes.

The coalescence is a completely three dimensional, dynamical event and effects of general relativity become so large that the post-Newtonian approximation may not be good at least in the last three milliseconds of coalescence, when two stars come into contact and finally merge into a single black hole. Thus general relativistic (GR) calculations are needed in order to know the characteristics of the radiation in the last three milliseconds although a 3D GR code requires great powers of computers.

For a 3D relativistic calculation, however, one needs nearly 300^3 grid points to calculate radiation fields at wavezone; it requires more than 15GBytes of memory and 50 GFLOPS or more of peak performance. With such a supercomputer, it takes nearly 20 CPU seconds per time step, that is, 100 CPU hours up to 20,000 time steps. A vector-parallel supercomputer with a hundred processor elements or a massively parallel supercomputer with thousands processor elements can provide performance and memory of this

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order of magnitude. In 1994, we could use a new supercomputer, with memories of tens of GBytes and a peak speed of nearly a hundred GFLOPS. Then we started to make a 3D, relativistic code for coalescence of binary neutron stars.

In the followings, we show preliminary results of 3D relativistic simulation of coalescence of binary neutron stars.

2 Basic Equations

In order to follow time evolution of space-time and matter, the Einstein equation should be reduced to a system of evolution equations. Then we use (3+1)-formalism of the Einstein equation.

2.1 Initial Value Equations

In the (3+1)-formalism of the Einstein equation, the line element takes the form

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \quad (1)$$

where α , β^i and γ_{ij} are the lapse function, the shift vector and the intrinsic metric of 3-space, respectively. The initial data should satisfy the constraint equations

$$R + K^2 - K_{ij}K^{ij} = 16\pi\rho_H, \quad (2)$$

$$D_j(K^j_i - \delta^j_i K) = 8\pi J_i, \quad (3)$$

where R is the 3-dimensional Ricci scalar curvature, K_{ij} is the extrinsic curvature,

$$K_{ij} = -\frac{1}{2\alpha}(\partial_t \gamma_{ij} - D_i \beta_j - D_j \beta_i), \quad (4)$$

K is its trace and the symbol D_i denotes the covariant derivative with respect to γ_{ij} . The quantities ρ_H and J_i are, respectively, the energy density and the momentum density of the matter measured by the normal line observer. The relation of these quantities to the stress-energy tensor is given by Eq.(15) below.

Now we assume that on the Cauchy slice the trace $K = 0$ and γ_{ij} is conformally flat, that is,

$$\gamma_{ij} = \phi^4 \tilde{\gamma}_{ij}, \quad (5)$$

where $\tilde{\gamma}_{ij}$ is the flat metric. Defining the conformal transformation as

$$\begin{aligned} \tilde{K}_{ij} &\equiv \phi^2 K_{ij}, \quad \tilde{K}_i^j \equiv \phi^8 K_i^j, \quad \tilde{K}^{ij} \equiv \phi^{10} K^{ij} \\ \rho_B &\equiv \phi^6 \rho_H \quad \text{and} \quad \tilde{J}_i \equiv \phi^6 J_i, \end{aligned} \quad (6)$$

Eq.(3) can be expressed as

$$\tilde{D}_j \tilde{K}^j_i = 8\pi \tilde{J}_i, \quad (7)$$

where $\tilde{D}_i$ is the covariant derivative with respect to $\tilde{\gamma}_{ij}$ [1]. The traceless extrinsic curvature can be decomposed with the transverse traceless part $\tilde{K}_{ij}^{TT}$ and the longitudinal traceless part $(LW)_{ij}$ [2];

$$\tilde{K}_{ij} = \tilde{K}_{ij}^{TT} + (LW)_{ij}, \quad (8)$$

where

$$(LW)_{ij} = \tilde{D}_i W_j + \tilde{D}_j W_i - \frac{2}{3} \tilde{\gamma}_{ij} \tilde{D}^l W_l. \quad (9)$$

Assuming $\tilde{K}_{ij}^{TT} = 0$, Eq.(7) becomes

$$\tilde{\Delta} W_i + \frac{1}{3} \tilde{D}_i \tilde{D}^j W_j = 8\pi \tilde{J}_i, \quad (10)$$

where $\tilde{\Delta} \equiv \tilde{D}^i \tilde{D}_i$. The method of solving this equation and the boundary condition of W_i will be discussed a subsequent section.

The conformal transformation of the scalar curvature is

$$R = \phi^{-4} \tilde{R} - 8\phi^{-5} \tilde{\Delta} \phi, \quad (11)$$

where $\tilde{R}$ is the scalar curvature with respect to $\tilde{\gamma}_{ij}$. Since $\tilde{R} = 0$ for a conformally flat metric, Eq.(2) is reduced to

$$\tilde{\Delta} \phi = -2\pi\phi^{-1} \rho_D - \frac{1}{8}\phi^{-7} \tilde{K}_{ij} \tilde{K}^{ij}. \quad (12)$$

The boundary condition for ϕ is

$$\phi = 1 + \frac{M_G}{2r} + O\left(\frac{1}{r^3}\right), \quad (13)$$

where M_G is the gravitational mass of the system.

2.2 Relativistic Hydrodynamics

We assume the perfect fluid stress-energy tensor, which is given by

$$T_{\mu\nu} = (\rho + \rho\epsilon + p)u_\mu u_\nu + pg_{\mu\nu}, \quad (14)$$

where ρ , ϵ and p are the proper mass density, the specific internal energy and the pressure, respectively, and u_μ is the four-velocity of the fluid. The energy density ρ_H , the momentum density J_i and the stress tensor S_{ij} of the matter measured by the normal line observer are, respectively, given by

$$\rho_H \equiv n^\mu n^\nu T_{\mu\nu}, \quad J_i \equiv -h_i^\mu n^\nu T_{\mu\nu} \quad \text{and} \quad S_{ij} \equiv h_i^\mu h_j^\nu T_{\mu\nu}, \quad (15)$$

where n_μ is the unit timelike four-vector normal to the spacelike hypersurface and $h_{\mu\nu}$ is the projection tensor into the hypersurface defined by

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \quad (16)$$

The relativistic hydrodynamics equations are obtained from the conservation of baryon number, $\nabla_\mu(\rho u^\mu)$, and the energy-momentum conservation law, $\nabla_\nu T_\mu{}^\nu = 0$. In order to obtain equations similar to the Newtonian hydrodynamics equations, we define ρ_N and u_i^N as

$$\rho_N \equiv \sqrt{\gamma} \alpha u^0 \rho \quad \text{and} \quad u_i^N = \frac{J_i}{\alpha u^0 \rho}, \quad (17)$$

respectively, where $\gamma = \det(\gamma_{ij})$. Then the equation for the conservation of baryon number takes the form

$$\partial_t \rho_N + \partial_i (\rho_N V^i) = 0, \quad (18)$$

where

$$V^i = \frac{u^i}{u^0} = \frac{\alpha J^i}{p + \rho_H} - \beta^i. \quad (19)$$

The equation for momentum conservation is

$$\begin{aligned} \partial_t (\rho_N u_i^N) + \partial_i (\rho_N u_i^N V^i) &= -\sqrt{\gamma} \alpha \partial_i p - \sqrt{\gamma} (p + \rho_H) \partial_i \alpha \\ &\quad + \frac{\sqrt{\gamma} \alpha J^k J^i}{2(p + \rho_H)} \partial_i \gamma_{kt} + \sqrt{\gamma} J_t \partial_i \beta^i. \end{aligned} \quad (20)$$

The equation for internal energy conservation becomes

$$\partial_t (\rho_N \epsilon) + \partial_i (\rho_N \epsilon V^i) = -p \partial_\nu (\sqrt{\gamma} \alpha u^\nu). \quad (21)$$

To complete hydrodynamics equations, we need an equation of state,

$$p = p(\epsilon, \rho). \quad (22)$$

2.3 Time Evolution of the Metric Tensor

The evolution equation for the extrinsic curvature takes the form

$$\partial_t K_{ij} - \partial_t (K_{ij} \beta^t) = (SK)_{ij}^{\text{main}} + (SK)_{ij}^{\text{NL}} + (SK)_{ij}^\beta, \quad (23)$$

where

$$(SK)_{ij}^{\text{main}} = \alpha \{ R_{ij} - 8\pi [S_{ij} + \frac{1}{2} \gamma_{ij} (\rho_H - S^t_t)] \} - D_i D_j \alpha \quad (24)$$

$$(SK)_{ij}^{\text{NL}} = \alpha (K K_{ij} - 2K_{it} K^t_j), \quad (25)$$

$$(SK)_{ij}^\beta = K_{mi} \partial_j \beta^m + K_{mj} \partial_i \beta^m - K_{ij} \partial_m \beta^m, \quad (26)$$

and R_{ij} is the 3-dimensional Ricci tensor. From the definition of the extrinsic curvature, we obtain the time evolution equation of the 3-metric γ_{ij} ;

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i. \quad (27)$$

Now we define the conformal factor ϕ as

$$\phi \equiv (\det(\gamma_{ij}))^{\frac{1}{12}} \quad (28)$$

and $\tilde{\gamma}_{ij}$ as

$$\tilde{\gamma}_{ij} = \phi^{-4} \gamma_{ij}. \quad (29)$$

The conformal factor ϕ is used for the lapse function in conformal slicing described in §2.4. From the Hamiltonian constraint equations, ϕ obeys

$$\tilde{\Delta} \phi = -\frac{\phi^5}{8} (16\pi \rho_H + K_{ij} K^{ij} - K^2 - \phi^{-4} \tilde{R}), \quad (30)$$

where $\tilde{\Delta}$ and $\tilde{R}$ is the Laplacian and the scalar curvature with respect to $\tilde{\gamma}_{ij}$, respectively. From Eq.(27), we obtain

$$\begin{aligned} \partial_t \tilde{\gamma}_{ij} &= -2\alpha \left(\tilde{K}_{ij} - \frac{1}{3} \tilde{\gamma}_{ij} K \right) + \tilde{D}_i \tilde{\beta}_j + \tilde{D}_j \tilde{\beta}_i - \frac{2}{3} \tilde{\gamma}_{ij} \tilde{D}_t \beta^t \\ &\equiv A_{ij}^T, \end{aligned} \quad (31)$$

where

$$\tilde{K}_{ij} = \phi^{-4} K_{ij}, \quad \tilde{\beta}_i = \phi^{-4} \beta_i \quad (32)$$

and $\tilde{D}_i$ denotes the covariant derivative with respect to $\tilde{\gamma}_{ij}$. Note that the definition of $\tilde{K}_{ij}$ here is different from that in Eq.(6). Moreover Eq.(31) can be written as

$$\partial_t \tilde{\gamma}_{ij} - \partial_t (\tilde{\gamma}_{ij} \beta^t) = -2\alpha \left(\tilde{K}_{ij} - \frac{1}{3} \tilde{\gamma}_{ij} K \right) + \tilde{\gamma}_{it} \partial_j \beta^t + \tilde{\gamma}_{jt} \partial_i \beta^t - \frac{5}{3} \tilde{\gamma}_{ij} \partial_t \beta^t \quad (33)$$

We will solve Eqs.(30) and (33) instead of Eq.(27).

2.4 Coordinate Conditions

One of the most important problems for numerical relativity is the control of the space and time coordinates. Difficulties arise from (a) coordinate singularities caused by strong gravity and the dragging effect, where coordinate lines may be driven towards or away from each other, and (b) spacetime singularities which appear when a black hole is formed. We should demand coordinate conditions so that these singularities can be avoided. The first difficulty is handled mainly as choice of the shift vector β^i and the second is as choice of the lapse function α .

2.4.1 Spatial coordinates — the shift vector β^i

First we consider how to determine the shift vector. The right hand side of Eq.(31), A_{ij}^T , is trace-free. If we demand that A_{ij}^T should be transverse, that is,

$$D^j A_{ij}^T = 0, \quad (34)$$

the shift vector β^i will be determined by the minimal distortion condition proposed by York and Smarr [3]. This must be a good choice of the spatial coordinate, but it is too complicated to be solved. Instead we demand the divergence with respect to the flat-space metric of A_{ij}^T should vanish;

$$\partial_i A_{ij}^T = 0. \quad (35)$$

We call this condition as the pseudo-minimal distortion condition. Equation(35) is reduced to the equation for the shift vector β^i ,

$$\begin{aligned} \nabla^2 \beta^i + \frac{1}{3} \partial_i (\partial_t \beta^t) = \\ \partial_j \left[2\alpha \left(\tilde{K}_{ij} - \frac{1}{3} \tilde{\gamma}_{ij} K \right) \right] \\ - \partial_j \left[\tilde{\gamma}_{jt} \partial_i \beta^t + \tilde{\gamma}_{it} \partial_j \beta^t - \frac{2}{3} \tilde{\gamma}_{ij} \partial_t \beta^t + \beta^t \partial_t \tilde{\gamma}_{ij} \right], \end{aligned} \quad (36)$$

where ∇^2 is the simple flat-space Laplacian and $\hat{\gamma}_{ij} = \tilde{\gamma}_{ij} - \delta_{ij}$. Equation (36) is the similar to Eq.(10) while β^i appears in the right hand side, too.

2.4.2 Time Slicing — the lapse function α

As for time slicing, we use conformal slicing. In conformal slicing, the lapse function α is given by

$$\alpha = \exp \left[-2 \left(\hat{\phi} + \frac{\hat{\phi}^3}{3} + \frac{\hat{\phi}^5}{5} \right) \right], \quad (37)$$

where $\hat{\phi} = \phi - 1$. In conformal slicing, the space outside the central matter quickly approaches the Schwarzschild metric. Since the non-wave parts decrease as $O(r^{-4})$ for large r , we can catch gravitational waves numerically with $O((M/r_c)^3)$ accuracy, where M is mass of the system and r_c is distance from the origin of the position where gravitational waves are evaluated. Conformal slicing is likely to have singularity avoidance property, since density and curvature increase and then $\hat{\phi}$ becomes large when a singularity is formed; the lapse function α decreases very rapidly as $\hat{\phi}$ increases.

3 Numerical Methods

The system of equations consist of elliptic partial differential equations (PDEs) and hyperbolic PDEs.

Equation (10) is a coupled elliptic PDE but can be reduced to four decoupled Poisson equations by means of one of the following procedures: (1) Define a vector potential A_i and a scalar potential ψ such that they satisfy

$$\Delta A_i = 8\pi \tilde{J}_i \quad (38)$$

and

$$\Delta \psi = 8\pi \tilde{J}_i x^i, \quad (39)$$

where Δ is the Laplacian in the flat space. Then W_i is given by

$$W_i = A_i + \frac{1}{8} \partial_i (\psi - A_k x^k). \quad (40)$$

(2) Defining $\chi = \partial_i W_i$, it satisfies

$$\Delta \chi = 6\pi \partial_i \tilde{J}_i \quad (41)$$

and then Eq.(10) becomes

$$\Delta W_i = 8\pi \tilde{J}_i - \frac{1}{3} \partial_i \chi. \quad (42)$$

To find boundary conditions for Eqs.(38), (39), (41) and (42), consider a vector Poisson equation

$$\Delta A_i = 8\pi \hat{P}_i \delta(r), \quad (43)$$

where $\hat{P}_i$ is a constant vector. Since its solution is

$$A_i = -\frac{2\hat{P}_i}{r}, \quad (44)$$

the solution of Eq.(38) is given by

$$A_i(\mathbf{x}) = -\int \frac{2\tilde{J}_i(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3 x'. \quad (45)$$

Since

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{r} + \frac{x^j x'^j}{r^3} + O\left(\frac{1}{r^3}\right) \quad (46)$$

for large r , we have

$$A_i = -\frac{2P_i}{r} - \frac{2M_{ij}x^j}{r^3} + O\left(\frac{1}{r^3}\right) \quad (47)$$

for large r , where

$$P_i = \int \tilde{J}_i d^3 x \quad (48)$$

$$M_{ij} = \int \tilde{J}_i x^j d^3 x. \quad (49)$$

Similarly the boundary condition of ψ is given by

$$\psi = -\frac{2M_{ii}}{r} + O\left(\frac{1}{r^2}\right) \quad (50)$$

for large r . For the boundary condition of W_i , consider

$$\Delta W_i + \frac{1}{3} \partial_i \partial_j W_j = 8\pi \hat{P}_i \delta(r). \quad (51)$$

From Eq.(47) and Eq.(40) and the fact that $\psi = 0$ if $\tilde{J}_i = \hat{P}_i \delta(r)$, we can find that its solution is given by

$$W_i = -\frac{x^i x^k \hat{P}_k}{4r^3} - \frac{7\hat{P}_i}{4r}, \quad (52)$$

thus the solution of Eq.(42) is given by

$$W_i = -\int \frac{(x^i - x'^i)(x^k - x'^k)\tilde{J}_k(\mathbf{x}')}{4|\mathbf{x} - \mathbf{x}'|^3} d^3 x' - \int \frac{7\tilde{J}_i(\mathbf{x}')}{4|\mathbf{x} - \mathbf{x}'|} d^3 x'. \quad (53)$$

Using the relation

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|^3} = \frac{1}{r^3} + \frac{3x^j x'^j}{r^5} + O\left(\frac{1}{r^5}\right), \quad (54)$$

we have

$$W_i = -\frac{P_k x^k x^i}{4r^3} - \frac{7P_i}{4r} - \frac{(7M_{ij} - M_{ji} - M_{kk}\delta_{ij})x^j}{4r^3} - \frac{3M_{jk}x^j x^k x^i}{4r^5} + O\left(\frac{1}{r^3}\right) \quad (55)$$

for large r . Since $\chi = \partial_i W_i$,

$$\chi = \frac{3P_i x^i}{2r^3} - \frac{3M_{ii}}{2r^3} + \frac{9M_{ij}x^i x^j}{2r^5} + O\left(\frac{1}{r^4}\right). \quad (56)$$

Once the vector potential W_i and then the extrinsic curvature $\tilde{K}_{ij}$ are obtained, a non-linear Poisson equation Eq. (12) should be solved. This kind of equation, namely,

$$\Delta\phi = F(\phi), \quad (57)$$

where $F(\phi)$ is a non-linear function of ϕ , is solved by an iterative method such as

$$\phi^{(I+1)} = \Delta^{-1} F(\phi^{(I)}); \quad (58)$$

the loop is repeated for $I = 0, 1, 2, \dots$ until a norm of the change in ϕ , $\|\phi^{(I+1)} - \phi^{(I)}\|$, becomes small enough.

In addition, we should solve Eqs.(30) and (36) on each time step. These equations have the similar structure to Eqs.(12) and (10), while the right-hand side of Eq.(36) depends on β^i . These equation should be solved by an iterative method. Iterations for solving these equations on each time step, however, consume CPU seconds, so that writing them as

$$\begin{aligned} \Delta\phi &= -\left[\tilde{\Delta}\phi - \Delta\phi + \frac{\phi^5}{8} (16\pi\rho_H + K_{ij}K^{ij} - K^2 - \phi^{-4}\tilde{R})\right] \\ &\equiv -4\pi\rho_\phi(\phi) \end{aligned} \quad (59)$$

and

$$\Delta\beta^i + \frac{1}{3}\partial_i(\partial_t\beta^t) = 8\pi J_\beta(\beta^k), \quad (60)$$

where Δ is the Laplacian in flat space and $\tilde{\Delta}$ is the Laplacian with respect to $\tilde{\gamma}_{ij}$, we use ϕ and β^k at the previous time step to calculate the source terms $\rho_\phi(\phi)$ and $J_\beta(\beta^k)$. With this approximation, we need not perform iterations. It will give fairly good solutions for ϕ and β^i if the time step Δt is small enough and/or the second term of the right-hand side of Eq.(36) is small in comparison with the first term.

Poisson equations are solved using a pre-conditioned conjugate gradient method and hyperbolic PDEs, Eqs. (18), (20), (21), (23) and (33), are integrated using a scheme based on var Leer's scheme with second-order accuracy in space [4].

4 Numerical Results

Now we show numerical results of simulation for coalescence binary neutron stars. Here we use a polytropic equation of state $p = (\Gamma - 1)\rho\epsilon$ with $\Gamma = 2$ and units of

$$M = M_\odot, \quad L = \frac{GM_\odot}{c^2} = 1.5\text{km}, \quad T = \frac{GM_\odot}{c^3} = 5 \times 10^{-6}\text{sec}. \quad (61)$$

The grid size is $201 \times 201 \times 201$ and it covers $[-100, 100]$ in the x , y and z directions.

We place as initial data two spherical neutrons stars of rest mass $M = 1.04$ and radius $r_0 = 6$ at $(\pm x_0, 0, 0)$ with $x_0 = 7.2$. We add a rigid rotation with angular velocity Ω and an approaching velocity

$v_a = \Omega x_0$ such that

$$v_x^N = \begin{cases} -\Omega y - v_a & \text{if } x > 0, \\ -\Omega y + v_a & \text{if } x < 0, \end{cases} \quad (62)$$

$$v_y^N = \Omega x. \quad (63)$$

We perform two simulations, BI1-1 and BI1-2, with $\Omega = 0.0264$ and 0.0396 . For BI1-1, the total gravitational mass $M_g = 1.84$ and (the angular momentum divided by M_g^2) $\equiv q = 0.99$ and for BI1-2 $M_g = 1.88$ and $q = 1.44$. Figure 1 shows the evolution of the density on the $x-y$, $y-z$ and $x-z$ planes of BI1-2.

In the conformal time slicing given by Eq.(37), the time-variable parts of $\tilde{\gamma}_{ij}$ can be considered as gravitational radiation parts, since the space outside the central matter approaches the Schwarzschild metric quickly [5]. Thus the total energy of the gravitational waves is given by

$$E_{\text{GW}} = \int dt \int_{r \rightarrow \infty} d\Omega \frac{r^2}{32\pi} (A_{ij}^{\text{TT}})^2, \quad (64)$$

where A_{ij}^{TT} is the transverse-traceless part of the time derivative of $\tilde{\gamma}_{ij}$,

$$A_{ij}^{\text{TT}} = (\partial_t \tilde{\gamma}_{ij})^{\text{TT}} \quad (65)$$

and the time derivative of $\tilde{\gamma}_{ij}$ is given by Eq.(31). Now we define the “energy density of the gravitational waves”, although it cannot be defined locally, as

$$\rho_{\text{GW}} = \frac{1}{32\pi} (A_{ij}^{\text{TT}})^2 = \frac{1}{32\pi} (A^{\text{TT}})_{ij} (A^{\text{TT}})^{ij}. \quad (66)$$

Figure 2 shows “energy density of the gravitational waves” $r^2 \rho_{\text{GW}}$.

Figures 3 and 4 show the luminosity $L_{\text{GW}}(t, r)$ and the energy $E_{\text{GW}}(t, r)$ of gravitational waves for BI1-2, respectively, where

$$L_{\text{GW}}(t, r) = \int r^2 \rho_{\text{GW}}(t, r) d\Omega \quad (67)$$

and

$$E_{\text{GW}}(t, r) = \int^t L_{\text{GW}}(t', r) dt' \quad (68)$$

We calculated $L_{\text{GW}}(t, r)$ and $E_{\text{GW}}(t, r)$ at $r = 30 \sim 50$. The dependence on the retarded time $t - r$ is the same for all r but the value of $L_{\text{GW}}(t, r)$ at $t - r \sim 10$ is smaller for the smaller r . The radius of $r = 30$ is too small to estimate the gravitational radiation, since the wave length of the gravitational waves is approximately 40 for BI1-2. The amplitude of $L_{\text{GW}}(t, r)$ is almost saturated for $r \gtrsim 45$ although it is comparable with the wave length. The total energy of the gravitational radiation up to $t = 40$ is 4×10^{-3} , which is about 0.2% of the gravitational mass of the system.

5 Conclusion

Now we constructed the first version of 3D code of numerical relativity including matter. With a $201 \times 201 \times 201$ grid, it requires about a 15GB memory and about 4 CPU hours up to $t = 80$ using 15 processor elements of VPP300/16R at National Astronomical Observatory, Japan. At present, our code is on a way of construction. The numerical boundary at $r = 100$ is too near and the grid separation $\delta x = 1$ is too large. For calculation accurate enough, we need more than 1TB of memory if our code is simply enlarged. So that we may have to implement technique such as an adaptive mesh. We made some approximation to the solution of β^i and ϕ for $t > 0$, which should be revised. Moreover, the conformal time slice is likely to lead instability of ϕ and α . Then we should use the different time slicing such as maximal slicing, which leads another non-linear elliptic PDE. It is in progress to make the next version of the code including some of the above modification. We hope that the completion of the 3D code will be done near future.

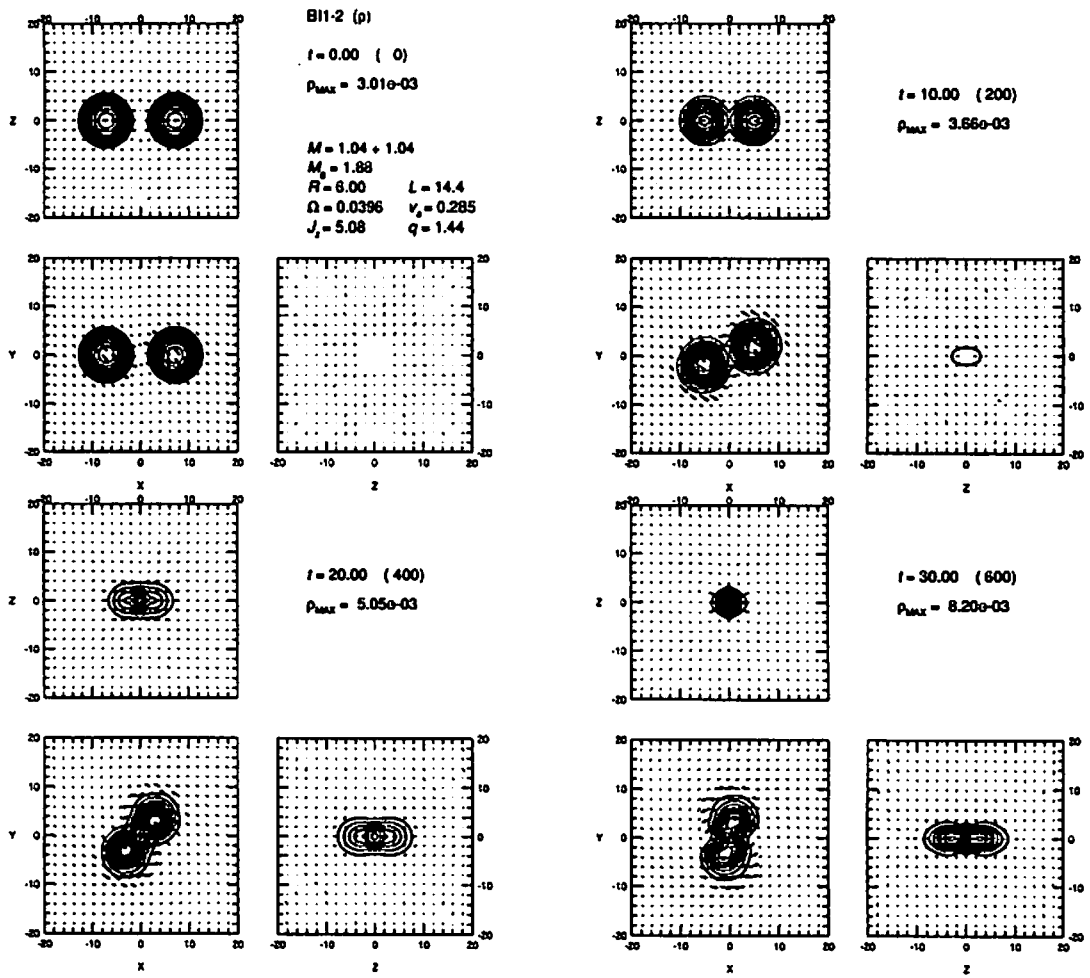


Figure 1: (continued)

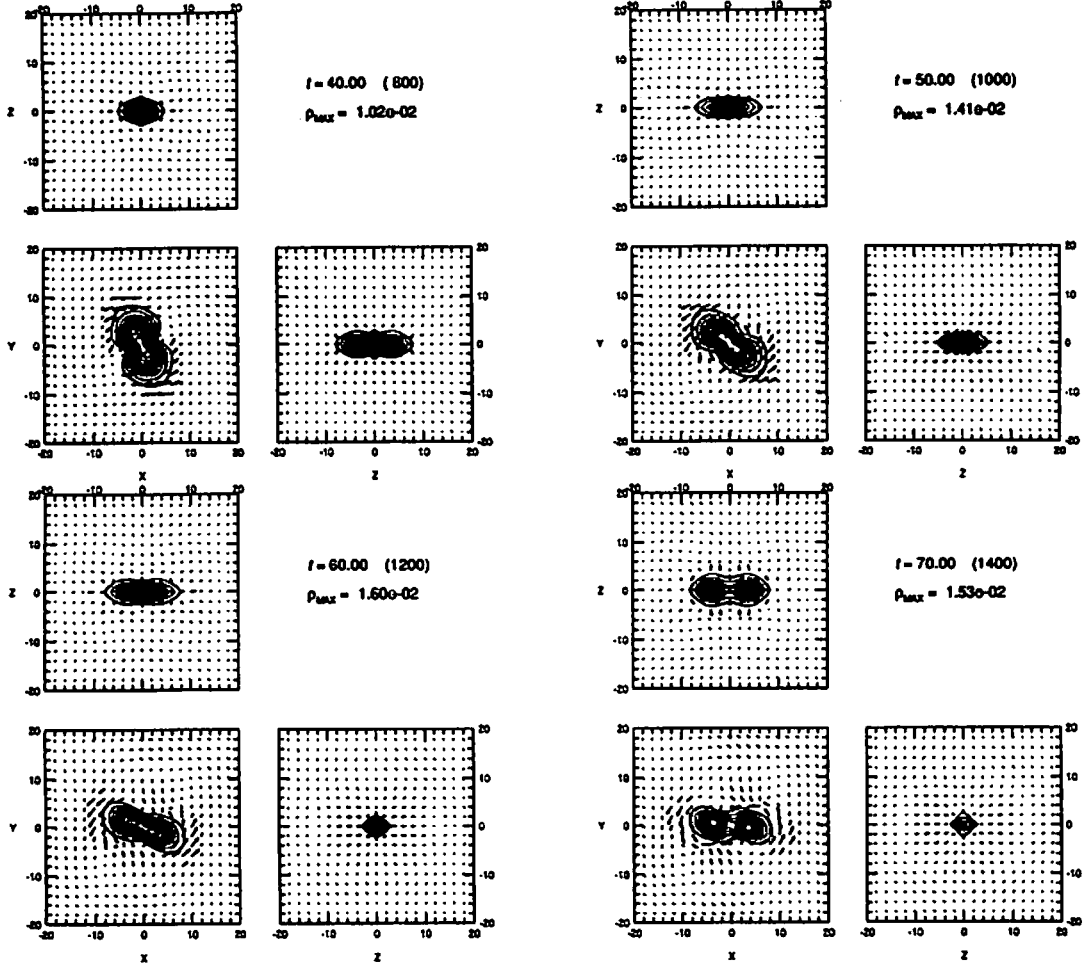


Figure 1: Mass density ρ on the $x-y$, $y-z$ and $x-z$ planes for BI1-2. Arrows indicate the velocity vector V^i .

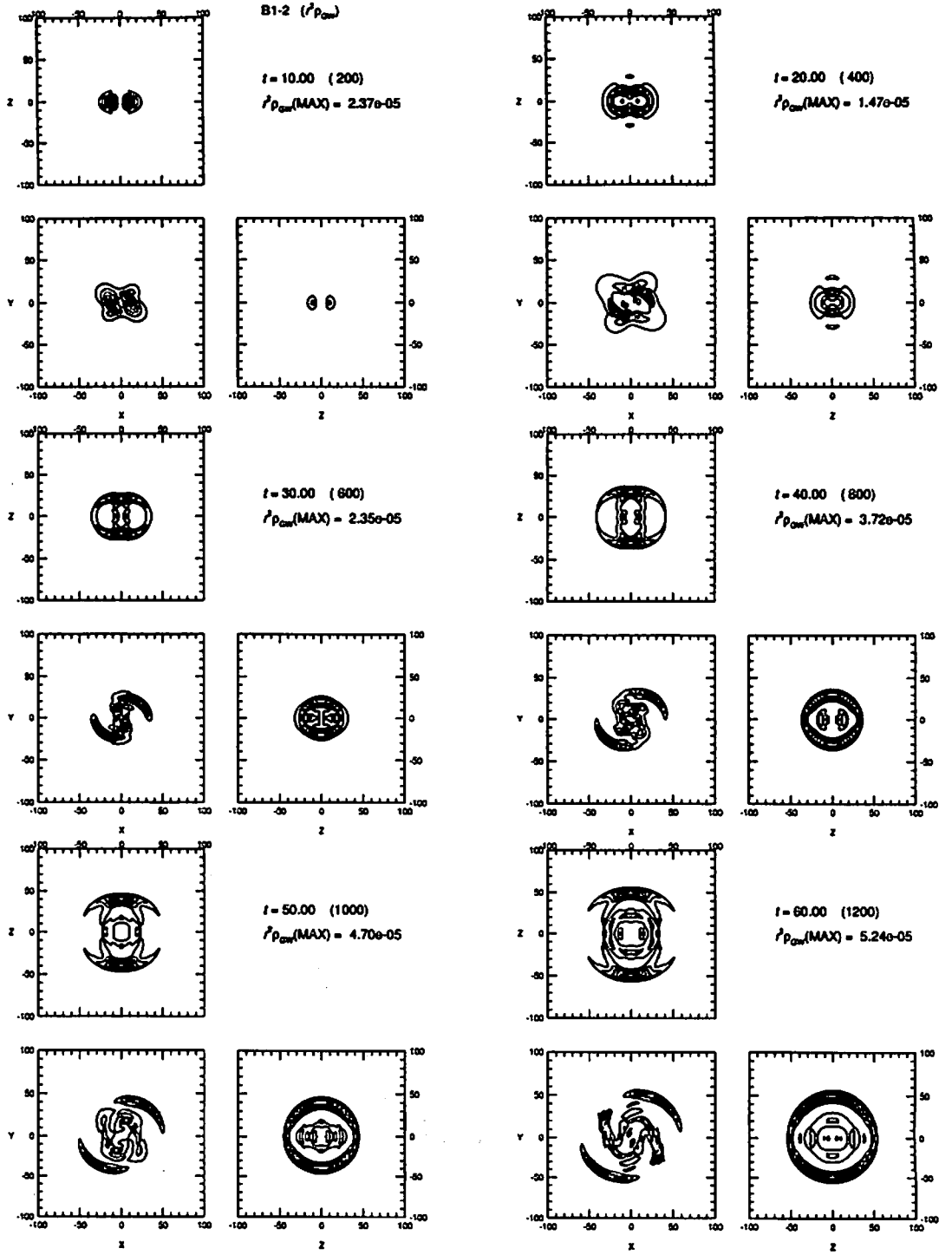


Figure 2: “Energy density of the gravitational waves” $r^2 \rho_{\text{GW}}$ for B11-2.

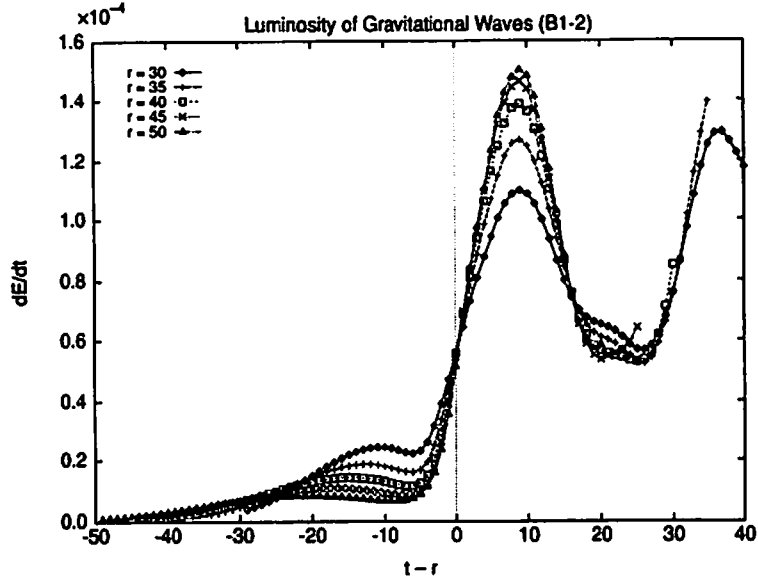


Figure 3: Luminosity of gravitational waves L_{GW} as a function of the retarded time $t - r$ calculated at $r = 30 \sim 50$.

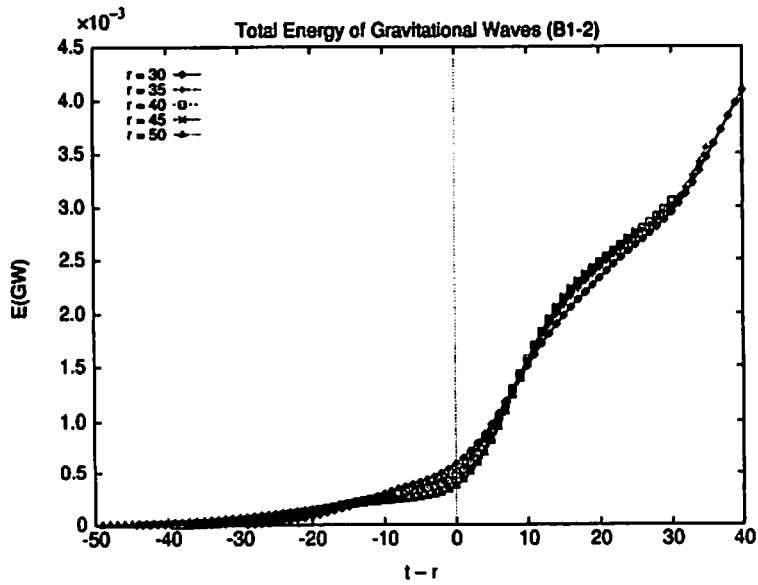


Figure 4: Energy of gravitational waves E_{GW} as a function of the retarded time $t - r$ calculated at $r = 30 \sim 50$.

This work was in part supported by the Grant-in-Aid of Ministry of Education, Culture, Science and Sports (08NP0801, 08237210 and 09740336) and the Supercomputer Project (No.97-22) of High Energy Accelerator Research Organization(KEK). Numerical computations were carried out on VPP300/16R and VX/4R at the Astronomical Data Analysis Center of the National Astronomical Observatory, Japan and on VPP500/80 at High Energy Accelerator Research Organization(KEK).

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Stability of coalescing binary stars against gravitational collapse : A relativistic hydrodynamical simulation

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Abstract

We perform simulations of relativistic binary stars in post-Newtonian gravity to investigate their dynamical stability against gravitational collapse in a tidal field. While our equations are only accurate to first post-Newtonian order in general, they are in fact exact for spherical, static stars. Thus, we may regard them as a formulation to investigate dynamical stability of fully general relativistic spherical stars due to first post-Newtonian tidal effects from their companion star. As initial data we consider both corotational and (approximately) irrotational equilibrium configurations for binary in circular orbits. We demonstrate that, independent of the internal stellar velocity profiles, the tidal field from a binary companion at this order stabilizes a star against gravitational collapse. This result has been predicted from other recent stability analyses, e.g., by Baumgarte et al.; Flanagan; and Thorne, but our result is the first confirmation of post-Newtonian tidal stabilization using a hydrodynamical evolution code.

1 Introduction

Wilson, Mathews and Maronetti (WMM) have carried out approximate general relativistic simulations for obtaining quasi-equilibrium states of coalescing binary neutron stars of equal mass[1]. To make their computation tractable, and also to obtain quasi-equilibrium neglecting radiation reaction of gravitational waves, they adopt an approximation in which three dimensional spatial metric is conformal flat. Results they have obtained are very surprising because their results suggest star-crushing effect that the central densities of the stars increase as their orbital separation decreases and that massive neutron stars near their maximum mass in binary collapse to be black holes prior to merging. To explain their numerical results, furthermore surprisingly, they concluded that gravitational field from binary companion acts to destabilize each star against gravitational collapse.

After they published their results, several researchers have claimed that their numerical results were incorrect, by means of many different approaches. Lai showed that tidal interaction effect between two Newtonian stars tends to stabilize against gravitational collapse[2]. Wiseman showed in first post-Newtonian approximation that gravitational interaction (without tidal interaction) never changes the central density[3]. Similar conclusion was deduced by Brady and Hughes using perturbation study[4]. Baumgarte et al. have performed numerical computation for obtaining equilibrium states of corotational binary neutron stars using the same approximation as that WMM adopted.[5]. They have shown that relativistic gravitational interaction (in conformal flat approximation) tends to stabilize against gravitational collapse for the case of corotational velocity field. Flanagan and Thorne, independently, have extended the Newtonian analysis by Lai to general relativistic case using matched asymptotic expansion technique[6][7], and shown that tidal field acts as a stabilization in general relativity. The conclusion the above researchers have obtained can be summarized as follows: Central density of stars in binary is affected only by *tidal field* among gravitational effects from their companion[6][7], and the tidal field acts to *stabilize* stars against gravitational collapse instead of destabilization not only in Newtonian theory[2], but also in post-Newtonian gravity or general relativity[7][5]. As a result, maximum allowed mass of stars in binary is larger than that for isolated stars[7][5]. This conclusion is deduced by firm analyses, so that it is much more reliable than conclusion by WMM. Thus, it seems unlikely that star-crushing effect claimed by WMM does not occur in reality.

In this paper, we perform numerical simulation for binary stars in an approximate relativistic gravity. In our present approximation, we use a formalism in which (1) all first post-Newtonian terms are taken

into account, and (2) fully general relativistic spherical, static stars can be yielded. Thus, in general case, results obtained are accurate only up to first post-Newtonian order, but for binaries of two (nearly) spherical stars, the formalism will be suitable for investigating stability of relativistic stars perturbed by first post-Newtonian tidal field (see section 2).

Our purpose in this paper is to demonstrate in numerical simulation that in first post-Newtonian order, tidal field from binary companion really acts as stabilization against gravitational collapse, and star-crushing phenomena never happen. These conclusions have been already predicted by the above researchers, in particular, by Baumgarte et al.[5] and Thorne[7] clearly. However, in their analyses, they made some assumptions. For example, in the analysis of Baumgarte et al., they restricted their attention only to the case of corotational velocity field. From their study, they did not get any information with regard to dynamical stability of each star in non-corotational binaries. In the matched asymptotic analysis by Thorne[7], he assumes that orbital separation of binary a is much smaller than radius of each star R and hence that tidal field is weak enough. Someone can claim that his conclusion may not be always valid for $a \gtrsim R$. Of course, we believe that their conclusions never change even if such assumptions were not made. However, we feel it important to show in dynamical simulation without such assumptions that their conclusions really do not change. This is the main motivation for our present study.

The paper is organized as follows. In section 2, we present an approximate general relativistic formulation adopted in this paper. In section 3 and 4, we perform numerical simulations using the formulation explained in section 2. After we demonstrate that our numerical code can yield correct, expected results with regard to dynamical stability of spherical stars against gravitational collapse in section 3, we give numerical results for investigation of dynamical stability against gravitational collapse of stars in binary system in section 4. We show that first post-Newtonian gravitational effects from their companion stars do not destabilize stars in binary, but stabilize them. Section 5 is devoted to summary. Throughout this paper, we use units in which speed of light and gravitational constant are unity.

2 Formulation

In this section, we give an approximate formulation of general relativity. Following previous approximate general relativistic works[1][5], first, we adopt the conformal flat approximation in which the three metric is conformal flat, i.e., $\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} = \psi^4 \delta_{ij}$, and write the line element as

$$ds^2 = (-\alpha^2 + \beta_k \beta^k) dt^2 + 2\beta_i dx^i dt + \psi^4 \delta_{ij} dx^i dx^j, \quad (1)$$

where α , β_i and ψ are the lapse function, shift vector and conformal factor, whose equations are derived below. We note that we use the Cartesian coordinate $x^i = (x, y, z)$ throughout this paper; i.e., the covariant derivative with respect to $\tilde{\gamma}_{ij}$ becomes equal to the partial derivative, $\partial/\partial x^k$ or $_{,k}$.

As the energy momentum tensor, we adopt

$$T^{\mu\nu} = \rho \left(1 + \varepsilon + \frac{P}{\rho} \right) u^\mu u^\nu + P g^{\mu\nu}, \quad (2)$$

where ρ , ε , P , and u^μ denote the rest mass density, specific internal energy, pressure, and four velocity, respectively. We assume the polytropic equation of state $P = (\Gamma - 1)\rho\varepsilon$, where $\Gamma = 1 + 1/n$ and n is the polytropic index.

The continuity equation for ρ is

$$\frac{\partial \rho_\star}{\partial t} + \frac{\partial(\rho_\star v^i)}{\partial x^i} = 0, \quad (3)$$

where $\rho_\star = \rho \alpha u^0 \psi^6$ and $v^i = u^i / u^0$.

The Euler equation and the energy equation are

$$\frac{\partial(\rho_\star \tilde{u}_i)}{\partial t} + \frac{\partial(\rho_\star \tilde{u}_i v^j)}{\partial x^j} = -\alpha \psi^6 P_{,i} - \rho_\star \alpha \tilde{u}^0 \alpha_{,i} + \rho_\star \tilde{u}_j \beta^j_{,i} + \frac{2\rho_\star \tilde{u}_k \tilde{u}_k}{\psi^5 \tilde{u}^0} \psi_{,i}, \quad (4)$$

$$\frac{\partial e_\star}{\partial t} + \frac{\partial(e_\star v^j)}{\partial x^j} = 0, \quad (5)$$

where

$$\tilde{u}_j = (1 + \Gamma\epsilon)u_j, \quad (6)$$

$$\tilde{u}^0 = (1 + \Gamma\epsilon)u^0, \quad (7)$$

$$e_* = (\rho_* \epsilon_*)^{1/\Gamma}; \epsilon_* = \epsilon(\alpha u^0 \psi^6)^{\Gamma-1}, \quad (8)$$

$$v^i = -\beta^i + \frac{u_j}{\psi^4 u^0}. \quad (9)$$

Note that $\beta^i = \psi^{-4}\beta_i$ in the conformal flat approximation. u^0 is obtained by solving the following algebraic equation,

$$(\alpha u^0)^2 = 1 + \frac{u_i u_i}{\psi^4} = 1 + \frac{\tilde{u}_i \tilde{u}_i}{\psi^4} \left[1 + \Gamma \frac{\epsilon_*}{(\alpha u^0 \psi^6)^{\Gamma-1}} \right]^{-2}. \quad (10)$$

In numerical simulation, we adopt ρ_* , $\tilde{u}_i$, and e_* as the basic variables for hydrodynamical equations to be solved.

Equations for α , β^i and ψ are derived from the Einstein equation in the following way. α is determined by the slice condition, for which we choose the maximal slicing condition $\text{tr}K = 0$, where $\text{tr}K$ is the tracefree part of the extrinsic curvature K_{ij} . Then, we get

$$\Delta(\alpha\psi) = 2\pi\alpha\psi^5 \left(E + 2S_{ij}\delta^{ij}\psi^{-4} \right) + \frac{7}{8}\alpha\psi^5 K_{ij}K^{ij}, \quad (11)$$

where $E = \rho(1 + \Gamma\epsilon)(\alpha u^0)^2 - P$, $S_{ij} = T_{ij}$, and Δ is the flat Laplacian.

From assumption for three metric as $\gamma_{ij} = \psi^4 \delta_{ij}$, i.e., from assumption of time derivative of $\tilde{\gamma}_{ij}$ is zero, we get

$$0 = -2\alpha\psi^{-4}K_{ij} + \delta_{ij}\beta^l{}_{,j} + \delta_{ji}\beta^l{}_{,i} - \frac{2}{3}\delta_{ij}\beta^l{}_{,l}. \quad (12)$$

Then, using the momentum constraint, $(\psi^6 K^i{}_j)_{,i} = 8\pi J_j \psi^6$, where $J_i = \rho_* \tilde{u}_i \psi^{-6}$, we get the equation for β^i as

$$\Delta\beta^i + \frac{1}{3}\beta^j{}_{,ji} = \left[\ln\left(\frac{\alpha}{\psi^6}\right) \right]_{,j} \left(\beta^i{}_{,j} + \beta^j{}_{,i} - \frac{2}{3}\delta_{ij}\beta^l{}_{,l} \right) + 16\pi\alpha J_i. \quad (13)$$

Here, we note that the above K_{ij} does not satisfies its evolution equation in general.

Finally, equation for ψ is derived from the Hamiltonian constraint as

$$\Delta\psi = -2\pi\psi^5 E - \frac{K_{ij}K^{ij}\psi^5}{8}. \quad (14)$$

In the conformal flat formulation, we can always obtain exact solutions for initial value problems of Einstein equation, but it does not yield exact dynamical solutions in general. In the case where the system has spherical symmetry, it always yields exact general relativistic solutions, but otherwise it yields only an approximate solution. From the viewpoint of post-Newtonian approximation[8][9], equations in this formulation contain all the Newtonian and first post-Newtonian terms consistently, but as for second post-Newtonian order, some terms are neglected. Thus, we can regard a result in numerical simulation using this approximation as that accurate only up to first post-Newtonian order. Due to this fact, we simplify equations neglecting terms of second post-Newtonian order in the Poisson equations of $\alpha\psi$, β_i and ψ such as $7\alpha\psi^5 K_{ij}K^{ij}/8$, $[\ln(\alpha\psi^{-6})]_{,j}(\beta^i{}_{,j} + \beta^j{}_{,i} - 2\delta_{ij}\beta^l{}_{,l}/3)$, and $K_{ij}K^{ij}\psi^5/8$. Then, we obtain slightly simple Poisson equations as

$$\Delta(\alpha\psi) = 2\pi\alpha\psi^5 \left(E + 2S_{ij}\delta^{ij}\psi^{-4} \right) \equiv 4\pi S_{\alpha\psi}, \quad (15)$$

$$\Delta\beta^i + \frac{1}{3}\beta^j{}_{,ji} = 16\pi\alpha J_i, \quad (16)$$

$$\Delta\psi = -2\pi\psi^5 E \equiv 4\pi S_{\psi}. \quad (17)$$

We further decompose the equation for β^i using

$$\beta^i = 4B_i - \frac{1}{2} \left[\chi_{,i} + (B_k x^k)_{,i} \right]. \quad (18)$$

Then, the equations for B_i and χ become

$$\Delta B_i = 4\pi\alpha J_i, \quad (19)$$

$$\Delta\chi = -4\pi\alpha J_i x^i. \quad (20)$$

Thus, equations for the gravitational field we need to solve are six Poisson equations for $\alpha\psi$, ψ , B_i and χ which are solved under the boundary condition at outer grids as

$$\alpha\psi = 1 - \frac{1}{r} \int S_{\alpha\psi} dV + O(r^{-3}), \quad (21)$$

$$\psi = 1 - \frac{1}{r} \int S_{\psi} dV + O(r^{-3}), \quad (22)$$

$$B_x = -\frac{x}{r^3} \int \alpha J_x x dV - \frac{y}{r^3} \int \alpha J_x y dV + O(r^{-4}), \quad (23)$$

$$B_y = -\frac{x}{r^3} \int \alpha J_y x dV - \frac{y}{r^3} \int \alpha J_y y dV + O(r^{-4}), \quad (24)$$

$$B_z = -\frac{z}{r^3} \int \alpha J_z z dV + O(r^{-4}), \quad (25)$$

$$\chi = \frac{1}{r} \int \alpha J_i x^i dV + O(r^{-3}). \quad (26)$$

Note that we do not omit any terms in hydrodynamical equations. Changes were made only in Poisson equations for gravitational potentials.

A reward due to this simplification is that the source terms of Poisson equations become compact. This means that we do not have to attach the boundary condition far from binary stars, and hence need not take a large number of grid points in order to guarantee the accuracy in numerically solving Poisson equations. With a smaller number of grid points than that in previous papers[5], we can guarantee the accuracy well.

As mentioned above, in the present formulation, we neglect several general relativistic terms, but first post-Newtonian terms are consistently included. Hence, the present approximation is regarded as an alternative version of first post-Newtonian approximation. Also, we may consider it as a formulation of wider applicability: In the usual post-Newtonian case, we assume not only that velocity is much smaller than velocity of light, but also that compactness of each star is small enough (i.e., ratio of star's gravitational mass to its radius is much smaller than unity). If we are concerned about compact stars, the latter assumption is not appropriate in all sorts of problems. However, in the present type of approximation, we may think in some cases that the latter assumption is not made because it can derive exact general relativistic solutions of spherical, static stars. The present formulation is promising for investigation of the following objects.

First trivial example is a binary in which the separation between two stars are sufficiently large and also stars have no spin. This is because in this case, both stars have spherical symmetric configurations and geometries whether or not they are very compact.

Second example is a single, perturbed, and relativistic spherical star for which we want to investigate dynamical stability against gravitational collapse. This is because the dynamical stability of relativistic spherical stars is determined by a first post-Newtonian effect[10]. In the next section, we perform that, and show that expected results are derived.

Third and most interesting example is a binary in which the orbital separation, a , is appropriately larger than the stellar radius, R . In this case, we may regard gravitational effects from companion stars and tidal deformation due to them as a small perturbation in general, and solution of geometric variables for Einstein equation around one of binary stars will be expanded as[6][7]

$$\alpha = {}_{(0)}\alpha + \epsilon_{(2)}^2\alpha + \epsilon_{(4)}^4\alpha + \epsilon_{(5)}^5\alpha + \epsilon_{(6)}^6\alpha + \epsilon_{(7)}^7\alpha + \dots, \quad (27)$$

$$\psi = {}_{(0)}\psi + \epsilon_{(2)}^2\psi + \epsilon_{(4)}^4\psi + \epsilon_{(5)}^5\psi + \epsilon_{(6)}^6\psi + \epsilon_{(7)}^7\psi + \dots, \quad (28)$$

$$\beta^i = \epsilon_{(1)}^1\beta^i + \epsilon_{(3)}^3\beta^i + \epsilon_{(5)}^5\beta^i + \epsilon_{(6)}^6\beta^i + \epsilon_{(7)}^7\beta^i + \epsilon_{(8)}^8\beta^i + \dots, \quad (29)$$

$$\tilde{\gamma}_{ij} = \delta_{ij} + \epsilon_{(2)}^2h_{ij} + \epsilon_{(4)}^4h_{ij} + \epsilon_{(5)}^5h_{ij} + \dots, \quad (30)$$

where we assume $v^i \sim u_i \sim (M/a)^{1/2} \lesssim (R/a)^{1/2} \sim \epsilon$, and $({}_0)\alpha$ and $({}_0)\psi$ denote the spherical symmetric solutions. We note that in Eqs. (27)–(30), pure gauge terms are included. From post-Newtonian studies in the (3+1) formalism[9], we can infer the following facts for examples; $({}_2)\alpha$, $({}_2)\psi$, $({}_1)\beta^i$, and $({}_2)h_{ij}$ does not affect dynamics of stars; effects due to tidal deformation of spherical stars appear from $n = 6$ in $({}_n)\alpha$ and $({}_n)\psi$, from $n = 7$ in $({}_n)\beta^i$, and from $n = 8$ in $({}_n)h_{ij}$; effects of gravitational radiation reaction are included in $({}_n)\alpha$ ($n \geq 7$), $({}_n)\beta^i$ ($n \geq 8$), $({}_n)\psi$ ($n \geq 7$), and $({}_n)h_{ij}$ ($n \geq 5$).

Applicability of the present formulation to the above two-body problems is grasped as follows. In the present formulation, we can calculate $({}_0)\alpha$ and $({}_0)\psi$ exactly. Newtonian and first post-Newtonian terms in $({}_n)\alpha$, $({}_n)\psi$ ($n \geq 2$), and $({}_n)\beta^i$ ($n \geq 1$) are also taken into account consistently for all order in ϵ , although the second post-Newtonian terms in them and $({}_n)h_{ij}$ ($n \geq 2$) are not included. Thus, the present approximation is appropriate when we investigate stability of fully general relativistic spherical stars due to Newtonian and first post-Newtonian effects from their companion star[11]. This problem is just what we investigate in this paper.

Before closing this section, we briefly summarize numerical methods. Simulations were carried out using Eulerian method. Numerical methods for solving the hydrodynamical equations and the Poisson equations are the same as those given in a previous paper[12]. We assume symmetry with respect to equatorial plane, and adopt $(2N + 1, 2N + 1, N + 1)$ grid covering $-L \leq x, y \leq L$ and $0 \leq z \leq L$ where L is location of outer boundaries. Grid number adopted is $N = 50$ for the spherical symmetric case, and $N = 50, 60$, and 75 for binary cases.

3 Dynamical stability of spherical static stars

In this section, we exhibit numerical results on dynamical stability of general relativistic spherical stars due to first post-Newtonian (approximately) spherical perturbation. It is well-known that dynamical stability of spherical stars can be simply investigated by deriving the relation between the proper mass (i.e., conserved baryon mass) and central density (see Fig. 1(a)) without numerical simulation[13]. Thus, the following numerical simulations simply demonstrate a trivial fact. However, showing that our numerical code has an ability to investigate dynamical stability as well as clarifying accuracy of its numerical results are important first steps toward study of dynamical stability of stars in binary.

First of all, we give the relation among the mass, central density and radius of spherical stars in equilibrium states. As equation of state, we set

$$P = K \rho^\Gamma, \quad (31)$$

and choose $K = 1$ for simplicity. In this paper, we pay attention only to the case for $\Gamma = 1.4$ ($n = 2.5$), i.e., soft equation of state. This is because we are interested in dynamical stability of spherical stars or each star in binary and for such a soft equation of state, the star will be unstable even for the case where compactness of stars is small, i.e., where the approximation in this paper is good.

In Fig. 1(a), we show the relation between the central density ρ_c and proper mass M_p which is defined as

$$M_p = \int \rho_c dV. \quad (32)$$

We note that the relation agrees with fully general relativistic one. As is well-known, there exists the maximum of M_p at $\rho_c = \rho_{\text{crit}} (\simeq 1.2 \times 10^{-4})$, and for $\rho_c > \rho_{\text{crit}}$ ($\rho_c < \rho_{\text{crit}}$), the star is unstable (stable). In Fig. 1(b), we also show the relation between the central density and the coordinate radius.

We perform numerical simulation using the spherical stars obtained above as initial conditions. Among them, we adopted five spherical stars which have the proper mass slightly less than the maximum and we mark them with A, B, C, D and E in Figs. 1(a) and (b). We prepare two types of initial conditions. One is the equilibrium condition obtained above. While, in the other, we reduce the internal energy by 0.5% at the initial state (i.e., we set $K = 0.995$ initially). This is because we are mainly interested in whether the star collapses or not, and to test that, it is one of the simplest method. Note that from dimensional analysis, we can show that reducing K by a factor of δ is equivalent to adding proper mass by a factor of $n\delta/2$ fixing K to be unity. Hence, if we add proper mass by a factor of 0.625% fixing $K = 1$ instead of reducing K , we will get essentially the same result. We arrange the grid spacing to cover a radius of initial spherical stars by $25 - 35$ grid points. Following results exhibited are those for 30 grid case.

In Fig. 2, we show time evolutions of ρ_c at the center (ρ_{*c}) for models A–E. Throughout this paper, the time is shown in units of $t\rho_0^{1/2}$ where ρ_0 is ρ_c of equilibrium stars. Solid and dotted lines show results in the case where we set $K = 0.995$ and $K = 1$ at their initial condition, respectively.

Numerical results obtained here agree with what we expect, as shown in the following. For models A and B, ρ_{*c} simply oscillates even when we reduce K initially. In particular, for model A, the oscillation time scale is roughly equal to their known approximate formula as[13]

$$2\pi \left[\frac{3(\Gamma - 1)M^2}{(5\Gamma - 6)RI} \left(3\Gamma - 4 - 6.75 \frac{M}{R} \right) \right]^{-1/2}, \quad (33)$$

where M is the Newtonian mass, R is the Newtonian radius of star, and

$$I = \int \rho r^2 dV. \quad (34)$$

(Note that $\rho_c \simeq 5.588M/R^3$ and $I \simeq 0.1677MR^2$ for polytrope of $\Gamma = 1.4$.) Eq. (33) also suggests why the period of the oscillation for model B is longer than that for model A in units of $t\rho_0^{1/2}$. These facts support that our numerical simulation gives correct results, and allow us to conclude that stars in models A and B are stable.

For model C, if we start simulation with $K = 1$, the central density purely oscillates, but if we start with $K = 0.995$, the central density monotonically increases. This indicates that the stability of the star C is marginal against small perturbations, $\delta K/K \lesssim 0.5\%$. The result is reasonable because proper mass of star in model C is only 0.05% smaller than the maximum mass.

For models D and E, the star monotonically expands or collapses, and never oscillate: If we start with $K = 0.995$, the star collapses, and if $K = 1$, the star expands by a large factor[14]. Here, we must note that if our code for simulation yielded completely accurate results, we could get the following results; for $K > 1$, the star expands, for $K < 1$, the star collapses, and for $K = 1$, the configuration of the star does not change. Unfortunately, accuracy of our code is restricted. It is found that the boundary value for distinguishing between the collapse and expansion is $K \simeq 0.997$ for the case where radius of stars are initially covered by 30 grid points. This is the reason why the star with $K = 1$ expands. However, we also found that the boundary value of K systematically approached to (deviated from) unity when the grid became finer (coarser), so that we recognize that the deviation simply originates from the error due to finite differencing. Thus, we can mention that except for this numerical effect, the numerical simulations clearly demonstrate that the configurations for models D and E are unstable.

From the above results, we conclude that our code for simulation have an ability to investigate dynamical stability of spherical stars within certain small numerical error which depends on resolution.

4 Dynamical stability of stars in binary systems

Next, we exhibit numerical results on dynamical stability of stars in binary systems. In this paper, we assume that two stars in binaries have equal mass, and are in circular orbits. We prepared three types of initial conditions. One is the binary in corotational equilibrium states. The equilibrium configurations are numerically obtained by methods of Baumgarte et al.[5] or Shibata[15]. In other two cases, we chose spherical stars obtained in section 3 as each star of binaries. In particular, we adopted models B and D as the spherical stars. As the velocity fields, we simply gave as[16]

$$(u_x, u_y, u_z) = (-\Omega y, \Omega x, 0) \quad (35)$$

or

$$(u_x, u_y, u_z) = \left(0, \Omega x_0 \frac{x}{|x|}, 0 \right), \quad (36)$$

where Ω is an appropriately determined angular velocity. In both cases, the center of mass for each star is located at $(\pm x_0, 0, 0)$ where $x_0 > 0$. Here, we simply set Ω as $(M_T/a^3)^{1/2}$, where M_T is the sum of the proper mass of two spherical stars (i.e., $M_T = 2M_p$) and a is the coordinate separation between

two stars. We note that the reason why we gave u_i is simply because we can solve B_i without iteration for computation of initial conditions. (If we gave v^i , we needed iteration to determine B_i , χ and hence β^i .) Hereafter, we call the former and latter velocity fields as nearly corotational and nearly irrotational velocity fields, respectively. Also, we call binary models, in which spherical stars in models B and D are used, as models Bb and Db for nearly corotational case and models Bc and Dc for nearly irrotational case, respectively. We summarize the initial conditions for numerical simulation in Table I.

At initial condition, we often reduce the internal energy (i.e., K) appropriately. One of the reasons is that we are interested in whether the star collapses or not. To investigate it adding small perturbation, reduction of K is the simplest method. Another reason is that for models Bb, Db, Bc, and Dc, the binaries are in highly non-equilibrium states at their initial stage unless we reduce K , because each star is stretched due to tidal force from its companion. The reduction factor of K is changed with change of their initial conditions such as orbital separation, internal velocity field, and so on. For appropriate choices of K , binaries in these models are expected to settle down to nearly equilibrium states. Hence, if stars in binary systems are dynamically stable, they will relax to some approximate equilibrium states. (This speculation is shown to be correct below.)

4.1 Corotational equilibrium binary cases

Among infinite possible states of corotational equilibrium binaries, we pay particular attention to binaries where two surfaces come into contact ($z_A = 0$ in the terminology in ref.[5]). This is the most suitable setting for the present purpose because in this case, the tidal effect from each companion is largest. In Fig. 3, we plot the proper mass of each star as a function of the central density for binaries in contact (filled circles). We also show the same relation for single spherical stars (solid line). As pointed out in ref.[5], it is found that the maximum proper mass for binaries is slightly larger than that for single spherical stars.

We note that these equilibria were obtained using grids with $N = 40, 50, 60$, and 75 , and we found that proper mass is second order convergent as $M_p = M_0 + M_2 d^2$, where d denotes the grid spacing and M_0 should be the exact proper mass. (On the other hand, it hardly depends on L .) In Fig. 3, M_0 was plotted, not bare M_p . In Table II, we also show M_p calculated with different resolutions as well as M_0 . It is found that error for estimation of M_p is not small; for $N = 60$ and 75 grid cases, the errors are $\sim 0.6\%$ and $\sim 0.4\%$, respectively. The error is generated due to finite differencing of non-linear equations. This implies that in the following numerical simulation, we had better expect numerical errors at least of similar magnitude.

We guess that for $\rho_c \lesssim 10^{-4}$, each star is dynamically stable, and otherwise, unstable. To investigate the stability, we choose the equilibrium binary of $\rho_c = 5 \times 10^{-5}$, and $\rho_c = 1.5 \times 10^{-4}$ as initial conditions of numerical simulation. (We mark them with Ba, and Da in Fig. 3.) Note that the orbital periods are ~ 50 in units of $t_{p0}^{1/2}$ in both models.

In Fig. 4, we show time evolution of the maximum value of ρ_* ($\rho_{\max}$) for models Ba and Da. In order to see convergence of numerical results, simulations are performed using grids with $N = 50, 60$, and 75 . In all cases, location of the outer boundary (i.e., L) is fixed for a given model (see Table I), and grids cover major axis of each star with $\sim (N/2 - 5)$. Solid, dotted and dashed lines denote results for $N = 75, 60$ and 50 , respectively. For model Ba, we always set $K = 1$ initially; i.e., we did not add any perturbation besides those generated numerically. It is found that $\rho_{\max}$ for model Ba purely oscillates and hence the configurations are stable. Here, we note that $\rho_{\max}$ oscillates although we did not add any perturbation. The reason for it is simply because the numerical grid is not fine enough and the error due to finite differencing affects the results. In reality, as shown in Fig. 4, the oscillation amplitude converges with improve of resolution.

For model Da, we perform simulations changing initial K appropriately. The value of K at initial condition is written in Fig. 4 for $K \neq 1$ (see also the figure caption). Stars in model Da collapse or expand, but never oscillate. This indicates that they are dynamically unstable and can never exist in reality as an inspiraling binary.

Here, we should note again that if our numerical code for simulation as well as for preparing initial conditions yielded completely accurate results, the boundary value of K for distinguishing between collapse and expansion becomes unity. Unfortunately, accuracy is limited and as a result, the boundary

value of K becomes less than unity. The deviation, however, converges with improve of resolution as shown in Fig. 4. For example, the boundary value is $K < 0.99$ for $N = 50$ grid simulation, $K \sim 0.992$ for $N = 60$ grid simulation, and $K \sim 0.995$ for $N = 75$ grid simulation. This seems to imply that the deviation is simply due to the error due to finite differencing which will converge with improve of resolution. Thus, we conclude that as in the case of spherical symmetric stars, stars in corotational equilibrium binary systems whose central maximum density exceeds a critical value are dynamically unstable, but otherwise stars are dynamically stable.

In Figs. 5(a) and (b), we show contour lines of ρ_* and velocity field of (v^x, v^y) in the equatorial plane for model Ba and model Da for which we set initially $K = 0.994$ (for the case $N = 75$). Contour lines are drawn for $\rho_*/\rho_{*j} = 10^{-0.3j}$ where ρ_{*j} denotes the maximum at initial conditions and $j = 0, 1, 2, \dots, 10$. Arrows denote the local velocity field. We can explicitly see the results on dynamical stability. The binary of model Ba keeps an almost steady orbit in more than one orbital period, while that of model Da collapses. It is also found that for model Ba, the velocity fields are kept to be almost corotational one, and there are no signs that a curious double vorticity feature which was reported in simulation by WMM[1].

4.2 Nearly corotational non-equilibrium case

Next, we give numerical results with regard to dynamical stability of stars in nearly corotational, non-equilibrium binary systems. The main purpose here is to show that we can get essentially the same results of dynamical stability as those in corotational equilibrium cases even when we start simulation from non-equilibrium states.

Simulations in this and next subsections were carried out in the following setting. Grid number adopted is $N = 60$. Location of the outer boundary (L) is adjusted to cover radius of each star with ~ 25 grid points, and as a result, $L = 125$ for models Bb and Bc, and $L = 96$ for models Db and Dc (see Table I). Mass center of each star is initially located at $(x, y, z) = (\pm L/2, 0, 0)$. Hence, the approximate orbital period becomes ~ 40 in units of $t\rho_0^{1/2}$ irrespective of models. Note that orbital coordinate separations for Bb and Bc, or Db and Dc are slightly smaller than those of Ba or Da.

We initially set $K = 0.98, 0.975$ and 0.97 for model Bb, and $K = 0.98$ and 0.975 for model Db. In Fig. 6, we show time evolution of ρ_{*max} for models Bb and Db. Dotted, solid and dashed lines denote results for which initial value of K is $0.98, 0.975$ and 0.97 , respectively. It is found from Fig. 6 that as in corotational equilibrium cases, the central density of stars in binary for model Db monotonically increases or decreases. On the other hand, that for model Bb of $K \geq 0.97$ purely oscillates. Thus, the results imply that stars in binary for model Db is always unstable, while those for model Bb with $K \gtrsim 0.97$ is stable.

As mentioned in section 3, reducing K by a factor of δ is approximately equivalent to adding proper mass by a factor of $n\delta/2$ fixing K to be unity. Results that model Bb with $K \geq 0.97$ is stable imply that model Bb with $\delta M_p/M_p \lesssim 3.7\%$ will be also stable, but otherwise unstable. Since proper mass for spherical model B is about 1.2% smaller than the maximum mass for spherical static stars, the above fact means that the maximum mass for nearly corotational binaries of orbital separation $a \sim L$ is about 2.5% larger than the maximum mass of spherical star. Taking into account the fact that orbital separation for this case is slightly smaller than that for corotational equilibrium cases investigated in the previous subsection, as well as uncertainty due to numerical error of $\lesssim 1\%$ for $N = 60$ grid, this result seems to be consistent with that for corotational equilibrium case.

In Fig. 7, we show contour lines of ρ_* and velocity field of (v^x, v^y) in the equatorial plane for stable model Bb where $K = 0.975$ initially. Rules for drawing contour lines and allow for velocity field are the same as those in Figs. 5. We find that the velocity fields for model Bb are kept to be almost corotational one, and no signs for double vorticity reported by WMM[1] can be seen.

Here, we emphasize that with time evolution, the density contour, maximum density, and the velocity field in model Bb do not change so much. Furthermore, comparing the results for corotational non-equilibrium case with those for corotational equilibrium case, we recognize that they are quite similar. Thus, we may consider that binary stars in model Bb with $K = 0.975$ or 0.98 are almost in equilibrium state at the initial condition. This fact suggests to us that for an appropriate initial reduction of K , we

will be able to construct approximate equilibrium states for nearly irrotational binaries, models Bc and Dc.

4.3 Nearly irrotational non-equilibrium case

Up to this section, we pay attention only to dynamical stability of stars in corotational (or nearly corotational) binaries. However, if we intend to apply the present results to the case of coalescing binary neutron stars, assumption of corotation is not appropriate because velocity fields of real binary neutron stars are expected to be nearly irrotational one as pointed out by Kochanek, and Bildsten and Cutler[17]. Although Bonazzola, Gourgoulhon and Marck[18] and Uryu and Eriguchi[19] recently proposed numerical methods for computation of equilibrium configurations of irrotational binary stars, computation of irrotational binaries carrying out their methods is not as easy as that of corotational binaries because we must construct a numerical code for the special purpose. Hence in this paper, we use an approximate irrotational binary stars whose velocity field is denoted by Eq. (36) as initial conditions for simulation. Reminding the fact in subsection 4.2 that results in nearly corotational case are essentially the same as those in corotational equilibrium case, we believe that initial conditions we give here are in approximate equilibrium states.

We initially set $K = 0.985$ and 0.98 for model Bc, and also $K = 0.985$ and 0.98 for model Dc. In Fig. 8, we show time evolution of ρ_{max} for models Bc and Dc. Dotted and solid lines denote results for which initial value of K is 0.985 and 0.98 , respectively. As in corotational cases, the central density of each star for model Dc monotonically increases or decreases, while that for model Bc of $K \geq 0.98$ purely oscillates. Thus, their stability is the same as that in corotational cases.

Here, we point out that reduction factor of K for model Bc to make an approximately equilibrium state is slightly (about 1%) smaller than that for nearly corotational model Bb. (Compare model Bc for which initial value of K is 0.985 with model Bb for which initial value of K is 0.975 .) This is because in irrotational cases, spin of stars (with respect to distant inertial observer) is approximately zero, but in corotational cases, certain spin exists, and it acts as the centrifugal force, weakening the self-gravity of each star. In other words, the internal velocity field could slightly change the maximum mass of stars against gravitational collapse: The maximum mass of stars for corotational binaries will be $\sim 1\%$ larger than that for irrotational binaries. The result indicates that internal velocity field of stars does not change the qualitative feature with regard to stability, that is, for $\rho < \rho_c$ ($\rho > \rho_c$), stars are stable (unstable), but it quantitatively changes the maximum mass at $\rho = \rho_c$ slightly. This point should be carefully investigated by investigating irrotational binaries in equilibrium states.

In Fig. 9, we show contour lines of ρ_* and velocity field of (v^x, v^y) in the equatorial plane for model Bc of $K = 0.985$ initially. Rules for drawing contour lines and allow for velocity field are the same as those in Figs. 5. We find that the velocity fields are kept to be almost irrotational one, and no signs for a curious double vorticity feature reported in simulation by WMM[1] can be seen again.

5 Summary

We have performed a relativistic simulation, in which we investigate dynamical stability of stars in binary against gravitational collapse in first post-Newtonian gravity. As a result, we have obtained the following conclusions.

- Star-crushing effect in binary systems reported by WMM[1] does not occur due to first post-Newtonian effects as predicted by previous researchers. If the maximum density of stars in equilibrium binaries is less than a critical density ρ_c , they are dynamically stable and never collapse due to first post-Newtonian effects.
- Contrary to their report of WMM, the first post-Newtonian tidal effect stabilizes against gravitational collapse as predicted by Baumgarte et al.[5], Flanagan[6] and Thorne[7]. For example, its effects increase the maximum proper mass by $\sim 1 - 2\%$ for the case where two surfaces come into contact.

- The above facts do not seem to change, whether the velocity field is corotational or irrotational. However, this does not mean that the internal velocity field plays no roles in determining equilibrium states. In the case of corotational velocity field, each star has a spin angular momentum with respect to distant inertial observer, and it acts as the centrifugal force weakening the self gravity. On the other hand, in the irrotational case, spin of each star is approximately zero. As a result, the maximum mass of stars for corotational binaries becomes larger than that for irrotational binaries.

Acknowledgments

Numerical computations were performed on FACOM VX/4R in data processing center of National Astronomical Observatory of Japan. The work presented here was initiated during author stayed in University of Illinois, and is now in progress in collaboration with Thomas Bamgarte and Stu Shapiro. He wishes to express his gratitude to Stu Shapiro, Thomas Bamgarte, and Luciano Rezzola for their hospitality during his stay in Illinois.

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Table I. Initial conditions of binaries for numerical simulation.

Model	$\rho_c/10^{-5}$	velocity field	each star	L
Ba	5	corotation	equilibrium	141
Da	15	corotation	equilibrium	97.5
Bb	5	corotation	spherical	125
Db	15	corotation	spherical	96
Bc	5	irrotation	spherical	125
Dc	15	irrotation	spherical	96

Table II. Proper mass M_p of each star in corotational equilibrium binaries. Computations were carried out using different grid numbers $(2N + 1, 2N + 1, N + 1)$, where $N = 40, 50, 60$, and 75 , but fixing location of outer boundary (i.e., fixing L). In the last line, proper mass for spherical stars of the same central density is shown.

$\rho_c/10^{-5}$	2	5	10	15
$N = 40$	1.1954	1.2347	1.2469	1.2451
$N = 50$	1.2004	1.2402	1.2526	1.2509
$N = 60$	1.2031	1.2431	1.2556	1.2539
$N = 75$	1.2051	1.2455	1.2581	1.2565
M_0	1.209	1.250	1.263	1.261
spherical	1.1933	1.2344	1.2482	1.2476

Figure captions

- Fig. 1. (a) proper mass M_p as a function of central density ρ_c , and (b) coordinate radius as function of ρ_c for spherical and polytropic relativistic stars of $\Gamma = 1.4$.
- Fig. 2. Time evolution of ρ_* at center of stars in numerical simulation for spherical star, models A–E.
- Fig. 3. Proper mass as a function of maximum density for each star in corotational equilibrium binaries (filled circles). Solid line denotes proper mass for spherical stars.
- Fig. 4. Time evolution of ρ_{*max} in numerical simulations of corotational equilibrium binaries, models Ba and Da. Solid, dotted and dashed lines denote results carried out using grid numbers of $N = 75, 60$, and 50 , respectively. For model Ba, we set K to be unity initially. For model Da, initial value of K is 1, 0.995 and 0.994 for $N=75$, 1, 0.992 and 0.99 for $N=60$, and 1 and 0.99 for $N=50$.
- Fig. 5. Contour lines of ρ_* and velocity field of (v^x, v^y) in the equatorial plane for model Ba (a) and model Da (b) for the case where $K = 0.994$ initially. Contour lines are drawn for $\rho_*/\rho_{*i} = 10^{-0.3j}$ where ρ_{*i} denotes the maximum at initial conditions and $j = 0, 1, 2 \dots 10$. Arrows denote the local velocity field.
- Fig. 6. Time evolution of ρ_{*max} in numerical simulations of corotational non-equilibrium binaries, models Bb and Db. Initial value of K is 0.98(dotted line), 0.975(solid line) and 0.97(dashed line) for model Bb, and 0.98 (dotted line) and 0.975 (solid line) for model Db. Note that in every case, orbital period is approximately 40 in units of $t_{p0}^{1/2}$.

- Fig. 7. The same as Figs. 5, but for model Bb in the cases where $K = 0.975$ initially.
- Fig. 8. Time evolution of ρ_{max} in numerical simulation of irrotational non-equilibrium binaries, models Bc and Dc. Initial value of K is 0.985 and 0.98 for model Bc, and 0.98 and 0.985 for model Dc. Solid and dotted lines denote results for $K = 0.98$ and 0.985, respectively. Note that in every case, orbital period is approximately 40 in units of $t\rho_0^{1/2}$.
- Fig. 9. The same as Figs. 5, but for model Bc in the case where $K = 0.985$ initially.

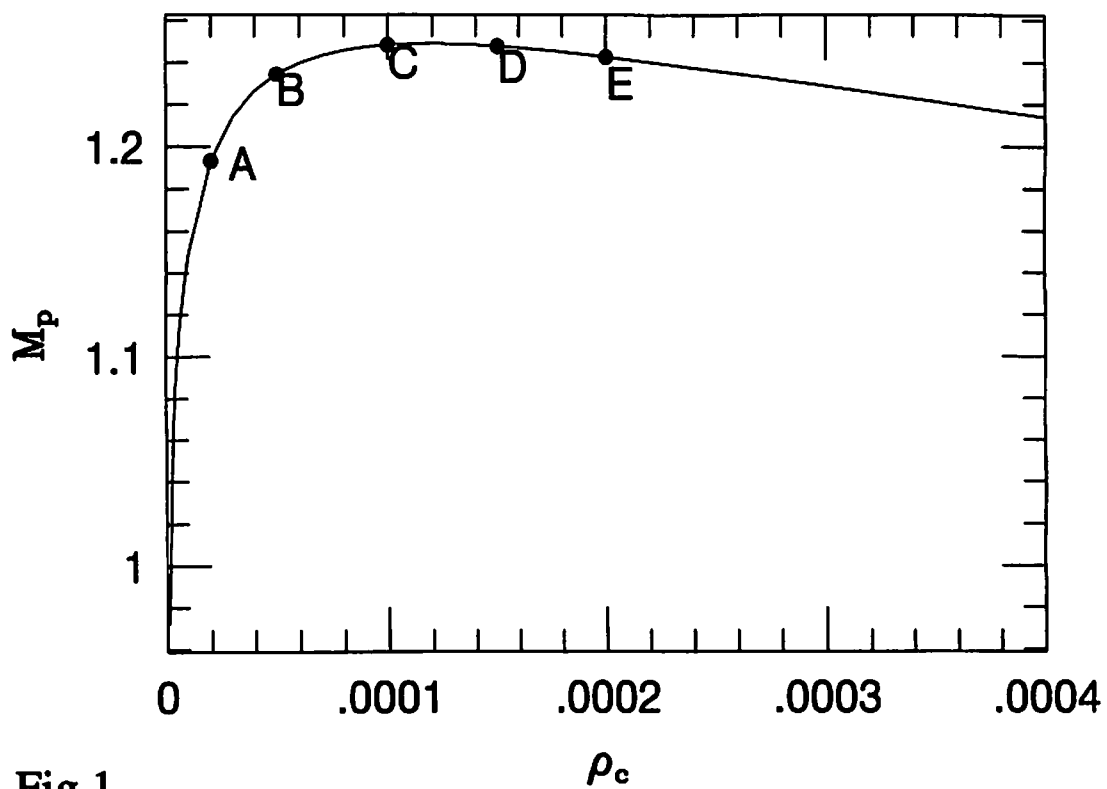
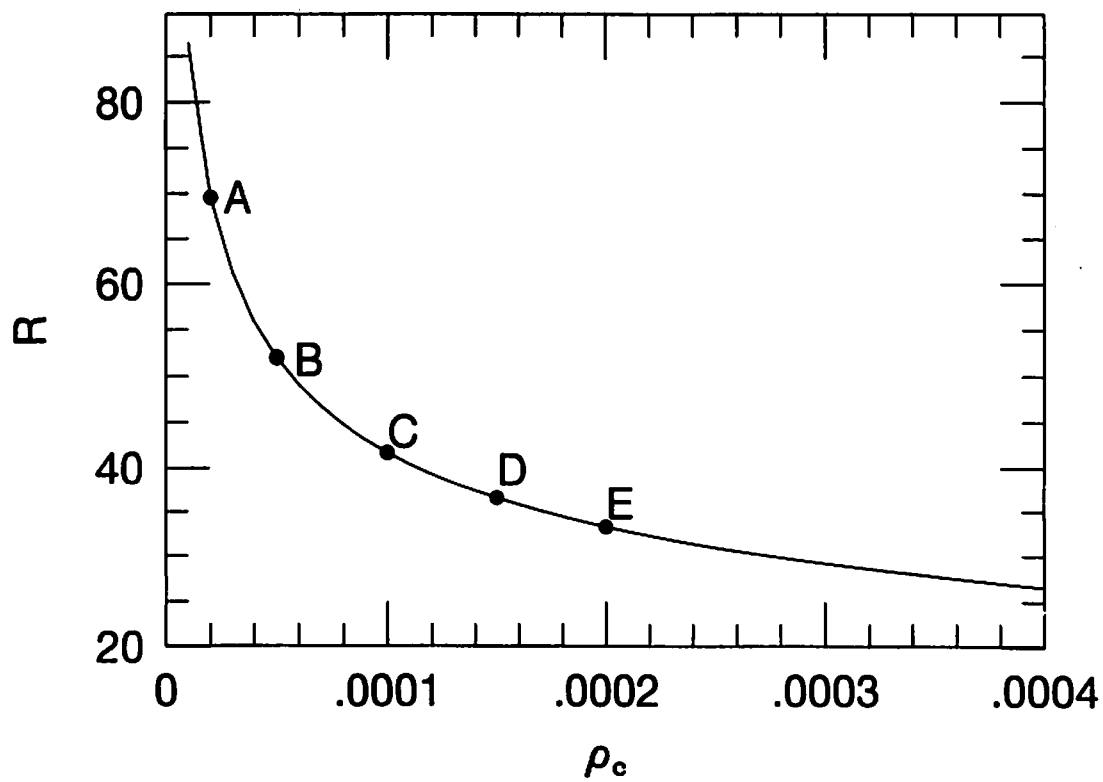


Fig.1



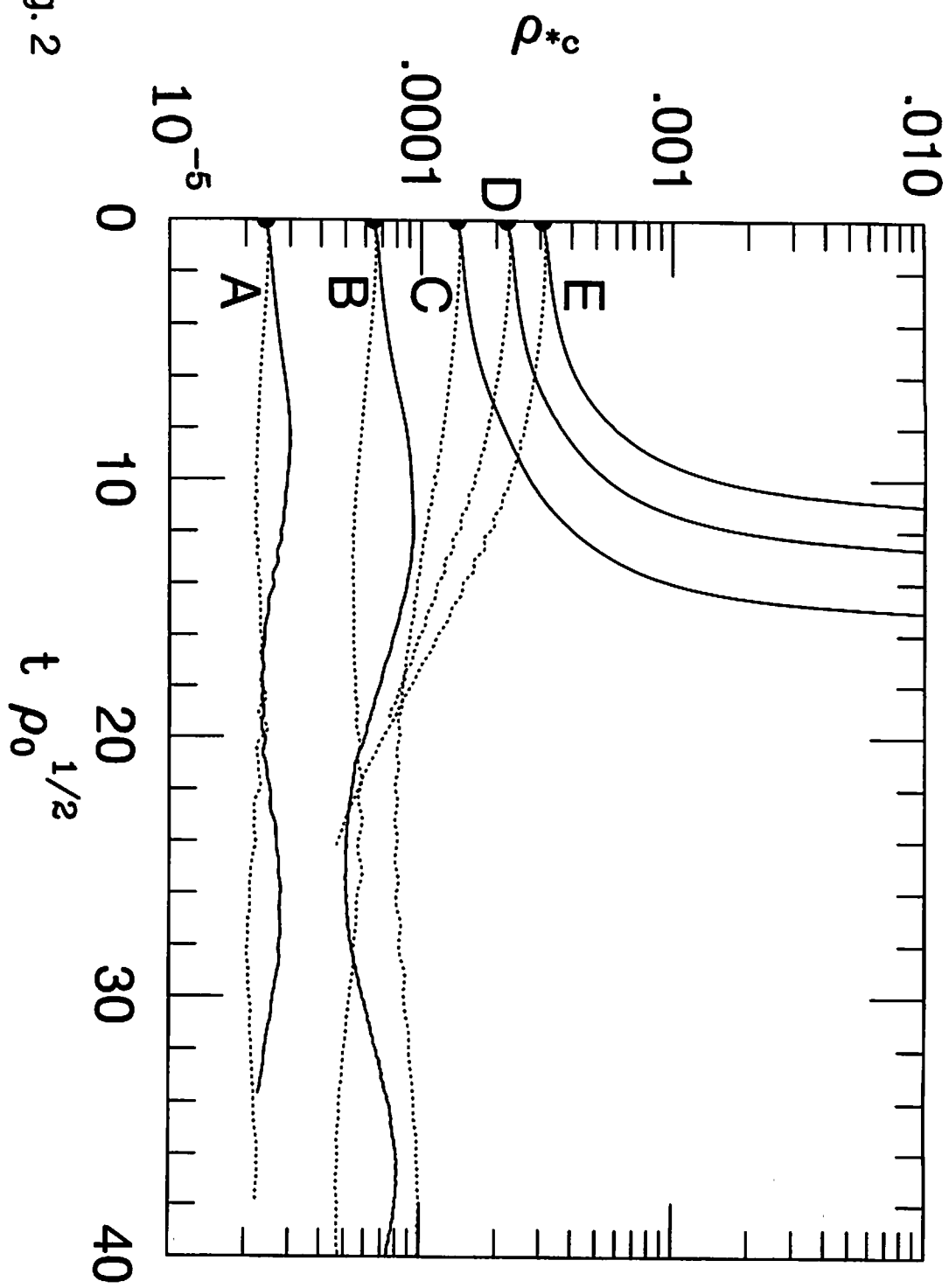


Fig. 2

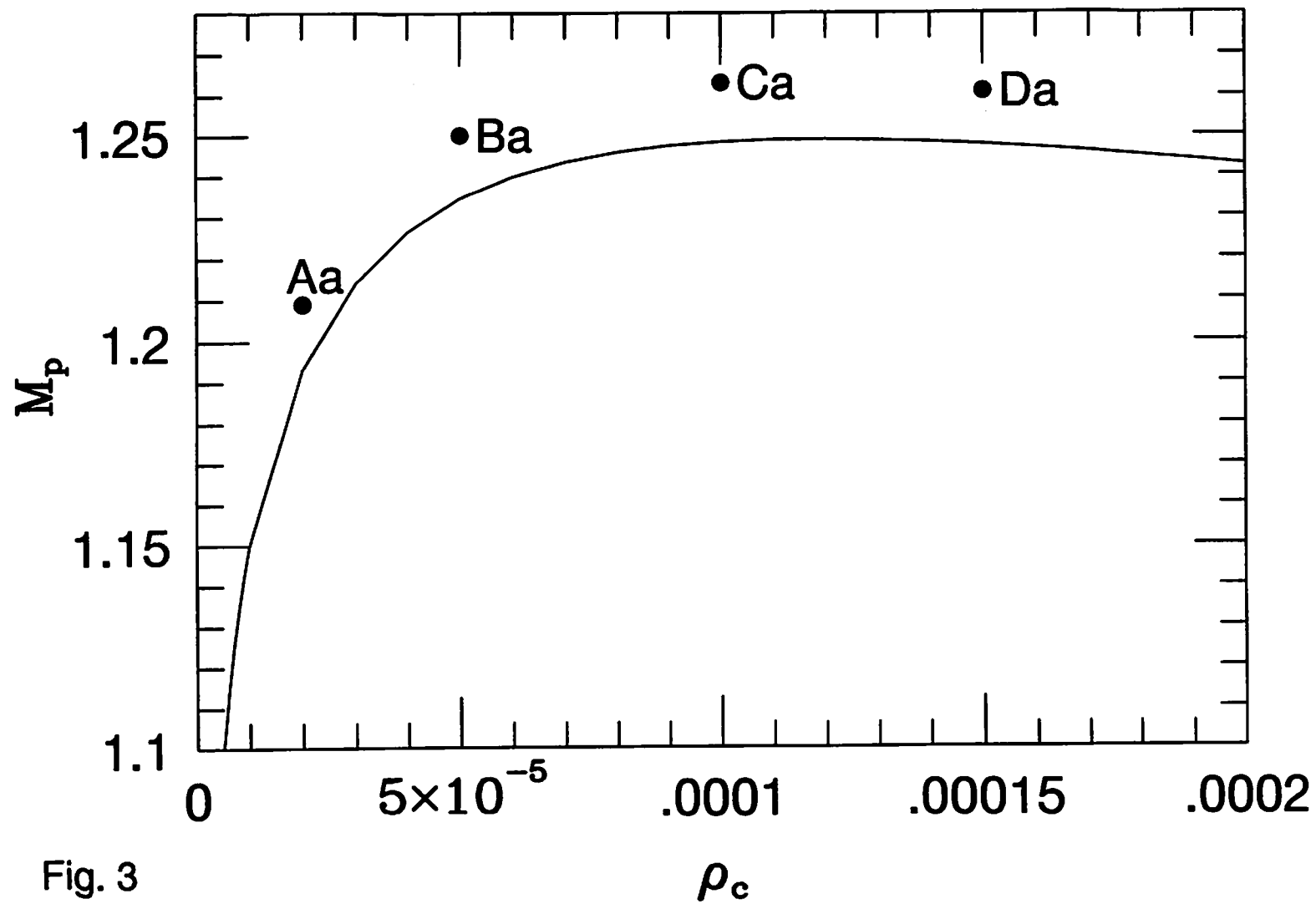


Fig. 3

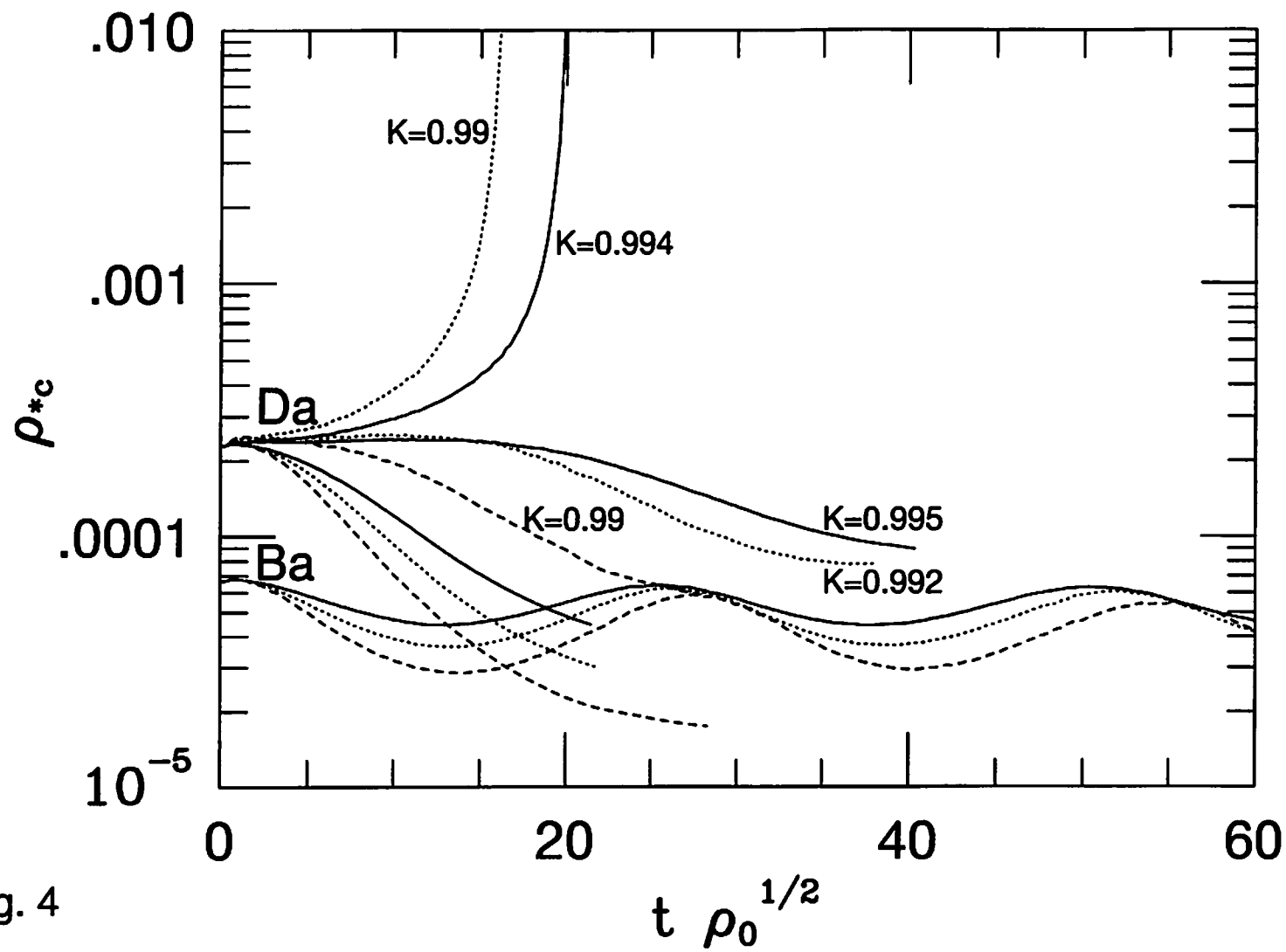


Fig. 4

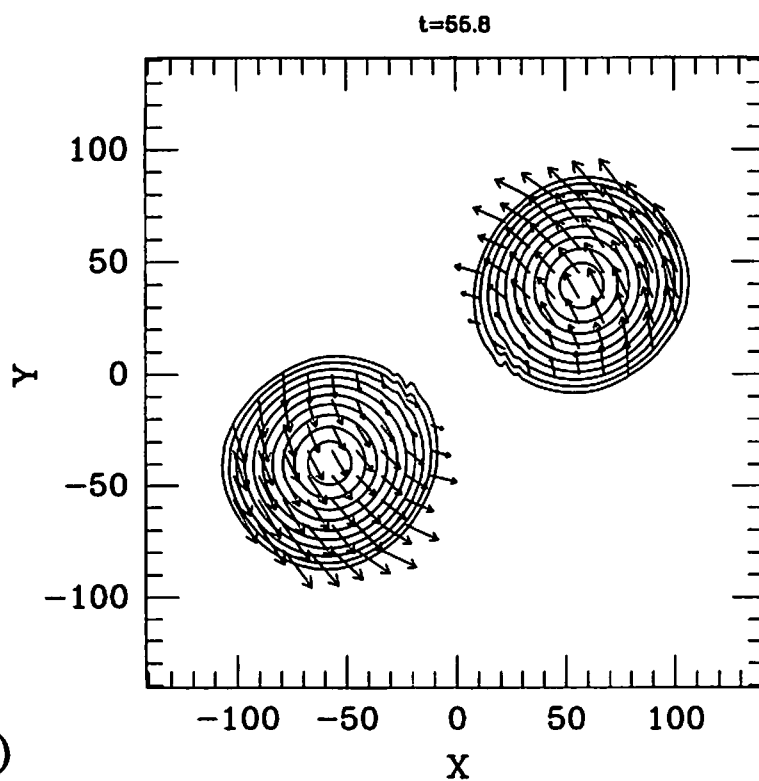
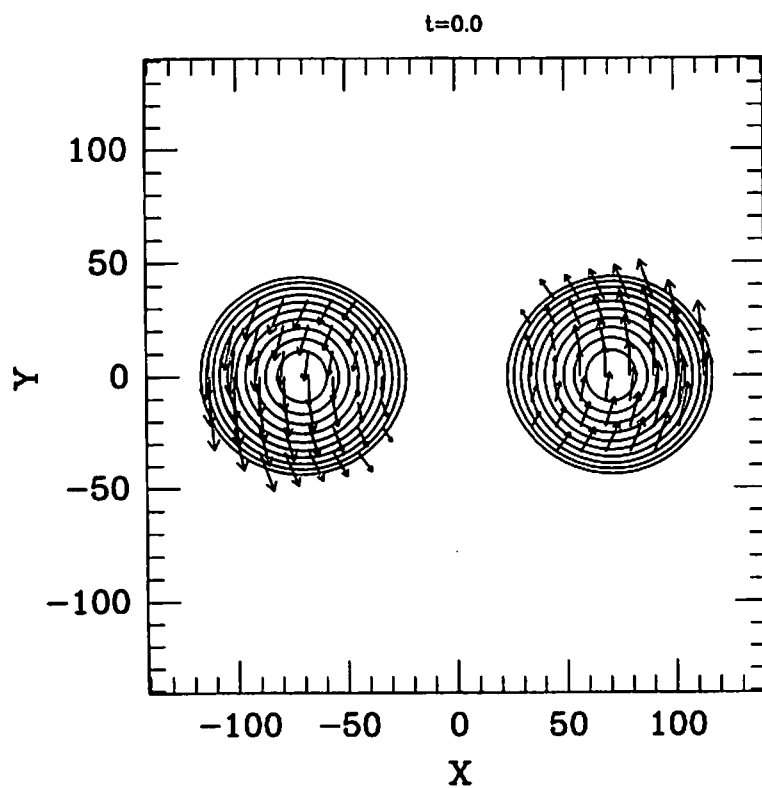


Fig.5(a)

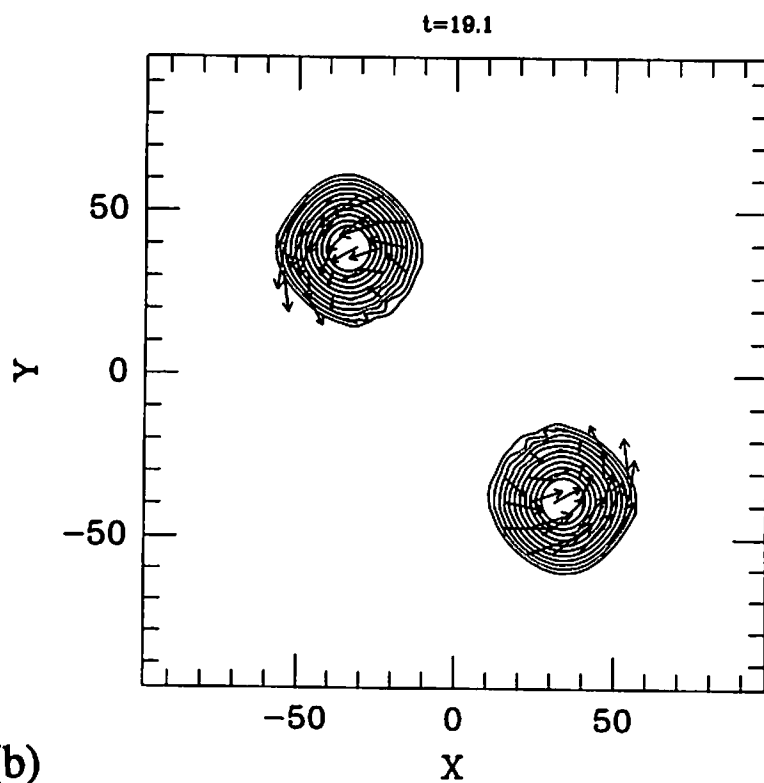
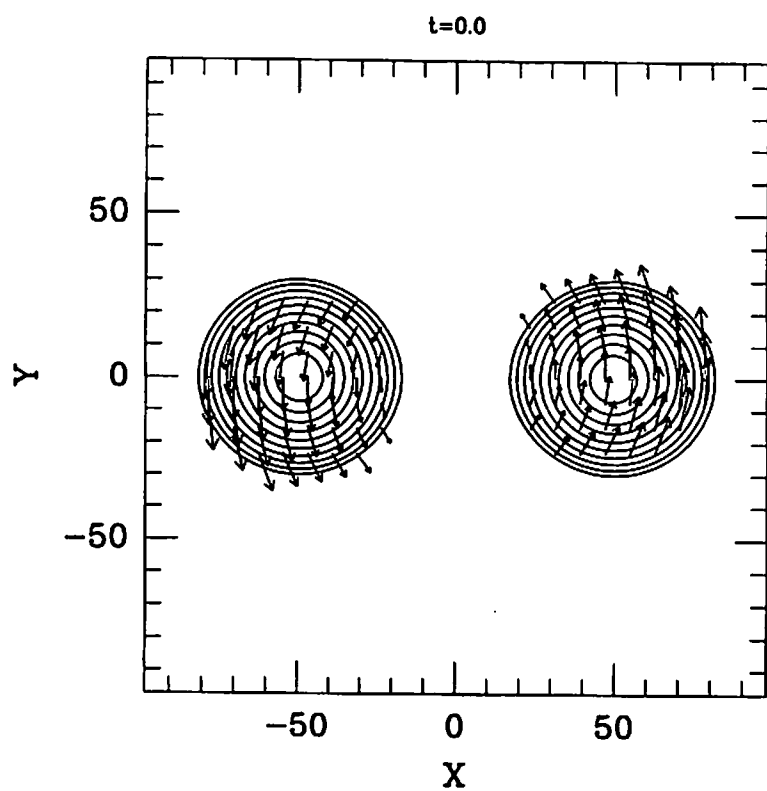


Fig.5(b)

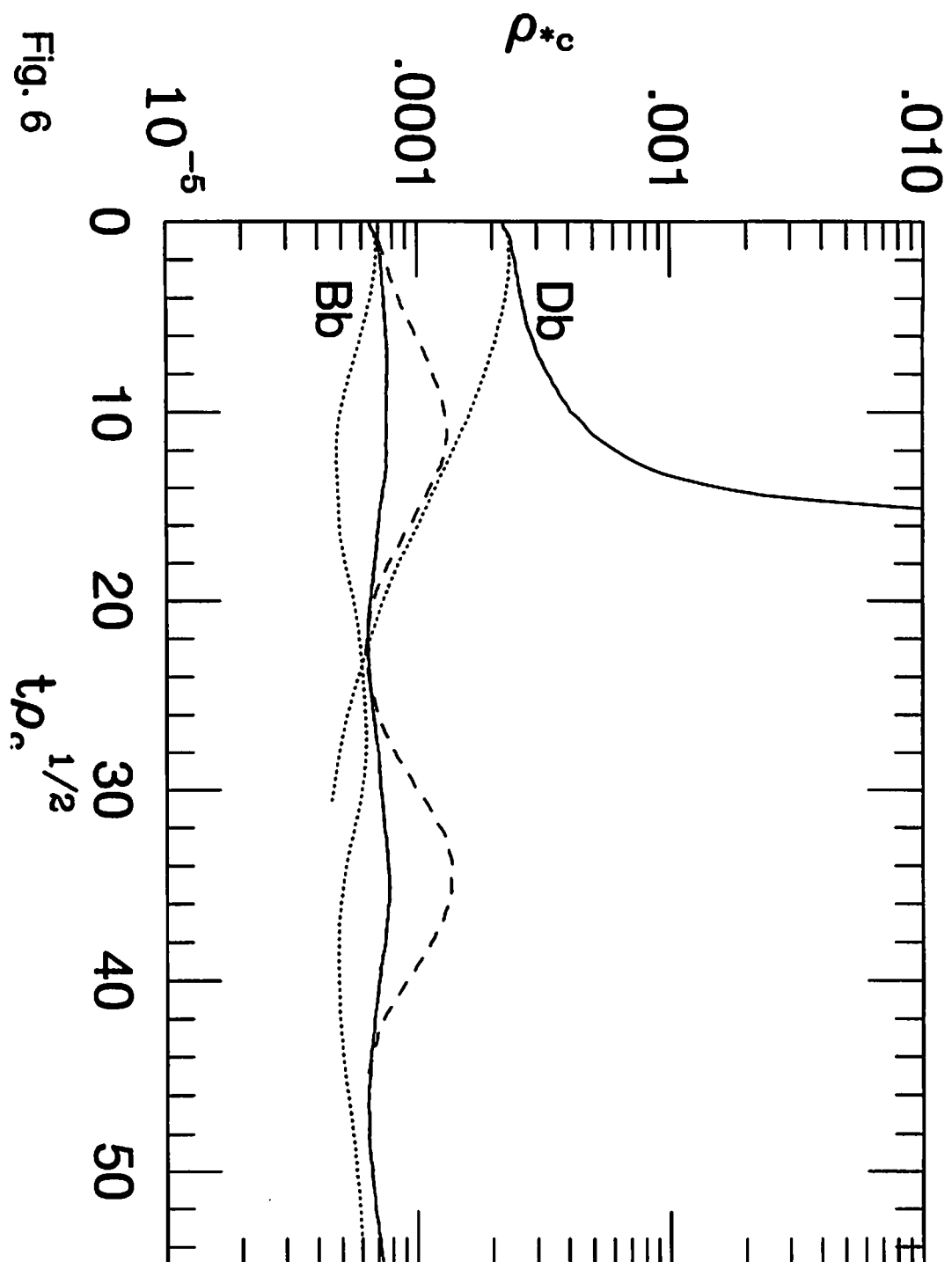


Fig. 6

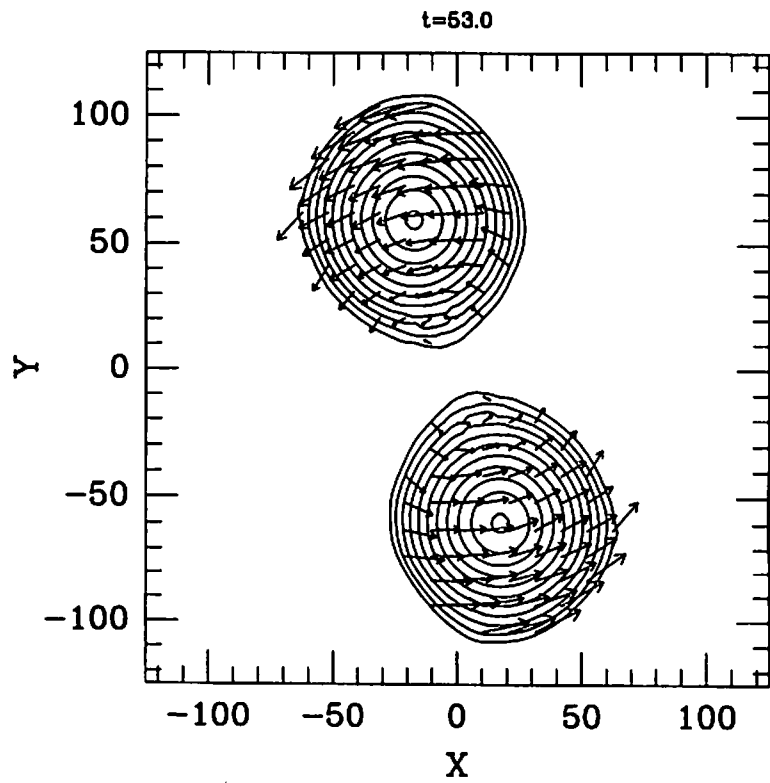
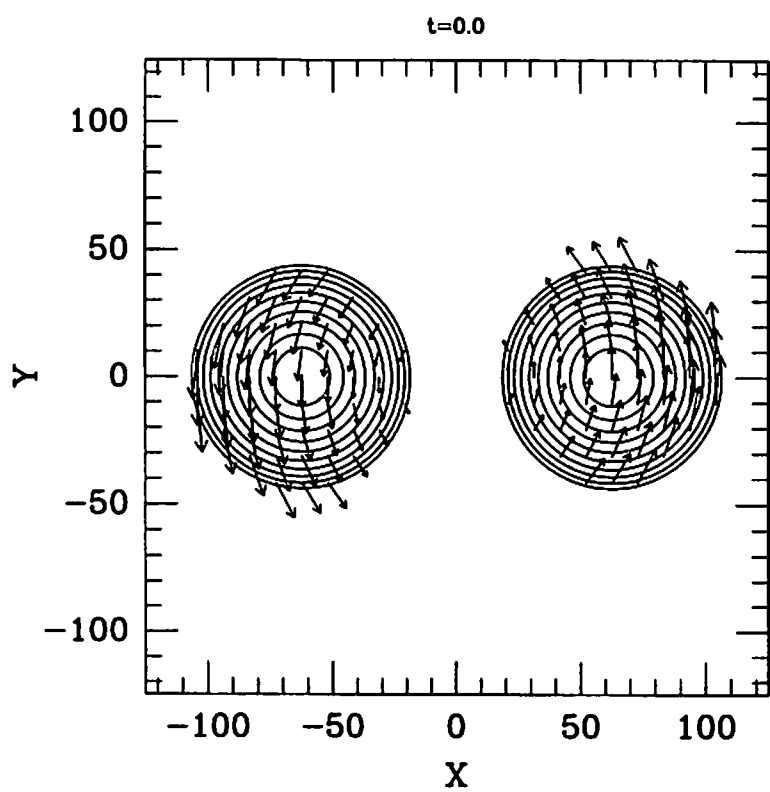


Fig.7

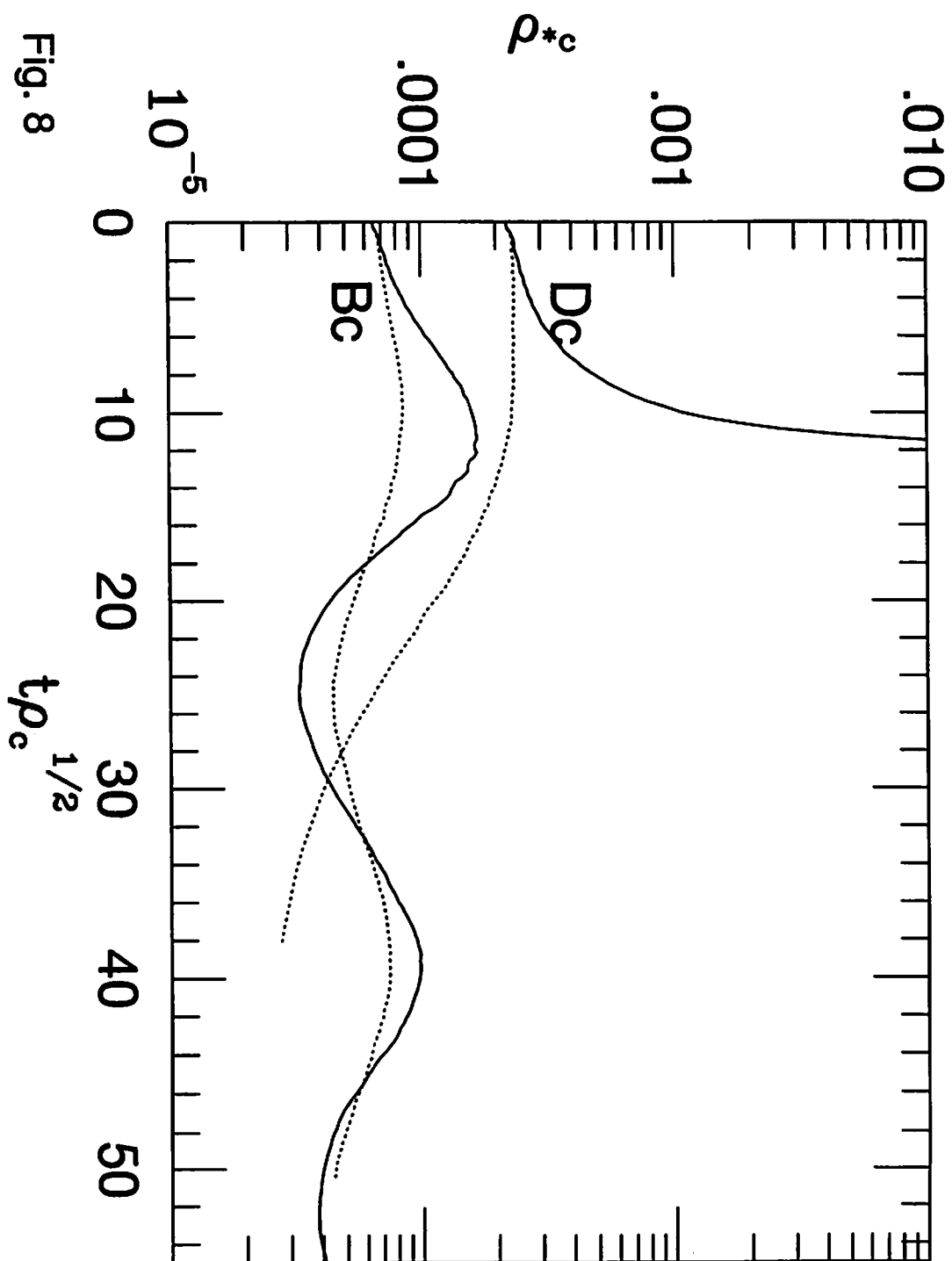


Fig. 8

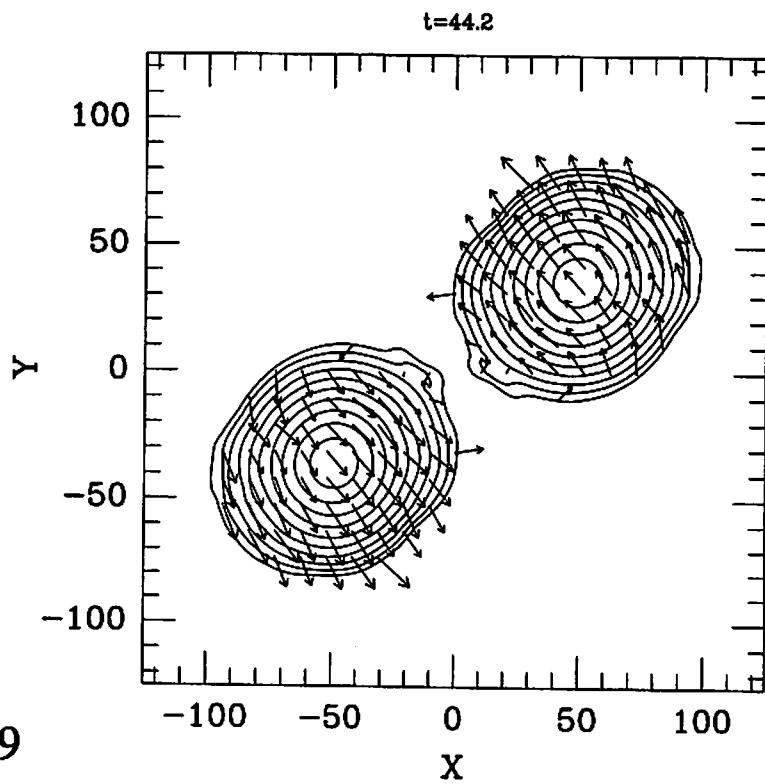
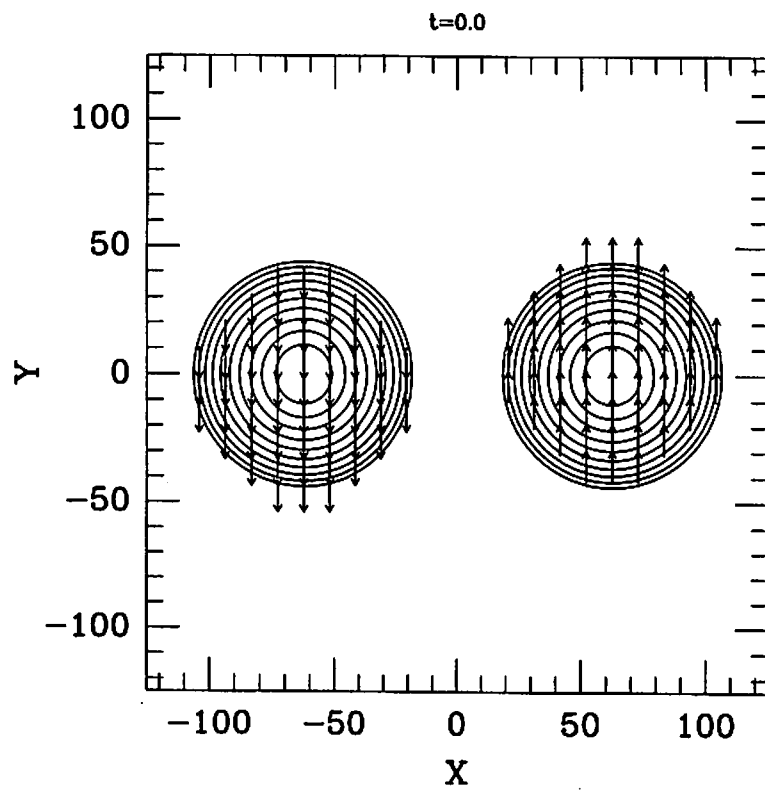


Fig.9

Gravitational waves in the vicinity of a naked singularity

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Abstract

The motion of a spherical dust cloud is described by the Lemaitre-Tolman-Bondi solution and is completely specified by initial data for distributions of a rest mass density and a specific energy of the dust fluid. From generic initial data, there appears a naked singularity at the symmetric center in the course of the gravitational collapse of the dust cloud and hence this seems to be a counter example for the cosmic censorship hypothesis. To investigate the genericity of this example, we examine stability of ‘nakedness’ of this singularity against odd-parity modes of non-spherical linear perturbations for the metric, i.e., linear gravitational waves. We find that the perturbations do not diverge but are well-behaved even in the neighborhood of the central naked singularity. This means that the naked singularity formation process is marginally stable against the odd-parity modes of linear gravitational waves.

1 INTRODUCTION

The naked singularity means that it is observable and a singularity is a boundary of a space-time. In order to obtain a solution of hyperbolic field equations for matters, gauge fields and space-time itself in the causal future of a naked singularity, we need to impose a boundary condition on it. However, we have not yet known the physically reasonable boundary conditions on singularities and hence, to avoid this difficulty, the cosmic censorship hypotheses (CCH) proposed by Penrose [1] is often adopted in the analysis of the physical phenomena of the strong gravitational field. For the latest two decades several researchers have shown that the Lemaitre-Tolman-Bondi (LTB) space-time[2] is one of the candidates for a counter example for the CCH [3]. This space-time describes the motion of a spherically symmetric inhomogeneous dust cloud. These results are summarized as follows; in this space-time, a naked singularity appears from generic initial data for *spherically symmetric* configurations of the rest mass density and specific energy of the dust fluid. In order to recognize this example as a serious counter example for the CCH, we should examine its genericity. That is to say, there is a possibility that the naked singularity is due to physically unrealistic conditions, e.g., assumptions of the spherical symmetry, dust matter and so on.

We concentrate our attention to the issue whether the spherical symmetry is essential to the occurrence of the shell focusing naked singularity. For this purpose, we consider odd-parity modes of non-spherically symmetric perturbations in the marginally bound LTB space-time. Here we consider only the metric perturbations, i.e., linear gravitational waves. We obtain a partially differential equation for a gauge-invariant variable corresponding to the odd-parity metric perturbations of the LTB space-time. We analyze this equation numerically. Then we shall discuss the stability of the LTB space-time with the central naked singularity from the results of these analyses. We adopt the geometrized units, $c = G = 1$. There are full explanations and derivations in Ref.[5]

2 BASIC EQUATIONS

Using the synchronous coordinate system, the line element of LTB space-time is expressed in the form

$$d\bar{s}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu \equiv -dt^2 + A^2(t, r) dr^2 + R^2(t, r) (d\theta^2 + \sin^2 \theta d\phi^2). \quad (1)$$

The energy-momentum tensor for the dust fluid is

$$\bar{T}^{\mu\nu} = \bar{\rho}(t, r) \bar{u}^\mu \bar{u}^\nu, \quad (2)$$

where $\bar{\rho}(t, r)$ is the rest mass density and $\bar{u}^\mu = \delta_0^\mu$ is the 4-velocity of the dust fluid.

Then the Einstein equations and the equation of motion for the dust have two arbitrary functions of r , $f(r)$ and $F(r)$. We know that $F(r)$ is related to the Misner-Sharp mass function[6], $m(r)$, of the dust cloud in the manner $F(r) = 2m(r)$ and that $f(r)$ is recognized as the specific energy of the dust fluid. The motion of the dust cloud is completely specified by the function, $F(r)$, and the specific energy, $f(r)$. Restricting to the case that the symmetric center is initially regular, the central shell focusing singularity is naked if and only if $\partial_r^2 \bar{\rho}|_{r=0} < 0$ is initially satisfied for the marginally bound collapse, $f(r) = 0$ [7].

The perturbation method used in this article is the gauge-invariant one formulated by Gerlach and Sengupta [4] for spherically symmetric background. The perturbed metric tensor is expressed in the form

$$g_{\mu\nu} dx^\mu dx^\nu = \bar{g}_{\mu\nu} dx^\mu dx^\nu + h_a(t, r) \Phi_l^m{}_B dx^a dx^B + h_2(t, r) \chi_l^m{}_{AB} dx^A dx^B, \quad (3)$$

where $\bar{g}_{\mu\nu}$ is a metric tensor of a spherically symmetric background space-time, and where the sub- and superscripts, $a, b, \dots$ represent t and r while $A, B, \dots$ represent θ and ϕ , and where $\Phi_l^m{}_B$ and $\chi_l^m{}_{AB}$ are odd-parity vector and tensor harmonics[8]. The energy-momentum tensor is written in the form

$$T_{\mu\nu} dx^\mu dx^\nu = \bar{T}_{ab} dx^a dx^b + \frac{1}{2} \bar{T}_B^B R^2(t, r) d\Omega^2 + t_a(t, r) \Phi_l^m{}_B dx^a dx^B + t_2(t, r) \chi_l^m{}_{AB} dx^A dx^B, \quad (4)$$

where $\bar{T}_{\mu\nu}$ is a background quantity. In the LTB space-time, we obtain a single decoupled wave equation,

$$\partial_t \left(\frac{A}{R^2} \partial_t (R^4 \psi_s) \right) - \partial_r \left(\frac{1}{AR^2} \partial_r (R^4 \psi_s) \right) + (l-1)(l+2) A \psi_s = -16\pi \partial_r \left(\frac{1}{AR^2} \frac{dJ}{dr} \right), \quad (5)$$

where $J(r)$ is an arbitrary function and we have introduced gauge-invariant variable, ψ_s , defined by

$$\psi_s \equiv \frac{1}{A} \left[\partial_t \left(\frac{k_1}{R^2} \right) - \partial_r \left(\frac{k_0}{R^2} \right) \right], \quad (6)$$

where k_a is a gauge-invariant metric variable introduced by Gerlach and Sengupta.

From the analysis of the regularity at the center, we find that ψ_s behaves near the center as

$$\psi_s \longrightarrow \psi_{s,c}(t) r^{l-2} + O(r^l) \quad \text{for } l \geq 2, \quad (7)$$

$$\psi_s \longrightarrow \psi_{s,c}(t) r + O(r^3) \quad \text{for } l = 1. \quad (8)$$

We note that only the quadrupole mode of ψ_s does not vanish at the center. From the regularity conditions for the perturbations of the Riemann tensor, the central naked singularity formation is affected only by the quadrupole mode up to the linear order. Therefore, we shall consider the quadrupole mode only.

3 NUMERICAL RESULTS AND DISCUSSIONS

We shall perform numerical integration of Eq.(5) for the quadrupole mode. Here we restrict our investigation to the case of no matter perturbations, namely the right hand side of Eq.(5) vanishes. Further, for calculational simplicity, we consider only the marginally bound case, $f(r) = 0$, as background space-time

Then, the wave equation (5) becomes as follows,

$$\begin{aligned} \frac{\partial^2 \psi_s}{\partial t^2} - \frac{1}{(\partial_r R)^2} \frac{\partial^2 \psi_s}{\partial r^2} &= \frac{1}{(\partial_r R)^2} \left(6 \frac{\partial_r R}{R} - \frac{\partial_r^2 R}{\partial_r R} \right) \frac{\partial \psi_s}{\partial r} - \left(6 \frac{\partial_t R}{R} + \frac{\partial_t \partial_r R}{\partial_r R} \right) \frac{\partial \psi_s}{\partial t} \\ &\quad - 4 \left[\left(\frac{\partial_t \partial_r R}{\partial_r R} \right) \frac{\partial_t R}{R} + \left(\frac{\partial_t R}{R} \right)^2 + \frac{\partial_t^2 R}{R} \right] \psi_s. \end{aligned} \quad (9)$$

We solve this partially differential equation numerically.

3.1 Background Density Profile and Numerical Procedure

The background space-time and the motion of the marginally bound dust cloud are completely determined by the initial rest mass density profile, $\bar{\rho}(0, r)$. We adopt the following initial rest mass density profile so that the central naked singularity appears;

$$\bar{\rho}(0, r) = \begin{cases} \rho_0 [1 - 2(r/r_b)^2 + (r/r_b)^4] & \text{for } 0 \leq r \leq r_b \\ 0 & \text{for } r > r_b, \end{cases} \quad (10)$$

where ρ_0 is a positive constant and r_b denotes the radial coordinate at the surface of the dust cloud. The total (gravitational) mass of the dust cloud is

$$M = m(r_b) = \frac{32\pi}{105} \rho_0 r_b^3. \quad (11)$$

The time of the central naked singularity formation is $t_0(0) = 1/\sqrt{6\pi\rho_0}$. Whether the naked singularity is global or not is determined by a non-dimensional constant $\rho_0 r_b^2$. It is known that the singularity is globally naked for sufficiently small $\rho_0 r_b^2$ [7]. We shall consider the case of $\rho_0 r_b^2 = 3 \times 10^{-2}$, which corresponds to the globally naked singularity. The initial radius of the dust cloud and the time of the central naked singularity formation are given by $R(0, r_b)/M = r_b/M \cong 34.8$ and $t_0(0)/M \cong 46.3$, respectively.

Next, we describe the procedure of our numerical calculation. Instead of the (t, r) coordinate system, we introduce a single-null coordinate system, (u, r') , where u is an out-going null coordinate and chosen so that it agrees with t at the symmetric center and we choose $r' = r$. We perform the numerical integration along two characteristic directions. Using this coordinate system, Eq.(9) is expressed in the form

$$\begin{aligned} \frac{d\phi_s}{du} &= -\frac{\alpha}{R} \left[3\partial_r R + \frac{1}{2} R(\partial_t R)\partial_t \partial_r R - (\partial_t R)^2 \partial_r R + \frac{1}{2} R(\partial_r R)\partial_t^2 R \right] \psi_s \\ &\quad - \frac{\alpha}{2} \left[\frac{\partial_r^2 R}{(\partial_r R)^2} - \frac{2}{R} (1 - \partial_t R) \right] \phi_s, \end{aligned} \quad (12)$$

$$\partial_{r'} \psi_s = \frac{1}{R} \phi_s - 3 \frac{\partial_r R}{R} (1 + \partial_t R) \psi_s, \quad (13)$$

where the ordinary derivative in the left hand side of Eq.(12) is defined by

$$\frac{d}{du} = \partial_u + \frac{dr'}{du} \partial_{r'} = \partial_u - \frac{\alpha}{2\partial_r R} \partial_{r'}, \quad (14)$$

and we have introduced a new variable, ϕ_s , defined by Eq.(13).

The procedure of the numerical integration is as follows. First, we prepare initial data corresponding to imploding waves for ϕ_s at each grid point on the initial null hypersurface labeled by $u = u_0 = \text{constant}$. Using this ϕ_s , ψ_s is obtained at each grid point on $u = u_0$ by the integration of Eq.(13). Next, in order to obtain ϕ_s at each grid point on $u = u_0 + \Delta u$, we integrate Eq.(12) by using values of ϕ_s and ψ_s on $u = u_0$. Then, ψ_s on $u = u_0 + \Delta u$ is obtained from Eq.(13) by using of ϕ_s on $u = u_0 + \Delta u$. We repeat this procedure and obtain a solution outside the Cauchy horizon associated with the central naked singularity.

In the above procedure, we should impose a boundary condition at the center to perform the numerical integration of Eq.(13). From Eqs.(7), in the case of $l = 2$ mode, ψ_s behaves in the manner

$$\psi_s \longrightarrow \psi_{s0} + O(r^2) \quad \text{for} \quad r \longrightarrow 0. \quad (15)$$

Hence, we have to numerically make ψ_s near the center so that Eq.(15) is guaranteed on a surface of $t = \text{constant}$, and this leads to the boundary condition for ψ_s at the center.

3.2 Initial Conditions and Numerical Results

The initial conditions we use are the Gaussian-shaped wave packet with respect to the coordinate, r' ,

$$\psi_s|_{u=u_0} = \psi_s^i \exp \left[-\frac{(r' - r_c')^2}{2\sigma^2} \right], \quad (16)$$

where ψ_s^i , σ , and r_c' are constants and characterize the amplitude, width and initial position of the initial wave packet, respectively. The initial null hypersurface, $u = u_0$, is chosen so that it includes a world point $(t, r) = (0, 0)$, except for the analysis of the scattered waves which will be discussed in this section.

We shall investigate three kinds of initial positions of the wave packet on the initial null hypersurface. In Case 1, the wave packet reaches the center of the dust cloud before the formation of the central naked singularity. In Case 2, a significant portion of the wave packet hits the central naked singularity. In Case 3, the packet does not hit the central naked singularity but reaches Cauchy horizon associated with it.

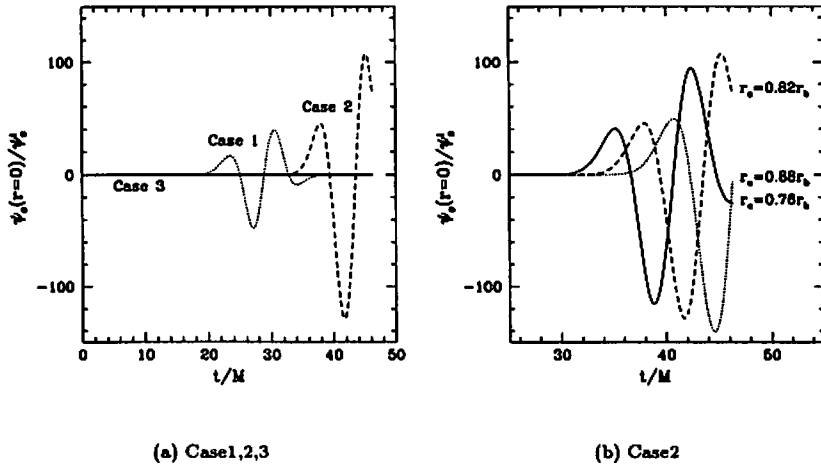


Figure 1: Plots of the gauge-invariant variable, ψ_s , at the center, $r = 0$, with an initial width $\sigma = 0.05r_b$ for the Cases 1-3 for globally naked cases. In (a), the dotted line denotes Case1 ($r'_c = 0.5r_b$), the broken line shows Case2 ($r'_c = 0.82r_b$), and the solid line denotes Case3 ($r'_c = 1.2r_b$). In (b), the results of Case2 are shown in more detail. The broken line in (b) is the same as the broken line in (a). The solid and dotted lines show the cases that the wave packets were put on the initial surface at $r'_c = 0.76r_b$ and at $r'_c = 0.88r_b$, respectively.

In each case, the value of ψ_s at the center is plotted as a function of the coordinate time, t , in Fig.1. Note that it is impossible to perform the numerical calculation in the causal future of the central naked singularity. Therefore we plot ψ_s at the center only before the occurrence of the central naked singularity. Although such a difficulty exists, we find that violent growth of the amplitude of ψ_s is not observed near the central naked singularity and Cauchy horizon associated with it.

Next we shall show the dependence of ψ_s at the center on the width of initial wave packet. Fig.2 depicts ψ_s at the center for various widths of packets in Case 2. It is found from this figure that the amplitude of ψ_s with smaller initial width becomes larger at the center. The relation between the width, σ , and the maximal value of $|\psi_s|$ at the center is shown in Fig.3. We can see that there is the following power-law relation

$$|\psi_s|_{r=0} \propto \sigma^{-3}. \quad (17)$$

We also observe the time dependence of ψ_s along the line of a constant circumferential radius outside of the dust cloud. Since we would like to see the effect of the central naked singularity on ψ_s , we consider the globally naked case only. We set up an

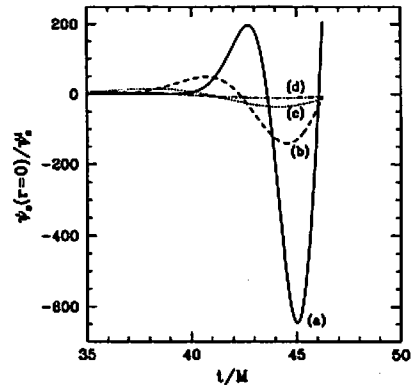


Figure 2: Plots of the ψ_s at the center in the LTB space-time for various widths of initial wave packets. The wave packets are put at $r = 0.88r_b$ on the initial null surface. The solid line (a) is the wave form for the initial width, $\sigma = 0.03r_b$. The broken line (b) is that of $\sigma = 0.05r_b$ while the dotted line (c) is that of $\sigma = 0.08r_b$. The broken dotted line (d) is that of $\sigma = 0.12r_b$.

initial wave packet of $\sigma = 0.05r_c$ at $R = 100M$ on the initial null hypersurface which does not include the space-time point $(t, r) = (0, 0)$ but is chosen so that the wave packet will reach the neighborhood of the central naked singularity.

The results are shown in Fig.4 in which ψ_s at $R = 100M$ is plotted as a function of t . Note that the point, $R = 100M$, is located in the vacuum region which is the Schwarzschild space-time by Birkhoff's theorem. Hence the value of t along the curve of $R = 100M$ agrees with that of the usual static time coordinate of the Schwarzschild space-time. In Fig.4(a), the solid line corresponds to Case 1 while the broken line is for Case 2. The dotted line denotes the result for Case 3. In Case 3, the right-hand peak does not exist and this is because, in this case, almost all portions of incident waves enter into the Cauchy horizon associated with the central naked singularity. Fig.4(b) shows detailed behavior of the scattered ψ_s for Case 2. It is the most important fact seen in these figures that the amplitude of scattered waves is almost the same as the initial incident waves in Cases 1 and 2.

3.3 Minkowski Case

Here we shall investigate the behavior of ψ_s in the Minkowski space-time and compare it with the results of the LTB space-time obtained in the above in order to reveal the effects of the existence of the dust cloud and the central naked singularity on the dynamics of ψ_s . In the Minkowski case, Eq.(9) becomes

$$\partial_t^2 \psi_s - \partial_r^2 \psi_s = \frac{6}{r} \partial_r \psi_s. \quad (18)$$

The solution of this equation which is regular at $r = 0$ is obtained in the form

$$\psi_s = 3 \frac{f(t-r) - f(t+r)}{r^5} + 3 \frac{f^{(1)}(t-r) + f^{(1)}(t+r)}{r^4} + \frac{f^{(2)}(t-r) - f^{(2)}(t+r)}{r^3}, \quad (19)$$

where $f(x)$ is an arbitrary function and $f^{(n)}(x)$ denotes the n -th order derivative of $f(x)$ with respect to x . We shall set the following initial wave packet on the null hypersurface, $t = r$,

$$\psi_s = \exp \left[-\frac{(r-r_c)^2}{2\sigma^2} \right]. \quad (20)$$

Using the above solution, we shall compare the evolution of wave forms in the Minkowski space-time with that in the LTB space-time. Since the circumferential radius, R , is tightly related with the behavior of the amplitude of the wave, we should set the same initial data with respect to the R both for the LTB and Minkowski cases. Hence first we plot the initial wave packet (16) as a function of R/M on the initial null hypersurface and then the value of the σ and r_c in Eq.(20) is adjusted so that the initial wave form fits well with that of the LTB case.

First we consider the evolution of ψ_s at the center. Using Eqs.(19) and (20), we obtain ψ_s at the center in the form

$$\begin{aligned} \psi_s(t, 0) = & \left[1 - \frac{1}{\sigma^2} \left(\frac{t}{2} - r_c \right) \frac{t}{2} - \frac{1}{4\sigma^2} \left(\frac{t}{2} \right)^2 + \frac{1}{4\sigma^4} \left(\frac{t}{2} - r_c \right)^2 \left(\frac{t}{2} \right)^2 \right. \\ & \left. + \frac{1}{20\sigma^4} \left(\frac{t}{2} - r_c \right) \left(\frac{t}{2} \right)^3 - \frac{1}{60\sigma^6} \left(\frac{t}{2} - r_c \right)^3 \left(\frac{t}{2} \right)^3 \right] \exp \left[-\frac{1}{2\sigma^2} \left(\frac{t}{2} - r_c \right)^2 \right]. \quad (21) \end{aligned}$$

The parameters, σ and r_c , in Eq.(20) are chosen so that the initial wave packets fit well with those of the Cases 1 and 2 of Fig1. The results are given in Fig.5(a) and (b), respectively. It should be noted that there is scarcely any difference between the wave forms of the Minkowski and LTB cases.

Next we consider the behavior of ψ_s at a finite circumferential radius. Here the wave form is obtained numerically by the same procedure as in the LTB case. The result is shown in Fig.6. We find that there is a little difference of the phase between the Minkowski and LTB cases. However, the behavior of ψ_s in the Minkowski case is basically the same as that in the LTB case. The effect due to the dust cloud and the existence of the central naked singularity on the propagation of ψ_s is rather small.

We consider the relation between the maximum value of $|\psi_s|$ at the center and the width of an initial wave packet in the Minkowski space-time. This relation is obtained from Eq.(21). The results are also depicted in Fig.3. The power-law relation Eq.(17) is also valid in the Minkowski case. From Eq.(19), ψ_s is approximately proportion to $1/r^3$ except for the region of $r \lesssim \sigma$. If the initial amplitude of the wave packet has a value ψ_i at $r = r_c$, then the value of ψ_s at $r = \sigma$ is roughly estimated as $\psi_i \times (\sigma/r_c)^{-3}$. This will be why the relation (17) holds in the Minkowski space-time. As we have discussed above, ψ_s behaves outside the Cauchy horizon of the LTB space-time in approximately the same manner as in the Minkowski space-time. Therefore this would be also why the relation (17) holds in the LTB space-time.

As a result, we conclude that even in the neighborhood of the central naked singularity and of the Cauchy horizon associated with it, the metric perturbation, ψ_s , does not show any peculiar behavior. However, we should note that ψ_s does not vanish in the neighborhood of the central naked singularity although it is well-behaved. Therefore, the formation process of the central naked singularity is *marginally stable* against the odd-parity metric perturbations.

4 CONCLUSIONS

We have investigated the behavior of the odd-parity linear perturbations in the LTB space-time. We have derived the wave equation for the gauge-invariant variable, ψ_s . Then we have performed numerical experiments how a Gaussian-shaped incident wave packet behaves under this wave equation for $l = 2$ mode without matter perturbations. From those numerical experiments, we have obtained the following results. When this wave packet approaches to the center, its amplitude becomes larger but finite. The amplitude at the center depends on the width of the initial wave packet according to the power law. On the other hand, when the incident wave packet initially located outside the dust cloud returns back to the same circumferential radius as the initial one, the amplitude of the returned wave is almost equal to that of the incident one.

In order to reveal the characteristic effects of the LTB space-time on the behavior of ψ_s , we have also investigated ψ_s in the Minkowski space-time. Then we have found that in the outside of the Cauchy horizon associated with the central naked singularity, the behavior of ψ_s in the LTB space-time seems not to be so different from that in the Minkowski space-time at least except for the extreme neighborhood of the naked singularity. Therefore the power-law dependence in the LTB space-time described above is basically realized by the analytical discussion about the case of the Minkowski space-time. Further the propagation effect due to the existence of the dust cloud and occurrence of the central naked singularity is rather small. In other words, there is no peculiar behavior of ψ_s even in the neighborhood of the central naked singularity. However, it should be noted that the odd-parity metric perturbation does not vanish in the neighborhood of the central naked singularity and Cauchy horizon associated with it. As a result, we conclude that the central naked singularity formation in the LTB space-time is 'marginally' stable against the odd-parity metric perturbations.

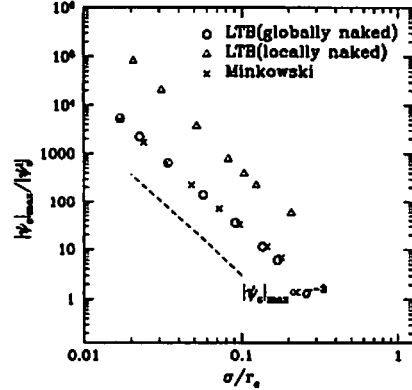


Figure 3: The relation between the widths of initial wave packets and the maximal values of $|\psi_s|$ at the center. The results for the case of the LTB space-time with globally naked singularity are marked by open circles. The locally naked case are marked by the triangles. The results of the Minkowski space-time are marked by cross marks. The broken line denotes the relation, $|\psi_s|_{\max} \propto \sigma^{-3}$.

ACKNOWLEDGMENTS

We would like to thank T. Nakamura, M. Sasaki, M. Shibata, T. Tanaka, M. Siino and T. Chiba for helpful and useful discussions. We are also grateful to H. Sato and colleagues in the theoretical astrophysics group in Kyoto University for useful comments and encouragement.

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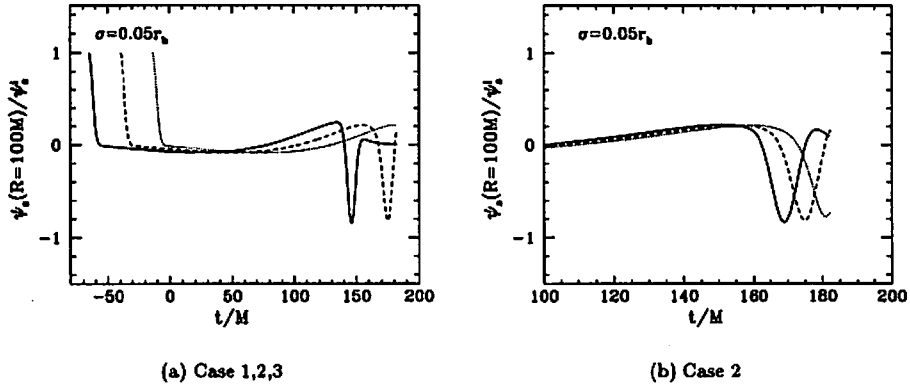


Figure 4: Plots of ψ_* with an initial width $\sigma = 0.05r_s$ at $R = 100M$ as a function of time, t , in the LTB space-time. In (a), the solid line shows the result for the case with an initial time when we put the wave packet on the initial surface, $t_i/M = -65.310$ (Case1), the broken line is for $t_i/M = -38.529$ (Case2), the dotted line is for $t_i/M = -13.610$ (Case3). (b) depicts the details of the Case2. The solid line shows the plot of ψ_* with the initial time, $t_i/M = -34.336$, the broken line is for $t_i/M = -38.529$, the dotted line is for $t_i/M = -44.677$. We find no diverging tendency of the gauge-invariant, ψ_* , when it approaches the Cauchy horizon.

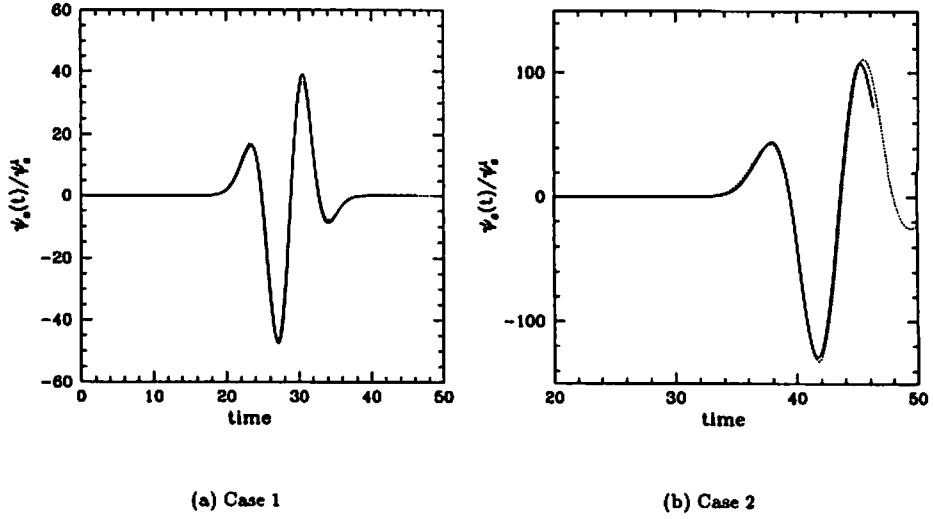


Figure 5: Wave forms at the center both for the Minkowski and LTB cases are shown. In (a), the solid line shows the LTB case, as a function of t/M , that is identical with Case 1 in Fig.1(a). The dotted line shows the corresponding one of the Minkowski case, as a function of t , where $\sigma = 1.18$ and $r_c = 13.9$. In (b), the solid line is the LTB case and identical with Case 2 in Fig.1(a). The dotted line is the Minkowski case where $\sigma = 1.25$ and $r_c = 21.4$.

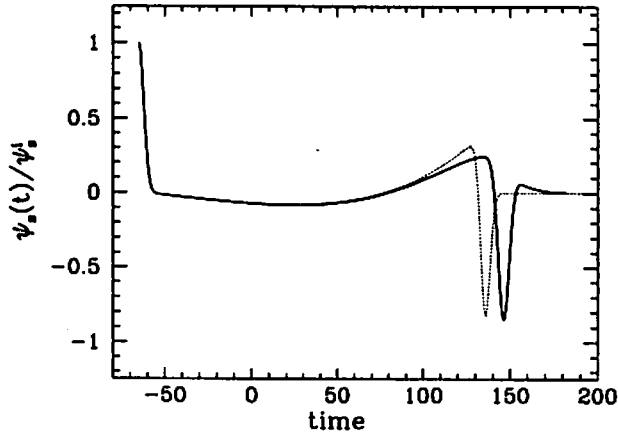


Figure 6: Wave forms along the constant circumferential radius both for the Minkowski and LTB cases are plotted. The solid line shows the LTB case, as a function of t/M , that is identical with Case 1 in Fig.4 (a). The dotted line shows the corresponding one of the Minkowski case, as a function of t , where $\sigma = 1.33$ and $r_c = 100$.

Neutral Points of CFS Instability of Rapidly Rotating Stars in General Relativity – Relativistic Cowling Approximation –

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Abstract

Relativistic Cowling approximation in which all metric perturbations are omitted is applied to non-axisymmetric infinitesimal oscillations of uniformly rotating general relativistic polytropes.

Frequencies of lower order f-modes which are important in analysis of secular instability driven by gravitational radiation are investigated and neutral points of the mode along equilibrium sequences of rotating polytropes are determined. Since this approximation becomes more accurate as stars are more relativistic and/or as they rotate more rapidly, we will be able to analyze how a rotation period of a neutron star may be limited by this instability.

Possible errors in determining neutral points caused by omitting metric perturbations are also estimated.

1 Introduction

Since the discovery of non-axisymmetric secular instability of rotating stars driven by gravitational radiation (so called *Chandrasekhar-Friedman-Schutz (CFS) instability*; see [1, 3, 4]), its possibility to limit a rotational period of a neutron star has been pointed out by several authors. As the instability sets in at *neutral points* in the absence of viscosity, it would become the main issue to determine these neutral points for rapidly rotating stars in general relativity. However, thus far, only several investigations in Newtonian gravity [5, 9] and one post-Newtonian investigation [2] have been made, because it has been very difficult to manage general relativistic perturbations of rotating stars completely.

In such a situation, Yoshida & Kojima [14] have examined the accuracy of the relativistic version of the Cowling approximation for oscillations of slowly rotating relativistic stars. Comparing the results by the *relativistic Cowling approximation* (hereafter RCA) with ‘exact’ eigenfrequencies obtained by fully relativistic analysis, they argued that the approximation works well for slowly rotating relativistic stars. This suggests that we can determine with a fairly good accuracy the neutral points by this approximation.

We here investigate the “counter-rotating” f-mode oscillations of rapidly rotating relativistic stellar models by the RCA. Though, as seen in the Newtonian results [10], errors of f-mode eigenfrequencies are larger than those of p- or g- modes, absolute values of errors arising from the RCA are likely to become smaller for *relativistic* and *rotating* stars compared to the Newtonian values (see below). Furthermore we will estimate roughly the amount of the corrections to the results of the RCA.

2 Formulation

Here we investigate only *uniformly rotating polytropic stars*. Polytropic relation is defined by $(p/\epsilon_c) = \kappa(\epsilon/\epsilon_c)^{1+1/N}$, where ϵ_c is the central energy density, κ is a measure of strength of gravity of the star and other symbols have their usual meanings.

The stationary axisymmetric equilibrium states are obtained by the KEH scheme [7]. The space coordinates (r, θ, φ) and the time coordinate t are used in this Letter. As a parameter characterizing the stellar rotation we adopt the ratio of the rotational energy to the gravitational energy $T/|W|$ (see [7]).

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In the present study, all the Eulerian perturbations of the metric components are omitted. Thus our basic equations consist of (1) the perturbation equation of the baryon number conservation and (2) the perturbation equation of the energy momentum conservation. The perturbed Eulerian variables are assumed to have harmonic dependences on the time and the azimuthal angular coordinate: $\sim e^{-i\sigma t + im\varphi}$, where m is an integer and σ is the frequency which is normalized by $\sqrt{4\pi\epsilon_c}$ in our numerical calculation. The boundary condition of the problem is that *the Lagrangian perturbation of the pressure vanishes at the original stellar surface*. Since the basic equations closely resemble those of Newtonian gravity, they are solved by the same scheme developed by us for the Newtonian perturbations [13].

3 Results

3.1 Test Calculations

We have checked our RCA code by applying it to two cases, i.e. spherical stars in general relativity and rotating stars in Newtonian gravity. First our frequencies for relativistic spherical stars agree well with those of other RCA calculations of Yoshida & Kojima [14]. As for rapidly rotating Newtonian stars, by comparing our results with those of the Newtonian Cowling approximation which are obtained by the modified code of Yoshida & Eriguchi [13], we have found that our RCA code can reproduce the neutral points of f-modes with enough accuracy, i.e. within errors of 2%.

3.2 Neutral Points along Sequences

A neutral point of an oscillation mode of an equilibrium sequence for fixed parameters (N, κ) are found as the intersection of $T/|W| - \sigma$ curve with $\sigma = 0$ (Fig.1)

In the framework of Newtonian gravity critical values of $T/|W|$ for the $m = 2$ 'bar' mode have been considered to be universal, i.e. $T/|W| \sim 0.14$ irrespective of compressibility and/or the rotation laws [12]. However, even for $N = 0.5$ polytropes we could not find a neutral point against this bar mode in weak gravity limit, i.e. for small κ . This may results from the crudeness of the Cowling approximation for lower order modes. In the Newtonian Cowling approximation, smaller frequencies are obtained than those calculated from the full treatment. As a consequence, the critical values of $T/|W|$ at neutral points tend to be raised in this approximation. Thus even if $N < 0.808$ neutral points for the bar mode may not be reached before the termination of equilibrium sequence due to mass-shedding.

On the other hand, neutral points for the bar mode could be found for stronger gravity, although the value of $T/|W|$ does not coincide with the Newtonian universal value 0.14. Remarkable is that even for soft equations of state ($N > 0.808$) with sufficiently strong gravity, neutral points for the bar mode do appear. This is not, however, so surprising because general relativistic effect on equilibrium states becomes significant.²

Our numerical computational results can be summarized as follows: (a) for sequences with N and κ kept fixed, values of $T/|W|$ at neutral points become smaller as the value of m increases; (b) for sequences with N and m kept fixed, values of $T/|W|$ at neutral points become smaller as the value of κ increases; (c) for sequences with m and κ kept fixed, values of $T/|W|$ at neutral points becomes smaller as the value of N increases.

In Newtonian gravity, normalized eigenfrequency scales as: $\frac{\sigma}{\sqrt{4\pi\rho_c}} \sim \sqrt{l} \frac{\bar{\rho}}{\rho_c}$, where ρ_c is the mass density at the centre, $\bar{\rho}$ is the mean density of the star and l is the zenithal quantum number. From this relation, normalized frequencies become smaller for models with higher mass concentration toward the centre. Because of strong gravity, matter distributions are concentrated towards the central region for relativistic models. In other words, relativistic stars can be considered to become softer effectively. Consequently frequencies become lower for larger values of κ .

4 Discussion – How Far is the RCA Reliable ?

²After submitting this paper, we received a preprint of Stergioulas & Friedman [11], where they also report to have obtained neutral points of $m = 2$ mode for softer equations of state with taking metric perturbations into consideration.

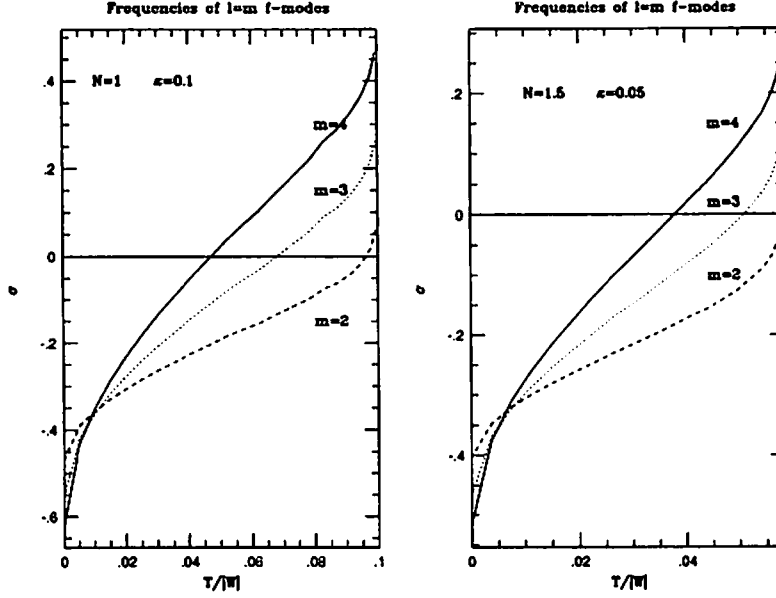


Figure 1: Normalized eigenfrequency σ is plotted against rotational parameter $T/|W|$.

Before applying the RCA to realistic neutron stars, it is important to note how far this approximation can be reliably applied. In particular we need to estimate how the strength of relativity and the amount of rotation affect the results.

It should be noted that eigenfunctions of energy density perturbation for f-modes are peaked near the surface. It implies that surface regions are important for f-modes. On the other hand, as is known in Newtonian models, relative importance of gravitational perturbations reduces as the amount of mass participating in oscillations decreases.

Consequently relativistic gravity can be considered to improve the approximation because the effective concentration of the matter due to relativity makes the role of metric perturbations less important. Furthermore perturbed density distribution of rotating models is more sharply peaked toward the surface of the star as the stars are spun up. Therefore the accuracy of the Cowling approximation can be relatively increased for rotating models than spherical ones. Thus these two factors can be said qualitatively to have advantages in deciding neutral points.

However, since there have been no investigations on perturbations of rotating stars in full general relativity, the influence of rotation on the errors due to the RCA cannot be quantitatively estimated by direct comparison with the results of full calculations. Thus we need some device to have quantitative estimation about the errors.

Our proposal is as follows. We assume that qualitative dependence of the mode on stellar rotation in general relativity is similar to that of the Newtonian counterpart. In Newtonian theory, differences between the eigenfrequencies obtained by the Cowling approximation and those by the full analysis including gravitational perturbations decrease monotonically as stars are spun up. If this tendency is the same for general relativistic rotating stars, the maximal error of the RCA in determining neutral points can be estimated.

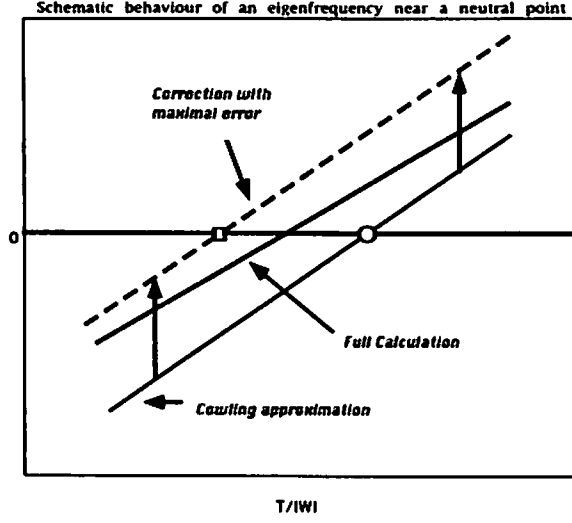


Figure 2: A schematic behaviour of an eigenfrequency near the neutral point. ‘Full calculation’ means values obtained from full inclusion of gravitational perturbations. ‘Correction with maximal error’ line is the line with the maximal error denoted by vertical arrows and extrapolated from a spherical model. The open circle and square denote approximated neutral points.

Let us select one $T/|W| - \sigma$ curve for a certain equilibrium sequence computed by the RCA. Since the maximum error of this curve is assumed to be that of the spherical model, the amount of the maximum error can be calculated by comparing our results with exact values of spherical models in general relativity. A parallel displacement of this $T/|W| - \sigma$ curve by this amount of the maximum error yields a curve with the maximal error in determination of neutral points (Fig. 2). Since the amount of displacement required to obtain the correct eigenfrequency from the value by the Cowling approximation decreases as the value of $T/|W|$ is increased, the ‘true eigenfrequency curve’ with gravitational perturbations passes between the curves of the Cowling approximation and of the curve of the maximal error defined above.

Therefore if the error of the eigenfrequency for the spherical model and the inclination of $T/|W| - \sigma$ curve of the RCA at the approximated neutral point are known, the maximal error in determining a neutral point will be determined by:

$$\text{Maximal Error of } \frac{T}{|W|} = \Delta\sigma_0 \cdot \frac{d\left(\frac{T}{|W|}\right)}{d\sigma}, \quad (1)$$

where $\Delta\sigma_0$ is the maximum error of the frequency which is the error of RCA eigenfrequency in the non-rotating model.

For $N = 1$ polytropes we have obtained the maximal errors of $T/|W|$ at neutral points for several modes (Fig. 3). We also display the post-Newtonian values of $T/|W|$ at neutral points computed by Cutler & Lindblom [2]. Two results agree within the maximal errors estimated here except for $m = 2$ modes which are not available for the post-Newtonian calculations.

5 Concluding Remarks

Here we have shown that the RCA works rather well in evaluating eigenfrequencies of rotating stars in general relativity despite its naiveness. As eigenmode analysis in the RCA is much easier than in the post-Newtonian treatment, this approximation can be extensively used in determining oscillatory frequencies of realistic neutron stars and/or determining their neutral points.

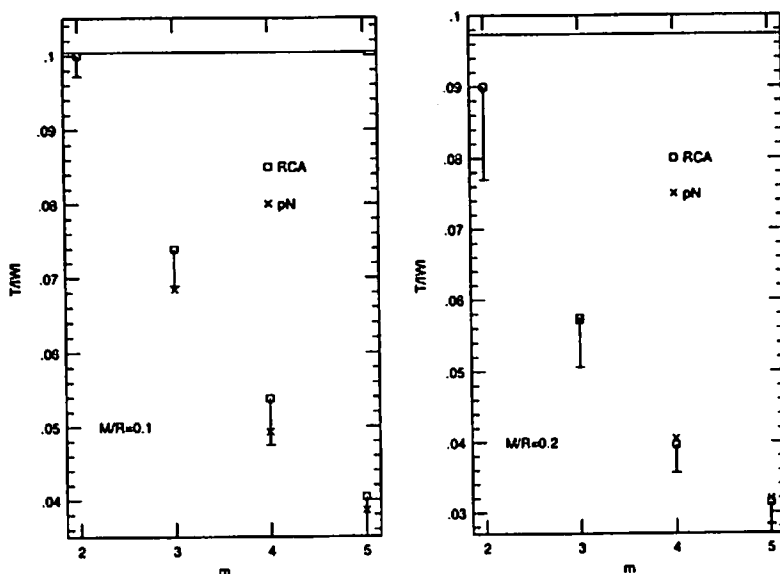


Figure 3: Critical values of $T/|W|$ at neutral points for several orders of f-modes for $N = 1$ polytrope. In this figure, as a measure of strength of gravity, the ratio of the mass to the radius in Schwarzschild coordinate M/R evaluated for spherical models is adopted. Error bars with open squares for both sides show maximal errors expected for the RCA. Integers m are the orders of the mode. Crosses are values of post-Newtonian results. Horizontal lines in the upper region denote critical values where equilibrium sequences terminate due to mass-shedding.

The main drawback of this approximation is that it is hard to include the effect of gravitational radiation which is responsible for damping or growing of corresponding modes. The post-Newtonian treatment can, in principle, systematically include this effect by increasing further the order of approximation. In contrast the RCA in itself cannot include its effect consistently. Some modification such as an estimation of effective damping due to gravitational radiation by using eigenfunctions obtained by the RCA may be needed.

Acknowledgments

We are grateful to Dr. Shijun Yoshida for discussions and Professor John Friedman for sending us a copy of their preprint.

This paper is presented as a part of the Ph.D. thesis of the first author to the Department of Astronomy, University of Tokyo. This research has been supported by JSPS Research Fellowships for Young Scientists and Grant-in-Aid for Scientific Research of the Japanese Ministry of Education, Science and Culture(08-03989).

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Multipole Moments of Almost Stationary Spacetime

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Abstract

First, we review briefly the construction of multipole moments of stationary spacetimes given by Geroch and Hansen (GH). Next, in order to describe the almost stationary spacetime, almost Killing vectors proposed by Matzner are re-advocated. In practice, these almost Killing vectors may be obtained by solving the second-order differential equations as eigen value problems. However, it is not straightforward to relate them with GH's conformal approach to define multipole moments. Thus, there still remain problems to be solved to find out the way to define non-perturbatively multipole moments of almost stationary spacetimes.

1 Introduction

Motivated by the ongoing construction of Kilometer-size interferometric gravitational wave detectors, such as LIGO[1], VIRGO[2] and TAMA[3], the importance of theoretical studies of gravitational waves has been recently increased. Coalescing binary neutron stars are among the most promising sources for such detectors [4, 5]. The main reasons are that (1) we can expect to detect the signal of coalescence of binary neutron stars about several times per year [6], and (2) the waveform from coalescing binaries can be predicted with a high accuracy compared with other sources [1, 4, 5]. Informations carried by gravitational waves tell us not only various physical parameters of neutron stars[4], but also the cosmological parameters[7, 8, 9] if and only if we can make a detailed comparison between the observed signal with theoretical prediction during the epoch of the so-called inspiraling phase where the orbital separation is much larger than the radius of component stars [10].

To obtain such a large amount of informations of the gravitational waves signal, we must understand the theoretical mechanism of merging and construct templates for signal a priori. When the orbital separation of binary neutron stars is $\leq 10GM/c^2$, where M is the total mass of binary neutron stars and c is the light velocity, they move approximately in circular orbits because the timescale of the energy loss due to gravitational radiation t_{GW} is much longer than the orbital period P as

$$\frac{t_{GW}}{P} \approx 15 \left(\frac{dc^2}{10GM} \right)^{5/2} \left(\frac{M}{4\mu} \right), \quad (1)$$

where μ and d are the reduced mass and the separation of binary neutron stars [11]. Thus, binary neutron stars evolve adiabatically radiating gravitational waves. When the orbital separation approaches $6GM/c^2$, however, they cannot be described by the circular orbital motion because of instabilities due to the general relativistic gravity and/or the tidal field. As a consequence of such instabilities, binary neutron stars will plunge to merge. Thus, before plunging, binary neutron stars may be considered as quasi-equilibrium configurations. In the post-Newtonian approximation, the radiation reaction appears at the 2.5 PN order [12, 13]. Therefore, the stationary binary neutron stars can exist in the second post-Newtonian approximation [14].

For understanding some properties of such a quasi-stationary system, multipole moments may be useful (For instance, [15]). This paper is organized as follows. We review how to define multipole moments of stationary spacetime in section 2, and Matzner's proposal for defining almost Killing vectors in section 3. In section 4, we discuss difficulties on defining *multipole moments of almost stationary spacetimes* as the straightforward extension of the above procedures.

Greek and Latin indices take 0, 1, 2, 3 and 1, 2, 3, respectively.

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2 Multipole moments of stationary spacetime

In this section, we briefly review the Geroch and Hansen's approach [16, 17, 18]. Let us consider the Newtonian case in the flat space [16]. Even in this case, there are many ways of defining multipole moments; one is the use of integrals of the density over the matter, other is the expansion of the Newtonian potential. In general relativity, it is often convenient to consider outside the matter, especially for the compact object. This is why, we consider the expansion of potentials. First, we consider the Laplace equation

$$D^2\phi = 0, \quad (2)$$

which can express the potential outside the matter, This potential is expanded as

$$\phi = \frac{Q}{r} + \frac{Q_a n^a}{r^2} + \frac{1}{2} \frac{Q_{ab} n^a n^b}{r^3} + \dots, \quad (3)$$

where n^a denotes the unit directional vector and multipole moments can be defined as the coefficients of the expansion of the potential at the far zone. Here, we consider the conformal transformation defined by

$$\Omega = r^{-2}, \quad (4)$$

$$\bar{x}^a = \Omega x^a, \quad (5)$$

$$\bar{\phi} = \Omega^{-1/2} \phi, \quad (6)$$

where $\bar{x}^a = 0$ at the spatial infinity (i^0). Then, $\bar{\phi}$ becomes

$$\bar{\phi} = Q + Q_a \bar{x}^a + \frac{1}{2} Q_{ab} \bar{x}^a \bar{x}^b + \dots, \quad (7)$$

Thus, multipole moments can be defined as

$$Q = \bar{\phi}|_{i^0}, \quad (8)$$

$$Q_a = \bar{D}_a \bar{\phi}|_{i^0}, \quad (9)$$

$$\dots \dots \dots \quad (10)$$

As well-known, these definitions depend on the choice of the origin of the coordinate. If one consider the system of the center of mass (COM) for instance, we shall obtain $Q_a = 0$. This coordinate choice closely relates with the choice of the conformal factor: For an arbitrary but small parameter ω such as

$$|\omega - 1| \ll 1, \quad (11)$$

multipole moments defined by Eq.(10) changes as

$$Q'_{a_1 \dots a_s} = Q_{a_1 \dots a_s} - \frac{1}{2} s(s-1) STF [Q_{a_1 \dots a_{s-1}} \bar{D}_{a_s} \omega], \quad (12)$$

where STF denotes the symmetric and tracefree parts.

By taking Eq.(12) as the central idea, multipole moments can be defined for *asymptotically flat* curved spacetimes [17]: First, let us consider the static spacetime. This spacetime can be described fully by the potential ϕ satisfying

$$D^2\phi - \frac{1}{8} \mathcal{R}\phi = 0, \quad (13)$$

where D^2 and $\mathcal{R}$ denote the covariant Laplacian and the scalar curvature of the space respectively. We consider the conformal transformation,

$$P = \bar{\phi}, \quad (14)$$

$$\dots \dots \dots \quad (15)$$

$$P_{a_1 \dots a_{s+1}} = STF \left[\bar{D}_{a_1} P_{a_2 \dots a_{s+1}} - \frac{1}{2} s(s+1) \bar{\mathcal{R}}_{a_1 a_2} P_{a_3 \dots a_{s+1}} \right], \quad (16)$$

where $\bar{R}_{a_1 a_2}$ denotes the three-dimensional Ricci tensor. Then, for the infinitesimal transformation $\Omega' = \Omega\omega$, multipole moments defined by Eq.(16) obeys the "transformation law" (12) [17].

Next, we shall mention the stationary case [18], where there exists a timelike Killing vector ξ^a . From ξ^a , we can define

$$\lambda = -\xi^a \xi_a, \quad (17)$$

$$\omega_a = \epsilon_{abcd} \xi^b \nabla^c \xi^d, \quad (18)$$

where ϵ_{abcd} denotes the alternating tensor. Moreover, in the vacuum spacetime, there exists a scalar ω such as $\omega_a = \nabla_a \omega$ [19]. From scalars λ and ω , two potentials $\phi_M(\lambda, \omega)$ and $\phi_J(\lambda, \omega)$ are constructed in the algebraic manner [19]. The former $\phi_M(\lambda, \omega)$ becomes the Newtonian potential in the Newtonian limit and the latter $\phi_J(\lambda, \omega)$ vanishes in the static case. Using these potentials, multipole moments can be defined, so that these multipole moments satisfy the transformation law (12) for the infinitesimal transformation [18]. In practice, multipole moments of the Kerr solution were obtained by Hansen [18].

3 Almost Killing vector

In describing the quasi-stationary spacetime, it might be important to consider the *almost* timelike Killing vector, if such a vector field would be obtained. In the physical intuition, such an *almost* timelike Killing vector must satisfy

$$\xi_{(\alpha;\beta)} \approx 0, \quad (19)$$

where () denote the symmetrization. However, the expression (19) is unclear, since it does not describe the deviation from 0. In other words, we can not know, from Eq.(19), the difference between the stationary spacetime and the almost stationary spacetime which we wish to consider. As for this respect, Matzner presented a proposal for the *almost* Killing vector. In his proposal, the following integral plays a central role in defining almost Killing vectors;

$$\mu[\xi] = \frac{\int dV \xi^{(\alpha;\beta)} \xi_{(\alpha;\beta)}}{\int dV \xi^\alpha \xi_\alpha}, \quad (20)$$

where $\int dV$ denotes the integral over the whole spacetime. Then, we can define almost Killing vectors as ξ^α minimizing the integral (20).

For the bounded space, this minimization of Eq.(20) means the following eigen value problem: It is necessary to search ξ^α which minimizes the eigen value μ_m for the equation

$$\xi^{(\alpha;\beta)}_{;\beta} + \mu_m \xi^\alpha = 0. \quad (21)$$

In this proposal, $\mu[\xi]$ can be considered as the measure of the deviation of the vector ξ^α from the Killing vector. Indeed, $\mu[\xi]$ exactly vanishes for the Killing vector.

4 multipole moments for almost stationary spacetime ?

If one could make potentials ϕ_M and ϕ_J from almost Killing vectors, it might be possible to define tensors $P_{a_1 \dots a_n}$. Obviously, these tensors become multipole moments in the stationary spacetime. However, one key point in the vacuum stationary case is as follows [19]: If ξ^α is a Killing vector, we obtain

$$\begin{aligned} D_{[a} \omega_{b]} &= -\epsilon_{abcd} \xi^c R^d_e \xi^e \\ &= 0, \end{aligned} \quad (22)$$

where [] denotes the anti-symmetrization, and we used the Killing equation in the first line and the vacuum condition in the second line. Thus, ω such as $\omega_a = D_a \omega$ can exist.

However, in the case of almost stationary spacetimes, we may consider $\xi_{(\alpha;\beta)} = O(\mu)$, where μ expresses the deviation from the stationary spacetime. Then, we obtain

$$D_{[a} \omega_{b]} = O(\mu). \quad (23)$$

Consequently, one can find

$$\omega_a = D_a \omega + O(\mu). \quad (24)$$

This means that we cannot produce potentials ϕ_M and ϕ_J in the same manner as in the stationary case. Thus, the conformal approach does not work for the almost stationary case.

5 Discussion

We considered how to introduce multipole moments of almost stationary spacetimes by using an *almost* Killing vector proposed by Matzner. However, we find many difficulties in this course.

The following issues might remain:

1. Almost Killing vectors:

In Matzner's proposal, the almost Killing vector for a given spacetime is defined as the vector minimizing $\mu[\xi]$. However, it is unclear that this minimization can determine the vector field uniquely in the Lorentzian spacetime. Thus, it seems important to search other ways to define the almost Killing vector.

2. How to define multipole moments:

In this respect, perturbative definitions are presented [15]. However, this definition depends on the special choice of coordinate systems. In the stationary case, the relation between Geroch and Hansen's multipole moments and Thorne's moments was investigated at the infinity [20]. An analysis of stationary external moments was performed by Thorne and Hartle [21]. In almost stationary cases, it might be interesting to investigate Thorne's moments. Such an extension was done by Zhang [22]. These works may become guidelines to study the way to define multipole moments of almost stationary spacetimes.

3. How to construct quasi-stationary spacetimes:

In above, we have not mentioned the quasi-stationary spacetime itself. In most cases, one must construct quasi-stationary spacetimes before finding out the almost Killing vector. Since it seems difficult and physically less important to consider the general case of the almost stationary spacetime, let us consider the compact binary as a physically important example. There are two timescales t_{GW} and P in this system, as described in the Introduction. We obtain in the large separation limit

$$\frac{P}{t_{GW}} \ll 1. \quad (25)$$

Thus, this parameter may be used as the expansion parameter for the Einstein equation. In some sense, this expansion becomes an approximation of the slow orbital motion, where the weak field is not assumed. As a result, such a treatment may have an advantage over the usual post-Newtonian approximation which assumes both the slow motion and the weak field. It might be important to investigate whether the approximation of the slow orbital motion is really useful to construct the *quasi-stationary* compact binary.

Acknowledgements

For useful conversations, we would like to thank the Japanese group of researchers of gravitational waves.

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Solutions of gauge invariant cosmological perturbations in long-wavelength limit

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Abstract

We investigate gauge invariant cosmological perturbations in a spatially flat Friedman-Robertson-Walker universe with scalar fields. It is well known that the evolution equation for the gauge invariant quantities has exact solutions in the long-wavelength limit. We find that these gauge invariant solutions can be obtained by differentiating the background solution with respect to parameters contained in the background system. This method is very useful when we analyze the long-wavelength behavior of cosmological perturbation with multiple scalar fields.

1 introduction

has exact The theory of gauge invariant cosmological perturbations[1, 2, 3] is an important tools when we investigate the origin and the evolution of the structure in our universe. In the context of general relativity, the treatment of linear perturbation of any matter field requires great care because these perturbations are in general not gauge invariant. We must evaluate gauge invariant quantities to extract physical meaning.

One of the main application of the gauge invariant perturbation is the theory of inflationary universe, in which the gauge invariant treatment of perturbations is crucial. The inflation predicts that the seed of density fluctuation, which evolves to the structures in our universe, is generated as quantum fluctuation of the inflaton field. The created fluctuations is stretched by deSitter expansion and their wavelength exceeds the Hubble horizon scale. As the gauge dependence of perturbations becomes conspicuous on super horizon scale, we must use gauge invariant perturbation to handle such a situation. But the derivation of the gauge invariant equation and evaluation of its solution needs tedious calculation in general, especially for the model with multiple scalar fields which is considered in the context of hybrid or extended inflation and scalar-tensor theories.

We here pay attention to the Einstein gravity with minimally coupled scalar fields ϕ . As a background space, we assume a flat Friedman-Robertson-Walker(FRW) universe of which dynamical variable is a scale factor $a = e^\alpha$. We use a following gauge invariant combination(Mukhanov's variable)[4]:

$$Q \equiv \delta\phi - \frac{\dot{\phi}}{H}\varphi, \quad (1)$$

where $\delta\phi$ is a perturbation of the scalar field, φ is a perturbation of three curvature and $H = \dot{\alpha}$ is a Hubble parameter. The evolution equation for Q is given by

$$\ddot{Q} + 3H\dot{Q} + \left[\left(\frac{k}{a} \right)^2 + V_{\phi\phi} - \frac{8\pi G}{e^{3\alpha}} \left(\frac{e^{3\alpha}}{H} \dot{\phi}^2 \right) \right] Q = 0. \quad (2)$$

It is known that this equation has exact solutions in the long-wavelength limit $k \rightarrow 0$:

$$Q = c_1 \frac{\phi}{H} + c_2 \frac{\dot{\phi}}{H} \int^t \frac{H^2}{e^{3\alpha} \dot{\phi}^2} dt, \quad (3)$$

where c_1, c_2 are arbitrary constants. Therefore if we can obtain

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the background solution $(\alpha(t), \phi(t))$, it is possible to predict the behavior of the long-wavelength gauge invariant perturbation without solving the evolution equation of the perturbation.

Our question is why we can have the exact solution for $k \rightarrow 0$ (0-mode) gauge invariant perturbation. The 0-mode perturbation is a perturbation of homogeneous mode, it must be contained in a homogeneous background system. So we expect that the evolution equation for the 0-mode perturbation can be derived by the analysis of the mini-super space model which has no inhomogeneous mode.

In this paper, we aim at clarifying the relation between the gauge invariant 0-mode solution and the background solution. We will show that the evolution equation and the solution of the gauge invariant 0-mode perturbation can be obtained within the perturbation of mini-super space model. In Sec.II, we use one dimensional autonomous system to demonstrate that the perturbed equation and its solution is obtained by differentiating the background solution with respect to parameters contained in the background solution. In Sec.III, we treat a mini-super space model with scalar fields and derive the 0-mode gauge invariant perturbation equation(Mukhanov's equation) and its solutions. We use Hamilton-Jacobi method which is used by several authors[5, 6] to derive the evolution equation of the gauge invariant cosmological perturbation. This method does not need to specify any gauge condition during calculation, it is suitable to apply to gauge invariant perturbation. Sec.IV is devoted to summary. We use the unit in which $c = \hbar = 8\pi G = 1$ and denote the partial derivative of a function F with respect to a some variable A by F_A throughout the paper.

2 toy model example

As a demonstration, we consider a one dimensional autonomous system with a Hamiltonian

$$\mathcal{H} = \frac{1}{2}p^2 + V(x). \quad (4)$$

Equation of motion is

$$\ddot{x} + V'(x) = 0. \quad (5)$$

The solution of this equation is written as $x = x(t + t_0)$ where t_0 is an origin of time and an arbitrary constant. The equation for the perturbation is obtained by splitting $x = x^{(0)} + \delta x$ and linearize with respect to δx :

$$\ddot{\delta x} + V''(x^{(0)})\delta x = 0. \quad (6)$$

This equation has the same form as the equation of $\dot{x}$ which is obtained by differentiating the background equation (5) with respect to time variable t : $(\dot{x})'' + V''(x)\dot{x} = 0$. As $\dot{x} = \dot{x}_{t_0}$, a solution of the perturbation equation is obtained by differentiating a background solution with respect to the parameter t_0 .

We explain this result more rigorously using Hamilton-Jacobi(H-J) method which is suitable to treat a cosmological model in next section. The H-J equation for a generating function S is

$$\frac{\partial S}{\partial t} + \frac{1}{2}S_x^2 + V(x) = 0, \quad (7)$$

and the evolution equation is

$$\dot{x} = S_x. \quad (8)$$

Eq.(7) is solved by $S = -Et + W(x)$ (E is a separation constant):

$$\begin{aligned} W_x^2 + 2V(x) &= 2E, \\ \dot{x} &= W_x. \end{aligned} \quad (9)$$

The solution is

$$t + t_0 = \int^x \frac{dx}{W_x} \equiv \tau(x; E), \quad (10)$$

where t_0 is an arbitrary constant. From this we get $x = x(t + t_0; E)$ or $E = E(x, t + t_0)$. The background solution is specified by the two parameters (t_0, E) and the perturbed solution is defined by the small change of the constants $(t_0 + \delta t_0, E + \delta E)$.

The generating function is written as $S = S^{(0)} + F$ where $S^{(0)}$ is a background part and F is a perturbed part which consists of quadratic terms of δt_0 and δE . F satisfies

$$\frac{\partial F}{\partial t} + S_x^{(0)} F_x + \frac{1}{2} F_x^2 = 0. \quad (11)$$

Now we change variables from (x, t) to $(\tau(x; E), t_0, E(x, t + t_0))$:

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial \tau}{\partial t} \frac{\partial}{\partial \tau} + \frac{\partial t_0}{\partial t} \frac{\partial}{\partial t_0} + \frac{\partial E}{\partial t} \frac{\partial}{\partial E} \\ &= -\frac{\partial}{\partial t_0} + E_x \frac{\partial}{\partial E}, \end{aligned} \quad (12)$$

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial \tau}{\partial x} \frac{\partial}{\partial \tau} + \frac{\partial t_0}{\partial x} \frac{\partial}{\partial t_0} + \frac{\partial E}{\partial x} \frac{\partial}{\partial E} \\ &= \tau_x \left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial t_0} \right) + E_x \frac{\partial}{\partial E}. \end{aligned} \quad (13)$$

The Eq.(11) becomes

$$\frac{\partial F}{\partial \tau} + \frac{1}{2} \left(\tau_x \left(\frac{\partial F}{\partial \tau} + \frac{\partial F}{\partial t_0} \right) + E_x \frac{\partial F}{\partial E} \right)^2 = 0. \quad (14)$$

As we are interested in linear perturbation due to the change of the parameters $(\delta t_0, \delta E)$, we replace $\frac{\partial}{\partial t_0} \rightarrow \frac{\partial}{\partial \delta t_0}$, $\frac{\partial}{\partial E} \rightarrow \frac{\partial}{\partial \delta E}$. Using a new variable $\delta t \equiv \delta t_0 + \tau_x \delta E$ and taking into account the quadratic form of F , the H-J equation becomes

$$\frac{\partial F}{\partial \tau} + \frac{1}{2 \dot{x}^2} \left(\frac{\partial F}{\partial \delta t} \right)^2 = 0, \quad (15)$$

From this equation, we get a Hamiltonian for the perturbation variable δt :

$$\mathcal{H}^{(2)} = \frac{1}{2 \dot{x}^2} P_{\delta t}^2, \quad (16)$$

where $P_{\delta t}$ is a conjugate momentum of δt . The equation of motion is

$$\ddot{\delta t} + 2 \frac{\ddot{x}}{\dot{x}} \dot{\delta t} = 0. \quad (17)$$

The equation for a variable $\delta x \equiv \dot{x} \delta t = x_{t_0} \delta t_0 + x_E \delta E$ becomes

$$\ddot{\delta x} + V''(x) \delta x = 0. \quad (18)$$

This is our desired result. The two independent solutions are given by

$$\delta x = x_{t_0}, x_E. \quad (19)$$

The explicit form of the solution is obtained by differentiating Eq.(10) with respect to t_0 and E . Regarding x as a function of $(t + t_0, E)$ and differentiating Eq.(10) with respect to t_0 and E ,

$$\begin{aligned} \frac{\partial}{\partial t_0} : 1 &= x_{t_0} \frac{1}{W_x}, \\ \frac{\partial}{\partial E} : 0 &= x_E \frac{1}{W_x} - \int^x \frac{W_{xE}}{W_x^2} dx, \end{aligned} \quad (20)$$

and we have

$$\begin{aligned} x_{t_0} &= W_x = \dot{x}, \\ x_E &= W_x \int^x \frac{dx}{W_x^3} = \dot{x} \int^t \frac{dt}{\dot{x}^2}. \end{aligned} \quad (21)$$

The perturbed solution is obtained by differentiating the background solution with respect to the parameters which specify the background solution.

3 gauge invariant long-wavelength solution from mini-super space solution

3.1 a single scalar field case

We consider a spatially flat FRW model with a scalar field. Dynamical variables are a scale factor $a = e^\alpha$ and a scalar field ϕ . The Hamiltonian for this system is

$$\mathcal{H} = Ne^{-3\alpha} \left[-\frac{1}{12}P_\alpha^2 + \frac{1}{2}P_\phi^2 + e^{6\alpha}V(\phi) \right], \quad (22)$$

where N is a lapse function, P_α and P_ϕ are conjugate momentum of α and ϕ , $V(\phi)$ is an arbitrary potential for the scalar field. The equations of motion are

$$\begin{aligned} \dot{\alpha} &= \frac{\partial \mathcal{H}}{\partial P_\alpha} = -\frac{N}{6}e^{-3\alpha}P_\alpha, \quad \dot{\phi} = \frac{\partial \mathcal{H}}{\partial P_\phi} = Ne^{-3\alpha}P_\phi, \\ \dot{P}_\alpha &= -\frac{\partial \mathcal{H}}{\partial \alpha} = Ne^{-3\alpha} \left[-\frac{1}{4}P_\alpha^2 + \frac{3}{2}P_\phi^2 - 3e^{6\alpha}V(\phi) \right], \\ \dot{P}_\phi &= -\frac{\partial \mathcal{H}}{\partial \phi} = Ne^{3\alpha}V'(\phi). \end{aligned} \quad (23)$$

The Hamilton-Jacobi equation follows from the Hamiltonian constraint with the canonical conjugate momentum replaced by derivative of a generating function S , $P_\alpha = \frac{\partial S}{\partial \alpha}$, $P_\phi = \frac{\partial S}{\partial \phi}$:

$$-\frac{S_\alpha^2}{12} + \frac{S_\phi^2}{2} + e^{6\alpha}V(\phi) = 0. \quad (24)$$

The evolution equations are

$$\frac{\dot{\alpha}}{N} = -\frac{e^{-3\alpha}}{6}S_\alpha, \quad (25)$$

$$\frac{\dot{\phi}}{N} = e^{-3\alpha}S_\phi. \quad (26)$$

The background solution is obtained by assuming $S^{(0)} = -2e^{3\alpha}H(\phi)$. Then

$$-3H^2 + 2H_\phi^2 + V(\phi) = 0, \quad (27)$$

$$\frac{\dot{\alpha}}{N} = H, \quad (28)$$

$$\frac{\dot{\phi}}{N} = -2H_\phi. \quad (29)$$

The solution of Eq.(27) has a constant of integration C and the solution can be written $H = H(\phi; C)$. By differentiating Eq.(27) with respect to C and using the evolution equation (28) and (29), we have

$$H_C = De^{-3\alpha} \equiv e^{-3(\alpha+\alpha_0)}, \quad (30)$$

where D and α_0 are arbitrary constants. From this, we can express $\alpha + \alpha_0$ as a function of (ϕ, C) :

$$\alpha + \alpha_0 \equiv \bar{\alpha}(\phi; C). \quad (31)$$

If we invert this relation, we get $C = C(\phi, \alpha + \alpha_0)$ which remains constant along the evolution of $(\alpha(t), \phi(t))$. The background solution is characterized by two parameters (α_0, C) and perturbation is defined by small change of these parameters $(\alpha_0 + \delta\alpha_0, C + \delta C)$.

We split the generating function into the background and the perturbed part: $S = S^{(0)} + F$. F satisfies

$$-\frac{S_\alpha^{(0)}}{6}F_\alpha + S_\phi^{(0)}F_\phi - \frac{F_\alpha^2}{12} + \frac{F_\phi^2}{2} = 0. \quad (32)$$

Change variables from (α, ϕ) to $(\tilde{\alpha}, \alpha_0, C)$:

$$\begin{aligned}\frac{\partial}{\partial \alpha} &= \frac{\partial \tilde{\alpha}}{\partial \alpha} \frac{\partial}{\partial \tilde{\alpha}} + \frac{\partial \alpha_0}{\partial \alpha} \frac{\partial}{\partial \alpha_0} + \frac{\partial C}{\partial \alpha} \frac{\partial}{\partial C} \\ &= -\frac{\partial}{\partial \alpha_0} + C_\alpha \frac{\partial}{\partial C},\end{aligned}\quad (33)$$

$$\begin{aligned}\frac{\partial}{\partial \phi} &= \frac{\partial \tilde{\alpha}}{\partial \phi} \frac{\partial}{\partial \tilde{\alpha}} + \frac{\partial \alpha_0}{\partial \phi} \frac{\partial}{\partial \alpha_0} + \frac{\partial C}{\partial \phi} \frac{\partial}{\partial C} \\ &= \tilde{\alpha}_\phi \left(\frac{\partial}{\partial \tilde{\alpha}} + \frac{\partial}{\partial \alpha_0} \right) + C_\phi \frac{\partial}{\partial C}.\end{aligned}\quad (34)$$

Then the H-J equation (32) becomes

$$e^{3\alpha} H \frac{\partial F}{\partial \tilde{\alpha}} - \frac{1}{12} \left(-\frac{\partial F}{\partial \alpha_0} + C_\alpha \frac{\partial F}{\partial C} \right)^2 + \frac{1}{2} \left(\tilde{\alpha}_\phi \left(\frac{\partial F}{\partial \tilde{\alpha}} + \frac{\partial F}{\partial \alpha_0} \right) + C_\phi \frac{\partial F}{\partial C} \right)^2 = 0. \quad (35)$$

To extract the linear perturbation part, we replace $\frac{\partial}{\partial \alpha_0} \rightarrow \frac{\partial}{\partial \delta \alpha_0}$, $\frac{\partial}{\partial C} \rightarrow \frac{\partial}{\partial \delta C}$. Using a new variable $\mathcal{R} \equiv \delta \alpha_0 + \tilde{\alpha}_C \delta C$ and assuming a quadratic form of the generating function F with respect to $\mathcal{R}$, we have

$$\frac{\partial F}{\partial \tilde{\alpha}} + \frac{H}{8e^{3\alpha} H_\phi^2} \left(\frac{\partial F}{\partial \mathcal{R}} \right)^2 = 0. \quad (36)$$

From this, we get a Hamiltonian for the variable $\mathcal{R}$:

$$\mathcal{H}^{(2)} = \frac{H}{8e^{3\alpha} H_\phi^2} P_{\mathcal{R}}^2, \quad (37)$$

where $P_{\mathcal{R}}$ is a conjugate momentum of $\mathcal{R}$. The equation of the motion for $\mathcal{R}$ becomes

$$\mathcal{R}_{\alpha\alpha} + \left(3 + \frac{H_\alpha}{H} + 2 \frac{\partial}{\partial \alpha} \ln \left(\frac{H_\phi}{H} \right) \right) \mathcal{R}_\alpha = 0. \quad (38)$$

By introducing a new variable $Q \equiv \phi_\alpha \mathcal{R} = \phi_{\alpha_0} \delta \alpha_0 + \phi_C \delta C$, Q satisfies

$$Q_{\alpha\alpha} + \left(3 + \frac{H_\alpha}{H} \right) Q_\alpha + \left(\frac{V_{\phi\phi}}{H^2} - \frac{1}{e^{3\alpha} H} (e^{3\alpha} H \phi_\alpha^2)_\alpha \right) Q = 0. \quad (39)$$

This is nothing but a Mukhanov equation for the long-wavelength limit. The introduced variable $\mathcal{R}$ represents a curvature perturbation in the long-wavelength limit and corresponds to the Bardeen's parameter ζ .

The two independent solutions are given by

$$Q = \phi_{\alpha_0}, \phi_C. \quad (40)$$

The explicit form of these solutions can be obtained using Eq.(27), (28), (29). Assuming that the lapse N is a function of the scalar field ϕ , we have

$$t + t_0 = -\frac{1}{2} \int^\phi \frac{d\phi}{NH_\phi}. \quad (41)$$

Regarding ϕ as a function of $(t + t_0, C)$ and differentiating the both side of this equation with respect to t_0 and C ,

$$\begin{aligned}\frac{\partial}{\partial t_0} : 1 &= -\frac{\phi_{t_0}}{2NH_\phi}, \\ \frac{\partial}{\partial C} : 0 &= -\frac{\phi_C}{2NH_\phi} + \int^\phi \frac{H_{\phi C}}{2NH_\phi^2} d\phi.\end{aligned}\quad (42)$$

Choosing the scale factor as a time parameter($t \equiv \alpha, N \equiv 1/H$),

$$\begin{aligned}\phi_{\alpha_0} &= -2\frac{H_\phi}{H} = \phi_\alpha, \\ \phi_C &= NH_\phi \int^\phi \frac{H_\phi C}{NH_\phi^2} d\phi = -6\phi_\alpha \int^\alpha \frac{e^{-3(\alpha+\alpha_0)}}{\phi_\alpha^2 H} d\alpha.\end{aligned}\quad (43)$$

We therefore obtained the gauge invariant 0-mode solution by differentiating the background solution with respect to two parameters α_0 and C .

3.2 two scalar fields case

We consider a flat FRW universe with two scalar fields ϕ_1 and ϕ_2 . The Hamilton-Jacobi equation is

$$-\frac{S_\alpha^2}{12} + \frac{S_{\phi_1}^2}{2} + \frac{S_{\phi_2}^2}{2} + e^{6\alpha} V(\phi_1, \phi_2) = 0. \quad (44)$$

The background solution is obtained by $S^{(0)} = -2e^{3\alpha} H(\phi_1, \phi_2)$,

$$-3H^2 + 2H_{\phi_1}^2 + 2H_{\phi_2}^2 + V(\phi_1, \phi_2) = 0, \quad (45)$$

$$\frac{\dot{\alpha}}{N} = H, \quad (46)$$

$$\frac{\dot{\phi}_1}{N} = -2H_{\phi_1}, \quad \frac{\dot{\phi}_2}{N} = -2H_{\phi_2}. \quad (47)$$

The solution of Eq.(45) has two constants of integration C_1, C_2 and can be written $H = H(\phi_1, \phi_2, C_1, C_2)$. By differentiating the H-J equation (45) with respect to these constants and using the evolution equation (46) and (47), it can be shown

$$e^{3\alpha} H_{C1} = D_1, \quad e^{3\alpha} H_{C2} = D_2, \quad (48)$$

where D_1, D_2 are constants. Let us define new constants α_0 and f by $D_1 = e^{-3\alpha_0}, D_2 = e^{-3\alpha_0} f$. Then

$$H_{C1} = e^{-3(\alpha+\alpha_0)}, \quad (49)$$

$$\frac{H_{C2}}{H_{C1}} = f. \quad (50)$$

From this we have

$$\alpha + \alpha_0 = \tilde{\alpha}(\phi_1, \phi_2, C_1, C_2), \quad (51)$$

$$f = f(\phi_1, \phi_2, C_1, C_2). \quad (52)$$

We split the generating function into the background and the perturbed part: $S = S^{(0)} + F$. F satisfies

$$-\frac{S_\alpha^{(0)}}{6} F_\alpha + S_{\phi_1}^{(0)} F_{\phi_1} + S_{\phi_2}^{(0)} F_{\phi_2} - \frac{F_\alpha^2}{12} + \frac{F_{\phi_1}^2}{2} + \frac{F_{\phi_2}^2}{2} = 0. \quad (53)$$

Change variables from (ϕ_1, ϕ_2, α) to $(\tilde{\alpha}, \alpha_0, C_1, f, C_2)$:

$$\frac{\partial}{\partial \alpha} = -\frac{\partial}{\partial \alpha_0} + C_{1\alpha} \frac{\partial}{\partial C_1} + C_{2\alpha} \frac{\partial}{\partial C_2}, \quad (54)$$

$$\frac{\partial}{\partial \phi_1} = \tilde{\alpha}_{\phi_1} \left(\frac{\partial}{\partial \tilde{\alpha}} + \frac{\partial}{\partial \alpha_0} \right) + C_{1\phi_1} \frac{\partial}{\partial C_1} + f_{\phi_1} \frac{\partial}{\partial f} + C_{2\phi_1} \frac{\partial}{\partial C_2}, \quad (55)$$

$$\frac{\partial}{\partial \phi_2} = \tilde{\alpha}_{\phi_2} \left(\frac{\partial}{\partial \tilde{\alpha}} + \frac{\partial}{\partial \alpha_0} \right) + C_{1\phi_2} \frac{\partial}{\partial C_1} + f_{\phi_2} \frac{\partial}{\partial f} + C_{2\phi_2} \frac{\partial}{\partial C_2}. \quad (56)$$

The H-J equation becomes

$$\begin{aligned}
e^{3\alpha} H \frac{\partial F}{\partial \tilde{\alpha}} &= \frac{1}{12} \left(-\frac{\partial F}{\partial \alpha_0} + C_{1\alpha} \frac{\partial F}{\partial C_1} + C_{2\alpha} \frac{\partial F}{\partial C_2} \right)^2 \\
&+ \frac{1}{2} \left(\tilde{\alpha}_{\phi_1} \left(\frac{\partial F}{\partial \tilde{\alpha}} + \frac{\partial F}{\partial \alpha_0} \right) + C_{1\phi_1} \frac{\partial F}{\partial C_1} + f_{\phi_1} \frac{\partial F}{\partial f} + C_{2\phi_1} \frac{\partial F}{\partial C_2} \right)^2 \\
&+ \frac{1}{2} \left(\tilde{\alpha}_{\phi_2} \left(\frac{\partial F}{\partial \tilde{\alpha}} + \frac{\partial F}{\partial \alpha_0} \right) + C_{1\phi_2} \frac{\partial F}{\partial C_1} + f_{\phi_2} \frac{\partial F}{\partial f} + C_{2\phi_2} \frac{\partial F}{\partial C_2} \right)^2 = 0
\end{aligned} \tag{57}$$

To extract a linear perturbation part, we replace $\frac{\partial}{\partial \alpha_0} \rightarrow \frac{\partial}{\partial \delta \alpha_0}$, $\frac{\partial}{\partial f} \rightarrow \frac{\partial}{\partial \delta f}$, $\frac{\partial}{\partial C_1} \rightarrow \frac{\partial}{\partial \delta C_1}$, $\frac{\partial}{\partial C_2} \rightarrow \frac{\partial}{\partial \delta C_2}$. Using a new variable $\mathcal{R} \equiv \delta \alpha_0 + \tilde{\alpha}_{C_1} \delta C_1 + \tilde{\alpha}_{C_2} \delta C_2$ and assuming a quadratic form of F with respect to $\mathcal{R}$ and δf , we have

$$e^{3\alpha} H \frac{\partial F}{\partial \tilde{\alpha}} + \frac{1}{2} \left(\alpha_{\phi_1} \frac{\partial F}{\partial \mathcal{R}} + f_{\phi_1} \frac{\partial F}{\partial \delta f} \right)^2 + \frac{1}{2} \left(\alpha_{\phi_2} \frac{\partial F}{\partial \mathcal{R}} + f_{\phi_2} \frac{\partial F}{\partial \delta f} \right)^2 = 0. \tag{58}$$

Define 2×2 matrices

$$\hat{X} = \begin{bmatrix} \phi_{1\alpha} & \phi_{1f} \\ \phi_{2\alpha} & \phi_{2f} \end{bmatrix}, \quad \hat{A} = \frac{1}{e^{3\alpha} H} (\hat{X}^{-1}) (\hat{X}^{-1})^T. \tag{59}$$

The H-J equation becomes

$$\frac{\partial F}{\partial \tilde{\alpha}} + \frac{1}{2} \begin{bmatrix} \frac{\partial F}{\partial \mathcal{R}} & \frac{\partial F}{\partial \delta f} \end{bmatrix} \hat{A} \begin{bmatrix} \frac{\partial F}{\partial \mathcal{R}} \\ \frac{\partial F}{\partial \delta f} \end{bmatrix} = 0. \tag{60}$$

We get a Hamiltonian for the perturbation variables $(\mathcal{R}, \delta f)$:

$$\mathcal{H}^{(2)} = \frac{1}{2} \begin{bmatrix} P_{\mathcal{R}} & P_{\delta f} \end{bmatrix} \hat{A} \begin{bmatrix} P_{\mathcal{R}} \\ P_{\delta f} \end{bmatrix}, \tag{61}$$

where $P_{\mathcal{R}}$ and $P_{\delta f}$ are conjugate momentum of $\mathcal{R}$ and δf , respectively. The evolution equation becomes

$$\begin{bmatrix} \mathcal{R} \\ \delta f \end{bmatrix}_{\alpha\alpha} + \hat{A} (\hat{A}^{-1})_{\alpha} \begin{bmatrix} \mathcal{R} \\ \delta f \end{bmatrix}_{\alpha} = 0, \tag{62}$$

Introducing a new variable

$$\tilde{Q} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \equiv \hat{X} \begin{bmatrix} \mathcal{R} \\ \delta f \end{bmatrix}, \tag{63}$$

$\tilde{Q}$ satisfies

$$\tilde{Q}_{\alpha\alpha} + \left(3 + \frac{H_{\alpha}}{H} \right) \tilde{Q}_{\alpha} + \hat{M} \tilde{Q} = 0, \tag{64}$$

where $\hat{M}$ is a 2×2 matrix defined by

$$(\hat{M})_{ij} = \frac{V_{\phi_i \phi_j}}{H^2} - \frac{1}{e^{3\alpha} H} (e^{3\alpha} H \phi_{i\alpha} \phi_{j\alpha})_{\alpha}. \tag{65}$$

This is the Mukhanov equation for two scalar field system in the long-wavelength limit[7]. The four independent solutions of this equation are

$$\tilde{Q} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_{\alpha_0}, \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_f, \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_{C_1}, \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_{C_2}. \tag{66}$$

These solutions can be written in another form:

$$\tilde{Q} = \hat{X} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \hat{X} \int^{\alpha} \frac{1}{e^{3\alpha} H} (\hat{X}^{-1}) (\hat{X}^{-1})^T d\alpha \begin{bmatrix} d_3 \\ d_4 \end{bmatrix}, \tag{67}$$

where d_1, d_2, d_3, d_4 are arbitrary constants. Using the Mukhanov's variable, $\mathcal{R}$ and δf can be expressed as

$$\begin{bmatrix} \mathcal{R} \\ \delta f \end{bmatrix} = \dot{X}^{-1} Q = \begin{bmatrix} \alpha_{\phi_1} Q_1 + \alpha_{\phi_2} Q_2 \\ f_{\phi_1} Q_1 + f_{\phi_2} Q_2 \end{bmatrix}. \quad (68)$$

This gives a relation between the curvature perturbation $\mathcal{R}$ which corresponds to the Bardeen's parameter ζ and the Mukhanov's variable Q_1, Q_2 in the long-wavelength limit.

4 summary

We have shown that the solution and the evolution equation of the gauge invariant perturbation in the long-wavelength limit can be derived within the mini-super space model. The solution is obtained by differentiating the background solution with respect to parameters contained in the background system. These parameters completely specify the background solutions. If we need to know the behavior of the long-wavelength perturbation only, this method is very useful because we do not have to solve the perturbation equation. What we have to do is to solve the background system and identify the parameters contained in the background solution. The analysis of the long-wavelength perturbation in the model that involves multiple scalar fields becomes easier. As an application, a model of reheating after inflation was considered[8]. The system consists of a oscillating inflaton field and a massless scalar field which interacts with the inflaton. The behavior of the long-wavelength curvature perturbation was obtained using the method presented in this paper.

As a straightforward extension, it is possible to treat spatially closed and open FRW universe and anisotropic Kasner type universe. Instead of scalar fields, perfect fluid can be included. These subjects are left as a future exercise.

This work is supported by the Grant-In-Aid for Scientific Research of the Ministry of Education, Sports and Culture of Japan(09740196).

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Evolution of Cosmological Perturbations in the Long Wavelength Limit

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Abstract

The relation between the long wavelength limit of solutions to the cosmological perturbation equations and the perturbations of solutions to the exactly homogeneous background equations is investigated for scalar perturbations on spatially flat cosmological models. It is shown that a homogeneous perturbation coincides with the long wavelength limit of some inhomogeneous perturbation only when the former satisfies an additional condition corresponding to the momentum constraint if the matter consists only of scalar fields. In contrast, no such constraint appears if the fundamental variables describing the matter contain a vector field as in the case of a fluid. Further, as a byproduct of this general analysis, it is shown that there exist two universal exact solutions to the perturbation equations in the long wavelength limit, which are expressed only in terms of the background quantities. They represent adiabatic growing and decaying modes, and correspond to the well-known exact solutions for perfect fluid systems and scalar field systems.

§1 Introduction

In the current standard scenario based on the gravitational instability theory, the present large scale structures of the universe are formed through the following four stages: production of seed fluctuations in the early universe, their linear evolution on superhorizon scales, the subsequent linear modulation after they enter the horizon, and the final non-linear evolutions. In the inflationary universe models the seed fluctuations in the first stage are produced from quantum fluctuations on the Hubble horizon scales during inflation, and we now have universal formulae to determine their nature such as the amplitudes and the spectrum for a wide variety of inflation models. We can now also easily handle the evolution during the third stage because the matter content of the universe during this stage is rather simple and restricted. Of course the actual behavior of perturbations on subhorizon scales are quite complicated even in the linear stage, and we need numerical computations to determine their details.

On the other hand universal formulae to determine the evolution of perturbations during the second stage have not been established yet, although it is generally believed that the so-called Bardeen parameter is conserved with good accuracy during this stage and this conservation law essentially determines the amplitudes and the spectrum of perturbations when they reenter the horizon, from those at the first stage[1]. Of course the conservation of the Bardeen parameter during the Friedmann stage is well established including the case in which the equation of state of cosmic matter changes slowly [2, 3]. The simplicity comes from that fact that the evolution of perturbations on superhorizon scales are determined

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with good accuracy by that on the long wavelength limit [4, 5, 6]. In this limit the evolution of adiabatic modes is determined by a set of simple 1st-order ordinary differential equations for two gauge-invariant variables, from which follows the exact conservation of the Bardeen parameter [2, 3]. However, for some important situations, such as the reheating stage of inflation models, this simple analysis does not work because the adiabatic mode may produce entropy modes which feed back to the behavior of the adiabatic modes[7]. In these situations the knowledge on the evolution of perturbations of all the components of matter in the long wavelength limit is required to determine the behavior of the Bardeen parameter.

In this connection Nambu and Taruya recently wrote an interesting paper [8] in which they stated that solutions to the gauge-invariant perturbation equations for a multi-component scalar field system on expanding universe in the long wavelength limit are obtained as derivatives of exactly homogeneous solutions to the Einstein equations with respect to the solution parameters. If this result is correct and all the solutions to the perturbation equations can be obtained in this way, it provides a very powerful tool to analyze the behavior of perturbations during inflation and reheating including the problem of the conservation of the Bardeen parameter. However, no proof nor explanation on their statement was given in their paper. Further, direct calculation shows that the perturbation equations in the long wavelength limit do not coincide with those for exactly homogeneous perturbations.

In order to see whether Nambu and Taruya's statement is true or not as well as to obtain a deeper understanding on the behavior of perturbations on superhorizon scale perturbations, in the present proceeding, we investigate the relation between the solutions to the perturbation equations in long wavelength limit and the exactly homogeneous solutions to the Einstein equations for multi-component systems on spatially flat Robertson-Walker universe. Our arguments are quite general except for the assumptions that the fundamental variables describing matter are scalar fields and/or vector fields, and that the amplitude of anisotropic stress perturbations vanishes rapidly enough in the long wavelength limit. We will clarify general conditions under which the perturbation solutions in the long wavelength limit coincide with some exactly homogeneous perturbations to the homogeneous solutions to the Einstein equations. In particular, we will show that for the multi-component scalar field system not all the exactly homogeneous perturbations directly correspond to perturbation solutions in the long wavelength limit. Further, as a byproduct of our analysis, we will show that the perturbation equations in the long wavelength limit have two universal adiabatic solutions which can be expressed explicitly as time integrals of known background quantities.

The proceeding is organized as follows. First in the next section we give basic definitions of perturbation variables used in the present proceeding and the perturbation equations for them. By inspecting the dependence of these quantities and equations on the wave number k of perturbations, in §3 we derive the conditions on the exactly homogeneous perturbations to coincide with the $k \rightarrow 0$ limit of some solutions to the perturbation equations with $k \neq 0$. In §4, as a special application of the argument in §3, we show the existence of two universal adiabatic modes in the $k \rightarrow 0$ limit. Section 5 is devoted to summary and discussions.

§2 Perturbation Equations

In this section we recapitulate the definitions of basic perturbation variables and their equations in the framework of the gauge-invariant perturbation theory. We adopt the notations used in the review article by Kodama and Sasaki[2].

We only consider perturbations on a spatially flat($K = 0$) Robertson-Walker universe throughout the paper. Hence the background metric is given by

$$ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2, \quad (2.1)$$

and its perturbation by

$$d\bar{s}^2 = -(1 + 2AY)dt^2 - 2aBY_j dt dx^j + a^2 [(1 + 2H_L Y)\delta_{jk} + 2H_T Y_{jk}] dx^j dx^k, \quad (2.2)$$

where Y , Y_j and Y_{jk} are harmonic scalar, vector and tensor for a scalar perturbation with wave vector $\mathbf{k}$ on flat 3-space:

$$Y := e^{i\mathbf{k} \cdot \mathbf{x}}, \quad Y_j := -i \frac{k_j}{k} Y, \quad Y_{jk} := \left(\frac{1}{3} \delta_{jk} - \frac{k_j k_k}{k^2} \right) Y. \quad (2.3)$$

Under the infinitesimal gauge transformation

$$\delta t = TY, \quad \delta x^j = LY^j, \quad (2.4)$$

the metric perturbation variables in Eq.(2.2) transform as

$$\bar{A} = A - \dot{T}, \quad (2.5)$$

$$\bar{B} = B + a\dot{L} + \frac{k}{a}T, \quad (2.6)$$

$$\bar{H}_L = H_L - \frac{k}{3}L - HT, \quad (2.7)$$

$$\bar{H}_T = H_T + kL \quad (2.8)$$

where the overdot represents the derivative with respect to the proper time t and H is the cosmic expansion rate $\dot{a}/a$. From this it follows that the spatial curvature perturbation $\mathcal{R}$ and the shear σ_g defined by

$$\mathcal{R} := H_L + \frac{1}{3}H_T, \quad (2.9)$$

$$\sigma_g := \frac{a}{k}\dot{H}_T - B \quad (2.10)$$

transform as

$$\bar{\mathcal{R}} = \mathcal{R} - HT, \quad \bar{\sigma}_g = \sigma_g - \frac{k}{a}T. \quad (2.11)$$

Hence we obtain the following two independent gauge-invariant combinations:

$$\mathcal{A} := A - (\mathcal{R}/H)', \quad (2.12)$$

$$\Phi := \mathcal{R} - \frac{aH}{k}\sigma_g. \quad (2.13)$$

All the other gauge-invariant combinations of metric perturbation variables are written as linear combinations of $\mathcal{A}$ and Φ .

Here note that for the exactly homogeneous perturbations corresponding to $k = 0$, the vector-like and the tensor-like perturbation variables should vanish. Hence for the metric perturbation, $B = H_T = 0$. Correspondingly, the gauge freedom for them is just the time reparametrization T , and L should vanish in the gauge transformation. Therefore there exists only one gauge-invariant combination for the exactly homogeneous metric perturbations, which coincides with $\mathcal{A}$. The variable Φ has no counterpart for them. This point is very important in the argument on the correspondence between the perturbations in the $k \rightarrow 0$ limit and the exactly homogeneous ones.

Next we consider the matter perturbations. In this paper we assume that the fundamental variables describing matter are scalar and vector quantities. This assumption is satisfied in most of the realistic applications. Let us denote the corresponding perturbation variables by χ_I and v_P where I and P are indices labeling components. Then they transform under the gauge transformation as

$$\bar{\chi}_I = \chi_I - \dot{S}_I T, \quad \bar{v}_P = v_P + a\dot{L}, \quad (2.14)$$

where S_I is the background quantity corresponding to the perturbation χ_I . Hence the corresponding gauge-invariant variables are given by

$$X_I := \chi_I - \frac{\dot{S}_I}{H}\mathcal{R}, \quad (2.15)$$

$$V_P := v_P - \frac{a}{k}\dot{H}_T. \quad (2.16)$$

In the exactly homogeneous system the matter is described only by the scalar quantities S_I . Hence its perturbation gives only the scalar-type gauge-invariant variables X_I as in the case of the metric perturbation.

The Einstein equations relate these matter variables with the metric variables through the energy-momentum tensor. For scalar perturbations its generic form is written as

$$\tilde{T}_0^0 = -(\rho + \delta\rho Y), \quad (2.17)$$

$$\tilde{T}_j^0 = a(\rho + p)(v - B)Y_j, \quad (2.18)$$

$$\tilde{T}_k^j = (p\delta_k^j + \delta p\delta_k^j + \Pi Y_k^j), \quad (2.19)$$

where ρ and p are the background values of the energy density and the pressure, respectively, and follow the equations,

$$H^2 = \frac{\kappa^2}{3}\rho, \quad (2.20)$$

$$\dot{\rho} = -3(\rho + p)H. \quad (2.21)$$

In the present paper we only consider the case in which the anisotropic stress perturbation Π , which is gauge-invariant by itself, vanishes faster than k^2 in the $k \rightarrow 0$ limit.

Applying the general argument on the matter perturbation above, we can construct from the density perturbation $\delta\rho$, the velocity perturbation v and the isotropic stress perturbation δp the following three gauge-invariant combinations:

$$\rho\Delta_g := \delta\rho + 3(\rho + p)\mathcal{R}, \quad (2.22)$$

$$V := v - \frac{a}{k}\dot{H}_T, \quad (2.23)$$

$$p\Gamma := \delta p - c_s^2\delta\rho, \quad (2.24)$$

where $c_s^2 := \dot{p}/\dot{\rho}$ is the square of the sound velocity.

Now in terms of these gauge-invariant variables the perturbation of the Einstein equations are written as

$$\tilde{G}_0^0 = \kappa^2\tilde{T}_0^0 : \quad \mathcal{A} + \frac{1}{2}\Delta_g = \frac{k^2}{3a^2H^2}\Phi, \quad (2.25)$$

$$\tilde{G}_j^0 = \kappa^2\tilde{T}_j^0 : \quad k\left[\mathcal{A} + \frac{3}{2}(1+w)Z\right] = 0, \quad (2.26)$$

$$\tilde{G}_j^j = \kappa^2\tilde{T}_j^j : \quad H\dot{\mathcal{A}} + (2\dot{H} + 3H^2)\mathcal{A} = \frac{\kappa^2}{2}(p\Gamma + c_s^2\rho\Delta_g) - \frac{\kappa^2}{3}\Pi, \quad (2.27)$$

$$\tilde{G}_k^j - \tilde{G}_l^j\delta_k^l = \kappa^2(\tilde{T}_k^j - \tilde{T}_l^j\delta_k^l) : \quad k^2\left[\mathcal{A} + \frac{1}{a}\left(\frac{a}{H}\Phi\right)\right] = -\kappa^2a^2\Pi, \quad (2.28)$$

where $w = p/\rho$ and Z is the Bardeen parameter defined by

$$Z := \Phi - \frac{aH}{k}V = \mathcal{R} - \frac{aH}{k}(v - B). \quad (2.29)$$

If we eliminate Δ_g from Eq.(2.27) using Eq.(2.25), we obtain

$$\left(\frac{\mathcal{A}}{1+w}\right)' = \frac{c_s^2}{1+w}\frac{1}{H}\frac{k^2}{a^2}\Phi + \frac{H}{\rho+p}\left(\frac{3}{2}p\Gamma - \Pi\right). \quad (2.30)$$

Note that we have not multiplied any power of k in deriving these equations from the Einstein equations. Hence putting $k = 0$ in these equations yields the equations for the exactly homogeneous perturbations if one takes account of the fact that the terms multiplied by inverse powers of k vanishes identically for these perturbations as mentioned above.

In general these equations must be supplemented by the expressions for Δ_g , Z , Γ and Π in terms of the fundamental gauge-invariant matter variables X_I and V_P and their evolution equations. We do not write them explicitly here because the details of these equations are not necessary until we specialize the general arguments to specific models.

§3 $k = 0$ vs $k \rightarrow 0$ limit

In this section we clarify under what conditions the $k \rightarrow 0$ limit of solutions to Eqs.(2.25)-(2.28) coincide with some solutions to the corresponding equations for $k = 0$. For that purpose first note that the gauge-invariant variables introduced in the previous section are classified into two groups.

The first one consists of the gauge-invariant variables whose expressions in terms of the gauge-variant perturbation variables do not contain inverse powers of k . $\mathcal{A}$, Δ_g , Γ and X_I belong to this group. These variables always have well-defined finite $k \rightarrow 0$ limits which coincide with the corresponding gauge-invariant variables for the exactly homogeneous perturbations. Of course in taking $k \rightarrow 0$ limit the conditions

$$B, H_T, v_P, v, \Pi/k^2 \rightarrow 0 \quad (3.1)$$

should be taken into account.

On the other hand the second group consists of the gauge-invariant variable which contain inverse powers of k when expressed in terms of gauge-variant variables. Φ , Z (or V) and V_P belong to this group. These variables have no counterpart for the exactly homogeneous perturbations, and their $k \rightarrow 0$ limits depend on how fast the quantities in Eq.(3.1) vanish in the limit.

With these points in mind, let us first consider the $k \rightarrow 0$ limit of the perturbation of the equations for the fundamental matter variables. When expressed in terms of the original gauge-variant perturbation variables χ_I , v_P , A , B , H_L and H_T , they do not contain inverse powers of k . We can write them in terms of gauge-invariant variables in the following way. First eliminate χ_I and A by replacing them by X_I , $\mathcal{A}$ and $\mathcal{R}$. Second eliminate v_P using $V_{PQ} = v_P - v_Q$ and Z . These procedures do not produce terms with inverse powers of k , and the coefficient of Z contains a positive power of k . Now since all the gauge-variant matter variables and A are eliminated, the remaining terms should be proportional to Φ (and its time derivatives). This implies that in the final expressions Z and Φ appears only in the forms $k^m Z$ ($m \geq 1$) and $k^n \Phi$ ($n \geq 2$). Further the other terms, which are proportional to $\mathcal{A}$, X_I or V_{PQ} , contain no inverse power of k . Therefore, taking into account the condition (3.1), the $k \rightarrow 0$ limit of a solution to the perturbation equations satisfies the corresponding equations for $k = 0$ if and only if

$$k^2 \Phi \rightarrow 0, \quad (3.2)$$

$$kZ \rightarrow 0, \quad (3.3)$$

$$V_{PQ} \rightarrow 0. \quad (3.4)$$

Next let us examine the Einstein equations. Among the four equations it is easy to see that Eq.(2.25) and Eq.(2.27) reduce to the perturbation of Eq.(2.20) and Eq.(2.21) in the $k \rightarrow 0$ limit under the condition $k^2 \Phi \rightarrow 0$. On the other hand Eq.(2.26) and Eq.(2.28) become trivial for the exactly homogeneous perturbations. However, for $k \neq 0$, these equations yield non-trivial perturbation equations

$$\mathcal{A} + \frac{3}{2}(1+w)Z = 0, \quad (3.5)$$

$$\mathcal{A} + \frac{1}{a} \left(\frac{a}{H} \Phi \right)' = 0, \quad (3.6)$$

where we have put $\Pi = 0$ because it does not affect the arguments on the $k \rightarrow 0$ limit under the assumption $\Pi/k^2 \rightarrow 0$ adopted in the present paper.

The second equation of these can be solved in terms of Φ as

$$\Phi = \frac{H}{a} \left(C - \int_{t_0} a(t) \mathcal{A}(t) dt \right), \quad (3.7)$$

where C is an integration constant and t_0 is an initial time. This equation can be regarded as an equation to determine the $k \rightarrow 0$ limit of Φ from a solution for $\mathcal{A}$ to the perturbation equations with $k = 0$. In this viewpoint the consistency condition (3.2) is simply replaced by the condition on the k -dependence of C ,

$$k^2 C(k) \rightarrow 0. \quad (3.8)$$

On the other hand the condition (3.3) is always satisfied under Eq.(3.5). Further the condition (3.4) just select a subset of solutions to the perturbation equations with $k \neq 0$, and does not give any restriction

on the exactly homogeneous perturbations. Therefore starting from any solution to the perturbation equations with $k = 0$, one can always construct the gauge-invariant quantities representing the $k \rightarrow 0$ limit of a solution to the perturbation equations with $k \neq 0$ by supplementing $\mathcal{A}$ and X_I for the exactly homogeneous perturbation with Φ determined by Eq.(3.7), provided that Eq.(3.5) do not yield any additional constraint on the seed exactly homogeneous perturbation.

In the case in which the fundamental variables describing matter contains a dynamical vector field, the $k \rightarrow 0$ limit of Z depends on the value $\lim_{k \rightarrow 0} (v - B)/k$ which cannot be determined from the information of the exactly homogeneous perturbations. Hence Eq.(3.5) can be simply regarded as the equation to determine the $k \rightarrow 0$ limit of Z .

In contrast, in the case in which the matter is described only by scalar fields, $v - B$ in $\tilde{T}_j^0$ should be written as a combination of the spatial derivatives of χ_I . Hence it must be proportional to a positive power of k , and Z has a well-defined $k \rightarrow 0$ limit which is written only in terms of X_I . Thus the condition Eq.(3.5) yields a restriction on the seed exactly homogeneous perturbation in order for it to be a $k \rightarrow 0$ limit of some solution to the perturbation equations with $k \neq 0$:

$$\lim_{k \rightarrow 0} W \left(\frac{\dot{\phi}}{H}, X \right) = 0, \quad (3.9)$$

where

$$W(X_1, X_2) := X_1 \cdot \dot{X}_2 - \dot{X}_1 \cdot X_2. \quad (3.10)$$

For a N -component scalar field system $(2N - 1)$ exactly homogeneous perturbations satisfying Eq.(3.9) correspond to the $k \rightarrow 0$ limit of solutions to the perturbation equation with $k \neq 0$. The remaining one independent solution is the adiabatic decaying mode which does not correspond to the exactly homogeneous perturbations. See the detail for the reference [10].

§4 Universal Adiabatic Solutions

Let $a(t)$ and $S_I(t)$ be an exactly homogeneous solution describing the background. Then, since the scale factor $a(t)$ comes into the evolution equations only through $H = \dot{a}/a$, $(1 + \lambda)a(t)$ and $S_I(t)$ also gives an exactly homogeneous solution to the Einstein equation where λ is a constant. If we regard this solution as an exactly homogeneous perturbation, all the corresponding gauge-variant perturbation variables for matter vanish, and the metric perturbation is given by

$$A = B = 0, \quad \mathcal{R} = H_L = \lambda. \quad (4.1)$$

From this we obtain

$$\mathcal{A} = - \left(\frac{\mathcal{R}}{H} \right)' = - \frac{3}{2} (1 + w) \lambda. \quad (4.2)$$

Equation (3.7) determines Φ from this as

$$\Phi = C \frac{H}{a} + \frac{3}{2} \lambda \frac{H}{a} \int_{t_0} (1 + w) a(t) dt. \quad (4.3)$$

If there exists a dynamical vector field describing matter, this equation and Eq.(3.5) determines Z as

$$Z = - \frac{2}{3(1 + w)} \mathcal{A} = \lambda. \quad (4.4)$$

On the other hand, if the matter is described only by scalar fields, $v - B$ is written as $v - B = k\chi$ where χ is a combination of the scalar field perturbations χ_I which does not contain a negative power of k . Since the matter perturbation vanishes in the present case, this term should vanish, which implies that $Z = \lambda - aH\chi = \lambda$. Hence the condition (3.5) is satisfied.

Thus we find that the perturbation equations have always two solution for which the $k \rightarrow 0$ limits of $\mathcal{A}$, Φ and Z are given by Eqs.(4.2)-(4.4). Since the matter is not perturbed, $\Gamma = 0$ for them. Hence they represent adiabatic modes in the $k \rightarrow 0$ limit. Clearly the solution proportional to C is a decaying mode,

while that proportional to λ is a growing mode because the Bardeen parameter Z is a non-vanishing constant. Note that these solutions are universal in the sense that they are valid for any matter contents and interactions.

This universal solutions cover most of the exact solutions in the $k \rightarrow 0$ limit found so far, and the general argument developed here explains why such exact solutions were found for large variety of matter content. For example, for the case in which matter consists of a single-component perfect fluid, there exist two adiabatic modes. The above solutions just give their $k \rightarrow 0$ limit, and in terms of the standard variables $\Delta = \delta\rho/\rho + 3(1+w)aH(v-B)/k$ and V they are expressed as

$$\Delta = \frac{2k^2}{3a^2H^2}\Phi = \frac{2k^2}{3Ha^3} \left[C + \frac{3}{2}\lambda \int_{t_0} (1+w)adt \right], \quad (4.5)$$

$$V = \frac{k}{aH}(\Phi - Z) = \frac{k}{a^2} \left[C - \lambda \left(a(t_0)/H(t_0) + \int_{t_0} adt \right) \right], \quad (4.6)$$

which recover the well-known exact solutions[2].

Another example is the case in which matter consists of a multi-component scalar field $\phi = (\phi_I)$. In terms of the gauge-invariant variable for the scalar field perturbation defined by

$$X_I = \delta\phi_I - \frac{\dot{\phi}_I}{H}\mathcal{R}, \quad (4.7)$$

the above solution gives

$$X_I = -\frac{\dot{\phi}_I}{H}\lambda, \quad (4.8)$$

and

$$\Phi = C\frac{H}{a} + \lambda\frac{\kappa^2}{2}\frac{H}{a} \int \frac{a\dot{\phi}^2}{H^2}dt. \quad (4.9)$$

These give the extension of the well-known exact solution in the $k \rightarrow 0$ limit for the single component scalar field case[5, 6, 9].

§5 Summary and Discussions

In this proceeding we have investigated the relation between the $k \rightarrow 0$ limit of solutions to the perturbation equations with $k \neq 0$ and the exactly homogeneous perturbations ($k = 0$) for quite general multi-component systems on a spatially flat Robertson-Walker universe. The main result is summarized as follows.

First for the case in which the fundamental variables describing cosmic matter contain a dynamical vector field, for any solution to the perturbation equations with $k = 0$ there exists a solution to the perturbation equations with $k \neq 0$ which coincides with the former in the $k \rightarrow 0$ limit. On the other hand if cosmic matter consists only of scalar fields, such a correspondence holds if and only if the exact homogeneous perturbation satisfies an additional constraint corresponding to the momentum constraint which is expressed by a linear equation on the initial condition for the homogeneous perturbation. This implies that for the N -component scalar field system one can obtain the $k \rightarrow 0$ limit of $2N-1$ independent solutions to the perturbations equations if one knows all the solutions to the homogeneous background equations.

We have also shown that from trivial exactly homogeneous perturbations one can always construct two adiabatic solutions to the perturbation equations in the $k \rightarrow 0$ limit which can be explicitly expressed in terms of integrals of background quantities, irrespective of the content and the interactions of cosmic matter. One of them corresponding to the vanishing homogeneous perturbation represents a decaying mode for which the Bardeen parameter vanishes. The other corresponds to a simple constant scaling of the scale factor and gives a growing mode whose Bardeen parameter is a non-vanishing constant. This constant scaling is not a gauge transformation, but an extra symmetry of the background equations which holds only for spatially homogeneous Robertson-Walker universes. Thus this extra symmetry is

the hidden reason why exact solutions in the $k \rightarrow 0$ limit have been so far found for various systems in spite of the non-trivial structure of their perturbation equations.

For the multi-component scalar field system these universal adiabatic modes and the solutions obtained from the non-trivial exact homogeneous perturbations exhaust all the solutions to the perturbation equations in the $k \rightarrow 0$ limit. Hence the evolution of perturbations on superhorizon scales can be determined only from the knowledge on the homogeneous solutions to the background equations in the lowest-order approximation. However, if one wants to know the higher-order correction in k , one must solve the perturbation equations $k \neq 0$ iteratively [6]. In this procedure one needs the $k \rightarrow 0$ limit of all the solutions to the perturbation equations for the gauge-invariant variable describing the perturbation of the scalar fields. As we have shown, this information can be also obtained from the exactly homogeneous solutions to the perturbation equations. Thus the behavior of superhorizon perturbations of this system can be determined with any accuracy from the knowledge on the homogeneous solutions.

As for the application of the general argument given in this proceeding to the multi-component scalar field system and the multi-component perfect fluid system, see the reference [10]. This argument can be also directly extended to the scalar field system coupled with radiation, for which all the $k \rightarrow 0$ limit of the solutions to the perturbation equations are directly determined from the homogeneous background solutions. This correspondence may provide a very powerful method to analyze the evolution of perturbations during reheating in the inflationary models because the structure of the equations for the homogeneous scalar fields decaying to radiation is much simpler than that for the perturbation equations with energy transfer terms.

Acknowledgments

The authors would like to thank K. Nakao, Y. Nambu, M. Sasaki, T. Tanaka, A. Taruya and K. Tomita for valuable discussions and comments, without which this paper would not have emerged. T. H. would like to thank Prof. H. Sato for continuous encouragements. H. K. is supported by the Grant-In-Aid of for Scientific Research (C) of the Ministry of Education, Science, Sports and Culture in Japan(05640340). T. H. is partly supported by the Yukawa Foundation in Osaka University.

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Perturbative analysis of non-singular cosmological model

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Abstract

A stability analysis is made for a non-singular pre-big-bang like cosmological model based on 1-loop corrected string effective action. Its homogeneous and isotropic solution realizes non-singular transition from de Sitter universe to Friedmann-like universe, via super inflation phase. We are interested in whether the non-singular nature of the solution would be stable or not in more general inhomogeneous case. Perturbative analysis is made for scalar, vector, and tensor linear perturbations, and instability is found for tensor-type perturbation.

1 Introduction

One of the most remarkable predictions of General Relativity is the existence of singularities. Particularly at the center of the black hole and in the very early universe, one might interpret the singularities as a result of the breakdown of General Relativity. If so, the behavior of the system “near” the singularity should be explained by some more general theory, preferably including GR as an effective theory in particular situation. In this sense, searches for singularity-free cosmological models have been made by many authors [8]. Recently non-singular cosmological models based on superstring theory[1] were presented, and among these, we will here investigate the model presented in [13], which is derived from the superstring effective action with 1-loop string correction [11]

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} R - \frac{1}{4} (D\Phi)^2 - \frac{3}{4} (D\sigma)^2 + \frac{1}{16} [\lambda_1 e^\Phi - \lambda_2 \xi(\sigma)] R_{GB}^2 \right\} \quad (1)$$

where R , Φ and σ are the Ricci scalar curvature, the dilaton, and the modulus field, respectively. The Gauss-Bonnet curvature is

$$R_{GB}^2 = R^{\mu\nu\kappa\lambda} R_{\mu\nu\kappa\lambda} - 4R^{\mu\nu} R_{\mu\nu} + R^2, \quad (2)$$

and $\xi(\sigma)$ is a function of modulus field expressed with Dedekind η function:

$$\xi(\sigma) = -\ln[2e^\sigma \eta^4(i e^\sigma)]. \quad (3)$$

Since the non-singular nature essentially depends on the behavior of modulus field σ , we will concentrate on the modulus field only and actually use the effective action appearing in [13]:

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} R - \frac{1}{2} (D\varphi)^2 - \frac{\lambda}{16} \xi(\varphi) R_{GB}^2 \right\}. \quad (4)$$

From this action we will derive the back ground equations of motion and see the behavior of the solution in the next section. In the section 3, the method of perturbation is explained and equations of motion for perturbative variables are derived, and also the stability of this cosmological model is discussed. The results are summarized in the section 4.

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2 Model and the background solution

We will assume the homogeneous and isotropic metric with conformal time η ,

$$ds^2 = a(\eta)^2(-d\eta^2 + \gamma_{ij}dx^i dx^j), \quad (5)$$

$$\gamma_{ij} = \frac{1}{1 + \frac{1}{4}\mathcal{K}(x^2 + y^2 + z^2)}\delta_{ij} \quad (6)$$

where $\mathcal{K} = 0, +1, -1$ for flat, closed, open universe, respectively. From the action (4) one can derive the equations of motion as

$$\varphi'^2 = 6(\mathcal{H}^2 + \mathcal{K})(1 - \frac{\lambda}{2a^2}\mathcal{H}\xi') \quad (7)$$

$$(\mathcal{H}' + \mathcal{H}^2 + \mathcal{K})(1 - \frac{\lambda}{2a^2}\mathcal{H}\xi') + (\mathcal{H}^2 + \mathcal{K})(1 - \frac{\lambda}{4a^2}\xi'') = 0 \quad (8)$$

$$\varphi'' + 2\mathcal{H}\varphi' + \frac{3\lambda}{2a^2}(\mathcal{H}^2 + \mathcal{K})\mathcal{H}'\xi_{,\varphi} = 0 \quad (9)$$

Here, the conformal Hubble parameter

$$\mathcal{H} := \frac{a'}{a} \quad (10)$$

is used and prime (') denotes differentiation with respect to conformal time η . We denote the derivative with respect to the physical time t as dot ($\dot{}$), and usual Hubble parameter as $H := \frac{\dot{a}}{a}$. Since the action (4) is invariant under the change of φ 's sign, we can choose the initial condition of φ so that φ increases in time (at least near the initial point). Using

$$\xi = \frac{1}{2}\varphi^2 \quad (11)$$

instead of (3), which is a good approximation near $\varphi = 0$, the solutions of the equations of motion which continue to Friedmann like phase ($H > 0$, $\dot{H} < 0$) at $t = 0$ are shown in the fig.1. In these solutions φ increases monotonically, so larger φ corresponds to later time. As can be seen from the figure, there are two classes of such solutions — singular and non-singular solutions. Solutions a and b in the fig.1 are singular at $\varphi = 0$, whereas c, d, e and f continues beyond the initial singularity. One may also notice that these non-singular solutions approaches de Sitter like solution ($H \simeq \text{const} \neq 0$) as $\varphi \rightarrow -\infty$, $t \rightarrow -\infty$. This is peculiar to the form of the potential (11). If we use (3), the behavior of the solutions near $\varphi = 0$ are quite similar to those shown in the fig.1, but H finally approaches zero in the infinite past. Although the asymptotic behavior is quite different, we will use (11) as ξ in the actual numerical calculation since it is simple, and we are only interested in the behavior of the solution near the singularity.

3 Perturbative analysis

3.1 Method of perturbation

We will consider perturbation of our model(4) to analyze the stability of the non-singular nature. To avoid the gauge ambiguity, we here use the gauge-invariant perturbation method [14]. The metric is decomposed into the back ground part and the perturbed part, and the perturbed part into scalar-, vector-, and tensor-perturbation part:

$$g_{\mu\nu} = {}^{(0)}g_{\mu\nu} + \delta^S g_{\mu\nu} + \delta^V g_{\mu\nu} + \delta^T g_{\mu\nu}. \quad (12)$$

Also, the scalar field φ (or the potential ξ) is decomposed into its background and perturbed part:

$$\begin{aligned} \varphi &= {}^{(0)}\varphi + \delta\varphi, \\ \xi &= {}^{(0)}\xi + \delta\xi. \end{aligned} \quad (13)$$

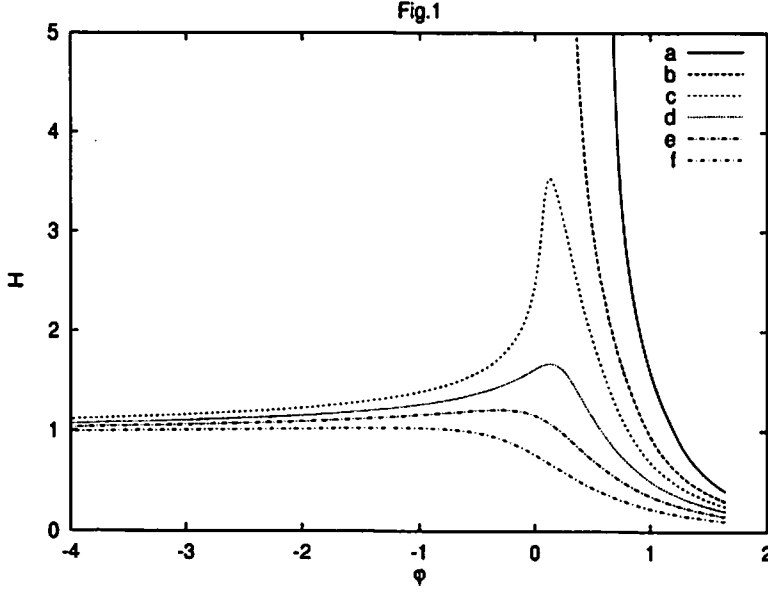


Figure 1: Behavior of Hubble parameter versus scalar field ϕ . Time flows from left to right since in these solutions $\dot{\phi}$ is always positive. There are two classes of solutions: singular solutions (a and b), and non-singular solutions (c,d,e,f). We can see that the initial singularity is avoided when ϕ crosses zero. For detailed discussions see [13].

Change of perturbed values under gauge transformations is given by Lie derivatives as:

$$\delta Q \rightarrow \delta Q + \mathcal{L}_X^{(0)} Q. \quad (14)$$

We will introduce such combinations of perturbed variables that the effects of gauge transformation cancel. These gauge invariant variables are used to describe the dynamics of the perturbation. Equations of motion for perturbed variables are obtained as perturbed Einstein-like equation:

$$\delta G^\mu{}_\nu = \delta T^\mu{}_\nu + \frac{\lambda}{2} \delta K^\mu{}_\nu. \quad (15)$$

Each term is also decomposed into scalar, vector, and tensor part. Perturbed Klein-Gordon equation is not necessary since it is derived from (15).

3.2 Scalar part

Scalar part of the perturbed metric is written in terms of 4 scalar functions ϕ , ψ , B , and E ,

$$\delta^S g_{\mu\nu} = a^2(\eta) \begin{pmatrix} -2\phi & B_{|i} \\ B_{|j} & -2(\psi\gamma_{ij} - E_{|ij}) \end{pmatrix}, \quad (16)$$

and following combination of gauge-dependent variables are shown to be unchanged under gauge transformations:

$$\Phi = \phi + \frac{1}{a} [a(B - E')]', \quad (17)$$

$$\Psi = \psi - \frac{a'}{a} (B - E'), \quad (18)$$

$$\delta\varphi^{(gi)} = \delta\varphi + {}^{(0)}\varphi'(B - E'), \quad (19)$$

$$\delta\xi^{(gi)} = \delta\xi + {}^{(0)}\xi'(B - E'). \quad (20)$$

Introducing a variable

$$\Theta := \Psi + \frac{\lambda}{2a^2} \left\{ \frac{1}{2}(\mathcal{H}^2 + \mathcal{K})\delta\xi^{(si)} - \mathcal{H}\xi'\Psi \right\}, \quad (21)$$

scalar part of the perturbed Einstein equation (15) gives two independent equations:

$$\begin{aligned} & \nabla^2 \Theta + \left[\frac{3\lambda\xi'}{4a^2\alpha}(\mathcal{H}^2 + \mathcal{K}) - 3\mathcal{H} \right] \Theta' \\ & + \left[-6\mathcal{H}' - 12\mathcal{H}^2 - 6\mathcal{K} - \frac{3\lambda\xi'}{2a^2\alpha}\mathcal{H}(2\mathcal{H}' + \mathcal{H}^2 + \mathcal{K}) + \frac{3\lambda^2\xi'^2}{8a^4\alpha^2}(\mathcal{H}^2 + \mathcal{K})\mathcal{H}' \right] \Theta \\ & - \frac{3\lambda^2\xi'}{16a^4\alpha}(\mathcal{H}^2 + \mathcal{K})^2\delta\xi^{(si)'} - \frac{1}{2}\varphi'\delta\varphi^{(si)'} \\ & + (\mathcal{H}^2 + \mathcal{K}) \left[\frac{3\lambda}{4a^2}\mathcal{H}' + \frac{3\lambda}{4a^2\alpha}(3\mathcal{H}' + 2\mathcal{H}^2 + 2\mathcal{K}) - \frac{3\lambda^3\xi'^2}{32a^6\alpha^2}(\mathcal{H}^2 + \mathcal{K})\mathcal{H}' \right] \delta\xi^{(si)} = 0 \end{aligned} \quad (22)$$

$$\begin{aligned} & \Theta' + \left[\mathcal{H} + \frac{\lambda\xi'}{4a^2\alpha}(4\mathcal{H}' + 3\mathcal{H}^2 + 3\mathcal{K}) \right] \Theta \\ & - \frac{3\lambda^2\xi'}{16a^4\alpha}(\mathcal{H}^2 + \mathcal{K})(2\mathcal{H}' + \mathcal{H}^2 + \mathcal{K})\delta\xi^{(si)} - \frac{1}{2}\varphi'\delta\varphi^{(si)} = 0 \end{aligned} \quad (23)$$

where α is

$$\alpha = 1 - \frac{1}{2}\lambda\mathcal{H}\xi'. \quad (24)$$

Eliminating $\delta\varphi^{(si)}$ from (22) and (23), we can obtain one wave function for Θ . However, for finding numerical solutions it is more convenient to directly integrate (22) and (23). Decomposing Θ into Fourier mode and using (11) for ξ , we obtain the numerical solutions shown in fig.2. Although the amplitude of the perturbation is somewhat enhanced near the peak of Hubble parameter, there is no growing mode appearing in the metric perturbation.

Fig. 3 shows the effective density contrast defined by:

$$\frac{\delta\rho}{\rho} := \frac{\delta^S G^0_0}{G^0_0}. \quad (25)$$

We notice that the small scale effective density contrast grows in Friedmann phase. In late time in Friedmann phase, the effect of the Gauss-Bonnet term becomes negligible and the evolution of the density contrast can be considered as in the ordinary Friedmann universe. Thus the growth of the density contrast is the consequence of the free scalar field remaining in the Friedmann universe. This result is analytically understood as follows. Putting $\lambda = 0$ in equations (22) and (23), we have a equation for Θ :

$$\Theta'' - \nabla^2 \Theta + 6\mathcal{H}\Theta' + (10\mathcal{H}' + 20\mathcal{H}^2)\Theta = 0. \quad (26)$$

Using back ground equations (7)(8)(9) with $\lambda = 0$, and introducing

$$u = \frac{a\Theta}{\varphi'} \quad (27)$$

$$\theta = \frac{\mathcal{H}}{a\varphi'}, \quad (28)$$

the equation can be written in a simple form[14]:

$$u'' - \nabla^2 u - \frac{\theta''}{\theta}u = 0 \quad (29)$$

Regarding the perturbation as plane wave, the solution for this equation can be easily found in both large and small wave number limits. For large enough wave number ($k^2 \gg \theta''/\theta$), the solution of the equation(29) is

$$u \propto \exp(\pm i k \eta).$$

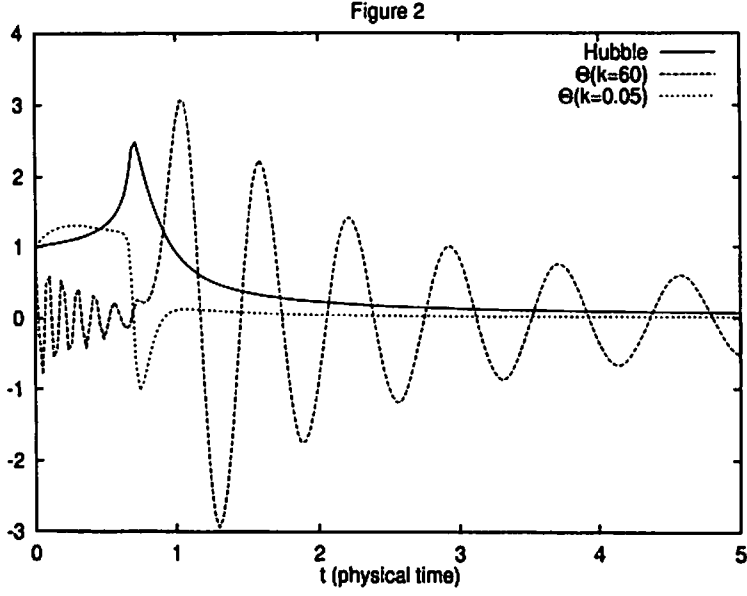


Figure 2: Behavior of scalar perturbation. Perturbations with different wave numbers are shown here, and there is no growing mode.

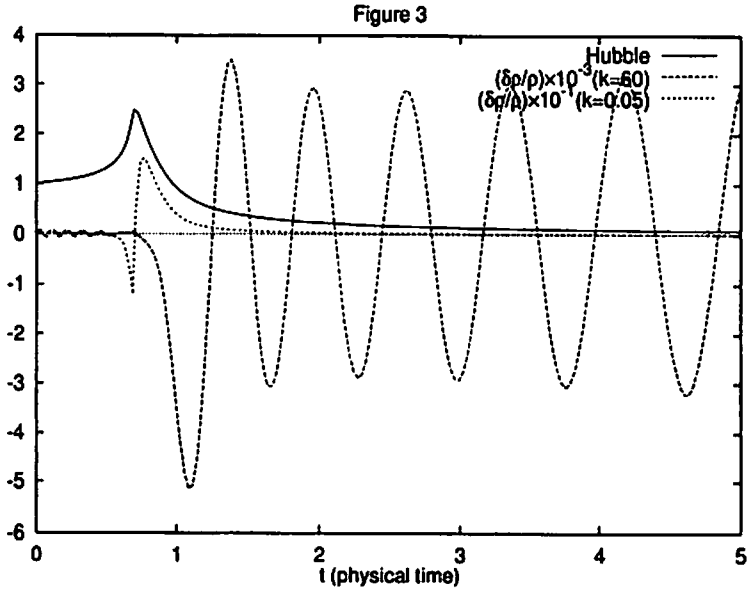


Figure 3: Density contrast calculated using the solution shown in Fig.2. Small scale density contrasts grow as $\frac{\delta\rho}{\rho} \propto t^{1/3}$ due to the free scalar field in the Friedmann universe.

Transforming this back into the gauge-invariant perturbative variables, for density contrast we have

$$\frac{\delta\rho}{\rho} \propto a \propto \eta^{1/2} \propto t^{1/3}. \quad (30)$$

Thus the density contrast grows like $t^{1/3}$.

3.3 Vector part

The perturbed metric for vector perturbation is defined as:

$$\delta^V g_{\mu\nu} = a^2(\eta) \begin{pmatrix} 0 & -S_i \\ -S_j & F_{ij} + F_{j|i} \end{pmatrix} \quad (31)$$

Vector function S_i and F_i satisfy $S_i^{|i} = F_i^{|i} = 0$, and the combination $P_i = S_i + F_i'$ is unchanged under the gauge transformation. Vector part of the perturbed Einstein-like equation (15) gives 2 non-trivial equations:

$$(1 - \frac{\lambda}{2a^2} \mathcal{H}\xi')(\frac{1}{2}\nabla^2 + \mathcal{K})P_i = 0, \quad (32)$$

$$\{(1 - \frac{\lambda}{2a^2} \mathcal{H}\xi')(P_{i|j} + P_{j|i})\}' + 2\mathcal{H}(1 - \frac{\lambda}{2a^2} \mathcal{H}\xi')(P_{i|j} + P_{j|i}) = 0. \quad (33)$$

Equation (32) shows the spatial behavior of the vector perturbation, indicating the fluctuation of curvature scale. Equation (33) can be easily integrated to give

$$P_{i|j} + P_{j|i} \propto a^{-2}(1 - \frac{\lambda}{2a^2} \mathcal{H}\xi')^{-1} = \frac{1}{a^2\alpha}. \quad (34)$$

which indicates that the vector perturbation decreases as the universe expands.

3.4 Tensor part

The tensor part of the metric perturbation is defined as:

$$\delta^T g_{\mu\nu} = a^2(\eta) \begin{pmatrix} 0 & 0 \\ 0 & h_{ij} \end{pmatrix}, \quad (35)$$

which satisfies the constraint $h^i_i = 0$, $h_{ij}^{|j} = 0$. Tensor part of (15) gives one non-trivial equation for i-j part:

$$\{(1 - \frac{\lambda}{2a^2} \mathcal{H}\xi')h^i_{j'}\}' + 2\mathcal{H}(1 - \frac{\lambda}{2a^2} \mathcal{H}\xi')h^i_{j'} - (1 + \frac{\lambda}{2a^2} \mathcal{H}\xi' - \frac{\lambda}{2a^2} \xi'')(\nabla^2 - 2\mathcal{K})h^i_j = 0$$

Expanding h_{ij} with transverse-traceless basis tensors as

$$h_{ij} = h_+(k, \eta)e_{+ij}(k) + h_\times(k, \eta)e_{\times ij}(k). \quad (36)$$

for each polarization mode the function $h(k)$ satisfies the equation of motion using the physical time:

$$\ddot{h} + (3H + \frac{\dot{\alpha}}{\alpha})\dot{h} + \frac{2\mathcal{K} + k^2}{a^2\alpha}(1 - \frac{\lambda}{2}\ddot{\xi})h = 0 \quad (37)$$

Numerical solutions for $h(k)$ with different wave numbers, in the same background as scalar and vector perturbations, are shown in the fig.4. There is an instability in the de Sitter-like phase, which results from large positive value of $\ddot{\xi}$ in the equation (37).

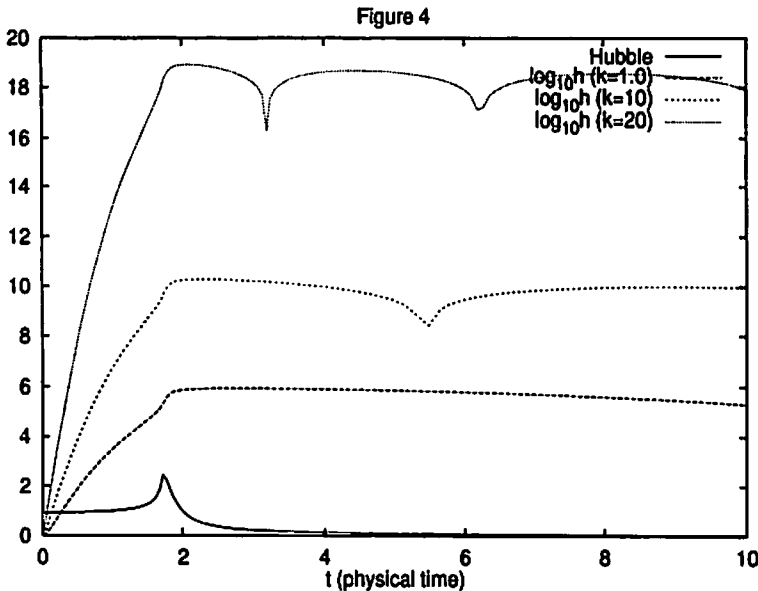


Figure 4: Tensor perturbation in the same background. Note that the tensor perturbations are plotted with logarithmic scale. Growing mode appears in the initial de Sitter-like phase. The smaller the perturbation scale is, the larger the growth rate becomes.

4 Conclusion

We have made a perturbative analysis of the cosmological model presented by Rizo and Tamvakis [13], to investigate the stability of the model. The perturbative equations of motion are solved numerically, and we found that the system is unstable under tensor perturbation. This instability appears in the de Sitter-like stage, and kinetic-driven inflation is thought to be responsible for the growing mode of the perturbation. Since the form of the function ξ coming from superstring effective action gives different asymptotic behavior of H , from this calculation alone we cannot conclude that this instability continues from the infinite past. But anyway, even with the modulus-geometry coupling of the form (3), there exists a tensor-part instability at least right before the Hubble parameter peak. Taking this model more seriously, we have to point out some problems. First of all, we have to check the validity of the effective action (1). Since the inclusion of the 1-loop effect predicts somewhat different character of the solution, it is certainly possible that the higher loop correction might change the nature of the cosmological model drastically. In this analysis we have ignored the effects of dilaton, but of course we must take these into account. Viewing this model as a realistic model of our universe, we also have to find a mechanism to create the ordinary matter, i.e. there have to be something like reheating in the inflationary universe.

Acknowledgements

We would like to thank D. Wands and H. K. Jassal for useful discussions.

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Renormalization group, a perturbation technique for differential equations and non-linear dynamics in cosmology

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Abstract

We review the renormalization group (RG) approach as a tool of asymptotic analysis of the differential equations. We see that the RG approach gives a simpler and more useful approximation method than the conventional approach known as the singular perturbation methods. As an application of this method, we investigate the evolution equations for the cosmological perturbations during reheating stage after inflation. The non-trivial behavior of the metric fluctuations appears due to the non-linearity of the background scalar fields. Using the RG method, we get the analytic expression for the background solution and the behavior of the metric fluctuations during reheating can be studied.

1 Introduction

Many of the problems in theoretical physics and applied mathematics involve the analysis of differential equations. In general relativity and cosmology, the main task is to solve the Einstein equations. Although we cannot solve the equations exactly in many cases, asymptotic and perturbative analyses often shed light on the problems. There often exists a parameter ϵ smaller than unity in the differential equations (DEs) or the boundary conditions when we are faced with the problems. Using ϵ as an expansion parameter, we expand the equation in powers of ϵ and get the solutions order by order. There are various perturbation techniques such as WKB, the method of asymptotic matching, the multiple scale analysis, the method of strained coordinate and so on. They are known as the *singular perturbation methods*. Applying one of the most appropriate perturbation techniques to the DE, it is possible to construct a globally valid approximation.

Recently, a new method based on the *renormalization group* (RG) has been proposed [1][2]. The idea of RG itself is well known in subject of quantum field theory and statistical physics. The important point is that the RG approach can be used for mathematical problems as a tool of solving the DEs. It has been recognized that the method of RG provides an unified approach to the perturbation techniques. Furthermore this method is simple and widely applicable for solving the various DEs.

The purpose of this review article is to present the following two things: (1) introduction of the RG method and (2) its application to the problem faced on cosmology. In Sec.2, treating simple examples, we shall explain the RG method. Comparing with the singular perturbation methods, we demonstrate that the RG approach is simpler and more useful than the conventional approach. We also see that the RG method is applicable to the various types of DEs. In Sec.3, we apply the RG method to a problem in cosmology. We analyze the evolution of cosmological perturbation during reheating after inflation. The non-linearity of the background scalar fields can affect the metric fluctuations associated with the large-scale structure in the universe. We can study the metric fluctuations with the help of the RG analysis of the background system. Sec.4 is devoted to the summary and the discussion.

2 Renormalization Group and Perturbation Methods

In this section, we review the RG method by treating simple examples. For brevity, we restrict ourselves to the analysis on the ordinary DEs throughout the paper. The first example is boundary layer problem. In the singular perturbation methods, it is known that the method of asymptotic matching is appropriate to this problem[5, 6]. To understand that the RG method provides a unified approach for the conventional

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techniques, we begin to solve the problem by *matched asymptotic expansion* and then introduce the RG approach. We learn that the RG method is simpler and more powerful than the matched asymptotic expansion. The second example is the Mathieu equation. Analyzing the stability/instability behavior of the Mathieu equation, we see that the RG approach is also applicable to the other type of DEs.

2.1 Boundary Layer Problem

For the first example, we consider the following equations:

$$\epsilon \ddot{y} + (1 + \epsilon)\dot{y} + y = 0; \quad \epsilon \searrow 0, \quad (1)$$

where ϵ is a small parameter with a positive sign. We solve the above equation with boundary conditions $y(0) = 0$ and $y(1) = 1$. In this case, the problem is solved exactly and we obtain

$$y(t) = \frac{e^{-t} - e^{-t/\epsilon}}{e^{-1} - e^{-1/\epsilon}}. \quad (2)$$

We observe that the solution (2) has two characteristic behavior distinguished by a boundary at $t \simeq \epsilon$:

$$\begin{aligned} y &\propto e^{-t/\epsilon} & ; & (0 \leq t \lesssim \epsilon), \\ y &\propto e^{-t} & ; & (\epsilon \lesssim t \leq 1). \end{aligned}$$

We shall describe the time interval $0 \leq t \lesssim \epsilon$ as the inner region, $\epsilon \lesssim t \leq 1$ as the outer region hereafter.

We examine the above problem by using the matched asymptotic expansion. In this method, we construct the inner and the outer approximations which are valid in the inner and outer region, respectively. Combining them at the overlapping (intermediate) region where the both approximations are applicable, we can obtain a globally valid approximation which can be used in the inner and outer regions.

We start to construct the outer approximation. Regarding ϵ as the expansion parameter, we expand y as

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots \quad (3)$$

Substituting (3) into (1) and collecting the terms of each order of ϵ , we have

$$\begin{aligned} \mathcal{O}(\epsilon^0) &: \dot{y}_0 + y_0 = 0, \\ \mathcal{O}(\epsilon^1) &: \dot{y}_1 + y_1 = -(\ddot{y}_0 + \dot{y}_0), \end{aligned}$$

We will get the solutions satisfying the condition $y_0(1) = 1$, $y_1(1) = 0$. The solutions become

$$y_0(t) = e^{1-t}, \quad y_1(t) = 0. \quad (4)$$

Note that (4) cannot be applicable at $t \rightarrow 0$. For the inner approximation, we introduce the new variable $\tau \equiv t/\epsilon$ and rewrite (1) in terms of τ . Expanding (1) in powers of ϵ yields

$$\mathcal{O}(\epsilon^0) : y_0'' + y_0' = 0, \quad (5)$$

$$\mathcal{O}(\epsilon^1) : y_1'' + y_1' = -(y_0' + y_0), \quad (6)$$

where prime denotes the derivative with respect to τ . The solution satisfying $y_0(\tau = 0) = 0$, $y_1(\tau = 0) = 0$ becomes

$$y_0(\tau) = A_0(1 - e^{-\tau}), \quad y_1(\tau) = A_1(1 - e^{-\tau}) - A_0\tau, \quad (7)$$

where A_0 , A_1 are the unknown constants which can be determined by comparing with the outer approximation.

Let us consider the overlapping region to determine A_0 , A_1 . We define the intermediate variable η as

$$\eta \equiv \frac{t}{\epsilon^\alpha} = \frac{\tau}{\epsilon^{\alpha-1}}; \quad 0 < \alpha < 1. \quad (8)$$

Fixing the variable η and taking the limit $\epsilon \rightarrow 0$, t and τ approach 0, ∞ respectively. Therefore we can determine the unknown variables by equating the outer and inner approximations. This process is called matching. Rewriting (4) (7) in terms of η , the expansion series become

$$y(t) \simeq e \left(1 - \epsilon^\alpha \eta + \frac{1}{2} \epsilon^{2\alpha} \eta^2 + \dots \right) \quad (9)$$

and

$$y(\tau) \simeq A_0 - \epsilon^\alpha A_0 \eta + \epsilon A_1 + \dots \quad (10)$$

Comparing (9) with (10), we get $A_0 = e$, $A_1 = 0$.

The globally valid approximation is constructed from the outer and inner approximations although the both approximations cannot be used at the whole region $0 \leq t \leq 1$. Adding the inner and the outer approximation and subtracting their common form in the overlapping region, we finally obtain

$$\lim_{\epsilon \searrow 0} y \Big|_t + \lim_{\epsilon \searrow 0} y \Big|_\tau - \lim_{\epsilon \searrow 0} y \Big|_\tau = e(e^{-t} - e^{-t/\epsilon}) \quad (11)$$

(11) is called a composite approximation[6]. We see that (11) from the exact solution (2) by the factor $1/(1 - e^{1-1/\epsilon})$. Thus the matched asymptotic expansion is valid unless the parameter ϵ increases to 1 (see Fig.1).

Now we introduce the RG method and apply it to the problem (1). The important point of the RG method is presence of the secular terms which appear in the higher order expansion. When we expand $y(\tau)$ and construct the perturbation series in naive treatment, we may obtain the secular terms which grow in proportion to τ or $\log \tau$. In the case of matched asymptotic expansion, we observe that the secular term appears in the inner approximation (7). Because of the secular term, $|y_1|$ becomes greater than $|y_0|$ at $\epsilon\tau \gtrsim 1$. Thus the inner approximation cannot be applicable to the outer region.

The RG approach provides a method to improve the perturbation series by eliminating the divergence of the secular terms. The resulting improved perturbation gives a globally valid approximation which can be used at $\epsilon\tau \gtrsim 1$. The general procedure of the RG method is simple and tractable[1][2]. We begin to construct the perturbation series expanded by ϵ in naive treatment:

$$y(\tau) = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \dots \quad (12)$$

We obtain the solution expressed as $y(A_0, B_0; \tau, \tau_0)$. The 0th order solution y_0 has the constants A_0, B_0 which depend on the initial condition given at $\tau = \tau_0$. In the higher order solution $y_1, y_2, \dots$, we only have to evaluate the inhomogeneous part of the solutions. The secular terms in the higher order solution are chosen so that they vanish at $\tau = \tau_0$. The next step is to introduce an arbitrary parameter μ . Splitting the secular term $(\tau - \tau_0)$ as $(\tau - \mu + \mu - \tau_0)$ and absorbing the terms containing $\mu - \tau_0$ into the renormalization constants Z_A, Z_B defined by

$$A_0 = A(\mu)Z_A(\tau_0, \mu), \quad B_0 = B(\mu)Z_B(\tau_0, \mu). \quad (13)$$

This treatment is always possible for each order of expansion and we obtain the μ dependent solution without changing the form of the perturbation. Since μ does not appear in the original equations, the solution should not depend on μ . We require

$$\left(\frac{\partial y}{\partial \mu} \right)_\tau = 0, \quad \text{for } \forall \tau. \quad (14)$$

We will call the above equation the RG equation. The μ dependence of $A(\mu), B(\mu)$ are determined by this equation. After solving the RG equation, we set $\mu = \tau$. The secular terms are eliminated and we can obtain the globally valid approximation $y(A(\tau), B(\tau); \tau)$.

Following the above procedure, let us try to solve the problem (1) again. We pay an attention on the inner region. From (5)(6), the naive perturbation up to $\mathcal{O}(\epsilon^1)$ are obtained as follows:

$$\begin{aligned} y &= y_0 + \epsilon y_1 + \dots \\ &= A_0 + B_0 e^{-\tau} - \epsilon(\tau - \tau_0)A_0 + \mathcal{O}(\epsilon^2). \end{aligned} \quad (15)$$

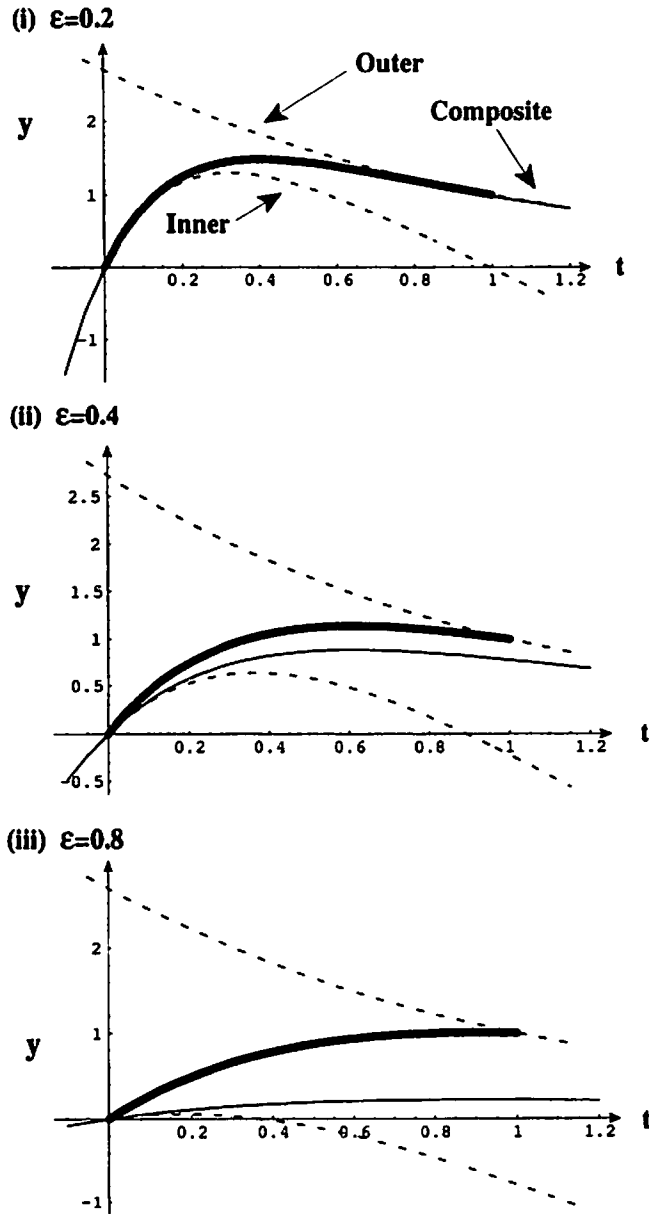


Figure 1: Comparison between the exact solution and the approximation by the matched asymptotic expansion (MAE): The dashed lines denote the inner and outer approximation obtained by MAE respectively and the thin solid lines are the composite approximation (11). As the parameter ϵ increases to 1, the approximation deviates from the exact solution (2) denoted by the thick solid lines.

Introducing the renormalization constants Z_A , Z_B , we replace the initial time τ_0 with a parameter μ . Taking $Z_A = 1 - \epsilon(\tau_0 - \mu) + \mathcal{O}(\epsilon^2)$, $Z_B = 1 + \mathcal{O}(\epsilon^2)$ and substituting (13) into (15), we have

$$y = A(\mu) + B(\mu)e^{-\tau} - \epsilon(\tau - \mu)A(\mu) + \mathcal{O}(\epsilon^2). \quad (16)$$

The condition (14) yields

$$0 = \frac{\partial y}{\partial \mu} = \left(\frac{dA}{d\mu} + \epsilon A \right) + \frac{dB}{d\mu}e^{-\tau} - \epsilon(\tau - \mu)\frac{dA}{d\mu} + \mathcal{O}(\epsilon^2). \quad (17)$$

Since $dA/d\mu \sim \mathcal{O}(\epsilon)$, we obtain

$$\frac{dA}{d\mu} + \epsilon A = \mathcal{O}(\epsilon^2), \quad \frac{dB}{d\mu} = \mathcal{O}(\epsilon^2). \quad (18)$$

Ignoring the terms proportional to $\mathcal{O}(\epsilon^2)$, the solutions of the above equations become

$$A(\mu) = A(0)e^{-\epsilon\mu}, \quad B(\mu) = B(0). \quad (19)$$

We now eliminate the secular term by setting $\mu = \tau$. This means that the constants A_0 , B_0 have time dependence as a result of *renormalization*. Requiring the boundary conditions $y(\tau = 0) = 0$, $y(\epsilon\tau = 1) = 1$ in (16), we get $A(0) = B(0) = (e^{-1} - e^{-1/\epsilon})^{-1}$. The final expression becomes

$$y(\tau) = \frac{e^{-\epsilon\tau} - e^{-\tau}}{e^{-1} - e^{-1/\epsilon}}. \quad (20)$$

Using the definition $t = \epsilon\tau$, we see that (20) coincides with the exact solution (2). It is remarkable that the solution obtained by the perturbation method recovers the exact solution.

2.2 Mathieu equation

For the second example using the RG method, we consider the Mathieu equation:

$$\ddot{y} + [\delta + 2\epsilon \cos(2t)]y = 0. \quad (21)$$

Different from the boundary layer problem, this equation has periodically varying potential. The solutions of (21) can be classified as stable or unstable solutions by the coefficients (ϵ, δ) . For the unstable solutions, amplitude of the solution is unbounded and grows exponentially $y \propto e^{\kappa t}$, where κ denotes the characteristic exponent which can be determined by (ϵ, δ) .

In case of the small coefficient $\epsilon \lesssim 1$, the solutions have the fastly oscillating phase and the slowly varying amplitude. Regarding ϵ as a small parameter, we can investigate such behavior from the perturbative analysis. In the singular perturbation techniques, the *multiple scale method* is appropriate [5][6]. To apply this method, it is necessary to introduce multiple time $\tau_n \equiv \epsilon^n t$, ($n = 0, 1, \dots$) and treat them as independent variables. Together with the expansion of y in powers of ϵ , the derivative with respect to t is expanded as

$$\frac{d}{dt} = D_0 + \epsilon D_1 + \epsilon^2 D_2 + \dots; \quad D_n = \frac{\partial}{\partial \tau_n}. \quad (22)$$

Using this method, we can determine the stable/unstable behavior of (21) in slow time scales.

It is not necessary for the RG method to assume the multiple scales. All necessary time scales arise naturally from the RG equation. Let us study the behavior of (21) near the coefficient $\delta \simeq 1$. For the perturbative approach, δ should be expanded as follows:

$$\delta = 1 + \epsilon\delta_1 + \dots. \quad (23)$$

Taking the perturbation series as $y = y_0 + \epsilon y_1 + \dots$ and substituting (23) into (21), we obtain

$$\begin{aligned} \mathcal{O}(\epsilon^0) : \quad & \ddot{y}_0 + y_0 = 0, \\ \mathcal{O}(\epsilon^1) : \quad & \ddot{y}_1 + y_1 = -[\delta_1 + \cos(2t)]y_0, \end{aligned}$$

and so on. The straightforward perturbation up to $\mathcal{O}(\epsilon)$ is given by

$$y(t) = A_0 e^{it} + A_0^* e^{-it} + \epsilon \left[\frac{i}{2} (t - t_0) (\delta_1 A_0 + A_0^*) e^{it} + \frac{1}{8} A_0 e^{i3t} + \text{c.c.} \right] + \mathcal{O}(\epsilon^2), \quad (24)$$

where A_0 is the amplitude at the initial time t_0 and c.c. means the complex conjugate. We observe the secular term in proportion to $(t - t_0)$ appears in the solution y_1 . To eliminate the secular term, we repeat the procedure described in Sec.2.1. For the multiplicative renormalization constant $Z_A = 1 + \sum_n^\infty \epsilon^n z_n$ defined by $A_0 = A(\mu) Z_A(t_0, \mu)$, we choose $z_1 = i(t_0 - \mu)(\delta_1 A + A^*)/(2A)$. We successfully replace t_0 with μ without changing the form of the perturbation:

$$y(t) = \left[A(\mu) + \frac{i}{2} \epsilon (t - \mu) \{ \delta_1 A(\mu) + A^*(\mu) \} \right] e^{it} + \frac{\epsilon}{8} A(\mu) e^{i3t} + \text{c.c.} + \mathcal{O}(\epsilon^2). \quad (25)$$

The RG equation $\partial y / \partial \mu = 0$ for any time t to $\mathcal{O}(\epsilon)$ gives

$$\frac{dA}{d\mu} - \frac{i}{2} \epsilon (\delta_1 A + A^*) = \mathcal{O}(\epsilon^2). \quad (26)$$

Rewriting $A = (u + iv)/2$ (u, v are real functions), the above equation becomes

$$\frac{d}{d\mu} \begin{pmatrix} u \\ v \end{pmatrix} = \frac{\epsilon}{2} \begin{pmatrix} 0 & 1 - \delta_1 \\ 1 + \delta_1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \quad (27)$$

By setting $\mu = t$, the renormalized regular solution up to $\mathcal{O}(\epsilon)$ can be obtained:

$$y(t) \simeq u(t) \cos t - v(t) \sin t + \frac{\epsilon}{8} [u(t) \cos(3t) - v(t) \sin(3t)]. \quad (28)$$

From the RG equation (27), the time dependent functions $u(t), v(t)$ behave as

$$u(t), v(t) \propto e^{\pm (\epsilon/2) \sqrt{1 - \delta_1^2} t}. \quad (29)$$

We see that the slow time scale ϵt appears. We also observe that the unstable behavior arises when $|\delta_1| < 1$ and the characteristic exponent is given by $\kappa = (\epsilon/2) \sqrt{1 - \delta_1^2}$. $|\delta_1| = 1$ provides the marginal transition curves between the stable and the unstable regions in the stability/instability chart of the Mathieu equation.

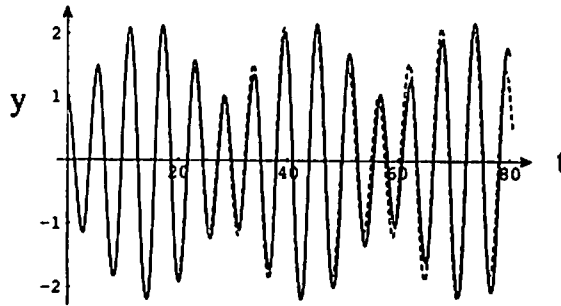
For comparison, we plotted the numerical solution of (21) and the result of the RG method (28) with the same initial conditions (Fig.2). In both the stable and unstable cases, the RG results trace the behavior of the Mathieu equation accurately beyond the time interval $0 \leq \epsilon t \leq 1$.

3 Scalar field dynamics and cosmological perturbation during reheating

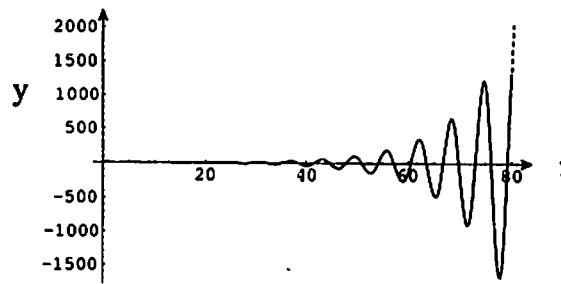
In previous section, we saw that the RG approach is useful and it gives a simple approximation method for solving the DEs. Although the examples we treated in previous section are the linear equations, the RG method can also be applied to the non-linear equations.

As an application of the RG method, we shall investigate the evolution of the cosmological perturbations during reheating[18]. A recent development in a theory of reheating after inflation is one of the most interesting topics in cosmology. It has been recognized that the dynamics of the oscillating scalar fields plays a significant role in the reheating process[7][8]. The non-linear interaction between the scalar fields amplifies the fluctuations of the scalar fields by the effect of parametric resonance and the energy of the oscillating field is transferred to the fluctuation of the massless field by catastrophic particle production. From the viewpoint of the structure formation of our universe, it is an important question whether the growth of fluctuations of the matter field during reheating affects the large-scale inhomogeneities in the universe or not. To answer this question, we must study the evolution of the metric and the matter fluctuation in general relativistic treatment.

(i) Stable case ($\delta=1.3, \varepsilon=0.2$)



(ii) Unstable case ($\delta=1.0, \varepsilon=0.2$)



(iii) Unstable case ($\delta=1.0, \varepsilon=0.5$)

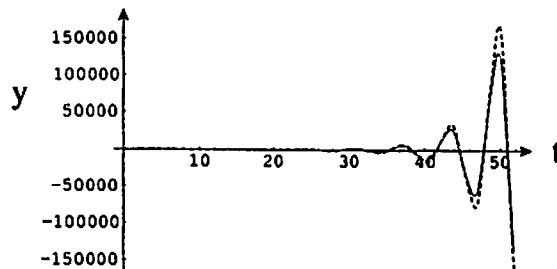


Figure 2: Comparison between the numerical solution and the solution by the RG method: The solid lines represent the numerical calculation of the Mathieu equation (21) and the broken lines are the RG result (28).

In this section, following the paper[18] by the present author and Y.Nambu, we analyze the cosmological perturbation of the scalar field model to clarify whether the dynamics of coherently oscillating scalar field affects the metric perturbation or not. The model we treat here contains two scalar fields ϕ, χ with the potential

$$V(\phi, \chi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}g^2\phi^2\chi^2, \quad (30)$$

where ϕ is inflaton and χ is a massless boson field. (30) describes a typical model of the reheating which is studied by many authors[9]. We begin to give the basic quantities and the evolution equations for the cosmological perturbations. As is shown below, the long wavelength mode of the fluctuations can be expressed by using the background quantities. We will concentrate on the background dynamics during reheating and apply the RG method.

3.1 Basic equations

We consider a homogeneous, isotropic and flat cosmological model with scalar type metric fluctuations. In longitudinal (Newtonian) gauge, the metric is given by

$$ds^2 = -(1 + 2\Phi)dt^2 + a^2(t)(1 - 2\Psi)\delta_{ij}dx^i dx^j. \quad (31)$$

Φ, Ψ correspond to a gauge-invariant potential, which characterize the physical mode of the metric perturbations. To study the metric fluctuations, we shall analyze the Bardeen parameter which represents the fluctuation of the spatial curvature on the uniform energy density slice in super-horizon scale given by [10, 11, 12]

$$\zeta = \Phi - \frac{H^2}{H}(\Phi + H^{-1}\dot{\Phi}); \quad H \equiv \frac{\dot{a}}{a}. \quad (32)$$

For the later analyses, we use the following dimensionless variables:

$$\tau = mt, \quad h = \frac{H}{m}, \quad \varphi_1 = \sqrt{\frac{4\pi}{3}} \frac{\phi}{m_{pl}}, \quad \varphi_2 = \sqrt{\frac{4\pi}{3}} \frac{\chi}{m_{pl}}, \quad \lambda = \frac{3}{4\pi} \left(\frac{gm_{pl}}{m} \right)^2. \quad (33)$$

In terms of these variables, the background Einstein equations are given by

$$h^2 = \varphi_1'^2 + \varphi_2'^2 + 2U(\varphi_1, \varphi_2); \quad U = \frac{1}{2}\varphi_1^2 + \frac{1}{2}\lambda\varphi_1^2\varphi_2^2, \quad (34)$$

$$\varphi_i'' + 3h\varphi_i' + \frac{\partial U}{\partial \varphi_i} = 0 \quad (i = 1, 2), \quad (35)$$

where $(') = d/d\tau$. We use the following gauge-invariant variable to describe the fluctuation:

$$Q_i = \delta\varphi_i + \frac{\varphi_i'}{h}\Psi, \quad (36)$$

where $\delta\varphi_i$ is fluctuation of the scalar field φ_i . The variable Q_i was first introduced by Mukhanov [14] and all the other gauge-invariant quantities can be expressed by these variables. Using the fact $\Phi = \Psi$ in the present model, the Bardeen parameter defined by (32) is written by

$$\zeta = \frac{h}{\varphi_1'^2 + \varphi_2'^2}(\varphi_1'Q_1 + \varphi_2'Q_2). \quad (37)$$

The evolution equation for Q_i is simply expressed by [15]

$$Q_i'' + 3hQ_i' + \left(\frac{k}{ma} \right)^2 Q_i + \sum_{j=1}^2 \left[\frac{\partial^2 U}{\partial \varphi_i \partial \varphi_j} - \frac{6}{a^3} \left(\frac{a^3}{h} \varphi_i' \varphi_j' \right)' \right] Q_j = 0. \quad (38)$$

This is our basic equation.

3.2 Solution in long wavelength limit

As we are interested in the formation of the large-scale structure in our universe, it is important to investigate the metric fluctuation outside the scale of the Hubble horizon. Fortunately, we can obtain the exact solution of eq.(38) in the long wavelength limit ($k \rightarrow 0$) and the result is expressed in terms of the background quantities.

Let us suppose the solution of the background field is written as $\varphi_i(\alpha, C)$, where $\alpha \equiv \log a$ and C is the parameter given by a suitable choice of the integration constants. α plays a role of the time parameter and we can trace the time evolution of the trajectory of the background solution in configuration space (φ_1, φ_2) using this parameter. C distinguishes trajectories in this space and remains constant on a each trajectory.

Using the time parameter α , the field equation (35) is rewritten as

$$\frac{d^2 \varphi_i}{d\alpha^2} + \left(3 + h^{-1} \frac{dh}{d\alpha}\right) \frac{d\varphi_i}{d\alpha} + h^{-2} \frac{\partial U}{\partial \varphi_i} = 0. \quad (39)$$

Differentiating (39) with respect to α, C , we can show that the equation of motion for the tangent vectors $d\varphi_i/d\alpha, d\varphi_i/dC$ becomes (38) in the limit $k \rightarrow 0$. Therefore these vectors are the independent solutions of (38) in long wavelength limit.

Since (38) are the second order coupled differential equations, we have four independent solutions. The remaining two solutions can be obtained as follows: define the component of a 2×2 matrix X as

$$X_{11} = \frac{d\varphi_1}{d\alpha}, \quad X_{12} = \frac{d\varphi_1}{dC}, \quad X_{21} = \frac{d\varphi_2}{d\alpha}, \quad X_{22} = \frac{d\varphi_2}{dC}. \quad (40)$$

Each row vectors in X satisfy (38). Expressing the remaining solutions by the matrix Y and putting $Y = X \cdot P$ yields

$$P'' + (3h + 2X^{-1}X')P' = 0. \quad (41)$$

We can obtain $P' = X^{-1}(X^{-1})^T/a^3$ by using the fact that $X' \cdot X^{-1}$ is the symmetric matrix. Thus the general solution of (38) in long wavelength limit is given by

$$Q_i = \sum_{j=1}^2 \left(c_j^+ X_{ij} + c_j^- \left[X \int \frac{d\tau}{a^3} X^{-1} (X^{-1})^T \right]_{ij} \right), \quad (42)$$

where $(^T)$ represents transpose of the matrix and $c_j^\pm (j = 1, 2)$ are arbitrary constants.

3.3 Background dynamics in reheating phase

We can know the behavior of the long wavelength fluctuation from (42) if we obtain the background solution. As we pay our attention to the reheating phase of the inflaton dominated universe, we assume the condition $\varphi_1, \varphi_2 \lesssim 1$ and $\lambda\varphi_2^2 \lesssim 1$. The former is the condition for the oscillation of the scalar fields. The latter condition implies that the potential energy is dominated by the massive inflaton.

We first solve the background equations in a naive treatment. Because $\lambda\varphi_2^2 \lesssim 1$, the evolution equations (34)(35) can be solved separately and we get oscillatory behavior of the inflaton $\varphi_1 \simeq \bar{\varphi}(\tau) \cos \tau$, where $\bar{\varphi} \propto 1/\tau$. Substituting this into (35) again, we find that the equation for the field φ_2 becomes the same form as the Mathieu equation:

$$(a^{3/2}\varphi_2)'' + [A + 2q \cos(2\tau)](a^{3/2}\varphi_2) = 0; \quad a \propto \tau^{2/3}, \quad (43)$$

where the coefficients A and q are time dependent functions given by $A = 2q = \lambda\bar{\varphi}^2(\tau)/2$. The stability/instability chart of the Mathieu equation says that φ_2 has unstable behavior at $1/3 \leq q \leq 1$, in the region of the first resonance band of the Mathieu function.

To investigate the Bardeen parameter during the reheating, we must study the background evolution more precisely. We can show that the RG method is a powerful tool for analyzing the background dynamics. To apply the RG method, we identify the small expansion parameter ϵ as the amplitude of the scalar fields $\varphi_i \sim \mathcal{O}(\epsilon) < 1$. Since we want to take into account of the non-linear interaction, we assume

$\tilde{\lambda} \equiv \epsilon\lambda \sim \mathcal{O}(1)$ which implies λ is not small parameter. Then the background quantities are expanded as follows:

$$\varphi_i = \epsilon\varphi_{i(1)} + \epsilon^2\varphi_{i(2)} + \dots, \quad h = \epsilon h_{(1)} + \epsilon^2 h_{(2)} + \dots \quad (44)$$

The crucial treatment in the present model is to insert the mass term $\omega^2\varphi_2$ in the equation of φ_2 (35). We put $\omega^2 = 1 + \epsilon\sigma$ and take the limit $\epsilon\sigma \rightarrow -1$ after the end of calculation. This treatment is necessary to extract the effect of parametric resonance in φ_2 field.

In appendix, we apply the RG method to the background equations (34)(35). The final form of the solution up to $\mathcal{O}(\epsilon)$ becomes

$$\varphi_1 \simeq h \cos \Theta \cos(\tau + \theta), \quad \varphi_2 \simeq h \sin \Theta \cos(\tau + \psi), \quad (45)$$

where we replace $\epsilon h_{(1)}$ with h . The variables Θ, θ, ψ and h are time dependent and determined by so-called RG equations (58)-(61) which are the result of the renormalization of the initial amplitude.

We further impose the condition $\varphi_2 \ll \varphi_1$ ($\Theta \ll 1$), which makes our analysis easier. We will restrict our analysis in the case of $\Theta \ll 1$ hereafter. (60) becomes $\theta' \simeq 0$ and we can set $\theta = 0$ without loss of generality. Eqs.(58)(59)(61) become

$$\Theta' \simeq \frac{1}{2}q(\tau) \sin \gamma \cdot \Theta, \quad \gamma' \simeq -1 + q(\tau)(2 + \cos \gamma), \quad (46)$$

$$h' = -\frac{3}{2}h^2, \quad (47)$$

where we define the time dependent parameter

$$q(\tau) = \frac{1}{4}\lambda h^2. \quad (48)$$

Since (47) yields $h \propto 1/\tau$, $q(\tau)$ is the decreasing function of time. Defining the new variables u, v by $u = \Theta \cos(\gamma/2)$, $v = \Theta \sin(\gamma/2)$, (46) is rewritten as follows:

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 1-q \\ 3q-1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \quad (49)$$

The eigenvalue of the matrix in the right hand side of (49) is real for $1/3 \leq q \leq 1$ and Θ can grow during this epoch. We see that the function q corresponds to the coefficient q in the Mathieu equation (43). Recall that we have $\varphi_1 \simeq h \cos \tau$, $\varphi_2 \simeq h \Theta \cos(\tau + \gamma/2)$ from (45), we recognize that the amplitude of φ_2 has growing behavior as a result of the parametric resonance. Soon after q becomes smaller than $1/3$, Θ reaches a constant value. Fig.3 shows the evolution of the background fields obtained by numerical integration of (34)(35). The field φ_2 (the solid line) is amplified during the short period denoted by the horizontal arrow, which corresponds to the resonance epoch $1/3 \leq q \leq 1$. We can evaluate how much φ_2 is amplified through the resonance epoch. From (46), we have

$$\Theta \simeq \Theta_0 \exp \left[\frac{1}{2} \int_{\tau_i}^{\tau_f} d\tau' q \sin \gamma \right]. \quad (50)$$

Using this expression, we get the upper bound

$$\frac{\Theta(\tau_f)}{\Theta(\tau_i)} \lesssim \exp(c_* \sqrt{\lambda}); \quad c_* = \frac{1}{6} \left(1 - \frac{1}{\sqrt{3}} \right) \simeq 0.07, \quad (51)$$

where we used the relation $q(\tau_f) = 1/3$, $q(\tau_i) = 1$. The above expression is valid if the condition $\Theta \ll 1$ is satisfied throughout the resonance epoch.

3.4 Evolution of the Bardeen parameter

Behavior of the background solution obtained by RG method is translated to the metric fluctuation ζ using the long wavelength solution (42). The reduced system (46) has the two integration constants: The initial values of γ, Θ . During the resonance epoch, γ approaches a constant value which does not depend on

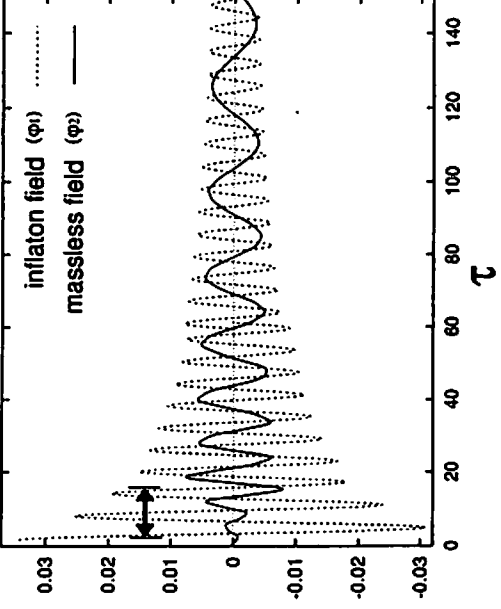


Figure 3: The amplitude of the background fields φ_1 and φ_2 is shown as a function of time τ by solving (34)(35) numerically. The broken and the solid lines represent the inflaton(φ_1) and the massless field(φ_2) respectively. During the period denoted by the horizontal arrow, the field φ_2 is amplified by the effect of parametric resonance. We used the initial values which correspond to the expression (45) at $\tau = 0$ with the parameters $\lambda = 4000$, $h = 0.035$, $\Theta = 0.01$, $\gamma = 0.4$, $\theta = 0.0$.

the initial value. We can regard Θ_0 (the initial value of Θ) as the parameter C described in Sec.3.2. That is, Θ_0 is the parameter which distinguishes the trajectories in the reduced system (Θ, γ) . Substituting (45) into (42) and evaluating the integral adiabatically, we find that the dominant contribution to the Bardeen parameter ζ comes from the solution $Q_i^{qd} \equiv d\varphi_i/d\alpha$ and

$$Q_1^{iso} \equiv \frac{d\varphi_1}{dC} \simeq -h\Theta_{,C} \sin \Theta \cos \tau, \quad Q_2^{iso} \equiv \frac{d\varphi_2}{dC} \simeq h\Theta_{,C} \cos \Theta \cos \left(\tau + \frac{\gamma}{2}\right), \quad (52)$$

where $\Theta_{,C} = \exp \left[\int d\tau' q/2 \cdot \sin \gamma \right]$. The other independent solutions are recognized as the decaying modes.

The solution Q_i^{qd} is referred to as the adiabatic growing mode[17]. Substituting Q_i^{qd} into (37), the Bardeen parameter remains constant in time. The constancy of ζ arises in the hydrodynamical perturbation without the entropy fluctuation [12]. For the solution Q_i^{iso} , (37) gives

$$\zeta^{iso} \simeq h\Theta_{,C} \sin \Theta \cos \Theta \cdot \frac{\sin \tau \cos \tau - \sin \left(\tau + \frac{\gamma}{2}\right) \cos \left(\tau + \frac{\gamma}{2}\right)}{\cos^2 \Theta \sin^2 \tau + \sin^2 \Theta \sin^2 \left(\tau + \frac{\gamma}{2}\right)}. \quad (53)$$

We see that ζ^{iso} has the periodically sharp peaks around the zero points of φ_1' . This behavior also appears in the single field case[16]. Except for the short interval of the peaks, ζ^{iso} traces the amplitude of Q_1^{iso} . Therefore ζ^{iso} deviates from zero when $\Theta_{,C}$ grows due to the parametric resonance. Note that ζ^{iso} vanishes when the background field φ_2 is zero ($\Theta = 0$). Because the initial amplitude of the curvature perturbation ζ^{iso} is vanishingly small, we can identify Q_i^{iso} is the contribution of the iso-curvature fluctuation.

We confirm the behavior of ζ^{iso} by solving (34)(35)(38) numerically. In Fig.4, we show the time evolution of the iso-curvature mode. In contrast to the constancy of the adiabatic growing mode (the dashed line), we observe that ζ^{iso} (the solid line) is amplified during the resonance epoch when the amplitude of the background φ_2 field grows (see Fig.1). After the resonance epoch, ζ^{iso} approaches zero

due to the Hubble damping which can be seen by the analytic result (53). Using (51)(53), we can estimate the growth of ζ^{iso} except for the sharp peak:

$$\left| \frac{\zeta^{iso}(\tau_f)}{\zeta^{iso}(\tau_i)} \right| \approx \frac{h(\tau_f)}{h(\tau_i)} \cdot \exp(2c_s \sqrt{\lambda}). \quad (54)$$

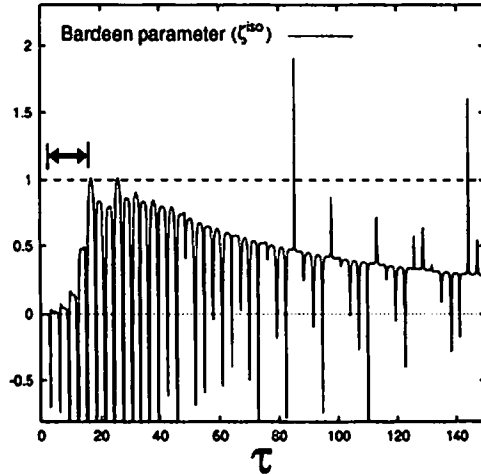


Figure 4: The Bardeen parameter ζ^{iso} is plotted as a function of time τ by solving (34)(35)(38) numerically. We calculate (34)(35) with the same initial condition as described in Fig.1. As for the fluctuations, we set the initial values corresponding to the expression (52) at $\tau = 0$. The figure shows that ζ^{iso} (the solid line) is amplified during the resonance epoch denoted by the horizontal arrow and approaches zero due to the Hubble damping. We also plotted the Bardeen parameter for the adiabatic growing mode (the dashed line) obtained by the substitution of Q_i^{ad} into (37).

4 Summary

In this article, We have reviewed the RG method as a tool for solving DEs and described its application to cosmology. In the first part of this article, treating the simple examples, we introduced the RG method. Comparing with the conventional perturbation methods, we demonstrated that the RG approach gives a useful and simple unified procedure in the perturbation techniques. In the second part, we have presented the analysis of cosmologication perturbation with two scalar fields in reheating after inflation. Using the fact that the long wavelength mode of the fluctuations can be written in terms of the background quantities, we applied the RG method to investigate the non-linear background dynamics in the coherently oscillating stage. We found that the massless field gets the effect of parametric resonance and this leads to the amplification of the Bardeen parameter. The results implies that the Bardeen parameter may deviate from the initial amplitude in reheating phase even though the initial amplitude determined in the inflationary stage is dominated by the adiabatic mode. To evaluate how much the amplitude deviates in the realistic cosmological setting, we should analyze the evolution of cosmological perturbations more quantitatively. This is our next task.

The application of the RG method is not limited to the problem of the ordinary DEs. It is known that RG method is also useful to analyze the partial DEs[1][4]. The perturbative RG approach to the partial DEs is similar to the *reductive perturbation theory* which extracts the characteristic behavior in the dynamical system[3]. The RG method can also reduce the complicated system to a simple dynamical

system which is described by the RG equation. Thus we can expect that the RG approach also provides a useful approximation for more complicated problems in general relativity and cosmology.²

In cosmology, the important non-linear problems are

- (i) Structure formation in Newton cosmology,
- (ii) Dynamics of relativistic inhomogeneous cosmology.

To resolve the problems analytically, various approximation methods are studied. Comparing with these methods, we notice that some of them are similar to the RG approach. In (i), the present author discussed that adhesion approximation in Newton cosmology may be understood as the RG approach [20]. The adhesion approximation is known as a nonperturbative extension of the Zel'dovich approximation adding a viscosity term [21]. The viscosity term is introduced phenomenologically in order to mimic the adhesive effects of nonlinear gravity to the collisionless particles. The adhesion model is described by the three-dimensional generalization of Burger's equation and the analytic study of this equation gives a good explanation to the result of N-body simulation. The present author pointed out that if we treat the collisionless particles as the irrotational dust with the small pressure ($P = w\rho$; $w \ll 1$), the evolution equation for velocity perturbation can be reduced to the 3D Burger's equation using the RG method. In (ii), there also exists the non-linear perturbation methods similar to the RG approach. For instance, Isaacson expansion, the averaged Lagrangian method and gradient expansion (anti-Newton approximation)[22][23][24, 25]. It may be expected that they are understood as the RG approach in a unified way. To clarify this thing, we should compare these approximations with the RG method in detail.

Acknowledgements

The author would like to thank M.Egi, K.Imai, T.Konishi, K.Matsuba, K.Nakamura, H.Yamada, and Y.Y. Yamaguti for useful discussions and critical comments about the RG method. The author would like to thank M.Sakagami for teaching him the references of RG approach. The author would also acknowledge Y.Nambu for careful reading of the manuscript.

Appendix: Applying the RG method to the background system

In this appendix, we apply the RG method to the background system (34)(35). Following the treatment described in Sec.3.3, we substitute (44) to (34)(35) and expand in powers of ϵ :

$$\begin{aligned} \mathcal{O}(\epsilon^1) : \quad & \varphi''_{i(1)} + \varphi_{i(1)} = 0, \quad (i = 1, 2) \\ & h_{(1)} = \sqrt{\varphi'^2_{1(1)} + \varphi'^2_{1(1)} + \varphi'^2_{2(1)} + \varphi'^2_{2(1)}}, \\ \mathcal{O}(\epsilon^2) : \quad & \varphi''_{1(2)} + \varphi_{1(2)} = -3h_{(1)}\varphi'_{1(1)} - \bar{\lambda}\varphi^2_{2(1)}\varphi_{1(1)}, \\ & \varphi''_{2(2)} + \varphi_{2(2)} = -3h_{(1)}\varphi'_{2(1)} - \sigma\varphi_{2(1)} - \bar{\lambda}\varphi^2_{1(1)}\varphi_{2(1)}. \end{aligned}$$

We first solve the above equations order by order. The solutions up to $\mathcal{O}(\epsilon^2)$ become

$$\begin{aligned} \varphi_1 = \epsilon & \left[A_0 + \epsilon(\tau - \tau_0) \left\{ -\frac{3}{2}h_{(1)}A_0 + i\frac{\bar{\lambda}}{2} (2|B_0|^2 A_0 + B_0^2 A_0^*) \right\} \right] e^{i\tau} \\ & + \epsilon^2 \frac{\bar{\lambda}}{8} B_0^2 A_0 e^{i3\tau} + c.c. + \mathcal{O}(\epsilon^3) \end{aligned}$$

and

$$\begin{aligned} \varphi_2 = \epsilon & \left[B_0 + \epsilon(\tau - \tau_0) \left\{ -\frac{3}{2}h_{(1)}B_0 + i\frac{\bar{\lambda}}{2} (2|A_0|^2 B_0 + A_0^2 B_0^*) + \frac{i}{2}\sigma B_0 \right\} \right] e^{i\tau} \\ & + \epsilon^2 \frac{\bar{\lambda}}{8} A_0^2 B_0 e^{i3\tau} + c.c. + \mathcal{O}(\epsilon^3), \end{aligned}$$

² An alternative RG approach which uses the scale invariance of the system has been explored[19].

where c.c. means complex conjugate and $h_{(1)}$ is given by

$$h_{(1)} = 2\sqrt{|A_0|^2 + |B_0|^2}. \quad (55)$$

A_0, B_0 are the amplitudes at the initial time $\tau = \tau_0$. Because the secular terms which grow as $(\tau - \tau_0)$ makes the perturbative expansion break down. it is necessary to improve the perturbation series. Following the RG manner, we succeed to get the following μ dependent solutions:

$$\begin{aligned} \varphi_1 = & \epsilon \left[A + \epsilon(\tau - \mu) \left\{ -\frac{3}{2}h_{(1)}A + i\frac{\tilde{\lambda}}{2}(2|B|^2A + B^2A^*) \right\} \right] e^{i\tau} \\ & + \epsilon^2 \frac{\tilde{\lambda}}{8} B^2 A e^{i3\tau} + \text{c.c.} + \mathcal{O}(\epsilon^3), \end{aligned} \quad (56)$$

$$\begin{aligned} \varphi_2 = & \epsilon \left[B + \epsilon(\tau - \mu) \left\{ -\frac{3}{2}h_{(1)}B + i\frac{\tilde{\lambda}}{2}(2|A|^2B + A^2B^*) + \frac{i}{2}\sigma B \right\} \right] e^{i\tau} \\ & + \epsilon^2 \frac{\tilde{\lambda}}{8} A^2 B e^{i3\tau} + \text{c.c.} + \mathcal{O}(\epsilon^3). \end{aligned} \quad (57)$$

The requirement $(\partial\varphi_i/\partial\mu)_\tau = 0$ for arbitrary τ yields the following RG equations:

$$\begin{aligned} \frac{dA}{d\mu} &= \epsilon \left[-\frac{3}{2}h_{(1)}A + i\frac{\tilde{\lambda}}{2}(2|B|^2A + B^2A^*) \right] + \mathcal{O}(\epsilon^2), \\ \frac{dB}{d\mu} &= \epsilon \left[-\frac{3}{2}h_{(1)}B + \frac{i}{2}\sigma B + i\frac{\tilde{\lambda}}{2}(2|A|^2B + A^2B^*) \right] + \mathcal{O}(\epsilon^2). \end{aligned}$$

By setting $\mu \equiv \tau$ in (56)(57), the secular terms are eliminated and we obtain the regular perturbation. Rewriting the variables A, B by

$$A = \frac{1}{2}h_{(1)} \cos \Theta e^{i\theta}, \quad B = \frac{1}{2}h_{(1)} \sin \Theta e^{i\psi},$$

and using (55), we get the solution (45) and the above RG equations reduce to

$$\Theta' = \epsilon \frac{\tilde{\lambda}}{16} h_{(1)}^2 \sin \gamma \sin 2\Theta, \quad (58)$$

$$\gamma' = -1 + \epsilon \frac{\tilde{\lambda}}{4} h_{(1)}^2 \cos 2\Theta (2 + \cos \gamma), \quad (59)$$

$$\theta' = \epsilon \frac{\tilde{\lambda}}{8} h_{(1)}^2 \cos \gamma \sin^2 \Theta, \quad \gamma = 2(\psi - \theta), \quad (60)$$

$$h'_{(1)} = -\epsilon \frac{3}{2} h_{(1)}^2, \quad (61)$$

where we set $\epsilon\sigma = -1$.

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Renormalization Group Approach in Newtonian Cosmology

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Abstract

We apply the renormalization group method to examine the observable scaling properties in the Newtonian cosmology. The original scaling properties of the equations of motion in our model are modified for averaged observables on the constant time slices. In the flow diagram of the renormalization group, we find three robust fixed points: Quiescent, Einstein-de Sitter, and Milne fixed points. The stability/instability property of them does not change under the effect of fluctuations. Inspired by the inflationary scenario in the early Universe, we set the Einstein-de Sitter fixed point with small amount of fluctuations, as the boundary condition at the horizon scale. We solve the RG equations under this boundary condition toward the smaller scales and obtain general behavior of observables; reducing density parameter Ω and increasing Hubble parameter H . The quantitative scaling properties are analyzed by calculating the characteristic exponents around each fixed point.

1 Introduction

Observed physical variables such as the Hubble parameter H or the density parameter Ω seem to vary depending on the observational methods and the distance scales. Since the real universe is clumpy and inhomogeneous at least in much smaller scale than present horizon scale, the observed H and Ω may be deviated from the homogeneous Friedmann background. In fact, the density parameter Ω , defined by the ratio of the real matter density ρ to the critical density $\rho_{cr} = 3H^2/8\pi G$, linearly increases toward the larger scale as $\Omega \propto L$ up to the scale about 10Mpc [1],[2].

On the other hand, several observations such as density-size relation for galaxies or the observed two-point correlation function $\xi(r)$ for galaxies and clusters seem to support a scaling property in our present universe at least up to 10 Mpc. Moreover, although it is still controversial, some groups discuss a fractal structure of the Universe and claim that the fractal dimension is about two up to the scale of 1000 Mpc[3],[4].

Conventionally, the above scaling properties are understood as an interplay of the scale invariant initial condition (Harrison-Zel'dovich spectrum) and the gravitational instability [5]. Here in this paper,

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we explore systematic physics, if any, which governs the scaling properties of observables in the present Universe. The scaling properties in observables may reflect any scaling law in the equations of motion for the Universe. Therefore we start our analysis with the equations of motion for fluid in the Newtonian Universe neglecting the pressure term. This set of equations is invariant under certain scale transformation. The scale dependence of real observed physical variable may not necessarily reflect their original scaling property. In order to reflect the real observational variables, we have to take into account at least the following two factors other than the scaling of basic equations.

First, an observable region is limited by causality. For example, we can only observe a light signal from a galaxy located on the past light-cone of us. In Newtonian cosmology, the observable quantities are located on a hyper-slice of constant time. Because in general the above scale transformation also requires a change of the time variable, we have to adjust the scale transformation by its time evolution in the past direction. Second, we may just observe averaged quantities in certain region in space. For example, the density parameter Ω is a dimensionless quantity and is invariant under the scale transformation. However, the observed density parameter can be, since it is obtained from two independent or separated measurements; i.e, the direct measurement of averaged matter density $\langle \rho \rangle$ and the cosmic expansion parameter H . In this sense, we need to take into account the observational way and the averaging effect of each physical variable carefully.

Thus in general, two observables at different scales would be related with each other by the naive scaling (S), compensating time evolution (T), and the inevitable averaging procedure (A). We have to apply each infinitesimal operations iteratively to obtain finite operations since they are non-linear transformations. The most sophisticated method so far to deal with such procedure on the scaling properties would be the renormalization group (RG) method[6]. This is a general method to obtain the observable response of the system against the scale change and is widely used in various fields in physics. In general relativity and cosmology, it has been applied to the critical behavior of a self-similar solution during the gravitational collapse of a fluid[7] or the analysis of the averaging effect of physical quantities on the fixed time slice in inhomogeneous universe [8]. The former is reflecting the aspects of (S) and (T) and the latter is reflecting (S) and (A).

In this paper, we try to apply this RG method including the full aspects of (S),(T), and (A) to the Newtonian Cosmology. The RG equations we obtain give the effective scale change of averaged observables, including the information about actual observations and the effect of spatial fluctuations of observables.

2 Derivation of Renormalization Group Equations

Here we begin with the following dynamical flow equations for irrotational dust in Newtonian cosmology, described by the spatial derivative of the velocity field decomposed into expansion $\theta = \nabla \cdot \mathbf{v}$, shear σ^i_j , and rotation ω^i_j ,

$$\frac{d\rho}{dt} = -\rho\theta, \quad (1)$$

$$\frac{d\theta}{dt} = -\frac{1}{3}\theta^2 - 2\sigma^2 - 4\pi G\rho, \quad (2)$$

$$\frac{d\sigma_{\pm}}{dt} = -\frac{2}{3}\theta\sigma_{\pm} - \sigma_{\pm}^2 + \frac{2}{3}\sigma^2 - E_{\pm}, \quad (3)$$

where $\frac{d}{dt}$ denotes the Lagrange full time-derivative and $\sigma^2 = 3(\sigma_+^2 + \sigma_-^2)$, respectively. We have assumed $\omega^i_j = 0$ and set

$$\begin{aligned} \sigma^i_j &= \text{diag}(\sigma_+ + \sqrt{3}\sigma_-, \sigma_+ - \sqrt{3}\sigma_-, -2\sigma_+), \\ E^i_j &= \text{diag}(E_+ + \sqrt{3}E_-, E_+ - \sqrt{3}E_-, -2E_+). \end{aligned} \quad (4)$$

The tensor $E^i_j = \nabla^i \nabla_j \phi - \delta^i_j \Delta \phi / 3$ denotes the tidal force which comes from the spatial difference of the gravitational acceleration. In pure Newtonian dynamics, this is a slaving variable and there is no equation of motion for it. Fortunately, several excellent approximation schemes have been developed to

deal with this highly non-local quantity. In this paper, we mainly follow the local tidal approximation (LTA) [9], in which the equation of motion for $E_{\pm}$ is given by

$$\frac{dE_{\pm}}{dt} = -\theta E_{\pm} - 4\pi G \rho \sigma_{\pm}. \quad (5)$$

The set of equations of motions (1) - (3), (5) are invariant under the following scaling transformation:

$$\begin{aligned} t &\rightarrow t' \equiv e^{s\Delta\tau} t \\ \mathbf{x} &\rightarrow \mathbf{x}' \equiv e^{\Delta\tau} \mathbf{x} \\ \theta &\rightarrow \theta' \equiv e^{-s\Delta\tau} \theta \\ \sigma_{\pm} &\rightarrow \sigma'_{\pm} \equiv e^{-s\Delta\tau} \sigma_{\pm} \\ E_{\pm} &\rightarrow E'_{\pm} \equiv e^{-s\Delta\tau} E_{\pm} \\ \rho &\rightarrow \rho' \equiv e^{-2s\Delta\tau} \rho, \end{aligned} \quad (6)$$

where s is a constant free parameter and $\Delta\tau = \tau' - \tau$ measures a change of scale. Here we do not directly treat a scaling solution for these equations. we would rather treat the dynamical equations for averaged variables derived from taking the average for (1) - (3), (5) on a finite spatial domain $\mathcal{D}(t)$, since it seems plausible to expect that some averaged values around an observer, rather than bare physical variables, possess a scaling property. Such a scaling is feasible at every spatial point if we assume a kind of fractal property in the matter distribution. It would be consistent with the cosmological principle even for highly inhomogeneous cosmological model. We assume a scaling property as (6) not only for averaged values of the variables in the above equations but also for those of the fluctuations such as ρ^2 .

We define a spatial average (...) of an observable $f(t, \mathbf{x})$ at time t as

$$\langle f \rangle_{\mathcal{D}(t)} = \frac{1}{V(t)} \int_{\mathcal{D}(t)} d^3x f(t, \mathbf{x}), \quad (7)$$

where $\mathcal{D}(t)$ is a spatial domain at time t and $V(t)$ is its volume defined by $V(t) = \int_{\mathcal{D}(t)} d^3x$. The observable $f(t, \mathbf{x})$ can be any function of $\theta, \rho, \sigma_{\pm}, E_{\pm}$. We derive the expression for the infinitesimal scale change of the averaged observable on the same time slice, which are decomposed into two parts:

- (i) a change $\Delta_S(f)$ associated with a scale transformation from $\mathcal{D}(t)$ to $\mathcal{D}'(t')$,
- (ii) a change $\Delta_T(f)$ associated with a time evolution from $\mathcal{D}'(t')$ to $\mathcal{D}'(t)$.

That is,

$$\begin{aligned} \frac{\Delta \langle f \rangle_{\mathcal{D}(t)}}{\Delta \ell} &= \frac{1}{\Delta \ell} [(\langle f \rangle_{\mathcal{D}'(t')} - \langle f \rangle_{\mathcal{D}(t)}) + (\langle f \rangle_{\mathcal{D}'(t)} - \langle f \rangle_{\mathcal{D}'(t')})] \\ &\equiv \frac{1}{\Delta \ell} (\Delta_S \langle f \rangle + \Delta_T \langle f \rangle). \end{aligned} \quad (8)$$

where the parameter ℓ measures the true scale change on this time slice, which is defined by

$$e^{\Delta \ell} \equiv e^{\ell' - \ell} = \left(\frac{V'(t)}{V(t)} \right)^{1/3}, \quad (9)$$

and is different from the naive scaling parameter τ .

The first part $\Delta_S(f)$ comes directly from the scaling law of the observable $\langle f \rangle_{\mathcal{D}(t)}$ and the second part $\Delta_T(f)$ comes from the time evolution of $\langle f \rangle_{\mathcal{D}(t)}$ obtained by Buchert and Ehlers [10]. To obtain the true scale change on the same time slice, we have to transform it back to the original time slice by the time evolution (ii). Thus we obtain the relation between l and ℓ including $\langle \theta \rangle$, $\Delta \ell = \Delta \tau - \frac{\langle \theta \rangle}{3} \Delta t$. The time interval $\Delta t = t' - t$ is given by the scale transformation (6) as $\frac{\Delta t}{\Delta \tau} = st$. Hereafter we identify this fixed time t as the present cosmic time t_0 , which is set to be 1. From these relations, we obtain the differential equation for the averaged observable $\langle f \rangle_{\mathcal{D}(t)}$:

$$\frac{d \langle f \rangle_{\mathcal{D}(t)}}{d \ell} = S \left(\alpha \langle f \rangle_{\mathcal{D}(t)} + \langle \theta \rangle_{\mathcal{D}(t)} \langle f \rangle_{\mathcal{D}(t)} - \langle \theta f \rangle_{\mathcal{D}(t)} - \left\langle \frac{df}{dt} \right\rangle_{\mathcal{D}(t)} \right), \quad (10)$$

where $S = s/(1 - s\langle\theta\rangle_{\mathcal{D}(t)}/3)$. The parameter s -dependence appears solely through this factor S .

The equation (10) is not yet sufficient to obtain a closed set of RG equations because of the non-linearity of the basic equations. Differential equations for the n -th order averaged quantities are not closed up to the n -th order. They necessarily contain $(n + 1)$ -th order averaged quantities. This is the famous BBGKY hierarchy in statistical physics. The ordinary method to obtain the closed set of equations is to truncate this hierarchy at some order. In our case, we include the effect of fluctuations at the lowest non-trivial level. Then we will neglect all the intrinsic fluctuations (cumulants) higher than the third order, finding the following truncation formula:

$$\langle fgh \rangle \rightarrow \langle f \rangle \langle gh \rangle + \langle g \rangle \langle fh \rangle + \langle h \rangle \langle fg \rangle - 2\langle f \rangle \langle g \rangle \langle h \rangle. \quad (11)$$

To write down the basic equations, we shall use the second order cumulant $\langle fg \rangle_c$ instead of the averaged quadratic quantity $\langle fg \rangle$, where $\langle fg \rangle_c \equiv \langle fg \rangle - \langle f \rangle \langle g \rangle$. Applying this reduction, we obtain a closed set of 27 differential equations for averaged observables up to second order.

3 Analysis of RG Equations : Fixed Points and RG Flow

We now examine the RG equations. First, we analyze the property of the fixed points of RG equations. We then examine the property of the eigenvalues derived from linearizing the RG equations around these fixed points. Next, we study a stability of those fixed points in various parameter spaces. Finally in this section, we set fixed point E with small amount of fluctuations, as the boundary condition at the horizon scale in our present Universe and examine a scale dependence of averaged variables toward smaller scale.

3.1 Fixed points and their stabilities of RG equations.

First we analyze the structure of our RG equations, which are written in a vector form as

$$\frac{d\langle F \rangle}{d\ell} = S(A[\langle F \rangle] + C \cdot \langle FF \rangle_c) \quad (12)$$

$$\frac{d\langle FF \rangle_c}{d\ell} = S B[\langle F \rangle] \cdot \langle FF \rangle_c, \quad (13)$$

where $\langle F \rangle$ and $\langle FF \rangle_c$ are 6- and 21-dimensional vectors defined as $\langle F \rangle \equiv (\langle f_i \rangle)$ and $\langle FF \rangle_c \equiv (\langle f_i f_j \rangle_c)$ with f_i 's ($i = 1 \sim 6$) denoting $\rho, \theta, \sigma_+, \sigma_-, E_+$ and E_- . $A[\langle F \rangle]$, $B[\langle F \rangle]$, and C are 6-dimensional vector of quadratic functions of $\langle f_i \rangle$, 21×21 matrix of linear functions of $\langle f_i \rangle$, and 6×21 constant matrix, respectively. Setting $\langle FF \rangle_c = 0$, we get the first order RG equations,

$$\frac{d\langle F \rangle}{d\ell} = SA[\langle F \rangle]. \quad (14)$$

These equations guarantee that the fixed points $\langle F \rangle^*$ in the first order RG equations are always those for the second order RG equations with $\langle FF \rangle_c^* = 0$, although new additional fixed points appear in the second order.

The stability around each fixed point is determined by the eigenvalues of the matrix derived from linearizing the RG equations around the fixed point. From the form of the linearized equations, it is guaranteed that the eigenvalues of the first order matrix $\mathcal{W}^{[1]}$ are always those of the second order matrix $\mathcal{W}^{[2]}$. The other eigenvalues of the second order matrix are constructed by a sum of any two eigenvalues of the first order matrix. The positive (negative) eigenvalue of the matrix $\mathcal{W}^{[2]}$ (or $\mathcal{W}^{[1]}$) shows the instability (stability) of the fixed point $(\langle F \rangle^*, \langle FF \rangle_c^*)$ (or $\langle F \rangle^*$) toward the larger scale, i.e. in the large ℓ -direction. The stability and the instability are exchanged if we follow the RG flow in the opposite direction, i.e. toward the smaller scale.

The eigenvalues fully characterize the scale dependence of all physical variables in the vicinity of each fixed point. Although each observable is scaled according to its dimension, the effect of dynamics changes the scaling property from the original bare scaling law (6), so that the scaling of each observable

is determined by the superposition of several eigen functions as follows:

$$\begin{aligned} \langle f_i \rangle &= \langle f_i \rangle_* + \sum_{m=1}^N \epsilon_{i,m} e^{\lambda_m \ell} \\ \lambda_{\min} &= \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N = \lambda_{\max}, \end{aligned} \quad (15)$$

where $\langle f_i \rangle_*$ is the value of $\langle f_i \rangle$ at a fixed point, $\epsilon_{i,m}$ is some small constant, and λ_m ($m = 1 \sim N$) are eigenvalues. $N = n(n+3)/2$ for the second order RG equations, while $N = n$ for the first order RG equations, with n being the number of observables. Since the maximum (minimum) eigenvalue, $\lambda_{\max}(\lambda_{\min})$ becomes the most relevant toward larger (smaller) scale, the scaling property of each observable around fixed point is determined by $\lambda_{\max}(\lambda_{\min})$.

3.2 Fixed points and the flow of RG equations

Since our model of the Universe may be quite inhomogeneous, an inclusion of fluctuations of the physical variables is inevitably important, when we compare our results with those for the real Universe. Hence, we add fluctuations and see how the fluctuations affect the RG flow diagram. We just include second order cumulants to the RG equations and truncate this hierarchy there by ignoring all intrinsic fluctuations (cumulants) higher than the third order.

For the case without shear and tidal force as well as without fluctuations, we have simple RG equations (14) with

$$A[\langle F \rangle] = \begin{pmatrix} -\langle \theta \rangle + \frac{1}{3}\langle \theta \rangle^2 + 4\pi G\langle \rho \rangle \\ (\langle \theta \rangle - 2)\langle \rho \rangle \end{pmatrix} \quad (16)$$

for two averaged variables $\langle F \rangle = (\langle \rho \rangle, \langle \theta \rangle)$. We easily find three fixed points $\langle F \rangle_{(k)}^*$ ($k = 1, 2, 3$), which are listed in Table 1. All eigenvalues of $\mathcal{W}^{[1]}$ for three fixed points are given in Table 1.

As seen from Table 1, the fixed point (1), which we call Q (quiescent), is stable toward larger scale. For increasing scale, the Universe approaches this fixed point, at which both the energy density and expansion vanish. The second fixed point (2), which we call E, is saddle. It represents the ‘‘Einstein-de Sitter’’ Universe because $\langle \theta \rangle_{(2)}^* = \theta_{\text{EdS}}$ and $\langle \rho \rangle_{(2)}^* = \rho_{\text{EdS}}$. The third fixed point (3), which we call M, is unstable in any direction and corresponds to the ‘‘Milne’’ Universe. This Universe is reduced to the ‘‘Minkowski’’ space by adjusting the time slice appropriately. These three fixed points (1)~(3) still remain and their stabilities are also unchanged, even after the fluctuations, shear and tidal forces are included (See Table 2 as the example including fluctuations). Since all new fixed points are saddle points, global properties of RG flows toward both larger and smaller scales are solely determined by the fixed points Q and M, respectively.

As we mentioned, we find that the eigenvalue in the second order RG equations $\{\lambda_m^{[2]} | m = 1, \dots, n(n+3)/2\}$ are constructed by those in the first order $\{\lambda_m^{[1]} | m = 1, \dots, n\}$ as

$$\begin{aligned} \lambda_m^{[2]} &= \lambda_m^{[1]} \quad \text{for } m = 1, \dots, n \\ \lambda_m^{[2]} &= \lambda_l^{[1]} + \lambda_{l'}^{[1]} \quad \text{for } m = n+1, \dots, n(n+3)/2 \quad \text{with } l, l' = 1, \dots, n \end{aligned} \quad (17)$$

where n is the number of observables. Especially, the minimum eigenvalues at fixed points in the first order, in particular the fixed points Q, E and M, are given by

$$\lambda_{\min}^{[2]} = 2\lambda_{\min}^{[1]}, \quad (18)$$

if $\lambda_{\min}^{[1]} < 0$, otherwise

$$\lambda_{\min}^{[2]} = \lambda_{\min}^{[1]}. \quad (19)$$

We examine the RG flow for the case including fluctuations. Here we choose the fixed point E as a plausible boundary condition at the present horizon scale and beyond. This boundary condition would be inferred from an inflationary scenario in the early Universe, which predicts the flat FRW Universe with tiny fluctuations as our real Universe.

With this boundary condition, we solve the RG equations toward smaller scale and study the scale change of various observables. In solving the RG equations toward the smaller scale, we must remember that (i) the positive (negative) eigenvalue of the matrix $W^{[2]}$ at a fixed point denotes stability (instability). (ii) The direction of the integration is opposite to that of flow. It may be more comprehensive to show our results if we introduce the “dispersions” Δ_{ij} instead of the cumulants $\langle f_i f_j \rangle_c$. For the diagonal cumulants $\langle f_i^2 \rangle_c$, we define them as

$$\Delta_\rho^2 = \langle \rho^2 \rangle_c / \langle \rho \rangle^2, \quad \Delta_\theta^2 = \langle \theta^2 \rangle_c / \langle \theta \rangle^2, \quad \bar{\Delta}_{\sigma_\pm}^2 = \langle \sigma_\pm^2 \rangle_c / \langle \theta \rangle^2, \quad \bar{\Delta}_{E_\pm}^2 = \langle E_\pm^2 \rangle_c / \langle \theta \rangle^4, \quad (20)$$

where for shear and tidal force, $\bar{\Delta}_{\sigma_\pm}$ and $\bar{\Delta}_{E_\pm}$, which are normalized by $\langle \theta \rangle$, have been used rather than the conventional dispersions because those averaged variables sometimes vanish even if their cumulants are finite, resulting in a divergence of the dispersions.

Here we choose the boundary condition as follows. For the first order observables, we use the exact values of the fixed point E, i.e. $\langle \theta \rangle = 2$, $6\pi G(\rho) = 1$, $\langle \sigma_\pm \rangle = \langle E_\pm \rangle = 0$, which gives $\Omega = 1$. Since we expect some fluctuations even in an inflationary model, we set small finite fluctuations at the fixed point E, i.e. $\Delta_{\rho,0}$, $\Delta_{\theta,0}$, $\bar{\Delta}_{\sigma_\pm,0}$, $\bar{\Delta}_{E_\pm,0} \ll 1$. We also assume that the off-diagonal fluctuations have initially no intrinsic fluctuations, i.e. $\Delta_{ij,0} = 0$ ($i \neq j$) at $\ell = 0$.

We observe qualitatively different behavior of the RG flow of our Universe. If the fluctuations of shear and tidal force are sufficiently smaller than those of the density and the expansion, i.e. $\bar{\Delta}_{\sigma_\pm,0}/\Delta_0 \sim \bar{\Delta}_{E_\pm,0}/\Delta_0 < R_{cr} \approx 0.15$, the RG flow monotonically converges to fixed point M. Otherwise, i.e. if $\bar{\Delta}_{\sigma_\pm,0}/\Delta_0 \sim \bar{\Delta}_{E_\pm,0}/\Delta_0 > R_{cr}$, the RG flow monotonically diverges to infinity in the flow diagram as $H \rightarrow -\infty$ and $\Omega \rightarrow \infty$. In between there is another class of flow which initially approaches the fixed point M, but turns around before reaching it and eventually diverges to infinity. Hence, the first class is naturally realized as long as the shear and tidal force are sufficiently smaller than the density fluctuations at the horizon scale. This situation is strongly suggested by recent observations [11], i.e. the shear is strictly restricted around $\sigma/H < 10^{-9}$ by CMB data. In this case, the asymptotic values of observables such as Ω or H are given by the values at the fixed points E and M.

Is there any observational data which may suggest a scale dependence of either Ω or density? In discussion of dark matter, when we plot the mass-luminosity ratio in terms of the scale of the objects, we find a similar curve to Fig.1. If we assume that the mass-luminosity ratio curve is what we are discussing here, we find some constraint of s as $0.3 \lesssim s \lesssim 0.7$.

4 Conclusion

The renormalization group method has been applied to Newtonian cosmology and the scale dependence of the averaged observables on a fixed time slice has been studied.

To close our dynamical system and apply the RG method, here we have adopted an approximation of tidal force a method (LTA) and considered only fluctuations up to the second order. The higher order cumulants of averaged observables are ignored. We have found three robust fixed points of the RG equations toward larger scale; (1) stable fixed point Q (“Quiescent” Universe). (2) saddle point E (Einstein-de Sitter Universe) (3) unstable fixed point M (“Milne” Universe).

These fixed points do not change their stability even if we extend the “phase” space including fluctuations as well as shear and tidal force. Any other fixed points are saddle, and therefore the scaling property of the global RG flow toward larger (smaller) scale is determined by fixed point Q (M). To analyze the RG equations, we have imposed the boundary condition that our universe is close to the fixed point E with tiny fluctuations at the horizon scale. The inflationary scenario in the early Universe would imply such a boundary condition. We have solved the RG equation toward smaller scale from this boundary and found that, toward smaller scale, the expansion H defined by $\langle \theta \rangle/3$ monotonically increases from $2/3$ to 1 , and the density parameter $\Omega = 8\pi G(\rho)/3H^2$ monotonically reduces from 1 to 0 , provided the fluctuations of shear and tidal force are much smaller than the density fluctuations, which would be supported by the inflationary scenario and observations of CMB.

We have two free parameters, s and Δ_0 . These values might be fixed by some of the following arguments: (i) The scale dependence of the M/L ratio of astrophysical objects. (ii) A systematic decrease of the Hubble parameter H , if any, toward the larger scale. (iii) The scale dependence of the amplitude of

the two point correlation functions. (iv) Some of the inflationary models predict a fractal-like structure beyond the horizon scale [12].

Several comments related with our future work are in order. (1) We definitely need the general relativistic generalization of our work. This is because the large scale structures at high redshift and beyond the horizon scale would be adequately described only by general relativity. (2) So far we considered only fluctuations of variables up to the second order. Though this would be reasonable when we consider the vicinity of horizon scale, we ultimately need to incorporate higher order fluctuations as well to check a consistency of our model. In fact, if the Universe has a fractal structure, we may have to take into account infinite series of higher order correlations. (3) In our analysis, we assume a scaling property of the averaged observables and their fluctuations. However we do not know whether such a solution is dynamically stable or not. We may have to analyze a dynamical stability of our system. We would like to report the analysis on these issues in the near future.

Acknowledgments

We would like to thank Paul Haines, Kamila Piotrkowska, Naoshi Sugiyama and Kenji Tomita for many useful discussions. This work was supported partially by the Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Science and Culture (Specially Promoted Research No. 08102010), and by the Waseda University Grant for Special Research Projects.

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	$6\pi G\langle\rho\rangle^*$	$\langle\theta\rangle^*$	S^*	two eigenvalues
(1) Q	0	0	s	$-2S^*, -S^*$
(2) E	1	2	$\frac{3s}{3-2s}$	$-\frac{2}{3}S^*, S^*$
(3) M	0	3	$\frac{s}{1-s}$	S^*, S^*

Table 1. The list of three fixed points and their eigenvalues in the case with neither shear and tidal force nor fluctuations.

	$6\pi G\langle\rho\rangle^*$	$\langle\theta\rangle^*$	$36\pi^2 G^2\langle\rho^2\rangle_c^*$	$\langle\theta^2\rangle_c^*$	$6\pi G\langle\rho\theta\rangle_c^*$	S^*	five eigenvalues
(1) Q	0	0	0	0	0	s	$-4S^*, -3S^*, -2S^*, -2S^*, -S^*$
(2) E	1	2	0	0	0	$\frac{3s}{3-2s}$	$-\frac{4}{3}S^*, -\frac{2}{3}S^*, \frac{1}{3}S^*, S^*, 2S^*$
(3) M	0	3	0	0	0	$\frac{s}{1-s}$	$S^*, S^*, 2S^*, 2S^*, 2S^*$

Table 2. The list of original fixed points and their eigenvalues in the case without shear and tidal force including fluctuations. All of the other fixed points are unphysical.

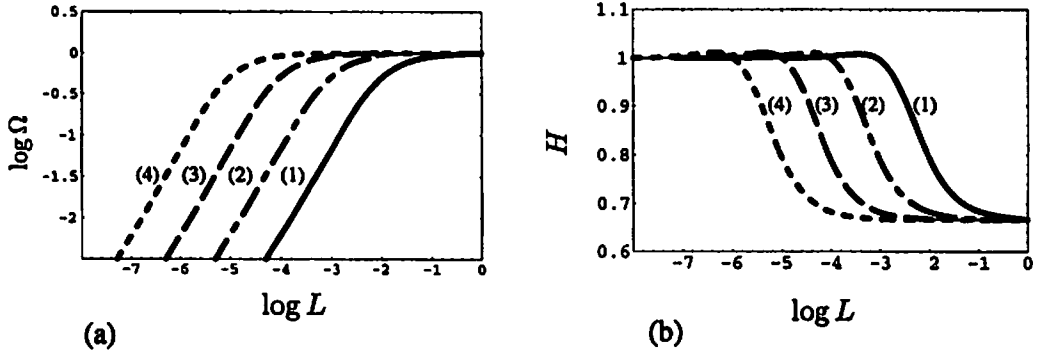


Figure 1: The scale dependence of (a) $\log \Omega$ and (b) H in shear-free case with $s = 0.5$ and $\Delta_0 =$ (1) 10^{-1} , (2) $10^{-1.5}$, (3) 10^{-2} , (4) $10^{-2.5}$. We solve the RG flows from the fixed point E toward smaller scale. $L \equiv \exp \ell$ represents the ratio of the scale of averaged region to the horizon scale. Every flow monotonically converges to M as H becomes larger ($H \nearrow$) and Ω gets smaller ($\Omega \searrow$) toward smaller scale. The critical scale where Ω and H deviate from the EdS values depends on the magnitude of Δ_0 .

New varieties of Gowdy spacetimes

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Abstract

Gowdy spacetimes are generalized to admit two commuting spatial *local* Killing vectors, and new varieties of them, for nilgeometry and solvegeometry, are presented.

1 Introduction

Spatially compact inhomogeneous spacetimes admitting two commuting spatial Killing vectors are known as Gowdy spacetimes [1], which are recently paid large attention as favorable models for the studies of the asymptotic behavior toward the cosmological initial singularity [2, 3, 4, 5, 6]. Since Gowdy spacetimes provide the simplest inhomogeneous cosmologies, it seems natural to use them in such a kind of studies first developed based on Bianchi homogeneous cosmologies [7] by Belinski et.al. [8]. This article's aim is not, however, to elucidate the dynamics of Gowdy spacetimes, but to point out some new features pertaining to their varieties, which may open new windows to look at the initial singularity.

As is well known, Bianchi homogeneous cosmologies are classified by the local structures of simply transitive three dimensional groups, Bianchi I~IX groups. Spatially compact Bianchi cosmologies [9, 10, 11] are more diverse than the open ones, since possible compact topologies are in general very diverse for each Bianchi (universal covering) cosmology. Nevertheless, Gowdy spacetimes having lower symmetry than these homogeneous ones are known to have very limited varieties, i.e., only T^3 , $S^2 \times S^1$, S^3 , and lens spaces as their possible spatial topologies. Why is their diversity so poor? In fact, all the Bianchi cosmologies except types VIII and IX admit two commuting Killing vectors, and moreover, compactifications are possible except for types IV and $VI_{a \neq 0}$ [12, 9]. Since these compact Bianchi models can be thought of as, if exist, homogeneous limits of compact inhomogeneous models which admit two commuting Killing vectors, it seems that Gowdy models should be more diverse. The solution to this paradox is in whether the Killing vectors are local or global. That is, the restriction for the possible topologies of Gowdy spacetimes is a consequence of the definition that the two commuting Killing vectors must be *globally* defined, while spatially compact Bianchi cosmologies admit in general only *local* Killing vectors.

If we consider Killing vectors for the simplification of Einstein's equation, which is local in nature, the imposition of the globality of the Killing vectors is evidently unnecessary. We therefore generalize Gowdy spacetimes in this article to admit two commuting *local* Killing vectors, and will find that there actually exist rich (topologically infinitely many) varieties of Gowdy spacetimes.

As is well known in theory of three dimensional topology, Thurston [13] considered eight types of homogeneous 3-manifolds, called *model geometries*, S^3 , E^3 , H^3 , $S^2 \times E^1$, $H^2 \times E^1$, $\widetilde{SL}(2, \mathbb{R})$, nilgeometry(Nil), and solvegeometry(Sol), and proved in essence that any compact three dimensional manifold which admit a locally homogeneous Riemannian metric is a compact quotient of one of these eight types of homogeneous manifolds. (See Sec.3.8 of Ref.[13].) We effectively utilize these model geometries, as in the compact locally homogeneous cases [9, 10].

In the next section, we show how we can apply Thurston's theorem to find (generalized) Gowdy spacetimes, and in the subsequent two sections present two new types of Gowdy spacetimes as examples. We also comments on the dynamics of them in the final section.

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2 Diversity of the possible topologies

To find possible topologies of Gowdy spacetimes, we consider a 'homogenization' of a compact Riemannian 3-manifold (M, h_{ab}) which admit two commuting local Killing vectors. Namely, suppose we can smoothly deform the metric h_{ab} preserving the two local Killing vectors and can take a locally homogeneous limit (M, h_{ab}^{lh}) of the Gowdy space (M, h_{ab}) . Since the universal cover $(\tilde{M}, \tilde{h}_{ab}^{lh})$ of (M, h_{ab}^{lh}) is homogeneous, it must be one of the BKS types, i.e., Bianchi homogeneous 3-manifolds and the Kantowski-Sachs-Nariai (KSN) homogeneous 3-manifold [14, 15]. On the other hand, since the homogenization (M, h_{ab}^{lh}) is compact, (M, h_{ab}^{lh}) must be homeomorphic to a compact quotient of one of the eight Thurston's model geometries.

A (model) geometry is the pair $(\tilde{M}, G)$ of a manifold $\tilde{M}$ and a group G of diffeomorphisms on $\tilde{M}$, such that G acts transitively on $\tilde{M}$ with compact isotropy subgroups. Since G acts transitively on $\tilde{M}$, $(\tilde{M}, G)$ can be thought of as an equivalence class of the homogeneous manifolds whose isometry group is isomorphic to (or includes a subgroup isomorphic to) G [10]. We for convenience call $(\tilde{M}, G)$ a *subgeometry* of $(\tilde{M}, G')$ if G is a subgroup of other transitive group G' with compact isotropy subgroups. Moreover, if geometry $(\tilde{M}, G)$ is not a subgeometry of any geometry, then we call $(\tilde{M}, G)$ a *maximal* geometry, and if $(\tilde{M}, G)$ does not have any subgeometry, then we call $(\tilde{M}, G)$ a *minimal* geometry. While Thurston's eight geometries are maximal geometries, all the BKS types except Bianchi IV and $VI_{a \neq 0}$ can be thought of as the minimal geometries of Thurston's geometries [9, 12]. This correspondence helps apply Thurston's geometries to spacetime models.

The group G of the geometry $(\tilde{M}, G)$ which corresponds to $(\tilde{M}, \tilde{h}_{ab}^{lh})$ should include a subgroup isomorphic to one of $\mathbb{R}^2$, $\mathbb{R} \times U(1)$, and $U(1) \times U(1)$, corresponding to the commuting two Killing vectors. All Thurston's geometries satisfy this condition. More precisely, what include $\mathbb{R}^2$ are geometries E^3 , H^3 , $H^2 \times E^1$, $\tilde{SL}(2, \mathbb{R})$, Nil, and Sol. Geometry $S^2 \times E^1$ includes $\mathbb{R} \times U(1)$, and S^3 does $U(1) \times U(1)$. Moreover, all the minimal geometries of these Thurston's geometries except Bianchi VIII ($\tilde{SL}(2, \mathbb{R})$ as the maximal one) and IX (S^3 , similarly) also satisfy this condition. Any compact quotient for all the homogeneous 3-manifolds except Bianchi VIII and IX can therefore be a homogenization of a Gowdy space. Even for Bianchi VIII and IX, they can be a homogenization if imposing a fourth Killing vector on them.

This observation tells what homogeneous manifold can be the universal cover of a homogeneous limit of a Gowdy space. Once fixed such a homogenization with isometry group G , the universal cover of the corresponding Gowdy space can be determined as follows. That is, we may find the subgroup H of G such that the actions are smooth along the two commuting Killing vectors, but discrete along the third one. The H -invariant metric is the universal cover metric of the Gowdy space (or spacetime), if the fundamental group can be represented in H . In fact, the H -invariance is a necessary condition for the universal cover to admit a (spatially) compact quotient, so that we need to check that. As we will see, it is easy to write such an H -invariant metric if using the invariant 1-forms for the corresponding Bianchi type. The KSN type has already been discussed in Ref.[1].

In the following, we show how these ideas work by presenting two examples, which are all new.

3 Gowdy Nil $\times\mathbb{R}$

As a first example let us consider Nil, which possesses Bianchi II as its minimal geometry. Note that the Bianchi II homogeneous spaces are characterized by the following commutation relations for the three Killing vectors

$$[\xi_1, \xi_2] = -\xi_3, [\xi_2, \xi_3] = 0, [\xi_3, \xi_1] = 0. \quad (1)$$

In terms of coordinate basis they can be represented by

$$\xi_1 = \frac{\partial}{\partial x} + y \frac{\partial}{\partial z}, \quad \xi_2 = \frac{\partial}{\partial y}, \quad \xi_3 = \frac{\partial}{\partial z}. \quad (2)$$

For future use, we take this opportunity to write the finite actions generated by these ξ_i 's

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a+x \\ b+y \\ c+z+ay \end{pmatrix}, \quad (3)$$

where the component vectors

$$\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \equiv e^{c\xi_3} e^{b\xi_2} e^{a\xi_1} \mid a, b, c \in \mathbb{R} \right\} \quad (4)$$

form Bianchi II group G_{II} . ($e^{a\xi_i}$ denotes the one-parameter group of diffeomorphisms generated by ξ_i .) Note that the "spatial point" (x, y, z) is the image of the origin $(0, 0, 0)$ by the "group element" (x, y, z) , so that Eq.(3) itself gives the multiplication rule in Bianchi II group.

From the commutation relations (1), we find two possible choices of obtaining a Gowdy spacetime model, i.e., one (Type 1) is to keep ξ_2 and ξ_3 as the two commuting Killing vectors and consider inhomogeneity along ξ_1 , and the other (Type 2) is to keep ξ_1 and ξ_3 as Killing vectors and consider inhomogeneity along ξ_2 . We consider the Type 1 first.

Note that the invariant 1-forms of Bianchi II, given by

$$\sigma^1 = dx, \sigma^2 = dy, \sigma^3 = dz - xdy \quad (5)$$

are globally defined on $\tilde{M} = \mathbb{R}^3$, so that we can expand any metric on $\tilde{M}$ by these 1-forms

$$dl^2 = h_{ij} \sigma^i \sigma^j, \quad (6)$$

with globally defined metric functions h_{ij} . Note also that the subgroup H_{II} of G_{II} of which action is discrete along ξ_1 is formed by

$$\left\{ \begin{pmatrix} 2m\pi \\ b \\ c \end{pmatrix} \mid m \in \mathbb{Z}, b, c \in \mathbb{R} \right\}. \quad (7)$$

Since the invariant 1-forms (5) are invariant under H_{II} (and G_{II} by definition), the metric (6) is invariant under H_{II} iff so are the metric functions h_{ij} . This requirement is equivalent to that h_{ij} 's depend only on x and are periodic with period 2π , i.e.,

$$h_{ij} = h_{ij}(x) = h_{ij}(x + 2\pi). \quad (8)$$

(The choice of the period is in this case arbitrary. Our choice of 2π is just for definiteness.) The homogeneous limit can be achieved when $h_{ij} = \text{constants}$. So, we have found the inhomogeneous metric of (the universal cover of) a Gowdy space, given by Eq.(6) with the boundary condition (8).

Now, we can write down the "appropriate" spacetime metric by [1] imposing the two-surface orthogonality and choosing the isothermal coordinates for the reference surface. After all, we obtain the spacetime metric

$$ds^2 = e^{-\lambda/2} t^{-1/2} (-dt^2 + (\sigma^1)^2) + R[e^P(\sigma^2)^2 + 2e^P Q \sigma^2 \sigma^3 + (e^P Q^2 + e^{-P})(\sigma^3)^2], \quad (9)$$

where P, Q, R and λ are functions of t and x and are periodic in x with period 2π . We have followed the parameterization of a recent paper [6] to make comparisons easier. The isometry group for the spacetime is unchanged, given by H_{II} (7).

This metric should be considered as the universal cover metric $\tilde{g}_{ab}$ of a Gowdy spacetime $(M \times \mathbb{R}, g_{ab})$. Any topology of M is possible if the fundamental group $\pi_1(M)$ can be represented in H_{II} . One can choose such a $\pi_1(M)$ in the list of compact quotients modeled on Nil, presented in Ref.[9]. For example, choose the manifold " $b/1(n)$ ", characterized by

$$\pi_1(M_n) = \langle g_1, g_2, g_3; [g_1, g_2] = g_3^n, [g_1, g_3] = 1, [g_2, g_3] = 1 \rangle, \quad (10)$$

where n is a positive integer parameterizing the family $b/1$ of topologies on Nil. Then using the multiplication rule (3), we can find the representation up to conjugations

$$\begin{aligned} \Gamma_n &= \{g_1, g_2, g_3\} \\ &= \left\{ \begin{pmatrix} 2p\pi \\ g_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2q\pi \\ g_2^2 \\ g_2^3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \frac{2\pi}{n}(pg_2^2 - qg_1^2) \end{pmatrix} \right\}, \end{aligned} \quad (11)$$

where $p, q \in \mathbb{Z}$, and $g_1^2, g_2^2, g_3^2 \in \mathbb{R}$. Thus, we have obtained the Gowdy spacetime $(M_n \times \mathbb{R}, g_{ab}) = (\tilde{M}_n \times \mathbb{R}, \tilde{g}_{ab})/\Gamma_n$, where the action of Γ_n is defined by Eq.(3). We may think of the real parameters (g_1^2, g_2^2, g_3^2) as the moduli parameters for the spacetime.

The Type 2 can be obtained by interchanging the roles of σ^1 and σ^2 . For example, the metric can be obtained from Eq.(9) by transforming $\sigma^1 \rightarrow \sigma^2$, $\sigma^2 \rightarrow \sigma^1$, $P(t, x) \rightarrow P(t, y)$, etc.. The isometry group thereof is formed by the actions

$$\left\{ \begin{pmatrix} a \\ 2m\pi \\ c \end{pmatrix} \middle| m \in \mathbb{Z}, a, c \in \mathbb{R} \right\}. \quad (12)$$

To write the vacuum Einstein equation for the Type 1 (and Type 2), note that the metric (9) belongs to the class of metrics Gowdy gave [16]. Moreover, as in Ref.[1], R describes the areas of the group orbits, consisting of flat T^2 's [17]. The area function R therefore satisfies the same key equation presented in Ref.[1], i.e.,

$$\partial_{tt} R - \partial_{xx} R = 0. \quad (13)$$

This equation can also be checked by a direct substitution of the metric (9) into the Einstein equation. Since the group orbits do not degenerate everywhere (except for the initial singularity), R can be taken as $t (= e^{-\tau})$ [1]. (The compactness of the spatial manifold M therefore helps simplify Einstein's equation.) $t = 0$ ($\tau = +\infty$) corresponds to the initial singularity. With this choice, the remaining independent Einstein's equations are found to be

$$\begin{aligned} \ddot{P} - e^{-2\tau} P'' - e^{2P} \dot{Q}^2 + e^{-2\tau} [e^{2P} (Q' \pm Q^2)^2 \pm 2Q' - e^{-2P}] &= 0, \\ \ddot{Q} - e^{-2\tau} Q'' + 2\dot{P}\dot{Q} - 2e^{-2\tau} [P'Q' \pm (P' \mp Q)(e^{-2P} + Q^2)] &= 0, \end{aligned} \quad (14)$$

and

$$\begin{aligned} \lambda' - 2(P' \mp 2Q)\dot{P} - 2[e^{2P} (Q' \pm Q^2) \mp 1]\dot{Q} &= 0, \\ \dot{\lambda} - \dot{P}^2 - e^{2P} \dot{Q}^2 - e^{-2\tau} [e^{2P} (Q' \pm Q^2)^2 + P'^2 + 2Q^2 \mp 2(Q' + 2P'Q) + e^{-2P}] &= 0, \end{aligned} \quad (15)$$

where dot and dash denote τ and x (or y for the Type 2) derivatives, respectively. The upper and lower signs are for the Type 1 and Type 2, respectively. Note that P and Q are not constrained, since λ does not appear in Eqs.(14). This is the advantage of the Gowdy models. The integrability condition for the constraint equations (15) for λ is automatically satisfied with Eqs.(14).

4 Gowdy Sol $\times\mathbb{R}$

Next example is Sol possessing Bianchi VI₀ as its minimal geometry. As in the previous example, we first observe the commutation relations of the three Killing vectors of Bianchi VI₀

$$[\xi_1, \xi_2] = 0, [\xi_2, \xi_3] = -\xi_1, [\xi_3, \xi_1] = \xi_2. \quad (16)$$

In terms of coordinate basis,

$$\xi_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad \xi_2 = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}, \quad \xi_3 = \frac{\partial}{\partial z} - x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}. \quad (17)$$

The finite actions of ξ_i 's are given by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a + e^{-c}x \\ b + e^c y \\ c + z \end{pmatrix}, \quad (18)$$

where the component vectors

$$\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \equiv e^{\frac{1}{2}(\xi_1 + \xi_2)} e^{\frac{1}{2}(\xi_1 - \xi_2)} e^{c\xi_3} \middle| a, b, c \in \mathbb{R} \right\} \quad (19)$$

form Bianchi VI₀ group G_{VI_0} . As in the Bianchi II case, the identification of the spatial points and the group elements is understood.

From the commutation relations (16), we find sole possibility of obtaining the Gowdy space, the one inhomogeneous along ξ_3 . Using the invariant 1-forms

$$\sigma^1 = \frac{1}{\sqrt{2}}(e^z dx + e^{-z} dy), \quad \sigma^2 = \frac{1}{\sqrt{2}}(-e^z dx + e^{-z} dy), \quad \sigma^3 = dz, \quad (20)$$

and the periodic functions

$$\begin{aligned} P &= P(\tau, z) = P(\tau, z + z_0), \quad Q = Q(\tau, z) = Q(\tau, z + z_0), \\ \lambda &= \lambda(\tau, z) = \lambda(\tau, z + z_0), \end{aligned} \quad (21)$$

we can immediately write the spacetime metric as

$$ds^2 = e^{-\lambda/2} e^{\tau/2} (-e^{-2\tau} d\tau^2 + (\sigma^3)^2) + e^{-\tau} [e^P (\sigma^1)^2 + 2e^P Q \sigma^1 \sigma^2 + (e^P Q^2 + e^{-P}) (\sigma^2)^2]. \quad (22)$$

We do not take the period z_0 to 2π , since as we will see below it cannot be arbitrarily specified. The isometry group H_{VI_0} of the spacetime metric (22) is formed by the discrete actions along ξ_3

$$\left\{ \begin{pmatrix} a \\ b \\ mz_0 \end{pmatrix} \middle| a, b \in \mathbf{R}, \quad m \in \mathbf{Z} \right\}. \quad (23)$$

We can take any topology of M of the Gowdy spacetime $(M \times \mathbf{R}, g_{ab})$ if we can represent the fundamental group $\pi_1(M)$ into H_{VI_0} (23) together with, if needed, the disconnected components defined with the discrete isometry

$$h : (x, y, z) \rightarrow (-x, -y, z). \quad (24)$$

As an explicit example, we consider the major sequence of compact quotients presented in Ref.[9]. The fundamental groups are parameterized by an integer n such that $|n| > 2$, and given by

$$\pi_1(M_n) = \langle g_1, g_2, g_3; [g_1, g_2] = 1, g_3 g_1 g_3^{-1} = g_2, g_3 g_2 g_3^{-1} = g_1^{-1} g_2^n \rangle. \quad (25)$$

(This does not, however, exhaust all the compact quotients [18].) We find that the representations are, up to conjugations, given by

$$\Gamma_n = \left\{ \begin{pmatrix} \alpha u_1 \\ \beta u_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha v_1 \\ \beta v_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_0 \end{pmatrix}, \right\} \quad (26)$$

for $n > 2$, and

$$\Gamma_n = \left\{ \begin{pmatrix} \alpha u_1 \\ \beta u_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha v_1 \\ \beta v_2 \\ 0 \end{pmatrix}, h \circ \begin{pmatrix} 0 \\ 0 \\ z_0 \end{pmatrix}, \right\} \quad (27)$$

for $n < -2$, with $\alpha, \beta \in \mathbf{R}$. In these representations, (u_1, v_1) , (u_2, v_2) , and z_0 are determined in such a way that $\text{sign}(n)e^{-z_0}$ and $\text{sign}(n)e^{z_0}$ are the eigenvalues of matrix $\begin{pmatrix} 0 & 1 \\ -1 & n \end{pmatrix}$, and the corresponding normalized eigenvectors are (u_1, v_1) and (u_2, v_2) , respectively. In particular, $e^{z_0} = |n + \sqrt{n^2 - 4}|/2$. We thus have obtained the Gowdy spacetime $(M_n \times \mathbf{R}, g_{ab}) = (\tilde{M}_n \times \mathbf{R}, \tilde{g}_{ab})/\Gamma_n$, where the universal cover metric $\tilde{g}_{ab}$ is given by Eq.(22).

In the metric (22), we wrote the area function of the group orbit as $e^{-\tau} = t$, since, as in the Nil case, the group orbit does not degenerate except for the initial singularity. Einstein's equations are then found to be

$$\begin{aligned} \ddot{P} - e^{-2\tau} P'' - e^{2P} \dot{Q}^2 + e^{-2\tau} [e^{2P} (Q' + Q^2)(Q' + Q^2 - 2) + 2Q' + e^{2P} - e^{-2P}] &= 0, \\ \ddot{Q} - e^{-2\tau} Q'' + 2\dot{P}\dot{Q} - 2e^{-2\tau} [P'Q' + (P' - Q)(e^{-2P} + Q^2 - 1)] &= 0, \end{aligned} \quad (28)$$

and

$$\begin{aligned}\lambda' &- 2(P' - 2Q)\dot{P} - 2[e^{2P}(Q' + Q^2 - 1) - 1]\dot{Q} = 0, \\ \lambda &- \dot{P}^2 - e^{2P}\dot{Q}^2 - e^{-2\tau} [e^{2P}(Q' + Q^2)(Q' + Q^2 - 2) + \\ & (P' - 2Q)^2 - 2(Q' + Q^2 - 1) + e^{2P} + e^{-2P}] = 0,\end{aligned}\tag{29}$$

where dot and dash denote, respectively, τ and z derivatives. The integrability condition for the constraint equations (29) for λ is automatically satisfied with Eqs.(28).

5 Concluding Remarks

We have constructed two new Gowdy spacetimes, on $\text{Nil} \times \mathbb{R}$, and on $\text{Sol} \times \mathbb{R}$. Any other types can be constructed in similar ways [19]. For example, if we start with Bianchi VII₀, we will have another $T^3 \times \mathbb{R}$ model, which is not identical to the one in the literature. An extension of our method to the so-called $U(1)$ models [20] would also be possible.

Finally, we comment on the dynamics near the initial singularity of our two models. The (conventional) Gowdy spacetime on $T^3 \times \mathbb{R}$ is conjectured [3] to be asymptotically velocity term dominated (AVTD) [2]. Recent investigation by Berger and Garfinkle [6] supports this, *except for measure-zero nongeneric spatial points*. They succeeded to explain the phenomenon by a potential picture, and showed the nongeneric points correspond to the points where $Q' = 0$. In our Nil and Sol models, such points correspond to points such that $Q' + Q^2 = 0$ and $(Q' + Q^2)(Q' + Q^2 - 2) + 1 = 0$, respectively. Note that the points where $Q' = 0$ is inevitable, since Q is periodic, but our two conditions are not necessarily satisfied in any spatial point. One may therefore expect that the Gowdy $\text{Nil} \times \mathbb{R}$ and $\text{Sol} \times \mathbb{R}$ models are AVTD *everywhere*.

Acknowledgments

The author acknowledges financial support from the Japan Society for the Promotion of Science and the Ministry of Education, Science and Culture.

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$$ds^2 = e^{-\lambda/2} t^{-1/2} (-dt^2 + dx^2) + R[e^P dy^2 + 2e^P Q dydz + (e^P Q^2 + e^{-P}) dz^2],$$

the present metric is formally obtained by substituting

$$P \rightarrow P + \ln[(1 - xQ)^2 + x^2 e^{-2P}], \quad Q \rightarrow \frac{Q(1 - xQ)^2 - x e^{-2P}}{(1 - xQ)^2 + x^2 e^{-2P}}.$$

In this sense, the two metrics are in the same class. Note, however, that the boundary conditions, i.e., the periodicities of the metric functions are not preserved by this transformations.

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Note on Averaging Problem in Cosmology

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Abstract

I give a simple definition of average of tensors on a Riemannian manifold and derive the Ricci-Hamilton flow for the Riemannian metric.

1 Introduction

It is widely assumed in cosmology that the universe is homogeneous and isotropic at scales larger than several hundred megaparsecs, although there are many strongly self-gravitating structures at smaller scales. In other words, the universe is considered to be a Friedman-Lemaître-Robertson-Walker (FLRW) space-time on average. To establish or examine this theoretically, several authors have been presented different definitions of an “average” of the universe [1, 2, 3, 4].

Among them Carfora and Piotrkowska [4] motivated that the average of a Riemannian manifold (M, g) (which is considered to be a time slice of a space-time manifold) should be described by the Ricci (semi)flow considered by Hamilton [5],

$$\frac{\partial g}{\partial \tau} = -2\text{Ric} + \frac{2}{3}\langle R \rangle g, \quad (1)$$

where $\text{Ric}[g]$ is the Ricci curvature of g and $\langle R \rangle$ is the average of a scalar curvature on M , by reinterpreting the variation of an average of a function caused by change of the averaging radius as a variation caused by change of metric g . One reason of this may be absence of a unique and natural definition of average of the metric g . A nice feature of the flow is that it is spatially covariant.

In this talk I define an (local) average for functions, vectors and tensors, and give a simple derivation of the Ricci flow equation as one which the metric satisfies. Rather, the motivation here is to know intuitively in what sense the Ricci flow can be considered as an average of the metric.

If we want to define a local average of a tensor field at each point of a manifold M , we have to take sums of its values at different points. One such way is to transport the tensor at some point to another by parallel transport. However, this does not define an average of the metric, since the metric at a point and one parallelly transported from another point is always the same. Alternatively we could use some map on M and pull back the tensor, or, equivalently, use some vector field and Lie drag the tensor. To define an average around a point p , it is natural to use a vector field diverging from p . However, we cannot Lie drag the tensors around p to p because p is a fixed point. A natural map which is defined solely by differentiable and Riemannian structures is an exponential map $\exp : T_p M \rightarrow M$, which is not a map on M . I will make use of this map.

2 Definition

It is desirable that we treat the averaged quantity $\langle X \rangle_\eta$, where X is a function, a vector field, or a tensor field on M , on the equal footing as non-averaged quantity X if we want to consider averaged quantities as effective tensor fields. Namely, the average should be a map from a tensor field to another. This can be said to be renormalization group philosophy. To be precise, we require

1. Average $\langle T \rangle_\eta$ forms a one-parameter (semi)group of transformations, namely, $\langle X \rangle_{\eta+\eta'} = \langle \langle X \rangle_\eta \rangle_{\eta'}$, where η is related to the scale of the region over which T is averaged.

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It is also natural to require the following.

2. For a scalar field f , the average on infinitesimally small geodesic ball V_s agrees with usual volume average, $\langle f \rangle_\eta = \int_{V_s} f d\mu$, where s is the radius of the ball and $d\mu$ is the volume element.

The local average around point p of M is determined if we define how to carry a tensor at a point q near p to p . We do this by the tangent map $d\phi$ of the exponential map $\phi : T_p M \rightarrow M$ at p . We can do even though tangent map $(d\phi)_v$ at $v \in T_p M$ is not $T_p M \rightarrow T_{\phi(v)} M$ but $T_v T_p M \rightarrow T_{\phi(v)} M$, because there is a natural identification $I_{p,v} : T_v T_p M \rightarrow T_p M$ of a linear space $T_p M$ and its tangent space $T_v T_p M$. A basis $\{e_\alpha\}$ of $T_p M$ as a vector space defines coordinates $\{x^\alpha\}$ on $T_p M$. $I_{p,v}$ can be written as

$$T_v T_p M \ni w^\alpha \left(\frac{\partial}{\partial x^\alpha} \right)_v \mapsto w^\alpha e_\alpha \in T_p M. \quad (2)$$

By defining dually the map $\phi_v^* : T_{\phi(v)}^* M \rightarrow T_v^* T_p M$ for 1-forms and defining $\phi_v^* := (d\phi)_v^{-1}$ for vectors as usual, one has the pull-back for tensors of any type. A similar extension is understood for $I_{p,v}$.

Let us give another description which might be more intuitive. Let X be a vector field, and consider a geodesics emanating from p which are parametrized by length s from p . We decompose X_q at q into parts Y_q parallel to the geodesic and Z_q normal to it. We drag the vector Y_q by parallel transport to p . We drag the vector Z_q by using *Jacobi fields*: Letting s be the geodesic distance of p and q and $\tilde{Z}_q(s')$ be the Jacobi field at geodesic distance s' satisfying $\tilde{Z}_q(0) = 0$ and $\tilde{Z}_q(s) = Z_q$, we define Z_q carried to p as $\lim_{s \rightarrow 0} \tilde{Z}_q(s')/(s'/s)$.

Now we give the definition of the average of a tensor field X (of any type) over an infinitesimal geodesic ball with center p ,

$$\langle X \rangle_\eta := \frac{\int_{v \in B_s} [\phi_v^* d\mu_{\phi(v)} \otimes I_{p,v}(\phi_v^* X_{\phi(v)})]}{\int_{v \in B_s} \phi_v^* d\mu_{\phi(v)}} + o(s^2), \quad (3)$$

where B_s is a ball of radius s in the tangent space $T_p M$, $\phi_v^* d\mu_{\phi(v)}$ is the volume element pulled back from point $\phi(v) \in M$ to $v \in T_p M$, and $o(x)$ is a term with $\lim_{x \rightarrow 0} o(x)/x \rightarrow 0$. Note that $I_{p,v}(\phi_v^* X_{\phi(v)})$ is always an element of (a tensorial generalization of) $T_p M$ and can be summed over $v \in T_p M$. To have a nontrivial flow, we set $\eta = s^2/(n+2)$. (Choice of multiplicative factor is arbitrary.)

3 Equation for the flow

For a scalar field f one easily has

$$\langle f \rangle_\eta = f + \frac{s^2}{2(n+2)} \Delta f + O(s^4), \quad (4)$$

where $n := \dim M$, Δ is the Laplacian with respect to g . Then we have

$$\frac{d\langle f \rangle_\eta}{d\eta} = \Delta \langle f \rangle_\eta. \quad (5)$$

If the manifold is flat, and the average of tensor fields is such that it does not change the flat metric (which is the case for the average defined later), we can integrate this equation to have

$$\langle f \rangle_\eta = \int_{V_s} K(x - x') f(x) d^3 x, \quad (6)$$

where K is the usual heat kernel. Thus our average for scalar fields is a generalization of smoothing with a heat kernel to curved spaces.

For a tensor field we have

$$\frac{d(\langle X \rangle_\eta)^{a_1 \dots a_l}_{b_1 \dots b_m}}{d\eta} = \Delta X + \frac{1}{3} \sum_{i=1}^l \text{Ric}[(g)_\eta]^{a_i}_c (\langle X \rangle_\eta)^{a_1 \dots c \dots a_l}_{b_1 \dots b_m} \quad (7)$$

$$- \frac{1}{3} \sum_{j=1}^m \text{Ric}[(g)_\eta]^c_{b_j} (\langle X \rangle_\eta)^{a_1 \dots a_l}_{b_1 \dots c \dots b_m}. \quad (8)$$

where Latin sub- or super-scripts are abstract indices.

For a metric

$$\frac{d(\langle g \rangle_\eta)_{ab}}{d\eta} = -\frac{2}{3} \sum_{j=1}^m \text{Ric}[\langle g \rangle_\eta]_{ab}. \quad (9)$$

This is the Ricci flow. The factor on the right hand side depends on a definition of η .

If we want to take average of both tensor fields and geometry, the equations for the flows should be considered as a *coupled equations*.

4 Discussion

Our Ricci flow does not have a normalization term which is proportional to the global average of the scalar curvature. It is natural that we cannot expect a global normalization term, which requires global information of the manifold, for a local average. Its inclusion may be considered to be ad hoc. A local normalization could be defined by replacing the right hand side of (9) by a traceless Ricci tensor. However, it implies a backward diffusion equation for the scalar curvature, as pointed out in [5].

It follows from our Ricci flow (9) that a positive curvature space such as a sphere shrinks and negative curvature one expands. Thus, even we include the normalization term, because it changes only the overall volume, the average pays more attention to negative curvature regions and less to positive curvature regions. Whether the Ricci flow is physically reasonable definition of the average of the metric might depend on problems in consideration.

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Distance-redshift relation in an Inhomogeneous Universe

- FRW vs Dustshell universe -

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Abstract

Luminosity distance-redshift (d_L - z) relation in an inhomogeneous universe is studied. We solve the Raychaudhuri equation on a spherically symmetric "dustshell universe" where large inhomogeneities of density distribution exist on small scales. We obtain a surprising result that the relation is fitted well by that of a FRW model if we set an appropriate initial condition. This supports the "averaging hypothesis" that a universe looks like a FRW model if the universe is homogeneous when averaged on large scales.

1 Introduction

For a long time, we have regarded that a FRW (Friedmann-Robertson-Walker) model as a large-scale "average" of a locally inhomogeneous universe. Even though the observational data are consistent with the picture that our universe is described well by a FRW metric with small inhomogeneities of gravitational potential on it, we are not sure how to derive a background FRW model from the inhomogeneous universe by any "averaging" procedure. This remains a non-trivial question to which we have to give a clear answer.

Many approaches have been made to study how small-scale inhomogeneities affect the global behavior when "averaged" on larger scales[1-4]. Buchert and Ehlers[1] studied this averaging problem in the framework of Newtonian cosmology. They showed that inhomogeneities do not influence the overall expansion in spatially compact models. In other cases, the significance of inhomogeneities depends on the cosmology one adopts.

In the framework of general relativity, Futamase[2] introduced Isaacson averaging and derived a FRW geometry in the linear approximation. He studied the behavior of inhomogeneities in a simple model, concluding that they work as a negative spatial curvature term in the FRW model so that they tend to *extend* the age of the universe. Russ et.al.[3] studied the influence of inhomogeneities on the global expansion factor quantitatively using the relativistic Zel'dovich approximation. They showed that the inhomogeneities tend to *shorten* the age of the universe, though the effect is very small.

Another interesting approach was made by Carfora and Piotrkowska[4], in which 3-dimensional manifold is deformed according to the Ricci-Hamiltonian flow which they say would be equivalent to changing the scale of averaging.

One might think there is some discrepancy in the conclusions of the "backreaction" of inhomogeneities to the background spacetime. Actually, their results are not inconsistent when we correctly consider the difference in the definition of "backreaction" of inhomogeneities[5], but we cannot tell which is the best way to describe the behavior of an inhomogeneous universe since none of these works have treated the observable quantities directly. For example, none of the previous authors has shown how to relate the observed quantities such as redshifts for galaxies in our inhomogeneous universe with the averaged "Hubble constant" defined differently by each author.

Thus, to understand clearly what effect inhomogeneities produce, it is necessary to relate them with physical quantities which we can observe.

In this study, we investigate observable quantities in an isotropic and inhomogeneous universe. As an inhomogeneous model, we take a spherically symmetrical dustshell model, in which light rays travel from each dustshell toward an observer at the center. We obtain a luminosity distance-redshift relation, with which we can determine the cosmological parameters of a FRW model. Then it enables us to discuss

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what an inhomogeneous universe looks like when we "observe" it and interpret it using a FRW model. Difference with FRW models and its implications are discussed, showing how inhomogeneities affect the dynamics of the universe and the travel of light rays.

2 Formulation of Dustshell universe

2.1 Motion of dustshell

We put a number of spherically symmetrical shells whose center is common. Each shell is infinitely thin, characterized by the energy-momentum tensor of the form

$$S^{ab} \equiv \sigma u^a u^b \quad (1)$$

where we defined the surface density of the shell σ and u^a is the 4-velocity of an observer rest on the shell. The conserved baryonic mass is defined by

$$m_a = 4\pi\sigma r^2 \quad (2)$$

where r is the circumferential radius of the shell.

Between the shells, the space-time is described by the Schwarzschild metric

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \quad (3)$$

where $r_g = 2Gm$ is the Schwarzschild radius and we will refer m as the gravitational mass.

The expansion law of the i -th shell which lies between the i -th region and the $(i+1)$ -th region is [6,7]

$$\left(\frac{\dot{r}_i}{r_i}\right)^2 = G \frac{2M_+(i)}{r_i^3} + \frac{1}{r_i^2} \left\{ \left(\frac{M_-(i)}{m_a(i)}\right)^2 - 1 \right\} + \frac{G^2 m_a^2(i)}{4r_i^4} \quad (4)$$

where

$$M_+(i) = \frac{m_i + m_{i+1}}{2}, \quad (5)$$

$$M_-(i) = m_{i+1} - m_i. \quad (6)$$

2.2 Redshift

We emit a photon from each shell toward an observer at the center.

The observed redshift of the photon emitted from the N -th shell is

$$1 + z_N = \frac{f(1/1)}{f(1/0)} \frac{f(2/2)}{f(2/1)} \cdots \frac{f(N/N)}{f(N/N-1)} \quad (7)$$

with

$$f(n/m) = \sqrt{\left(1 - \frac{r_{gn}}{r_m}\right)^{-1} + \left(1 - \frac{r_{gn}}{r_m}\right)^{-2} \dot{r}_m^2 + \left(1 - \frac{r_{gn}}{r_m}\right)^{-1} \dot{r}_m}. \quad (8)$$

2.3 Luminosity distance

In order to obtain the luminosity distance, we first derive the expansion at each shell. Evolution of the expansion $\theta \equiv \nabla_a k^a$ is described by the Raychaudhuri equation

$$\frac{d\theta}{d\lambda} = -R_{ab} k^a k^b - \frac{1}{2}\theta^2 \quad (9)$$

where k^a is the null vector of the photon. We omit the shear and vorticity by the symmetry. The affine parameter of the null geodesic λ satisfies

$$\frac{d}{d\lambda} = k^a \partial_a \quad . \quad (10)$$

Let n^a be a unit vector normal to u^a directing outward. Defining the project vector P_a^b as

$$P_a^b = \delta_a^b + n_a n^b \quad , \quad (11)$$

we can rewrite the l.h.s. of equation(9) as

$$\frac{d\theta}{d\lambda} = P_a^b k^a \partial_b \theta - n^b n_a k^a \partial_b \theta \quad . \quad (12)$$

Integrating equation(9) and (12) over an infinitesimal range along n^a , we obtain the difference of the expansion between outside(+) and inside(-) the shell

$$\theta^+ - \theta^- = 8\pi G S_{ab} \frac{k^a k^b}{n_c k^c} \quad (13)$$

where we have used the Einstein equation

$$R_{ab} k^a k^b = 8\pi G T_{ab} k^a k^b \quad . \quad (14)$$

Substituting the expressions of k^a, n_a and S_{ab} into equation(13), we can simply write equation(13) as

$$\theta^+ - \theta^- = 8\pi G \sigma w \quad (15)$$

where w is the frequency of the photon at the shell.

Between the shells, the space is vacuum so that

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 \quad . \quad (16)$$

Then the integration from the inside of the i -th shell to the outside of the $(i-1)$ -th shell gives

$$\frac{1}{\theta_{i-1}^+} = \frac{1}{2} \frac{r_i - r_{i-1}}{k_i} + \frac{1}{\theta_i^-} \quad . \quad (17)$$

The results above with the initial condition $\theta_N = \infty$ gives the expansion at each shell. Utilizing these, we can calculate the luminosity distance d_L from the observer to the shell.

The result is

$$d_L = \frac{\theta_1^- \theta_2^-}{\theta_0^+ \theta_1^+} \cdots \frac{2w_N}{\theta_{N-1}^+} (1 + z_N) \quad . \quad (18)$$

2.4 Initial condition

We choose the initial condition for the motion of shells such that the behavior of d_L - z relation in our model should agree with that of an appropriate FRW model when we put an enough number of shells. At the initial slice, we set

$$\frac{2M_+(i)}{r_i^3} = \text{independent of } i \quad , \quad (19)$$

i.e.,

$$\frac{M_+(i)}{r_i^3} = \frac{4}{3}\pi\rho \left(\frac{r_i(t_{\text{init}})}{r_i(t)} \right)^3 \quad (20)$$

with a constant energy density ρ . Besides, we define $E(i)$ as

$$\left(\frac{M_-(i)}{m_s(i)}\right)^2 \equiv E^2(i) \quad . \quad (21)$$

Then the expansion law of a dustshell is

$$\left(\frac{\dot{r}_i}{r_i}\right)^2 = \frac{8}{3}\pi G\rho \left(\frac{r_i(t_{\text{init}})}{r_i}\right)^3 + \frac{1}{r_i^2}(E^2(i) - 1) + \frac{G^2 m_s^2(i)}{4r_i^4} \quad . \quad (22)$$

We can see that the first term behaves like a nonrelativistic matter term in the Hubble equation of FRW models, the second and the third like a curvature and a radiation source term. We set $E(i) \equiv 1$ mimicking a spatially flat model. For $r_i(t_{\text{init}})$, we put

$$r_i(t_{\text{init}}) = li \quad (23)$$

with a constant interval l .

The initial hypersurface is defined as follows. First, we take a unit spacelike 4-vector which is directed outward and normal to the 4-velocity of first shell. Taking this vector as a starting tangential vector, we extend a geodesic curve until its radial coordinate reaches the circumferential radius of the second shell. This geodesic curve defines the simultaneous hypersurface in the region between the first shell and the second shell. Then we set a new 4-vector normal to the 4-velocity of the second shell, and extend a geodesic curve as before. Repeating this process, we complete the whole initial hypersurface. With this choice, the coordinate system coincide with the comoving coordinate system in a spatially flat FRW universe when we increase the number density of the shell to infinity.

3 Results and Discussion

3.1 Luminosity distance – redshift relation

We calculated d_L - z relation under the initial condition described above. We take the initial interval of shell $l = \tilde{H}_i^{-1} N^{-1}$ where the "Hubble constant" on the initial slice $\tilde{H}_i$ is defined as

$$\tilde{H}_i^2 = \frac{8\pi G}{3}\rho \quad (24)$$

with a constant "energy density" ρ , and N is the total number of shell within the "Hubble horizon." In Fig.1 we plot the luminosity distance versus the redshift of each shell for three models (a), (b) and (c) with a common $\tilde{H}_i = 1$.

The total number of shell N is (a) 10, (b)30 and (c)300. We emit a photon toward the center from each shell so that every photon reaches at the center simultaneously. The number of the outermost shell from which a photon is emitted is (a) 9, (b)27 and (c)270. We take $c = 1$ and $G = 1$.

We obtain a quite surprising result that all the data can be fitted by just one curve (plotted in panel(a)) *irrelevantly* to the number of shell. We confirm this in panel(d) where we plot all the three models in one panel. This supports "averaging hypothesis" that large-scale behaviour of a locally inhomogeneous universe can be well described by a FRW model.

In the next section, we discuss the implication of this result.

3.2 Comparison with FRW model

We compare the d_L - z obtained above with that of a FRW model and discuss how to interpret the calculated data in terms of the FRW model. We employ a spatially flat FRW model filled with nonrelativistic matter since in the r.h.s. of the expansion law of a dustshell (22), the curvature term vanishes and the radiation term is of order N^{-2} relative to the first term.

The Hubble equation in this model is

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho_{\text{FRW}} \quad (25)$$

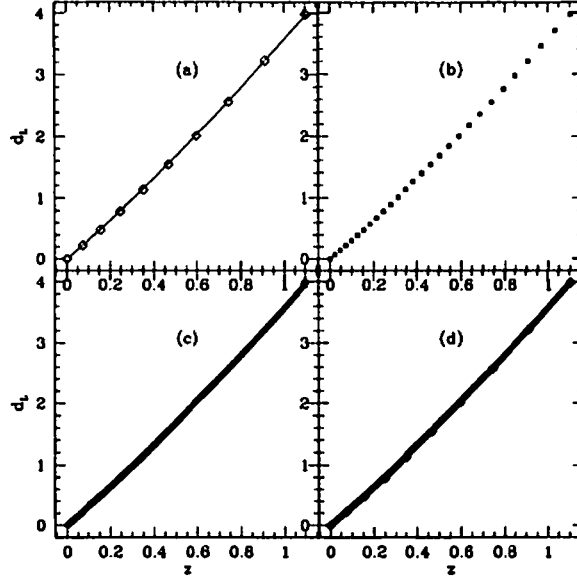


Figure 1: Luminosity distance-redshift relation in dustshell universe. Total number of shell in each model is $N =$ (a) 10, (b) 30 and (c) 300. Solid line in panel(a) is a curve obtained in a FRW model.

with $\rho_{FRW} \propto a^{-3}$. Remembering the relation between the redshift and the scale factor $1 + z = a_0/a$ where subscript 0 denotes a value at present, we rewrite the Hubble equation as

$$H_i^2 = H_0^2(1 + z)^3 \quad (26)$$

Let us consider what we would obtain when we estimate the "Hubble constant" at the initial slice $\tilde{H}_i$ in the dustshell universe using this relation with the observed H_0 and the redshift of the outermost shell z_{om} and identifying H_i with $\tilde{H}_i$. Actually, the value of $\tilde{H}_i$ we have put is reproduced by either model with an error less than a few per cents by this procedure. That is, starting from the initial condition we have set, we can construct a universe which looks like "an almost FRW universe" using dustshells though large inhomogeneities of density distribution exist.

One would like to relate the estimated H_i with the "averaged density" of the dustshell universe on the initial slice. Averaged density of dustshell universe, however, is not well-defined even if we define an initial slice. Since the distribution of mass is discrete, the averaged density is discontinuous at the position of each shell. Then we cannot assign an "averaged density" at the shell, i.e., at the scale of observed redshift. In the limit of $N \rightarrow \infty$, however, the ambiguity must vanish. We are checking this "averaged density" limit agrees with the constant density which determines the expansion law of dust shell.

We here place several questions under study: (1) What would we obtain when we start from another hypersurface with the same initial expansion law for each shell? Will we obtain a distance-redshift relation significantly different from that of a FRW model? (2) If so, which quantity on the initial surface characterize the difference? Which quantity is homogenous when a universe looks like a FRW universe? (3) How the relation changes when we observe the universe again after some interval?

4 Summary

We have studied the behaviour of d_L - z relation in a dustshell universe and see what happens when we interpret it in terms of a FRW model.

We have seen that d_L - z relation is fitted by that of a FRW model with high accuracy. When we estimate the initial "Hubble constant" in terms of a FRW model, we can reproduce almost the right value even if the number of shell is so small as ~ 10 within a Hubble horizon scale.

In order to make a clear statement on the effect of inhomogeneities on the parameters of a FRW model, however, we have to study more various patterns of inhomogeneities, which will be achieved in our forthcoming paper.

Acknowledgements

This study is done with K. Nakao and T. Harada at Kyoto University. The author is supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists, No.3167.

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Cosmological Backreaction and Cosmological Principle

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Abstract

The averaging problem and the cosmological backreaction problem are investigated in the context of the cosmological principle. From the Newtonian and Anti-Newtonian results, we clarified the condition for the dynamical cosmological principle to hold.

1 Introduction

Recent controversy of averaging problem in cosmology is intimately related to the backreaction problem. It is important to distinguish the different kind problems clearly, although they are deeply connected with each other. Therefore, we will almost ignore the local backreaction effects in this paper. The motivation of the research on the averaging problem is the following naive question. In what sense can we consider the Friedmann-Robertson-Walker universe (FRW) as an approximation of the real clumpy universe? Some of them try to attack this problem from the renormalization group point of view, i.e., the deformation of generic manifold by smearing procedure is considered as the renormalization group flow. However, FRW universe can not be the global fixed point of this kind of flow. Hence, an alternative approach seems to be appropriate.

The basis of the FRW universe is of course the cosmological principle. Namely, on average, our universe is homogeneous and isotropic. However, usually, the cosmological principle does not refer the dynamics of the universe. At each timeslicing, we can consider our universe as the homogeneous and isotropic one, if we average over the sufficiently large length scale. Is it correct to conclude that the dynamical evolution of our universe can be represented by that of FRW universe? So let us make our stand point clear. We would like to propose to extend the cosmological principle to the dynamical cosmological principle. Certainly, some backreaction effects exist at least locally. Futamase has been studying this problem. [1] Here, we ignored this kind of problem, except for the gravitational wave contribution. We have tried to make the problem concrete as possible as we can. Consequently, the result itself might appear trivial. If so, my purpose has been achieved. Namely, the problem has disappeared.

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2 Newtonian Limit

If we consider the inhomogeneity whose typical length scale is below the Horizon scale, then Newtonian approximation is fairly valid. The basic equations of Newtonian cosmology are

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} = -\vec{\nabla} \phi, \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \quad (2)$$

$$\nabla^2 \phi = 4\pi G \rho, \quad (3)$$

where $\vec{v}$, ρ , and ϕ are velocity field, density field, and gravitational potential, respectively. In terms of $\theta = \vec{\nabla} \cdot \vec{v}$, we can rewrite the above equation as

$$\frac{d\theta}{dt} + \frac{1}{3}\theta^2 = -4\pi G \rho, \quad (4)$$

$$\frac{d\rho}{dt} = -\rho\theta. \quad (5)$$

If we assume the maximal symmetry, the above equation leads to FRW equation. Behind this assumption, the averaging in some sense is hidden. However, in general, the time evolution and averaging do not commute. This is the point people worry about. Later, we will come back this problem.

Once the background space-time is fixed, the standard linear analysis is viable. By separating Hubble flow and the peculiar velocity part

$$\vec{v} = H\vec{r} + \vec{u}, \quad (6)$$

we get the perturbation equations:

$$\frac{\partial \vec{u}}{\partial \tau} + H\vec{u} = -\frac{1}{a}\nabla_x \phi \quad (7)$$

$$\frac{\partial \delta\rho}{\partial \tau} + 3H\delta\rho + \frac{1}{a}\rho\vec{\nabla}_x \cdot \vec{u} = 0 \quad (8)$$

$$\nabla^2 \phi = 4\pi G \delta\rho \quad (9)$$

Solving the above equations yields the standard results of structure formation.

Now, we shall take into account the non-linear effect and examine if the FRW picture is valid. Our starting equation is

$$3\dot{H} + 3H^2 + 4\pi G \langle \rho \rangle + \left\langle \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{u} + H\vec{r} \cdot \vec{\nabla} \vec{\nabla} \cdot \vec{u} + 2H\vec{\nabla} \cdot \vec{u} \right\rangle = - \langle \vec{\nabla} \cdot [\vec{u} \cdot \vec{\nabla} \vec{u}] \rangle. \quad (10)$$

Here, we have used eq.(6) in eq.(1) and taken the average specifield below. The above equation is apparently different from FRW equation. If we renormalize the expansion

$$\langle \theta \rangle = 3H(t) + \langle \vec{\nabla} \cdot \vec{u} \rangle \quad (11)$$

we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \langle \theta \rangle + \frac{1}{3} \langle \theta \rangle^2 + 4\pi G \langle \rho \rangle + \langle \vec{v} \cdot \vec{\nabla} \theta \rangle + \langle (\vec{\nabla} \cdot \vec{u})^2 \rangle - \langle \vec{\nabla} \cdot \vec{u} \rangle^2 \quad (12) \\ = -\frac{2}{3} \langle \vec{\nabla} \cdot \vec{u} \rangle^2 + \langle \vec{\nabla} \cdot [\vec{u} \vec{\nabla} \cdot \vec{u} - \vec{u} \cdot \vec{\nabla} \vec{u}] \rangle . \end{aligned}$$

To proceed further, we must specify the averaging procedure. We will adopt the averaging over the comoving volume, D :

$$\langle A \rangle = \frac{1}{V} \int_D d^3x A . \quad (13)$$

This definition gives the commutation rule

$$\vec{v} \cdot \vec{\nabla} \langle \theta \rangle = \langle \vec{v} \cdot \vec{\nabla} \theta \rangle + \langle \theta^2 \rangle - \langle \theta \rangle^2 . \quad (14)$$

Furthermore, let us define the Lagrangian derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} . \quad (15)$$

Finally, we obtain the result due to Buchert and Ehlers [2]

$$\frac{d}{dt} \langle \theta \rangle + \frac{1}{3} \langle \theta \rangle^2 + 4\pi G \langle \rho \rangle = -\frac{2}{3} \langle \vec{\nabla} \cdot \vec{u} \rangle^2 + \langle \vec{\nabla} \cdot [\vec{u} \vec{\nabla} \cdot \vec{u} - \vec{u} \cdot \vec{\nabla} \vec{u}] \rangle . \quad (16)$$

Apparently, the right hand side reduces to the surface integral. Hence, if the universe is topologically closed or the periodic, it gives zero. Except for the special cases, the short wavelength fluctuation gives negligible contribution. The dynamics of the universe can be represented by FRW universe on average. Thus, the dynamical cosmological principle holds.

3 Anti-Newtonian Limit

In the previous section, we reviewed the argument by Buchert and Ehlers from our point of view. There, in the limit of the large averaging scale, we have shown the condition for the dynamical cosmological principle. However, in that limit, the relativistic correction can not be neglected. This section is devoted to this theme.

A natural approach to the relativistic problem is to use the perturbative expansion of the metric: [3]

$$g = g^{(0)} + g^{(1)} + g^{(2)} + \dots , \quad (17)$$

where the superscript denotes the order of the perturbation. Einstein equation can also be expanded as

$$G[g] = G[g^{(0)}] + \frac{\delta G}{\delta g} g^{(1)} + \frac{\delta G}{\delta g} g^{(2)} + \frac{1}{2} \frac{\delta^2 G}{\delta g^2} g^{(1)2} + \dots . \quad (18)$$

In principle, we can solve the above equation order by order. However, there is the notorious gauge problem. Under the gauge transformation $\bar{x} = x + \xi$, each variables transform as

$$\bar{g}^{(0)} = g^{(0)}, \bar{g}^{(1)} = g^{(1)} - \mathcal{L}_\xi g^{(0)}, \bar{g}^{(2)} = g^{(2)} - \mathcal{L}_\xi g^{(1)} + \frac{1}{2} \mathcal{L}_\xi^2 g^{(0)}, \quad (19)$$

where $\mathcal{L}_\xi$ denotes the Lie derivative along the vector field $\tilde{\xi}$. Apparently, the averaging space-time

$$\langle g \rangle = g^{(0)} + \langle g^{(2)} \rangle + \dots \quad (20)$$

becomes gauge dependent. More worse, it is not even a solution of the averaging Einstein equation. Hence, the Isaacson-like approach in cosmology should be considered carefully. One possible attitude to this problem is to fix the gauge completely. We will adopt the synchronous and comoving gauge. Of course, this choice is possible only in the case of dust fluid. Therefore, in the following discussion, the dust fluid case will be investigated.

In the synchronous gauge, the metric takes the following form:

$$ds^2 = -dt^2 + \gamma_{ij} dx^i dx^j. \quad (21)$$

In this case, Einstein equation can be reduced to a simple form. The relevant one to our purpose is

$$\frac{\partial K}{\partial t} + K_j^i K_i^j = -4\pi G \rho, \quad (22)$$

where the extrinsic curvature is defined by

$$K_{ij} = \frac{1}{2} \frac{\partial \gamma_{ij}}{\partial t}. \quad (23)$$

The above equation corresponds to eq.(4) in the Newtonian theory. To specify the problem, we need to fix the averaging procedure. Let us define the average

$$\langle A \rangle = \frac{1}{V} \int d^3x \sqrt{h} A \quad (24)$$

and the expansion

$$\theta = K. \quad (25)$$

Then, we obtain the commutation rule

$$\frac{\partial}{\partial t} \langle A \rangle = \left\langle \frac{\partial A}{\partial t} \right\rangle + \langle A \theta \rangle - \langle A \rangle \langle \theta \rangle. \quad (26)$$

If we put

$$\langle \theta \rangle = 3H + \langle V_k^k \rangle, \quad (27)$$

eq.(22) leads to

$$\frac{\partial}{\partial t} \langle \theta \rangle + \frac{1}{3} \langle \theta \rangle^2 + 4\pi G \langle \rho \rangle = -\frac{2}{3} \langle V_k^k \rangle^2 + \langle (V_k^k)^2 - V_j^i V_i^j \rangle \quad (28)$$

To evaluate the right hand side of the above equation, we shall use the anti-Newtonian approximation initiated by Tomita. [4] The situation we have in mind is that the typical wavelength of the inhomogeneities, L , is larger than the Hubble horizon scale,

$$aL \gg H^{-1} . \quad (29)$$

In this case, in the lowest order, we can ignore the spatial gradient term. Then, the problem reduces to the local one and subsequent iteration implies that the metric can be expanded in the similar manner with the Taylor expansion: [5]

$$\gamma_{ij} = a^2 h_{ij} + f(t) R_{ij} + \frac{1}{3} g(t) R h_{ij} , \quad (30)$$

where $f(t)$ and $g(t)$ are some functions of t and are determined from Einstein equations and R_{ij} is the spatial Ricci tensor constructed from h_{ij} . It is straightforward to obtain

$$K_j^i = \frac{\dot{a}}{a} \delta_j^i + \frac{3}{20} t^{-\frac{3}{2}} (R \delta_j^i - 4 R_j^i) + \dots . \quad (31)$$

In principle, we can obtain higher order corrections in a similar manner. Eq.(31) gives

$$\langle (V_k^k)^2 - V_j^i V_i^j \rangle = -16 \left(\frac{3}{20} \right)^2 t^{-\frac{3}{2}} \langle R^{lm} R_{lm} - \frac{3}{8} R^2 \rangle . \quad (32)$$

To get the feeling of the meaning of the above eq.(32), we will examine the non-gravitational wave contribution first. In the conformal flat case

$$h_{ij} = (1 + \Psi) \delta_{ij} , \quad (33)$$

the scalar curvature becomes

$$R = -2 \Psi_{,m}^m , \quad (34)$$

where Ψ is a function of the spatial coordinates. Thus, the first term is found to be the total derivative,

$$\lim_{L \rightarrow \infty} \langle V_k^k \rangle = 0 . \quad (35)$$

The second term can be evaluated as

$$R^{lm} R_{lm} - \frac{3}{8} R^2 = \frac{1}{4} [\Psi^{,i} \Psi_{,i}^m - \Psi_{,m}^i \Psi^{,m}],_m , \quad (36)$$

thus again we obtain

$$\lim_{L \rightarrow \infty} \langle (V_k^k)^2 - V_j^i V_i^j \rangle = 0 . \quad (37)$$

These results agree with the Newtonian results. [6]

In order to examine the general case, we consider the second rank tensor

$$M^{ij} = \frac{1}{4} R^{ij} + \frac{1}{4} \Delta R h^{ij} - \Delta R^{ij} . \quad (38)$$

Notice that this is the transverse traceless tensor within this order of approximation. By using the above tensor and ignoring the total derivative, we get

$$R^{lm} R_{lm} - \frac{3}{8} R^2 = \frac{1}{2} M^{ij} \frac{1}{\Delta^2} M_{ij} . \quad (39)$$

Therefore, the second term gives the negative definite contribution

$$\langle (V_k^k)^2 - V_j^i V_i^j \rangle = -16 \left(\frac{3}{20}\right)^2 t^{-\frac{3}{2}} \frac{1}{2V} \int d^3x M^{ij} \frac{1}{\Delta^2} M_{ij} . \quad (40)$$

The above term can be interpreted as the contribution from the gravitational wave degrees of freedom.

4 Conclusion

We would like to claim that the dynamical cosmological principle

$$\lim_{L \rightarrow \infty} \langle G_{\mu\nu}[g] \rangle = G_{\mu\nu}[g_{\text{FRW}}] + O\left(\frac{1}{L}\right) \quad (41)$$

holds, if the non-linear inhomogeneities are sufficiently short range and consequently does not contribute the surface integral.

The Newtonian argument by Buchert and Ehlers [2] support the above formulation of the dynamical cosmological principle. And also, in the anti-Newtonian limit, we have shown that the above formulation is valid provided that the gravitational wave contribution is included into the definition of the energy momentum tensor.

Of course, the analysis presented in this paper is far from complete. We have analyzed only the Newtonian limit and the anti-Newtonian limit. The local backreaction effects have not been investigated thoroughly.

Acknowledgements

We would like to thank Masaaki Sakagami for useful discussion.

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Formation and Evolution of Global Strings in 2+1 Dimensions

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Abstract

We simulate the formation and the evolution of global strings in 2+1 dimensions taking into account the expansion of the universe and the associated change of the effective potential governing the phase transition. Starting from the thermal equilibrium state, we run simulations for 100 sets of different initial conditions. It is shown that evolution of the number density can be fitted by simple functions.

1 Introduction

Ever since the possibility of the symmetry restoration in the very early universe was suggested [1], the effect of the cosmological phase transitions [2] have been extensively investigated. As Kibble first pointed out, topological defects may be formed in consequence of the cosmological phase transition [3] (see also [4] for review). If the symmetry group G breaks into the group H and the coset space G/H is not simply-connected, strings should be formed. Strings are cosmologically attractive because they may have radiated axions from the Peccei-Quinn breaking scale to the QCD scale [5]. Also, in the condensed-matter system analogous vortexes are observed [6].

Studies on strings can roughly be divided into their formation and subsequent evolution. As for the formation, Monte-Carlo simulations have been used. Vachaspati and Vilenkin [7] first examined the initial configuration of the string network. They ensured that string network has the statistical property of the Brownian random walk (see also [8]), and found that string network is made of the scale-invariant system of closed loops and infinite strings occupy 80% of the total string length. Since then, many authors reexamined the problem in the more realistic situations than the original one, especially paying attention to the existence and the fraction if at all, of the infinite string [9].

However, as seen from the fact that the above papers were based on the Monte-Carlo simulations, little work has been done on the dynamical aspects of the string formation. One exception is Borrill's work [10], where he simulated the bubble nucleation process in the first order phase transition by the lattice-free method. However, in his approach the subsequent evolution of the string network could not be followed. In ref. [11], the Langevin-equation approach was adopted. But the effect of the expansion of the universe was considered only partially, which was far from the real dynamics. In ref. [12], the numerical simulation was made in the background of the expanding universe, in which the Mexican-hat potential was adopted from the outset and the initial condition was given by hand. Also, because of smallness of the lattice number, the whole system enters one Hubble horizon and the evolution could not be followed for sufficient time to judge whether the network reached a *scaling* regime. ⁴ For local(gauged) strings, their evolution has been investigated by using the Nambu-Goto action. There are many analytical

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⁴Roughly speaking, the length scale characterizing the large scale behavior of the string network grows in proportion to the horizon scale. In particular, the ratio between the energy density of infinite strings and that of radiation is constant throughout their evolution.

approach [13] and numerical one [14] and it is assured that the string networks obey the scaling solution. On the other hand, for global strings, the Kalb-Ramond action must be used to examine the evolution of the global string system, instead of the Nambu-goto action. But it is hard to deal with in simulation. Therefore, the evolution of global strings have hardly investigated.

In this paper, we simulate both formation and evolution of the global string network. For this purpose, we directly simulate the dynamics of the complex scalar field. Starting from a symmetry-restored state, we follow the dynamics of the complex scalar field, fully considering the expansion of the universe and the attendant change of the effective potential, which shows the spontaneous symmetry breaking below the critical temperature.

In the next section, we establish the formulation for the numerical simulation. In Sec. 3, we show the result of the numerical simulation and give the interpretation of the results.

2 Formulation

We consider the following Lagrangian density for a complex scalar field $\Phi(x)$,

$$\mathcal{L}[\Phi] = \partial_\mu \Phi \partial^\mu \Phi^\dagger - V[\Phi], \quad (1)$$

where the potential $V[\Phi]$ is given by

$$V[\Phi] = \frac{\lambda}{2} (\Phi \Phi^\dagger - \eta^2)^2. \quad (2)$$

Then the one-loop finite temperature effective potential $V_{\text{eff}}[\Phi]$ in the high temperature limit is written as

$$V_{\text{eff}}[\Phi] = \frac{\lambda}{2} (\Phi \Phi^\dagger - \eta^2)^2 + \frac{\lambda}{3} T^2 \Phi \Phi^\dagger, \quad (3)$$

which has a $U(1)$ symmetry. For $T > T_c = \sqrt{3}\eta$, the potential V_{eff} has a minimum at $\Phi = 0$ and the $U(1)$ symmetry is restored, on the other hand, new minima $|\Phi|_{\text{min}} = \eta \sqrt{1 - (T/T_c)^2}$ appear and the symmetry is broken for $T < T_c$. In this case the phase transition is second order.

In the expanding universe effective Lagrangian density $\mathcal{L}_{\text{eff}}[\Phi]$ is given by

$$\mathcal{L}_{\text{eff}}[\Phi] = g_{\mu\nu} (\partial^\mu \Phi) (\partial^\nu \Phi)^\dagger - V_{\text{eff}}[\Phi], \quad (4)$$

where $g_{\mu\nu}$ is identified with the Robertson-Walker metric. Then the equation of motion is given by

$$\ddot{\Phi}(x) + 3H\dot{\Phi}(x) - \frac{1}{a(t)^2} \nabla^2 \Phi(x) = -V'_{\text{eff}}[\Phi], \quad (5)$$

where $' = \partial/\partial\Phi^\dagger$ and $a(t)$ is the scale factor. The Hubble parameter $H = \dot{a}(t)/a(t)$ and cosmic time t are given by

$$\begin{aligned} H^2 &= \frac{8\pi}{3M_{\text{pl}}^2} \frac{\pi^2}{30} g_* T^4, \\ t &= \frac{1}{2H} \equiv \frac{\xi}{T^2}, \end{aligned} \quad (6)$$

where M_{pl} is the Plank mass, g_* is the total number of freedom for the relativistic particles, and the radiation-dominated universe is assumed. We define a dimensionless parameter ζ as

$$\zeta \equiv \frac{\xi}{\eta} = \left(\frac{45M_{\text{pl}}^2}{16\pi^3 g_* \eta^2} \right)^{1/2}. \quad (7)$$

In our simulation, we take $\zeta = 10$, which corresponds to $\eta = 10^{15} - 10^{16} \text{ GeV}$. The energy density of each point is written as

$$\rho(x) = \dot{\Phi}(x)\dot{\Phi}^\dagger(x) + \frac{1}{a(t)^2} \nabla\Phi(x) \cdot \nabla\Phi^\dagger(x) + V_{\text{eff}}[\Phi]. \quad (8)$$

We simulate the system in $2 + 1$ dimensions from the initial time $t_i = \frac{1}{4} t_c$, that is, $T_i = 2T_c$ to the final time $t_f = 150 t_i = 37.5 t_c$, that is, $T_f \sim T_c/6.1$. Since the $U(1)$ symmetry is restored at the initial time $t = t_i$, we adopt as the initial condition the thermal equilibrium state with the mass squared,

$$m^2 = \frac{d^2 V_{\text{eff}}[|\Phi|]}{d|\Phi|^2} \Big|_{|\Phi|=0}, \quad (9)$$

which is the inverse curvature of the potential at the origin at $t = t_i$. In the thermal equilibrium state, Φ and $\dot{\Phi}$ are Gaussian distributed with the following correlation function,

$$\langle \beta |\Phi(x)\dot{\Phi}^\dagger(y)|\beta \rangle_{\text{equal-time}} = \int \frac{dk}{(2\pi)^3} \frac{1}{2\sqrt{k^2 + m^2}} \coth \frac{\beta\sqrt{k^2 + m^2}}{2} e^{ik \cdot (x-y)}, \quad (10)$$

$$\langle \beta |\dot{\Phi}(x)\dot{\Phi}^\dagger(y)|\beta \rangle_{\text{equal-time}} = \int \frac{dk}{(2\pi)^3} \frac{\sqrt{k^2 + m^2}}{2} \coth \frac{\beta\sqrt{k^2 + m^2}}{2} e^{ik \cdot (x-y)}. \quad (11)$$

Φ and $\dot{\Phi}$ has no correlation except at the same point. We generate these fields as the initial condition in the momentum space because the Fourier transform $\tilde{\Phi}(\mathbf{k})$, $\dot{\tilde{\Phi}}(\mathbf{k})$, have no correlations.

Introducing the dimensionless variables $\tilde{\Phi} = \Phi t_i$, $\tilde{t} = t/t_i$, $\tilde{x} = x/t_i$, $\tilde{\rho} = \rho t_i^4$, the equation of motion and the total energy density are given by

$$\ddot{\tilde{\Phi}}(\mathbf{x}) + \frac{3}{2\tilde{t}} \dot{\tilde{\Phi}}(\mathbf{x}) - \frac{1}{\tilde{t}} \nabla^2 \tilde{\Phi}(\mathbf{x}) = - \left(|\tilde{\Phi}|^2 + \frac{25}{9\tilde{t}} - \frac{25}{36} \right) \tilde{\Phi}, \quad (12)$$

$$\rho(\mathbf{x}) = \dot{\tilde{\Phi}}(\mathbf{x})\dot{\tilde{\Phi}}^\dagger(\mathbf{x}) + \frac{1}{\tilde{t}} \nabla\tilde{\Phi}(\mathbf{x}) \cdot \nabla\tilde{\Phi}^\dagger(\mathbf{x}) + \frac{1}{2} \left(|\tilde{\Phi}|^2 - \frac{25}{36} \right)^2 + \frac{25}{9\tilde{t}} |\tilde{\Phi}|^2, \quad (13)$$

where λ is set to be unity⁵ and tildes are omitted. The scale factor $a(t)$ is normalized as $a(1) = 1$. We use 512×512 lattices. Then, the lattice spacing δx is set to be 0.1 and its physical length δx_{phys} grows as $a(t)\delta x = 0.1 t^{1/2}$. The time step δt is chosen to be 0.01 for both cases. At sufficiently low temperatures, the typical width of the string d is nearly $1/\eta$. Then,

$$\frac{H^{-1}}{\delta x_{\text{phys}}} = 20 t^{1/2} \quad (\sim 245 \text{ at } t = t_f) \quad (14)$$

$$\frac{d}{\delta x_{\text{phys}}} = 12 t^{-1/2} \quad (\sim 0.98 \text{ at } t = t_f) \quad (15)$$

Even at the final time t_f , the system includes the enough number of the Hubble horizons and the resolution of the simulation is good enough.

Under these circumstances, we simulate the system using the second order leap-frog method and the Crank-Nicholson scheme over 100 runs with different initial conditions.

3 Results and Discussion

First, we show how the strings are generated as the symmetry is spontaneously broken and both the gradient and the kinetic energies are sufficiently suppressed. In Figure 1, the potential energy density for each lattice point is depicted at $t = 1, 150$ for one realization. One can see that initially the field fluctuates very much and the gradient and the kinetic energies dominate over the potential energy. But

⁵Strictly speaking, the high temperature approximation is valid only when λ is much smaller than unity. But, the change of λ only scale the physical quantity and is not relevant to the essence.

as the universe expands, they rapidly get suppressed so that the false vacuum energy becomes dominant and the string configurations are stably formed.

Next, we also show the time development of the string number per the Hubble horizon. First of all, we decide the way the string should be identified in our simulations. Since in our simulation the spacetime is discretized, the point with $\Phi = 0$ is not necessarily mounted on the lattice. In the worst case, the point with $\Phi = 0$ lies in the center of the plaquette. Therefore we consider that a lattice becomes a part of a string if the potential energy density at the lattice is larger than that which corresponds to the field value of string solution at $r = \delta x_{phys}/\sqrt{2}$. If we assume the string configuration as $\Phi(r, \theta) = q(r)e^{i\theta}$, the string solution is obtained by solving the equation,

$$\frac{d^2 q}{dr^2} + \frac{1}{r} \frac{dq}{dr} - \frac{q}{r^2} - (q^2 + \frac{25}{9t} - \frac{25}{36})q = 0. \quad (16)$$

with the boundary conditions given by

$$q(0) = 0, \quad q(r \rightarrow \infty) = \frac{5}{6} \left(1 - \frac{4}{t}\right)^{1/2}. \quad (17)$$

Also, we regard the connected component of lattices satisfying the above condition as one string. Figure 2 depicts time development of string number per the horizon averaged over 100 realizations. The simulation results can be well fitted by the following fitting formula,

$$n(H^{-1})^2 = 0.381 \ln(t/0.351). \quad (18)$$

In this case the number does not approach the constant value asymptotically but continues increasing. The reader may wonder if other formulas, especially, those converging to the constant value asymptotically fit the simulation results. In fact, the following formula also fits the simulation results well,

$$n(H^{-1})^2 = \frac{3.06t}{(\sqrt{t} + 1.92)^2}. \quad (19)$$

In order to clarify the asymptotical behavior of the simulation, we make another simulation from the initial condition where the field is satisfied with the checkerboard of the string and the anti-string, and the string number per the Hubble horizon is set to be 100. Also, the time-derivative of the field is set to zero on all lattices and we run 100 realizations from the different initial conditions. The result is depicted in Figure 3. The log formula can fit well the new simulation result if the time is rescaled. On the other hand, the asymptotically constant formula cannot. Therefore we can conclude that the string number per the Hubble horizon continue increasing and, for example, the above log formula fits the result well.

Acknowledgements

MY is grateful to Professor K. Sato for his continuous encouragement and to Dr. T. Shiromizu and Professor M. Morikawa for their useful comments.

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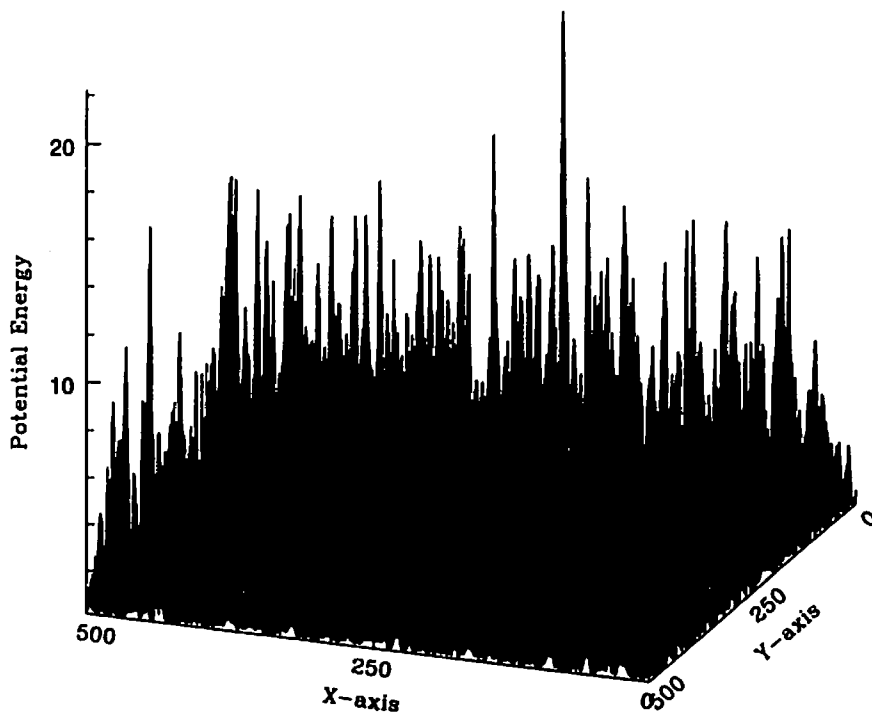


Figure 1(a): The potential energy density for each lattice point is depicted at $t = t_i = 1$.

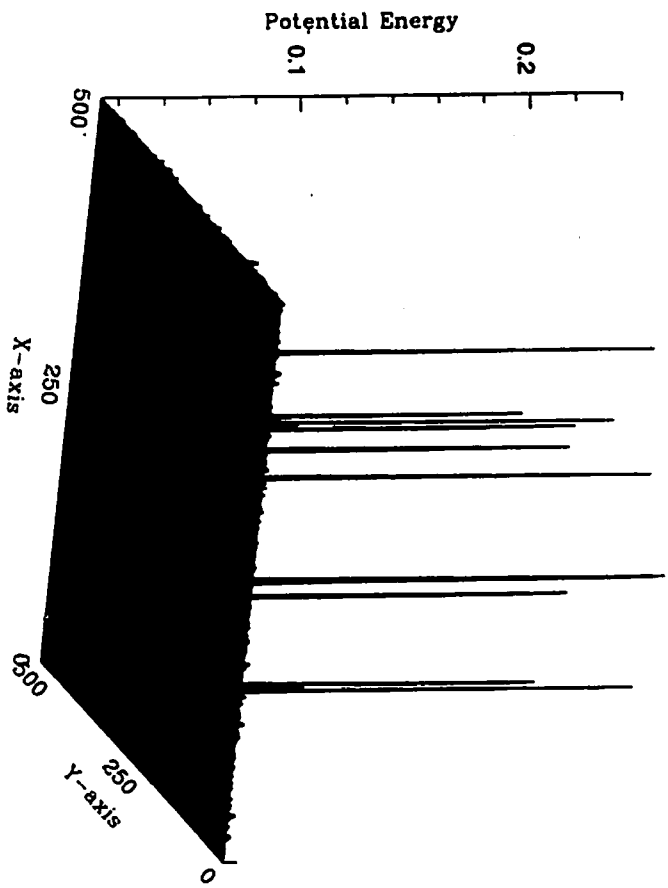


Figure 1(b): That at $t = 150$.

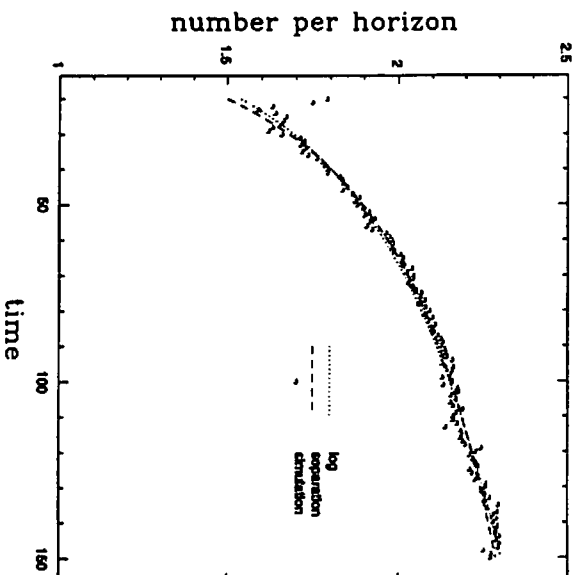


Figure 2: Time development of string number per the horizon averaged over 100 realizations. Dots represents the string number given by the simulation. The dotted line represents the log fitting formula and the dashed line the other one.

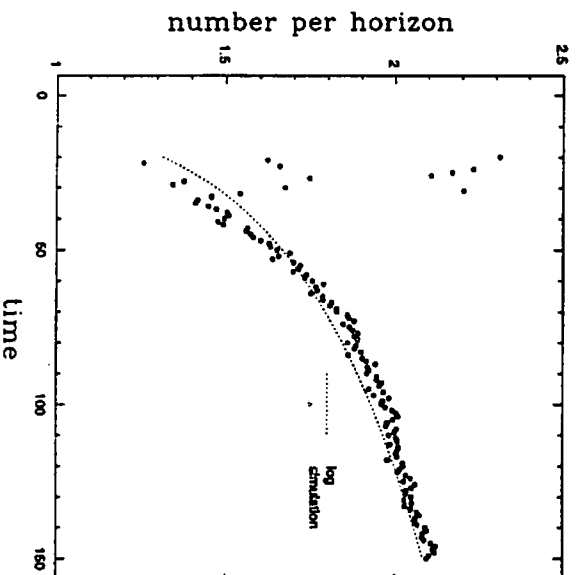


Figure 3: That from the checkerboard initial conditions. Dots represents the string number given by the simulation. The dotted line represents the fitting formula which is the same as the above log formula except the time rescaled.

One problem in quantum field theory in curved space

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Abstract

Here we give rise one question on the standard theory of quantum field in curved space. The standard strategy to quantize a free field in curved space is based on the decomposition by using spatial harmonic functions. This decomposition allows us to treat the field as an ensemble of infinite number of decoupled harmonic oscillators. Then these degrees of freedom can be quantized following the standard procedure of quantum mechanics. We find, however, in open universe, this method restricts the variety of quantum states that can be realized. That is, we find that the correlation beyond the spatial curvature scale is always exponentially suppressed. In this paper, we explain that this restriction is physically unacceptable and, hence, we suggest that some extension of the standard theory of quantum fields in curved space is necessary.

1 Introduction

To study the early universe, quantum field theory in curved space[1] is a very important tool. It is considered to be an approximation to the theory that unifies general relativity and quantum field theory. Here we discuss one aspect relating to the internal consistency of this effective theory.

Recently, the creation scenario of an open universe is extensively discussed by many authors[2]. Most of them are in the context of “the one-bubble open inflation scenario”. In this scenario, the predicted spectrum is found to be almost independent of details of the model. Hence, this model can be tested by future observations. The spacetime diagram of the one-bubble open inflation universe is shown in Fig.1(a). The shaded region represents an open universe, which is created inside a nucleated bubble. Although we use the name, the one-bubble open inflation, the whole spacetime can be also understood as a closed universe. A time constant hypersurface in the open universe has infinite volume, but at the same time this spacetime allows foliations by compact closed hypersurfaces.

One may think that there may be an alternative scenario to create an open universe. The simplest idea is that an open universe is created from the initial singularity. This singularity is governed by some unknown physics which guarantees the homogeneity of the universe. The spacetime diagram of this scenario is shown in Fig.1(b). Let us refer to this scenario as “the no-bubble open universe scenario”. Compared with the one-bubble open inflation scenario, this scenario is less attractive because it has less predictive power. Since nobody knows the physics that governs the initial singularity at present, any initial condition can be assumed as one wishes. Nevertheless, there seems no rigorous reason to deny this possibility. Hence, we cannot claim that the one-bubble open inflation is the unique scenario to create an open universe. Although, at present, we can tell little about the no-bubble open universe scenario, we can point out one important thing that there appears a fundamental difficulty in discussing it by applying the standard quantum field theory in curved space. We discuss this issue in this paper.

2 Quantum field in curved space

Here we briefly review the well-known theory of quantum fields in curved space[1]. Let us consider a free massive scalar field in a given background spacetime, whose metric is denoted by $g_{\mu\nu}$. The Heisenberg equation for the field operator is given by

$$[\nabla^2 - m^2] \hat{\phi}(x) = 0. \quad (1)$$

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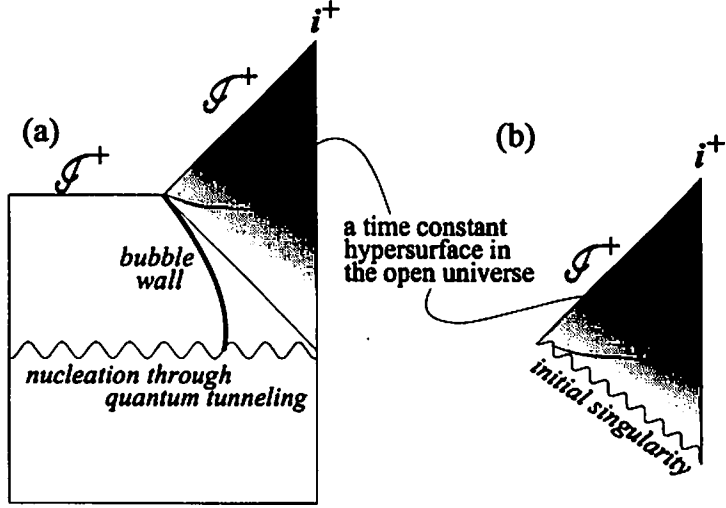


Figure 1: Conformal diagrams (a) for the one-bubble open inflation scenario and (b) for the no-bubble open universe scenario.

Let us introduce a set of mode functions which satisfy the field equation,

$$[\nabla^2 - m^2] u_\Lambda(x) = 0, \quad (2)$$

and the normalization condition,

$$(u_\Lambda, u_{\Lambda'}) := -i \int_\Sigma d\Sigma^\mu \sqrt{g^{(3)}} (u_\Lambda \partial_\mu u_{\Lambda'}^* - (\partial_\mu u_\Lambda) u_{\Lambda'}^*) = \delta_{\Lambda, \Lambda'}, \quad (3)$$

where Λ is the label to distinguish the mode functions and $\delta_{\Lambda, \Lambda'}$ should be understood as a Kronecker's delta for discrete labels and as a Dirac's delta function for continuous labels. Then, expanding the field operator as

$$\hat{\phi}(x) = \sum_\Lambda (\hat{a}_\Lambda u_\Lambda(x) + \hat{a}_\Lambda^\dagger u_\Lambda^*(x)), \quad (4)$$

we find that the system reduces to a set of decoupled harmonic oscillators.

To quantize this field, we set the commutation relation between $\hat{\phi}(x)$ and its conjugate momenta $\hat{\pi}(x)$:

$$[\hat{\phi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{x}')] = i\delta^3(\mathbf{x} - \mathbf{x}'). \quad (5)$$

This commutation relation is equivalent to

$$[\hat{a}_\Lambda, \hat{a}_{\Lambda'}] = \delta_{\Lambda, \Lambda'}. \quad (6)$$

We can specify a vacuum state associated with a set of annihilation operators by

$$\forall a_\Lambda |O_u\rangle = 0. \quad (7)$$

Since the annihilation operators are defined by the expansion (4), we find there is an one-to-one relation between a set of mode functions and a corresponding vacuum state.

In the cosmological perturbation theory, the most important quantity directly related to observations is the vacuum two point function defined by

$$G^+(x, x') := \langle O_u | \hat{\phi}(x) \hat{\phi}(x') | O_u \rangle = \sum_\Lambda u_\Lambda(x) u_\Lambda^*(x'). \quad (8)$$

3 Quantum field in an open universe

The metric of an open Friedmann universe is given by

$$ds^2 = -dt^2 + a^2(t) [dr^2 + \sinh^2 r d\Omega^2]. \quad (9)$$

Here we again consider a free massive scalar field. Decomposing the mode function as

$$u_{p\ell m} = a^{-1} \chi_p(t) f_{p\ell}(r) Y_{\ell m}(\Omega), \quad (10)$$

where $Y_{\ell m}(\Omega)$ is the ordinary spherical harmonics, the equation in the r -direction becomes

$$\left[\frac{1}{\sinh^2 r} \frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{\sinh^2 r} + p^2 + 1 \right] f_{p\ell}(r) = 0. \quad (11)$$

Then $f_{p\ell}(r) Y_{\ell m}(\Omega)$ gives the spatial harmonic functions in an open universe. The equation (11) can be solved imposing the regular boundary condition at $r = 0$. For simplicity, here we consider the case with $\ell = 0$. Then we find that the solution of Eq. (11) is easily written down as

$$f_{p0}(r) = N_{p0} \sqrt{\frac{2}{\pi}} \frac{\sin pr}{\sinh r}, \quad (12)$$

where N_{p0} is a normalization constant. Since the volume of the time constant hypersurface in an open universe is infinite, some trick is necessary to normalize the spatial harmonic functions. For this purpose, we take the finite box prescription. We first consider a sphere with radius R . After fixing the normalization by setting the boundary condition: $\dot{\phi} = 0$ at the boundary, we take the $R \rightarrow \infty$ limit.

From the boundary condition, $f_{p0}(R) = 0$, we have the discrete spectrum for p as,

$$p_n R = \pi n, \quad (13)$$

where n is a positive integer. The difference between the neighboring eigen values is given by

$$\Delta p = \frac{\pi}{R}. \quad (14)$$

The normalization is determined by

$$\int_0^R |f_{p_n 0}(r)|^2 \sinh^2 r dr = |N_{p_n 0}|^2 \frac{R}{\pi} = 1. \quad (15)$$

Hence, we should take $N_{p_n 0} = \sqrt{\pi/R}$. To take the $R \rightarrow \infty$ limit, let us consider the $\ell = 0$ mode contribution to the two point function:

$$\sum_n f_{p_n 0}(r) f_{p_n 0}(r') = \sum_n \frac{\pi}{R} \left(\sqrt{\frac{2}{\pi}} \frac{\sin p_n r}{\sinh r} \right) \left(\sqrt{\frac{2}{\pi}} \frac{\sin p_n r'}{\sinh r'} \right). \quad (16)$$

If we take the $R \rightarrow \infty$ limit in this expression, $\sum_n \frac{\pi}{R} \equiv \sum_n \Delta p$ is to be replaced by the integration over the eigen value p , $\int_0^\infty dp$. Thus we find the above expression reduces to

$$\longrightarrow \int_0^\infty dp \left(\sqrt{\frac{2}{\pi}} \frac{\sin pr}{\sinh r} \right) \left(\sqrt{\frac{2}{\pi}} \frac{\sin pr'}{\sinh r'} \right). \quad (17)$$

This means that there only exist the continuous spectrum with positive p^2 .

4 Discrete mode

In the previous section, we find it is concluded that only the continuous spectrum with positive p^2 exist in an open universe, as long as we follow the standard quantum field theory in curved space with the *finite box* prescription. However, through the recent study of the one-bubble open inflation scenario, it was also clarified that some discrete modes with negative p^2 must be included to represent a class of physically allowed quantum states in an open universe[4]. We here explain this fact very briefly. For simplicity, let us consider a massive scalar field in a pure de Sitter space. As shown in the conformal diagram in Fig. 2, a de Sitter space is covered by 4 disconnected open charts and one unfamiliar chart given by an analytic continuation of the open chart. This unfamiliar chart covers the central diamond region. The metric in the open chart is given by

$$ds^2 = H^{-2} (-dt^2 + \sinh^2 t (dr^2 + \sinh^2 r d\Omega^2)). \quad (18)$$

While the chart that covers the central region is given by

$$ds^2 = H^{-2} (dt^2 + \cos^2 t (-dr^2 + \cosh^2 r d\Omega^2)). \quad (19)$$

In the central region, the roles of t and r are interchanged. As is easily understood from Fig. 2, the $r = \text{constant}$ hypersurface in the central region is a compact global Cauchy surface of the whole de Sitter space.

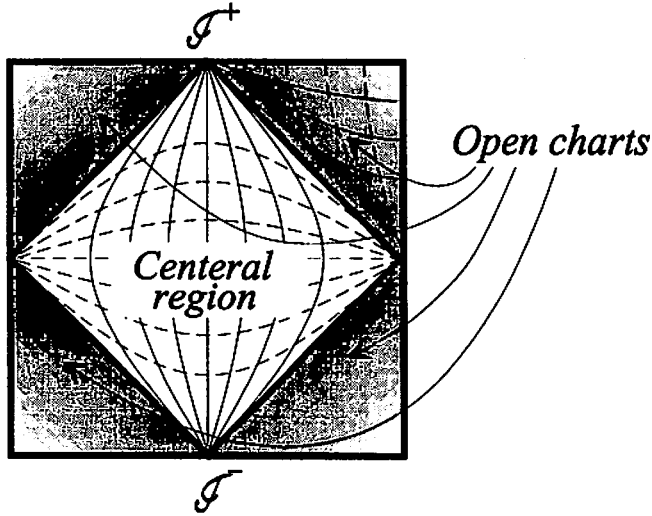


Figure 2: The conformal diagram for a pure de Sitter space. $t(r) = \text{constant}$ lines are drawn by rigid (dashed) lines.

In the central region, the spatial harmonic functions are determined by solving the equation in the t -direction instead of that in the r -direction. This equation is given by

$$\left[-\frac{\partial^2}{\partial \eta^2} + \frac{2 - \frac{m^2}{H^2}}{\cosh^2 \eta} \right] \chi_{p\ell}(\eta) = p^2 \chi_{p\ell}(\eta), \quad (20)$$

where $d\eta := dt/a(t)$. Since this equation has the form of the Schrödinger equation, the complete set of spatial harmonic functions can be obtained by the way analogous to finding eigen states in one-dimensional quantum mechanics. Noting that the potential $V(\eta) := \frac{2 - (m^2/H^2)}{\cosh^2 \eta}$ goes to zero at both ends, we have continuous spectrum for positive p^2 . Two spatial harmonic functions for each p^2 exist

corresponding to left and right moving waves. Since only one spatial harmonic function exists for each p^2 in an open universe, one may think that the state determined in the central region cannot be described in the language of an open universe. However, it is not true. The central region evolves into two left and right open charts. Since each chart has one spatial harmonic function for each positive p^2 , the number of degrees of freedom is consistent. Here we note that a vacuum state defined in the central region is no more a vacuum state in the left or right open universe. The observer in the right open universe cannot observe any information that belongs to the left open universe. Hence, if we want to represent this vacuum state in the language of the right open universe, the degrees of freedom belonging to the left open universe must be traced out. Then, since there is a correlation between the degrees of freedom in the left open universe and those in the right open universe in general, the state after this trace-out is no more a pure state but becomes a mixed state. Anyway, as long as only the continuous spectrum is concerned, any quantum state can be represented both in the languages of the open and central charts.

The real problem is in the discrete spectrum with negative p^2 , which corresponds to the bound state in the analogue to quantum mechanics. Since the potential is positive definite for $m^2 > 2H^2$, there is no discrete spectrum. While, for $0 < m^2 < 2H^2$, there is one discrete mode.

To make sure that this discrete mode is necessary, we can show that this discrete mode must be included to represent the so-called Bunch-Davies vacuum state. As was mentioned above, the two point function can be written in terms of the summation of products of mode functions. After the summation over ℓ and m , we have

$$a(t)a(t')G^+(x, x') = \frac{1}{2\pi^2} \int_0^\infty dp \frac{p \sin p\xi}{\sinh \xi} \sum_{\sigma=\pm} \chi_{p\sigma}(t) \chi_{p\sigma}^*(t'), \quad (21)$$

where $\cosh \xi := \cosh r \cosh r' - \sinh r \sinh r' \cos \Theta$ and Θ is the angle between the two spatial vectors, x and x' . Taking the inverse Fourier transformation of this expression, we have

$$\sum_{\sigma} \chi_{p\sigma}(t) \chi_{p\sigma}^*(t') = \frac{4\pi^2 a(t)a(t')}{p} \int_0^\infty d\xi \sin p\xi \sinh \xi G^+(t, t'; \xi). \quad (22)$$

This integral diverges if $G^+(t, t')$ decreases slower than $e^{-\xi}$. While, it is known that the two point function in the Bunch-Davies vacuum state behave as

$$G^+ \sim \exp \left[\left(-\sqrt{\frac{9}{4} - \frac{m^2}{H^2}} + \frac{3}{2} \right) \xi \right], \quad (23)$$

for large ξ . Hence, for $m^2 < 2H^2$, the Bunch-Davies vacuum cannot be described by the contribution only from the continuous spectrum.

5 Paradox?

As discussed in the preceding section, we need to take into account some discrete spectrum to represent some quantum states in an open universe. We observed these discrete modes play an essential role to represent large scale correlations. The necessity of these modes could also be convinced by considering a spacetime which allows an extension with a compact global Cauchy surface. As an example, we considered a pure de Sitter space. The spectrum is determined by the distribution of eigen values of normalizable spatial harmonic functions. Determining this distribution is analogous to solving a potential problem in quantum mechanics. We found there appear discrete modes as an analogue to bound states in quantum mechanics. However, as discussed in Sec. 3, in the no-bubble open universe scenario with an initial singularity which prohibits such an extension, we were lead to the conclusion that any discrete spectrum cannot be included, if we follow the standard quantum field theory in curved space adopting the finite box prescription. This means that the long scale correlation cannot be realized in the no-bubble open universe scenario. However, the no-bubble open universe begins with the initial singularity that is governed by some unknown physics. Hence, it is quite strange to get any constraint to the initial condition from the low-energy effective theory. There seems no reason why the same initial condition that is prepared in the one-bubble open inflation scenario cannot be realized in the context of no-bubble open universe scenario.

Hence, we doubt on the prescription to control infinite spatial volume in an open universe. The Minkowski space also has infinite spatial volume. It is known that the finite box prescription works well in this case. The only difference from the case of an open universe seems to be the structure at the spatial infinity, $r \rightarrow \infty$. In the case of the Minkowski space, there is the space-like infinity, i_0 . While, in the case of the open universe with the initial singularity, there is no space-like infinity. In the Minkowski case, the spatial volume grows in a power of the radius r , while in the open universe it grows exponentially.

6 Discussion

We observed the evidence that the standard quantum field theory in curved space may break down in the no-bubble open universe scenario. At least, some physically acceptable states in the one-bubble open inflation scenario cannot be represented in the no-bubble open universe scenario within the standard theory. I think there are two possibilities to mend this controversial situation. One is that the standard prescription is not complete and it should be extended. If so, the way of extension must be unique. One approach might be to construct a theory based on the two point and/or higher correlation functions without using the decomposition by means of mode functions². The other possibility is that a class of spacetime including the no-bubble open universe is prohibited to be created by some unknown mechanism. It is very likely that a universe with infinite volume is forbidden to be realized. It is also probable that the realizable universe must be foliated by a set of compact spacelike hypersurface. Then, we will be happy to find that the inconsistency disappears.

Before closing this short paper, I want to stress one point. Here we pointed out that the standard quantum field theory in curved space has an inconsistency when the no-bubble open universe scenario is considered. However, this issue is nothing to do with the one-bubble open inflation scenario. In this scenario, we do not meet any inconsistency and we do not need any extension of the standard quantum field theory in curved space.

Acknowledgments

I would like to thank M. Sasaki for valuable discussions.

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²U. Moschella and R. Schaeffer made interesting comments on this issue in their paper[5]. But they discussed this issue as if it were related to the one-bubble open inflation scenario, and hence their arguments seem a little bit misleading.

Renormalization Group Approach to Einstein Equation in Cosmology

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Abstract

The renormalization group method has been adapted to the analysis of the long-time behavior of nonlinear partial differential equation and has demonstrated its power in the study of critical phenomena of gravitational collapse. In the present work we apply the renormalization group to the Einstein equation in cosmology and carry out detailed analysis of renormalization group flow in the vicinity of the scale invariant fixed point in the spherically symmetric and inhomogeneous dust filled universe model.

1 Introduction

Recently the renormalization group (RG) idea is applied to study the long-time asymptotics of the nonlinear partial differential equations [1, 2]. The RG transformation there is integration of the equation up to a finite time followed by a rescaling of the dependent and independent variables. The RG transformation together with the original differential equation gives a RG equation. Using the RG transformation, the problem on an infinite time is reduced to the problem on a finite time. A fixed point of the RG transformation corresponds to a scale invariant solution of the differential equation. We can obtain the long-time behavior of the equation by studying the flow around fixed points.

As an application of this RG method to the system of gravity, Koike, Hara and Adachi [3] analyzed the Einstein equation to understand the problem of critical behavior of black hole mass in gravitational collapse found by numerical study [4]. A pedagogical exposition of the RG method in the deterministic system is given by Tasaki [5] in a simple but very illustrative example of motion of a point particle in the Newtonian gravity.

Here we apply the RG to the Einstein equations in the cosmological situation. For simplicity, we shall consider a spherically symmetric universe filled with dust. We shall study the flow near the fixed points of the RG equations, which have self-similarity.

The astronomical observations indicate that the present universe has the hierarchical structure such as galaxies, clusters of galaxies, and super clusters, and that the two-point correlation function of the galaxies and of the clusters of galaxies can be expressed roughly by a single power law [6]. Recently the temperature fluctuations of about $13\mu\text{K}$ have been detected by the COBE in the 2.7 K CMBR over angular scales larger than 7° and are shown a fractal structure with fractal dimension $\sim 1.43 \pm 0.07$ [7]. These suggest the possibility that the present universe has a scale invariance up to a very large scale $\sim 1000h^{-1}$ Mpc and might be described by a fractal universe rather than the Friedmann-Robertson-Walker model. If it is really the case, it is necessary to explain such a fractal universe which has a scale invariance, from the viewpoint of the dynamics of the universe.

In this talk, we apply the RG method to the Einstein equations in the cosmological context. In Sec.2, we illustrate the application of the RG method to the heat equation with the nonlinear term. In Sec.3, we apply the RG method to the Einstein equations. Section 4 is devoted to the summary and discussions.

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2 Renormalization group transformation

In this section, we review the RG method for studying the long-time asymptotics of nonlinear partial differential equations [1].

First we consider the heat equation with the nonlinear term as a simple example:

$$\frac{\partial u(x, t)}{\partial t} = \frac{1}{2} [u''(x, t) + \lambda u^2(x, t)], \quad (1)$$

where the prime denotes the spatial derivative and λ is a coupling constant.

The equation (1) has scale invariance under the following scale transformation:

$$x \longrightarrow Lx, \quad (2)$$

$$t \longrightarrow L^2 t, \quad (3)$$

$$u(x, t) \longrightarrow L^2 u(Lx, L^2 t), \quad (4)$$

where L is a parameter of the scale transformation and is taken to be larger than 1. Namely, if $u(x, t)$ is a solution of Eq.(1), the scaled function,

$$u^{(L)}(x, t) = L^2 u(Lx, L^2 t), \quad (5)$$

is also a solution of Eq.(1). We can thus obtain a one-parameter family of solutions, provided that $u(x, t)$ is a solution.

Here we define the RG transformation $\mathcal{R}_L$ of a function of x by

$$\mathcal{R}_L u(x, 1) = u^{(L)}(x, 1). \quad (6)$$

In short, the $\mathcal{R}_L$ is a map from a set of initial data to another. It is convenient to take the initial time to be $t = 1$. The RG transformation $\mathcal{R}_L$ has a semi-group property:

$$\mathcal{R}_{L^n} = \mathcal{R}_{L^{n-1}} \circ \mathcal{R}_L. \quad (7)$$

Letting $t = 1$ and then $L^2 = t$ in Eqs.(5) and (6), we can express an arbitrary solution $u(x, t)$ as an initial data $u(\cdot, 1)$ transformed by RG transformations:

$$u(x, t) = t^{-1/2} u^{(t^{1/2})}(xt^{-1/2}, 1). \quad (8)$$

The large L means the late time. Repeating the RG transformation (6), we can see the long-time behavior of the solution $u(x, t)$ in Eq.(1).

Denoting $L = e^\tau$, we have from Eq.(5) that

$$\frac{d^{(L)}u}{d\tau} = L \frac{d^{(L)}u}{dL} = 2^{(L)}u + xu' + 2 \frac{\partial^{(L)}u}{\partial t}. \quad (9)$$

Using the original partial differential equation (1), we have

$$\frac{d^{(L)}u}{d\tau} = 2^{(L)}u + \lambda u^2 + xu' + u''. \quad (10)$$

This is the equation satisfied by the scaled function $u^{(L)}$, which we call the RG equation. We note that the equation (10) has no explicit scale L dependence because of the scale invariance of the original equation (1).

We investigate the fixed point of the RG equation (10). The fixed point u^* is defined by

$$\mathcal{R}_L u^* = u^* \quad (11)$$

for any $L > 1$. This condition means that the field profile is unchanged after time evolution followed by suitable rescaling. In general this condition is equivalent to

$$\frac{du^*}{d\tau} = 0. \quad (12)$$

From Eq.(10), u^* satisfies the following equation:

$$2u^* + \lambda u^{*2} + xu^{*'} + u^{*''} = 0. \quad (13)$$

In the homogeneous case, we can easily obtain the fixed points:

$$u^* = 0 \quad \text{and} \quad -\frac{2}{\lambda}. \quad (14)$$

To investigate the character of the fixed point Eq.(14), we consider the linear perturbation around the fixed point Eq.(14). The perturbed quantity $\delta u^{(L)}$ is defined by

$$u^{(L)} = u^* + \delta u, \quad (15)$$

where $\delta u^{(L)}$ is assumed small. Substituting Eq.(15) into Eq.(10) and neglecting the second order term $\delta u^{(L)2}$, we obtain the linearized equation for $\delta u^{(L)}$:

$$\frac{d\delta u^{(L)}}{d\tau} = \begin{cases} 2\delta u^{(L)} + x\delta u^{(L)'} + \delta u^{(L)''} & (u^* = 0) \\ -2\delta u^{(L)} + x\delta u^{(L)'} + \delta u^{(L)''} & (u^* = -\frac{2}{\lambda}) \end{cases}. \quad (16)$$

We require the boundary condition:

$$\delta u^{(L)} \rightarrow 0 \quad (|x| \rightarrow \infty). \quad (17)$$

We are going to find the normal modes with the ansatz:

$$\delta u^{(L)} = f(x)e^{-\frac{x^2}{2} + \omega\tau}, \quad (18)$$

where $f(x)$ is a function to be determined below and ω is a constant. From Eqs.(16) and (18), we have

$$\begin{aligned} f'' - xf' - (\omega - 1)f &= 0 \quad (u^* = 0), \\ f'' - xf' - (\omega + 3)f &= 0 \quad (u^* = -\frac{2}{\lambda}). \end{aligned} \quad (19)$$

The regularity at $x = 0$ and the boundary condition at $|x| = \infty$ imply

$$f(x) = H_n(x), \quad (20)$$

$$\omega = \begin{cases} 1 - n & (u^* = 0) \\ -3 - n & (u^* = -\frac{2}{\lambda}) \end{cases}, \quad (21)$$

where $H_n(x)$ is the Hermite polynomial and $n = 0, 1, 2, \dots$.

From Eq.(21), $u^* = -2/\lambda$ is an attractor because all ω 's are negative. On the other hand, $u^* = 0$ has only one relevant mode ($n = 0$).

We can discuss the long-time behavior of a solution of the nonlinear diffusion equation (1), if $u(x, 1)$ is sufficiently close to the self-similar profile, u^* . From Eq.(14), we obtain the two self-similar profiles.

Suppose the initial spatial profile of $\delta u^{(L)}$ is expressed as a superposition of the normal modes $H_n e^{-x^2/2}$. As we have seen from Eq.(21), if $u(x, 1)$ is sufficiently close to the fixed point $u^* = -2/\lambda$, the solution approaches to $-2t^{-1}/\lambda$ in course of time because all modes of perturbation are irrelevant. On the other

hand, there is the only one growing mode ($n = 0$) of perturbation around the fixed point $u^* = 0$. As time goes on, the behavior of the solution near this fixed point is dominated by the relevant mode $n = 0$. This relevant mode corresponds a gaussian distribution $t^{-1/2}e^{-x^2/(2t)}$.

From this instructive example, we see that if the perturbation around the fixed point has a finite number of relevant modes or no relevant modes, we have some prediction power for the long-time behavior of nonlinear partial differential equation.

Finally, we consider the universality of the long-time behavior of a solution of above differential equation (1). In order to do so, we investigate the following equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{1}{2} [u''(x,t) + \lambda u^2(x,t) + F(u(x,t))], \quad (22)$$

where $F(u(x,t))$ is a function of $u(x,t)$. This equation (22) can be seen the original equation (1) including some phenomenological term such as viscosity. Using the scale transformation (2)-(4), we easily understand the universality class of the long-time behavior of a solution in Eq.(1). Under the scale transformation (2)-(4), we assume $F(u)$ scales as

$$F(u(x,t)) \longrightarrow L^n F(u(Lx, L^2t)), \quad (23)$$

where n is a constant. If $F(u) = 0$, the equation (22) is scaled up by L^2 under the scale transformation (2)-(4). Since $L > 1$ means the late time, if $n \geq 2$, the additional term $F(u)$ is proportioned to the other term or dominates in course of time. The long-time behavior of a solution in Eq.(22) is different from one in Eq.(1). On the other hand, if $n < 2$, the additional term is negligible in course of time. In this case, a solution in Eq.(22) has the same long-time behavior as one in Eq.(1). In short, the long-time behavior of the system described by Eq.(1) and Eq.(22) belong to the same universality class. Thus we can easily see how universal the long-time behavior of a solution in Eq.(1) is by use of the scale transformation.

3 Renormalization group for Einstein equations

In this section, we apply the RG method, which is explained in the previous section, to the Einstein equations.

We consider the spherically symmetric *inhomogeneous* case where the line element is

$$\begin{aligned} ds^2 &= -dt^2 + \gamma_{ij} dx^i dx^j. \\ &= -dt^2 + A^2(r,t) dr^2 + B^2(r,t) (d\theta^2 + \sin^2 \theta d\phi^2). \end{aligned} \quad (24)$$

As a simple case, we investigate the universe filled with dust.

The Einstein equations have scale invariance under the following scale transformation:

$$r \longrightarrow Lr, \quad (25)$$

$$t \longrightarrow L^\alpha t, \quad (26)$$

$$A(r,t) \longrightarrow \overset{(\alpha)}{A}(r,t) \equiv L^{-\alpha} A(Lr, L^\alpha t), \quad (27)$$

$$B(r,t) \longrightarrow \overset{(\alpha)}{B}(r,t) \equiv L^{-\alpha} B(Lr, L^\alpha t), \quad (28)$$

$$\rho(r,t) \longrightarrow \overset{(\alpha)}{\rho}(r,t) \equiv L^{2\alpha} \rho(Lr, L^\alpha t), \quad (29)$$

where ρ is a energy density of dust, L is a parameter of the scale transformation, and α is an arbitrary constant because the coordinate transformation, $r \rightarrow r^\theta$, yields substitution of α/β for α in Eqs.(25)-(29). Without loss of generality, we take α to be positive. If $\overset{(\alpha)}{A}(r,t)$ and $\overset{(\alpha)}{B}(r,t)$ are solutions of the Einstein equations, the scaled functions $\overset{(\alpha)}{A}$ and $\overset{(\alpha)}{B}$ are also solutions. We applied the following RG transformation $\mathcal{R}_L$,

$$\mathcal{R}_L A(r, t) = \overset{(\alpha)}{A}(r, t), \quad \mathcal{R}_L B(r, t) = \overset{(\alpha)}{B}(r, t), \quad (30)$$

to study its long-time asymptotics. Letting $t = 1$ and then $L^\alpha = t$ in Eqs.(27)–(29), we can express arbitrary solutions $A(r, t)$ and $B(r, t)$ as initial data $A(\cdot, 1)$ and $B(\cdot, 1)$ transformed by RG transformation:

$$A(r, t) = t^{(\alpha-1)/\alpha} A^{(1/\alpha)}(rt^{-1/\alpha}, 1), \quad (31)$$

$$B(r, t) = t^{(1/\alpha)} B^{(1/\alpha)}(rt^{-1/\alpha}, 1). \quad (32)$$

The fixed point of the RG transformation (A^*, B^*) are defined by

$$\mathcal{R}_L A^* = A^*, \quad \mathcal{R}_L B^* = B^*. \quad (33)$$

At the fixed point,

$$A(r, t) = t^{(\alpha-1)/\alpha} A^*(rt^{-1/\alpha}, 1) = t^{(\alpha-1)/\alpha} \times (\text{function of } rt^{-1/\alpha} \text{ only}) \quad (34)$$

and

$$B(r, t) = t B^*(rt^{-1/\alpha}, 1) = t \times (\text{function of } rt^{-1/\alpha} \text{ only}) \quad (35)$$

is a self-similar solution with scale invariance of the Einstein equations. This fixed point is the Tolman-Bondi solution [8] with self-similarity, which includes the flat Friedmann universe. The precise form of these are

For $c = 0$:

$$A^*(r, 1) = \frac{\alpha r^{\alpha/3-1}(1-3pr^\alpha)}{3(1-pr^\alpha)^{1/3}}, \quad (36)$$

$$B^*(r, 1) = r^{\alpha/3}(1-pr^\alpha)^{2/3}, \quad (37)$$

$$\kappa\rho^*(r, 1) = \frac{4}{3(1-pr^\alpha)(1-3pr^\alpha)}, \quad (38)$$

For $c > 0$:

$$A^*(r, 1) = \frac{\alpha}{(1+c)^{1/2}r} \left[\frac{2}{9c}(\cosh \eta - 1)r^\alpha - c^{1/2} \frac{\sinh \eta}{\cosh \eta - 1} \right], \quad (39)$$

$$B^*(r, 1) = \frac{2}{9c} r^\alpha (\cosh \eta - 1), \quad (40)$$

$$\sinh \eta - \eta = \frac{9c^{3/2}}{2} (r^{-\alpha} - p), \quad (41)$$

$$\kappa\rho^*(r, 1) = \frac{9c^2}{r^\alpha (\cosh \eta - 1)^2 \left[\frac{2r^\alpha}{9c} (\cosh \eta - 1) - c^{1/2} \frac{\sinh \eta}{\cosh \eta - 1} \right]}, \quad (42)$$

For $c < 0$:

$$A^*(r, 1) = \frac{\alpha}{(1-|c|)^{1/2}r} \left[\frac{2}{9|c|} (1 - \cos \eta) r^\alpha - |c|^{1/2} \frac{\sin \eta}{1 - \cos \eta} \right], \quad (43)$$

$$B^*(r, 1) = \frac{2}{9|c|} r^\alpha (1 - \cos \eta), \quad (44)$$

$$\eta - \sin \eta = \frac{9|c|^{3/2}}{2} (r^{-\alpha} - p), \quad (45)$$

$$\kappa\rho^*(r, 1) = \frac{9c^2}{r^\alpha (1 - \cos \eta)^2 \left[\frac{2r^\alpha}{9|c|} (1 - \cos \eta) - |c|^{1/2} \frac{\sin \eta}{1 - \cos \eta} \right]}, \quad (46)$$

where c and p are constants.

The constant c can be interpreted as the total energy of the universe in the analogy of the Newtonian mechanics. By the signature of the constant c , these fixed points are classified into the following three. The universe with $c = 0$ is similar to the flat Friedmann universe. The universes with a positive c

or a negative c are similar to the open and closed Friedmann universes, respectively. Especially when $c = p = 0$, the above fixed point coincides with the flat Friedmann universe and the spacetime becomes homogeneous.

In the context of the RG, we can treat the time evolution of the field variables as the map from a set of initial data to another. If the initial data is taken to be the above fixed point A^* , B^* , and ρ^* , the spacetime will evolve into the Tolman-Bondi solution with self-similarity. In the cases of $c = 0$ and $c > 0$, the fixed point is not regular if $p > 0$. For $c < 0$, the fixed point has singularities irrespective of the signature of p because the spacetime is similar to the closed Friedmann universe and will recollapse. Since we should set a regular initial data in the physical situation, we investigate only the case of $c = 0$ ($p \leq 0$) and $c > 0$ ($p \leq 0$) where the fixed point is everywhere regular.

To study the behavior of the flow in the RG around the fixed point, we consider the linear perturbation around the fixed point. The linear perturbation around the self-similar Tolman-Bondi solution has also been discussed by Tomita [9] in a different context from ours. For simplicity, we concentrate on the spherical modes of linear perturbation. The perturbed quantities $\delta A^{(L)}$ and $\delta B^{(L)}$ are defined by

$$\begin{aligned} A^{(L)} &= A^* + A^* \delta A^{(L)}, & B^{(L)} &= B^* + B^* \delta B^{(L)}. \end{aligned} \quad (47)$$

We assume the spatial metric variables for $\delta A^{(L)}$ and $\delta B^{(L)}$ in the following form,

$$\delta A^{(L)} = a(r) e^{\omega \tau}, \quad \delta B^{(L)} = b(r) e^{\omega \tau}. \quad (48)$$

The perturbed quantities $a(r)$ and $\delta \rho^{(L)}$ are expressed by $b(r)$;

$$\frac{a}{A^*} = \frac{b'}{B'^*} - \frac{c_1}{2(1+c)} r^\omega, \quad \frac{\delta \rho^{(L)}}{\rho^*} = e^{\omega \tau} \left[\frac{9(\omega + \alpha)}{4\alpha} c_2 r^\omega - 2 \frac{b}{B^*} - \frac{b'}{B'^*} \right], \quad (49)$$

where c_1 and c_2 are arbitrary constants.

As for the spherical modes of the perturbation, we can easily obtain the solutions.

For $c = 0$ ($p < 0$), we have

$$b(r) = r^\omega \left[\frac{9}{20} c_1 r^{-\alpha/3} (1 - pr^\alpha)^{4/3} + \frac{3}{4} c_2 r^{\alpha/3} (1 - pr^\alpha)^{2/3} - \frac{2}{3} c_3 r^{4\alpha/3} (1 - pr^\alpha)^{-1/3} \right], \quad (50)$$

where c_3 is another arbitrary constant. The density contrast is

$$\begin{aligned} \frac{\delta \rho^{(L)}}{\rho^*} &= e^{\omega \tau} r^\omega \left\{ -\frac{9}{20\alpha} c_1 r^{-2\alpha/3} (1 - pr^\alpha)^{2/3} (1 - 3pr^\alpha)^{-1} [3\omega + \alpha - 3(\omega + 3\alpha)pr^\alpha] \right. \\ &\quad \left. - \frac{9\omega}{2\alpha} c_2 pr^\alpha (1 - 3pr^\alpha)^{-1} + \frac{2}{\alpha} c_3 r^\alpha (1 - pr^\alpha)^{-1} (1 - 3pr^\alpha)^{-1} [\omega + 2\alpha - (\omega + 3\alpha)pr^\alpha] \right\}. \end{aligned} \quad (51)$$

In the expression for the linear perturbation Eq.(51) there are three terms corresponding to c_1 , c_2 , and c_3 so that there should be a gauge mode hidden in Eq.(51) because the number of physical modes has to be two. Actually there remains a gauge freedom corresponding to the coordinate transformation of r . The gauge mode is given by

$$\frac{\delta \rho_g^{(L)}}{\rho^*} = f r^{\omega+1} e^{\omega \tau} \frac{\rho'^*}{\rho^*} = e^{\omega \tau} r^\omega f [2\alpha pr^\alpha (2 - 3pr^\alpha) (1 - pr^\alpha)^{-1} (1 - 3pr^\alpha)^{-1}], \quad (52)$$

with f being an arbitrary constant. Since we should fix the freedom of gauge we choose f so that $(\delta \rho^{(L)} + \delta \rho_g^{(L)})/\rho^*$ behave as nicely as possible at $r = \infty$ because we are interested in the perturbation modes which are finite at $r = \infty$.

We use the following condition as a convenient gauge condition.

$$f = -\frac{9(\omega + 3\alpha)}{40\alpha^2} p^{2/3} c_1 + \frac{3\omega}{4\alpha^2} c_2 - \frac{\omega + 3\alpha}{3\alpha^2 p} c_3. \quad (53)$$

We fix the gauge mode by the above condition and obtain the physical perturbation $\delta \bar{\rho}^{(L)} = \delta \bar{\rho}^{(L)} + \delta \bar{\rho}_g^{(L)}$ as follows.

$$\frac{\delta \bar{\rho}^{(L)}}{\rho^*} = e^{\omega \tau} r^\omega \left\{ \Delta_1 (1 - 3pr^\alpha)^{-1} \left[p^{-2/3} r^{-2\alpha/3} (1 - pr^\alpha)^{2/3} [3\omega + \alpha - 3(\omega + 3\alpha)pr^\alpha] \right. \right. \\ \left. \left. + (\omega + 3\alpha)pr^\alpha (2 - 3pr^\alpha)(1 - pr^\alpha)^{-1} \right] + \Delta_2 pr^\alpha (1 - pr^\alpha)^{-1} (1 - 3pr^\alpha)^{-1} \right\}, \quad (54)$$

where

$$\Delta_1 = -\frac{9p^{2/3}}{20\alpha} c_1, \quad \Delta_2 = -\frac{3\omega}{2\alpha} \left(c_2 - \frac{4}{9} c_3 \right). \quad (55)$$

Since we consider only the case of $p < 0$, the coordinate r can be taken from 0 to ∞ . We demand the regularity condition that $\delta \bar{\rho}^{(L)}/\rho^*$ should be finite at the boundary, $r = 0$ and $r = \infty$. This condition implies that Δ_1 modes with $2\alpha/3 \leq \omega \leq \alpha$ and Δ_2 modes with $-\alpha \leq \omega \leq \alpha$ are allowed. The mode with $\omega = 0$ corresponds to change of p in the self-similar solution Eqs.(36)–(38) and ρ^* remains constant independent of τ in the direction. Although this fixed point is not a repeller, it has many relevant modes, Δ_1 with $0 < \omega \leq \alpha$ and Δ_2 with $0 < \omega \leq \alpha$. Note that a suitable linear combination of the Δ_1 and Δ_2 modes will have an asymptotic behavior $\approx r^{\omega-2\alpha}$ at $r = \infty$. For such modes, $2\alpha/3 \leq \omega \leq 2\alpha$ is allowed. These modes which satisfy the regularity condition are not discrete but continuous. We suppose that this special feature arises because our matter is assumed to be dust. To summarize, the possible value of ω ranges from $-\alpha$ to 2α .

Similarly, we obtain the possible value of ω in the case $c = 0$ ($p = 0$) and $c > 0$ ($p \leq 0$). We summarize the results in the following table.

parameters in fixed point		possible value of ω
$c = 0$	$p = 0$	$\omega = -\alpha, 2\alpha/3$
	$p < 0$	$-\alpha \leq \omega \leq 2\alpha$
$c > 0$	$p = 0$	$\times$
	$p < 0$	$0 \leq \omega \leq 2\alpha$

4 Summary and Discussions

We considered the spherically symmetric but inhomogeneous universe filled with dust, where the Einstein equations have scale invariance Eqs.(25)–(29) and applied the renormalization group (RG) method to study its long-time asymptotics. The fixed point of the RG transformation is a self-similar solution with scale invariance of the Einstein equations. In order to study the flow of the RG around this fixed point, the linear perturbation analysis is used. We impose the perturbation on the regularity at the boundary where the radial coordinate r equals zero or infinity. This boundary means that the area radius equals zero or infinity in the case of $c = 0$, on the other hand, in the case of $c > 0$, it equals a finite or infinity. The fixed point is the Tolman-Bondi solution with self-similarity, which includes the flat Friedmann universe. The behavior of the fixed point is separated into two types. Both types have many relevant modes of the perturbation. The fixed points of the RG flow are self-similar solutions of the Einstein equations, which are worth studying in their own right and have been studied by many people[10, 11]. Our approach has an obvious advantage that the fixed point is in general not always spherical symmetric like the Tolman-Bondi solution of the dust filled universe. More importantly, we can systematically treat the *dynamics* of the universe near the self-similar solution adopting the scale as a “time” of the evolution. The RG flow near the fixed points is a new aspect in the study of the self-similarity in the universe[12].

The Tolman-Bondi solution with self-similarity is unstable against almost all spherical modes of linear perturbation. The spacetime will deviate from this fixed point. It is necessary to study the non-spherical mode of perturbation to say something more definite. In the cosmological problem, only the statistical quantities are meaningful if we think of comparison with observations. There are some works in the RG approach [13, 14, 15] on the universe which has a hierarchical structure. We may contemplate further development of the RG approach to cosmology formulated in the present work by introducing some

kind of volume or statistical average for observables like energy density and the Hubble constant of the universe. The statistical concept is needed not only for comparison with observations but also for us to proceed further in the analysis of the RG equation because we have continuously many relevant (growing) modes around the fixed points. That is, the long-time behavior of the universe is sensitive to the initial configuration which we have no a priori control and we have to consider statistical likelihood of the initial values. We remark that the introduction of volume average in a finite region of the universe potentially introduces the scale invariance violation by hand because the exact scale invariance holds only for an infinite space. Note that in quantum field theories and statistical physics of the second order phase transition the scale invariance violations are hidden in the form of cut-off of the spectrum of physical modes. We shall elaborate our present observation in our future work.

The self-similar solution given by Eqs.(31) and (32) through the fixed point of the RG equation is essentially a function of $t^{-1/\alpha_r} r = t^{1-1/\alpha_r}/t$, which is roughly the fraction of physical distance to the horizon scale of the Friedmann universe. Also note that in the case of nonlinear diffusion equation Eq.(8) implies the self-similar solution is a function of the ratio of the distance x to the diffusion length $\sqrt{t}$. In the both cases, the self-similar solution is a function of the distance in the unit of physically relevant time dependent scale. We believe this is a general phenomenon and the physical background of the RG equation which governs how dynamical variables deviate from the self-similar solution.

Acknowledgements

O. I. would like to thank Professor H. Ishihara and K. Nakamura for discussions. The research is supported in part by Japan Society for the Promotion of Science (O. I.). This work is partially supported by grant-in-aid by the Ministry of Education, Science, Sports, and Culture of Japan (A. H., 09640341 and T. K., 012-10096097)

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Equivalence of black hole thermodynamics between Einstein theory and the generalized theories of gravity

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Abstract

We analyze black hole thermodynamics in the generalized theories of gravity whose Lagrangian is an analytic function of the metric, the Ricci tensor and a scalar field. We convert the theory into the Einstein frame via a Legendre transformation, and calculate black hole thermodynamical variables both in the original frame and in the Einstein frame. To calculate the thermodynamical variables in the original frame, we follow Iyer-Wald's definition which satisfies the first law of thermodynamics. We show that all the thermodynamical variables are the same between the original frame and the Einstein frame. This result may be helpful in the analysis of the second law of black hole thermodynamics in the generalized theories.

1 Introduction

The unsolved problems related to black hole evaporation, such as an information loss problem, could be explained by quantum gravity. A recent development of a string theory, which is one of the most promising candidates of quantum gravity, unveiled that the number of states of a D-brane is exactly the same as the entropy of the corresponding black hole. However, such coincidence is true only for extreme black holes because of their BPS nature. Further development of a string theory would be required to understand the origin of the entropy of a generic black hole.

Together with a string theory, investigations based on the field theory including stringy effects may be also helpful to understand such microscopic black holes and their properties. One possible method to study the stringy effects is to consider effective theories, in which corrections to the Einstein gravity are expected, e.g., higher curvature terms and a dilaton coupling. Since the effective theories may not be fundamental, we can only discuss physics of sufficiently larger black holes than Planck scale and those corrections should be regarded as perturbative effects. However, such corrections will modify black hole thermodynamics, which includes one of the most important properties of microscopic black holes, from that in the Einstein theory. By investigating the generalized theories of gravity, which may come from a string theory, we may extract an universal feature of black hole thermodynamics, which cannot be understood by the analysis within the Einstein theory.

The thermodynamical variables of a black hole in the generalized theory of gravity can be defined such that they satisfy the first law of thermodynamics[1, 2]. However, the second law has not been established so far in the most generic theories. For the theories which can be transformed into the Einstein frame via a conformal transformation, the entropy is shown to be the same with that in the Einstein frame, and the second law is proved by showing that the null energy condition in the Einstein frame is satisfied[3]. Equivalence between the entropy in the original frame and that in the Einstein frame is also discussed in the theory whose Lagrangian includes a specific form of higher curvature terms[4], based on the fact that the transformation into the Einstein frame does not alter the asymptotic structure of the spacetime, such as the mass and the angular momentum of the black hole. It is not the case, however, when the theory contains a scalar field which couples non-minimally to gravity.

In this paper, we generalize those results into much wider class of theories. The Lagrangian we consider is an analytic function of the metric, the Ricci tensor, a scalar field and its derivative. This

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class of theories includes the higher order terms of the Ricci tensor as well as a non-minimal coupling of a scalar field. We convert the theory into the Einstein frame via a Legendre transformation[5], and show the equivalence of the thermodynamical variables.

We first describe a Legendre transformation in the next section. In Section 3, we review the definition of the thermodynamical variables of a black hole in the generalized theory of gravity formulated in Ref.[1, 2]. By following their definition, we show, in Section 4, the equivalence between the thermodynamical variables in the original theory and those in the Einstein frame. The final section is devoted to the conclusion and the discussion.

2 Legendre transformation

The action we consider is written as

$$I = \frac{1}{16\pi} \int d^4x \sqrt{-g} f(g^{\alpha\beta}, R_{\alpha\beta}, \phi, \nabla_\alpha \phi), \quad (2.1)$$

where f is an analytic function of the metric $g^{\mu\nu}$, the Ricci tensor $R_{\mu\nu}$, a scalar field ϕ and its derivative. The equations of motion derived from the above action are the forth-order differential equations, in general. We transform the action(2.1) into the Einstein frame via a Legendre transformation[5], in which we find the second-order differential equations.

To do this, we first define a new "metric" tensor $\bar{g}^{\mu\nu}$ by

$$\bar{g}^{\mu\nu} \equiv \frac{\partial(\sqrt{-g}f)}{\partial R_{\mu\nu}} \frac{1}{\sqrt{-\det(\partial(\sqrt{-g}f)/\partial R_{\alpha\beta})}} = \frac{\sqrt{-g}}{\sqrt{-\bar{g}}} \frac{\partial f}{\partial R_{\mu\nu}}. \quad (2.2)$$

Then $\bar{g}^{\mu\nu}$ is a function of $R_{\mu\nu}$ as well as of $g^{\mu\nu}$, ϕ and $\nabla_\mu \phi$, if

$$\frac{\partial^2 f}{\partial R_{\mu\nu} \partial R_{\rho\sigma}} \neq 0, \quad (2.3)$$

which we assume here. Thus, we can write as $\bar{g}^{\mu\nu} = \bar{g}^{\mu\nu}(g^{\alpha\beta}, R_{\alpha\beta}, \phi, \nabla_\alpha \phi)$ and solve this relation for $R_{\mu\nu}$ as

$$R_{\mu\nu} = R_{\mu\nu}(g^{\alpha\beta}, \bar{g}^{\alpha\beta}, \phi, \nabla_\alpha \phi). \quad (2.4)$$

Next, we define a new action $\bar{I}$ by

$$\begin{aligned} \bar{I} \equiv & \frac{1}{16\pi} \int d^4x \sqrt{-\bar{g}} \left[\bar{g}^{\mu\nu} R_{\mu\nu}(g^{\alpha\beta}, \partial_\gamma g^{\alpha\beta}, \partial_\gamma \partial_\lambda g^{\alpha\beta}) - \bar{g}^{\mu\nu} R_{\mu\nu}(g^{\alpha\beta}, \bar{g}^{\alpha\beta}, \phi, \nabla_\gamma \phi) \right. \\ & \left. + \frac{\sqrt{-g}}{\sqrt{-\bar{g}}} f(g^{\alpha\beta}, R_{\alpha\beta}(g^{\rho\sigma}, \bar{g}^{\rho\sigma}, \phi, \nabla_\rho \phi), \phi, \nabla_\alpha \phi) \right], \end{aligned} \quad (2.5)$$

where independent variables are $g^{\mu\nu}$, $\bar{g}^{\mu\nu}$ and ϕ . The Ricci tensor in the first term of the right hand side of Eq.(2.5) is given by the metric and its derivatives as usual, whereas those in the second and the third terms are given by the relation Eq.(2.4). We find, by varying Eq.(2.5), the Euler-Lagrange equation for $\bar{g}^{\mu\nu}$ gives the same relation as Eq.(2.2), and the total variation of I and $\bar{I}$ are the same when we substitute Eq.(2.2).

Finally, we rewrite Eq.(2.5) into the form

$$\begin{aligned} \bar{I} = & \frac{1}{16\pi} \int d^4x \sqrt{-\bar{g}} \left[\bar{R}(\bar{g}^{\alpha\beta}, \partial_\gamma \bar{g}^{\alpha\beta}, \partial_\gamma \partial_\lambda \bar{g}^{\alpha\beta}) + \bar{g}^{\mu\nu} (\bar{\Gamma}_{\rho\sigma}^\mu \bar{\Gamma}_{\mu\nu}^\sigma - \bar{\Gamma}_{\mu\sigma}^\rho \bar{\Gamma}_{\rho\nu}^\sigma) \right. \\ & \left. - \bar{g}^{\mu\nu} R_{\mu\nu}(g^{\alpha\beta}, \bar{g}^{\alpha\beta}, \phi, \bar{\nabla}_\alpha \phi) + \frac{\sqrt{-g}}{\sqrt{-\bar{g}}} f(g^{\alpha\beta}, R_{\alpha\beta}(g^{\rho\sigma}, \bar{g}^{\rho\sigma}, \phi, \bar{\nabla}_\rho \phi), \phi, \bar{\nabla}_\alpha \phi) \right], \end{aligned} \quad (2.6)$$

by using the relation

$$R_{\mu\nu}(g^{\alpha\beta}, \partial_\gamma g^{\alpha\beta}, \partial_\gamma \partial_\lambda g^{\alpha\beta}) = \bar{R}_{\mu\nu}(\bar{g}^{\alpha\beta}, \partial_\gamma \bar{g}^{\alpha\beta}, \partial_\gamma \partial_\lambda \bar{g}^{\alpha\beta}) + \bar{\Gamma}_{\rho\sigma}^\mu \bar{\Gamma}_{\mu\nu}^\sigma - \bar{\Gamma}_{\mu\sigma}^\rho \bar{\Gamma}_{\rho\nu}^\sigma + \bar{\nabla}_\rho \bar{\Gamma}_{\mu\nu}^\rho - \bar{\nabla}_\mu \bar{\Gamma}_{\rho\nu}^\rho, \quad (2.7)$$

where $\bar{\Gamma}_{\mu\nu}^\rho \equiv \frac{1}{2}g^{\rho\sigma}(\bar{\nabla}_\mu g_{\nu\sigma}^{-1} + \bar{\nabla}_\nu g_{\mu\sigma}^{-1} - \bar{\nabla}_\sigma g_{\mu\nu}^{-1})$, $^\dagger \bar{\nabla}_\nu$ is the covariant derivative associated to $\bar{g}^{\mu\nu}$ and the total divergence is omitted in Eq.(2.6). Now $\bar{g}^{\mu\nu}$ plays a role of the metric, and the first term in the right hand side of Eq.(2.6) is the Einstein-Hilbert action of $\bar{g}^{\mu\nu}$. The second term is regarded as a kinetic term of an exotic spin-2 matter field $g^{\mu\nu}$, since it is bilinear in the first derivatives of $g^{\mu\nu}$. The rest consists of a kinetic term of the scalar field ϕ , potential and interaction of the matter fields. Hence $\bar{I}$ is regarded as the action of the Einstein gravity with exotic matters, which generates the equivalent equations of motion to those derived from the original action I .

3 Definition of thermodynamical variables

The thermodynamical variables of black holes in the generalized theories of gravity are defined such that they satisfy the first law of black hole thermodynamics[1, 2]. In Ref.[2], the first law is established for an arbitrary perturbation off from a regular background spacetime of a black hole, for which the following assumptions are made: (1) it is stationary and axisymmetric, and hence it possesses two Killing vectors $\xi_{(t)}^\mu$ and $\xi_{(\varphi)}^\mu$ associated to the respective symmetry, (2) it is asymptotically flat, and (3) it possesses a bifurcate Killing horizon. Definition of the thermodynamical variables is based on the covariance of the Lagrangian density $\epsilon_{\mu\nu\rho\sigma}L$ under a diffeomorphism, where L is a scalar function of the field variables which we denote as $\varphi_{(i)}$ collectively. The variation of $\epsilon_{\mu\nu\rho\sigma}L$ is described as

$$\delta(\epsilon_{\mu\nu\rho\sigma}L) = \epsilon_{\mu\nu\rho\sigma}E^{(i)}\delta\varphi_{(i)} + \epsilon_{\mu\nu\rho\sigma}\nabla_\beta\Theta^\beta[\varphi_{(i)}, \delta\varphi_{(i)}], \quad (3.1)$$

where $E^{(i)} = 0$ is the equation of motion for $\varphi_{(i)}$, and $\Theta^\beta[\varphi_{(i)}, \delta\varphi_{(i)}]$ in the total divergence is a functional of the field variables $\varphi_{(i)}$ and their variations $\delta\varphi_{(i)}$. When we identify the variation $\delta\varphi_{(i)}$ with a general coordinate transformation, $\mathcal{L}_\zeta\varphi_{(i)}$, induced by an arbitrary vector field ζ^μ , we obtain

$$\nabla_\mu(\zeta^\mu L) = -E^{(i)}\mathcal{L}_\zeta\varphi_{(i)} + \nabla_\beta\Theta^\beta[\varphi_{(i)}, \mathcal{L}_\zeta\varphi_{(i)}]. \quad (3.2)$$

Then the vector field $J^\beta[\varphi_{(i)}, \zeta^\rho, \mathcal{L}_\zeta\varphi_{(i)}] \equiv \Theta^\beta[\varphi_{(i)}, \mathcal{L}_\zeta\varphi_{(i)}] - \zeta^\beta L$ is divergence-free, $\nabla_\beta J^\beta = 0$, when the equations of motion $E^{(i)} = 0$ are satisfied. In addition, we can find an antisymmetric potential $Q^{\beta\alpha}[\varphi_{(i)}, \zeta^\rho, \nabla_\lambda\zeta^\rho]$ of the current $J^\beta[\varphi_{(i)}, \zeta^\rho, \mathcal{L}_\zeta\varphi_{(i)}]$, such that $J^\beta = \nabla_\alpha Q^{\beta\alpha}$. $\Theta^\beta[\varphi_{(i)}, \delta\varphi_{(i)}]$ and $Q^{\beta\alpha}[\varphi_{(i)}, \zeta^\rho, \nabla_\lambda\zeta^\rho]$ are calculated in a straightforward manner for a given action. In the case of the action(2.1), those are given by

$$\begin{aligned} \Theta^\beta &= \frac{1}{32\pi} [(\nabla_\nu P_\mu^\beta) \delta g^{\mu\nu} - g_{\mu\nu}^{-1} (\nabla_\alpha P^{\alpha\beta}) \delta g^{\mu\nu} + (\nabla_\mu P_\alpha^\mu) \delta g^{\alpha\beta} - (\nabla^\beta P_{\mu\nu}) \delta g^{\mu\nu} \\ &\quad + g_{\mu\nu}^{-1} P^{\alpha\beta} \nabla_\alpha \delta g^{\mu\nu} - P_\alpha^\nu \nabla_\nu \delta g^{\alpha\beta} - P_\nu^\beta \nabla_\mu \delta g^{\mu\nu} + P_{\mu\nu} \nabla^\beta \delta g^{\mu\nu} + 2v^\beta \delta\phi] \end{aligned} \quad (3.3)$$

$$Q^{\beta\alpha} = \frac{1}{16\pi} [P^{\nu[\alpha} \nabla_{\nu} \zeta^{\beta]} + P^{\nu[\beta} \nabla_{\nu} \zeta^{\alpha]} \zeta_\nu + 2(\zeta_\nu \nabla^{[\beta} P^{\alpha]\nu} + \zeta^{[\alpha} \nabla_\nu P^{\beta]\nu})], \quad (3.4)$$

where $P^{\mu\nu} \equiv \partial f / \partial R_{\mu\nu}$ and $v^\beta \equiv \partial f / \partial (\nabla_\beta \phi)$, while those for the action(2.6) are calculated as

$$\begin{aligned} \bar{\Theta}^\beta &= \frac{1}{16\pi} \left[\bar{g}_{\mu\nu}^{-1} \bar{\nabla}^\beta \delta \bar{g}^{\mu\nu} - \bar{\nabla}_\nu \delta \bar{g}^{\nu\beta} + H_{\mu\nu}^\beta \delta g^{\mu\nu} + \bar{v}^\beta \delta\phi \right. \\ &\quad \left. + H_{\mu\alpha}^\beta g_\nu^\mu \delta \bar{g}^{\alpha\nu} - H_{\mu\nu\alpha} g^{\beta\mu} \delta \bar{g}^{\alpha\nu} - H_{\mu\nu}^\beta g_\alpha^\mu \delta \bar{g}^{\alpha\nu} \right] \end{aligned} \quad (3.5)$$

$$\bar{Q}^{\beta\alpha} = \frac{1}{8\pi} \left[\bar{\nabla}^{[\alpha} \zeta^{\beta]} + \zeta^\nu H_{\mu\nu}^{[\alpha} g^{\beta]\mu} + \zeta^\nu H_{\mu\nu}^{[\alpha\beta]} g_\nu^\mu + \zeta^\nu H_{\mu\nu}^{[\alpha} g^{\beta]\mu} \right], \quad (3.6)$$

where $H_{\mu\nu}^\rho$ and $\bar{v}^\beta$ are defined by $H_{\mu\nu}^\rho \equiv \partial \bar{f} / \partial (\bar{\nabla}_\rho g^{\mu\nu})$ and $\bar{v}^\beta \equiv \partial \bar{f} / \partial (\bar{\nabla}_\beta \phi)$, respectively, with $\bar{f}$ being the Lagrangian of the matter fields in the Einstein frame, i.e.,

$$\begin{aligned} \bar{f} &= \bar{g}^{\mu\nu} \left(\bar{\Gamma}_{\rho\sigma}^\rho \bar{\Gamma}_{\mu\nu}^\sigma - \bar{\Gamma}_{\mu\sigma}^\rho \bar{\Gamma}_{\rho\nu}^\sigma \right) - \bar{g}^{\mu\nu} R_{\mu\nu}(g^{\alpha\beta}, \bar{g}^{\alpha\beta}, \phi, \bar{\nabla}_\alpha \phi) \\ &\quad + \frac{\sqrt{-\bar{g}}}{\sqrt{-g}} f(g^{\alpha\beta}, R_{\alpha\beta}(g^{\rho\sigma}, \bar{g}^{\rho\sigma}, \phi, \bar{\nabla}_\rho \phi), \phi, \bar{\nabla}_\alpha \phi). \end{aligned} \quad (3.7)$$

[†] We denote the inverse matrix of $g^{\mu\nu}$ by $g_{\mu\nu}^{-1}$, which conventional expression is $g_{\mu\nu}$, in order to distinguish $g_{\mu\nu}$ from $\bar{g}_{\mu\alpha} \bar{g}_{\nu\beta} g^{\alpha\beta}$.

When we set ζ^μ to be the Killing vector $\xi^\mu \equiv \xi_{(t)}^\mu + \Omega_H \xi_{(\varphi)}^\mu$, where Ω_H is the angular velocity of the black hole, we obtain the relation

$$\begin{aligned} & \delta \int_H \frac{1}{2} \epsilon_{\beta\alpha\mu\nu} Q^{\beta\alpha} [\varphi_{(i)}, \xi^\rho, \nabla_\lambda \xi^\rho] dx^\mu dx^\nu \\ &= \delta \int_\infty \frac{1}{2} \epsilon_{\beta\alpha\mu\nu} Q^{\beta\alpha} [\varphi_{(i)}, \xi^\rho, \nabla_\lambda \xi^\rho] dx^\mu dx^\nu + \int_\infty \xi^\beta \epsilon_{\beta\alpha\mu\nu} \Theta^\alpha [\varphi_{(i)}, \delta\varphi_{(i)}] dx^\mu dx^\nu, \end{aligned} \quad (3.8)$$

where δ denotes a variation from the background solution. Integration of the left hand side in Eq.(3.8) is taken over the bifurcation surface, whereas that of the right hand side is over a 2-dimensional surface in the asymptotically flat region. If there exists a vector $B^\mu[\varphi_{(i)}]$ which satisfies

$$\delta \int_\infty \xi^\beta \epsilon_{\beta\alpha\mu\nu} B^\alpha [\varphi_{(i)}] dx^\mu dx^\nu = \int_\infty \xi^\beta \epsilon_{\beta\alpha\mu\nu} \Theta^\alpha [\varphi_{(i)}, \delta\varphi_{(i)}] dx^\mu dx^\nu, \quad (3.9)$$

Eq.(3.8) gives the relation between the variation of an integral of a geometrical quantity on the horizon and that over the 2-surface at infinity, i.e.,

$$\begin{aligned} & \delta \int_H \frac{1}{2} \epsilon_{\beta\alpha\mu\nu} Q^{\beta\alpha} [\varphi_{(i)}, \xi^\rho, \nabla_\lambda \xi^\rho] dx^\mu dx^\nu \\ &= \delta \int_\infty \frac{1}{2} \epsilon_{\beta\alpha\mu\nu} \left(Q^{\beta\alpha} [\varphi_{(i)}, \xi^\rho, \nabla_\lambda \xi^\rho] + 2\xi^{[\beta} B^{\alpha]} [\varphi_{(i)}] \right) dx^\mu dx^\nu. \end{aligned} \quad (3.10)$$

Then the thermodynamical variables are defined as

$$\frac{\kappa}{2\pi} S \equiv \int_H \frac{1}{2} \epsilon_{\beta\alpha\mu\nu} Q^{\beta\alpha} [\varphi_{(i)}, \xi^\rho, \nabla_\lambda \xi^\rho] dx^\mu dx^\nu, \quad (3.11)$$

$$M \equiv \int_\infty \frac{1}{2} \epsilon_{\beta\alpha\mu\nu} \left(Q^{\beta\alpha} [\varphi_{(i)}, \xi_{(t)}^\rho, \nabla_\lambda \xi_{(t)}^\rho] + 2\xi_{(t)}^{[\beta} B^{\alpha]} [\varphi_{(i)}] \right) dx^\mu dx^\nu, \quad (3.12)$$

$$J \equiv - \int_\infty \frac{1}{2} \epsilon_{\beta\alpha\mu\nu} \left(Q^{\beta\alpha} [\varphi_{(i)}, \xi_{(\varphi)}^\rho, \nabla_\lambda \xi_{(\varphi)}^\rho] + 2\xi_{(\varphi)}^{[\beta} B^{\alpha]} [\varphi_{(i)}] \right) dx^\mu dx^\nu, \quad (3.13)$$

where κ is the surface gravity of the black hole. The existence of B^μ is closely related to asymptotical flatness. In Ref.[2], it is argued that the condition on the asymptotic behavior of the field variables should be imposed to ensure the existence of B^μ , because the mass and the angular momentum are naturally defined in an asymptotically flat spacetime.

The first law of thermodynamics,

$$\frac{\kappa}{2\pi} \delta S = \delta M - \Omega_H \delta J \quad (3.14)$$

follows by substituting Eq.(3.11), (3.12) and (3.13) into Eq.(3.10), and noting that $Q^{\beta\alpha}[\varphi_{(i)}, \xi^\rho, \nabla_\lambda \xi^\rho]$ is linear in ξ^μ and $\nabla_\nu \xi^\mu$.

4 Equivalence of thermodynamical variables

Now, following the definition in the previous section, we show[6] the equivalence of the thermodynamical variables between the original frame and the Einstein frame, where the actions are given by Eq.(2.1) and Eq.(2.6), respectively. First we have to mention further assumptions in addition to those in the previous section. Then, after showing the equivalence of the surface gravity and the angular velocity, we prove that the entropy, the mass and the angular momentum are the same in both frames.

4.1 Assumptions

Since we do not specify our model, it is not guaranteed that the spacetime in the Einstein frame will be regular and asymptotically flat, in general, where we cannot discuss black hole thermodynamics. However, when we consider the effective theory of the string theory, $\bar{g}^{\mu\nu}$ is typically given as

$$\bar{g}^{\mu\nu} = e^{-\alpha\phi} (g^{\mu\nu} + \beta l_{\text{Pl}}^2 g^{\mu\nu} R + \gamma l_{\text{Pl}}^2 R^{\mu\nu} + \dots), \quad (4.1)$$

where dots denote the higher order terms of curvature, l_{Pl} is the Planck length, and α, β, γ are numerical constants. In this case, $\bar{g}^{\mu\nu}$ approaches $g^{\mu\nu}$ in the asymptotically flat region in the original frame, if the asymptotic behavior of the scalar field is appropriately specified, and hence the spacetime in the Einstein frame is also asymptotically flat. If the spacetime and the scalar field in the original frame are regular, $\bar{g}^{\mu\nu}$ is expected to be regular at least outside the horizon for black holes much larger than Planck scale, since $\bar{g}^{\mu\nu}$ is expressed in terms of the field variables in the original frame as Eq.(4.1), and the corrections by the higher curvature terms are sufficiently small. Thus, the spacetime in the Einstein frame will be regular in this case, since we are concerned only with large black holes, as we mentioned in the introduction.

In the followings, we assume: (1) the dominant contribution in the asymptotically flat region to the Lagrangian f in the original action (2.1) is proportional to the scalar curvature, where $g^{\mu\nu} \rightarrow \eta^{\mu\nu} + \mathcal{O}(r^{-1})$, and (2) the transformation into the Einstein frame is regular in the sense that the spacetime in the Einstein frame is regular if it is regular in the original frame.

The first assumption ensures that the spacetime in the Einstein frame is also asymptotically flat and $\bar{g}^{\mu\nu} \rightarrow g^{\mu\nu} + \mathcal{O}(r^{-1})$ at infinity, whereas it does not indicate the mass and the angular momentum are the same between the original frame and the Einstein frame, in contrast to Ref.[4]. From those assumptions, it is shown that the signature of the metric $\bar{g}^{\mu\nu}$ in the Einstein frame is still Lorentzian, since the spacetime is regular and $\bar{g}^{\mu\nu}$ approaches $\eta^{\mu\nu}$ in the asymptotically flat region.

4.2 Surface gravity and angular velocity

Here we show equality of the surface gravity and the angular velocity in two frames[‡]. For this purpose, we first show the Killing horizon in the Einstein frame appears at the same place as in the original frame.

When we consider, in the Einstein frame, the same vector field as the Killing vector ξ^μ in the original frame, fixed points of ξ^μ in the Einstein frame exist at the same points as in the original frame. Then, by introducing a coordinate system (θ, φ) on the bifurcation surface in the original frame such that $\xi^\mu_{(\varphi)} \equiv \partial/\partial\varphi$ and denoting the basis vector of the coordinate θ by $\theta^\mu \equiv \partial/\partial\theta$, we have $\bar{g}_{\mu\nu}^{-1}\xi^\mu\xi^\nu = 0$, $\bar{g}_{\mu\nu}^{-1}\xi^\mu\theta^\nu = 0$ and $\bar{g}_{\mu\nu}^{-1}\xi^\mu\xi^\nu_{(\varphi)} = 0$ at those points in the Einstein frame. In addition, we notice that the Killing vectors, $\xi^\mu_{(t)}$, $\xi^\mu_{(\varphi)}$ and ξ^μ , in the original frame are also Killing vectors in the Einstein frame, because the field variables in the Einstein frame are composed of those in the original frame, and hence the Lie derivatives of the field variables in the Einstein frame with respect to the Killing vector vanish. Then we have $\mathcal{L}_\xi(\bar{g}_{\mu\nu}^{-1}\xi^\mu\xi^\nu) = 0$, $\mathcal{L}_\xi(\bar{g}_{\mu\nu}^{-1}\xi^\mu\theta^\nu) = 0$ and $\mathcal{L}_\xi(\bar{g}_{\mu\nu}^{-1}\xi^\mu\xi^\nu_{(\varphi)}) = 0$, because ξ^μ commutes with θ^μ and $\xi^\mu_{(\varphi)}$. Therefore, we find that $\bar{g}_{\mu\nu}^{-1}\xi^\mu\xi^\nu = 0$, $\bar{g}_{\mu\nu}^{-1}\xi^\mu\theta^\nu = 0$ and $\bar{g}_{\mu\nu}^{-1}\xi^\mu\xi^\nu_{(\varphi)} = 0$ along the orbits of ξ^μ through the fixed points of ξ^μ . Thus, ξ^μ is null and hypersurface orthogonal along that orbit of ξ^μ , and hence generate a Killing horizon also in the Einstein frame. Since ξ^μ and the location of the bifurcation surface are the same, we see the horizon appears at the same place as in the original frame.

The surface gravity $\bar{\kappa}$ in the Einstein frame is calculated by $2\bar{\kappa}^2 = (\bar{\nabla}_\mu\xi^\nu)(\bar{\nabla}_\nu\xi^\mu)$ from hypersurface orthogonality of ξ^μ , as is the surface gravity κ in the original frame by $2\kappa^2 = (\nabla_\mu\xi^\nu)(\nabla_\nu\xi^\mu)$. Then we see $\bar{\kappa}$ is equal to κ at the bifurcation surface, by noting that $\bar{\nabla}_\mu\xi^\nu = \nabla_\mu\xi^\nu + \xi^\rho\Gamma^\nu_{\mu\rho}$ is equal to $\nabla_\mu\xi^\nu$ at the bifurcation surface, where $\Gamma^\rho_{\mu\nu}$ is defined by $\Gamma^\rho_{\mu\nu} \equiv \frac{1}{2}\bar{g}^{\rho\sigma}(\bar{\nabla}_\mu\bar{g}_{\nu\sigma}^{-1} + \bar{\nabla}_\nu\bar{g}_{\mu\sigma}^{-1} - \bar{\nabla}_\sigma\bar{g}_{\mu\nu}^{-1})$. In addition, the zero-th law, i.e., the constancy of $\bar{\kappa}$ all over the horizon, is proved whenever the Killing horizon is bifurcate[7]. Therefore, the surface gravity $\bar{\kappa}$ in the Einstein frame is the same as that in the original frame κ .

The angular velocity $\bar{\Omega}_H$ in the Einstein frame is given by $\bar{g}_{\mu\nu}^{-1}(\xi^\mu_{(t)} + \bar{\Omega}_H\xi^\mu_{(\varphi)})\xi^\nu_{(\varphi)} = 0$, and we have $\bar{g}_{\mu\nu}^{-1}(\xi^\mu_{(t)} + \Omega_H\xi^\mu_{(\varphi)})\xi^\nu_{(\varphi)} = 0$ from hypersurface orthogonality of ξ^μ . Hence we see the angular velocities in two frames are the same, $\bar{\Omega}_H = \Omega_H$.

4.3 Entropy, mass and angular momentum

The equivalence of the entropy, the mass and the angular momentum is shown if the integrand of Eq.(3.11) on the bifurcation surface and the asymptotic form of the integrand of Eq.(3.12) and Eq. (3.13) at infinity are the same between two frames.

[‡]The most part of this subsection has been already noted in Ref.[4].

At the bifurcation surface, the integrand of Eq.(3.11) in two frames are given, from Eq.(3.4) and Eq.(3.6), by

$$\frac{1}{2}\epsilon_{\beta\alpha\mu\nu}Q^{\beta\alpha} = -\frac{1}{16\pi}\epsilon_{\alpha\beta\mu\nu}P^{\alpha\gamma}\nabla_{\gamma}\xi^{\beta}, \quad (4.2)$$

in the original frame, and

$$\frac{1}{2}\bar{\epsilon}_{\beta\alpha\mu\nu}\bar{Q}^{\beta\alpha} = -\frac{1}{16\pi}\bar{\epsilon}_{\alpha\beta\mu\nu}\bar{g}^{\alpha\gamma}\bar{\nabla}_{\gamma}\xi^{\beta} \quad (4.3)$$

in the Einstein frame, respectively. Substituting Eq.(2.2) into Eq.(4.3) and noticing again that $\bar{\nabla}_{\mu}\xi^{\nu} = \nabla_{\mu}\xi^{\nu}$ at the bifurcation surface, we find the tensor components of Eq.(4.3) are equal to those of Eq.(4.2). Then, since $\bar{\kappa} = \kappa$, the entropy is the same in two frames.

The mass and the angular momentum are also the same in two frames, if the integrand of Eq.(3.12) and Eq.(3.13) are the same up to the order of $\mathcal{O}(r^{-2})$ in the asymptotically flat region. Since $\bar{\epsilon}_{\alpha\beta\mu\nu}$ approaches $\epsilon_{\alpha\beta\mu\nu}$ as $r \rightarrow \infty$, and the integrands in the both frames fall as r^{-2} , it is sufficient to show that

$$Q^{\beta\alpha} + 2\chi^{[\beta}B^{\alpha]} = \bar{Q}^{\beta\alpha} + 2\chi^{[\beta}\bar{B}^{\alpha]} + \mathcal{O}(r^{-3}), \quad (4.4)$$

where χ^{μ} is taken to be $\xi_{(t)}^{\mu}$ and $\xi_{(\varphi)}^{\mu}$ to show the equivalence of the mass and the angular momentum, respectively.

In the original frame, we have $\delta g^{\mu\nu} \sim \mathcal{O}(r^{-1})$, $\nabla_{\mu}\chi^{\nu} \sim \mathcal{O}(r^{-2})$ and $P^{\mu\nu} \rightarrow g^{\mu\nu} + \mathcal{O}(r^{-1})$ in the asymptotically flat region. Then, the asymptotic forms of $Q^{\beta\alpha}$ and Θ^{β} are calculated, from Eq.(3.4) and (3.3), as

$$Q^{\beta\alpha} \rightarrow \frac{1}{8\pi} \left[\nabla^{[\alpha}\chi^{\beta]} + \chi_{\nu}\nabla^{[\beta}P^{\alpha]\nu} + \chi^{[\alpha}\nabla_{\nu}P^{\beta]\nu} \right] + \mathcal{O}(r^{-3}) \quad (4.5)$$

$$\begin{aligned} \Theta^{\beta} &\rightarrow \frac{1}{16\pi} \left[g_{\mu\nu}^{-1}\nabla^{\beta}\delta g^{\mu\nu} - \nabla_{\alpha}\delta g^{\alpha\beta} + v^{\beta}\delta\phi \right] + \mathcal{O}(r^{-3}) \\ &= \frac{1}{16\pi} \left[g_{\mu\nu}^{-1}g^{\alpha\beta}\partial_{\alpha}\delta g^{\mu\nu} - \partial_{\alpha}\delta g^{\alpha\beta} + v^{\beta}\delta\phi \right] + \mathcal{O}(r^{-3}) \end{aligned} \quad (4.6)$$

to the order of $\mathcal{O}(r^{-2})$, where we have used the fact that $\nabla_{\alpha}\delta g^{\mu\nu} = \partial_{\alpha}\delta g^{\mu\nu} + \mathcal{O}(r^{-3})$. If there exists a vector field C^{β} such that $\delta C^{\beta} \rightarrow v^{\beta}\delta\phi \sim \mathcal{O}(r^{-2})$ at most in the asymptotically flat region, then $\Theta^{\beta} \rightarrow \delta B^{\beta} + \mathcal{O}(r^{-3})$ as $r \rightarrow \infty$, where

$$B^{\beta} = \frac{1}{16\pi} \left[g_{\mu\nu}^{-1}g^{\alpha\beta}\partial_{\alpha}g^{\mu\nu} - \partial_{\alpha}g^{\alpha\beta} + C^{\beta} \right]. \quad (4.7)$$

C^{β} is expected to exist in the asymptotically flat spacetime, so that B^{β} exists, and hence the mass and the angular momentum are well-defined. In the usual cases, where the kinetic term of ϕ is the form of $(\nabla_{\mu}\phi)(\nabla^{\mu}\phi)$, $v^{\beta}\delta\phi \sim \mathcal{O}(r^{-3})$, if $\phi \rightarrow \text{const.}$ in the asymptotically flat region, and then C^{β} does not contribute to B^{β} in this case.

If we take into account that $\bar{g}^{\mu\nu} \rightarrow g^{\mu\nu} + \mathcal{O}(r^{-1})$ and $H_{\mu\nu}{}^{\rho} = H_{\nu\mu}{}^{\rho}$, $\bar{Q}^{\beta\alpha}$ in the Einstein frame given by Eq.(3.6) is calculated as

$$\begin{aligned} \bar{Q}^{\beta\alpha} &\rightarrow \frac{1}{8\pi} \left[\bar{\nabla}^{[\alpha}\chi^{\beta]} + \chi^{\nu}H_{\mu}^{[\alpha}\bar{g}^{\beta]\mu} + \chi^{\nu}H_{\mu}^{[\alpha\beta]}\bar{g}^{\mu}_{\nu} + \chi^{\nu}H_{\nu\mu}^{[\alpha}\bar{g}^{\beta]\mu} \right] + \mathcal{O}(r^{-3}) \\ &= \frac{1}{8\pi} \bar{\nabla}^{[\mu}\chi^{\beta]} + \mathcal{O}(r^{-3}) \end{aligned} \quad (4.8)$$

in the asymptotically flat region. We rewrite Eq.(4.8), in terms of the tensors in the original frame, by using the relations

$$\bar{\nabla}_{\nu}\chi^{\beta} = \nabla_{\nu}\chi^{\beta} - \frac{1}{2}\chi^{\alpha} \left(\bar{g}_{\nu\rho}^{-1}\nabla_{\alpha}\bar{g}^{\rho\beta} + \bar{g}_{\alpha\rho}^{-1}\nabla_{\nu}\bar{g}^{\rho\beta} - \bar{g}^{\beta\gamma}\bar{g}_{\alpha\rho}^{-1}\bar{g}_{\nu\sigma}^{-1}\nabla_{\gamma}\bar{g}^{\rho\sigma} \right), \quad (4.9)$$

and

$$\bar{g}^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{1}{\sqrt{-\det(P^{\alpha\beta})}} P^{\mu\nu}, \quad (4.10)$$

into

$$\begin{aligned}\bar{Q}^{\beta\alpha} &= \frac{1}{8\pi} \left[\nabla^{[\alpha} \chi^{\beta]} + \chi_\nu \nabla^{[\beta} P^{\alpha]\nu} + \frac{1}{2} \chi^{[\beta} \nabla^{\alpha]} P \right] + \mathcal{O}(r^{-3}) \\ &= Q^{\beta\alpha} + \frac{1}{8\pi} \left[\chi^{[\beta} \nabla_\nu P^{\alpha]\nu} + \frac{1}{2} \chi^{[\beta} \nabla^{\alpha]} P \right] + \mathcal{O}(r^{-3}),\end{aligned}\quad (4.11)$$

where $P \equiv g_{\mu\nu} P^{\mu\nu}$. Since $H_{\mu\nu} \sim \mathcal{O}(r^{-2})$, $\delta\bar{g}^{\mu\nu} \sim \mathcal{O}(r^{-1})$ and hence $\bar{\nabla}_\mu \delta\bar{g}^{\alpha\beta} = \partial_\mu \delta\bar{g}^{\alpha\beta} + \mathcal{O}(r^{-3})$ in the asymptotically flat region, the asymptotic form of $\bar{\Theta}^\beta$ is given, from Eq.(3.5), by

$$\begin{aligned}\bar{\Theta}^\beta &= \frac{1}{16\pi} \left[\bar{g}_{\mu\nu}^{-1} \bar{\nabla}^\beta \delta\bar{g}^{\mu\nu} - \bar{\nabla}_\nu \delta\bar{g}^{\nu\beta} + \bar{v}^\beta \delta\phi \right] + \mathcal{O}(r^{-3}) \\ &= \frac{1}{16\pi} \left[g_{\mu\nu}^{-1} g^{\alpha\beta} \partial_\alpha \delta\bar{g}^{\mu\nu} - \partial_\nu \delta\bar{g}^{\nu\beta} + \bar{v}^\beta \delta\phi \right] + \mathcal{O}(r^{-3}).\end{aligned}\quad (4.12)$$

We see from Eq.(4.8) and Eq.(4.12) that the exotic matter $g^{\mu\nu}$ does not contribute to the mass and the angular momentum. $\bar{v}^\beta$ in Eq.(4.12) is related to v^β by $\bar{v}^\beta = \sqrt{-g} v^\beta / \sqrt{-\bar{g}}$, so $\bar{v}^\beta \delta\phi \rightarrow \delta C^\beta + \mathcal{O}(r^{-3})$. Then we find $\bar{B}^\beta$ such that $\bar{\Theta}^\beta \rightarrow \delta\bar{B}^\beta + \mathcal{O}(r^{-3})$, which is given by

$$\begin{aligned}\bar{B}^\beta &= \frac{1}{16\pi} \left[g_{\mu\nu}^{-1} g^{\alpha\beta} \partial_\alpha \delta\bar{g}^{\mu\nu} - \partial_\nu \delta\bar{g}^{\nu\beta} + C^\beta \right] + \mathcal{O}(r^{-3}) \\ &= \frac{1}{16\pi} \left[g_{\mu\nu}^{-1} g^{\alpha\beta} \partial_\alpha \delta g^{\mu\nu} - \partial_\nu \delta g^{\nu\beta} + C^\beta - \left(\nabla_\nu P^{\nu\beta} + \frac{1}{2} \nabla^\beta P \right) \right] + \mathcal{O}(r^{-3}) \\ &= B^\beta - \frac{1}{16\pi} \left(\nabla_\nu P^{\nu\beta} + \frac{1}{2} \nabla^\beta P \right) + \mathcal{O}(r^{-3}),\end{aligned}\quad (4.13)$$

where we have rewritten the equation in terms of the tensors in the original frame.

Then, from Eq.(4.11) and Eq.(4.12), we obtain Eq.(4.4) for an arbitrary Killing vector χ^μ , thus, the mass and the angular momentum are the same between the original frame and the Einstein frame.

5 Conclusion and Discussion

In summary, we consider a generalized theory of gravity whose Lagrangian is an analytic function of the metric, the Ricci tensor, a scalar field and its derivative. By converting the theory into the Einstein frame via a Legendre transformation and following the definition of the thermodynamical variables formulated in Ref.[1, 2], we have shown that all the thermodynamical variables are the same both in the original frame and the Einstein frame. The assumptions we made are that the asymptotically flat region in the Einstein frame corresponds to that in the original frame, and the transformation into the Einstein frame is regular.

The above result is a generalization of the works by Jacobson, Kang and Myers[3, 4]. They showed the entropy in the theory whose Lagrangian is given by either

$$I = \frac{1}{16\pi} \int d^4x \sqrt{-g} f(R, \phi, \nabla_\alpha \phi) \quad (5.14)$$

or

$$I = \frac{1}{16\pi} \int d^4x \sqrt{-g} [R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu}] \quad (5.15)$$

is the same as that in the Einstein frame. In the latter case, they used the fact in their proof that the asymptotic structure, such as the mass and the angular momentum, in the Einstein frame is the same as that in the original frame. However, if we consider a non-minimal coupling of a scalar field to gravitation, which appears, e.g., in the Brans-Dicke theory and the effective theory of a string theory, the scalar field contributes to the asymptotic structure, generally. Actually, there is an example[8] where the gravitational mass in the Einstein frame is different from that in the Brans-Dicke frame, whereas the ‘‘thermodynamical’’ mass defined by Eq.(3.12) is the same between two frames, which is equal to the gravitational mass in the Einstein frame.

The possibility of further generalization into the cases where Lagrangian includes Wyle tensor and the derivatives of Riemann tensor is not clear at present. The Legendre transformation in those cases is discussed by Magnano et al.[9]. However, the theory is not transformed into the Einstein frame in this case.

As for an application of our result, we are planning to investigate the second law of black hole thermodynamics, which has not been established yet, except few special cases[3, 10]. Our result will be useful in the analysis of the second law, since we can make use of Hawking's area theorem in the Einstein theory, if the energy condition is satisfied.

Acknowledgment

This work was supported partially by the Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Science and Culture (No. 08102010), and by Waseda University Grant for Special Research Projects.

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Entanglement thermodynamics

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Abstract

Entanglement entropy is a statistical entropy measuring information loss due to coarse-graining corresponding to a spatial division of a system. In this paper we construct a thermodynamics (entanglement thermodynamics) which includes the entanglement entropy as the entropy variable, for a massless scalar field in Minkowski, Schwarzschild and Reissner-Nordström spacetimes to understand the statistical origin of black-hole thermodynamics. It is shown that the entanglement thermodynamics in Minkowski spacetime differs significantly from black-hole thermodynamics. On the contrary, the entanglement thermodynamics in Schwarzschild and Reissner-Nordström spacetimes has close relevance to black-hole thermodynamics.

1 Introduction

In the classical theory of black-hole there appears a thermodynamical relation among the parameters describing black-holes. We call it black-hole thermodynamics [1]. For example, for a one-parameter family of Schwarzschild black-holes parameterized by their mass M , the same relation as the first law of thermodynamics holds:

$$dE_{BH} = T_{BH} dS_{BH} , \quad (1)$$

where E_{BH} , S_{BH} and T_{BH} are defined by

$$\begin{aligned} E_{BH} &\equiv M , \\ S_{BH} &\equiv \frac{A_H}{4l_p^2} , \\ T_{BH} &\equiv \frac{1}{4\pi R_H} . \end{aligned} \quad (2)$$

Here $R_H = 2Ml_p^2$ and $A_H = 4\pi R_H^2$ are the area radius and the area of the event horizon. It is well-known that in general the area of the event horizon does not decrease in time classically. Hence S_{BH} defined by (2) does not decrease in time as thermodynamical entropy (the second law of black-hole). It can also be shown [2] that sum of S_{BH} and entropy of matter field outside the horizon does not decrease in time semi-classically (the generalized second law). Thus S_{BH} can be regarded as a entropy of a black-hole. Moreover, a black-hole with surface gravity κ emits thermal radiation at the so-called Hawking temperature $\kappa/2\pi$ [3]. For the Schwarzschild black-hole this Hawking temperature is given by T_{BH} . From these facts S_{BH} and T_{BH} are called black-hole entropy and black hole temperature, respectively.

There have been many attempts to understand the statistical origin of the black-hole entropy [4]. Among them a strong candidate is the so-called entanglement entropy [5, 6, 7]. In this paper we attempt to construct a thermodynamics which includes the entanglement entropy as the entropy. We call it thermodynamics of entanglement or entanglement thermodynamics [8].

2 Basic ingredients of the entanglement thermodynamics

First we review the basic concepts of entanglement thermodynamics. Let $\mathcal{F}$ be a Hilbert space constructed from two Hilbert spaces $\mathcal{F}_1$ and $\mathcal{F}_2$ as

$$\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 , \quad (3)$$

where $\bar{\otimes}$ denotes a tensor product followed by a suitable completion. From an element u of $\mathcal{F}$ with unit norm we construct an operator ρ ('density operator') by

$$\rho v = (u, v)u \quad \forall v \in \mathcal{F}, \quad (4)$$

where (u, v) is the inner product which is antilinear with respect to u . From ρ we define so-called reduced density operators $\rho_{1,2}$ by

$$\begin{aligned} \rho_1 y &= \sum_{i,j} e_i (e_i \otimes f_j, \rho(y \otimes f_j)) \quad \forall y \in \mathcal{F}_1, \\ \rho_2 z &= \sum_{i,j} f_j (e_i \otimes f_j, \rho(e_i \otimes z)) \quad \forall z \in \mathcal{F}_2, \end{aligned} \quad (5)$$

where $\{e_i\}$ and $\{f_j\}$ are orthonormal bases of $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. The entanglement entropy is defined by $S_{ent} = S[\rho_1]$ or $S_{ent} = S[\rho_2]$ or $S_{ent} = S[\rho_1 \otimes \rho_2]$, where

$$\begin{aligned} S[\rho_{1,2}] &= -\text{Tr}[\rho_{1,2} \ln \rho_{1,2}], \\ S[\rho_1 \otimes \rho_2] &= S[\rho_1] + S[\rho_2]. \end{aligned} \quad (6)$$

Since it can be shown that $S[\rho_1] = S[\rho_2]$ in general [8], the three options for S_{ent} are identical up to the factor 2. For definition of the entanglement energy we consider the following four options: $E_{ent} = \langle : H_1 : \rangle$ or $E_{ent} = \langle : H_2 : \rangle$ or $E_{ent} = \langle : H_1 : \rangle + \langle : H_2 : \rangle$ or $E_{ent} = \langle : H_{tot} : \rangle_{\rho_1 \otimes \rho_2}$, where

$$\begin{aligned} \langle : H_{1,2} : \rangle &= \text{Tr}[\rho : H_{1,2}], \\ \langle : H_{tot} : \rangle_{\rho_1 \otimes \rho_2} &= \text{Tr}[(\rho_1 \otimes \rho_2) : H_{tot}]. \end{aligned} \quad (7)$$

Here the total hamiltonian H_{tot} is assumed to be decomposed as $H_{tot} = H_1 + H_2 + H_{int}$, and $: \dots :$ denotes the normal ordering. Finally entanglement temperature T_{ent} is obtained by imposing the first law of entanglement thermodynamics:

$$T_{ent} dS_{ent} = dE_{ent}. \quad (8)$$

3 Discretized theory of a scalar field

In this paper we consider a massless real scalar field described by the action

$$I = -\frac{1}{2} \int dx^4 \sqrt{-g} \partial^\mu \phi \partial_\mu \phi, \quad (9)$$

where the background geometry is fixed to be a spherically symmetric static spacetime with the metric

$$ds^2 = -N(\rho)^2 dt^2 + d\rho^2 + r(\rho)^2 (d\theta + \sin^2 \theta d\psi^2). \quad (10)$$

For this system we calculate the entanglement entropy and the entanglement energy to construct entanglement thermodynamics using the methods developed in Ref. [5] and Ref. [8, 9]. Those methods are both based on the following form of the Hamiltonian describing a discrete system $\{q^A\}$ ($A = 1, 2, \dots, n_{tot}$):

$$H_0 = \frac{1}{2a} \delta^{AB} p_A p_B + \frac{1}{2} V_{AB} q^A q^B, \quad (11)$$

where p_A is a momentum conjugate to q^A and a is a fundamental length of the system. For this discrete system it is easy to divide the whole Hilbert space $\mathcal{F}$ into the form (3): $\mathcal{F}$ is defined as a Fock space constructed from $\{q^A\}$ ($A = 1, 2, \dots, n_{tot}$); $\mathcal{F}_1$ is defined as a Fock space constructed from $\{q^a\}$ ($a = 1, 2, \dots, n_B$); $\mathcal{F}_2$ is defined as a Fock space constructed from $\{q^\alpha\}$ ($\alpha = n_B + 1, \dots, n_{tot}$). In order to apply this scheme to our problem we have to construct a discretized theory of the scalar field whose Hamiltonian is of the form (11).

First we expand the field ϕ in terms of the spherical harmonics as

$$\phi(\rho, \theta, \psi) = \sum_{l,m} \frac{N^{1/2}}{r} \phi_{lm}(\rho) Z_{lm}(\theta, \psi) , \quad (12)$$

where $Z_{lm} = \sqrt{2}Y_{lm}$ for $m > 0$, $\sqrt{2}iY_{lm}$ for $m < 0$, and $Z_{l0} = Y_{l0}$. Then the Hamiltonian corresponding to the Killing energy for the action (9) is decomposed into a direct sum of contributions from each harmonics component H_{lm} as

$$H = \sum_{lm} H_{lm} . \quad (13)$$

Here H_{lm} is given by

$$H_{lm} = \frac{1}{2} \int d\rho \left[\pi_{lm}^2 + N r^2 \left\{ \frac{\partial}{\partial \rho} \left(\frac{N^{1/2}}{r} \phi_{lm} \right) \right\}^2 + l(l+1) \left(\frac{N \phi_{lm}}{r} \right)^2 \right] , \quad (14)$$

where $\pi_{lm}(\rho)$ is a momentum conjugate to $\phi_{lm}(\rho)$.

Note that for any (l, m) (14) has the form

$$\tilde{H} = \frac{1}{2} \int d\rho p^2(\rho) + \frac{1}{2} \int d\rho d\rho' q(\rho) V(\rho, \rho') q(\rho') , \quad (15)$$

where the following algebra of Poisson brackets is understood:

$$\{q(\rho), p(\rho')\} = \delta(\rho - \rho') , \quad \{q(\rho), q(\rho')\} = 0 , \quad \{p(\rho), p(\rho')\} = 0 . \quad (16)$$

Each subsystem described by the Hamiltonian (15) can be discretized by the following procedure:

$$\begin{aligned} \rho &\rightarrow (A - 1/2)a , \\ \delta(\rho - \rho') &\rightarrow \delta_{AB}/a , \end{aligned} \quad (17)$$

where $A, B = 1, 2, \dots$ and a is a cut-off length. The corresponding Hamiltonian of the discretized system is of the form (11) with

$$\begin{aligned} q(\rho) &\rightarrow q^A , \\ p(\rho) &\rightarrow p_A/a , \\ V(\rho, \rho') &\rightarrow V_{AB}/a^2 . \end{aligned} \quad (18)$$

In this way we obtain a discretized system with the Hamiltonian (11) with the matrix V given by the direct sum

$$V = \oplus_{l,m} V^{(l,m)} , \quad (19)$$

where $V^{(l,m)}$ is explicitly expressed as

$$\begin{aligned} V_{AB}^{(l,m)} \phi_{lm}^A \phi_{lm}^B &= a \sum_{A=1}^{\infty} \left[N_{A+1/2} \left(\frac{x_{A+1/2}}{a} \right)^2 \left(\frac{N_{A+1}^{1/2}}{x_{A+1}} \phi_{lm}^{A+1} - \frac{N_A^{1/2}}{x_A} \phi_{lm}^A \right)^2 \right. \\ &\quad \left. + \frac{l(l+1)}{r_0^2} \left(\frac{N_A \phi_{lm}^A}{x_A} \right)^2 \right] . \end{aligned} \quad (20)$$

Here

$$\begin{aligned} x_A &= r(\rho = (A - 1/2)a)/r_0 , \\ x_{A+1/2} &= r(\rho = Aa)/r_0 , \\ N_A &= N(\rho = (A - 1/2)a) , \\ N_{A+1/2} &= N(\rho = Aa) , \\ \phi_{lm}^A &= \phi_{lm}(\rho = (A - 1/2)a) . \end{aligned} \quad (21)$$

In the matrix representation $V^{(l,m)}$ is given by the $n_{tot} \times n_{tot}$ matrix

$$\begin{aligned} \left(V_{AB}^{(l,m)} \right) &= \frac{2a}{r_0^2} \begin{pmatrix} \Sigma_1^{(l)} & \Delta_1 & & & \\ \Delta_1 & \Sigma_2^{(l)} & \Delta_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \Delta_{A-1} & \Sigma_A^{(l)} & \Delta_A \\ & & & \ddots & \ddots & \ddots \end{pmatrix}, \\ \Sigma_A^{(l)} &= \frac{1}{2} (r_0/a)^2 N_A x_A^{-2} \left[N_{A-1/2} x_{A-1/2}^2 + N_{A+1/2} x_{A+1/2}^2 \right] \\ &\quad + \frac{1}{2} l(l+1) N_A^2 x_A^{-2}, \\ \Delta_A &= -\frac{1}{2} (r_0/a)^2 N_A^{1/2} N_{A+1/2} N_{A+1}^{-1/2} x_A^{-1} x_{A+1/2}^2 x_{A+1}^{-1}, \end{aligned} \quad (22)$$

where we have imposed the boundary condition $\phi_{lm}^{n_{tot}+1} = 0$. In these expressions r_0 is an arbitrary constant, which we set to be the area radius R_B of the boundary in the following arguments.

4 Entanglement thermodynamics in Minkowski spacetime

In the case of Minkowski spacetime $r(\rho) = \rho$ and $N(\rho) = 1$. The discretized system $\{\phi_{lm}^A\}$ ($A = 1, \dots, n_{tot}$) is described by Hamiltonian of the form (11) with the potential given by (22).

We split the system $\{\phi_{lm}^A\}$ ($A = 1, \dots, n_{tot}$) into the two subsystems, $\{\phi_{lm}^a\}$ ($a = 1, \dots, n_B$) and $\{\phi_{lm}^\alpha\}$ ($\alpha = n_B + 1, \dots, n_{tot}$). For this splitting, the entanglement entropy S_{ent} and the entanglement energy E_{ent} for the ground state can be calculated numerically. The results are expressed as

$$\begin{aligned} S_{ent} &\propto \frac{R_B^2}{a^2}, \\ E_{ent} &\propto \frac{R_B^2}{a^3}, \end{aligned} \quad (23)$$

for all definitions of S_{ent} and E_{ent} , where R_B is the radius of the boundary defined by $R_B = n_B a$ [5, 8, 9]. Hence the entanglement temperature T_{ent} is proportional to a^{-1} :

$$T_{ent} \propto \frac{1}{a}. \quad (24)$$

Thus the entanglement thermodynamics in Minkowski spacetime is quite different from the black-hole thermodynamics [8] since $E_{BH} \propto R_H/l_{pl}^2$ and $T_{BH} \propto 1/R_H$.

5 Entanglement thermodynamics in Schwarzschild spacetime

In Schwarzschild spacetime the metric is given by

$$ds^2 = -\left(1 - \frac{R_H}{r}\right) dt^2 + \left(1 - \frac{R_H}{r}\right)^{-1} dr^2 + r^2 (d\theta + \sin^2 \theta d\psi^2), \quad (25)$$

where R_H is the area radius of the horizon. As the radial coordinate ρ we take the proper distance from the horizon:

$$\rho = \frac{R_H}{2} \left[\sqrt{y^2 - 1} + \ln \left(y + \sqrt{y^2 - 1} \right) \right], \quad (26)$$

where the variable y is defined by $y = 2r/R_H - 1$.

As in the case of the Minkowski spacetime we split the system $\{\phi_{lm}^A\}$ ($A = 1, \dots, n_{tot}$) into the two subsystems, $\{\phi_{lm}^a\}$ ($a = 1, \dots, n_B$) and $\{\phi_{lm}^\alpha\}$ ($\alpha = n_B + 1, \dots, n_{tot}$), and calculated the entanglement entropy and the entanglement energy E_{ent} for the ground state numerically. Note that we consider only

degrees of freedom corresponding to positive A (see Ref. [9] for the reason why we do not consider those corresponding to $A \leq 0$).

The results are expressed as

$$\begin{aligned} S_{ent} &\propto \frac{R_B^2}{a^2} , \\ E_{ent} &\propto \frac{R_B}{a^2} , \end{aligned} \quad (27)$$

for all definitions of S_{ent} and E_{ent} , where R_B is the area radius of the boundary defined by $R_B = r(\rho = n_B a)$ with $n_B = O(1)$ [9]. Here note that the proportionality in (27) holds for a fixed value of n_B . Hence the entanglement temperature T_{ent} is proportional to R_B^{-1} :

$$T_{ent} \propto \frac{1}{R_B} . \quad (28)$$

Thus the entanglement thermodynamics in Schwarzschild spacetime has the same structure as that of the black-hole thermodynamics [9].

6 Entanglement thermodynamics in R-N spacetime

In Reissner-Nordström spacetime the metric is given by

$$ds^2 = - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 + r^2 (d\theta + \sin^2 \theta d\psi^2) , \quad (29)$$

where M and Q are the mass and the charge of the black-hole. The area radii of the outer and the inner horizon $R_{H\pm}$ are

$$R_{H\pm} = M \pm \sqrt{M^2 - Q^2} . \quad (30)$$

As the radial coordinate ρ we take the proper distance from the outer horizon

$$\rho = \sqrt{(M^2 - Q^2)(y^2 - 1)} + M \ln \left(y + \sqrt{y^2 - 1} \right) , \quad (31)$$

where the variable y is defined by

$$y = \frac{r - M}{\sqrt{M^2 - Q^2}} . \quad (32)$$

Now the numerical results for the entanglement entropy S_{ent} and the entanglement energy E_{ent} for the ground state are expressed as

$$\begin{aligned} S_{ent} &\propto \frac{R_B^2}{a^2} , \\ E_{ent} &\propto c(q) \frac{R_B}{a^2} , \end{aligned} \quad (33)$$

for all definitions of S_{ent} and E_{ent} , where R_B is the area radius of the boundary defined by $R_B = r(\rho = n_B a)$ with $n_B = O(1)$, and the coefficient $c(q)$ is a function of $q = Q/M$ given by

$$c(q) = \frac{\sqrt{1 - q^2}}{1 + \sqrt{1 - q^2}} . \quad (34)$$

The coefficient $c(q)$ approaches to zero in the limit $q \rightarrow 1$. Hence the entanglement temperature T_{ent} is proportional to R_B^{-1} :

$$T_{ent} \propto \frac{c(q)}{R_B} . \quad (35)$$

Note that T_{ent} is zero in the extremal limit of the background spacetime. Therefore as in the case of the Schwarzschild spacetime, the entanglement thermodynamics in the Reissner-Nordström spacetime has the same structure as that of the black-hole thermodynamics.

7 Conclusion

In this paper we have constructed entanglement thermodynamics for a massless scalar field in Minkowski, Schwarzschild and Reissner-Nordström spacetimes. The entanglement thermodynamics in Minkowski spacetime differs significantly from black-hole thermodynamics. On the contrary, the entanglement thermodynamics in Schwarzschild and Reissner-Nordström spacetimes has the same structure as that of black-hole thermodynamics. In particular, it has been shown that entanglement temperature in the Reissner-Nordström spacetime approaches zero in the extremal limit.

Finally we comment on possible extensions of the entanglement thermodynamics. The first is the inclusion of a charged field as matter. In particular, it will be valuable to analyze the entanglement thermodynamics for a charged field in Reissner-Nordström spacetime. In this case we can define the entanglement charge as an expectation value of the charge of the field for the coarse-grained state. This is now under investigation. The second is a generalization to non-spherically-symmetric spacetimes. For example it is expected that construction of entanglement thermodynamics in Kerr spacetime requires the introduction of a concept of an entanglement angular-momentum.

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Conformal Anomaly in 2D Dilatonic Gravity

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Abstract

Conformal anomaly in the dilaton-scalar system in 2 dimensional gravity is examined. We take the heat-kernel regularization for the ultraviolet divergences. Generally the Weyl anomaly is determined by the 2nd order differential (elliptic) operator of the system and the definition of the measure. We have the freedom of the operator choice caused by the arbitrariness of total divergences (surface terms) in the action. We examine the Weyl anomaly in connection with such points and the hermiticity of the system operator.

1 Introduction

Motivated by 2 dim or 4 dim black hole physics[1], Bousso and Hawking[2] recently have calculated the Weyl anomaly for the dilaton coupled scalar theory in 2 dim gravity.

$$S_m = \frac{1}{2} \int d^2x \sqrt{g} e^{-2\phi} g^{\mu\nu} \nabla_\mu f \cdot \nabla_\nu f, \quad (1)$$

where $g_{\mu\nu}$, f and ϕ are the metric, a scalar matter field and the dilaton respectively. It is invariant under the Weyl transformation.

$$\phi' = \phi, \quad f' = f, \quad g'_{\mu\nu} = e^{2\alpha(x)} g_{\mu\nu} \quad (2)$$

They have obtained, using the generalized zeta function, the Weyl anomaly due to the scalar matter quantum effect ($g_{\mu\nu}$ and ϕ are background fields), as

$$T = \sqrt{g} (q_1 R + q_2 (\nabla\phi)^2 + q_3 \nabla^2 \phi), \quad (3)$$
$$q_1 = \frac{1}{24\pi}, \quad q_2 = -\frac{1}{4\pi}, \quad q_3 = -\frac{1}{12\pi}.$$

They pointed out, in the induced effective action derived from the above result, the term $-2\phi R$ naturally appears from the q_3 term of eq.(3). The term was originally introduced by hand in [3] or from considerations of (Weyl) anomalies in [4]. The result shows an important role of the dilaton field. Stimulated by this work, some authors have examined the problem in different ways. The divergence structure of a wide class of theories, which includes the above model, was already examined by [5] in the dimensional regularization. Based on the work, Ref.[6] has calculated the Weyl anomaly of (1). Their result differs from eq.(3) in $q_3 = 1/4\pi$. They have also obtained it for a wide class of dilaton coupled scalar theories taking into account not only the scalar matter but also the dilaton quantum effect. Ref.[7, 8] has also examined the Weyl anomaly taking into account all fields (graviton, dilaton and scalar matter) quantum effects.

It is true that the dilaton field is becoming more and more important in accord with the rapid progress of the string theory. The string theory predicts very naturally the existence of the dilaton in its low energy (field theory) limit. We do not have much experience about the correct treatment of the dilaton field and its characteristic interaction $\exp(-2\phi)$. In this sense, the close analysis of the present problem looks important.

Some problems in the Weyl anomaly itself also motivate us to do the present analysis. Compared with the chiral anomaly (anomaly related to " γ_5 ") the general argument of Weyl anomaly is not so easy [9, 10, 11, 12]. It originates partly from the difficulty of clear characterization of Weyl-anomaly

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terms by a simple symmetry argument. The Weyl anomaly looks to be more influenced by the local dynamics and generally requires more complicated calculation than the chiral one. The present 2 dim model is a nice simple model, by which we can test some general "belief" about the Weyl anomaly. To say it more concretely, we focus on the following points: 1) General structure of the anomaly terms; 2) Regularization dependence of the Weyl anomaly; 3) The origin of the discrepancy in the value of q_3 ; 4) How the arbitrariness of the total divergence affects the Weyl anomaly. The quantization of the scalar field f is formally defined by the path-integral.

$$\exp \Gamma[g, \phi] = \int d\mu(f; g, \phi) \exp S_m[f; g, \phi] \quad , \quad (4)$$

where $d\mu(f; g, \phi)$ is the integration measure and requires the precise definition. which will become a key point in later analysis.

There are several ways of obtaining the Weyl anomaly: 1) the generalized zeta function ([2]); 2) the counterterm formula expressed in the dimensional regularization([6]); 3) the heat-kernel regularization([13]). We calculate the Weyl anomaly in the approach 3) and take the Fujikawa's standpoint[14, 15] about the anomaly. Compared with the counter-term calculation with the dimensional regularization, the advantage of the present approach is that, once the differential operator of the system is fixed, we have no ambiguity, in all stages of calculation, arising from the total divergences[16]. This is because the heat-kernel approach deals with the operator directly, not with the (effective) action. Ref.[6] hints that the discrepancy could come from total divergence ambiguity in the counterterm calculation. We will clarify this point in this paper. Ref.[6] has also pointed out that the way of fixing q_3 which ref.[2] has taken, is not unique because of the presence of ambiguity in the procedure of translating the Weyl anomaly result to the induced (nonlocal) action.

Before a detailed analysis, we fix the general form of the Weyl anomaly by the symmetry arguments. Generally the anomaly is caused by a conflict between some symmetries. In the present case, they are the (2 dim) general coordinate invariance (diffeomorphism) and the Weyl symmetry. We take the familiar convention of keeping the general coordinate invariance. Because of the quantization effect, the system cannot keep the Weyl symmetry. The general form is obtained as (3) with general values of q 's by the following requirements: 1) 2 dim general coordinate invariance;; 2) Weyl invariance (up to total divergences)[9, 10, 11, 12]; 3) Invariance under the constant shift of the dilaton field, $\phi \rightarrow \phi + \text{const.}$. The requirement 3) is necessary because the constant shift changes the action (1) only by an overall constant factor, which can be absorbed into a constant rescaling of the scalar f . Without the condition 3) we would have the arbitrariness of functionals $A(\phi)$ and $B(\phi)$ in two terms, $A(\phi)(\nabla\phi)^2$ and $B(\phi)\nabla^2\phi$, as the anomaly general form[17]. Among the three terms, q_3 -term is a "trivial term" in the terminology of the anomaly-cohomology because the (infinitesimal) Weyl transformation of the local term $\sqrt{g}\phi R$ shows

$$(\sqrt{g}\phi R)' = \sqrt{g}(\phi R + 2\alpha(x) \cdot \nabla^2\phi) + \text{total-derivative} + O(\alpha^2) \quad , \quad |\alpha| \ll 1 \quad . \quad (5)$$

2 Anomaly formula and hermiticity of system operator

First we rescale the scalar field, f , in order to adjust the kinetic term in the ordinary form: $f \exp(-\phi) = F$. Then eq.(1) is expressed as

$$\begin{aligned} S_m &= \frac{1}{2} \int d^2x \sqrt{g} (\nabla_\mu F + \nabla_\mu \phi \cdot F) (\nabla^\mu F + \nabla^\mu \phi \cdot F) \\ &= \frac{1}{2} \int d^2x \left[\sqrt{g} F \bar{D}_1 (\sqrt{g} F) + \sqrt{g} \{ \nabla_\mu (F \nabla^\mu F + a \nabla^\mu \phi \cdot F^2) \} \right] , \\ \bar{D}_1 &= \sqrt{g} \{ -\nabla^2 + 2(1-a) \nabla_\mu \phi \cdot \nabla^\mu - a \nabla^2 \phi + \nabla_\mu \phi \cdot \nabla^\mu \phi \} \frac{1}{\sqrt{g}} , \end{aligned} \quad (6)$$

where a free parameter, a , appears. It represents the freedom of the total derivative terms. A purpose of this section is to check the effect of the freedom. We can take the BRS-invariant measure for the matter scalar field by taking $\tilde{F} = \sqrt{g}F$ instead of F [18]. This is the reason for the factors $\sqrt{g}$, $\frac{1}{\sqrt{g}}$ in $\bar{D}_1$ above.

The 2nd order differential operator $\bar{D}_1$ is hermitian for $a = 1$ when we define the inner product and the hermitian conjugate as

$$\begin{aligned} \langle \bar{F}_1 | \bar{F}_2 \rangle &= \int d^2x \bar{F}_1^\dagger \bar{F}_2 = \int d^2x \sqrt{g} F_1^\dagger F_2 = \int d^2x \sqrt{g} e^{-2\phi} f_1^\dagger f_2, \\ &< \bar{D}_1 \bar{F}_1 | \bar{F}_2 \rangle = \langle \bar{F}_1 | \bar{D}_1^\dagger \bar{F}_2 \rangle. \end{aligned} \quad (7)$$

This choice is most natural for the operator $\bar{D}_1$. For the case of $a \neq 1$, $\bar{D}_1$ is not hermitian. Note here that the factor of $e^{-2\phi}$ appears in the inner product. This will be compared with the case of Sec.3.

The above definition of the inner product between the fields $\bar{F}$'s fixes the integration measure in the path integral as follows[14]. Let $\{u_n(x)\}$ be the complete and orthogonal set of the eigen functions of the operator $\bar{D}_1$.

$$\bar{D}_1 u_n = \lambda_n u_n, \quad \langle u_n | u_m \rangle = \int d^2x u_n^\dagger u_m = \delta_{nm}. \quad (8)$$

From the completeness, a general field $\bar{F} = \sqrt{g} F$ can be expressed as

$$\bar{F} = \sum_n c_n u_n, \quad c_n \equiv \langle u_n | \bar{F} \rangle, \quad (9)$$

where the orthogonality is used. Then the integration measure in the path-integral can be precisely defined as

$$\begin{aligned} \exp \Gamma_1[g, \phi] &= \int d\mu_1(f; g, \phi) \exp S_m^1[f; g, \phi], \quad S_m^1 = \frac{1}{2} \int d^2x \bar{F} \bar{D}_1 \bar{F}, \\ d\mu_1(f; g, \phi) &= \prod_n dc_n \equiv \mathcal{D}\bar{F} \end{aligned} \quad (10)$$

Following Fujikawa, let us evaluate the Jacobian J in the measure change due to Weyl transformation (2). We regard the Jacobian as the Weyl anomaly.

$$\begin{aligned} \bar{F}' &\equiv \sum_n c'_n u_n = e^{\alpha(x)} \bar{F}(x) = \sum_n c_n e^{\alpha(x)} u_n, \\ c'_n &= \sum_m c_m \langle u_n | e^{\alpha(x)} u_m \rangle = c_n + \sum_m c_m \langle u_n | \alpha(x) u_m \rangle + O(\alpha^2), \\ J &= \det \left(\frac{\partial c'_n}{\partial c_m} \right) = \exp \text{Tr} \ln (\delta_{nm} + \langle u_n | \alpha(x) u_m \rangle + O(\alpha^2)) \\ &= \exp \left[\int d^2x \alpha(x) \sum_n u_n^\dagger(x) u_n(x) + O(\alpha^2) \right]. \end{aligned} \quad (11)$$

Therefore the Weyl anomaly is obtained as

$$T_1 = \left. \frac{\delta J}{\delta \alpha(x)} \right|_{\alpha=0} = \sum_n u_n^\dagger(x) u_n(x). \quad (12)$$

This expression is generally divergent, and must be regularized. We take the heat-kernel regularization as

$$\begin{aligned} T_1 &= \lim_{t \rightarrow +0} \sum_n \text{tr} \langle x | n \rangle e^{-\bar{D}_1 t} \langle n | x \rangle = \lim_{t \rightarrow +0} \text{tr} \langle x | e^{-\bar{D}_1 t} | x \rangle = \lim_{t \rightarrow +0} \text{tr} G_1(x, x; t), \\ &= \lim_{t \rightarrow +0} \sum_n \text{tr} \langle x | n \rangle e^{-\bar{D}_1 t} \langle n | x \rangle = \lim_{t \rightarrow +0} \text{tr} \langle x | e^{-\bar{D}_1 t} | x \rangle = \lim_{t \rightarrow +0} \text{tr} G_1(x, x; t), \end{aligned} \quad (13)$$

We note some general facts about the heat-kernel $G(x, y; t)$ [13]. It is rigorously defined by the heat equation in terms of an 2nd order differential operator $\bar{D}$,

$$\left(\frac{\partial}{\partial t} + \bar{D} \right) G(x, y; t) = 0. \quad (14)$$

and can be generally solved perturbatively even for the non-hermitian operator under the condition that the operator $\bar{D}$ asymptotically goes to $-\delta_{\mu\nu} \partial_\mu \partial_\nu$ in the weak field limit. In the present model, the weak field means $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}| \ll 1$, $|\phi| \ll 1$. We present a useful anomaly formula valid for most

2 dim gravity-matter theories. For a general theory, specified by general background fields $\mathcal{N}_\mu$ and $\mathcal{M}$ (which are general covariants) as below, the Weyl anomaly T is given by

$$\begin{aligned}\bar{D} &= \sqrt[4]{g}(-\nabla^2 + \mathcal{N}_\mu \nabla^\mu + \mathcal{M}) \frac{1}{\sqrt[4]{g}}, \\ T &= \text{tr } G(x, x; t)|_{t=0} = \frac{1}{4\pi} \sqrt{g} \text{Tr} \left(\frac{1}{6} R - \mathcal{M} + \frac{1}{2} \nabla_\mu \mathcal{N}^\mu \right).\end{aligned}\quad (15)$$

"Tr" in the above equation means "trace" over all field indices: $i, j = 1, 2, \dots, N$; $\text{Tr} \mathcal{M} = \mathcal{M}^{ii}$, etc. This formula is easily derived from two facts: 1) the formula must be general invariant, 2) for the weak field gravity case, it leads to eq.(27) of ref.[13] or eq.(40) of ref.[19].

In the present special case of $\bar{D}_1$ ($\mathcal{N}_\mu = 2(1-a)\nabla_\mu \phi$, $\mathcal{M} = -a\nabla^2 \phi + \nabla_\mu \phi \cdot \nabla^\mu \phi$, $N = 1$), we obtain the Weyl anomaly as

$$\begin{aligned}T_1 &= \frac{1}{4\pi} \sqrt{g} \left(\frac{1}{6} R - \nabla_\mu \phi \cdot \nabla^\mu \phi + \nabla^2 \phi \right), \\ q_1 &= \frac{1}{24\pi}, \quad q_2 = -\frac{1}{4\pi}, \quad q_3 = \frac{1}{4\pi}.\end{aligned}\quad (16)$$

The final result does not depend on the free parameter a and agrees with that of Ref.[6]. [20] The result of a -independence is naturally accepted because the Weyl anomaly is a local (ultraviolet) phenomenon.

3 Second Choice of Measure

The dilaton-scalar theory (1) can be modified to, up to total divergences,

$$S_m^2 = \frac{1}{2} \int d^2x \sqrt{g} e^{-2\phi} f (-\nabla^2 + 2\nabla_\mu \phi \cdot \nabla^\mu) f. \quad (17)$$

S_m^2 seems to give the same Weyl anomaly as S_m if the situation is the same as in Sect.2. In this section we take the following one

$$\bar{D}_2 \equiv \sqrt{g} e^{-2\phi} (-\nabla^2 + 2\nabla_\mu \phi \cdot \nabla^\mu) \frac{1}{\sqrt{g}}, \quad (18)$$

as the system operator. The operator $\bar{D}_2$ is hermitian only for the inner product and the hermitian conjugate defined by

$$\begin{aligned}\tilde{f} &\equiv \sqrt{g} f, \\ \langle \tilde{f}_1 | \tilde{f}_2 \rangle &\equiv \int d^2x \tilde{f}_1^\dagger(x) \tilde{f}_2(x) = \int d^2x \sqrt{g} f_1^\dagger(x) f_2(x), \\ \langle \bar{D}_2 \tilde{f}_1 | \tilde{f}_2 \rangle &= \langle \tilde{f}_1 | \bar{D}_2^\dagger \tilde{f}_2 \rangle.\end{aligned}\quad (19)$$

In contrast with the inner product (7) in Sect.2, the dilaton factor $\exp(-2\phi)$ does not appear in the above definition. The factor appears in the operator $\bar{D}_2$. As in the Sect.2, the path-integral measure must be taken as

$$\begin{aligned}\bar{D}_2 v_n &= \sigma_n v_n, \quad \langle v_n | v_m \rangle = \int d^2x v_n^\dagger v_m = \delta_{nm}, \\ \tilde{f} &= \sum_n b_n v_n, \quad b_n \equiv \langle v_n | \tilde{f} \rangle, \\ \exp \Gamma_2[g, \phi] &= \int d\mu_2(f; g, \phi) \exp S_m^2[f; g, \phi], \quad S_m^2 = \frac{1}{2} \int d^2x \tilde{f} \bar{D}_2 \tilde{f}, \\ d\mu_2(f; g, \phi) &= \prod_n db_n \equiv \mathcal{D} \tilde{f}.\end{aligned}\quad (20)$$

The same procedure as given in Sect.2 leads to the Weyl anomaly, using the measure defined by (19),

$$\begin{aligned} T_2 &= \lim_{t \rightarrow +0} \sum_n v_n^\dagger(x) v_n(x) e^{-\sigma_n t} = \lim_{t \rightarrow +0} \sum_n \langle n|x \rangle e^{-\tilde{D}_2 t} \langle x|n \rangle \\ &= \lim_{t \rightarrow +0} \sum_n \text{tr} \langle x|n \rangle \langle n|x \rangle e^{-\tilde{D}_2 t} = \lim_{t \rightarrow +0} \text{tr} \langle x|e^{-\tilde{D}_2 t}|x \rangle = \lim_{t \rightarrow +0} \text{tr} G_2(x, x; t) \quad , \\ G_2(x, y; t) &\equiv \langle x|e^{-\tilde{D}_2 t}|y \rangle \quad . \end{aligned} \quad (21)$$

We note the anomaly formula (15) can not be applied for $\tilde{D}_2$ because of the dilaton factor $\exp(-2\phi)$ in front of the second derivative term. We evaluate $G_2(x, x; t)$ in the following two steps.

i) The flat space

In the case of the flat space, $g_{\mu\nu} = \delta_{\mu\nu}$, $\tilde{D}_2$ reduces to

$$\tilde{D}_2 = e^{-2\phi}(-\partial^2 + 2\partial_\mu \phi \cdot \partial_\mu) \quad . \quad (22)$$

Now we consider the weak field case of the dilaton: $|\phi| \ll 1$. We fix the anomaly in a first few orders with respect to ϕ . $\tilde{D}_2$ is expressed as

$$\begin{aligned} \tilde{D}_2 &= -\partial^2 - W_{\mu\nu} \partial_\mu \partial_\nu - N_\mu \partial_\mu - M \quad , \\ W_{\mu\nu} &= (e^{-2\phi} - 1)\delta_{\mu\nu} = (-2\phi + 2\phi^2 + O(\phi^3)) \delta_{\mu\nu} \quad , \\ N_\mu &= -2e^{-2\phi} \partial_\mu \phi = -2(1 - 2\phi)\partial_\mu \phi + O(\phi^3) \quad , \\ M &= 0 \quad , \end{aligned} \quad (23)$$

where $W_{\mu\nu}$, N_μ and M are introduced for the general background fields and their explicit content, in the present case, are given above. From the definition of the heat-kernel (14), G_2 satisfy

$$\left(\frac{\partial}{\partial t} - \partial^2 \right) G_2(x, y; t) = (W_{\mu\nu} \partial_\mu \partial_\nu + N_\mu \partial_\mu + M) G_2(x, y; t) \quad , \quad (24)$$

which is used to solve the heat-kernel perturbatively with respect to the fields in the right-hand side (W, N, M) . The 1-st order result of $G_2(x, x; t)|_{t=0}$ was obtained in (27) of [13] as

$$\frac{1}{4\pi} \left(-\frac{1}{12} \partial^2 W_{\mu\mu} + \frac{1}{3} \partial_\mu \partial_\nu W_{\mu\nu} - \frac{1}{2} \partial_\mu N_\mu + M \right) \quad . \quad (25)$$

This formula gives

$$\frac{1}{4\pi} \left(\frac{2}{3} \partial^2 \phi - \frac{4}{3} (\partial\phi)^2 \right) \quad . \quad (26)$$

The 2nd order formula is similarly obtained as[22]

$$\begin{aligned} &\frac{1}{4\pi} \left\{ \frac{1}{48} \partial_\mu W_{\lambda\lambda} \cdot \partial_\mu W_{\sigma\sigma} - \frac{1}{12} \partial_\mu W_{\mu\nu} \cdot \partial_\nu W_{\lambda\lambda} - \frac{1}{4} \partial_\mu W_{\mu\nu} \cdot \partial_\lambda W_{\lambda\nu} \right. \\ &+ \frac{1}{24} \partial_\mu W_{\lambda\sigma} \cdot \partial_\mu W_{\lambda\sigma} + \frac{1}{12} \partial_\mu W_{\nu\lambda} \cdot \partial_\lambda W_{\mu\nu} - \frac{1}{4} N_\mu N_\mu + \frac{1}{2} \partial_\mu W_{\mu\nu} \cdot N_\nu \} \\ &+ (W \partial \partial W, W \partial N, W M \text{ terms}) \quad . \end{aligned} \quad (27)$$

This formula gives, for the present model,

$$\frac{1}{4\pi} \times \frac{1}{3} (\partial\phi)^2 \quad . \quad (28)$$

Therefore we have

$$q_2 = \frac{1}{4\pi} \left(-\frac{4}{3} + \frac{1}{3} \right) = -\frac{1}{4\pi} \quad , \quad q_3 = \frac{1}{4\pi} \times \frac{2}{3} \quad . \quad (29)$$

ii) Vanishing Dilaton

In the case of vanishing dilaton, $\phi = 0$, $\tilde{D}_2$ reduces to $-\sqrt{g} \nabla^2 \frac{1}{\sqrt{g}}$, and the Weyl anomaly is well known (for example, see (29) of [13]).

$$q_1 = \frac{1}{24\pi} \quad . \quad (30)$$

We note, in (29), that the value of q_3 differs from the result in Sect.2[23] whereas q_2 is the same. The q_3 value agrees with (16) of Ref.[21] with the choice $\psi(\phi) = 0$ (trivial measure) and $\varphi(\phi) = \phi$ in the notation of the reference where the zeta-function regularization is taken.

4 Discussion and Conclusion

Generally the way to analyse a quantum field theory using some background (external) fields is quite popular in the theoretical physics. A systematic approach is called the background field method[24, 25] or the effective action approach[26]. The anomaly formulae (15), (25) and (27) (or its 4 dim version in [13, 19]) are based on the formalism. It should be noted that the anomaly formula is different from the counter-term formula in that it is definite including total derivative terms.

Taking the heat-kernel regularization, we have analysed the Weyl anomaly with special interest in the freedom in the choice of the system operator, which is caused by the arbitrariness of the total divergence. The operator $\tilde{D}_2$ of Sect.3 was taken in [2], whereas $\tilde{D}_1$ of Sect.2 was essentially taken in [6]. Although they are related by the total divergence, the corresponding Weyl anomalies become different because of the measure difference. The most natural measures are fixed for each case from the requirement that the hermiticity of the system operators hold. This reminds us of a similar one in the non-abelian chiral anomaly[14, 15], where different choices of $\tilde{D}$ give different anomalies (consistent anomaly and covariant one, see Ref.[19]). In that case, the operators are not related by the total divergence. We find here a similar phenomenon in the Weyl anomaly. We might be able to regard it as the regularization dependence of the Weyl anomaly.

The heat-kernel calculation in Sect.3 is equivalent to the generalized zeta function ($\zeta_{\tilde{D}_2}(s)$) analysis taken in [2]. It is well known $\zeta_{\tilde{D}_2}(s)$ is related with $G_2(x, x; t)$ of (21) as

$$\begin{aligned} Y(t) &\equiv \int d^2x G_2(x, x; t) = \sum_n e^{-\sigma_n t} , \\ \zeta_{\tilde{D}_2}(s) &= \frac{1}{\Gamma(s)} \int_0^\infty Y(t) t^{s-1} dt = \sum_n \sigma_n^{-s} . \end{aligned} \quad (31)$$

Technically Ref.[2] evaluated $\zeta_{\tilde{D}_2}(s)|_{s=0}$, whereas this paper has evaluated $G_2(x, x; t)|_{t=0}$.

We have treated the metric and the dilaton as background (classical) fields. Their quantum effects can be straightforwardly taken into account by following the background formalism. (In fact they are evaluated in [6, 7, 8].) At present, however, it looks that the correct quantum treatment of the metric and the dilaton will be given by the string theory.

The Weyl anomaly is a key factor in the Hawking radiation phenomenon. In the present case of 2 dimension, the energy momentum tensor $T_{\mu\nu}$ is determined by the conformal anomaly and the conservation equation with an appropriate boundary condition.

$$\begin{aligned} \sqrt{g} g^{\mu\nu} T_{\mu\nu} &= \sqrt{g} \left(\frac{1}{24\pi} R - \frac{1}{4\pi} (\nabla\phi)^2 + q_3 \nabla^2 \phi \right) , \\ \nabla^\mu T_{\mu\nu} &= 0 \end{aligned} \quad (32)$$

The different values of q_3 should not affect the physical quantities. If the above equations are right as a physical quantity, they show the energy momentum tensor itself depends on q_3 . It seems to give some trouble about the physical interpretation of the energy momentum tensor and the radiation amplitude. We should notice, however, the following points: 1) we do not take into account the contribution of the quantum effect of the dilaton and the graviton; 2) the q_3 term can be eliminated by introducing a ϕR term in the gravity-dilaton lagrangian in advance: $\Delta\mathcal{L} = -\frac{1}{2}q_3\sqrt{g}\phi R$. This problem looks to involve graviton-dilaton dynamics and "renormalization" of its lagrangian. It is beyond the scope of the present paper.

Acknowledgements

The author thanks Prof.S.D.Odintsov and Prof.S.Noiri for informing their stimulating results and discussions about this subject. He also thanks Prof. K.Fujikawa for an encouraging comment and some corrections in the original manuscript.

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Evaluation of the Heat Kernel in Riemann-Cartan Space using the Covariant Taylor Expansion Method¹

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Abstract

An algorithm to evaluate asymptotic expansion coefficients of the heat kernel associated with a fermion of spin 1/2 in Riemann-Cartan space-time, by applying the covariant Taylor expansion method by Avramidi, is given. The fourth coefficient in the approach is manifestly computed.

1 Introduction

Motivated by studying all one-loop quantities (such as the effective action, zeta function, Green functions, anomalies) in quantum field theory, the heat kernel $K^{(d)}(x, x')$ in d dimensions defined by equations

$$\frac{\partial}{\partial t} K^{(d)}(x, x'; t) = -H K^{(d)}(x, x'; t), \quad (1)$$

$$K^{(d)}(x, x'; 0) = 1|h(x)|^{-1}|h(x')|^{-1}\delta^{(d)}(x, x'), \quad (2)$$

has been studied by many authors [1-8]. Here $\delta^{(d)}(x, x')$ is the d dimensional invariant δ -function, and $1 = \{\delta^A_B\}$, $h = \det h^a_\mu$, where h^a_μ is a vielbein. The most general elliptic second order differential operator H for a fermion $\psi = \{\psi^A(x)\}$ of spin 1/2 in Riemann-Cartan space is expressed in the following form,

$$H = D_\mu D^\mu + 2Q^\mu D_\mu + Z \quad (3)$$

with

$$\begin{aligned} D_\mu &= \partial_\mu + \frac{1}{4}\tilde{\omega}_{ab\mu}\gamma^{ab} + iA_\mu, & \tilde{\omega}_{ab\mu} &= \omega_{ab\mu} + \frac{1}{2}h_a^\alpha h_b^\beta C_{\alpha\beta\mu}, \\ \{\gamma^a, \gamma^b\} &= 2\eta^{ab}, & \gamma^{ab} &= \frac{1}{2}[\gamma^a, \gamma^b], & \gamma^\mu &= h_a^\mu \gamma^a, \end{aligned} \quad (4)$$

where $\omega_{ab\mu}$ is the Ricci coefficient of rotation, $C_{\alpha\beta\mu}$ a totally antisymmetric torsion tensor, and A_μ an arbitrary vector gauge field. The local quantities Q^μ and Z stand for matrices for the spinor with non-differential operator. The metric tensor is given by $g_{\mu\nu} = h_a^\mu h_b^\nu \eta_{ab}$ with $\eta_{ab} = \text{diag}(-1, -1, \dots, -1)$. We assume that $D_\mu h^a_\nu = \partial_\mu h^a_\nu + \tilde{\omega}^a_{b\mu} h^b_\nu - \tilde{\Gamma}^\lambda_{\nu\mu} h^a_\lambda = 0$, where $\tilde{\Gamma}^\lambda_{\nu\mu} = \Gamma^\lambda_{\nu\mu} + (1/2)C^\lambda_{\nu\mu}$, and $\Gamma^\lambda_{\nu\mu}$ is the Christoffel symbol. Then we note that $D_\mu \gamma_\nu = 0$.

The differential operator (3) can be rewritten in the following form [5],

$$H^{(s)} = \tilde{D}_\mu^{(s)} \tilde{D}^{(s)\mu} + X, \quad (5)$$

where

$$\begin{aligned} X &= Z - (D_\mu Q^\mu) - Q_\mu Q^\mu, \\ \tilde{D}_\mu^{(s)} &= D_\mu^{(s)} + (1-s)C_\mu + Q_\mu, & C_\mu &= \frac{1}{8}C_{\alpha\beta\mu}\gamma^{\alpha\beta}, \\ D_\mu^{(s)} &= \partial_\mu + \frac{1}{4}(\omega_{ab\mu} + \frac{s}{2}h_a^\alpha h_b^\beta C_{\alpha\beta\mu})\gamma^{ab} + iA_\mu, \end{aligned} \quad (6)$$

and we assume $D_\mu^{(s)} h^a_\nu = 0$. The commutation relation of $\tilde{D}_\mu^{(s)}$ changes for the parameter s :

$$[\tilde{D}_\mu^{(s)}, \tilde{D}_\nu^{(s)}]I = (\tilde{\Lambda}_{\mu\nu} + sC^\rho_{\mu\nu}\tilde{D}_\rho^{(s)})I, \quad (7)$$

¹This article is that the result of [a4] is added to parts of ref. [5].

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where

$$\begin{aligned}\tilde{\Lambda}_{\mu\nu} &= \Lambda_{\mu\nu} - (C+Q)_{\mu,\nu} + (C+Q)_{\nu,\mu} + [(C+Q)_{\mu}, (C+Q)_{\nu}], \\ \Lambda_{\mu\nu} &= \frac{1}{4}\gamma^{\alpha\beta}R_{\alpha\beta\mu\nu} + iF_{\mu\nu}, \quad F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + i[A_{\mu}, A_{\nu}], \\ R^{ab}{}_{\mu\nu} &= \partial_{\mu}\omega^{ab}{}_{\nu} - \partial_{\nu}\omega^{ab}{}_{\mu} + \omega^{ac}{}_{\mu}\omega_c{}^b{}_{\nu} - \omega^{ac}{}_{\nu}\omega_c{}^b{}_{\mu}.\end{aligned}\quad (8)$$

Here we note that the factor $(C+Q)_{\mu}$ is independent of s . We can choose to be $s=1$ without loss of generality, since $H^{(s)}$ can reduce to $H^{(1)}$ by redefining C_{μ} and Q_{μ} . Hereafter, we focus on the case of $s=1$ ($\tilde{D}_{\mu} \equiv \tilde{D}_{\mu}^{(1)}$).

One can solve (1) by the following expression according to the De Witt's ansatz which reproduces (2) as $t \rightarrow 0$ [2],

$$K^{(d)}(x, x'; t) \sim \frac{\Delta^{1/2}(x, x')}{(4\pi t)^{d/2}} \exp\left(\frac{\sigma(x, x')}{2t}\right) \sum_{q=0}^{\infty} a_q(x, x') t^q, \quad (9)$$

where $\sigma(x, x')$ and $\Delta(x, x')$ are the geodesic interval and the Van Vleck-Morette determinant between x and x' , respectively, and the coefficients $a_q(x, x')$ satisfy the recursive relations³

$$\tilde{D}I \equiv \sigma^{\mu} \tilde{D}_{\mu} I = 0, \quad I(x, x') \equiv a_0(x, x'), \quad (10)$$

$$(q + \tilde{D})a_q = -\Delta^{-1/2} H \Delta^{1/2} a_{q-1}. \quad (11)$$

2 Technique for the calculation of a_q

We give an algorithm for the calculation of the covariant Taylor expansion of a_q according to refs. [3, 5]. The formal operator solution of (11) is written as

$$a_q = I(q + \tilde{D})^{-1} \mathcal{H}(q - 1 + \tilde{D})^{-1} \mathcal{H} \dots (1 + \tilde{D})^{-1} \mathcal{H}, \quad (12)$$

where

$$\mathcal{H} = I^{-1}(-\Delta^{-1/2} H \Delta^{1/2}) I. \quad (13)$$

Expanding a_q in a covariant Taylor series

$$a_q = I \sum_{n=0}^{\infty} |n\rangle \langle n| a_q, \quad (14)$$

where the compact notations⁴

$$|0\rangle \equiv 1, \quad |n\rangle \equiv |\mu'_1 \dots \mu'_n\rangle = \frac{(-1)^n}{n!} \sigma^{\mu'_1} \dots \sigma^{\mu'_n}, \quad \langle n| a_q \rangle = [\tilde{D}_{(\mu_1} \dots \tilde{D}_{\mu_n)} a_q] \quad (15)$$

is used, and noting that $\tilde{D}|n\rangle = n|n\rangle$, one can obtain

$$\begin{aligned}\langle n| a_q \rangle &= \sum_{\substack{\{n_i\} \geq 0 \\ N_q = n}} (q+n)^{-1} (q-1+n_{q-1})^{-1} \dots (1+n_1)^{-1} \\ &\quad \cdot \langle n| \mathcal{H} |n_{q-1}\rangle \langle n_{q-1}| \mathcal{H} |n_{q-2}\rangle \dots \langle n_1| \mathcal{H} |0\rangle,\end{aligned}\quad (16)$$

where $N_q = \sum_{i=1}^q n_i$, and $\langle m| \mathcal{H} |n\rangle$ are matrix elements of the operator $\mathcal{H}$

$$\begin{aligned}\langle m| \mathcal{H} |n\rangle &= \langle \mu_1 \dots \mu_m | \mathcal{H} | \nu_1 \dots \nu_n \rangle = \left[\tilde{D}_{(\mu_1} \dots \tilde{D}_{\mu_m)} \mathcal{H} \frac{(-1)^n}{n!} \sigma^{\nu'_1} \dots \sigma^{\nu'_n} \right] \\ &= - \binom{m}{n-2} 1 \delta_{(\mu_1}^{\nu_1} \dots \delta_{\mu_{n-2}}^{\nu_{n-2}} G^{\nu_{n-1} \nu_n}_{\mu_{n-1} \dots \mu_m}) + \binom{m}{n-1} \delta_{(\mu_1}^{\nu_1} \dots \delta_{\mu_{n-1}}^{\nu_{n-1}} U^{\nu_n}_{\mu_n \dots \mu_m}) \\ &\quad - \binom{m}{n} \delta_{(\mu_1}^{\nu_1} \dots \delta_{\mu_n}^{\nu_n} W_{\mu_{n+1} \dots \mu_m}),\end{aligned}\quad (17)$$

³Hereafter, the comma ", μ " means the Riemannian covariant differentiation $\nabla_{\mu} = \partial_{\mu} + (1/4)\omega_{ab\mu}\gamma^{ab} + iA_{\mu}$, the dot ". μ " the Riemann-Cartan covariant one D_{μ} , and the colon ": μ " the new covariant one $\tilde{D}_{\mu}$. We note that $\psi_{;\mu\nu} = \psi_{;\mu,\nu} + ((1/8)C_{\nu\alpha\beta}\gamma^{\alpha\beta} + Q_{\nu})\psi_{;\mu} - (1/2)C^{\rho}_{\mu\nu}\psi_{;\rho}$ and $\sigma_{;\mu\nu} = \sigma_{;\mu,\nu}$.

⁴Here and everywhere below, the square brackets "[]" mean that the coincidence limits ($x' \rightarrow x$) of two-point quantities are to be taken.

where $\langle m|\mathcal{H}|n\rangle = 0$ at $n > m + 2$ and $n = m + 1$, and

$$\begin{aligned}
G^{\mu\nu}_{\mu_1 \dots \mu_n} &= \sum_{k=0}^n \binom{n}{k} \theta^{(\mu}_{\alpha(\mu_1 \dots \mu_k} \theta^{\nu)\alpha}_{\mu_{k+1} \dots \mu_n)}, \\
U^\nu_{\mu_1 \dots \mu_n} &= -1 G^{\nu\alpha}_{\alpha\mu_1 \dots \mu_n} + 2 \sum_{k=0}^n \binom{n}{k} G^\nu_{\alpha(\mu_1 \dots \mu_k} \mathcal{A}^\alpha_{\mu_{k+1} \dots \mu_n)}, \\
W_{\mu_1 \dots \mu_n} &= X_{:(\mu_1 \dots \mu_n)} + \sum_{k=0}^n \binom{n}{k} \{ G_{\alpha\beta(\mu_1 \dots \mu_k} (-\mathcal{A}^{\alpha\beta}_{\mu_{k+1} \dots \mu_n}) + 1 \zeta^{\alpha\beta}_{\mu_{k+1} \dots \mu_n}) \\
&\quad + G^\alpha_{\beta\alpha(\mu_1 \dots \mu_k} (-\mathcal{A}^\beta_{\mu_{k+1} \dots \mu_n}) + 1 \zeta^\beta_{\mu_{k+1} \dots \mu_n}) \} \\
&\quad + \sum_{\substack{m,k \geq 0 \\ m+k \leq n}} \frac{n!}{k!m!(n-k-m)!} G_{\alpha\beta(\mu_1 \dots \mu_k} \\
&\quad \cdot \{ \mathcal{A}^\alpha_{\mu_{k+1} \dots \mu_{k+m}} \mathcal{A}^\beta_{\mu_{k+m+1} \dots \mu_n} - 1 \zeta^\alpha_{\mu_{k+1} \dots \mu_{k+m}} \zeta^\beta_{\mu_{k+m+1} \dots \mu_n} \}. \quad (18)
\end{aligned}$$

Here $\theta^\alpha_{\beta\mu_1 \dots \mu_n}$, $\zeta_{\mu_1 \dots \mu_n}$ and $\mathcal{A}_{\mu\mu_1 \dots \mu_n}$ are given by

$$\begin{aligned}
\theta^\alpha_{\beta\mu_1 \dots \mu_n} &= - \sum_{k=1}^n \sum_{\substack{\{n_i\} \geq 1 \\ N_k = n}} \frac{n!}{n_1! \dots n_k!} \beta^\alpha_{\lambda_1(\mu_1 \dots \mu_{n_1}} \dots \beta^{\lambda_k}_{\beta|\mu_{N_{k-1}+1} \dots \mu_n)}, \\
\zeta_{\mu_1 \dots \mu_n} &= \sum_{k=1}^n \frac{1}{2k} \sum_{\substack{\{n_i\} \geq 1 \\ N_k = n}} \frac{n!}{n_1! \dots n_k!} \beta^\alpha_{\lambda_1(\mu_1 \dots \mu_{n_1}} \dots \beta^{\lambda_k}_{\alpha|\mu_{N_{k-1}+1} \dots \mu_n)}, \\
\mathcal{A}_{\mu\mu_1 \dots \mu_n} &= \frac{n}{n+1} \left\{ \tilde{\Lambda}_\mu(\mu_1 \dots \mu_n) - \sum_{k=1}^{n-1} \binom{n-1}{k-1} \tilde{\Lambda}_\alpha(\mu_1 \dots \mu_k | \beta^\alpha_{\mu|\mu_{k+1} \dots \mu_n}) \right\}. \quad (19)
\end{aligned}$$

Therefore, all the quantities necessary are expressible in terms of $\beta^\alpha_{\beta\mu_1 \dots \mu_n}$, which has the form

$$\begin{aligned}
\beta^\alpha_{\beta\mu_1 \dots \mu_n} &= \sum_{k=1}^n (-1)^{k+1} \sum_{\substack{\{n_i\} \geq 1, \{s_{ij}\} \geq 0, \{p_i\} \geq 0 \\ N_k + S_{kp_k} = n}} N^{-1}(n_1 \dots n_k; p_1 \dots p_k; s_{11} \dots s_{kp_k}) \\
&\quad \cdot (C^\alpha_{\lambda_{kp_k}(\mu_n \dots \mu_{N_k + S_{kp_k} - s_{kp_k} + 1}} \dots C^{\lambda_{k2}}_{\lambda_{k1}|\mu_{N_k + S_{k-1p_{k-1}} + s_{k1}} \dots \mu_{N_k + S_{k-1p_{k-1}} + 1}} \\
&\quad \cdot K^{\lambda_{k1}}_{\lambda_{k0}|\mu_{N_k + S_{k-1p_{k-1}} \dots \mu_{N_{k-1} + S_{k-1p_{k-1}} + 1}}) \\
&\quad \dots (C^{\lambda_{20}}_{\lambda_{1p_1}|\mu_{n_1 + S_{1p_1}} \dots \mu_{n_1 + S_{1p_1} - s_{1p_1} + 1}} \dots C^{\lambda_{12}}_{\lambda_{11}|\mu_{n_1 + s_{11}} \dots \mu_{n_1 + 1}} K^{\lambda_{11}}_{\beta|\mu_{n_1} \dots \mu_1}) \quad (20)
\end{aligned}$$

where $S_{kp_k} = \sum_{i=1}^k \sum_{j=1}^{p_i} s_{ij}$, and

$$\begin{aligned}
N(n_1 \dots n_k; p_1 \dots p_k; s_{11} \dots s_{kp_k}) &= \frac{1}{(n-1)!} (n_1-1)! \dots (n_k-1)! \prod_{h=1}^k \{ (s_{h1}-1)! \dots (s_{hp_h}-1)! \} \\
&\quad \cdot \prod_{l=1}^{k-1} (N_l + S_{lp_l}) \prod_{m=1}^k \prod_{r=0}^{p_m} (N_m + S_{m-1p_{m-1}} + \sum_{j=1}^r s_{mj} + 1), \\
K^\mu_{\nu\mu_1 \dots \mu_n} &= (n-1) R^\mu_{(\mu_1|\nu|\mu_2 \dots \mu_n)} - \frac{n+1}{2} C^\mu_{\nu(\mu_1 \dots \mu_n)} \\
&\quad + \frac{n-1}{4} \sum_{r=1}^{n-1} \binom{n-2}{r-1} C^\mu_{\rho(\mu_1 \dots \mu_r} C^\rho_{\nu|\mu_{r+1} \dots \mu_n)}. \quad (21)
\end{aligned}$$

It is found by straightforward calculation that all terms containing the torsion tensor in $G^{\mu\nu}_{\mu_1 \dots \mu_n}$ to the ninth order and $\zeta_{\mu_1 \dots \mu_n}$ to the tenth order are cancelled out, and that $\mathcal{A}_{\mu\mu_1 \dots \mu_n}$ to the ninth order

$$\begin{aligned}
& + \frac{44}{15} R_{\rho\mu,\nu} \{ \bar{J}^\rho, \bar{\Lambda}^{\mu\nu} \} - \frac{22}{5} R_{(\rho}^{\mu}{}_{\nu}{}_{\sigma,\lambda)} \{ \bar{\Lambda}_\mu{}^\lambda, \bar{\Lambda}_\nu{}^{\rho\sigma} \} - \frac{22}{5} \mathcal{K}^{\mu\nu}{}_{\rho(2)} \{ \bar{\Lambda}_{\mu\sigma}, \bar{\Lambda}_\nu{}^{(\rho\sigma)} \} \\
& - \frac{64}{45} R_{\mu}^{\nu}{}_{\rho} \bar{\Lambda}_{\mu(\rho\sigma)} \bar{\Lambda}^{\nu\rho\sigma} - \frac{16}{15} R^{\mu\nu\rho\sigma} \{ \bar{\Lambda}_{\mu\nu\rho}, \bar{J}_\sigma \} - \frac{256}{45} R_{(\mu}^{\nu)}{}_{\rho} \bar{\Lambda}_\mu{}^{(\rho\lambda)} \bar{\Lambda}_\nu{}^{(\sigma\lambda)} - \frac{32}{45} R^{\mu\nu} \bar{J}_\mu \bar{J}_\nu \\
& - \left(\frac{6}{5} \mathcal{K}_{\mu\nu(4)} + \frac{17}{40} R_{\mu\tau\rho\sigma} R_{\nu}{}^{\tau\rho\sigma} + \frac{17}{60} R_{\mu\rho} R_{\nu}{}^{\rho}{}_{\lambda} \bar{\Lambda}^{\mu\lambda} \bar{\Lambda}^\nu{}_\lambda - \left(\frac{24}{5} \mathcal{K}_{\mu\nu\rho\sigma(2)} + \frac{17}{40} R_{\mu\tau} R_{\nu}{}^{\tau}{}_{\rho\sigma} \right. \right. \\
& + \frac{17}{40} R_{\nu\tau} R_{\sigma\mu\rho} + \frac{17}{30} R_{\mu\nu\kappa\lambda} R_{\rho\sigma}{}^{\kappa\lambda} + \frac{17}{60} R_{\mu\kappa\sigma\lambda} R_{\nu}{}^{\kappa}{}_{\rho}{}^\lambda + \frac{51}{80} R_{\mu\rho\kappa\lambda} R_{\nu\sigma}{}^{\kappa\lambda} \left. \right) \bar{\Lambda}^{\mu\rho} \bar{\Lambda}^{\nu\sigma} \\
& + 1 \left(-\frac{7}{18} \mathcal{M}_{(8)} + R^{\mu\nu} \left(-\frac{10}{63} \mathcal{K}_{\mu\nu(6)} + \frac{5}{7} \mathcal{M}_{\mu\nu(4)} \right) - \frac{20}{21} R^{\mu\rho\nu\sigma} \mathcal{K}_{\rho\sigma\mu\nu(4)} + \frac{1}{3} R^{\mu}{}_{\mu} \mathcal{M}_{\mu(4)} \right. \\
& + 2 R^{\mu\nu,\rho} \mathcal{M}_{\mu\nu\rho(2)} - \frac{5}{3} \mathcal{K}^{\rho\sigma\mu}{}_{(2)} \mathcal{K}_{\rho\sigma\mu(4)} - \frac{20}{9} R^{\rho\alpha\sigma\beta,\mu} \mathcal{K}_{\rho\sigma\mu\alpha\beta(2)} + \frac{18}{25} R_{(\rho\sigma,\mu\nu)} R^{\rho\sigma,\mu\nu} \\
& + \frac{27}{25} \mathcal{M}_{\mu\nu(2)} \mathcal{M}^{\mu\nu}{}_{(2)} - \frac{48}{25} \mathcal{K}_{\rho\sigma\mu\nu(2)} \mathcal{K}^{\rho\sigma\mu\nu}{}_{(2)} - \frac{6}{25} \mathcal{K}_{\mu\nu(4)} \mathcal{K}^{\mu\nu}{}_{(4)} - \frac{16}{25} R_{\rho(\alpha|\sigma|\beta,\mu\nu)} R^{\rho(\alpha|\sigma|\beta,\mu\nu)} \\
& + \mathcal{K}_{\mu\nu(4)} \left(\frac{179}{1575} R^\mu{}_\alpha R^{\nu\alpha} - \frac{68}{525} R^{\mu\alpha\beta\gamma} R^\nu{}_{\alpha\beta\gamma} + \frac{8}{105} R^{\mu\alpha\nu\beta} R_{\alpha\beta} \right) \\
& + \mathcal{M}_{\mu\nu(2)} \left(\frac{4}{5} R^\mu{}_\alpha R^{\nu\alpha} + \frac{6}{25} R^{\mu\alpha\beta\gamma} R^\nu{}_{\alpha\beta\gamma} - \frac{16}{25} R^{\mu\alpha\nu\beta} R_{\alpha\beta} \right) \\
& + \mathcal{K}_{\alpha\beta\mu\nu(2)} \left(\frac{4}{5} R^\mu{}_\alpha R^{\nu\beta} + \frac{16}{105} R^{\alpha\beta} R^{\mu\nu} + \frac{64}{105} R^\mu{}_\rho R^{\alpha\nu\beta\rho} - \frac{1088}{1575} R^\alpha{}_\rho R^{\mu\beta\nu\rho} \right. \\
& - \frac{544}{1575} R^{\alpha\rho\mu\sigma} R^\beta{}_\rho{}^\nu{}_\sigma - \frac{1088}{1575} R^{\alpha\mu\rho\sigma} R^{\beta\nu}{}_{\rho\sigma} \left. \right) + \frac{8}{25} R_{(\alpha\beta,\mu\nu)} R^\mu{}_\rho{}^\nu{}_\sigma R^{\alpha\rho\beta\sigma} \\
& + R_{\alpha(\mu|\beta|\nu,\rho\sigma)} \left(-\frac{1088}{1575} R^{\mu\alpha\nu}{}_\gamma R^{\rho\beta\sigma\gamma} + \frac{32}{105} R^{\mu\nu} R^{\rho\alpha\sigma\beta} \right) \\
& + R^\rho{}_\sigma \left(-\frac{10}{21} \mathcal{K}^{\sigma\alpha\beta}{}_{(2)} \mathcal{K}_{\rho\alpha\beta(2)} - \frac{20}{63} R_{(\mu}{}^\sigma{}_{\nu}{}_{\tau},{}_\alpha) R^\mu{}_\rho{}^\nu{}_\tau{}^\alpha + \frac{3}{7} R_{\mu(\rho|\nu|\sigma,\alpha)} \mathcal{K}^{\mu\nu\alpha}{}_{(2)} \right. \\
& + \frac{3}{14} \mathcal{K}_{\mu\nu\rho(2)} \mathcal{K}^{\mu\nu\sigma}{}_{(2)} + \frac{3}{7} R_{\mu(\alpha|\nu|\beta,\rho)} R^{\mu(\alpha|\nu|\beta,\sigma)} + \frac{1}{9} R_{,\rho} R^\sigma + \frac{1}{15} R^{\mu}{}_{\mu} \mathcal{K}^\sigma{}_{\rho\mu(2)} + \frac{2}{3} R^{\mu}{}_{\mu} \mathcal{K}_{\mu\rho(2)}^\sigma \\
& - \frac{7}{15} R_{\alpha\beta,\mu} R^{\alpha\sigma\beta}{}_\rho{}^\mu + 2 R_{(\mu\nu,\rho)} \mathcal{K}^{\sigma\mu\nu}{}_\rho{}_{(2)} + \frac{1}{2} R_{(\alpha\beta,\rho)} R^{(\alpha\beta,\sigma)} \left. \right) \\
& + R^\mu{}_{(\rho}{}^\nu{}_{\sigma)} \left(-\frac{40}{21} R_{(\mu}{}^\tau{}_{\nu},{}_\gamma) \mathcal{K}^{\sigma\tau}{}_{\gamma(2)} - \frac{20}{21} \mathcal{K}^{\sigma\tau}{}_{\mu(2)} \mathcal{K}^{\rho}{}_{\tau\nu(2)} - \frac{40}{21} R_{(\alpha}{}^\tau{}_{\beta,\mu)} \mathcal{K}^{\sigma\tau\alpha\beta}{}_\nu \right. \\
& + \frac{2}{15} R_{(\mu}{}^\sigma{}_{\nu},{}_\alpha) R^\alpha + \frac{2}{15} R_{,\mu} \mathcal{K}^{\rho\sigma}{}_\nu{}_{(2)} + \frac{6}{5} R_{(\alpha\beta,\mu)} \mathcal{K}^{\rho\sigma\alpha\beta}{}_\nu + \frac{3}{5} \mathcal{K}^{\rho\sigma\alpha}{}_{(2)} R_{(\mu\nu,\alpha)} - R_{(\mu\nu,\alpha)} R^{(\rho\sigma,\alpha)} \left. \right) \\
& + \frac{44}{525} R^\mu{}_\nu R^\nu{}_\alpha R^\alpha{}_\beta R^\beta{}_\mu - \frac{32}{315} R^\mu{}_\alpha R^{\nu\alpha} R^{\rho\sigma} R_{\mu\rho\nu\sigma} + \frac{302}{1575} R^\mu{}_\alpha R^{\nu\alpha} R_{\mu\lambda\rho\sigma} R_{\nu}{}^{\lambda\rho\sigma} \\
& + \frac{8}{225} R_{\mu\nu} R^{\alpha\beta} R^{\mu\rho\nu\sigma} R_{\alpha\rho\beta\sigma} - \frac{866}{4725} R_{\mu\nu} R_{\alpha\beta} R^{\mu\alpha\rho\sigma} R^{\nu\beta}{}_{\rho\sigma} + \frac{404}{4725} R^{\mu\nu} R_{\rho\sigma} R^\rho{}_{\alpha\mu\beta} R^{\sigma\alpha}{}_{\nu}{}^\beta \\
& - \frac{8}{175} R^{\mu\nu} R_{\mu\alpha\nu\beta} R^\alpha{}_{\lambda\rho\sigma} R^{\beta\lambda\rho\sigma} + \frac{16}{4725} R^\mu{}_\nu R_{\mu\rho\lambda\sigma} R^{\rho\alpha\sigma\beta} R^\nu{}_\alpha{}^\lambda{}_\beta - \frac{8}{675} R^\mu{}_\nu R_{\mu\lambda\rho\sigma} R^{\rho\sigma\alpha\beta} R^{\nu\lambda}{}_{\alpha\beta} \\
& - \frac{1}{105} R^\mu{}_{\lambda\rho\sigma} R^{\nu\lambda\rho\sigma} R^{\mu\alpha\beta\gamma} R^\nu{}_{\alpha\beta\gamma} - \frac{32}{4725} R^{\mu\rho\nu\sigma} R_{\rho\alpha\sigma\beta} (R_{\mu\gamma}{}^\alpha{}_\delta R_{\nu}{}^{\gamma\beta\delta} - R_{\mu\gamma\nu\delta} R^{\alpha\gamma\beta\delta}) \\
& - \frac{247}{9450} R^{\mu\nu}{}_{\alpha\beta} R^{\alpha\beta}{}_{\rho\sigma} R^{\rho\sigma}{}_{\gamma\delta} R^{\gamma\delta}{}_{\mu\nu} + \frac{232}{4725} R^{\alpha\beta\rho\sigma} R_{\mu\nu\rho\sigma} R^{\mu\gamma}{}_\alpha{}^\delta{}_\nu R^{\nu}{}_{\gamma\beta\delta} \\
& - \frac{299}{4725} R^{\alpha\beta\rho\sigma} R^{\mu\nu}{}_{\rho\sigma} R_{\alpha\mu\gamma\delta} R_{\beta\nu}{}^{\gamma\delta} \Big), \tag{22}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{K}_{\alpha\beta\mu_1\cdots\mu_n} &\equiv R_{\alpha(\mu_1|\beta|\mu_2,\mu_3\cdots\mu_n)}, & \mathcal{K}^{\alpha\beta}{}_{\mu_1\cdots\mu_n(2k)} &\equiv \mathcal{K}^{\alpha\beta}{}_{\mu_1\cdots\mu_n\lambda_1\cdots\lambda_k}{}^{\lambda_1\cdots\lambda_k}, \\
\mathcal{M}_{\mu_1\cdots\mu_n} &\equiv \mathcal{K}^{\rho\rho\mu_1\cdots\mu_n} = R_{\mu_1\mu_2,\mu_3\cdots\mu_n}, & \mathcal{M}_{\mu_1\cdots\mu_n(2k)} &\equiv \mathcal{M}_{\mu_1\cdots\mu_n\lambda_1\cdots\lambda_k}{}^{\lambda_1\cdots\lambda_k}, \\
\tilde{\Lambda}^\alpha{}_{\mu_1\cdots\mu_n(2k)} &\equiv \tilde{\Lambda}^\alpha{}_{\mu_1\cdots\mu_n\lambda_1\cdots\lambda_k}{}^{\lambda_1\cdots\lambda_k}, & X_{\mu_1\cdots\mu_n} &\equiv X_{!(\mu_1\cdots\mu_n)}, \tag{23}
\end{aligned}$$

and the exclamation mark "!" means a convenient covariant differentiation as follows⁵

$$\begin{aligned} I_{! \mu} &= (\nabla + C + Q)_{\mu} I = I_{, \mu} + (C + Q)_{\mu} I, \\ I_{! \mu \nu} &= (\nabla + C + Q)_{\nu} I_{! \mu} = I_{! \mu, \nu} + (C + Q)_{\nu} I_{! \mu}, \\ \tilde{\Lambda}_{\mu \nu ! \rho} &= [(\nabla + C + Q)_{\rho}, \tilde{\Lambda}_{\mu \nu}] = \tilde{\Lambda}_{\mu \nu, \rho} + [(C + Q)_{\rho}, \tilde{\Lambda}_{\mu \nu}]. \end{aligned} \quad (24)$$

The first two coefficients $[a_1]$ and $[a_2]$ in this approach agree with those calculated in [6, 7]. In the case that $C_{\mu} = Q_{\mu} = 0$, the coefficients $[a_3]$ and $[a_4]$ coincide with those calculated in [3, 8].

4 Discussion

The first derivative term $s C^{\rho}_{\mu \nu} \tilde{D}_{\rho}^{(s)} I$ in r.h.s. of the commutator (7) and the contorsion term of $\tilde{\Gamma}^{\lambda}_{\nu \mu}$ depend on the parameter s . However, it is shown by the above calculation that their terms don't influence the final forms of $[a_q]$ ($q = 1, 2, 3, 4$), because the contributions of those terms are cancelled each other in the covariant expansion coefficients of $G^{\mu \nu}$, ζ and $\mathcal{A}_{\mu}$. In the first four coefficients $[a_q]$, the torsion tensor is included only in $\tilde{\Lambda}_{\mu \nu}$, X and their derivatives, in the form of $(C + Q)_{\mu}$ except for X itself. Therefore, the coefficients $[a_q]$ obtained by starting from $H^{(s)}$ are equal to those obtained by starting from $H^{(0)}$, irrespective of s . The dependence on the torsion tensor for $\tilde{\Lambda}_{\mu \nu}$, X and the differential operator gives a rule that $[a_q]$ obtained by starting from $H^{(s)}$ in Riemann-Cartan space is derived from the final form of $[a_q]$ obtained by starting from $\nabla_{\mu} \nabla^{\mu} + Z$ in Riemannian space by the simple replacements,

$$Z \rightarrow X, \quad \Lambda_{\mu \nu} \rightarrow \tilde{\Lambda}_{\mu \nu}, \quad \nabla_{\mu} \rightarrow \tilde{D}_{\mu}^{(0)}. \quad (25)$$

The covariant Taylor expansion for a function $f(x')$ at x^{μ} with respect to $\sigma^{\mu'}(x, x')$ includes the expansion [6] for the function with respect to the geodesic normal coordinate $y^{\mu} = x'^{\mu} - x^{\mu}$ satisfying

$$\tilde{\Gamma}^{\mu}_{\alpha \beta}(y) y^{\alpha} y^{\beta} = \Gamma^{\mu}_{\alpha \beta}(y) y^{\alpha} y^{\beta} = 0, \quad (26)$$

and

$$y^{\mu} \tilde{\Omega}_{\mu}(x') \equiv y^{\mu} \left\{ \frac{1}{4} \gamma^{ab} \omega_{ab \mu} + C_{\mu} + Q_{\mu} + i A_{\mu} \right\} (y) = 0. \quad (27)$$

In the case that x is close to x' , the conditions (26) and (27) are obtained from $\tilde{D} \sigma^{\mu} = \sigma^{\mu}$ and $\sigma^{\mu} \tilde{D}_{\mu} I = 0$, respectively, because $\sigma^{\mu'} \approx y^{\mu}$, $I \approx 1$ and $I_{, \mu} \approx 0$. The geodesic normal coordinate expansion for $g^{\mu \nu}$, $\log |g|$ ($g = \det g_{\mu \nu}$) and $\tilde{\Omega}_{\mu}$ at x^{μ} with respect to y^{μ} is expressed by the covariant expansion coefficients for $G^{\mu \nu}$, ζ and $\mathcal{A}_{\mu}$, respectively,

$$\begin{aligned} g^{\mu \nu}(x') &= \sum_{n=0}^{\infty} \frac{1}{n!} G^{\mu \nu}_{\mu_1 \dots \mu_n}(x) y^{\mu_1} \dots y^{\mu_n}, \\ \log |g|(x') &= 4 \sum_{n=2}^{\infty} \frac{1}{n!} \zeta_{\mu_1 \dots \mu_n}(x) y^{\mu_1} \dots y^{\mu_n}, \\ \tilde{\Omega}_{\mu}(x') &= \sum_{n=1}^{\infty} \frac{1}{n!} \mathcal{A}_{\mu \mu_1 \dots \mu_n}(x) y^{\mu_1} \dots y^{\mu_n}. \end{aligned} \quad (28)$$

Therefore, even in Riemann-Cartan space, the expansion for $g^{\mu \nu}(x')$ and $\log |g|(x')$ doesn't depend on the torsion tensor, up to the ninth and the tenth orders with respect to y^{μ} , respectively. The torsion tensor is included only in $\tilde{\Lambda}_{\mu \nu}$ and its derivatives in the expansion coefficients of $\tilde{\Omega}_{\mu}(x')$ to the ninth order, in the form of $(C + Q)_{\mu}$. If the dependence on the torsion tensor for each covariant Taylor expansion coefficients for $G^{\mu \nu}$, ζ and $\mathcal{A}_{\mu}$ is held at all orders of the expansion, then we may conclude that the torsion tensor appears only in $\tilde{\Lambda}_{\mu \nu}$, X and their derivatives in the coefficients $< n | a_q >$, in the form of $(C + Q)_{\mu}$ except for X itself.

⁵Here the differentiation of "!" denotes that of " μ " or $\tilde{D}_{\mu}$ in the case that the torsion tensor in $\tilde{\Gamma}^{\lambda}_{\nu \mu}$ is eliminated. We note that, for example,

$$\begin{aligned} I_{! \mu \nu} &= I_{\mu \nu} - (1/2) C^{\rho}_{\mu \nu} I_{, \rho}, \\ \tilde{\Lambda}_{\mu \nu ! \rho} &= \tilde{\Lambda}_{\mu \nu, \rho} + (1/2) C^{\lambda}_{\mu} \tilde{\Lambda}_{\lambda \nu} - (1/2) \tilde{\Lambda}_{\mu \lambda} C^{\lambda}_{\nu \rho}. \end{aligned}$$

Acknowledgements

I am grateful to Professor V. P. Frolov for valuable suggestion. I would like to thank the members of the Institute for Theoretical Physics at the University of Alberta for their hospitality.

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Exact Solution for the Motion of Two-Charged Particles in (1+1)-dim Dilaton Gravity with a Cosmological Constant

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Abstract

The canonical formalism for dynamics of charged particles in (1+1)-dim dilaton gravity is developed. The reduced Hamiltonian for a system of particles is given as a solution to a kind of transcendental equation. The canonical equations of motion is proved to be identical with the geodesic equations given by a starting action. The formalism is extended to the theory with a cosmological constant, in which the Hamiltonian describes a conservative system in contrast to (3+1)-dim case.

1 Introduction

We extend our result [1] of the exact solution of the particles' motion in 1+1-dim dilaton gravity to the system of charged particles. This is based on the canonical formalism [2] developed for particle dynamics. For example, the Hamiltonian for the system of two identical particles was given by

$$H = \sqrt{p^2 + m^2} + \epsilon p \operatorname{sgn}(r) - 8 \frac{W\left(-\frac{\kappa}{8}(|r| \sqrt{p^2 + m^2} - \epsilon p r)\right) \exp\left[\frac{\kappa}{8}(|r| \sqrt{p^2 + m^2} - \epsilon p r)\right]}{\kappa|r|} \quad (1)$$

where W is Lambert W -function defined by

$$W(z)e^{W(z)} = z.$$

This result shows the following characteristics :

- (1) Not only the Hamiltonian but also the canonical equations are given by W function, since the derivative of W is expressed in terms of W itself.
- (2) It has a correct Newtonian limit.
- (3) It gives expansions to an arbitrary large order in either the coupling constant, the mass, or the center of inertia momentum.
- (4) It gives also an expression in strong coupling.
- (5) For large energy the trajectory in phase space becomes more S -shaped, which is described by the second branch of W function. This provides a new set of solutions which is not a relativistic perturbation of the Newtonian case.

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2 A system of charged particles

Since Einstein's theory is trivial in 1+1-dim, a general covariant version of Newtonian gravity in 1+1-dim necessitates to introduce a dilaton [3], though of course it was inspired also from the string theories [4]. We treat the 1+1-dim $R = T$ theory which contains many analogous characteristics with 3+1 gravity [3, 5].

The action integral for a system of charged particles is

$$I = \int dx^2 \left[\frac{1}{2\kappa} \sqrt{-g} g^{\mu\nu} \left\{ \Psi R_{\mu\nu} + \frac{1}{2} \nabla_\mu \Psi \nabla_\nu \Psi \right\} - A_\mu \mathcal{F}^{\mu\nu}{}_{,\nu} + \frac{1}{4\sqrt{-g}} \mathcal{F}^{\mu\nu} \mathcal{F}^{\alpha\beta} g_{\mu\alpha} g_{\nu\beta} + \frac{1}{2} \sqrt{-g} C \right. \\ \left. + \sum_a \int d\tau_a \left\{ -m_a \left(-g_{\mu\nu}(x) \frac{dz_a^\mu}{d\tau_a} \frac{dz_a^\nu}{d\tau_a} \right)^{1/2} + e_a \frac{dz_a^\mu}{d\tau_a} A_\mu(x) \right\} \delta^2(x - z_a(\tau_a)) \right] \quad (2)$$

Here we introduced a subtraction term $C = \frac{1}{4}(\sum_a e_a)^2$ to make the energy of electric field definite. This is necessary even in the flat (1+1)-dimension and this term does not contribute to the equations of motion. For a system of zero total charge the subtraction term vanishes. Later we relax the term C to be a free parameter, which is equivalent to introducing a cosmological constant.

Rewrite the action in the first order form

$$I = \int dx^2 \left\{ \sum_a p_a \dot{z}_a \delta(x - z_a(t)) + \pi \dot{\gamma} + \Pi \dot{\Psi} + N_0 R^0 + N_1 R^1 \right\} \quad (3)$$

where $\gamma \equiv g_{11}$, $N_0 \equiv (-g^{00})^{-1/2}$, $N_1 \equiv g_{10}$, and p_a, π and Π are the canonically conjugate momenta to the variables z_a, γ and Ψ , respectively, and

$$R^0 = -\kappa \sqrt{\gamma} \gamma \pi^2 + 2\kappa \sqrt{\gamma} \pi \Pi + \frac{1}{4\kappa \sqrt{\gamma}} (\Psi')^2 - \frac{1}{\kappa} \left(\frac{\Psi'}{\sqrt{\gamma}} \right)' - \frac{1}{2} \sqrt{\gamma} (E^2 - C) - \sum_a \sqrt{\frac{p_a^2}{\gamma} + m_a^2} \delta(x - z_a(t)) \quad (4)$$

$$R^1 = \frac{\gamma'}{\gamma} \pi - \frac{1}{\gamma} \Pi \Psi' + 2\pi' + \sum_a \frac{p_a}{\gamma} \delta(x - z_a(t)) \quad (5)$$

The electric field $E \equiv \mathcal{F}^{01}$ satisfies the constraint

$$\frac{\partial E}{\partial x} = \sum_a e_a \delta(x - z_a(t))$$

and is given by

$$E = \frac{1}{2} \sum_a e_a |x - z_a(t)|' \quad (6)$$

Transformation of generator and setting the coordinate conditions proceed just the same way in the case of no charge [2]. The coordinate condition we impose is

$$\gamma = 1 \quad \text{and} \quad \Pi = 0. \quad (7)$$

This condition is quite analogous to ADM condition in (3+1)-dim and plays an essential role in establishing particle dynamics.

The action (3) is reduced to

$$I_R = \int dx^2 \left\{ \sum_a p_a \dot{z}_a \delta(x - z_a) - \mathcal{H} \right\} \quad (8)$$

and the reduced Hamiltonian for the system is

$$H = \int dx \mathcal{H} = -\frac{1}{\kappa} \int dx \Delta \Psi. \quad (9)$$

Dilaton field Ψ satisfies the constraints

$$\Delta\Psi = \frac{1}{4}(\Psi')^2 - \kappa^2(\chi')^2 - \frac{\kappa}{2}(E^2 - C) - \kappa \sum_a \sqrt{p_a^2 + m_a^2} \delta(x - z_a) \quad (10)$$

$$\Delta\chi = -\frac{1}{2} \sum_a p_a \delta(x - z_a) \quad (\chi' = \pi) \quad (11)$$

For the integration (9) to be definite we set the boundary condition

$$\Psi^2 - 4\kappa^2\chi^2 = 0 \quad \text{in the outer regions } z_{1(2)} < x \text{ and } x < z_{2(1)} \quad (12)$$

which is inferred from the iterations.

3 Hamiltonian for two charged particles

For a system of two particles we can choose the center of inertia (C.I.) system ($p_1 = -p_2 = p$). Solution to (11) is

$$\chi = -\frac{1}{4}p(|x - z_1| - |x - z_2|) - \epsilon X x + \epsilon C_X. \quad (13)$$

By setting $\Psi = -4\log|\phi|$, Eq.(10) becomes

$$\Delta\phi - \frac{1}{4} \left\{ \kappa^2(\chi')^2 + \kappa V \right\} \phi = \frac{\kappa}{4} \left\{ \sqrt{p^2 + m_1^2} \phi(z_1) \delta(x - z_1) + \sqrt{p^2 + m_2^2} \phi(z_2) \delta(x - z_2) \right\} \quad (14)$$

where

$$V \equiv \frac{1}{2}(E^2 - C) = \begin{cases} 0 & (+) \text{ region} \\ -\frac{1}{2}\epsilon_1\epsilon_2 & (0) \text{ region} \\ 0 & (-) \text{ region} \end{cases}. \quad (15)$$

The solution in each region is (for $z_2 < z_1$)

$$\begin{cases} \phi_+(x) = A_+ e^{\frac{1}{2}\kappa X x} + B_+ e^{-\frac{1}{2}\kappa X x} \\ \phi_0(x) = A_0 e^{\frac{1}{2}K_0 x} + B_0 e^{-\frac{1}{2}K_0 x} \\ \phi_-(x) = A_- e^{\frac{1}{2}\kappa X x} + B_- e^{-\frac{1}{2}\kappa X x} \end{cases} \quad (16)$$

where

$$K_0 = \sqrt{\kappa^2(X - \frac{1}{2}\epsilon p)^2 - \frac{1}{2}\kappa\epsilon_1\epsilon_2}.$$

Here we assume that the product of two charges is smaller than $2\kappa(X - \frac{1}{2}\epsilon p)^2$. In the opposite case we have to choose the solution in (0) region as

$$\phi_0(x) = A_0 \sin \frac{1}{2} K_0 x + B_0 \cos \frac{1}{2} K_0 x$$

with $K_0 = \sqrt{\frac{1}{2}\kappa\epsilon_1\epsilon_2 - \kappa^2(X - \frac{1}{2}\epsilon p)^2}$.

From the matching conditions at $x = z_1, z_2$ the equation for X is derived

$$\begin{aligned} & \left(\kappa\sqrt{p^2 + m_1^2} - 2\tilde{K}_0 - 2\kappa X \right) \left(\kappa\sqrt{p^2 + m_2^2} - 2\tilde{K}_0 - 2\kappa X \right) \\ & = \left(\kappa\sqrt{p^2 + m_1^2} + 2\tilde{K}_0 - 2\kappa X \right) \left(\kappa\sqrt{p^2 + m_2^2} + 2\tilde{K}_0 - 2\kappa X \right) e^{K_0 r} \end{aligned} \quad (17)$$

where K_0 is replace by $\tilde{K}_0$

$$\tilde{K}_0 \equiv \sqrt{\kappa^2 \left(X - \frac{1}{2}\epsilon\tilde{p} \right)^2 - \frac{1}{2}\kappa\epsilon_1\epsilon_2}, \quad \tilde{p} \equiv p \frac{z_1 - z_2}{r}, \quad r \equiv |z_1 - z_2|.$$

Hamiltonian is given by

$$H = -\frac{1}{\kappa} \int dz \Delta \Psi = -\frac{1}{\kappa} [\Psi']_{-\infty}^{\infty} = 4X . \quad (18)$$

Then Eq.(17) is the determining equation for the Hamiltonian

$$\begin{aligned} & \left(2\kappa\sqrt{p^2 + m_1^2} - J - \kappa H \right) \left(2\kappa\sqrt{p^2 + m_2^2} - J - \kappa H \right) \\ & = \left(2\kappa\sqrt{p^2 + m_1^2} + J - \kappa H \right) \left(2\kappa\sqrt{p^2 + m_2^2} + J - \kappa H \right) e^{\frac{1}{2}Jr} . \end{aligned} \quad (19)$$

where

$$J \equiv \sqrt{\kappa^2(H - 2\epsilon\tilde{p})^2 - 8\kappa e_1 e_2} \quad (20)$$

It is generally and also explicitly proved that the canonical equations derived from the Hamiltonian (18) is identical with the geodesic equation derived from the starting action (2).

Though in the case of no charge the solution to (19) was expressed in terms of generalized W function [1], we need to define a further generalization of W . The equation (19) gives expansions in any of the parameters (coupling constant, mass, momentum, and charge). For example the expansion to the second order of κ is

$$\begin{aligned} H = & \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} - \frac{1}{2}e_1 e_2 r \\ & + \frac{\kappa}{4} \left\{ \sqrt{p^2 + m_1^2} \sqrt{p^2 + m_2^2} r - \epsilon p (\sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2}) (z_1 - z_2) + p^2 r \right. \\ & - \frac{1}{4}e_1 e_2 (\sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2}) r^2 + \frac{1}{2}\epsilon e_1 e_2 p r (z_1 - z_2) + \frac{1}{24}(e_1 e_2)^2 r^3 \left. \right\} \\ & + \frac{\kappa^2}{4^2} \left\{ \frac{1}{2 \times 4} r^2 (\sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} - 2\epsilon\tilde{p} - \frac{1}{2}e_1 e_2 r)^3 \right. \\ & + \frac{1}{3 \times 4^2} e_1 e_2 r^3 (\sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} - 2\epsilon\tilde{p} - \frac{1}{2}e_1 e_2 r)^2 \\ & - \frac{1}{6 \times 4^2} r^2 \left[12(\sqrt{p^2 + m_1^2} - \sqrt{p^2 + m_2^2})^2 + (e_1 e_2)^2 r^2 \right] (\sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} - 2\epsilon\tilde{p} - \frac{1}{2}e_1 e_2 r) \\ & + \frac{1}{3 \times 4^2} e_1 e_2 r^3 (\sqrt{p^2 + m_1^2} - \sqrt{p^2 + m_2^2})^2 - \frac{1}{15 \times 4^3} (e_1 e_2)^3 r^5 \left. \right\} \\ & + \mathcal{O}(\kappa^3) \end{aligned} \quad (21)$$

Since H is a constant of motion, a trajectory in the (r, p) plane is uniquely determined by setting $H = H_0$ in (18). For H_0 sufficiently small and small charge, the trajectory can be regarded as a relativistic perturbation of the Newtonian case. As e gets larger in a unlike-sign charge the phase space trajectories contract toward the origin. On the other hand in a like-sign charge situation, gravity and electric force compete against one another and the trajectories show the peculiar change as the magnitude of e increases.

4 Condition for static balance

In (3+1)-dim Einstein theory the problem of the exact balance condition between gravitational attractive force and electrical repulsive force has not yet been solved. There has been a controversy between two standpoints.

(I) Balance condition should be the same as one in a flat space-time [6]

$$\frac{1}{2}\kappa m_1 m_2 - e_1 e_2 = 0 . \quad (22)$$

However an exact solution under this condition has not been found.

(II) Perturbative solution in the post-Newtonian approximation leads to the balance condition [7]

$$e_i = \pm \sqrt{\frac{\kappa}{2}} m_i . \quad (23)$$

It is evident that this is a sufficient condition, but there is no proof of a necessary condition.

In (1+1)-dim the necessary and sufficient conditions for static balance are

$$p = 0 \quad \text{and} \quad \frac{\partial H}{\partial r} = 0 \quad (24)$$

which lead to (owing to (19))

$$\frac{1}{2} \kappa m_1 m_2 - e_1 e_2 = 0 \quad (H = m_1 + m_2) \quad (25)$$

5 Extension to the case of a cosmological constant

The canonical formalism of a gravitational field in an expanding universe was first developed by S. Nariani and T. Kimura in 1962 [8]. They treated the gravitational field in $k = 0$ universe in a fluid model and defined the Hamiltonian which is equivalent, in a linear approximation, to a system of damped harmonic oscillators and hence represents a non-conservative system.

In (1+1)-dim the situation is different, since no graviton exists. When a cosmological term $\frac{1}{2\kappa} \sqrt{-g} \Lambda_0$ is added to the starting action (2), we can formulate the Hamiltonian dynamics quite parallelly.

To show the effect of Λ_0 to the dynamics in our coordinate condition (7), let's first solve the field equations for a single source (mass M , charge $e = 0$)

$$\begin{aligned} \kappa N_0 (\pi - \Pi) + N'_1 &= 0 \\ \dot{\Psi} &= -2\kappa N_0 \pi + N_1 \Psi' \\ -\kappa \pi^2 + \frac{1}{4\kappa} (\Psi')^2 - \frac{1}{\kappa} \Delta \Psi + \frac{1}{2\kappa} \Lambda_0 - M \delta(x) &= 0 \\ \pi' &= 0 \\ \dot{\pi} + N_0 \left\{ \frac{3\kappa}{2} \pi^2 + \frac{1}{8\kappa} (\Psi')^2 - \frac{1}{4\kappa} \Lambda_0 \right\} + N_1 \pi' + \frac{1}{2\kappa} N'_0 \Psi' + N'_1 \pi &= 0 \\ \partial_1 \left(\frac{1}{2} N_0 \Psi' + N'_0 \right) &= 0 . \end{aligned} \quad (26)$$

The solutions to the above equations are

$$\begin{aligned} N_0 &= e^{\frac{\kappa M}{4} |x|}, \quad N_1 = \epsilon \frac{\sqrt{\frac{M^2}{16} + \frac{1}{2\kappa^2} \Lambda_0}}{\frac{M}{4}} \frac{x}{|x|} \left[e^{\frac{\kappa M}{4} |x|} - 1 \right] \\ \pi &= -\epsilon \sqrt{\frac{M^2}{16} + \frac{1}{2\kappa^2} \Lambda_0}, \quad \Psi = -\frac{\kappa M}{2} |x| + 2\kappa \epsilon \sqrt{\frac{M^2}{16} + \frac{1}{2\kappa^2} \Lambda_0} t \end{aligned} \quad (27)$$

Then the equations of motion for a test particle (mass μ) and its Hamiltonian are given by

$$\begin{aligned} \dot{p} &= -\sqrt{p^2 + \mu^2} \frac{\partial N_0(z)}{\partial z} + p \frac{\partial N_1(z)}{\partial z} \\ \dot{z} &= \frac{p}{\sqrt{p^2 + \mu^2}} N_0(z) - N_1(z) \end{aligned} \quad (28)$$

$$\begin{aligned}
H &= M + \sqrt{p^2 + \mu^2} N_0(z) - p N_1(z) \\
&= M + \sqrt{p^2 + \mu^2} e^{\frac{\kappa M}{4}|z|} - \epsilon p \sqrt{1 + \frac{8\Lambda_0}{\kappa^2 M^2}} \frac{z}{|z|} \left[e^{\frac{\kappa M}{4}|z|} - 1 \right] .
\end{aligned} \tag{29}$$

The ϕ equation for a system of two charged particles now becomes

$$\begin{aligned}
\Delta\phi - \frac{1}{4} \left\{ \kappa^2 (\chi')^2 + \frac{\kappa}{2} V - \frac{1}{2} \Lambda \right\} \phi \\
= \frac{\kappa}{4} \left\{ \sqrt{p_1^2 + m_1^2} \phi(z_1) \delta(x - z_1) + \sqrt{p_2^2 + m_2^2} \phi(z_2) \delta(x - z_2) \right\}
\end{aligned} \tag{30}$$

where $\Lambda \equiv \Lambda_0 - \kappa C$ is an effective cosmological constant.

In this case we have to choose the boundary condition

$$\Psi_{\pm}^2 - 4\kappa^2 \chi_{\pm}^2 + 2\Lambda x^2 = C_{\pm} x \quad \text{for } (+) \text{ and } (-) \text{ regions} \tag{31}$$

The determining equation and the Hamiltonian are

$$\begin{aligned}
&\left(\kappa \sqrt{p_1^2 + m_1^2} - 2K_0 - 2K_+ \right) \left(\kappa \sqrt{p_2^2 + m_2^2} - 2K_0 - 2K_- \right) \\
&= \left(\kappa \sqrt{p_1^2 + m_1^2} + 2K_0 - 2K_+ \right) \left(\kappa \sqrt{p_2^2 + m_2^2} + 2K_0 - 2K_- \right) e^{K_0 r}
\end{aligned} \tag{32}$$

and

$$H = \frac{2(K_+ + K_-)}{\kappa} \tag{33}$$

where

$$\begin{cases} K_+ = \sqrt{\kappa^2 \left(X + \frac{\epsilon}{4} (\tilde{p}_1 + \tilde{p}_2) \right)^2 - \frac{1}{2} \Lambda} & (+) \text{ region} \\ K_0 = \sqrt{\kappa^2 \left(X - \frac{\epsilon}{4} (\tilde{p}_1 - \tilde{p}_2) \right)^2 - \frac{\epsilon}{2} e_1 e_2 - \frac{1}{2} \Lambda} & (0) \text{ region} \\ K_- = \sqrt{\kappa^2 \left(X - \frac{\epsilon}{4} (\tilde{p}_1 + \tilde{p}_2) \right)^2 - \frac{1}{2} \Lambda} & (-) \text{ region} \end{cases} \tag{34}$$

In this general case the Hamiltonian includes many parameters as $\kappa, m_1, m_2, e_1, e_2$ and Λ . Nevertheless, we can get the explicit expansions of the Hamiltonian in any parameter and also the exact canonical equations. For example in the case of two identical particles with no charge the Hamiltonian in the perturbation of κ and Λ/κ^2 is

$$\begin{aligned}
H &= 2\sqrt{p^2 + m^2} + \frac{\kappa}{4} (\sqrt{p^2 + m^2} - \epsilon \tilde{p})^2 r + \frac{\kappa^2}{4^2} (\sqrt{p^2 + m^2} - \epsilon \tilde{p})^3 r^2 \\
&+ \frac{7\kappa^3}{6 \times 4^3} (\sqrt{p^2 + m^2} - \epsilon \tilde{p})^4 r^3 - \frac{\Lambda}{2\kappa} \cdot \frac{\epsilon \tilde{p}}{\sqrt{p^2 + m^2}} r - \frac{\Lambda}{16} \cdot \frac{\epsilon \tilde{p} m^2}{p^2 + m^2} r^2 \\
&+ \frac{\Lambda^2}{4\kappa^3} \cdot \frac{\epsilon \tilde{p}}{(p^2 + m^2)^{3/2}} r + \dots
\end{aligned} \tag{35}$$

It is important that in (1+1)-dim the Hamiltonian (??) describes a conservative system of charged particles.

6 Conclusion

We derived the determining equation for the Hamiltonian of charged particles in (1+1)-dim dilaton gravity theory with a cosmological constant. The Hamiltonian is expressed in terms of the space integral of the second derivative of the dilaton field which is given as a solution to the constraint equations. This Hamiltonian describes a conservative system and the canonical equations derived from the Hamiltonian are proved to be identical with the geodesic equations. All metric components are also determined.

Corresponding to the magnitudes and the signs of both charge and cosmological constant, the trajectories in phase space and configuration space show drastic changes which are not seen in the Newtonian case. Numerical analysis of these trajectories will be reported in another occasion.

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Gravitational effects on a light Dirac particle

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Abstract

We discuss the gravitational effects on a light Dirac particle which propagates in Schwarzschild geometry. We perform our calculations in the weak field limit. The two-component Weyl equations with the corrections due to the infinitesimal mass and the gravitational field are derived from the covariant Dirac equation. Our approach is applied to neutrino oscillations in the presence of the gravitational field. The relative phase of the two different mass eigenstates is calculated in radial propagation, and the result is compared with the previous work.

1 Introduction

It had been thought that the physical phenomena involving both quantum effects and gravitational effects were far beyond our reach, before Colella et al. made an elegant experiment [1] using a neutron interferometer. Their analysis was based on inserting the Newtonian gravitational potential into the non-relativistic Schrödinger equation. In spite of the simplicity, it is very interesting that the quantum effects and the gravitational effects appear simultaneously in the physical phenomenon. Recently, the gravitational effects on another physical phenomenon, neutrino oscillations, have been much discussed [2], [3], [4], [5], [6], [7], [8]. These two have common ground that the gravitational effects appear in the quantum interference. However, there are some differences between the two. In the former case, the spatial spread of the wave function plays a significant role, whereas in the latter case, the existence of the different mass eigenstates is important. Moreover, it is another important difference whether the related particle is non-relativistic or ultra-relativistic.

It seems that the controversy about the gravitationally induced neutrino oscillation phases arises. The authors of Ref. [2] state that the phases amount to approximately 20 % of the kinematical counterparts in the vicinity of a neutron star. Nevertheless, the definition of the neutrino energy was not clear in Ref. [2]. On the other hand, some groups [4], [5], [6], [8] have obtained similar results for a radially propagating neutrino; the results are evidently different from that in Ref. [2]. However, the authors of Ref. [4] assume that the different mass eigenstates are produced at the different times. This assumption seems to be questionable because the relative phase initially becomes arbitrary. These papers except Ref. [5] are based on previous paper [9] in which the classical action is taken as a quantum phase. Therefore, the effects due to the spin are not considered in these papers. Despite of the use of the covariant Dirac equation, the authors of Ref. [5] also calculate the classical action along the path in the end.

In this situation, we shall provide a simple framework different from the previous work for studying the gravitational effects on a Dirac particle with an infinitesimal mass such as a neutrino. (Neutrino mass might be zero.) In particular, we consider the propagation in Schwarzschild geometry. We do not merely calculate the classical action along the particle trajectory, but start with the covariant Dirac equation. Our approach will allow us to extend the discussion to the gravitational field arising from a rotating object. We shall perform our calculations in the weak field limit, and derive the two-component Weyl equations with the corrections due to the small mass and the gravitational field. Furthermore, we shall discuss neutrino oscillations in the presence of the gravitational field.

The plan of this paper is as follows. In Sec. 2, we start with the covariant Dirac equation, and derive the Hamiltonian in the weak field limit. Performing an unitary transformation, we shall obtain an effective Hamiltonian for a light Dirac particle. We discuss an application to neutrino oscillations in the

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presence of the gravitational field in Sec. 3. For simplicity, we consider a radially propagating neutrino. We then compare our result with the previous work. Conclusions are given in Sec. 4.

2 Propagation of a light Dirac particle in curved space-time

2.1 Covariant Dirac equation in Schwarzschild space-time

We now consider a light Dirac particle which propagates in Schwarzschild geometry. We restrict ourselves to the weak field limit. In this case, the Schwarzschild line element can be written as

$$ds^2 \simeq \left(1 + 2\frac{\phi}{c^2}\right) c^2 dt^2 - \left(1 - 2\frac{\phi}{c^2}\right) (dx^2 + dy^2 + dz^2), \quad (1)$$

where ϕ is the Newtonian gravitational potential, $\phi \equiv -GM/r$ with mass M and $r = \sqrt{x^2 + y^2 + z^2}$.

Let us start with the Dirac equation in the presence of the gravitational field. The covariant Dirac equation is given by (See, e.g., [10], [11])

$$[i\hbar\gamma^\mu (\partial_\mu - \Gamma_\mu) - mc]\Psi = 0, \quad (2)$$

where the covariant Dirac matrices γ^μ satisfy the relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} I_4, \quad (3)$$

with the 4×4 unit matrix I_4 , and Γ_μ is the spin connection.

Here we introduce the orthogonal tetrad $e^\mu_{(a)}$ defined as

$$g_{\mu\nu} e^\mu_{(a)} e^\nu_{(b)} = \eta_{ab}, \quad (4)$$

where $\eta_{ab} = \text{diag}(c^2, -1, -1, -1)$. Then constant matrices $\gamma^{(a)}$ defined as

$$\gamma^\mu = \gamma^{(a)} e^\mu_{(a)}, \quad (5)$$

satisfy the relations

$$\gamma^{(a)} \gamma^{(b)} + \gamma^{(b)} \gamma^{(a)} = 2\eta^{ab} I_4. \quad (6)$$

Using the tetrad and $\gamma^{(a)}$, the spin connection can be represented by

$$\Gamma_\mu = -\frac{1}{8} [\gamma^{(a)}, \gamma^{(b)}] g_{\lambda\nu} e^\lambda_{(a)} \nabla_\mu e^\nu_{(b)}. \quad (7)$$

We choose the tetrad

$$e^\mu_{(a)} \simeq \text{diag}\left(1 - \frac{\phi}{c^2}, 1 + \frac{\phi}{c^2}, 1 + \frac{\phi}{c^2}, 1 + \frac{\phi}{c^2}\right). \quad (8)$$

Since the tetrad is diagonal, the covariant Dirac equation (2) can be written as

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \Psi &= H \Psi \\ &= c^2 \left(e^0_{(0)}\right)^{-1} \gamma^{(0)} \left[\gamma^{(i)} e^i_{(i)} \bar{p}_i + i\hbar \gamma^{(a)} e^\mu_{(a)} \Gamma_\mu + mc \right] \Psi, \end{aligned} \quad (9)$$

where $\bar{p}_i$ is the momentum operator in flat space-time.

Following Ref. [12], we redefine the spinor and the Hamiltonian as

$$\Psi \rightarrow \gamma^{1/4} \Psi, \quad H \rightarrow \gamma^{1/4} H \gamma^{-1/4}, \quad (10)$$

where γ is the determinant of the spatial metric. In this case, the definition of the inner product becomes the same as in flat space-time. Under this redefinition, we have

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \Psi &= H \Psi \\ &= \left[\gamma^{(0)} \gamma^{(i)} c^2 \bar{p}_i + \gamma^{(0)} \gamma^{(i)} c^2 \left(\frac{\phi}{c^2} \bar{p}_i + \bar{p}_i \frac{\phi}{c^2} \right) + \gamma^{(0)} mc^3 + \gamma^{(0)} mc^3 \frac{\phi}{c^2} \right] \Psi. \end{aligned} \quad (11)$$

We adopt the Weyl representation as the Dirac matrices:

$$\gamma^{(0)} = \frac{1}{c} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^{(i)} = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad (12)$$

where I is the 2×2 unit matrix and σ_i are the Pauli spin matrices. Then in the case of a massless particle and flat space-time, we have the well-known two-component Weyl equations in the form

$$i\hbar \frac{\partial}{\partial t} \psi = \pm c \boldsymbol{\sigma} \cdot \vec{p} \psi, \quad (13)$$

where ψ is the two-component spinor.

2.2 Effective Hamiltonian for a light Dirac particle

As mentioned previously, we consider the situation that the particle has the infinitesimal mass, namely, the rest energy of the particle is much smaller than the kinematic energy. We perform an appropriate unitary transformation similar to the Foldy-Wouthuysen-Tani (FWT) [13], [14] transformation. Thereby we can transform the Hamiltonian derived in the last subsection to the "even" operator up to the order of our interest:

$$\begin{aligned} H \rightarrow H' &= U H U^\dagger \\ &= \begin{pmatrix} H_R & 0 \\ 0 & H_L \end{pmatrix} + O\left(\frac{m^3 c^6}{p^2 c^2}, \frac{\phi^2}{c^4}\right). \end{aligned} \quad (14)$$

The four-component spinor is also divided into each of the two-component spinors:

$$\Psi' = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \quad (15)$$

where subscript L and R correspond to the left-handed and right-handed component, respectively.

We restrict our attention to the left-handed component. The Hamiltonian for the left-handed component is given by

$$\begin{aligned} H_L &= - \left[c \boldsymbol{\sigma} \cdot \vec{p} + \frac{1}{c^2} (\phi c \boldsymbol{\sigma} \cdot \vec{p} + c \boldsymbol{\sigma} \cdot \vec{p} \phi) \right. \\ &\quad \left. + \frac{1}{2} m^2 c^3 \frac{\boldsymbol{\sigma} \cdot \vec{p}}{p^2} + \frac{1}{c^2} \frac{1}{8} m^2 c^3 \left\{ \phi \frac{\boldsymbol{\sigma} \cdot \vec{p}}{p^2} + \frac{\boldsymbol{\sigma} \cdot \vec{p}}{p^2} \phi - \left(\frac{1}{p^2} \phi \boldsymbol{\sigma} \cdot \vec{p} + \boldsymbol{\sigma} \cdot \vec{p} \phi \frac{1}{p^2} \right) \right\} \right]. \end{aligned} \quad (16)$$

The details of the calculations will be given in a full length paper to be published later. From this, it is obvious that H_L is hermitian. It is, however, sometimes convenient to rewrite Eq. (16) in the following form:

$$\begin{aligned} H_L &= - \left\{ \left[1 + \frac{1}{c^2} \left(\phi + \vec{p} \cdot \phi \vec{p} \frac{1}{p^2} + 2 \frac{GM}{r^3} \vec{L} \cdot \vec{S} \frac{1}{p^2} \right) \right] c \vec{p} \right. \\ &\quad \left. + \left\{ 1 + \frac{1}{c^2} \frac{1}{4} \left(\phi - \frac{1}{p^2} \phi p^2 + \frac{1}{p^2} \vec{p} \cdot \phi \vec{p} - \vec{p} \cdot \phi \vec{p} \frac{1}{p^2} \right. \right. \right. \\ &\quad \left. \left. \left. + 2 \frac{1}{p^2} \frac{GM}{r^3} \vec{L} \cdot \vec{S} - 2 \frac{GM}{r^3} \vec{L} \cdot \vec{S} \frac{1}{p^2} \right) \right\} \frac{m^2 c^3}{2p} \right] \frac{\boldsymbol{\sigma} \cdot \vec{p}}{p}, \end{aligned} \quad (17)$$

where $\vec{L} = \vec{x} \times \vec{p}$ is the angular momentum operator in flat space-time, and $\vec{S} = \hbar \boldsymbol{\sigma} / 2$ is the spin of the particle. In radial propagation, the angular momentum vanishes. Hence, there is no spin effect in this case. This consequence is consistent with the previous work [10].

Consequently we can obtain the two-component Weyl equations implying the corrections due to the infinitesimal mass and the general relativistic gravitational effects.

3 Application to neutrino oscillations

Finally, we shall consider the application of the two-component equation derived in the last section to neutrino oscillations in the presence of the gravitational field. (See, e.g., [15] for the analysis in flat space-time) In neutrino oscillations, the most important one is the phase difference of the two mass eigenstates. Hence we shall merely consider the phase shift of the particle. In particular, we shall investigate the radial propagation. Then the spin-orbit coupling vanishes.

We regard the terms due to the neutrino mass and the gravitational effects as perturbations, and consider the equation in the following form:

$$i\hbar \frac{\partial}{\partial t} \psi_L = (H_{0L} + \Delta H_L) \psi_L, \quad (18)$$

where H_{0L} denotes the Hamiltonian in the unperturbed system, and ΔH_L denotes the corrections due to the small mass and the gravitational effects. We now assume that the two-component spinor ψ_L is given by

$$\psi_L(x, t) = e^{i\Phi(t)} \psi_{0L}(x, t), \quad (19)$$

where ψ_{0L} satisfy the equation

$$i\hbar \frac{\partial}{\partial t} \psi_{0L} = H_{0L} \psi_{0L}. \quad (20)$$

Substituting Eq. (19) into Eq. (18), we have

$$\Phi = -\frac{1}{\hbar} \int^t \Delta H_L dt. \quad (21)$$

Furthermore, we assume

$$\frac{\sigma \cdot \bar{p}}{\bar{p}} \psi_{0L} = -\psi_{0L}, \quad (22)$$

and

$$\bar{p} \psi_{0L} = p \psi_{0L}. \quad (23)$$

Under these assumptions, from Eq. (17) we have

$$\begin{aligned} \Delta H_L = & \frac{1}{c^2} \left(\phi + \bar{p} \cdot \phi p \frac{1}{p^2} \right) cp \\ & + \frac{m^2 c^3}{2p} + \frac{1}{c^2} \left(\phi - \frac{1}{\bar{p}^2} \phi p^2 + \frac{1}{\bar{p}^2} \bar{p} \cdot \phi p - \bar{p} \cdot \phi p \frac{1}{p^2} \right) \frac{m^2 c^3}{8p}. \end{aligned} \quad (24)$$

However, we do not have a practical prescription for the inverse momentum operator $1/\bar{p}$. We here replace the remaining quantum numbers by the classical ones. This is a kind of semi-classical approximation. Thereby we have

$$\Delta H_L = 2 \frac{\phi}{c^2} cp + \frac{m^2 c^3}{2p}. \quad (25)$$

Let us consider the case that the neutrino is produced at a space-time point (r_A, t_A) , and propagate to a space-time point (r_B, t_B) . Using Eq. (21) and Eq. (25), we can obtain the phase the neutrino acquires in radial propagation:

$$\Phi = -\frac{1}{\hbar} \int_{t_A}^{t_B} \left(2 \frac{\phi}{c^2} cp + \frac{m^2 c^3}{2p} \right) dt. \quad (26)$$

From a perturbative point of view, it is natural that the integral in Eq. (26) is calculated along the path in the unperturbed system. Therefore, we can write the phase as follows:

$$\Phi = -\frac{1}{\hbar} \int_{r_A}^{r_B} \left(2 \frac{\phi}{c^2} cp + \frac{m^2 c^3}{2p} \right) \frac{1}{c} dr. \quad (27)$$

We pay attention to the term related to m^2 , because neutrino oscillations take place as the result of the mass square difference. This term reduces to

$$-\frac{m^2 c^3}{2\hbar pc}(r_B - r_A). \quad (28)$$

We now consider the case that the two different mass eigenstates having a common three-momentum p propagate along a same path. Then the relative phase $\Delta\Phi$ of the two different mass eigenstates is given by

$$\Delta\Phi = \frac{\Delta m_{ij}^2 c^3}{2\hbar pc}(r_B - r_A), \quad (29)$$

where Δm_{ij}^2 is the mass square difference between the mass eigenstate $|m_i\rangle$ and $|m_j\rangle$.

At first sight this result for the radially propagating case seems to be in agreement with that in Ref. [5], [6], [8], but we have obtained this with the quite different approach. The question is whether pc in our calculations corresponds to the definitions of the energy in Ref. [5], [6], [8]. Since p in our calculations is the momentum in unperturbed system, pc can be interpreted as the energy in flat space-time for a massless particle. On the other hand, E used in Ref. [5], [8] is the energy at infinity for a massive neutrino, and that in Ref. [6] for a massless neutrino. From this, we have an agreement with especially Ref. [6].

4 Conclusion

We have studied the gravitational effects on a light Dirac particle by deriving the two-component Weyl equations with the corrections due to the small mass and the general relativistic gravitational effects. We have performed our calculations in spherically symmetric, static Schwarzschild space-time, and restricted ourselves to the weak field limit. Furthermore, we have discussed the gravitationally induced neutrino oscillation phases for a radially propagating neutrino. Comparing our result with previous work, we have found that our result is consistent with that in Ref. [6].

We have not pursued the non-radial case in this paper. However, it is interesting to investigate this case because the spin-orbit coupling comes into play. Moreover, the extension into the gravitational field arising from a rotating object, which can be described by the Kerr metric, is another interesting topic. The merit of the simple framework we have provided in this paper is to make this extension straightforward. These issues will be discussed elsewhere [16].

Although it is difficult to provide the verification of these effects with current experimental detectability, we think that the investigation of these topics in which both quantum effects and gravitational effects come into play is important. Progress in technology may make the verification possible.

Acknowledgements

We would like to thank H. Asada for helpful discussions and comments.

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