

Proceedings of the Sixth Workshop
on

**GENERAL RELATIVITY
AND
GRAVITATION**

The International House, Tokyo Institute of Technology
December 2-5, 1996

edited by

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PREFACE

The Sixth Workshop on General Relativity and Gravitation was held at the International House, Tokyo Institute of Technology from December 2 to 5, 1996. The total number of the participants was about 120 and there were 60 talks.

In recent years, there has been remarkable progress in observational and theoretical investigations of the Universe. The interest of researchers on General Relativity and Gravitation broadened to a wide range of the related fields: Relativistic Astrophysics, Gravitational Waves, Black Holes, Large Scale Structures, Early Universe, Unified Theories including String Theories, Quantum Gravity and so on. The main purpose of this series of workshop is to review the latest works in these fields. We believe that there were stimulating discussions and interactions between the participants.

This workshop was supported by Grant-in-Aid for Scientific Research (A) (No.06302021) and Grant-in-Aid for Creative Basic Research (No.08NP0801) from the Japanese Ministry of Education, Science and Culture.

The organizing committee of the workshop consisted of:

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We hope this proceedings is useful for all researchers in General Relativity and Gravitation.

H.Ishihara

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Pair Plasma Production in Black Hole Magnetospheres

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Abstract

We qualitatively solve the problem of plasma supply to the stationary, axisymmetric, force-free magnetosphere of a rotating black hole. In the neighbourhood of the ‘null surface’ where the Goldreich-Julian charge density vanishes will be situated the pair creation zone, or ‘gap’ with strong parallel electric field $E_{\parallel}$ coming from shortage of charges and resulting acceleration of migratory electrons/positrons to the relativistic regime. Background photons are inverse Compton scattered by energetic electrons/positrons to γ -rays to interact with background photons, to produce plasma sufficiently to carry energy/angular momentum throughout the force-free magnetosphere. The width H of the gap is determined as an eigenvalue problem with a *free* boundary. It is demonstrated that sufficient amount of pair plasma can be supplied stationarily for the Blandford-Znajek process to work well especially when the background radiation field becomes relatively less luminous.

1 Introduction

The model of a stationary, plasma-filled magnetosphere around a rotating black hole has been evoked, to explain energy extracted and liberated in the vicinity of a black hole and going to the formation and maintenance of relativistic jets observed in active galactic nuclei ([Blandford 1976]; [Lovelace 1996]; [Blandford and Znajek 1977]). For the recent observation of high-energy γ -ray emission from AGNs the Blandford-Znajek process is described as a viable mechanism for energizing the γ -ray jets (see e.g. Montignony et al. 1995 for recent review of observations).

In the γ -ray emitting region, relativistic jets must be dominated by the kinetic energy flux of purely $e^{\pm}$ plasma over the electromagnetic flux originating from the central engine. A pair-cascade model of γ -ray jets has been proposed by Blandford and Levinson (1995; BL95): Once the bulk motion of the jets accelerates to the Lorentz factor Γ larger than the threshold energy above which the opacity to pair production on the background photons exceeds unity, copious pair production would ensue, leading to a sharp increase of the inertia and radiative drag acting upon the $e^{\pm}$ outflow. This is a phenomenon which is expected to occur above the annihilation radius r_{ann} for the $e^{\pm}$ plasma.

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On the other hand, in the region well below r_{ann} where rapid annihilation is dominating, the jet energy may well be regarded predominantly electromagnetic. The source of the electromagnetic power eventually resulting in the γ -ray jet may be the spin of the central black hole ([Blandford 1989], 1993). It is however the electric current that sustains the electromagnetic power in the field-dominated region of the magnetosphere, and it is charged particles, even though inertia-free, that carry the electric current. However, even though under the assumption of massless, it must be real charged particles that constitute the electric current. Therefore, electrons must flow into the horizon for the electric current to flow out, and positrons must flow into there for the electric current to flow in. In other words, there must be a plasma source between these two inflowing and outflowing regions. We thus have to contrive the process of plasma supply in the force-free magnetosphere.

We assume a stationary axisymmetric degenerate force-free magnetosphere around a rotating magnetized black hole surrounded by the accretion disk. In the magnetosphere, the plasma creation gap is possibly situated near the surface of $\rho_{\text{GJ}} = 0$ between the two light surfaces, where ρ_{GJ} is defined as ([Beskin et al. 1992])

$$\rho_{\text{GJ}} \equiv -\frac{1}{4\pi} \nabla \cdot \left(\frac{\Omega_{\text{F}} - \omega}{2\pi\alpha c} \nabla \Psi \right), \quad (1)$$

where α is the lapse function or the redshift factor, which vanishes at the horizon and becomes unity at large distances.

The surface of $\rho_{\text{GJ}} = 0$ (say at $r = r_0$) is penetrated by the field lines along which the poloidal electric current steadily flows into/out of the horizon. It will be in this layer that there at first happens the shortage of charged particles, which gives rise to strong electric field parallel to poloidal field lines, $E_{\parallel}$. If the gap becomes completely empty, the resulting field equals the amount by which to expel the Goldreich-Julian charge density out of the gap i.e. $\text{div} E_{\parallel} = -4\pi\rho_{\text{GJ}}$. In the (quasi-)stationary state we assume the existence of both the $E_{\parallel}$ field and the charge density ρ_e in the thin gap with width H , where ρ_e is the difference of $e^{\pm}$ number densities $n^{\pm}$ multiplied by unit charge e , i.e. $\rho_e = e(n^{+} - n^{-})$. The Poisson equation then reads in the gap

$$\nabla \cdot E_{\parallel} = 4\pi(\rho_e - \rho_{\text{GJ}}). \quad (2)$$

In spite of core significance in feeding the relativistic γ -ray jet from the central engine, the real process of supplying charged particles in the magnetosphere has not been substantiated for a realistic model of radiation field in AGNs until [Beskin et al. 1992]. For the typical parameters of the central radiation in AGN, they showed that the region of pair creation can actually be situated near the ‘null’ or ‘critical’ surface separating the regions of opposite signs of ρ_{GJ} along field lines.

The purpose of this paper is to further extend the analysis of Beskin et al. (1992) and to clarify more qualitatively the micro process of interaction of radiation – pair creation under the existence of parallel electric field. In the next section, we formulate basic equations describing pair production cascade in the magnetosphere. We then solve them in §3 and demonstrate that sufficient amount of plasma is supplied by the cascade so that hole’s rotational energy may be extracted effectively. In the final section, we sum up the results and discuss some implications.

2 Physical Processes and Basic Equations

As we have discussed in the previous section, the longitudinal electric field $E_{\parallel}$ can no longer be screened near the null surface from which both the inflows and the outflows originate, and a thin electrostatic potential gap will be formed there. We assume that the acceleration of $e^{\pm}$'s by $E_{\parallel}$ is perpendicular to the null surface; in this case, the acceleration works most effectively. We can then write the Poisson equation (2) into the form

$$\frac{dE_{\parallel}}{dx} = 4\pi[e(n^{+} - n^{-}) - \rho_{GJ}], \quad (3)$$

where x is the outwardly increasing coordinate along the field line.

In the potential gap, the $e^{\pm}$'s will lose their perpendicular momentum rapidly owing to Compton scatterings with ambient UV photons. However, their longitudinal motion will be maintained by the electrostatic acceleration due to $E_{\parallel}$ field. Therefore, the motion of a single e^{+} or e^{-} obeys

$$mc^2 \frac{d\Gamma}{dx} = eE_{\parallel} - \Gamma^2 \sigma_T U_b, \quad (4)$$

where Γ is the Lorentz factor of longitudinal motion, σ_T is the Thomson cross section and U_b is the energy density of the background radiation field. The $e^{\pm}$'s migrate *one-dimensionally* and their motion is *mono-energetic* with a single Lorentz factor $\Gamma(x)$, owing to the Compton drag represented in the second term. As Beskin et al. (1992) pointed out, the $e^{\pm}$'s emit high energy γ -ray photons neither by curvature radiation nor by synchrotron radiation. Nevertheless, these $e^{\pm}$'s can produce sufficient amount of high energy γ -ray photons via *inverse Compton scatterings* of background X-ray photons. As a result, $e^{\pm}$'s and γ -ray photons may move one-dimensionally along the magnetic field and the $e^{\pm}$ spectra may be mono-energetic. Under these conditions, assuming stationarity, we next formulate basic equations describing the pair production cascade.

First, let us consider the continuity equations of $e^{\pm}$'s. For simplicity, we shall suppose that positrons are migrating outwards while electrons inwards. Then the continuity equations become

$$\pm \frac{d}{dx} \left(n^{\pm}(x) \sqrt{1 - 1/\Gamma^2(x)} \right) = \int_0^{\infty} \eta_p(\epsilon_{\gamma}) \left[F^{+}(x, \epsilon_{\gamma}) + F^{-}(x, \epsilon_{\gamma}) \right] d\epsilon_{\gamma}, \quad (5)$$

where η_p is the angle-averaged pair production redistribution function (e.g., [Berestetskii et al. 1989]). $F^{\pm}(x, \epsilon_{\gamma})$ are the number density of γ -ray photons propagating in the $\pm x$ -directions, respectively, at a position x and in the energy interval $\epsilon_{\gamma} \sim \epsilon_{\gamma} + d\epsilon_{\gamma}$; $dN_s/d\epsilon_s$ refers to the number density of background soft photons in the energy interval $\epsilon_s \sim \epsilon_s + d\epsilon_s$. It must be noted that the γ -ray photons, which are produced by the inverse Compton scatterings, are highly aberrated to move in $\pm x$ direction.

We can easily see that the current, which is carried by $e^{\pm}$'s along a given field line inside the gap, must be conserved along x . From one combination of equations (5) we have

$$\left[n^{+}(x) + n^{-}(x) \right] \sqrt{1 - 1/\Gamma^2(x)} = \frac{j_0}{e} \equiv N_0 \quad (6)$$

where the current density $j_0 \equiv eN_0$ is constant along a field line. In order that the energy and angular momentum may be extracted effectively from a rotating supermassive black hole, j_0 must take the value of (e.g. [Thorne et al. 1986])

$$j_0 \sim 10^{-13} \left(\frac{a}{M} \right) \left(\frac{M}{10^8 M_\odot} \right) \left(\frac{B}{10^4 \text{G}} \right) \frac{\text{abampere}}{\text{cm}^2}. \quad (7)$$

in order of magnitude. Another combination of equations (5) gives

$$\frac{d}{dx} \left\{ [n^+(x) - n^-(x)] \sqrt{1 - 1/\Gamma^2(x)} \right\} = 2 \int_0^\infty \eta_p(\epsilon_\gamma) [F^+(x, \epsilon_\gamma) + F^-(x, \epsilon_\gamma)] d\epsilon_\gamma. \quad (8)$$

Instead of equations (5), we use equations (6) and (8) in the following.

We next derive the Boltzmann equations for the γ -ray photons. As we noted in the previous subsection, we may regard the γ -ray photons as directed only in the $\pm x$ -direction. Thus the γ -ray distribution functions $F^\pm$ obey

$$\pm \frac{d}{dx} F^\pm(x, \epsilon_\gamma) = \eta_c n^\pm \sqrt{1 - 1/\Gamma^2} - \eta_p F^\pm, \quad (9)$$

where $\eta_c(\epsilon_\gamma, \Gamma)$ is the Compton redistribution function (e.g., [Rybicki and Lightman 1979]).

To integrate the five equations (3), (8) and (9), we need to make some assumptions about the background radiation field, $dN_s/d\epsilon_s$. We shall suppose $dN_s/d\epsilon_s$ to be represented by a single power-law, i.e.

$$\frac{dN_s}{d\epsilon_s} \propto \epsilon_s^{-\alpha} \quad (\epsilon_{\min} = 10 \text{eV}/mc^2 < \epsilon_s < \epsilon_{\max} = 30 \text{keV}/mc^2), \quad (10)$$

The normalization factor is defined so that the energy density of the background radiation field may become U_b . For the Eddington background luminosity, U_b becomes $\sim 1.6 \times 10^6$ ergs/s.

2.1 Boundary Conditions

The purpose of this paper is to elucidate the physical processes of pair production cascade taking place in a magnetosphere surrounding a supermassive black hole in an AGN. To achieve this end, as a first step, we consider in this paper the case in which the functions $E_\parallel$, $n^\pm$, $F^\pm$ have the symmetric properties as follows:

$$E_\parallel(x) = E_\parallel(-x), \quad \Gamma(x) = \Gamma(-x), \quad n^+(x) = n^-(-x), \quad \text{and} \quad F^+(x) = F^-(-x). \quad (11)$$

The solutions obtained under these symmetry properties do not, of course, cover the whole solutions allowed in the black hole magnetosphere. However, we believe that the symmetric solutions represent the essential features of pair production cascade in the black hole magnetosphere. In view of these symmetry properties, it follows that it is sufficient for us to solve equations (3), (4), and (9) only in the range $0 \leq x \leq H$.

Let us now return to the discussion on boundary conditions at $x = 0$ and $x = H$. In the first place, from the first two equations in (11), we have

$$\frac{dE_{\parallel}}{dx} = 0 \quad \text{at } x = 0, \quad \text{and} \quad \frac{d\Gamma}{dx} = 0 \quad \text{at } x = 0. \quad (12)$$

In the second place, the third equation in (11) and (6) give

$$2n^+ \sqrt{1 - 1/\Gamma^2} = j_0/e \quad \text{at } x = 0. \quad (13)$$

Furthermore, the last equation in (11), we obtain another boundary condition

$$F^+ = F^- \quad \text{at } x = 0. \quad (14)$$

It must be noted that the first condition of (12) is automatically satisfied by (13).

Let us next consider the boundary conditions at the outer boundary of the gap. First, the *free* boundary, $x = H$, is defined so that $E_{\parallel}$ vanishes at this point. That is,

$$E_{\parallel} = 0 \quad \text{at } x = H. \quad (15)$$

Secondly, charge density distribution must be continuous at the boundary $x = H$. Outside of the gap, charge density $e(n^+ - n^-)$ must coincide with ρ_{GJ} . We therefore impose the condition

$$\frac{1}{4\pi} \frac{dE_{\parallel}}{dx} = \frac{j_0}{c\sqrt{1 - 1/\Gamma^2}} - Ax = 0 \quad \text{at } x = H. \quad (16)$$

Thirdly, any electron should not come out of the gap to the outside ($x > H$). This requires $n^-(H) = 0$, which combines with equation (6) to yield

$$n^+ \sqrt{1 - 1/\Gamma^2} = j_0/e \quad \text{at } x = H. \quad (17)$$

Finally, no γ -ray photons come into the gap from the outside; therefore we readily find

$$F^- = 0 \quad \text{at } x = H. \quad (18)$$

We shall seek the solutions satisfying the boundary conditions described (13)-(18), making use of a shooting method. The calculation starts from $x = 0$ with conditions (12) - (14), and is terminated at the

free boundary at $x = H$ with conditions (15), (17), and (18). In general, condition (16) is not satisfied. To solve this problem, in this paper, we adjust j_0 so that (16) may be satisfied.

In a realistic black hole magnetosphere, j_0 will be determined by a global requirement. That is, j_0 should be introduced as a free parameter in a local problem such as we are considering. Nevertheless, the physical importance of seeking the solution by adjusting j_0 cannot be overemphasized, because the 'adjusted' j_0 corresponds to the maximum allowed value and hence to the maximum energy extraction from the hole. For example, for very small values of j_0 , equation (3) gives $E_{\parallel} \approx E_{\parallel}(0) - 2\pi Ax^2$. Therefore, the boundary can always be defined at $x = \sqrt{E_{\parallel}(0)/2\pi A}$. ($dE_{\parallel}/dx$ never vanishes there.) As j_0 increases significantly from such a small value, the first term in the right-hand-side of (3) contributes especially near the boundary, because $n^+ - n^-$ monotonically increases with x . Therefore,

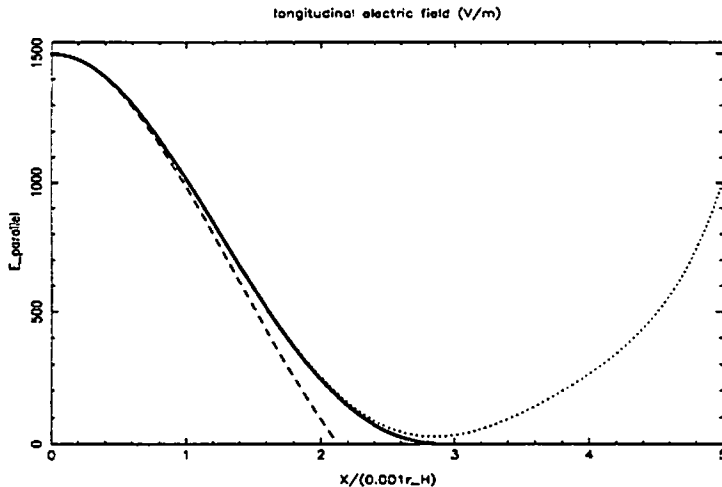


Figure 1: Variation of $E_{\parallel}$ as a function of the position x along the field line. The solid line represents the solution of which derivative vanishes at $x = H$ with the ‘critical’ current density $j_0 = j_{cr}$. The dashed line correspond to $j_0 = 0.992j_{cr}$, whereas the dotted line to $j_0 = 1.0004j_{cr}$.

the shape of $E_{\parallel}(x)$ deviates from the quadratic solution and have a ‘brim’ as indicated by the dashed line in Fig. 1; in this figure, we choose U_b to be 10^6 ergs/cm^3 and α to be 2.0. As j_0 increases, $dE_{\parallel}/dx$ decreases and attain the critical value j_{cr} at which $dE_{\parallel}/dx$ vanishes; it is realized at $j_0 = j_{cr} = 7.4145 \times 10^{-14} \text{ abampere/cm}^2$ as indicated by the solid line. Above the critical current density j_0 , condition (16) cannot no longer be satisfied, no matter what initial values of $n^-(0)$, $F^-(0)$ we choose. This situation is shown by the dotted line. To sum up, we shall seek the solutions satisfying (16) by adjusting j_0 to $j_{cr}(U_b, \alpha)$, which makes energy and angular momentum extraction from the hole works most effectively.

3 Structure of the Potential Gap

To grasp the rough feature, we first show some examples of the solutions of $E_{\parallel}(x)$ in Fig. 2. For a fixed value of $U_b = 10^6 \text{ ergs/cm}^3$, we depicted the three cases of $\alpha = 1.5, 2.0$, and 2.5 denoted by the dashed, solid, and dotted lines, respectively. This figure indicates that $E_{\parallel}$ is maximum at the symmetry point, $x = 0$, decreases monotonically with increasing x , and finally vanishes at the boundary, $x = H$. By such $E_{\parallel}$ fields, the Lorentz factor of $e^{\pm}$ becomes $\Gamma = 10^{2.5 \sim 4}$ at the center of the gap ($x = 0$) and $\Gamma \sim 60$ for $U_b = 10^6$ and $\Gamma \sim 400$ for $U_b = 10^6$.

As Fig. 2 indicates, the typical width of the potential gap is $H \sim 0.005r_H$ for the background radiation field of $U_b = 10^6 \text{ ergs/cm}^3$. Actually, H increases significantly with decreasing U_b . This is mainly because the less luminous background radiation field increases l_p , which essentially determines H .

Because $j_0 = j_{cr}(U_b, \alpha)$ is directly related with H by equation (3) at $x = H$, we can

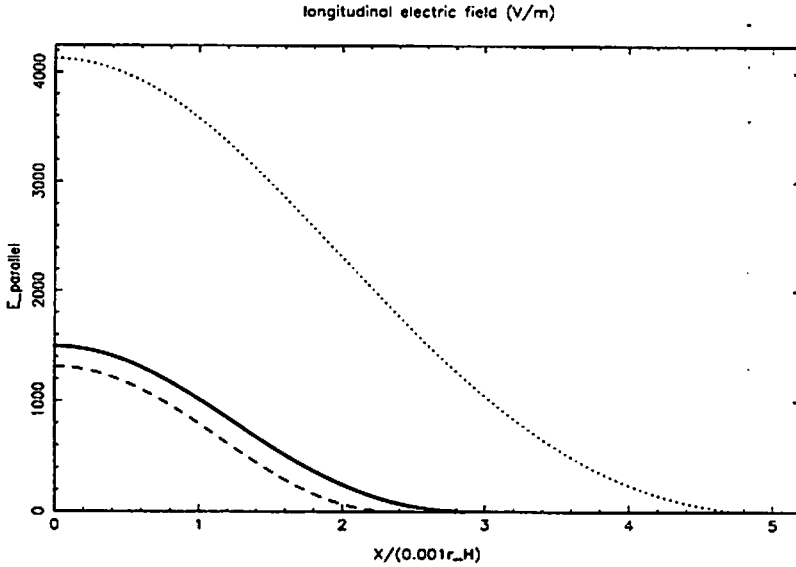


Figure 2: Solutions of $E_{\parallel}$ as a function of the position x . U_b is fixed to be 10^6 ergs/s. The solid line corresponds to the case of $\alpha = 2.0$, while the dotted and the dashed lines to $\alpha = 1.5$ and $\alpha = 2.5$, respectively.

depict $j_{cr}(U_b, \alpha)$ as Fig. 3. This figure reveals the fact that the pair production cascade provides sufficient current density ($\sim 10^{-13}$ abampere/cm²) for the effective energy and angular momentum extraction from a rotating supermassive black hole. Moreover, j_{cr} , which sets the maximum extractable energy due to the Blandford-Znajek process, becomes large when U_b decreases.

Finally, we shall investigate the number of γ -ray photons produced by a single electron or positron, N_{γ} . Evaluating N_{γ} at the boundary $x = H$, we summarized N_{γ} vs. α and U_b in Fig. 4. This figure shows the fact that one electron or positron produces typically $10^2 \sim 3$ γ -ray photons which are able to produce pairs. To put it more precisely, N_{γ} increases with increasing α , because the ‘seed’ photons increases when the spectrum is soft. It is interesting to note that N_{γ} increases with decreasing U_b . It looks like controversial; however, less luminous radiation field increases l_0 , and hence $E_{\parallel}(x)$ and $\Gamma(x)$. As a result, the energetic $e^{\pm}$ ’s upscatter more soft photons into energy range $\epsilon_{\gamma} > 1/\epsilon_{max}$, above which γ -ray photons can produce pairs. It is for this reason that N_{γ} increases with decreasing U_b .

4 Summary and Discussion

In this paper we qualitatively and self-consistently solved the process of pair production cascade in a thin gap in a rotating black hole magnetosphere. We summarize our results as follows:

1. Once a strong electric field ($E_{\parallel} = 10^{3 \sim 6}$ V/m) arises along the poloidal magnetic field line, it accelerates migratory or pair-created $e^{\pm}$ ’s into relativistic regimes with Lorentz factors $\Gamma = 10^{2.5 \sim 4}$.
2. The accelerated e^+/e^- ’s scatter background UV and X-ray photons to produce

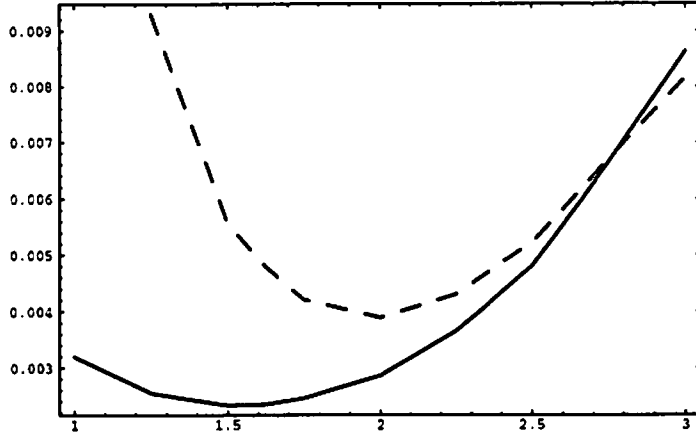


Figure 3: Gap width H/r_H vs. photon index α . The solid line corresponds to the case of $U_b = 10^6$ ergs/s, while the dashed line to $U_b = 10^5$ ergs/s.

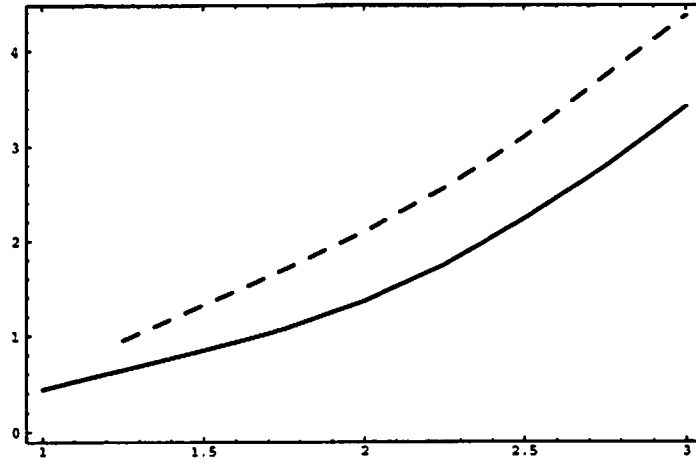


Figure 4: Number of γ -ray photons N_γ produced by a single e^+ or e^- vs. photon index α . The solid line corresponds to the case of $U_b = 10^6$ ergs/s, while the dashed line to $U_b = 10^5$ ergs/s.

$10^{1\sim 3}$ γ -ray photons.

3. Typically, one of such γ -ray photons (produced by a single e^+ or e^-) collides a background X-ray photon to produce an $e^\pm$ pair, which leads to pair production cascade.

4. The cascade supplies sufficient amount of pair plasma necessary in maintaining the electric current in the magnetosphere and thus ensures efficient extraction of the hole's rotational energy in the form of electromagnetic energy.

5. The width of the pair cascading layer is much less than the hole's radius, r_H .

It will be necessary to make clear the differences between this work and the previous work of Beskin et al. (1992). They implicitly assumed that all of the γ -ray photons created by the inverse Compton scatterings contribute to the cascade, and explicitly put the pair production mean free path l_p to be $l_p \sim l_c \sim H$. However, as seen in §3, l_p is much larger than l_c and H . This means that very small fractions of γ -ray photons are enough to produce the cascade, and most of them escape freely from the gap. This situation is similar to the pair production cascade which is expected to occur in pulsar magnetospheres ([Daugherty and Harding 1995]). In addition, we explicitly solved the γ -ray distribution functions by considering their spectrum, whereas in Beskin et al. (1992) the treatment is not explicitly mentioned. It is in fact important to take the γ -ray spectrum into account, because l_p , which essentially describes H together with N_γ , depends strongly upon γ -ray energies.

It might finally be interesting to investigate whether or not the same mechanism works around a stellar mass black hole, from which the rotational energy may be extracted, resulting in a galactic γ -ray jet (cf. [Levinson and Blandford 1996]). Around a supermassive black hole, which is the concern of this paper, the relaxation length l_0 at which $e^\pm$'s lose their perpendicular momenta owing to scatterings with background UV photons, is much smaller than l_p , l_c and H . Therefore, particle motion and γ -ray behaviour can be treated one-dimensionally; in addition, electron/positron spectra are approximated by mono-energetic ones. However, as Beskin et al. (1992) pointed out, around a stellar mass black hole, l_0 becomes comparable with l_c ; therefore, scatterings that lead to pair production can no longer be treated as one-dimensional. This issue will be our next target.

Acknowledgements

One of the authors (K.H.) wishes to thank Drs. S. Shibata and M. Takahasi for valuable comments and criticisms.

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The Dynamo Effect in Magneto-hydro-dynamic Accretion onto a Rotating Black Hole

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Abstract

(The elf-excitation mechanism of electro-magnetic fields called the dynamo action, in magneto-hydro-dynamic accretion onto a rotating black hole, are analytically studied. This mechanism is mainly supported by the coupling of the frame dragging angular velocity Ω_d of a rotating black hole and the plasma angular velocity Ω through the finite magnetic diffusivity η of the accretion plasma. The coupling effect can oppose the Ohmic dissipation, and has a possibility to dominate [1],[2]. If it dominates, the electro-magnetic field around black hole are not in the dumping mode of the Ohmic dissipation, but in the growing mode. In order to study such a criterion precisely, we develop the analytical perturbation method under the assumption that $\epsilon = \eta/\tau_h \ll 1$, where τ_h is the black hole radius, and our analysis is limited to the narrow boundary region to the horizon $(r - \tau_h)/\tau_h \sim O(\epsilon)$. We get a simple criterion for the growing mode of this process, and find that it is consistent with the condition that the outgoing Poynting flux from the horizon is positive. This means that the energy extracted from the black hole may be consumed by the dynamo effect.

1 Introduction

Many powerful extra-galactic radio sources with very active central regions are observed in the universe. Compared with the size of an associated galaxy, the active nucleus has an extraordinarily compact size and shows very rapid variability of the radiation flux. Therefore it is widely believed that a super-massive black hole works as an energetic engine of the active nucleus. Though active galactic nuclei are classified into various types with the luminosity L ranging from 10^{42} erg/sec to 10^{48} erg/sec, their fundamental

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energy-release process will be the liberation of gravitational energy of plasma accreting onto the central black hole. For an active nucleus with $L = 10^{46}$ erg/sec the black hole mass M is estimated to be 10^8 as large as the solar mass. The size of such a super-massive black hole is roughly equal to $2GM/c^2 = 3 \times 10^{13}$ cm, which corresponds to the size of the solar system in order of magnitude.

Further it is observed that the intense radio-emission region extended on length scales from parsecs to megaparsecs is linked by a jet of plasma to the central region [1]. These jets are highly collimated and are accelerated to relativistic speeds. Though their origin has not been fully understood as yet, the magnetic interaction may be crucial in the mechanism of jet formation. Then the central black hole driving magnetized jets should be immersed with magnetic fields, and the accretion process near the black hole should be also affected by the magnetic interaction including the angular momentum transport of plasma. In order to explain various observational features of active galactic nuclei in terms of the black hole magnetosphere, it is important to study basic roles of magnetic fields under the general-relativistic treatment.

In particular, we focus our attention on the purely general-relativistic effect due to the black hole rotation, and we briefly review in Sec.2 stationary and axisymmetric models of rotating magnetospheres. First, we give the vacuum model with only poloidal electromagnetic fields. Next, we study ideal magnetohydrodynamic interaction between accreting and rotating plasma and electromagnetic fields to see the generation of azimuthal magnetic field. In Sec.3 firstly we explain the idea of self-excitation process pointed out by Khanna and Camenzind [2], [3] and [4]⁴. This process is mainly supported by the frame dragging effect, which has the possibility to oppose the Ohmic dissipation. Next we consider the time development (self-excitation) of axisymmetric magnetic fields as the main topic in this paper. Now we use the generalized Ohm's law with a finite magnetic diffusivity η , which gives the diffusion timescale nearly equal to observed X-ray variability. To reduce the mathematical difficulty, the velocity field of plasma is assumed to be stationary. Brandenburg [5] presented numerical simulations and criticized the dynamo effect, because in his initial parameter space the growing modes were found only in the case that the dragging effect is artificially big. In Sec.4 we obtain a criterion for the dynamo action, by developing the analytical perturbation method under the approximation of small diffusivity $\epsilon \equiv \eta/c\tau_H \ll 1$, where τ_H is the black hole radius. Finally our results are discussed in comparison with the numerical simulations. We use units such that $c = G = 1$ in the following

2 Stationary magnetosphere

Any contribution of accreting plasma and electromagnetic fields around a rotating black hole to the gravitational field remains negligibly small. Therefore, we can always study their interaction under the fixed gravitational field described by the Kerr spacetime which

⁴They approached this problem numerically and found some self excitation modes. However recently their numerical solutions were found to contain some errors [4]. Correcting those errors, they could not find any growing modes on their initial condition.

has the line element of the form

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar \sin^2 \theta}{\Sigma} dt d\phi + \frac{A \sin^2 \theta}{\Sigma} d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2, \quad (1)$$

$$\Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta, \quad A = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta,$$

where M and a are the mass and angular momentum per unit mass of the black hole and r , θ and ϕ are the spherical coordinates. The event horizon is located at $r = r_H \equiv M + \sqrt{M^2 - a^2}$, and the frame dragging angular velocity around the black hole is $\Omega_d \equiv -g_{t\phi}/g_{\phi\phi}$. At the horizon we obtain the angular velocity $\Omega_d = \Omega_H \equiv a/2Mr_H$ of the Kerr black hole.

Let us begin with the simplest vacuum model of the black hole magnetosphere to see the so-called frame dragging effect. The general-relativistic Maxwell equations for the electromagnetic tensor field $F_{ab} \equiv \partial_a A_b - \partial_b A_a$ admit the stationary and axisymmetric solution with the asymptotically uniform magnetic field B_o [6], which has the form

$$A_t = (B_o/2)(2ag_{tt} + g_{t\phi}), \quad A_\phi = (B_o/2)(g_{\phi\phi} + 2ag_{t\phi}). \quad (2)$$

The poloidal magnetic field lines given by $A_\phi = \text{const.}$ are bent toward the central black hole owing to the strong gravity. The more remarkable point is that the electric field with the poloidal components given by $E_r \equiv F_{rt} = \partial_r A_t$ and $E_\theta \equiv F_{\theta t} = \partial_\theta A_t$ is generated as a result of the black hole rotation. In particular, we obtain the relation

$$F_{\theta t} = -\Omega_H F_{\theta\phi} \quad (3)$$

just at the horizon, which means the presence of a voltage drop along the surface.

For magnetohydrodynamic equilibrium between plasma flows with 4-velocity u^a and electromagnetic fields, we may require the condition of infinite conductivity as follows,

$$F_{ab} u^b = 0. \quad (4)$$

By virtue of the assumption of stationarity and axisymmetry the toroidal electric component $E_\phi \equiv F_{\phi t}$ always vanishes. Then, the poloidal parts of (4) leads to the results that poloidal flow lines are identical with poloidal magnetic field lines given by $A_\phi = \text{const.}$, and the angular velocity Ω_F of a poloidal field line is defined as follows,

$$\Omega_F \equiv -F_{tr}/F_{\phi r} = -F_{t\theta}/F_{\phi\theta}. \quad (5)$$

The angular velocity Ω_F is conserved along a poloidal flow line as an arbitrary function of A_ϕ . Different from the vacuum magnetosphere, the remaining part of (4) gives the azimuthal magnetic component $B_\phi \equiv F^{r\theta}$ as follows,

$$v^r F_{r\theta} = F_{\theta\phi}(\Omega - \Omega_F), \quad (6)$$

where $v^r = u^r/u^t$ is the radial component of the usual 3-dimensional velocity, and $\Omega = u^\phi/u^t$ is the angular velocity of plasma. At the horizon only the accretion flow with $v^r < 0$ is allowed, and Ω should be equal to Ω_H . The key point is that Ω_F may be different from

Ω_H . (See (3) for the vacuum model.) Thus we have the Poynting flux P_H at the horizon as follows,

$$P_H \sim E_\theta B_\phi \propto \Omega_F(\Omega_H - \Omega_F)(F_{\theta\phi})^2. \quad (7)$$

If the inequalities $0 < \Omega_F < \Omega_H$ hold, the positive Poynting flux $P_H > 0$ can occur at the horizon. (Exactly speaking, an ingoing flux with negative energy becomes possible.) This is an energy supply from the rotating black hole into the exterior magnetosphere via the generation of the toroidal magnetic component [7]. Some examples of accretion of magnetized fluids accompanied with the positive Poynting flux have been found [8]. However, it remains unclear how the Poynting flux is useful to activate the magnetosphere. One possible mechanism may be a self-amplification of the magnetic field near the black hole, which will be investigated in the following sections.

3 The dynamo effect

Khanna and Camenzind [2], [3] suggested that there is a possibility of a self-excitation mode of the axisymmetric electro-magnetic field around the rotating black hole, which is due to the coupling between the frame dragging angular velocity Ω_d and the finite magnetic diffusivity η . Their formulation is based on the general relativistic Ohm's law and the Maxwell equations. Khanna and Camenzind used the (3+1) split formalism, but now we use the covariant formulation for the later section,

$$F_{\alpha\beta}u^\beta = \eta \{ F_{\alpha\nu}{}^{;\nu} + (F^{\mu\nu}{}_{;\nu}u_\mu)u_\alpha \} \quad (8)$$

$$\partial_t F_{r\theta} + \partial_r F_{\theta t} + \partial_\theta F_{tr} = 0. \quad (9)$$

Now we consider time-dependent fields under the assumption of axisymmetry, which allows us to write the poloidal magnetic field by the single potential $\Psi \equiv A_\phi$. Then independent field variables are,

$$\Psi, F^{tr}, F^{t\theta}, F^{r\theta}. \quad (10)$$

Intuitively contours in the r - θ plane of $\Psi(t, r, \theta)$ represent poloidal magnetic field line at $t = t$, so the time evolution of the magnetic field can be seen by the time dependence of $\Psi(t, r, \theta)$. From the t component (8), we get such a information,

$$\partial_t \Psi + \frac{1}{u^t}(u^r \partial_r + u^\theta \partial_\theta)\Psi = -\frac{\eta g_{\phi\phi}}{u^t} \left\{ (F^{tr} \partial_r + F^{t\theta} \partial_\theta)\Omega_d - \frac{\partial_A (\sqrt{-g} g_{\phi\phi}^{-1} \Psi^{,A})}{\sqrt{-g}} + \eta(\dots) \right\}, \quad (11)$$

where, runs r and θ . In the above equation, the second term of the left hand-side represents the advection effect frozen in the plasma flow direction, which has essentially no contribution to increase or decrease of the number of field lines. The second term of the right hand-side represents Ohmic dissipation effect, which produces only dumping mode of Ψ . If we neglect the term related to η^2 , in the case of $\Omega_d = 0$, namely, the Schwarzschild case, Ψ is always in the dumping mode. However in the Kerr case, we can expect that there is a possibility for the first term related to the frame dragging effect Ω_d to dominate

the Ohmic dissipation term. To see the self-excitation process roughly, eliminating F^{tr} , $F^{t\theta}$ using r and θ component of (8), then (11) becomes,

$$\begin{aligned} \partial_t \Psi + \frac{1}{u^t} (u^r \partial_r + u^\theta \partial_\theta) \Psi &= -\frac{\eta g_{\phi\phi}}{u^t} \left\{ g^{tt} (\Omega - \Omega_d) \Psi^A \partial_A \Omega_d \right. \\ &\quad \left. + F^{r\theta} \frac{g^{tt}}{u^t} (u_\theta \partial_r - u_r \partial_\theta) \Omega_d - \frac{\partial_A (\sqrt{-g} g_{\phi\phi}^{-1} \Psi^{,A})}{\sqrt{-g}} + \eta(\dots) \right\}, \end{aligned} \quad (12)$$

where, the time evolution of $F^{r\theta}$ is described by (9).

$$\begin{aligned} \partial_t F^{r\theta} + \partial_\theta \left\{ \frac{u_\theta}{u^t} F_r^\theta \right\} + \partial_\theta \left\{ \frac{u_r}{u^t} F^\theta_r \right\} \\ = (\Psi_{,r} \partial_\theta - \Psi_{,\theta} \partial_r) \Omega + \eta \left\{ \partial_A \left[\frac{g_{BB} \partial_A (\sqrt{-g} F^{AB})}{u^t \sqrt{-g}} \right] + \eta(\dots) \right\}. \end{aligned} \quad (13)$$

Self-excitation sequence is as follows. We suppose initial Ψ and $F^{r\theta}$, then Ψ produce new $F^{r\theta}$ through (13) proportional to Ω plasma angular velocity. New $F^{r\theta}$ produce new Ψ through (12) proportional to Ω_d . So this chain is called $\Omega_d \Omega$ dynamo. Whether the electro-magnetic field is in the growing mode or dumping mode, is due to the balance between $\Omega_d \Omega$ dynamo effect and the Ohmic dissipation effect. The chain between the term related to Ω of (13) and the term related to $F^{r\theta}$ of (12), contributes the growing mode of the Ψ , so we call this chain as the $\Omega \Omega_d$ dynamo action.

4 Perturbation approach

The time variability of X-ray luminosity is a crucial feature observed in active galactic nuclei to test theoretical models. If the magnetosphere is powered by the above-mentioned dynamo action, the typical timescale of the X-ray variability should be of the order of the diffusion timescale τ_H^2/η . Because the observed timescale is much larger than the usual dynamical timescale estimated by r_H , we can assume the amplitude of η to be $\epsilon \equiv \eta/r_H \ll 1$. Our perturbation method is based on this small parameter ϵ , which is also assumed to be uniform in the magnetosphere near the black hole. In the numerical simulation given by Khanna and Camenzind it was claimed that very close to the event horizon, small poloidal flux islands are developed as the seeds for strong flux concentration observed in later stages. (Of course, Brandenburg did not find such a piling up of magnetic flux at the horizon.) We focus our attention on the time-dependent behavior in the narrow region $x \equiv (r - r_H)/r_H \sim \epsilon$ outside the event horizon which should occur with the diffusion timescale $t \sim \epsilon^{-1} r_H$. For this purpose it is convenient to use the dimensionless time coordinate $T \equiv \epsilon t/r_H$ and the dimensionless radial coordinate $R \equiv x/\epsilon$. Another point we must consider is infinite time delay ($t \rightarrow \infty$) just at the horizon: Any physical evolution of field variables near the horizon proceeds mainly through the advanced time coordinate $v \equiv t + m \ln((r/r_H) - 1)$ which remains finite even in the limit $r \rightarrow r_H$. (Here m is a constant defined by $m = M r_H / \sqrt{M^2 - a^2}$.) Then, we treat the field variables Ψ , F^{tr} , $F^{t\theta}$ and $F^{r\theta}$ as functions of R , diffusion time schale $V \equiv \epsilon v$ (instead of T) and θ . We

expand them as power series in the small parameter ϵ as follows,

$$\begin{aligned}
\Psi &= \sum_{n=0}^{\infty} \epsilon^n \Psi^{(n)}(V, R, \theta) , \\
F^{tr} &= \sum_{n=0}^{\infty} \epsilon^n f_r^{(n)}(V, R, \theta) , \\
F^{t\theta} &= (\epsilon R)^{-1} \sum_{n=0}^{\infty} \epsilon^n f_{\theta}^{(n)}(V, R, \theta) , \\
F^{r\theta} &= \sum_{n=0}^{\infty} \epsilon^n f_{\phi}^{(n)}(V, R, \theta) .
\end{aligned} \tag{14}$$

To estimate the order of each quantity in these expansions, the coordinate R is limited to the range $\epsilon \ll R \ll \epsilon^{-1}$.

Let us substitute these expressions of the field variables into (8) and (9). The metric g_{ab} and the velocity field u^a which are functions of $x = \epsilon R$ and θ are also expanded as power series in ϵ . Then, to the zeroth order, we obtain the solutions

$$\begin{aligned}
\Psi^{(0)} &= \Psi^{(0)}(V, \theta) , \quad f_{\phi}^{(0)} = f_{\phi}^{(0)}(V, \theta) , \\
f_r^{(0)} &= -u_{\theta}^{(0)} f_{\phi}^{(0)} / e , \quad f_{\theta}^{(0)} = -m f_{\phi}^{(0)} / r_H ,
\end{aligned} \tag{15}$$

where $e(\theta) = -(u_t^{(0)} + \Omega_H u_{\phi}^{(0)})$. The functions $\Psi^{(0)}$ and $f_{\phi}^{(0)}$ remains arbitrary, except that they are independent of R . Therefore, to reduce the degree of freedom, we apply the frozen-in condition (6) to the relation between the zeroth-order field variables as a boundary condition near the horizon, which leads to

$$f_{\phi}^{(0)} = (1/2M r_H \rho^2)(\Omega_F - \Omega_H) \partial_{\theta} \Psi^{(0)} , \tag{16}$$

where $\rho = (r_H^2 + a^2 \cos^2 \theta) / (r_H^2 + a^2)$. Though the effective angular velocity Ω_F introduced here may depend on θ , we will treat it as a constant parameter in the following calculations. Further, we limit our model to the radial velocity field with $u^{\theta} = 0$, and the specific energy and angular momentum of plasma to the zeroth order are assumed to be

$$-u_t^{(0)} = E \sin^2 \theta , \quad u_{\phi}^{(0)} = L \sin^2 \theta , \tag{17}$$

where L and E are constant. The effect of small η should appear in the zeroth-order field variables through their dependence on V , and following the idea of the reductive perturbation scheme, we will find the equation to determine $\Psi^{(0)}(V, \theta)$ from the first-order solutions .

Any one of the first-order field variables denoted by $f^{(1)}$ has the form $f^{(1)} = Rg(V, \theta) + h(V, \theta)$. The first term proportional to R is explained to be due to the Taylor expansion in the coordinate $x = \epsilon R \ll 1$ such that $f(x) \simeq f(0) + \epsilon R (\partial_x f)_0$, which may be present even in stationary fields. We expect that the second term h independent of R is generated by some dynamical process in the vicinity of the horizon. If so, the second term of the first-order solution should be directly related to the V dependence of the zeroth-order solution denoted by $f^{(0)}$. Note that the definition of V allows a small modification such

that $V \rightarrow V + V_1$, where V_1 is a constant of the order of ϵ . Then, we obtain the Taylor expansion $f^{(0)}(V + V_1) \simeq f^{(0)}(V) + V_1 \partial_V f^{(0)}(V)$ in the coordinate V , and it will be a plausible procedure to identify $h(V, \theta)$ with $(V_1/\epsilon) \partial_V f^{(0)}(V)$. In fact, this procedure is shown to be possible for the first-order solutions, if the following equation for the zeroth-order field variable $\Psi^{(0)}$ is satisfied:

$$\frac{d^2 f}{d\theta^2} + (H(\theta) \sin 2\theta - \cot \theta) \frac{df}{d\theta} + \lambda \rho^2 f = 0, \quad (18)$$

where the solution $\Psi^{(0)}$ is assumed to have the form

$$\Psi^{(0)} = \exp(\omega V) f(\theta), \quad (19)$$

and the parameter λ depending on the growth rate ω and the function $H(\theta)$ are given by

$$\lambda = \omega L / r_H^2 (\Omega_H - \Omega_F), \quad (20)$$

$$H(\theta) = (a^2 / 4 M r_H \rho) [1 - (L^2 / a^2) + (E - 2 \Omega_H L)^2 \sin^4 \theta]. \quad (21)$$

We must solve (14) under the boundary conditions at the polar region and the equatorial plane as an eigenvalue problem.

Of course, the eigenfunction $\Psi^{(0)}$ is insufficient to show any global shape of poloidal magnetic field lines. Nevertheless, it can determine the radial component $\partial_\theta \Psi^{(0)}$ and the zenithal component $-(\omega M / R r_H \sqrt{M^2 - a^2}) \Psi^{(0)}$ at least in the region very close to the horizon, and we obtain the useful information of topological (multipolar) structure of the poloidal field lines threading the horizon from the number of nodes of f in the range $0 < \theta < \pi$. For the regularity of $\Psi^{(0)}$ at the polar region we give the boundary condition such that $f = O(\theta^2)$ in the limit $\theta \rightarrow 0$. The additional boundary condition at the equatorial plane $\theta = \pi/2$ is that $\partial_\theta f = 0$ for dipolar (and higher multipolar antisymmetric) modes and $f = 0$ for quadrupolar (and higher multipolar symmetric) modes. In particular, the dipolar mode corresponds to the eigenfunction with no node and may be called antisymmetric because the radial magnetic field satisfies the condition $\partial_\theta f(\cos \theta) = -\partial_\theta f(-\cos \theta)$. For the dipolar growing mode with $\omega > 0$ the zenithal component ($B_\theta < 0$ for $\Psi^{(0)} > 0$) appears even at the equatorial plane. This can lead to the formation of a dipolar closed loop (drawn in Fig. 1), of which the neutral point will exist at a point with the distance R of the order of ϵ^{-1} . Some examples of the eigenfunctions are shown in Fig. 2 with the eigenvalue λ . Because λ should be always positive, we obtain the condition for $\omega > 0$ as follows,

$$L(\Omega_H - \Omega_F) > 0, \quad (22)$$

which clearly shows the crucial role of the parameter Ω_F for the existence of growing modes as seeds of the dynamo action.

Though in the present perturbation scheme based on the local analysis the parameter Ω_F remains arbitrary, any global analysis treating magnetic fields in a more extended region will determine it uniquely as a function of Ω_H , M , L and E . For a nonrotating black hole ($\Omega_H = 0$) we can expect that the azimuthal magnetic field is induced by the rotation of plasma and Ω_F is proportional to L . Then we have no growing modes. For the rotation

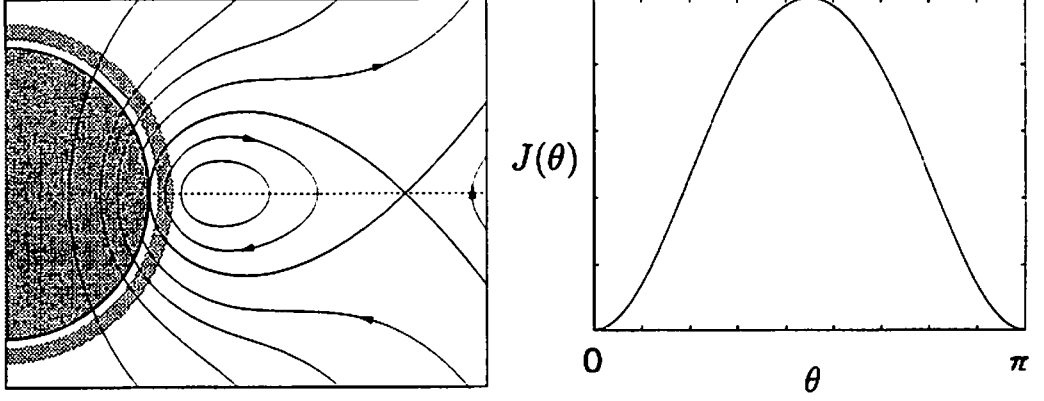


Figure 1: A Schematic field lines representing the formation of a dipolar closed loop. The magnetic fields in the thin shaded layer around the black hole are analyzed by the perturbation method.

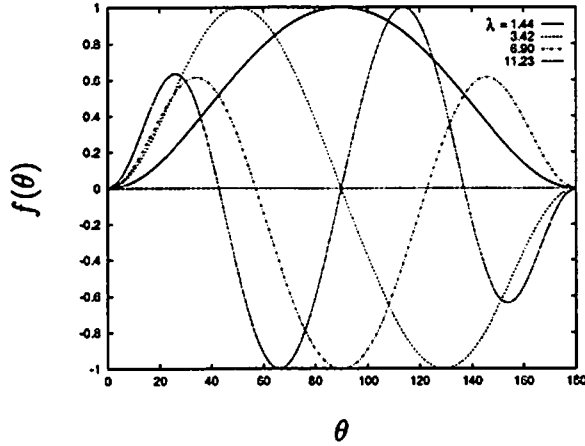


Figure 2: Examples of the eigenfunctions $f(\theta)$ classified by the number of nodes and the eigenvalue λ for the rotating black hole with $a = 0.95M$. The solid line denotes the dipolar mode, and the others with some nodes correspond to the quadrupolar and higher multipolar modes.

of plasma with a positive angular momentum ($L > 0$) it is interesting to note that the condition $0 < \Omega_F < \Omega_H$ for the positive Poynting flux in the stationary magnetosphere corresponds to $\omega > 0$. However, if the dynamo action is not effective as claimed by Brandenburg, such a range of Ω_F may not be allowed. Further, the growing time ω^{-1} turns out to decrease as the number of nodes increases. The smaller scale structure due to higher multipolar modes grows more rapidly near the horizon. The exponential amplification $e^{\omega V}$ must be suppressed in later stages, if any outward propagation of growing modes into a distant region is suitably taken into account. However, to treat such a magnetohydrodynamic propagation and to see how only the dipolar or quadupolar structure globally dominates, we must introduce another timescale different from V . The multi-timescale procedure as an extension of the framework of the present perturbation scheme remainsto future works.

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On Equilibrium of Two Kerr-Newman Black Holes

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Abstract

We consider the two-Kerr-Newman solution in an axisymmetric and stationary spacetime. As a first step to discuss the conditions for equilibrium of two Kerr-Newman sources we study how the conditions for equilibrium in the two-Kerr-NUT case are modified by the existence of electromagnetic field.

Recently many people have been interested in the solution that was given by Majumdar[1] and Papapetrou[2] (MP solution) in 1947. This solution is the multi-black-hole solution of the source-free Einstein-Maxwell equations and represents static equilibrium among extreme Reissner-Nordström black holes. We can easily obtain the MP solution if we assume the “conformastat” metric

$$ds^2 = -f(\vec{x})dt^2 + f(\vec{x})^{-1}(dx^2 + dy^2 + dz^2). \quad (1)$$

This metric is generalized to the “conformastationary” case

$$ds^2 = -f(\vec{x})[dt + A_x(\vec{x})dx + A_y(\vec{x})dy + A_z(\vec{x})dz]^2 + f(\vec{x})^{-1}(dx^2 + dy^2 + dz^2), \quad (2)$$

and in this metric we can discuss the solution describing Kerr-Newman sources in equilibrium[3]. However, it is shown that the superposition of Kerr-Newman solutions in the metric (2) gives rise to naked singularities[4].

In the axisymmetric case the equilibrium of two spinning sources in the superposition of Kerr-NUT solutions has been discussed[5,6]. It is also shown that the conditions for equilibrium are not compatible with those for the existence of event horizons. We studied the solution describing N Reissner-Nordström sources aligned along the symmetry axis and showed that their equilibrium is realized only when the solution becomes the MP solution[7].

In this report we consider the superposition of two Kerr-Newman solutions in an axisymmetric and stationary spacetime. As a first step to discuss the conditions for equilibrium of two Kerr-Newman sources we study how the conditions for equilibrium in the two-Kerr-NUT case are modified by the existence of electromagnetic field. To obtain the two-Kerr-Newman solution we first apply the inverse scattering method to solving the vacuum Einstein equation and then transform the obtained 4-soliton (two-Kerr-NUT) solution to the solution of the Einstein-Maxwell equations.

The metric is

$$ds^2 = -f(dt + Ad\phi)^2 + f^{-1}[e^k(d\rho^2 + dz^2) + \rho^2 d\phi^2], \quad (3)$$

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where f , A and k are functions of ρ and z . The source-free Einstein-Maxwell equations are given by

$$R_{\mu\nu} = 2 \left(F_{\mu}^{\alpha} F_{\nu\alpha} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right), \quad (4)$$

$$F^{\mu\nu}_{;\nu} = 0, \quad (5)$$

with

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \quad (6)$$

In the metric (3) Eqs.(4) and (5) are written as

$$\begin{aligned} & (\ln f)_{,\rho\rho} + (\ln f)_{,zz} + \rho^{-1}(\ln f)_{,\rho} + \rho^{-2}f^2(A_{,\rho}^2 + A_{,z}^2) \\ & = 2f^{-1}(1 + \rho^{-2}f^2A^2)(\chi_{,\rho}^2 + \chi_{,z}^2) + 2\rho^{-2}f(\psi_{,\rho}^2 + \psi_{,z}^2) \\ & \quad + 4\rho^{-2}fA(\chi_{,\rho}\psi_{,\rho} + \chi_{,z}\psi_{,z}), \end{aligned} \quad (7)$$

$$\begin{aligned} & A_{,\rho\rho} + A_{,zz} - \rho^{-1}A_{,\rho} + 2[A_{,\rho}(\ln f)_{,\rho} + A_{,z}(\ln f)_{,z}] \\ & = -4f^{-1}[A(\chi_{,\rho}^2 + \chi_{,z}^2) + \chi_{,\rho}\psi_{,\rho} + \chi_{,z}\psi_{,z}], \end{aligned} \quad (8)$$

$$\begin{aligned} & k_{,\rho} - \frac{\rho}{2}[(\ln f)_{,\rho}^2 - (\ln f)_{,z}^2] + \frac{f^2}{2\rho}(A_{,\rho}^2 - A_{,z}^2) \\ & = 2\rho[-f^{-1}(1 - \rho^{-2}f^2A^2)(\chi_{,\rho}^2 - \chi_{,z}^2) + \rho^{-2}f(\psi_{,\rho}^2 - \psi_{,z}^2) \\ & \quad + 2\rho^{-2}fA(\chi_{,\rho}\psi_{,\rho} - \chi_{,z}\psi_{,z})], \end{aligned} \quad (9)$$

$$\begin{aligned} & k_{,z} - \rho(\ln f)_{,\rho}(\ln f)_{,z} + \rho^{-1}f^2A_{,\rho}A_{,z} \\ & = 4\rho[-f^{-1}(1 - \rho^{-2}f^2A^2)\chi_{,\rho}\chi_{,z} + \rho^{-2}f\psi_{,\rho}\psi_{,z} \\ & \quad + \rho^{-2}fA(\chi_{,\rho}\psi_{,z} - \chi_{,z}\psi_{,\rho})], \end{aligned} \quad (10)$$

$$\begin{aligned} & \chi_{,\rho\rho} + \chi_{,zz} + \rho^{-1}\chi_{,\rho} - [\chi_{,\rho}(\ln f)_{,\rho} + \chi_{,z}(\ln f)_{,z}] \\ & - \rho^{-2}f^2[A(A_{,\rho}\chi_{,\rho} + A_{,z}\chi_{,z}) + A_{,\rho}\psi_{,\rho} + A_{,z}\psi_{,z}] = 0, \end{aligned} \quad (11)$$

$$\begin{aligned} & \psi_{,\rho\rho} + \psi_{,zz} - \rho^{-1}\psi_{,\rho} + \psi_{,\rho}(\ln f)_{,\rho} + \psi_{,z}(\ln f)_{,z} \\ & + A[\chi_{,\rho\rho} + \chi_{,zz} - \rho^{-1}\chi_{,\rho} + \chi_{,\rho}(\ln f)_{,\rho} + \chi_{,z}(\ln f)_{,z}] \\ & + A_{,\rho}\chi_{,\rho} + A_{,z}\chi_{,z} = 0, \end{aligned} \quad (12)$$

where we have written the non-vanishing components of the electromagnetic potential as $A_t = -\chi$ and $A_\phi = \psi$. To solve these coupled differential equations we introduce a new potential $\tilde{\chi}$ defined by

$$\tilde{\chi}_{,z} = \rho^{-1}f(A\chi_{,\rho} + \psi_{,\rho}), \quad (13)$$

$$\tilde{\chi}_{,\rho} = -\rho^{-1}f(A\chi_{,z} + \psi_{,z}), \quad (14)$$

and assume the relations

$$f = 1 - 2c\chi + \chi^2 + \tilde{\chi}^2, \quad (15)$$

$$A_{,\rho} = 2\rho f^{-2}[\tilde{\chi}\chi_{,z} - (\chi - c)\tilde{\chi}_{,z}], \quad (16)$$

$$A_{,z} = -2\rho f^{-2}[\tilde{\chi}\chi_{,\rho} - (\chi - c)\tilde{\chi}_{,\rho}], \quad (17)$$

where c is a constant. We have assumed Eqs.(15)-(17) so that the solutions we obtain here may be the Kerr-Newman type. Then Eqs.(7)-(12) reduce to

$$\chi_{,\rho\rho} + \chi_{,zz} + \rho^{-1}\chi_{,\rho} = 2f^{-1}[(\chi - c)(\chi_{,\rho}^2 + \chi_{,z}^2 - \tilde{\chi}_{,\rho}^2 - \tilde{\chi}_{,z}^2) + 2\tilde{\chi}(\chi_{,\rho}\tilde{\chi}_{,\rho} + \chi_{,z}\tilde{\chi}_{,z})], \quad (18)$$

$$\tilde{\chi}_{,\rho\rho} + \tilde{\chi}_{,zz} + \rho^{-1}\tilde{\chi}_{,\rho} = 2f^{-1}[\tilde{\chi}(\tilde{\chi}_{,\rho}^2 + \tilde{\chi}_{,z}^2 - \chi_{,\rho}^2 - \chi_{,z}^2) + 2(\chi - c)(\chi_{,\rho}\tilde{\chi}_{,\rho} + \chi_{,z}\tilde{\chi}_{,z})], \quad (19)$$

$$k_{,\rho} = 2\rho f^{-2}(c^2 - 1)(\chi_{,\rho}^2 - \chi_{,z}^2 + \tilde{\chi}_{,\rho}^2 - \tilde{\chi}_{,z}^2), \quad (20)$$

$$k_{,z} = 4\rho f^{-2}(c^2 - 1)(\chi_{,\rho}\chi_{,z} + \tilde{\chi}_{,\rho}\tilde{\chi}_{,z}). \quad (21)$$

From Eqs.(20) and (21) we find that the case $c^2 = 1$ corresponds to the conformastationary case (2).

Let us introduce new functions $\bar{f}$, $\bar{\omega}$ and $\bar{A}$ with the relations

$$\bar{f} = \frac{4\alpha^2 \bar{f}}{[(m + \alpha) - (m - \alpha)\bar{f}]^2 + (m - \alpha)^2 \bar{\omega}^2}, \quad (22)$$

$$\chi = \frac{e\{(1 - \bar{f})[(m + \alpha) - (m - \alpha)\bar{f}] + (m - \alpha)\bar{\omega}^2\}}{[(m + \alpha) - (m - \alpha)\bar{f}]^2 + (m - \alpha)^2 \bar{\omega}^2}, \quad (23)$$

$$\tilde{\chi} = \frac{2e\alpha\bar{\omega}}{[(m + \alpha) - (m - \alpha)\bar{f}]^2 + (m - \alpha)^2 \bar{\omega}^2}, \quad (24)$$

$$\bar{\omega}_{,\rho} = -\rho^{-1}\bar{f}^2\bar{A}_{,z}, \quad (25)$$

$$\bar{\omega}_{,z} = \rho^{-1}\bar{f}^2\bar{A}_{,\rho}, \quad (26)$$

where m , e and α are constants satisfying $m = ce$ and $\alpha = \sqrt{m^2 - e^2}$. Substituting Eqs.(22)-(24) in Eqs.(18)-(21) and using Eqs.(25) and (26) we obtain a set of differential equations for $\bar{f}$, $\bar{A}$ and k . Their expressions are the same as those for the vacuum Einstein equation in the metric (3) with $\bar{f}$ and $\bar{A}$ instead of f and A , respectively. Consequently we obtain the solutions for $\bar{f}$, χ and $\tilde{\chi}$ of the Einstein-Maxwell equations, firstly solving the vacuum Einstein equation to get $\bar{f}$ and $\bar{A}$, and then transforming them by means of Eqs.(22)-(26). As for the function A , we use the relations (16) and (17). Substituting the relations (22)-(26) in Eqs.(16) and (17) we have

$$A_{,\rho} = \frac{(m - \alpha)^2 \rho \bar{\omega}}{2\alpha^2 \bar{f}} \bar{f}_{,z} + \frac{(m + \alpha)^2 - (m - \alpha)^2 \bar{f}^2 + (m - \alpha)^2 \bar{\omega}^2}{4\alpha^2} \bar{A}_{,\rho}, \quad (27)$$

$$A_{,z} = -\frac{(m - \alpha)^2 \rho \bar{\omega}}{2\alpha^2 \bar{f}} \bar{f}_{,\rho} + \frac{(m + \alpha)^2 - (m - \alpha)^2 \bar{f}^2 + (m - \alpha)^2 \bar{\omega}^2}{4\alpha^2} \bar{A}_{,z}. \quad (28)$$

We note that the solution for k is directly given by solving the vacuum Einstein equation. This is important when we consider the conditions for equilibrium, as we see later. We can confirm the fact that we obtain the Kerr-Newman solution with the NUT parameter if we adopt the 2-soliton solution for $\bar{f}$ and $\bar{A}$.

We now consider the 4-soliton solution. The vacuum solutions are given by

$$\bar{f} = \left(|\Gamma| + \sum_{k=1}^4 |D_u^{(k)}| \right) |\Gamma|^{-1} \prod_{k=1}^4 (\mu_k \rho^{-1}), \quad (29)$$

$$\tilde{A} = \left(|\Gamma| + \sum_{k=1}^4 |D_{tt}^{(k)}| \right)^{-1} \sum_{k=1}^4 |D_{t\phi}^{(k)}|, \quad (30)$$

$$e^k = C_4 \left(|\Gamma| + \sum_{k=1}^4 |D_{tt}^{(k)}| \right) \prod_{k>l} [(\rho^2 + \mu_k \mu_l) \rho^{-1}]^2, \quad (31)$$

where

$$\mu_k = (w_k - z) + (-1)^{k-1} \sqrt{(w_k - z)^2 + \rho^2}, \quad (32)$$

$$|\Gamma| = \det \left[\frac{p_k p_l \rho^2 - q_k q_l \mu_k \mu_l}{\mu_k \mu_l (\rho^2 + \mu_k \mu_l)} \right], \quad (33)$$

and $|D_{tt}^{(k)}|$ and $|D_{t\phi}^{(k)}|$ are the determinants given by replacing the k -th row of $|\Gamma|$ with

$$\{N_t^{(1)} N_t^{(k)} \mu_1^{-1} \mu_k^{-1}, N_t^{(2)} N_t^{(k)} \mu_2^{-1} \mu_k^{-1}, N_t^{(3)} N_t^{(k)} \mu_3^{-1} \mu_k^{-1}, N_t^{(4)} N_t^{(k)} \mu_4^{-1} \mu_k^{-1}\},$$

and

$$\{N_\phi^{(1)} N_t^{(k)} \mu_1^{-1} \mu_k^{-1}, N_\phi^{(2)} N_t^{(k)} \mu_2^{-1} \mu_k^{-1}, N_\phi^{(3)} N_t^{(k)} \mu_3^{-1} \mu_k^{-1}, N_\phi^{(4)} N_t^{(k)} \mu_4^{-1} \mu_k^{-1}\},$$

respectively. In the above equations,

$$N_\phi^{(k)} = p_k \rho^2 \mu_k^{-1}, \quad N_t^{(k)} = q_k, \quad (34)$$

and w_k, p_k, q_k ($k = 1, \dots, 4$) and C_4 are constants. For w_k 's we assume

$$w_1 < w_2 < w_3 < w_4, \quad (35)$$

and then from the behaviors of the solutions (29)-(31) on the z -axis ($\rho \sim 0$) we find that there are two Kerr-NUT constituents along the z -axis in the regions $w_1 < z < w_2$ and $w_3 < z < w_4$. Therefore we parametrize the constants w_k 's, p_k 's and q_k 's as

$$w_1 = z_1 - \sigma_1, \quad w_2 = z_1 + \sigma_1, \quad w_3 = z_2 - \sigma_2, \quad w_4 = z_2 + \sigma_2, \quad (36)$$

$$p_1 p_2 + q_1 q_2 = \bar{\sigma}_1, \quad p_3 p_4 + q_3 q_4 = \bar{\sigma}_2, \quad (37)$$

$$p_1 p_2 - q_1 q_2 = \bar{m}_1, \quad p_3 p_4 - q_3 q_4 = \bar{m}_2, \quad (38)$$

$$p_1 q_2 - p_2 q_1 = \bar{a}_1, \quad p_3 q_4 - p_4 q_3 = \bar{a}_2, \quad (39)$$

$$p_1 q_2 + p_2 q_1 = -\bar{b}_1, \quad p_3 q_4 + p_4 q_3 = -\bar{b}_2, \quad (40)$$

so that we may have the usual Kerr-NUT solution with the mass parameter $\bar{m}_i$, the angular momentum parameter $\bar{a}_i$ and the NUT parameter $\bar{b}_i$ ($i = 1$ or 2) when one constituent vanishes. (For example, we have the Kerr-NUT solution with the parameters $\bar{m}_1, \bar{a}_1$ and $\bar{b}_1$, setting $\bar{\sigma}_1 = \sigma_1, \bar{\sigma}_2 = \sigma_2$ and $q_3 = q_4 = 0$, and then taking the limit $\sigma_2 \rightarrow 0$.) In addition we introduce new parameters α_i, a_i and b_i ($i = 1, 2$) by

$$\frac{\bar{m}_i}{\bar{\sigma}_i} \sigma_i = \alpha_i, \quad \frac{\bar{a}_i}{\bar{\sigma}_i} \sigma_i = a_i, \quad \frac{\bar{b}_i}{\bar{\sigma}_i} \sigma_i = b_i, \quad (i = 1, 2) \quad (41)$$

in order to obtain the usual Kerr-Newman solution when we apply the transformations (22)-(24). Among these parameters there are relations

$$\sigma_i^2 = \alpha_i^2 - a_i^2 + b_i^2. \quad (i = 1, 2) \quad (42)$$

We obtain the two-Kerr-Newman solution substituting the vacuum solutions (29) and (30) in the relations (22)-(28). We note that when $\alpha = m$ ($e = 0$) and $\sigma_i = \bar{\sigma}_i$ ($i = 1, 2$) the obtained solution becomes the two Kerr-NUT solution. In order to study the condition for equilibrium in the two-Kerr-Newman solution we first compute the asymptotic behaviors at $\sqrt{\rho^2 + z^2} \rightarrow \infty$ and get

$$f \sim 1 - \frac{m}{\alpha} \frac{Y_1}{X_1} \frac{1}{\sqrt{\rho^2 + z^2}}, \quad (43)$$

$$\chi \sim \frac{e}{2\alpha} \frac{Y_2}{X_1} \frac{1}{\sqrt{\rho^2 + z^2}}, \quad (44)$$

$$A \sim \frac{m}{\alpha} \frac{Y_3}{X_1} \frac{z}{\sqrt{\rho^2 + z^2}}, \quad (45)$$

with

$$X_1 = \bar{\sigma}_1 \bar{\sigma}_2 [d^2 + 2(\alpha_1 \alpha_2 - a_1 a_2 + b_1 b_2) - (\sigma_1^2 + \sigma_2^2)], \quad (46)$$

$$Y_1 = 2\bar{\sigma}_1 \bar{\sigma}_2 [(\alpha_1 + \alpha_2)d^2 + 2(a_2 b_1 - a_1 b_2)d - (\alpha_1 - \alpha_2)(\sigma_1^2 - \sigma_2^2)], \quad (47)$$

$$Y_2 = 2\bar{\sigma}_1 \bar{\sigma}_2 [(a_1 + a_2)d^2 + 2(\alpha_2 b_1 - \alpha_1 b_2)d - (a_1 - a_2)(\sigma_1^2 - \sigma_2^2)], \quad (48)$$

$$Y_3 = -2\bar{\sigma}_1 \bar{\sigma}_2 [(b_1 + b_2)d^2 + 2(\alpha_2 a_1 - \alpha_1 a_2)d - (b_1 - b_2)(\sigma_1^2 - \sigma_2^2)], \quad (49)$$

where $d = z_2 - z_1$. From Eqs.(43)-(45) we have the total mass m_{tot} , the total charge e_{tot} and the total NUT parameter b_{tot} , which are given by

$$m_{\text{tot}} = \frac{mY_1}{2\alpha X_1}, \quad e_{\text{tot}} = \frac{eY_1}{2\alpha X_1}, \quad b_{\text{tot}} = -\frac{mY_3}{2\alpha X_1}. \quad (50)$$

As for the mass, angular momentum and charge of each Kerr-Newman source we must calculate the Komar integrals. We next compute the behaviors on the z -axis. The Kerr-Newman sources exist along the axis in the regions $w_1 < z < w_2$ and $w_3 < z < w_4$ and their centers are z_1 and z_2 . We divide the z -axis into five segments and denote the regions $z < w_1$, $w_1 < z < w_2$, $w_2 < z < w_3$, $w_3 < z < w_4$ and $w_4 < z$ as I, II, III, IV and V, respectively. To consider the condition for equilibrium we concentrate in the regions I (or V) and III. For the metric function k we can use the vacuum solution and find that it vanishes in both the regions I and V if we choose the constant $C_4 = (16X_1^2)^{-1}$. In the region III we have

$$e^k|_{\text{III}} = \left(\frac{X_3}{X_1}\right)^2, \quad (51)$$

where

$$X_3 = \bar{\sigma}_1 \bar{\sigma}_2 [d^2 - 2(\alpha_1 \alpha_2 + a_1 a_2 + b_1 b_2) - (\sigma_1^2 + \sigma_2^2)]. \quad (52)$$

The metric function A in the region I (or V) is given by

$$A|_{I(V)} = -(+)\frac{m}{\alpha} \frac{Y_3}{X_1}, \quad (53)$$

in accordance with the asymptotic behavior (45). To obtain A in the region III we must integrate Eq.(28). This is not difficult because $\bar{A}$ is a step function along the z -axis. Then we have

$$A|_{III} - A|_I = \frac{(m + \alpha)^2}{4\alpha^2} (\bar{A}|_{III} - \bar{A}|_I) + \frac{(m - \alpha)^2}{4\alpha^2} \{ \bar{A}|_{II} [\bar{\omega}_0(w_1)^2 - \bar{\omega}_0(w_2)^2] + \bar{A}|_{III} \bar{\omega}_0(w_2)^2 - \bar{A}|_I \bar{\omega}_0(w_1)^2 \} \quad (54)$$

where $\bar{\omega}_0$ denotes the function $\bar{\omega}$ on the z -axis.

The condition for equilibrium is given by

$$e^k|_{III} = 1, \quad A|_{I(V)} = A|_{III} = 0. \quad (55)$$

In the two-Kerr-NUT case this condition is

$$e^k|_{III} = 1, \quad \bar{A}|_{I(V)} = \bar{A}|_{III} = 0. \quad (56)$$

The conditions for e^k in Eqs.(54) and (55) have the same expression but their meanings are different. For e^k in Eq.(55) we should replace the parameters σ_i , α_i , a_i and b_i in X_1 and X_3 with $\bar{\sigma}_i$, $\bar{m}_i$, $\bar{a}_i$ and $\bar{b}_i$, respectively. For $\bar{A}|_{I(V)}$ we have $\bar{A}|_{I(V)} = -(+)Y_3/X_1$ because we obtain the two-Kerr-NUT solution by setting $\alpha = m$. Also in this case we should replace the parameters in X_1 and Y_3 . From these facts we find how the conditions for equilibrium in the two-Kerr-NUT solution are modified in the two-Kerr-Newman case. From the conditions for k in the region III we find that the conditions imposed on the mass parameters $\bar{m}_i$ are changed into those on the parameters α_i . For example, if we assume $e^k|_{III} = 1$ for an arbitrary d then we have $\alpha_1\alpha_2 + b_1b_2 = 0$ in the two-Kerr-Newman case while $\bar{m}_1\bar{m}_2 + \bar{b}_1\bar{b}_2 = 0$ in the two-Kerr-NUT case. The parameters α_i must include the effect of the existence of electric charge. Also in the conditions for A and $\bar{A}$ in the region I or V we find only the same modification as far as $\alpha \neq 0$. (The case $\alpha = 0$ corresponds to the conformastationary case.) However, there is much difference in the condition imposed on the metric function A in the region III between the two cases, as is seen from Eq.(53). This might cause the possibility of a new condition for equilibrium of two charged black holes. The results and details will be published in a separate paper.

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Proca Black Hole in Brans-Dicke Theory

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Abstract

We discover a nontrivial black hole solution with non-Abelian Proca field in the Brans-Dicke theory. It is an extension of the Proca black hole in the Einstein theory, where a family of solutions shows a cusp structure in a mass-horizon radius ($M_g - r_h$) diagram. In our system, however, a cusp structure disappears in $M_g - r_h$ diagram. A cusp structure is recovered if we adopt the variables in the Einstein frame. We also discuss the reason from the point of view of black hole thermodynamics.

1 Introduction

The Brans-Dicke(BD) theory was proposed at 1961 by Brans and Dicke based on Mach's Principle[1]. Since then the difference between general relativity(GR) and the BD theory has been discussed in many aspects. Although the BD theory itself is strongly constrained by several experiments (the BD parameter $\omega \geq 500$), we believe that the BD theory may still be important from the following points of view.

1. The BD theory can be an effective field theory of a unified theory of fundamental forces, in particular a superstring theory where the BD scalar field appears as a dilaton field.
2. The BD theory is one of the simplest extension of GR. So if we wish to discuss something in a generalized theory of gravity, the BD theory can be the best model to see a difference from GR.

To see a difference from GR, a black hole(BH) is one of the best objects because it can exist only in strong gravitational field. However, a conventional BH such as a Kerr BH does not show any difference because of BH no-hair theorem[2]. Hence, here we discuss non-Abelian(NA) BH in the BD theory, which may not so far be studied so much. Here, we choose a massive Yang-Mills(YM) field(Proca field) as a NA field. We present the Proca BH numerically, study their properties, and discuss their stability via a catastrophe theory. We then interpret our results from the view point of BH thermodynamics. Throughout this paper we use units $c=\hbar=1$. We define the metric signature as $(-, +, +, +)$.

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2 Basic Equations

We consider a massive SU(2) YM field(Proca field) as an example of NA field. That is because the YM field provides only a colored BH even in the BD theory, which is already known. The action we discuss here is as

$$S = \int d^4x \sqrt{-g} \left[\frac{\phi}{2\kappa^2} R - \frac{\omega}{2\kappa^2 \phi} \phi_{,\alpha} \phi^{,\alpha} - \frac{1}{16\pi g_c^2} \text{Tr} F^2 - \frac{\mu^2}{8\pi g_c^2} \text{Tr} A^2 \right], \quad (1)$$

where $\kappa^2 \equiv 8\pi G$ and g_c is a gauge coupling constant. F is the YM field expressed by its potential A as $F = dA + A \wedge A$. μ is the mass of YM field. The dimensionless BD field ϕ is normalized to be unity at the present epoch. For the BD scalar field ϕ , the field equation becomes

$$\square \phi = \frac{\kappa^2}{2\omega + 3} T^\mu_\mu. \quad (2)$$

So if the right hand side of this equation vanishes as a massless gauge field, no difference from BH in GR may appear. If the YM field is massive, it makes a difference.

We search for a static and spherically symmetric solution, for which the metric is written as

$$ds^2 = - \left[1 - \frac{2Gm(r)}{r} \right] e^{-2\delta(r)} dt^2 + \left[1 - \frac{2Gm(r)}{r} \right]^{-1} dr^2 + r^2 d\Omega^2. \quad (3)$$

For the Proca field, we can write the vector potential as

$$A = a(r) \tau_r dt + \frac{b(r)}{r} \tau_r dr + \{d(r) \tau_\theta - (1 + w(r)) \tau_\phi\} d\theta + \{(1 + w(r)) \tau_\phi + d(r) \tau_\theta\} \sin \theta d\phi \quad (4)$$

from the Witten ansatz. And we adopt 't Hooft ansatz, i.e., $a \equiv 0$, which means that only a magnetic part of Yang-Mills field strength exists. We also set $b = 0^4$. In the static case, we can also set $d = 0$. Now, we have a potential as

$$A = -(1 + w(r)) \tau_\phi d\theta + (1 + w(r)) \tau_\theta \sin \theta d\phi. \quad (5)$$

We use the following dimensionless variables;

$$\bar{r} = r/r_h, \quad \bar{m} = Gm/r_h, \quad \lambda_h = r_h g_c / l_p, \quad \bar{\mu} = \mu / (g_c m_p), \quad (6)$$

where r_h denotes a radius of event horizon. l_p and m_p are the Planck length and mass, respectively. Finally we find the following basic equations.

$$\frac{d\bar{m}}{d\bar{r}} = \frac{1}{2\phi + \bar{r} \frac{d\phi}{d\bar{r}}} \left[\left(1 - \frac{2\bar{m}}{\bar{r}} \right) \left\{ \frac{\omega + 2}{2\phi} \left(\bar{r} \frac{d\phi}{d\bar{r}} \right)^2 + \frac{2}{\lambda_h^2} \left(\frac{dw}{d\bar{r}} \right)^2 \right\} - \bar{m} \frac{d\phi}{d\bar{r}} \right]$$

⁴We can always impose this condition via a gauge transformation, if the YM field is massless. However, in the present case this is just consistent with YM field equation.

$$+ \frac{1}{\lambda_h^2} \left(\frac{1-w^2}{\bar{r}} \right)^2 \left(1 + \frac{\bar{r}}{\phi} \frac{d\phi}{d\bar{r}} \right) + 2(1+w)^2 \bar{\mu}^2 \left\{ \frac{2\omega+1}{2\omega+3} + \frac{2(\omega+1)\bar{r}}{(2\omega+3)\phi} \frac{d\phi}{d\bar{r}} \right\} \Bigg] , \quad (7)$$

$$\begin{aligned} \frac{d\delta}{d\bar{r}} = & -\frac{\bar{r}}{\phi \left(2\phi + \bar{r} \frac{d\phi}{d\bar{r}} \right)} \left\{ (\omega+1) \left(\frac{d\phi}{d\bar{r}} \right)^2 + \frac{4}{\lambda_h^2} \frac{\phi}{\bar{r}^2} \left(\frac{dw}{d\bar{r}} \right)^2 \right\} \\ & - \frac{\bar{r}}{2\phi + \bar{r} \frac{d\phi}{d\bar{r}}} \left(1 - \frac{2\bar{m}}{\bar{r}} \right)^{-1} \left[\frac{1}{\lambda_h^2} \frac{\bar{r}}{\phi} \frac{d\phi}{d\bar{r}} \left\{ \left(\frac{1-w^2}{\bar{r}^2} \right)^2 + \frac{4(\omega+1)\bar{\mu}^2 \lambda_h^2}{2\omega+3} \left(\frac{1+w}{\bar{r}} \right)^2 \right\} \right. \\ & \left. - \frac{2}{\bar{r}} \frac{d\phi}{d\bar{r}} \left(1 - \frac{\bar{m}}{\bar{r}} \right) - \frac{4\bar{\mu}^2}{2\omega+3} \left(\frac{1+w}{\bar{r}} \right)^2 \right] , \end{aligned} \quad (8)$$

$$\begin{aligned} \frac{d^2\phi}{d\bar{r}^2} = & \frac{1}{\phi} \left(\frac{d\phi}{d\bar{r}} \right)^2 + \left(1 - \frac{2\bar{m}}{\bar{r}} \right)^{-1} \left[\frac{1}{\lambda_h^2} \frac{\bar{r}}{\phi} \frac{d\phi}{d\bar{r}} \left\{ \left(\frac{1-w^2}{\bar{r}^2} \right)^2 + \frac{4(\omega+1)\bar{\mu}^2 \lambda_h^2}{2\omega+3} \left(\frac{1+w}{\bar{r}} \right)^2 \right\} \right. \\ & \left. - \frac{2}{\bar{r}} \left(1 - \frac{\bar{m}}{\bar{r}} \right) \frac{d\phi}{d\bar{r}} - \frac{4\bar{\mu}^2}{2\omega+3} \left(\frac{1+w}{\bar{r}} \right)^2 \right] , \end{aligned} \quad (9)$$

$$\begin{aligned} \frac{d^2w}{d\bar{r}^2} = & \frac{1}{\phi} \frac{dw}{d\bar{r}} \frac{d\phi}{d\bar{r}} + \left(1 - \frac{2\bar{m}}{\bar{r}} \right)^{-1} \left[(1+w) \left\{ \bar{\mu}^2 \lambda_h^2 - \frac{w(1-w)}{\bar{r}^2} \right\} \right. \\ & \left. + \frac{dw}{d\bar{r}} \left\{ \frac{1}{\lambda_h^2} \frac{\bar{r}}{\phi} \left(\frac{1-w^2}{\bar{r}^2} \right)^2 - \frac{2\bar{m}}{\bar{r}^2} + \frac{4\bar{\mu}^2(\omega+1)\bar{r}}{(2\omega+3)\phi} \left(\frac{1+w}{\bar{r}} \right)^2 \right\} \right] . \end{aligned} \quad (10)$$

The boundary conditions for a BH solution at spatial infinity are

$$m(\infty) = M = \text{finite}, \quad \delta(\infty) = 0, \quad \phi(\infty) = 1 \quad \text{and} \quad w(\infty) = -1 . \quad (11)$$

And we call M a gravitational mass, because a test particle far from a BH feels this mass, and denote it M_g . For the existence of a regular horizon ($\bar{r} = 1$), we have

$$\bar{m}(1) \equiv \bar{m}_h = \frac{1}{2}, \quad \delta(1) \equiv \delta_h < \infty . \quad (12)$$

Moreover, the square brackets in (8), (9) and (10) must vanish at $\bar{r} = 1$ for the horizon to be regular. Hence

$$\left. \frac{dw}{d\bar{r}} \right|_{\bar{r}=1} = - \frac{\phi_h \lambda_h^2 (2\omega+3) (1+w_h) \{ \bar{\mu}^2 \lambda_h^2 - w_h (1-w_h) \}}{(2\omega+3) \{ (1-w_h^2)^2 - \phi_h \lambda_h^2 \} + 4\bar{\mu}^2 (\omega+1) \lambda_h^2 (1+w_h)^2} , \quad (13)$$

$$\left. \frac{d\phi}{d\bar{r}} \right|_{\bar{r}=1} = \frac{4\bar{\mu}^2 (1+w_h)^2 \lambda_h^2 \phi_h}{\{ (1-w_h^2)^2 (2\omega+3) + 4(\omega+1) \bar{\mu}^2 \lambda_h^2 (1+w_h)^2 \} - \phi_h \lambda_h^2 (2\omega+3)} , \quad (14)$$

where $w_h \equiv w(1)$ and $\phi_h \equiv \phi(1)$. w_h and ϕ_h are shooting parameters and should be determined iteratively so that the boundary conditions (11) are satisfied. We also require that nonexistence of a singularity outside the horizon, i.e., for $r > r_h$,

$$\bar{m}(\bar{r}) < \bar{r}/2 . \quad (15)$$

3 Solutions of Proca Black Hole in Brans-Dicke Theory

Here is an example of the field configurations of a BH solution. We choose the parameters as

$$\lambda_h = 1.0, \quad \bar{\mu} = 0.1, \quad \omega = 0, \quad (16)$$

and dotted lines are those in GR (i.e. $\lambda_h = 1.0, \quad \bar{\mu} = 0.1, \quad \omega = \infty$).

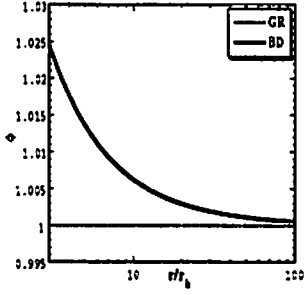


Figure 1: $\bar{r} - \phi$ diagram

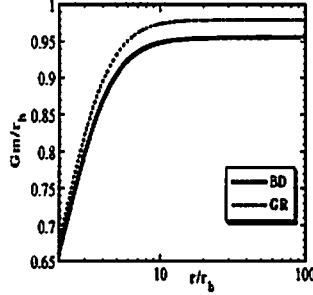


Figure 2: $\bar{r} - \bar{m}$ diagram

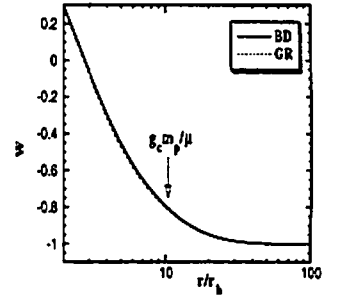


Figure 3: $\bar{r} - w$ diagram

Although $\omega = 0$ is not consistent with the present limit from experiments, we adopt this value because we wish to clarify a difference from the solution in GR. Fig.1 shows that the BD scalar field decreases monotonically as

$$\phi \sim 1 + \frac{\bar{\Sigma}}{\bar{r}}, \quad (17)$$

$$\frac{d\phi}{d\bar{r}} = -\frac{\bar{\Sigma}}{\bar{r}^2}, \quad (18)$$

where $\bar{\Sigma} = \Sigma/r_h$ is a constant and Σ is called a scalar charge. This behavior becomes important later when we discuss stability.

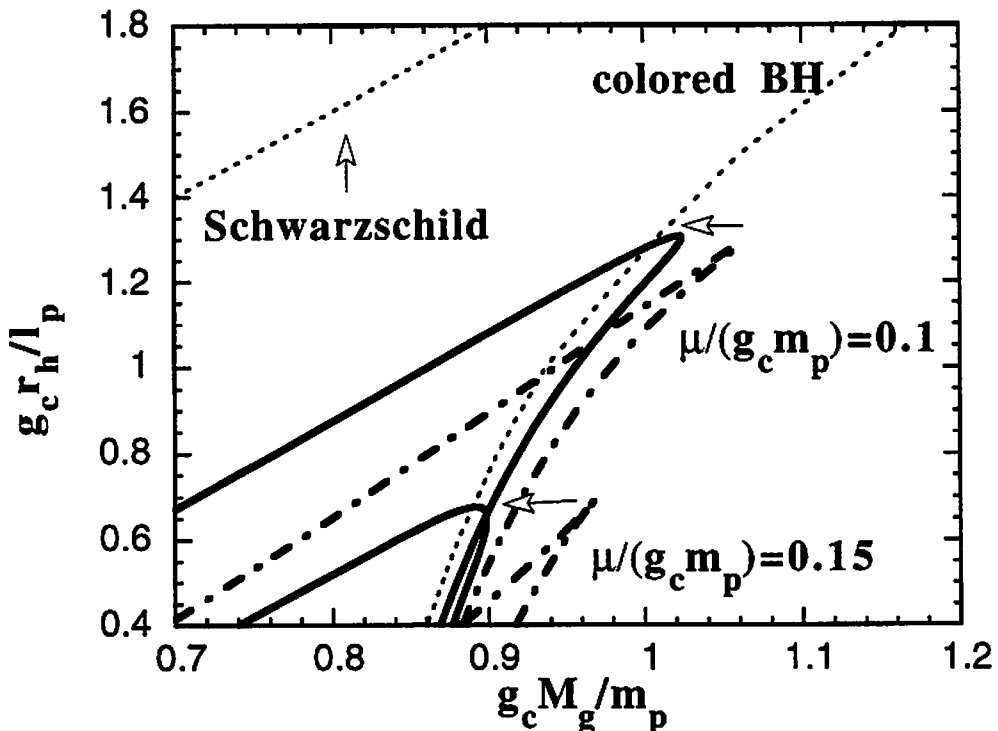


Figure 4: $M_g - r_h$ diagram of the Proca BHs with $\bar{\mu} = 0.1, 0.15$ in the BD theory ($\omega = 0$). The dot-dashed lines are Proca BHs in GR (i.e. $\omega = \infty$).

Fig.3 shows that NA field has a non-trivial structure on the scale of Compton wavelength of the Proca field ($\sim 1/\mu$). This is closely related to an existence of the maximum horizon radius (Fig.4). That is, beyond this critical horizon radius, the non-trivial structure of the NA field is swallowed into the horizon and then can't exist its outside, resulting in a Schwarzschild BH.

If the YM field is massless, a family of colored BHs exists as a non-trivial BH and has no maximum radius as in GR. The BD scalar field is $\phi = 1$. A family of Schwarzschild BHs also exists as a trivial solution ($m = M_g$, $\delta = 0$, $\phi = 1$, and $w = -1$). For massive YM field, Proca BH solutions in BD theory ($\omega = 0$) shift in the direction of arrow from those in GR. In GR, the solutions above have a cusp structure at a critical horizon radius which is important for stability analysis via a catastrophe theory[3]. However no cusp structure appears in the BD theory. This tendency becomes more clear for $\bar{\mu} = 0.15$ than

for $\bar{\mu} = 0.1$.

We also show the thermodynamical properties of Proca BHs in Figs.5 and 6. Note that the BH entropy S does not necessarily mean one quarter of BH area in the generalized theory of gravity[4, 5, 6]. In GR, maximum points of entropy and M_g coincide each other, i.e., the point C in Fig.5 and 6. In the BD theory, however those points, the point A (maximum point of entropy) and the point B (maximum point of M_g), are different with each other⁵. The temperature in the BD theory behaves qualitatively similar to that in GR, while the entropy is quite different. In particular, a cusp structure disappears in the BD theory as mentioned before, we may wonder whether a catastrophe theory can be applied to a stability analysis of NA BH in the BD theory, although it works quite well in GR[7].

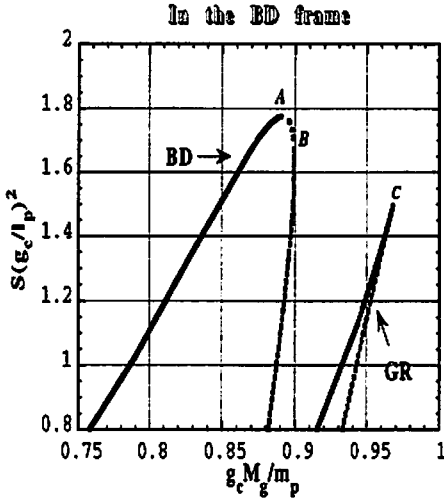


Figure 5: Comparing solutions in the BD theory and in GR in $M_g - S$ diagram. Mass of gauge field is $\bar{\mu} = 0.15$. In the BD theory ω is chosen to be 0.

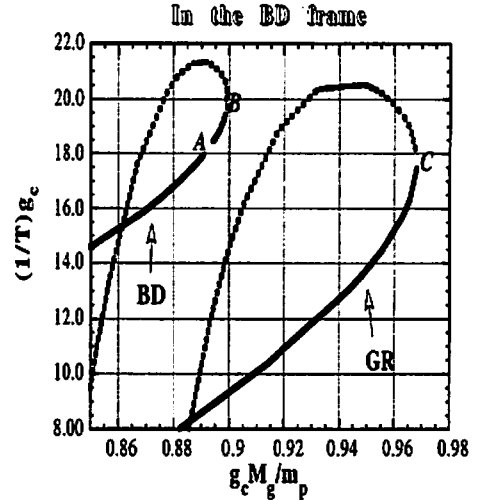


Figure 6: Comparing solutions in the BD theory and in GR in $M_g - 1/T$ diagram. Mass of gauge field is $\bar{\mu} = 0.15$. In the BD theory ω is chosen to be 0.

4 In Einstein Frame

To understand the behavior of NA BH in the BD theory better, it may be convenient to see the solution in the Einstein frame. The BD theory is equivalent to the Einstein

⁵The reader may think the selection of BD parameter $\omega = 0$ affect these results. So we calculated in other value of ω . The result is same in the point cusp no structure appeared in $M_g - S$ diagram.

theory via a conformal transformation $\hat{g}_{ab} = \phi g_{ab}$. The equivalent action becomes

$$\hat{S} = \int d^4\hat{x} \sqrt{-\hat{g}} \left[\frac{\hat{R}}{2\kappa^2} - \frac{1}{2} \varphi_{,\alpha} \varphi^{,\alpha} - \frac{1}{16\pi g_c^2} \text{Tr} F^2 - \frac{1}{8\pi g_c^2} \exp(-\kappa\beta\varphi) \mu^2 \text{Tr} A^2 \right], \quad (19)$$

$$\beta \equiv \left(\frac{2\omega + 3}{2} \right)^{-1/2} \quad \varphi = \frac{1}{\kappa\beta} \ln \phi. \quad (20)$$

For a BH solution, we find

$$\hat{M}_g = M_g + \frac{\Sigma}{2}, \quad \hat{r}_h = r_h \sqrt{\phi_h}, \quad (21)$$

where a caret $\hat{\cdot}$ denotes a variable in the Einstein frame. Equations (21) show that the entropy in the Einstein frame $\hat{S}$ is

$$\begin{aligned} \hat{S} = \frac{\hat{A}}{4} &= \pi \hat{r}_h^2 \\ &= \pi r_h^2 \phi_h = S. \end{aligned} \quad (22)$$

Using the variables in the Einstein frame, we drew the entropy S and the inverse temperature $1/T$ in terms of the gravitational mass $\hat{M}_g$ in Figs.7 and 8.

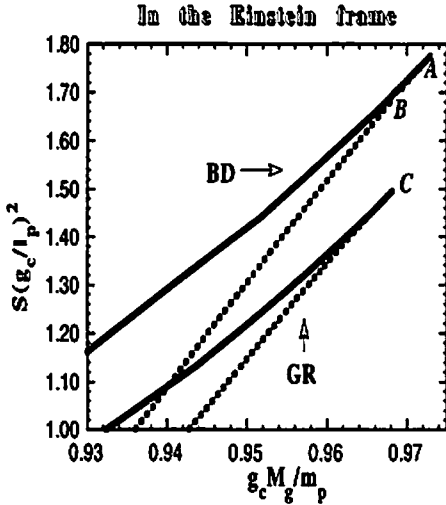


Figure 7: $\hat{M}_g - S$ diagram in the Einstein frame. It is connected to Fig.5 by conformal transformation.

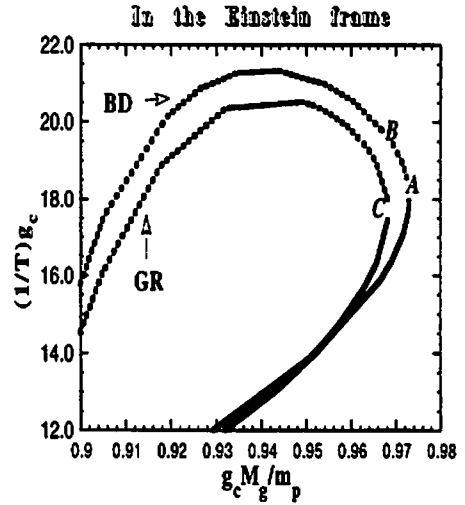


Figure 8: $\hat{M}_g - 1/T$ diagram in the Einstein frame. It is connected to Fig.6 by conformal transformation.

Now, we find a cusp structure even for the Proca BH in the BD theory. This suggests that a catastrophe theory[3] may also apply to a stability analysis of BH. We also understand how the BH solutions in the BD theory shift from those in GR. That is, in the

Einstein frame, the mass of YM field gets smaller by a factor $\exp(-\kappa\beta\varphi/2)$ than that in GR (μ), resulting in a shift in the direction of a Schwarzschild branch.

5 Black Hole Thermodynamics in Brans-Dicke Theory

How can we understand those results, in particular a stability of Proca BH, in the BD frame? In the Einstein frame, our results are same as those in GR, but in the BD frame those are quite different. For resolving this problem, we investigate the BH thermodynamics in the BD theory. Fortunately, for an arbitrary diffeomorphism invariant theory of gravity, the first law of BH thermodynamics is studied by Iyer and Wald[4]. In their papers, the canonical mass M_c and canonical entropy S_c are defined to satisfy the first law of BH thermodynamics.

Following their method, we have calculated them for the BD theory. The result is as follows

$$M_c = M + \frac{\Sigma}{2}, \quad (23)$$

$$S_c = \frac{A}{4}\phi_h = \pi r_h^2 \phi_h, \quad (24)$$

which are exactly those in the Einstein frame. A catastrophe theory seems to be applied to a stability of NA BH even in the BD theory, since a cusp structure appears if we adopt the variables in the Einstein frame, which are the same as their canonical variables.

For simplicity we discuss only an isolated system. From the point of view of catastrophe theory[3], stability changes at the point of cusp in control parameter-potential function diagram. In GR, if we assume that the control parameter is M_g and the potential function is entropy of BH, a catastrophe theory suggests that stability changes at the point C in Figs.5 and 7. However in the BD theory, a cusp structure does not appear in Fig.5, while appears in Fig.7. From Fig.7, a catastrophe theory suggests that stability change can occur at the point A . From Fig.5, however, no such prediction is possible. On the other hand, if we apply a turning point method, stability changes at the point $d(1/T)/dM = \infty$ [8]. However this M should satisfy the first law of BH thermodynamics. So in GR, we understand that stability change can occur at the point C in Figs.6 and 8 and this is consistent with a catastrophe theory[7]. While, in the BD theory, we should not apply it in Fig.6, because M_g in the BD frame does not satisfy the first law of BH thermodynamics. However, the turning point method can apply in Fig.8 because the first law is satisfied. Then we expect that stability change can occur at the point A and this is consistent with a catastrophe theory.

6 Conclusion and Discussion

We have analyzed NA BH (Proca BH) in the BD theory and shown some differences from those in GR. We find that in $M_g - S$ diagram, a cusp structure does not appear in the BD theory although it exists in GR. However, if we use the variables in the Einstein frame, a cusp structure appears. Since the variables in the Einstein frame are exactly the same as the canonical variables defined by Iyer and Wald, the BH thermodynamics may be also very important in stability analysis of BH.

Hence we may conjecture that we can apply a catastrophe theory to stability analysis of BH when we use the canonical thermodynamical variables. We shall give one conjecture from our analysis:

Conjecture: *The stability of a BH can be discussed by thermodynamics and a catastrophe theory or a turning point method may be one of the simplest ways to analyse.*

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A Static Model for Distorted Superdense Star by the Companion

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Abstract

An analytical model for the superdense star, which is slightly distorted by the tidal force of the companion is presented. This model is a modification of Schwarzschild's interior solution. Instead of his exterior solution, I have used a classically modified exterior solution, which is matched to my inner solution at the distorted surface.

1 Introduction

Using the Roche harmonics introduced by Kopal [1], Singh [2, 3] obtained the explicit expressions of Roche coordinates for tidally distorted stellar models, and Chakravarti and Singh [4] analyzed those associated with Roche tidal equipotential surfaces up to second-order terms of the tidal parameter “ q ”. As this case of Roche harmonics for tidally-distorted stellars, we should take tidal distortion into general relativistic Darwin's problem account.

In the past papers on analytical models for the interior solutions of superdense stars, many authors used a common set of the boundary conditions: The gravitational potentials and the metrics of the interior solutions are continuous to those of the Schwarzschild's exterior solution at the spheres of the stellar surfaces. And the density and the pressure are zero at the surfaces.

In this paper we have used a little different boundary condition as follows: The gravitational potential and the metrics of the interior solution are continuous to those of the modified Schwarzschild's exterior solution at the tidally distorted stellar surface. And the density and the pressure are zero at the surface too.

In order to construct the modified Schwarzschild's exterior solution, we have used the classical Roche potential, and have assumed the tidally distorted stellar surface to be one of the Roche equipotential as the following Section.

2 The classical potential

The total potential acting at an arbitrary point $P(x, y, z)$ outside the stars is

$$\psi' \cong -G \frac{M}{r'} - G \frac{m}{R'} - \frac{\omega'^2}{2} \left\{ \left(X' - \frac{m}{M+m} r_0 \right)^2 + y'^2 \right\}, \quad (1)$$

where M is the mass of the main star and m is that of its companion, and using the distance of the two stars r_0 , $r'^2 = X'^2 + Y'^2 + Z'^2$ and $R'^2 = (r_0 - X')^2 + y'^2 + Z'^2$.

Substituting $r' = r_0 r$, $R' = r_0 R$, $X' = r_0 X$, $y' = r_0 y$, $m = Mq$,

$$\omega' = \sqrt{GM/r_0^3} \omega, \text{ and } \psi' = GM\psi/r_0, \quad (2)$$

we obtain

$$\psi = -\frac{1}{r} - \frac{q}{R} - \frac{\omega^2}{2} \left\{ \left(X - \frac{q}{1+q} \right)^2 + y^2 \right\}. \quad (3)$$

Suppose ω' is identical with the Keplerian one,

$$\omega^2 = 1 + q. \quad (4)$$

If we use the spherical coordinate system:

$$x = r \cos \phi \sin \theta = rl$$

$$y = r \sin \phi \sin \theta = rm$$

$$z = r \cos \theta = rn$$

$$\psi_0 = 1 - \frac{K}{r} - Kq \left\{ \frac{1}{\sqrt{1 - 2lr + r^2}} - lr \right\} - \frac{K(1+q)r^2}{2} (1 - n^2) \quad (5)$$

where

$$\psi_0 = -\frac{q^2}{2(1+q)} - \psi. \quad (5')$$

Therefore we can assume the surface of the star to be the equipotential expressed as:

$$r \cong a + h = a + \frac{a^4 \{3ql^2 - (1+q)n^2\}}{2}. \quad (6)$$

Where a is the non-dimensional standard radius of the star, which is the distance from the center to the surface toward $n = l = 0$. Like the surface of the earth the shape of the star is estimated to be oval, and is bulged toward $l = 1, -1$; namely toward its companion and the opposite side. Note that the surface is also bulged toward the orbital plane by centrifugal force even in the case of $q = 0$, because the matter in the main star is assumed to be co-rotating with the companion like the moon to the earth.

3 The modified Schwarzschild's exterior solution

We have assumed the metric in the vicinity of the main star M to be of the form:

$$ds^2 = -e^{\lambda(r,\theta,\phi)} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) + e^{\nu(r,\theta,\phi)} dt^2. \quad (7)$$

Therefore the Ricci tensor which is not obviously zero is

$$R_{11} = \frac{\partial \Gamma_{1\alpha}^\alpha}{\partial x^1} - \frac{\partial \Gamma_{11}^\alpha}{\partial x^\alpha} + \Gamma_{1\alpha}^\beta \Gamma_{\beta 1}^\alpha - \Gamma_{11}^\beta \Gamma_{\beta\alpha}^\alpha$$

is small compared with 1.

$$K = \frac{r_0 C^2}{2GM}$$

where

$$e^{-\lambda(r,\theta,\phi)} = e^{\nu(r,\theta,\phi)} \approx 1 - \frac{r}{K} - Kq \left\{ 1 - \frac{2}{(3l^2 - 1)r^2} \right\} - \frac{2}{K(1+q)r^2} (1 - n^2), \quad (9)$$

or approximating within $O(l^2)$

$$e^{-\lambda(r,\theta,\phi)} = e^{\nu(r,\theta,\phi)} = 1 - \frac{r}{K} - Kq \left\{ \frac{1}{1 - 2lr + r^2} - lr \right\} - \frac{2}{K(1+q)r^2} (1 - n^2), \quad (9)$$

following form

Taking the physical situation far away from the main star into account [5], we get the

$$(8) \quad \begin{aligned} R_{44} = & \frac{\partial^2 x_4}{\partial t^2} - \frac{\partial^2 x_4}{\partial t^2} + \Gamma_{\theta}^{44} \Gamma_{\theta}^{44} - \Gamma_{\theta}^{44} \Gamma_{\theta}^{44} \\ & - \sin^2 \theta \cos \theta \left(\frac{1}{\partial \lambda} \frac{1}{\partial \nu} + \frac{2}{\partial \theta} \right), \\ & + \frac{1}{\partial^2 \nu} + \frac{2}{\partial^2 \lambda} + \frac{2}{\partial^2 \phi} + \frac{2}{\partial^2 \lambda} + \frac{4}{\partial^2 \phi} + \frac{4}{\partial^2 \phi} \left(\frac{\partial \phi}{\partial \lambda} \right)^2 \\ & = \sin^2 \theta e^{-\lambda} \left(\frac{1}{\partial \nu} \frac{\partial r}{\partial \lambda} + \frac{\partial r}{\partial \lambda} + 1 \right) - \sin^2 \theta \\ R_{33} = & \frac{\partial^2 x_3}{\partial t^2} - \frac{\partial^2 x_3}{\partial t^2} + \Gamma_{\theta}^{33} \Gamma_{\theta}^{33} - \Gamma_{\theta}^{33} \Gamma_{\theta}^{33} \\ & = \frac{2}{\partial \lambda} \frac{\partial r}{\partial \nu} - \frac{\partial r}{\partial \nu} + e^{-\lambda} - 1 + \frac{2}{\partial^2 \lambda} + \frac{2}{\partial^2 \nu} + \frac{2}{\partial^2 \theta} + \frac{4}{\partial^2 \phi} + \frac{4}{\partial^2 \phi} \left(\frac{\partial \theta}{\partial \nu} \right)^2 \\ R_{22} = & \frac{\partial^2 x_2}{\partial t^2} - \frac{\partial^2 x_2}{\partial t^2} + \Gamma_{\theta}^{22} \Gamma_{\theta}^{22} - \Gamma_{\theta}^{22} \Gamma_{\theta}^{22} \\ & + \frac{2}{(r^2 \sin^2 \theta)^{-1}} e^{\lambda} \left(\frac{1}{\partial \lambda} \frac{\partial \phi}{\partial \lambda} + \frac{\partial \phi}{\partial \lambda} + \frac{2}{\partial^2 \lambda} + \frac{2}{\partial^2 \phi} \right) \\ & + \frac{1}{\partial^2 \nu} e^{\lambda} \left(\frac{\partial^2 \lambda}{\partial \lambda} + \frac{2}{\partial \lambda} \frac{\partial \theta}{\partial \nu} + \frac{2}{\partial \theta} \right) + \frac{r}{\partial \nu} \cot \theta \\ & = \frac{1}{\partial^2 \nu} - \frac{1}{\partial \lambda} \frac{\partial r}{\partial \nu} + \frac{4}{\partial \nu} \frac{\partial r}{\partial \lambda} + \frac{4}{\partial \nu} \left(\frac{\partial r}{\partial \lambda} \right)^2 - \frac{r}{\partial \nu} \end{aligned}$$

4 The density distribution and the boundary conditions

The density of fluid inside the main star is assumed to be continuous and zero at the surface, therefore taking the physical situation in the main star into account [6], we can assume the variation of density as

$$\rho = \rho_c \left\{ 1 - r^2 / (a + h)^2 \right\}, \quad (10)$$

where ρ_c is the density at the center of the main star, and a is the standard radius of the star. As the gravitational potential and the metrics of the interior solution are continuous with those of the modified Schwarzschild's exterior solution, we must construct our interior solution to coincident with (9') at the surface (6).

5 The interior solution of the Einstein equation

Making the Einstein tensor from Eq.(8) and neglecting the small terms in the Einstein equation, we have been able to integrate it to satisfy the boundary conditions as follows

$$e^{-\lambda(r,\theta,\phi)} \cong 1 - \left[\frac{8\pi\rho_c}{3} + \frac{K \{3ql^2 - (1+q)n^2 + 1\}}{2} \right] r^2 + \frac{8\pi\rho_c}{5a^2} r^4. \quad (11)$$

And introducing

$$z = \log \left[\frac{r^2}{a^2} - \left(\frac{5}{6} + \frac{5K \{3ql^2 - (1+q)n^2 + 1\}}{32\pi\rho_c} \right) \sqrt{\frac{r^4}{a^4} - \left[\frac{5}{3} + \frac{5K \{3ql^2 - (1+q)n^2 + 1\}}{16\pi\rho_c} \right] \frac{r^2}{a^2} + \frac{5}{8\pi\rho_c a^2}} \right], \quad (12)$$

we have been able to integrate the Einstein equation to give

$$e^{\nu(r,\theta,\phi)} = \left(C_1 \sin \frac{z}{2} + C_2 \cos \frac{z}{2} \right)^2, \quad (13)$$

$$8\pi P = \frac{4}{a} \sqrt{\frac{2\pi\rho_c}{5} \left\{ 1 - \left[\frac{8\pi\rho_c}{3} + \frac{K \{3ql^2 - (1+q)n^2 + 1\}}{2} \right] r^2 + \frac{8\pi\rho_c}{5a^2} r^4 \right\}} \\ \left\{ \frac{C_1 \cos \frac{z}{2} - C_2 \sin \frac{z}{2}}{C_1 \sin \frac{z}{2} + C_2 \cos \frac{z}{2}} \right\} - \frac{16\pi\rho_c}{5} \left\{ \frac{5}{6} + \frac{5K \{3ql^2 - (1+q)n^2 + 1\}}{32\pi\rho_c} - \frac{r^2}{2a^2} \right\} \quad (14)$$

where C_1 and C_2 are constants determined by the boundary conditions; at $r = a + h$

$$e^{\nu(r,\theta,\phi)} = \left(C_1 \sin \frac{z_s}{2} + C_2 \cos \frac{z_s}{2} \right)^2 \\ \cong 1 - \left[\frac{8\pi\rho_c}{3} + \frac{K \{3ql^2 - (1+q)n^2 + 1\}}{2} \right] (a + h)^2 + \frac{8\pi\rho_c}{5a^2} (a + h)^4, \quad (15)$$

and $P = 0$, where z_s is the value of z at $r = a + h$ in Eq.(12).

From Eq.(14) and Eq.(15), we get C_1 and C_2 as,

$$C_1 = \sqrt{1 - \left[\frac{8\pi\rho_c}{3} + \frac{K \{3ql^2 - (1+q)n^2 + 1\}}{2} \right] (a+h)^2 + \frac{8\pi\rho_c}{5a^2} (a+h)^4} \sin \frac{z_s}{2} \\ + 2a \sqrt{\frac{2\pi\rho_c}{5}} \left\{ \frac{5}{6} + \frac{5K \{3ql^2 - (1+q)n^2 + 1\}}{32\pi\rho_c} - \frac{(a+h)^2}{2a^2} \right\} \cos \frac{z_s}{2} \quad (16)$$

$$C_2 = \sqrt{1 - \left[\frac{8\pi\rho_c}{3} + \frac{K \{3ql^2 - (1+q)n^2 + 1\}}{2} \right] (a+h)^2 + \frac{8\pi\rho_c}{5a^2} (a+h)^4} \cos \frac{z_s}{2} \\ - 2a \sqrt{\frac{2\pi\rho_c}{5}} \left\{ \frac{5}{6} + \frac{5K \{3ql^2 - (1+q)n^2 + 1\}}{32\pi\rho_c} - \frac{(a+h)^2}{2a^2} \right\} \sin \frac{z_s}{2}. \quad (17)$$

The value of z at $r = a + h$ z_s is

$$z_s = \log \left[\left(1 + \frac{h}{a} \right)^2 - \left(\frac{5}{6} + \frac{5K \{3ql^2 - (1+q)n^2 + 1\}}{32\pi\rho_c} \right) \right. \\ \left. + \sqrt{\left(1 + \frac{r}{a} \right)^4 - \left[\frac{5}{3} + \frac{5K \{3ql^2 - (1+q)n^2 + 1\}}{16\pi\rho_c} \right] \left(1 + \frac{h}{a} \right)^2 + \frac{5}{8\pi\rho_c a^2}} \right] \quad (18)$$

6 Discussion

Although we have neglected the higher derivatives of the metric functions by θ and ϕ in Einstein equation, the resultant pressure distribution is distorted by the density distribution and the boundary conditions. Therefore this calculation seems to be the good zeroth order approximation of the tidally distorted superdense star. However, the rotational condition used in this paper may be seem to be very special, and the angular velocity should be different from the orbital one. The reason why we have used this special rotational condition is that this is the only way to calculate the distorted superdense star without the detail of the dynamical property of the fluid in the star.

Moreover, because the external potential should be modified by this distorted density distribution, we must calculate the effect of the distortion to the external potential. This is the next step of our calculation.

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Gravitationally Induced Phase on the Neutrino Oscillation

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Abstract

Neutrino oscillation in the presence of gravitational field is examined. The quantum-mechanical phase mixing between different mass eigenstates is calculated under the weak gravity approximation. We also briefly review proposed test of the equivalence principle due to the neutrino flavor mixing, i.e., strict constraint of a long-range, flavor-changing tensor interaction, if exists.

1 Introduction

In the standard model of the particle physics, neutrinos are massless particles. The model has so far been very successful at low energy process. However, some particle physicists have been explored the new physics beyond the standard model. The possibility of massive neutrinos and the astrophysical/theoretical consequences have been extensively studied. See., e.g., [14] as for the comprehensive textbook of massive neutrinos. If the neutrino is a mixed state, the oscillation between different flavor states occurs. The flavor eigen-state is linear superposition of different mass eigen-states, which evolve in different way, so that the mixing rate changes with the increase of the path length L . The mixing phase angle for the relativistic neutrinos propagating in a flat space-time is given by $\varphi_0 = \Delta m^2 L / (4\hbar E)$, where E is energy and $\Delta m^2 = m_1^2 - m_2^2$. (See section 2.)

Recently, the neutrino oscillation phase due to the presence of gravity is much discussed [1], [2], [3], [4], [6], [9], [12]. Gravitational effect was not seriously studied so far. Gravity can be eliminated by choosing appropriate inertial frame locally. The propagating distance of the neutrinos is so long in some case, that gravitational effect may become important. Ahluwalia and Burgard [1] considered the gravitational field of a star with mass M adds a new contribution to the phase difference, denoted by $\varphi_G = -\Delta m^2 L \langle \phi \rangle / (4\hbar E)$, where $\langle \phi \rangle$ is defined by the average of gravitational potential over the semi-classical path, i.e., $\langle \phi \rangle = -(\int dLGM/r)/L$. This gravitationally induced phase can be estimated as $\varphi_G = -\langle \phi \rangle \varphi_0$. The phase becomes to the extent of roughly 20% of φ_0 , near neutron stars. They suggested that the new oscillation phase may be significant effect on the supernova explosions, since the extremely large fluxes of neutrinos are produced with different energies corresponding to the flavor states.

The gravitationally induced oscillation phase may have the important astrophysical consequences. However, their derivation and even the definitions such as energy were not clear in their original paper. After the work of Ahluwalia and Burgard [1], several groups have attacked this problem with various methods, [3], [4], [6], [9], [12]. Bhattacharya, Habib and Mottola [3] critically re-examined the quantum mechanical phase mixing. They calculated the phase difference for radially propagating particles, and found that the term of $\Delta m^2 L(\phi)/(\hbar E) \sim GM\Delta m^2 \log(r_B/r_A)/(\hbar E)$ is canceled out, where r_A and r_B are respectively the distances from the central astrophysical object with mass M to the created and detected positions. They showed that the possible gravitational effect appears at the higher order, $\Delta m^4/E^3$, and that the phase difference for radially propagating particles is $GM\Delta m^4 \log(r_B/r_A)/(4\hbar E^3)$. Numerically its magnitude is equal to $\sim 10^{-9} (M/M_\odot) (\Delta m^4/\text{eV}^4) (E/\text{MeV})^{-3} \log(r_B/r_A)$, which is completely negligible in typical astrophysical applications. Therefore, the conclusion of Ahluwalia and Burgard [1] seems to be incorrect for radially propagating case. Natural question is what happens in more general cases. Does the term of $GM\Delta m^2/E$ always disappear?

Furthermore, several authors have discussed the possibility of the violation of equivalence principle in the neutrino oscillation, e.g., [10], [13] and references therein. If the universality of the gravitational couplings to different flavors breaks down, additional phase difference appears. Very interesting interpretation of the solar and atmospheric neutrinos, and severe constraint from distant astrophysical sources are proposed. We will briefly review the effect of the violation of equivalence principle on the neutrino oscillation in section 4.

There are also additional factors of the neutrino mixing. If the neutrinos travel through a material medium, the patterns of the oscillation might be significantly affected (Mikheyev-Smirnov-Wolfenstein effect). If the neutrinos have non-vanishing magnetic moment, then the interaction with the external magnetic field may be important. We will not deal with these subjects in this paper. In section 2, we will review the mixing of different flavor states propagating in vacuum by the wave packet formalism. In section 3, we will extend to the propagation in curved space. Finally, some concluding remarks are given in section 5. This paper except sections 4 and 5 is mainly based on Ref. [12].

2 Flavor mixing in flat space-time

In this section, we use wave packet formalism [7], [11] to examine the evolution of mass eigenstates. Let us consider a weak flavor eigenstate $|\nu_\alpha\rangle$ ($\alpha = e, \mu$, or τ), created with a certain energy E . The flavor eigenstate is a linear superposition of mass eigenstates, which are represented by $|\nu_a\rangle$,

$$|\nu_\alpha\rangle = \sum_a U_{\alpha a} |\nu_a\rangle, \quad (1)$$

where $U_{\alpha a}$ is unitary mixing matrix. We assume that each mass eigenstate is created with the same energy as the Gaussian wave form in momentum space. The direction of the momentum is the same, but the mean value $\bar{p}$ is different due to different mass, i.e., $\bar{p}^2 = E^2 - m_a^2$. The wave packet for mass eigenstate, a can be written in coordinate space

as

$$\Psi_a(\vec{x}, t) = (2\pi\sigma^2)^{-3/4} \exp\left(\frac{iS_a}{\hbar}\right) \exp\left(-\frac{|\vec{x} - \vec{z}_a|^2}{4\sigma^2}\right), \quad (2)$$

where S is phase function and should satisfy $g^{\mu\nu}S_{,\mu}S_{,\nu} + m_a^2 = 0$ in geometrical optics limit. The size of the wave packet, σ is assumed to be much smaller than the curvature radius of the external gravitational field. The center of the wave packet, $\vec{z}_a(t)$ is determined by classical geodesic due to Ehrenfest's theorem.

The quantum mechanical transition probability from $|\nu_\alpha\rangle$ to $|\nu_\beta\rangle$ is

$$\begin{aligned} P_{\alpha\rightarrow\beta}(\vec{x}, t) &= \left| \sum_a U_{\beta a}^* \Psi_a(\vec{x}, t) U_{\alpha a} \right|^2 \\ &= (2\pi\sigma^2)^{-3/2} \sum_{ab} U_{\beta a} U_{\alpha a}^* U_{\beta b} U_{\alpha b}^* \exp\left(\frac{i}{\hbar}(S_a - S_b)\right) F, \end{aligned} \quad (3)$$

where

$$\begin{aligned} F(\vec{x}, t) &= \exp\left(-\frac{1}{4\sigma^2}(|\vec{x} - \vec{z}_a|^2 + |\vec{x} - \vec{z}_b|^2)\right) \\ &= \exp\left(-\frac{1}{2\sigma^2}|\vec{x} - \frac{1}{2}(\vec{z}_a + \vec{z}_b)|^2\right) \exp\left(-\frac{1}{8\sigma^2}|\vec{z}_a - \vec{z}_b|^2\right). \end{aligned} \quad (4)$$

The function F has a peak at $(\vec{z}_a + \vec{z}_b)/2$, so that the phase factor can be evaluated by taking the dominant contribution in the stationary point. The factor $\exp(-|\vec{z}_a - \vec{z}_b|^2/(8\sigma^2))$ represents that the coherence will be lost if two orbits separate much more than the size of the wave packet. The unitary matrix for the mixing between two generations is parameterized by θ ,

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \quad (5)$$

If the deviation of the orbit is negligible, then we have

$$P_{\alpha\rightarrow\beta}(t) = \sin^2(2\theta) \sin^2(\varphi), \quad (6)$$

where φ is the phase angle between two states, $\varphi = -\Delta S/(2\hbar) = (S_2 - S_1)/(2\hbar)$. In the opposite case, we have time-averaged one,

$$P_{\alpha\rightarrow\beta} = \frac{1}{2} \sin^2(2\theta). \quad (7)$$

We now estimate the split of the orbit, $\vec{z}_1 - \vec{z}_2$, and the phase difference $\Delta S = -2\hbar\varphi$, between two mass eigenstates. The classical trajectory of a particle with mass m_a in the flat space-time can be represented by the straight line,

$$\vec{z}_a = \left(\sqrt{1 - \frac{m_a^2}{E^2}} t, 0, 0 \right). \quad (8)$$

The phase S can be written with the four momentum, $p_\mu = (p_t, p_x, p_y, p_z) = (-E, \sqrt{E^2 - m_a^2}, 0, 0)$ as

$$\begin{aligned} S &= \int p_\mu dx^\mu \\ &= -E t + \sqrt{E^2 - m_a^2} x, \end{aligned} \quad (9)$$

where the propagating direction is chosen as the x -axis and the origin is the emitted point. The split of the trajectory Δx and the phase difference for relativistic particles, $E \gg m_a$ are given by

$$\Delta x \simeq -\frac{\Delta m^2}{2E^2} L, \quad (10)$$

and

$$\Delta S \simeq -\frac{\Delta m^2}{2E} L, \quad (11)$$

where we have used the path length $L = t$. The phase angle is given by,

$$\varphi_0 \simeq \frac{\Delta m^2 L}{4\hbar E}. \quad (12)$$

In this way, we have the standard result. In this section, we have assumed that two mass eigen-states have the same energy E , but it is possible to extend to the case of the energy distribution with the on-shell condition, $\bar{E}^2 - \bar{p}^2 = m_a^2$. However, we keep in the following sections for simplicity, the assumption that each mass eigen-state creates with the same energy.

3 Flavor mixing in the curved space-time

We will extend the results of the previous section to those in the curved space-time. In order to examine the gravitational effects explicitly, the calculation is limited to the order G^1 , i.e., the weak field approximation. The classical trajectory of a particle around non-rotating star with mass M can be described by the energy at infinity E and angular momentum J . They are conserved along the motion. The orbital plane may be chosen as the equatorial plane of the spherical coordinate. Then the trajectory leaving at $t = 0$ from $(r, \chi) = (r_A, 0)$ can be written as

$$\chi = \int_{r_A}^r \frac{1}{\sqrt{B}} \frac{J}{r^2} dr, \quad (13)$$

$$t = \int_{r_A}^r \frac{E}{\sqrt{B}} \left(1 - \frac{2GM}{r}\right)^{-1} dr, \quad (14)$$

where

$$B = E^2 - \left(1 - \frac{2GM}{r}\right) \left(m^2 + \frac{J^2}{r^2}\right). \quad (15)$$

We will consider unbounded orbit, i.e., $E \gg m$. It is convenient to use the turning point, r_0 , of the radial direction to calculate the above integrals. Eliminating J and expanding by G , we have

$$\chi = \left[-\sin^{-1} \frac{r_0}{r} + \frac{GM}{1-q} \frac{2r+r_0}{rr_0} \sqrt{\frac{r-r_0}{r+r_0}} - \frac{qGM}{1-q} \frac{\sqrt{r^2-r_0^2}}{rr_0} \right]_{r_A}^r, \quad (16)$$

$$\sqrt{1-qt} = \left[\sqrt{r^2-r_0^2} + \frac{GM}{1-q} \sqrt{\frac{r-r_0}{r+r_0}} + \frac{(2-3q)GM}{1-q} \log \left(\frac{r + \sqrt{r^2-r_0^2}}{r_0} \right) \right]_{r_A}^r, \quad (17)$$

where $q = m^2/E^2$, and $[f(r)]_{r_A}^r = f(r) - f(r_A)$. The first terms in eqs. (16) and (17) represent straight lines in the spherical coordinate. The first order terms in G are well known effects on the propagation of massless particles. The second term in (16) is relevant to deflection of light, and the logarithmic term in (17) is relevant to the time delay in the radar echo experiment. (See, e.g., [16].)

The phase S is given by the four momentum, p_μ conjugate to x^μ as,

$$S = \int p_\mu dx^\mu = \int (-Edt + p_i dx^i). \quad (18)$$

Both mass eigenstate are assumed to have the same conserved energy E , so that the first term $\int E dt$ is canceled out.¹ We only calculate the contribution from the second term. The phase function can be evaluated for the classical path of the particle with mass m and energy E as

$$S = \sqrt{E^2 - m^2} \times \left[\sqrt{r^2 - r_0^2} + GM \left(\frac{2-q}{1-q} \log \left(\frac{r + \sqrt{r^2 - r_0^2}}{r_0} \right) + \frac{1}{1-q} \sqrt{\frac{r-r_0}{r+r_0}} - 2 \frac{\sqrt{r^2 - r_0^2}}{r} \right) \right]_{r_A}^r. \quad (19)$$

Expanding by m^2/E^2 for the relativistic particles, we have

$$\begin{aligned} S \simeq & E \left[\sqrt{r^2 - r_0^2} + GM \left(2 \log \left(\frac{r + \sqrt{r^2 - r_0^2}}{r_0} \right) + \sqrt{\frac{r-r_0}{r+r_0}} - 2 \frac{\sqrt{r^2 - r_0^2}}{r} \right) \right]_{r_A}^r \\ & - \frac{m^2}{2E} \left[\sqrt{r^2 - r_0^2} - GM \left(\sqrt{\frac{r-r_0}{r+r_0}} + 2 \frac{\sqrt{r^2 - r_0^2}}{r} \right) \right]_{r_A}^r \\ & - \frac{m^4}{8E^3} \left[\sqrt{r^2 - r_0^2} - GM \left(2 \log \left(\frac{r + \sqrt{r^2 - r_0^2}}{r_0} \right) + 3 \sqrt{\frac{r-r_0}{r+r_0}} + 2 \frac{\sqrt{r^2 - r_0^2}}{r} \right) \right]_{r_A}^r \\ & + \dots \end{aligned} \quad (20)$$

It is clear that the logarithmic term never appears in order GMm^2/E , but in order GMm^4/E^3 . However, there is still another term in order GMm^2/E . We only consider the gravitational effect up to GMm^2/E , and neglect the higher order terms hereafter. The particle with $m^2 \pm \Delta m^2/2$ arrives at $(r \pm \Delta r, \chi \pm \Delta \chi/2)$ at the same coordinate time t . The split of the orbit can be calculated from eqs. (16) and (17). The phase difference ΔS can be calculated from eq.(20) at r . We explicitly show the differences in the path and phase due to mass square difference for the relativistic particles emitted to the radial and azimuthal directions, i.e., $\dot{\chi} = 0$ and $\dot{r} = 0$ initially. The turning points for these orbits can be chosen as $r_0 = 0$, and $r_0 = r_A$, respectively.

¹If the mixed state is a superposition of states with different energy, there is additional phase difference, $\Delta S = -\Delta E t = -\Delta m^2 t / (2E)$. We have the logarithmic term in ΔS by eq.(17), in converting from the coordinate time to the propagating distance. We will not consider such a term from the transformation between t and r hereafter.

(1) *radial orbits*

The orbital deviation and phase difference in the radial direction are given by:

$$\Delta r \simeq -\frac{\Delta m^2}{2E^2} \left\{ 1 - \frac{2GM}{r} \right\} L, \quad (21)$$

$$\Delta S \simeq -\frac{\Delta m^2}{2E} L, \quad (22)$$

where L is the path length in flat space, $L = r - r_A$. The phase difference eq.(22) is the same as obtained with different formalism, [4], [6]. Note that $L = r - r_A$ is not proper length of the path, and that E is not observed energy at r . One may wish to express the phase angle by the proper length L_p from r_A to r and the local energy E_{loc} . They are given by

$$\begin{aligned} L_p &\equiv \int_{r_A}^r \sqrt{g_{rr}} dr \\ &\simeq r - r_A + GM \log \left(\frac{r}{r_A} \right), \end{aligned} \quad (23)$$

$$E_{loc} \equiv \left(1 - \frac{2GM}{r} \right)^{-1/2} E. \quad (24)$$

Then we have [6]

$$\Delta S \simeq -\frac{\Delta m^2 L_p}{2E_{loc}} \left\{ 1 - GM \left(\frac{1}{L_p} \log \left(\frac{r}{r_A} \right) - \frac{1}{r} \right) \right\}. \quad (25)$$

(2) *transverse orbits*

For the particles emitted transversely, we have

$$\Delta r \simeq -\frac{\Delta m^2}{2E^2} \left\{ 1 + \frac{4GM}{r + r_A} - \frac{3GM}{r} \right\} \frac{L}{r}, \quad (26)$$

$$r \Delta \chi \simeq -\frac{\Delta m^2}{2E^2} \left\{ 1 + \frac{3GM}{r_A} + \frac{GM}{r + r_A} - \frac{2GM}{r} - \frac{2GM r}{r_A^2} \right\} \frac{r_A}{r} L, \quad (27)$$

$$\Delta S \simeq -\frac{\Delta m^2}{2E} \left\{ 1 - \frac{GM}{r + r_A} - \frac{2GM}{r} \right\} L, \quad (28)$$

where $L = \sqrt{r^2 - r_A^2}$ in this case. The differences increase with the propagating distance L , as expected. Gravitational corrections are shown in the braces $\{\}$. The coherent wave packet with the initial size σ will survive until $L \sim E^2 \sigma / \Delta m^2$. Gravitationally induced oscillation phase appears in order $G\Delta m^2/E$ except the purely radial motion, in which the gravitational correction is accidentally canceled as seen in eq.(20). The magnitude of the oscillation phase can be written as $\varphi_G \sim \langle \phi \rangle \varphi_0$, from the dimensional argument. However, the factor $\langle \phi \rangle$ is not the average of the gravitational potential over the semi-classical path. Rather, the correction can be regarded as the modification of the propagating distance.

4 Comment on violation of equivalence principle

The physics of the neutrino oscillation is exactly similar to that of strangeness oscillation in neutral kaons. Kaons are produced as either K^0 or $\bar{K}^0$. But they are superpositions of the physical states K_L and K_S . They are distinguished by their life-time, $\tau_L = 5.2 \times 10^{-8}$ s and $\tau_S = 8.9 \times 10^{-11}$ s. Good [8] discussed the argument against the anti-gravity using the neutral kaons. If the anti-particles would undergo the opposite sign of gravitational potential, then relative phase, $\exp(2im_K\phi t/\hbar)$, between K^0 and $\bar{K}^0$ could be produced, and the amplitudes for decay by two modes, K_L and K_S interfere. From the experiment, the anti-gravity can be excluded.

The same idea may hold in the neutrino oscillation. However, the superposition is not of particle and anti-particle, but of different flavor states. If the weak equivalence principle is violated between the flavor states, then additional phase of order G^1 is induced. Although there is no define model to include the flavor-dependent gravitational constant G_i , the following form may be plausible. The momentum for a relativistic particle in the radial motion is

$$[p_t, p_r] = \left[-E, E \left(1 + (\alpha + \gamma) \frac{GM}{r} \right) \right], \quad (29)$$

where α and γ are parameters used in the parameterized post-Newtonian gravity

$$ds^2 = - \left(1 - 2\alpha \frac{GM}{r} + \dots \right) dt^2 + \left(1 + 2\gamma \frac{GM}{r} + \dots \right) dr^2 + r^2 d\Omega^2. \quad (30)$$

The general relativity corresponds to $\alpha = \gamma = 1$, while the Brans-Dicke theory to $\alpha = 1$, $\gamma = (1 + \omega)/(2 + \omega)$.

The phase can be calculated as

$$S = E(r - t) + E(\alpha + \gamma)GM \int_{r_A}^r \frac{dr}{r}. \quad (31)$$

The difference of the phase can be written as,

$$-2\hbar\varphi_v = \Delta S = EGM\Delta f \log \left(\frac{r}{r_A} \right), \quad (32)$$

where $\Delta f = \Delta\alpha + \Delta\gamma$ is the difference of coupling to matter. If there is additional tensor-type interaction, we have $\Delta\alpha = \Delta\gamma \neq 0$. If the coupling constant ω is flavor dependent, then we have $\Delta\alpha = 0$, $\Delta\gamma \neq 0$.

The mixing phase φ_v due to the violation of equivalence principle is different in the energy dependence, $\varphi_v \sim E$, but $\varphi_0 \sim E^{-1}$. Therefore, the effect might be observed in the energy spectrum. Furthermore, Halprin et. al. [10] suggested that the atmospheric and solar neutrino puzzles may be resolved by the same parameter for the equivalence principle breaking. The phase φ_v becomes large for high energy particles. Therefore, the accuracy of the test of equivalence principle is significantly improved [13], if one can observe the ultra-high energy neutrinos ($E \sim 10^{15}$ eV) from distant active galactic nuclei. The constraint can be reduced to $\Delta f \sim 10^{-41}$.

Since the gravitational potential, or mass M is involved, which potential is relevant to? In the Table, values of the dimensionless gravitational potential $\phi = -GM/r$ are tabulated. We have more sever constraint from larger value of the potential.

Table: Values of the dimensionless gravitational potential GM/r on the Earth or solar system.

source	$ \phi $
Earth	6×10^{-10}
Sun	1×10^{-8}
Galaxy	6×10^{-7}
Virgo cluster	1×10^{-6}

5 Concluding remarks

In this paper, we have calculated the quantum-mechanical phase mixing in terms of the classical path in the weak gravitational field [3], [12]. The phenomenon is affected by the extreme relativistic nature of neutrinos. In order to demonstrate the semi-classical treatment, we apply it to the non-relativistic case. We will briefly derive the well-known result for the neutron interferometer experiment [5]. The neutron beam is emitted at A, split into two paths ($A \rightarrow B \rightarrow D$ and $A \rightarrow C \rightarrow D$), and detected at D, as shown in the Figure. Corresponding to the two paths, we have

$$S_{A-B-D} = -E \int dt + \int_A^B p_d dl + \int_B^D q dh, \quad (33)$$

and

$$S_{A-C-D} = -E \int dt + \int_A^C q dh + \int_C^D p_u dl, \quad (34)$$

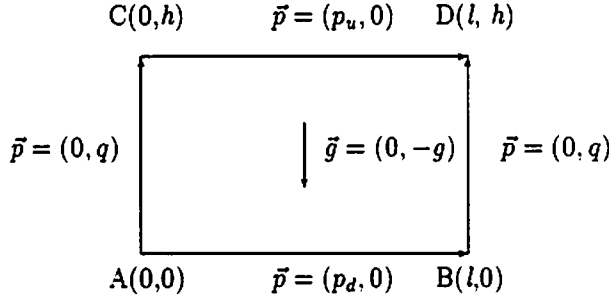
where E is energy and p_d, p_u and q are the components of the momenta as shown in the Figure. The energy and the momentum in the horizontal direction are conserved. Therefore, we have

$$p_u - p_d = -\frac{m^2 gh}{p} = -\frac{m^2 gh \lambda}{2\pi \hbar}, \quad (35)$$

where m is the neutron mass, g gravitational acceleration, and $p = (p_u + p_d)/2$. We neglect higher order of g , so that we may use the momentum of the input beam, p_d or that of the detected beam, p_u as p . The momentum is replaced by the de Broglie wave length, $\lambda = 2\pi \hbar/p$. The integrated values $\int_B^D q dh$ and $\int_A^C q dh$ are the same, so that the phase difference due to different paths is calculated as

$$\begin{aligned} 2\pi \hbar \varphi = -\Delta S &= -\int_A^B p_d dl + \int_C^D p_u dl \\ &= -\frac{m^2 gh l \lambda}{2\pi \hbar}. \end{aligned} \quad (36)$$

This is the well-known result as shown in the Sakurai's textbook [15].



It is often emphasized that it is not force but the potential that matters in a quantum mechanical context. The potential is different in the two paths in the neutron interferometer. The gravitational potential gh causes the phase difference. For the neutrino oscillation, however, the path, and hence the potential are the same. What is the origin of the phase difference? The phase difference comes only through the propagated path length. In this meaning, the gravitational correction can be regarded as the modification of the propagating distance as described in section 3. Or the clock associated with the flavor is different (“flavor oscillation clock”) and the clock suffers the red shift as required in general relativity.¹ In this case, the constant contribution to the gravitational potential may affect the red shift of the quantum clock, while it has no effect on the orbits of planets.

Acknowledgement

This work was supported in part by the Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Science and Culture of Japan (No.08640378).

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Distributional energy-momentum densities of Schwarzschild space-time

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Abstract

For the Schwarzschild space-time, distributional expressions of energy-momentum densities and of the scalar concomitants of the curvature tensors are examined for a class of coordinate systems which includes those of Schwarzschild and of Kerr-Schild as special cases. The energy-momentum density $\tilde{T}_\mu{}^\nu(x)$ of the gravitational source and the gravitational energy-momentum density $\tilde{t}_\mu{}^\nu$ have the expressions $\tilde{T}_\mu{}^\nu(x) = -Mc^2\delta_\mu^0\delta_0^\nu\delta^{(3)}(x)$ and $\tilde{t}_\mu{}^\nu = 0$, respectively. In expressions of the curvature squares for this class of coordinate systems, there are terms like $\delta^{(3)}(x)/r^3$, $[\delta^{(3)}(x)]^2$ as well as other terms which are singular at $x = 0$. It is pointed out that the well-known expression $R^{\rho\sigma\mu\nu}(\{\})R_{\rho\sigma\mu\nu}(\{\}) = 48G^2M^2/c^4r^6$ is not correct in a distribution theoretical sense.

1 Introduction

In general relativity, many investigations have been made on exact solutions of the Einstein equation and on the singularity structure of the space-time, but distribution theoretical treatment of them has not been developed sufficiently. This is so even for the universally-known Schwarzschild solution, which is given by, in the Schwarzschild coordinates $(\hat{x}^0, \hat{r}, \theta, \phi)$,

$$ds^2 = -\left(1 - \frac{a}{\hat{r}}\right)(d\hat{x}^0)^2 + \left(1 - \frac{a}{\hat{r}}\right)^{-1}(d\hat{r})^2 + \hat{r}^2[(d\theta)^2 + \sin^2\theta(d\phi)^2]. \quad (1)$$

Here, a is the Schwarzschild radius $a = 2GM/c^2$ with G, M and c being the Newton gravitational constant, mass of the source and the light velocity in vacuum Minkowski space-time, respectively. For this case, the total energy-momentum $\hat{P}_\mu$ and the energy-momentum density $\hat{\hat{t}}_\mu{}^\nu$ of the gravitational field have the expressions

$$\hat{P}_\mu = (-Mc^2, 0, 0, 0), \quad \hat{\hat{t}}_\mu{}^\nu = 0, \quad (2)$$

respectively. Thus, it is unambiguously expected that the energy-momentum density $\hat{\hat{T}}_\mu{}^\nu$ of the gravity source has the expression

$$\hat{\hat{T}}_\mu{}^\nu(\hat{x}) = -Mc^2\delta_\mu^0\delta_0^\nu\delta^{(3)}(\hat{x}). \quad (3)$$

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Nevertheless, this has not been given explicitly in any text book of general relativity, as far as the present authors know.

Recently, distributional expressions of energy-momentum densities of gravitational sources for the Schwarzschild and Kerr-Newmann space-times have been given by Balasin and Nachbagauer [1-4] for the Kerr-Schild coordinate system.

The purpose of this note is to examine, from the distribution theoretical point of view, the energy-momentum densities and the scalar concomitants $R^{\rho\sigma\mu\nu}(\{\})R_{\rho\sigma\mu\nu}(\{\})$, $R^{\mu\nu}(\{\})R_{\mu\nu}(\{\})$ and $R(\{\})$ of the Schwarzschild space-time. The examination is given for the coordinates $(\hat{x}^0, \hat{r}, \theta, \phi)$ and for the class of the coordinates (x^0, r, θ, ϕ) obtained by the transformation

$$\hat{x}^0 = \Omega x^0 + f(r), \quad \hat{r} = K(r) \quad (4)$$

with Ω being a constant and with f and K being functions of the coordinate r . (Note the following: We get Kerr-Schild coordinates when $f(r) = a \ln |a - r|$, $K(r) = r$ and $\Omega = 1$.)

2 Preliminary

In general relativity, the space-time is assumed to be a four-dimensional differentiable manifold endowed with the Lorentzian metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ ($\mu, \nu = 0, 1, 2, 3$). At each point p of the space-time, the metric can be diagonalized as $ds_p^2 = \eta_{\mu\nu}(dX^\mu)_p(dX^\nu)_p$ with $(\eta_{\mu\nu}) \stackrel{\text{def}}{=} \text{diag}(-1, 1, 1, 1)$, by choosing the coordinate system $\{X^\mu; \mu = 0, 1, 2, 3\}$ appropriately.

The energy-momentum density $\tilde{T}_\mu{}^\nu = \sqrt{-g}T_\mu{}^\nu$ is obtainable from the Einstein equation

$$G_\mu{}^\nu(\{\}) \stackrel{\text{def}}{=} R_\mu{}^\nu(\{\}) - \frac{1}{2}\delta_\mu{}^\nu R(\{\}) = \kappa T_\mu{}^\nu, \quad (5)$$

where κ is the Einstein gravitational constant $\kappa = 8\pi G/c^4$ and $g \stackrel{\text{def}}{=} \det(g_{\mu\nu})$. Here, we have defined

$$R(\{\}) \stackrel{\text{def}}{=} R^\mu{}_\mu(\{\}), \quad R_{\mu\nu}(\{\}) \stackrel{\text{def}}{=} R^\lambda{}_{\mu\lambda\nu}(\{\}), \quad (6)$$

where

$$R^\rho{}_{\sigma\mu\nu}(\{\}) \stackrel{\text{def}}{=} \partial_\mu \left\{ \begin{matrix} \rho \\ \sigma \nu \end{matrix} \right\} - \partial_\nu \left\{ \begin{matrix} \rho \\ \sigma \mu \end{matrix} \right\} + \left\{ \begin{matrix} \rho \\ \lambda \mu \end{matrix} \right\} \left\{ \begin{matrix} \lambda \\ \sigma \nu \end{matrix} \right\} - \left\{ \begin{matrix} \rho \\ \lambda \nu \end{matrix} \right\} \left\{ \begin{matrix} \lambda \\ \sigma \mu \end{matrix} \right\} \quad (7)$$

with $\left\{ \begin{matrix} \rho \\ \sigma \nu \end{matrix} \right\}$ being the Christoffel symbol.

The energy-momentum density $\tilde{t}_\mu{}^\nu$ of the gravitational field is defined by

$$\tilde{t}_\mu{}^\nu \stackrel{\text{def}}{=} \delta_\mu{}^\nu \bar{\mathbb{L}}_G - \frac{\partial \bar{\mathbb{L}}_G}{\partial g_{\sigma\tau,\nu}} g_{\sigma\tau,\mu} \quad (8)$$

with

$$\bar{\mathbb{L}}_G \stackrel{\text{def}}{=} \frac{1}{2\kappa} \mathbb{G} \stackrel{\text{def}}{=} \frac{1}{2\kappa} \sqrt{-g} g^{\mu\nu} \left[\left\{ \begin{matrix} \lambda \\ \mu \rho \end{matrix} \right\} \left\{ \begin{matrix} \rho \\ \nu \lambda \end{matrix} \right\} - \left\{ \begin{matrix} \lambda \\ \mu \nu \end{matrix} \right\} \left\{ \begin{matrix} \rho \\ \lambda \rho \end{matrix} \right\} \right]. \quad (9)$$

3 Schwarzschild Coordinates

Let us first consider the problem in Schwarzschild coordinates $(\hat{x}^0, \hat{r}, \theta, \phi)$ by which the Schwarzschild metric is expressed as (1). By using the Cartesian coordinates $(\hat{x}^0, \hat{x}^1, \hat{x}^2, \hat{x}^3)$, which is related to $(\hat{x}^0, \hat{r}, \theta, \phi)$ through the relation $\hat{x}^1 = \hat{r} \cos \phi \sin \theta$, $\hat{x}^2 = \hat{r} \sin \phi \sin \theta$, $\hat{x}^3 = \hat{r} \cos \theta$, the metric takes the form

$$ds^2 = \hat{g}_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu, \quad (10)$$

where $\hat{g}_{\mu\nu}$ is given by

$$\begin{aligned} \hat{g}_{00} &= -(1-h), \quad \hat{g}_{0\alpha} = 0, \quad \hat{g}_{\alpha\beta} = \delta^{\alpha\beta} + h(1-h)^{-1} \frac{\hat{x}^\alpha \hat{x}^\beta}{\hat{r}^2}, \\ \alpha, \beta &= 1, 2, 3 \end{aligned} \quad (11)$$

with $h \stackrel{\text{def}}{=} a/\hat{r}$.

Regularizing the function $h = a/\hat{r}$ as $a/\sqrt{\hat{r}^2 + \epsilon^2}$ with ϵ being a real number, we get

$$\hat{\mathbb{G}} = 0, \quad \hat{\hat{t}}_\mu^\nu = 0, \quad (12)$$

$$\begin{aligned} \kappa \hat{\hat{T}}_0^0(\hat{x}; \epsilon) &= -\frac{a\epsilon^2}{\hat{r}^2(\hat{r}^2 + \epsilon^2)^{3/2}}, \quad \kappa \hat{\hat{T}}_0^\alpha(\hat{x}; \epsilon) = 0, \quad \kappa \hat{\hat{T}}_\alpha^0(\hat{x}; \epsilon) = 0, \\ \kappa \hat{\hat{T}}_\alpha^\beta(\hat{x}; \epsilon) &= \delta_\alpha^\beta \frac{3a\epsilon^2}{2(\hat{r}^2 + \epsilon^2)^{5/2}} - \frac{\hat{x}^\alpha \hat{x}^\beta}{\hat{r}^2} \frac{a\epsilon^2}{(\hat{r}^2 + \epsilon^2)^{5/2}} \left(\frac{5}{2} + \frac{\epsilon^2}{\hat{r}^2} \right), \end{aligned} \quad (13)$$

where the hatted symbols $\hat{\mathbb{G}}, \hat{\hat{t}}_\mu^\nu, \hat{\hat{T}}_\mu^\nu(\hat{x}; \epsilon)$ stand for, respectively, the regularized $\mathbb{G}, \tilde{t}_\mu^\nu, \tilde{T}_\mu^\nu$ in the coordinate system $\{\hat{x}^\mu; \mu = 0, 1, 2, 3\}$. Equations (12) and (13) have been obtained by the use of (5) and (8).

From (13), the desired expression

$$\hat{\hat{T}}_\mu^\nu(\hat{x}) \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0} \hat{\hat{T}}_\mu^\nu(\hat{x}; \epsilon) = -Mc^2 \delta_\mu^0 \delta_0^\nu \delta^{(3)}(\hat{x}) \quad (3)$$

follows.

The regularized scalar quantities $\hat{R}(\{\}; \epsilon)$, $\hat{R}^{\mu\nu}(\{\}; \epsilon)$, $\hat{R}_{\mu\nu}(\{\}; \epsilon)$ and $\hat{R}^{\rho\sigma\mu\nu}(\{\}; \epsilon)$, $\hat{R}_{\rho\sigma\mu\nu}(\{\}; \epsilon)$ are calculated to be

$$\begin{aligned} \hat{R}(\{\}; \epsilon) &= -\frac{3a\epsilon^2}{(\hat{r}^2 + \epsilon^2)^{5/2}} + \frac{2a\epsilon^2}{\hat{r}^2(\hat{r}^2 + \epsilon^2)^{3/2}}, \\ \hat{R}^{\mu\nu}(\{\}; \epsilon) \hat{R}_{\mu\nu}(\{\}; \epsilon) &= \frac{1}{2} \left[\frac{3a\epsilon^2}{(\hat{r}^2 + \epsilon^2)^{5/2}} \right]^2 + 2 \left[\frac{a\epsilon^2}{\hat{r}^2(\hat{r}^2 + \epsilon^2)^{3/2}} \right]^2, \\ \hat{R}^{\rho\sigma\mu\nu}(\{\}; \epsilon) \hat{R}_{\rho\sigma\mu\nu}(\{\}; \epsilon) &= \frac{4a^2}{\hat{r}^2 + \epsilon^2} \left[\frac{2}{(\hat{r}^2 + \epsilon^2)^2} + \frac{1}{\hat{r}^4} \right] \\ &\quad - \frac{12a^2\epsilon^2}{(\hat{r}^2 + \epsilon^2)^4} + \frac{9a^2\epsilon^4}{(\hat{r}^2 + \epsilon^2)^5}, \end{aligned} \quad (14)$$

respectively. Thus, the scalar curvature has the well-defined limit:

$$\hat{R}(\{\}) \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0} \hat{R}(\{\}; \epsilon) = 4\pi a \delta^{(3)}(\hat{x}) . \quad (15)$$

But the quadratic scalars do not have well-defined limits, which can be written as

$$\begin{aligned} \hat{R}^{\mu\nu}(\{\}) \hat{R}_{\mu\nu}(\{\}) &\stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0} \hat{R}^{\mu\nu}(\{\}; \epsilon) \hat{R}_{\mu\nu}(\{\}; \epsilon) \sim 40\pi^2 a^2 \left[\delta^{(3)}(\hat{x}) \right]^2 \neq 0 , \\ \hat{R}^{\rho\sigma\mu\nu}(\{\}) \hat{R}_{\rho\sigma\mu\nu}(\{\}) &\stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0} \hat{R}^{\rho\sigma\mu\nu}(\{\}; \epsilon) \hat{R}_{\rho\sigma\mu\nu}(\{\}; \epsilon) \\ &\sim \frac{12a^2}{\hat{r}^6} + \frac{16\pi a^2}{3} \frac{1}{\hat{r}^3} \delta^{(3)}(\hat{x}) \\ &\quad + 16\pi^2 a^2 \left[\delta^{(3)}(\hat{x}) \right]^2 \neq \frac{12a^2}{\hat{r}^6} \end{aligned} \quad (16)$$

with $1/\hat{r}^6 \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0} 1/(\hat{r}^2 + \epsilon^2)^3$. Mathematics which describes singular quantities appearing in (16) is needed.

It is worth mentioning here that the spherically symmetric non-scalar quantities $\hat{r}^3 \hat{R}^{\mu\nu}(\{\}; \epsilon) \hat{R}_{\mu\nu}(\{\}; \epsilon)$ and $\hat{r}^3 \hat{R}^{\rho\sigma\mu\nu}(\{\}; \epsilon) \hat{R}_{\rho\sigma\mu\nu}(\{\}; \epsilon)$ have the well-defined limits:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \hat{r}^3 \hat{R}^{\mu\nu}(\{\}; \epsilon) \hat{R}_{\mu\nu}(\{\}; \epsilon) &= \frac{11}{4} \pi a^2 \delta^{(3)}(\hat{x}) , \\ \lim_{\epsilon \rightarrow 0} \hat{r}^3 \hat{R}^{\rho\sigma\mu\nu}(\{\}; \epsilon) \hat{R}_{\rho\sigma\mu\nu}(\{\}; \epsilon) &= 12a^2 \left[\frac{1}{\hat{r}^3} \right] + \frac{11}{2} \pi a^2 \delta^{(3)}(\hat{x}) \end{aligned} \quad (17)$$

with

$$\left[\frac{1}{\hat{r}^3} \right] \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0} \frac{\hat{r}^3}{(\hat{r}^2 + \epsilon^2)^3} . \quad (18)$$

From Eqs.(16) and (17), we see that the well-known expression $\hat{R}^{\rho\sigma\mu\nu}(\{\}) \times \hat{R}_{\rho\sigma\mu\nu}(\{\}) = 12a^2/\hat{r}^6$ is not correct in a distribution theoretical sense.

4 Generalization

We now consider the problem in the coordinate system $\{x^\mu; \mu = 0, 1, 2, 3\}$ with $x^1 = r \cos \phi \sin \theta$, $x^2 = r \sin \phi \sin \theta$, $x^3 = r \cos \theta$ related to the Schwarzschild coordinates $(\hat{x}^0, \hat{r}, \theta, \phi)$ through the relation (4). In this coordinate system, the metric has the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (19)$$

with $g_{\mu\nu}$ being given by

$$g_{00} = -A , \quad g_{0\alpha} = -D \frac{x^\alpha}{r} , \quad g_{\alpha\beta} = B \delta^{\alpha\beta} + C \frac{x^\alpha x^\beta}{r^2} , \quad (20)$$

where we have defined

$$\begin{aligned} A &\stackrel{\text{def}}{=} \Omega^2 \left(1 - \frac{a}{K} \right) , \quad B \stackrel{\text{def}}{=} \frac{K^2}{\rho^2} , \quad D \stackrel{\text{def}}{=} \Omega \left(1 - \frac{a}{K} \right) f' , \\ C &\stackrel{\text{def}}{=} \left(1 - \frac{a}{K} \right)^{-1} (K')^2 - \frac{K^2}{\rho^2} - \left(1 - \frac{a}{K} \right) (f')^2 . \end{aligned} \quad (21)$$

with $\rho \stackrel{\text{def}}{=} r$, $K' \stackrel{\text{def}}{=} dK/dr$ and $f' \stackrel{\text{def}}{=} df/dr$.

For $\tilde{t}_\mu^\nu$, it follows that

$$\tilde{t}_\mu^\nu = 0, \quad (22)$$

if (Case 1): $K(r) = \Lambda r$, or (Case 2): $K(r) = \Lambda r + a$, $f(r) = \Gamma r + \Xi$ with Λ, Γ and Ξ being real constants. (For (Case 1), $f(r)$ is arbitrary.)

In what follows, we restrict our considerations to the above two cases, (Case 1) and (Case 2) with positive³ Λ .

In order to treat the singularity at $x = 0$ in a distribution theoretical way, we employ the following regularization scheme: (R.1) First, replace K' with Λ in the expression of the function C . (R.2) Then, replace K' and K'' appearing in the process of calculation with $\Lambda\rho'$ and $\Lambda\rho''$, respectively. (R.3) Finally, replace the function ρ with $\sqrt{r^2 + \epsilon^2}$. (Case 1)

(1) The regularized energy-momentum density $\tilde{T}_\mu^\nu(x; \epsilon)$ is given by

$$\begin{aligned} \kappa \tilde{T}_0^0(x; \epsilon) &= -|\Omega| \frac{a\epsilon^2}{r^2(r^2 + \epsilon^2)^{3/2}}, \quad \kappa \tilde{T}_0^\alpha(x; \epsilon) = 0, \quad \kappa \tilde{T}_\alpha^0(x; \epsilon) = 0, \\ \kappa \tilde{T}_\alpha^\beta(x; \epsilon) &= |\Omega| \left[\delta_\alpha^\beta \frac{3a\epsilon^2}{2(r^2 + \epsilon^2)^{5/2}} - \frac{x^\alpha x^\beta}{r^2} \frac{a\epsilon^2}{(r^2 + \epsilon^2)^{5/2}} \left(\frac{5}{2} + \frac{\epsilon^2}{r^2} \right) \right]. \end{aligned} \quad (23)$$

This leads to

$$\tilde{T}_\mu^\nu(x) \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0} \tilde{T}_\mu^\nu(x; \epsilon) = -Mc^2 |\Omega| \delta_\mu^0 \delta_0^\nu \delta^{(3)}(x), \quad (24)$$

which reduces to $\tilde{T}_\mu^\nu(x) = -Mc^2 \delta_\mu^0 \delta_0^\nu \delta^{(3)}(x)$, when $|\Omega| = 1$. There is a relation $\tilde{T}_\mu^\nu(x) = \lim_{\epsilon \rightarrow 0} [\tilde{T}_\mu^\nu(x; \epsilon)]_\epsilon \neq [\tilde{T}_\mu^\nu(x)]_\epsilon$, where we have defined

$$[\tilde{T}_\mu^\nu(x; \epsilon)]_\epsilon \stackrel{\text{def}}{=} \frac{\partial(\hat{x})}{\partial(x)} \frac{\partial \hat{x}^\lambda}{\partial x^\mu} \frac{\partial x^\nu}{\partial \hat{x}^\rho} \hat{T}_\lambda^\rho(\hat{x}; \epsilon), \quad (25)$$

$$[\tilde{T}_\mu^\nu(x)]_\epsilon \stackrel{\text{def}}{=} \frac{\partial(\hat{x})}{\partial(x)} \frac{\partial \hat{x}^\lambda}{\partial x^\mu} \frac{\partial x^\nu}{\partial \hat{x}^\rho} \hat{T}_\lambda^\rho(\hat{x}). \quad (26)$$

From the above, we see the following: The correct expression for the energy-momentum density $\tilde{T}_\mu^\nu(x)$ is obtainable by transforming *first* the regularized density $\hat{T}_\mu^\nu(\hat{x}; \epsilon)$ and *then* taking the limit $\epsilon \rightarrow 0$. But, it cannot be, if the order of making coordinate transformation and taking the limit is exchanged.

In Ref. [2], the expression $\tilde{T}_\mu^\nu(x) = -Mc^2 \delta_\mu^0 \delta_0^\nu \delta^{(3)}(x)$ is given for Kerr-Schild coordinates. One might think that (3) can be easily obtained by transforming this to Schwarzschild coordinates. But, this is not the case; there is a situation similar to the one discussed above.

³The variable r cannot be "radial", if $\Lambda \leq 0$.

(2) For the regularized scalar quantities, we have

$$\begin{aligned} R(\{\}; \epsilon) &= \hat{R}(\{\}; \Lambda\epsilon) , \\ R^{\mu\nu}(\{\}; \epsilon) R_{\mu\nu}(\{\}; \epsilon) &= \hat{R}^{\mu\nu}(\{\}; \Lambda\epsilon) \hat{R}_{\mu\nu}(\{\}; \Lambda\epsilon) , \\ R^{\rho\sigma\mu\nu}(\{\}; \epsilon) R_{\rho\sigma\mu\nu}(\{\}; \epsilon) &= \hat{R}^{\rho\sigma\mu\nu}(\{\}; \Lambda\epsilon) \hat{R}_{\rho\sigma\mu\nu}(\{\}; \Lambda\epsilon) . \end{aligned} \quad (27)$$

(Case 2)

(1) The regularized energy-momentum density $\tilde{T}_\mu^\nu(x; \epsilon)$ is given by

$$\begin{aligned} \kappa \tilde{T}_0^0(x; \epsilon) &= |\Omega| \left[-\frac{6a\epsilon^2}{(r^2 + \epsilon^2)^{5/2}} + \frac{a\epsilon^2}{r^2(r^2 + \epsilon^2)^{3/2}} \right] , \\ \kappa \tilde{T}_0^\alpha(x; \epsilon) &= 0 , \quad \kappa \tilde{T}_\alpha^0(x; \epsilon) = -\frac{|\Omega|}{\Omega} \Gamma \frac{x^\alpha}{r} \left[\frac{6a\epsilon^2}{(r^2 + \epsilon^2)^{5/2}} \right] , \\ \kappa \tilde{T}_\alpha^\beta(x; \epsilon) &= |\Omega| \left[-\delta_\alpha^\beta \frac{3a\epsilon^2}{2(r^2 + \epsilon^2)^{5/2}} + \frac{x^\alpha x^\beta}{r^2} \frac{a\epsilon^2}{(r^2 + \epsilon^2)^{5/2}} \left(\frac{5}{2} + \frac{\epsilon^2}{r^2} \right) \right] , \end{aligned} \quad (28)$$

which gives the limit having the same form as (24).

For the quantities $[\tilde{T}_\mu^\nu(x; \epsilon)]_c$ and $[\tilde{T}_\mu^\nu(x)]_c$ defined in the same way as for (Case 1), we have

$$\lim_{\epsilon \rightarrow 0} [\tilde{T}_\mu^\nu(x; \epsilon)]_c = 0 \neq \tilde{T}_\mu^\nu(x) \neq [\tilde{T}_\mu^\nu(x)]_c . \quad (29)$$

The correct expression for $\tilde{T}_\mu^\nu(x)$ is *not* obtainable for (Case 2) by applying the coordinate transformation and the limiting procedure to the regularized density $\hat{\tilde{T}}_\mu^\nu(\hat{x}; \epsilon)$.

(2) For the regularized scalar quantities, we have

$$\begin{aligned} R(\{\}) &\stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0} R(\{\}; \epsilon) = 0 , \\ R^{\mu\nu}(\{\}; \epsilon) R_{\mu\nu}(\{\}; \epsilon) &\sim \frac{2a^2}{\Lambda^2(\Lambda r + a)^4} \frac{\epsilon^4}{r^4(r^2 + \epsilon^2)} \neq 0 , \\ R^{\rho\sigma\mu\nu}(\{\}; \epsilon) R_{\rho\sigma\mu\nu}(\{\}; \epsilon) &\sim \frac{12a^2}{(\Lambda r + a)^6} + \frac{4a^2}{\Lambda^2(\Lambda r + a)^4} \frac{\epsilon^4}{r^4(r^2 + \epsilon^2)} \\ &\neq \frac{12a^2}{(\Lambda r + a)^6} . \end{aligned} \quad (30)$$

Also, we can show that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} r^\omega R^{\mu\nu}(\{\}; \epsilon) R_{\mu\nu}(\{\}; \epsilon) &= 0 , \\ \lim_{\epsilon \rightarrow 0} r^\omega R^{\rho\sigma\mu\nu}(\{\}; \epsilon) R_{\rho\sigma\mu\nu}(\{\}; \epsilon) &= 0 , \quad \text{for } \omega > 1 . \end{aligned} \quad (31)$$

5 Comments

The following is worth mentioning:

- [A] When the condition $|\Omega| = 1$ is satisfied, the gravity is interpreted as produced by a point-like particle with mass M located at $\mathbf{x} = \mathbf{0}$ for both of (Case 1) and (Case 2), so far as energy-momentum densities are concerned. Also for the orbit of a testing point-like particle moving in the region $r > a$, this interpretation works well for (Case 1), if the additional condition $|f'|(1 - a/r) < 1$ is satisfied. But, for (Case 2), it does not. A testing point-like particle is expected to move as if the source were located at $r = -a/\Lambda$ for this case.

- [B] The fact that the gravitational energy-momentum density $\tilde{t}_\mu{}^\nu$ vanishes, which is natural and simple in the line of our argument, is often regarded as a deficiency. But, the following has to be noted here.
 - [B1] If we accept $P_\mu = (-Mc^2, 0, 0, 0)$ and $\tilde{T}_\mu{}^\nu = -Mc^2\delta_\mu^0\delta_0^\nu\delta^{(3)}(\mathbf{x})$, we are forced to accept that the energy density $-\tilde{t}_0^0$ is not positive definite.
 - [B2] There is no firm basis for the assertion that energy densities have to be positive definite.

- [C] Situations are simple and natural in (Case 1), as in Schwarzschild coordinates. This is reasonable, because the transformation (4) for this case is everywhere regular.

- [D] For (Case 2), the situation is rather curious, as is known from the statement [A]. The following is worth noting in this connection: The coordinate transformation (4) for this case is singular at $r = 0$ and at $\Lambda r + a = 0$. Thus, the coordinate systems of (Case 2) and of Schwarzschild are not equivalent to each other, and corresponding metrics describe different physical situations [5].

- [E] The regularization schemes employed in Sections 3 and 4 are crucial in giving our results. We have regularized the “dynamical part” only. If we had regularized the “kinematical factor” $x^\alpha x^\beta / \hat{r}^2$ in (11) as $x^\alpha x^\beta / (\hat{r}^2 + \epsilon^2)$ in addition to regularizing the “dynamical function” $h = a/\hat{r}$, for example, we would not have got (3), (15), etc. Also, if we had not followed the scheme prescribed by (R.1), (R.2) and (R.3) in Section 4, we might have been led to unwelcome results. But, as for this scheme, we cannot simply say that we have regularized “dynamical parts” only. At present, we cannot give a lucid interpretation to the scheme employed in Section 4.

- [F] Sometime, a singularity is said not to be a point in the space-time and is put outside of considerations. But, this way of coping is not appropriate, because then the Schwarzschild gravity, for example, has to be regarded as produced by zero energy-momentum [1]. A singularity is in the space-time manifold, although it is not a “place” in the space-time [6]. We believe it constructive to examine space-time singularities by extending the analysis on manifold in a distribution theoretical way, although we are not equipped with a satisfying mathematical machinery as yet.

The expressions for scalar concomitants of curvature tensor tell that mathematics capable of incorporating objects like $\delta^{(3)}(\boldsymbol{x})/r^3$ and $\left[\delta^{(3)}(\boldsymbol{x})\right]^2$ is needed for a precise description of singularities.

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Geometric Bremsstrahlung in the Expanding Universe

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Abstract

We shall discuss various kinds of quantum geometric bremsstrahlung processes in the spatially flat Robertson-Walker universe. Despite that the temperature of the universe is much higher than particle masses and the Hubble parameter, the transition probability of these processes do not vanish. It is also pointed out that explicit forms of the probability possess an expansion-contraction duality.

1 Introduction

Particles in the early universe undergo severe redshift as a result of the cosmic expansion and get decelerated extraordinarily in the comoving coordinates. Consequently radiation or massless particles may be emitted from the decelerated particle. This process induced by the geometry of the universe is regarded as a kind of bremsstrahlung. We call it geometric bremsstrahlung. This effect are prohibited kinematically in the flat spacetime.

Despite of its ease to recognize such a process in classical mechanics, there is a non-trivial aspect on existence of quantum geometric bremsstrahlung. Temperature in many interesting situations of the early universe is much higher than particle masses and the Hubble parameter. Since momentum of the particle is comparable with the temperature, one might naively think that we can neglect all the mass parameters even in calculation of the transition probabilities. Hence it might be expected that the probability of the process is equal to that calculated in massless theories in the flat spacetime, thus exactly zero. However this naive expectation is turned out to be wrong by careful analysis.

In [1], the first precise analysis has been performed in the four dimensional Robertson-Walker universe. They shows that even in a high momentum limit there remains the nonvanishing probability of photon emission via the geometric bremsstrahlung.

In this paper[2] we shall give an extended analysis on several kinds of processes of the geometric bremsstrahlung. It includes analyses on the ϕ^3 theory in arbitrary dimensional spacetimes, the theory with Yukawa interaction and the massive vector field theory. It will be shown that the high momentum limit, or the high temperature limit, does not terminate the geometric bremsstrahlung process in such a rather wide class of interactions and in arbitrary dimension. It is also stressed that a new type of duality can be found in the forms of the transition probabilities for renormalizable interactions.

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2 Robertson-Walker Universe with Past and Future Minkowskian Regions

Let us comment a little bit on our model of the Robertson-Walker universe. In this paper only the spatially flat model is argued, but we believe that our results will be extended to open and closed models. The geometry of the spacetime is expressed by a metric tensor whose form is written as $g_{\mu\nu} = a(t)^2 \eta_{\mu\nu}$, where $a(t)$ is a scale factor function and $\eta_{\mu\nu}$ is the Minkowskian metric. The argument of the scale factor, t , is conformal time and proper time τ can be defined by $d\tau = a(t)dt$. From the metric the Christoffel symbol is easily manipulated as follows.

$$\Gamma_{\beta\gamma}^{\alpha} = \delta_{\beta}^{\alpha} \partial_{\gamma} \ln a + \delta_{\gamma}^{\alpha} \partial_{\beta} \ln a - \eta_{\beta\gamma} \partial^{\alpha} \ln a, \quad (1)$$

where $\partial^{\alpha} = \eta^{\alpha\beta} \partial_{\beta}$. This yields an explicit form of the scalar curvature

$$R = -\frac{1}{a^2} \left[2(n-1) \frac{d^2}{dt^2} \ln a + (n-1)(n-2) \left(\frac{d}{dt} \ln a \right)^2 \right], \quad (2)$$

where n is the dimension of the spacetime. The Hubble parameter is defined as usual by the scale factor as

$$H(t) = \frac{1}{a} \frac{da}{d\tau} = \frac{1}{a^2} \frac{da}{dt}.$$

In later sections we must take account of interaction in the expanding universe. Then well-defined asymptotic free field is required to construct S matrix elements rigorously. However in general spacetimes the construction of asymptotic fields often encounters some difficulties [4].

For example the universe accompanied with the big-bang possesses a definite birth time and an initial singularity. Near at the birth time it needs more pieces of information (maybe quantum gravity) to specify how to define asymptotic in-field.

The simplest prescription to avoid such problems is to restrict ourselves on analysis in the universe equipped with Minkowskian past and future regions. Clearly this enables us to define both asymptotic in- and out- fields just like those in the flat spacetime. It should be emphasized that physically meaningful results derived from this model must be independent of artificial detail of the model, that is, how the universe gets into and out of the expanding era. It will be found that results obtained later satisfy this criterion.

Owing to appearance of the asymptotic flat regions the scale factor must satisfy two constraints. By virtue of the time rescaling invariance one of them is expressed, losing no generality, as

$$a(t \sim \infty) = 1.$$

Now let b denote ratio of initial scale factor to final one. Then another constraint is written as

$$a(t \sim -\infty) = b.$$

It is worth grasping qualitative behavior of Fourier components of $a(t)^n$,

$$F_n(\omega) = \int_{-\infty}^{\infty} dt a(t)^n e^{i\omega t}.$$

Now let $\omega_{min}^{(n)} < \omega_1^{(n)} < \omega_2^{(n)} < \dots < \omega_{max}^{(n)}$ denote typical frequencies characterizing detailed evolution of $a(t)^n$ with the following condition satisfied.

$$F_n(\omega_i^{(n)}) \sim O(1).$$

It is natural to think that all the $\omega_i^{(n)}$ are of order of the Hubble parameter in the expansion era.

For $\omega \ll \omega_{min}^{(n)}$ it can be shown that

$$F_n(\omega \sim 0) \sim \frac{i}{\omega}(1 - b^n). \quad (3)$$

Note that this relation (3) holds for rather arbitrary cosmic evolution.

On the other hand for $\omega \gg \omega_{max}^{(n)}$ F_n behaves as

$$F_n(\omega \sim \infty) \sim 0.$$

3 Free Fields in the Robertson-Walker Universe

Let us first review free scalar field in the expanding universe. The action with the conformal coupling term is written as

$$S_{scalar} = \int d^n x \sqrt{|g|} \left[\frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} \left(m^2 - \frac{n-2}{4(n-1)} R \right) \phi^2 \right]. \quad (4)$$

Next let us change the field variable ϕ into $\tilde{\phi} = a(t)^{n/2-1} \phi$. Then the action is reduced into

$$S_{scalar} = \int d^n x \left[\frac{1}{2} (\partial \tilde{\phi})^2 - \frac{1}{2} (ma(t))^2 \tilde{\phi}^2 \right]. \quad (5)$$

This is just a free field action with time dependent mass $ma(t)$ in the flat spacetime.

Boundary conditions for asymptotic fields can be described in terms of the rescaled field $\tilde{\phi}$. It is worthwhile to point out that spatially flat spacetimes possess isometry in the spatial section. Consequently the Fourier transformation can be used in equation of motion derived from eqn (5). Therefore what we need is a solution which form is such that

$$\tilde{\phi}(t, \vec{x}) = u_{\vec{p}}(t) e^{i\vec{p} \cdot \vec{x}}, \quad (6)$$

where $\vec{p}$ is conserved conformal momentum. Its related physical momentum is expressed as $\vec{p}_{phys} = \vec{p}/a(t)$. It is easily shown that this $u_{\vec{p}}$ satisfies a Schrödinger-type equation,

$$\left[-\frac{d^2}{dt^2} - m^2 a(t)^2 \right] u_{\vec{p}} = p^2 u_{\vec{p}}, \quad (7)$$

where $p = |\vec{p}|$. The in-mode function of this equation is specified with the following boundary condition,

$$u_{\vec{p}}^{in}(t \sim -\infty) = \frac{1}{\sqrt{(2\pi)^{n-1} 2\sqrt{p^2 + m^2 b^2}}} e^{-it\sqrt{p^2 + m^2 b^2}}. \quad (8)$$

On the other hand, the out-mode function satisfies another boundary condition,

$$u_{\vec{p}}^{out}(t \sim \infty) = \frac{1}{\sqrt{(2\pi)^{n-1} 2\sqrt{p^2 + m^2}}} e^{-it\sqrt{p^2 + m^2}}. \quad (9)$$

The in-(out-) mode function may have reflection wave terms in $t \rightarrow \infty(t \rightarrow -\infty)$ induced by the nontrivial potential term, $-m^2 a(t)^2$. Usually such existence of the reflection wave means particle creation from the vacuum state in the field theoretical context. Typical energy of the created particle is of order of the typical Hubble parameter. Hence the energy can be thought much smaller than temperature of the universe. This effect is of some interest, however it is not our target in this report. In fact we shall concentrate our attention on particles with high momentum nearly equal to the temperature. Consequently the reflection wave term is negligible in the following argument.

Now we can also exhibit results for a soluble example with a step scale factor $a(t) = b\Theta(-t) + \Theta(t)$. For example analytic form of the in-mode function is expressed as follows.

$$\begin{aligned} u_{\vec{p}}^{in}(t < 0) &= \frac{1}{\sqrt{(2\pi)^{n-1} 2\sqrt{p^2 + m^2 b^2}}} e^{-it\sqrt{p^2 + m^2 b^2}}, \\ u_{\vec{p}}^{in}(t > 0) &= \frac{1}{\sqrt{(2\pi)^{n-1} 2\sqrt{p^2 + m^2 b^2}}} \left[A e^{-it\sqrt{p^2 + m^2}} + B e^{it\sqrt{p^2 + m^2}} \right], \end{aligned}$$

where

$$A = \frac{1}{2} \left(1 + \sqrt{\frac{p^2 + m^2 b^2}{p^2 + m^2}} \right), \quad (10)$$

$$B = \frac{1}{2} \left(1 - \sqrt{\frac{p^2 + m^2 b^2}{p^2 + m^2}} \right). \quad (11)$$

This will be used in section 4. As it should be, the reflection coefficient B vanishes in $p \rightarrow \infty$.

Here we also comment on the WKB amplitudes of eqn(7). When a high momentum condition; $p^3 \gg m^2 a \frac{da}{dt}$ or $p_{phys}^3 \gg m^2 H$ is satisfied, the WKB approximation is validated enough in eqn(7). Then reflection coefficient can be neglected and both in- and out-mode functions are written in the same form as follows.

$$u_{\vec{p}} \sim \frac{1}{\sqrt{(2\pi)^3 2E(p, t)}} e^{-i \int dt E(p, t)}, \quad (12)$$

where $E(p, t) = \sqrt{p^2 + m^2 a(t)^2}$. In the early universe situation, this form gives us reliable estimation of the wavefunction.

4 Decay due to Three Point Vertex in Arbitrary Dimension

Let us consider the ϕ^3 theory with mass m in the n -dimensional spacetimes. The action reads

$$S = \int d^n x \sqrt{|g|} \left[\frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} \left(m^2 - \frac{n-2}{4(n-1)} R \right) \phi^2 - \frac{1}{3!} \lambda \Lambda^{3-n/2} \phi^3 \right], \quad (13)$$

where we have introduced a mass parameter Λ . Hence the coupling constant λ is dimensionless. In $n \leq 6$ the vertex is merely a renormalizable interaction. Meanwhile in $n > 6$, the interaction is not renormalizable and then Λ means some cutoff scale of the theory. Transition probability of a particle decay with conformal momentum p is straightforwardly given in the tree level as

$$\sum |S|^2 = (2\pi)^{2(n-1)} \lambda^2 \Lambda^{6-n} \int d^{n-1} q \left| \int dt a^{3-n/2} u_q^* u_{|\vec{p}-\vec{q}|}^* u_p \right|^2.$$

It is turned out soon that this expression itself converges for arbitrary fixed p . However taking $p \rightarrow \infty$, this grows up to infinity for $n > 6$. This comes from the fact that the vertex is nonrenormalizable for $n > 6$. However there exists no trouble in the real universe because p does not take infinite value but that of order the universe temperature. Thus in the following argument we treat p finite but large compared with m and the Hubble parameter. Thus the probability is also finite. To get a useful lower bound of the probability in $p \gg m$, we pick up only contribution from the phase space region with $0 \leq \theta \leq \theta_0 \ll 1$ and $p\epsilon \leq q \leq p(1-\epsilon)$. This implies that

$$\begin{aligned} \sum |S|^2 \geq W &= (2\pi)^{2(n-1)} \lambda^2 \Lambda^{6-n} \frac{2\pi^{n/2-1}}{\Gamma(n/2-1)} \\ &\times \int_{p\epsilon}^{p(1-\epsilon)} dq q^{n-2} \int_0^{\theta_0} d\theta \sin^{n-3} \theta \left| \int dt a^{3-n/2} u_q^* u_{|\vec{p}-\vec{q}|}^* u_p \right|^2. \end{aligned}$$

For $n \leq 6$ other phase space contributions vanishes in the high momentum limit and W gives us the probability itself, not merely a lower bound.

Here assuming that $p\epsilon \gg m$, we can use the WKB wavefunctions for both in- and out-mode functions. Replacing the integral variables like

$$q = py \quad (14)$$

$$\theta = \frac{m}{p} z \quad (15)$$

$$t = \frac{2p}{m^2} \eta, \quad (16)$$

we obtain

$$\begin{aligned} W &= \lambda^2 \left(\frac{m}{\Lambda} \right)^{n-6} \frac{\pi^{n/2-1}}{(2\pi)^{n-1} \Gamma(n/2-1)} \int_{\epsilon}^{1-\epsilon} dy \frac{y^{n-3}}{1-y} \int_0^{\theta_0} \frac{z}{m} dz z^{n-3} \\ &\times \left| \int d\eta a^{3-n/2} e^{i \int d\eta \left(a^2 \frac{1}{y} + \frac{y z^2 + a^2}{1-y} - a^2 \right)} \right|^2. \end{aligned}$$

By imposing $m^2/p \ll \omega_{\min}^{(3-n/2)}$ the replacement $a = b\Theta(-\eta) + \Theta(\eta)$ is validated[2]. Then the integration with respect to η can be easily performed and we obtain

$$W = \lambda^2 \left(\frac{m}{\Lambda}\right)^{n-6} \frac{\pi^{n/2-1}}{(2\pi)^{n-1}\Gamma(n/2-1)} \int_{\epsilon}^{1-\epsilon} dy y^{n-5} (1-y) \times \int_0^{\theta_0 \frac{p}{m}} dz z^{n-3} \left[\frac{1}{z^2 + C} - \frac{b^{3-n/2}}{z^2 + b^2 C} \right]^2, \quad (17)$$

where $C = A(m; m, m, y) = (1 - y + y^2)/y^2$.

Unfortunately for $n = 2$ this expression is not suitable and we must calculate separately. However the estimation is also possible analytically and it results in just

$$W(n = 2) = 0.$$

From $n = 3$ to $n = 6$ we can take $p \rightarrow \infty$ and $\epsilon \rightarrow 0$ in eqn (17). The results are as follows.

$$\begin{aligned} W(n = 3) &= \frac{\lambda^2}{8\pi} \left(\frac{4}{3} - \ln 3\right) \left(\frac{\Lambda}{m}\right)^3 \left[1 - \frac{2\sqrt{b}}{1+b}\right], \\ W(n = 4) &= \frac{\lambda^2}{8\pi^2} \left(\frac{4}{\sqrt{3}} \arctan\left(\frac{1}{\sqrt{3}}\right) - 1\right) \left(\frac{\Lambda}{m}\right)^2 \left[1 + \frac{b}{1-b^2} \ln b^2\right], \\ W(n = 5) &= \frac{\lambda^2}{32\pi^2} \left(\frac{5}{4} \ln 3 - 1\right) \frac{\Lambda}{m} \left[1 - \frac{2\sqrt{b}}{1+b}\right], \\ W(n = 6) &= \frac{\lambda^2}{384\pi^3} \left[\frac{1+b^2}{1-b^2} \ln \frac{1}{b^2} - 2\right]. \end{aligned}$$

Note that all these expressions are invariant under a transformation $b \rightarrow 1/b$.

As mentioned previously W diverges for $n > 6$ in $p \rightarrow \infty$. Keeping $\omega_{\min}^{(3-n/2)} > m^2/p$ in mind, a lower bound with $\theta_0 = m/p$ is obtained from eqn (17). The explicit form with $b \sim 0$ is as follows.

$$W(n > 6, b \sim 0) \geq \frac{\pi^{n/2-1}}{(2\pi)^{n-1}\Gamma(n/2-1)} \frac{\lambda^2}{(n-3)(n-4)(n-6)} \left(\frac{m}{\Lambda b}\right)^{n-6}.$$

Therefore after when the universe expands enough to satisfy

$$1/b \sim O\left(\lambda^{-\frac{3}{n-6}} \frac{\Lambda}{m}\right),$$

the particle gets decayed via the geometric bremsstrahlung.

After all we arrive at a conclusion that in spite of the high temperature of the universe the transition probability due to the geometric bremsstrahlung of this model does not vanish even in the n -dimensional spacetimes except $n = 2$.

5 Other models

We also analysed the theory with Yukawa interaction and the massive vector field theory in four dimension[2]. It will be shown that the high momentum limit, or the high temperature limit, does not terminate the geometric bremsstrahlung process in such a rather wide class of interactions[2]. It is also stressed that the forms of the transition probabilities are the same as those of the aforementioned model.

Acknowledgements

We would like to thank M. Hotta, H. Inoue, and I. Joichi who are collaborators of the present work.

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Instability in self-gravitating systems of particles with nearly critical charges

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1 Introduction

The physics of self-gravitating particle system has been investigated not only by numerical simulations but also by analytic studies. We can consider the case that there are other long-range forces acting between particles; for instance, the electromagnetic forces, and/or the force mediated by a massless scalar field. In general, elementary particles or states which appear in recent models of unified theory have various interactions other than gravitation. Thus, the study of the system of the particles with many kinds of “charges” may reveal some aspects of the very early stage, or beginning of our universe.

Further, recent studies on string theories and so-called M theory have shown that the elementary excitations correspond to BPS or nearly BPS black holes. Among the BPS black holes, static forces are cancelled out. There is, however, velocity-dependent forces among them. When the many particle system of (nearly) BPS states is studied, it must be important to take the velocity-dependent interactions into account.

In this talk, we consider the extension of the method of gaussian integration used by de Vega et al. [1] to the case in the presence of velocity-dependent interactions.

2 Review of gaussian integration technique

One can write the Hamiltonian of N -particle system with the Newtonian potential as

$$H_N = \sum_{t=1}^N \frac{p_t^2}{2m} - Gm^2 \sum_{t < j} \frac{1}{|\vec{q}_t - \vec{q}_j|}. \quad (1)$$

Then the grandcanonical partition function is written as

$$\begin{aligned} Z_G &= \sum_{N=0}^{\infty} \frac{z^N}{N!} \iint \prod_{t=1}^N \frac{d^3 p_t d^3 q_t}{(2\pi)^3} e^{-\beta H_N} \\ &= \sum_{N=0}^{\infty} \frac{1}{N!} \left[z \left(\frac{m}{2\pi\beta} \right)^{3/2} \right]^N \iint \prod_{t=1}^N d^3 q_t \exp \left(\beta G m^2 \sum_{t < j} |\vec{q}_t - \vec{q}_j|^{-1} \right), \end{aligned} \quad (2)$$

where z is the activity. Here we introduce the density function

$$\rho(\vec{r}) = \sum_{j=1}^{\infty} \delta(\vec{r} - \vec{q}_j) \quad (3)$$

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to perform Stratonovich-Hubbard transformation, by using the following gaussian integration:

$$\begin{aligned} & \exp \left(\frac{1}{2} \beta G m^2 \int \frac{d^3 x d^3 y}{|\vec{x} - \vec{y}|} \rho(\vec{x}) \rho(\vec{y}) \right) \\ &= \iint [D\xi] e^{-\frac{1}{2} \int d^3 x (\nabla \xi)^2 - 2m \sqrt{\pi G \beta} \int d^3 x \xi(\vec{x}) \rho(\vec{x})}. \end{aligned} \quad (4)$$

Then the grandcanonical partition function becomes

$$\begin{aligned} Z_G &= \sum_{N=0}^{\infty} \frac{z^N}{N!} \left(\frac{m}{2\pi\beta} \right)^{3N/2} \int [D\xi] e^{-\frac{1}{2} \int d^3 x (\nabla \xi)^2} \iint \prod_{\ell=1}^N d^3 q_{\ell} e^{-2m \sqrt{\pi\beta G} \sum_{\ell=1}^N \xi(\vec{q}_{\ell})} \\ &= \int [D\xi] e^{-\frac{1}{2} \int d^3 x (\nabla \xi)^2} \sum_{N=0}^{\infty} \frac{z^N}{N!} \left(\frac{m}{2\pi\beta} \right)^{3N/2} \left[\int d^3 q e^{-2m \sqrt{\pi\beta G} \xi(\vec{q})} \right]^N \\ &= \int [D\xi] \exp \left\{ -\int d^3 x \left[\frac{1}{2} (\nabla \xi)^2 - z \left(\frac{m}{2\pi\beta} \right)^{3/2} e^{-2m \sqrt{\pi\beta G} \xi} \right] \right\}. \end{aligned} \quad (5)$$

From the partition function, for example, we can derive the density of the particles in this system:

$$\rho = z \frac{\partial}{\partial z} \ln Z_G = z \left(\frac{m}{2\pi\beta} \right)^{3/2} \left\langle e^{-2m \sqrt{\pi\beta G} \xi} \right\rangle, \quad (6)$$

where $\langle \rangle$ means the field theoretic expectation value.

In this formulation, we can examine the properties of the physical quantities in the system by using the various field-theoretical method [1].

As an application, we can derive equations for the static equilibrium configuration. Such equations are obtained by the evaluation of the steepest descend, or the variation of the effective action. One can get the following equation from (5):

$$\nabla^2 \xi = z \left(\frac{m}{2\pi\beta} \right)^{3/2} 2m \sqrt{\pi\beta G} \exp \left(-2m \sqrt{\pi\beta G} \xi \right). \quad (7)$$

If we identify the gravitational potential ϕ as

$$\phi = 2 \sqrt{\frac{\pi G}{\beta}} \xi, \quad (8)$$

then we obtain

$$\nabla^2 \phi = z \left(\frac{m}{2\pi\beta} \right)^{3/2} 4\pi G m \exp(-\beta m \phi) = 4\pi G m \rho. \quad (9)$$

This equation is the well-known one for the gravitating isothermal sphere [2].

3 Incorporation of velocity-dependent interaction

Now we turn to study of the velocity-dependent interaction. First, we consider gravity only. The potential up to $O(v^2)$ is given by [3]

$$\begin{aligned}
 V = & - \sum_{\ell < j} \frac{Gm^2}{r_{\ell j}} \\
 & + \sum_{\ell < j} \frac{Gm^2}{2r_{\ell j}} [\vec{v}_\ell \cdot \vec{v}_j + (\vec{n}_{\ell j} \cdot \vec{v}_\ell)(\vec{n}_{\ell j} \cdot \vec{v}_j)] \\
 & - \sum_{\ell < j} \frac{3Gm^2}{2r_{\ell j}} |\vec{v}_\ell - \vec{v}_j|^2,
 \end{aligned} \tag{10}$$

where $r_{\ell j} = |\vec{q}_\ell - \vec{q}_j|$ and $\vec{n}_{\ell j} = (\vec{q}_\ell - \vec{q}_j)/r_{\ell j}$. What we must do is replacing the potential by V and integrating over momenta after rewriting v by p in V . The technique for evaluating the integration is shown in [4]. The result is, however, simply interpreted as follows. If we approximate the system as an ideal gas of the particles, we can see that the averaged value for the particle velocity is given by

$$\overline{v^2} = \frac{3T}{m}. \tag{11}$$

Therefore the averaged potential is

$$\overline{V} = - \sum_{\ell < j} \frac{Gm^2}{r_{\ell j}} - \sum_{\ell < j} \frac{9GmT}{r_{\ell j}}. \tag{12}$$

Using the result of the last section, we get

$$Z_G = \int [D\xi] \exp \left\{ - \int d^3x \left[\frac{1}{2} (\nabla \xi)^2 - z \left(\frac{m}{2\pi\beta} \right)^{3/2} \exp \left(-2m\sqrt{\pi G(\beta + 9/m)\xi} \right) \right] \right\}. \tag{13}$$

In the same manner, we can consider the case that particles have the electric charge q and the scalar charge σ . Then the averaged potential becomes

$$\overline{V} = - \sum_{\ell < j} \frac{Gm^2 - q^2 + \sigma^2}{r_{\ell j}} - \sum_{\ell < j} \frac{3T(3Gm^2 - \sigma^2)}{m r_{\ell j}}. \tag{14}$$

The partition function is now obtained as

$$\begin{aligned}
 Z_G = & \int [D\xi] \\
 & \exp \left\{ - \int d^3x \left[\frac{1}{2} (\nabla \xi)^2 - z \left(\frac{m}{2\pi\beta} \right)^{3/2} e^{-2\sqrt{\pi[\beta(Gm^2 - q^2 + \sigma^2) + 3(3Gm^2 - \sigma^2)/m]\xi}} \right] \right\}.
 \end{aligned} \tag{15}$$

If the critical relation (BPS condition) $Gm^2 - q^2 + \sigma^2 = 0$ is satisfied, the density is written as

$$\rho = z \frac{\partial}{\partial z} \ln Z_G = z \left(\frac{m}{2\pi\beta} \right)^{3/2} \left\langle \exp \left(-2\sqrt{\pi[3(3Gm^2 - \sigma^2)/m]\xi} \right) \right\rangle, \tag{16}$$

while the energy density is

$$\begin{aligned}\epsilon &= -\frac{\partial}{\partial\beta} \ln Z_G = \frac{3}{2}z \left(\frac{m}{2\pi\beta} \right)^{3/2} \frac{1}{\beta} \left\langle \exp \left(-2\sqrt{\pi[3(3Gm^2 - \sigma^2)/m]}\xi \right) \right\rangle \\ &= \frac{3}{2}\rho T.\end{aligned}\tag{17}$$

The relation between the density and the energy density is that of the ideal gas.

4 The equation for isothermal spheres

Our motivation is to study the properties of the system by the field theoretical method. In this talk, however, we concentrate ourselves on the equation for the dynamically equilibrium

configuration. Rescaling variables, we get the equation for an isothermal sphere:

$$\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} \psi = \mu^2 e^{-\psi}.\tag{18}$$

Here the characteristic scale (of mass) μ is

$$\mu^2 = 4\pi z \left(\frac{1}{2\pi} \right)^{3/2} Tm \left[(Gm^2 - q^2 + \sigma^2) \sqrt{\frac{m}{T}} + 3(3Gm^2 - \sigma^2) \sqrt{\frac{T}{m}} \right].\tag{19}$$

Without solving the equation, we can see that when $Gm^2 - q^2 + \sigma^2 > 0$ and $3Gm^2 - \sigma^2 < 0$, the static configuration is impossible at sufficiently high temperature because μ^2 becomes negative at high temperature.

If we regard the particles as charged dilatonic black holes [5], this situation can occur when $a^2 > 3$ (where a is the dilatonic coupling constant). This fact corresponds to the statement that black holes look like “elementary particles” if $a^2 > 3$ [6].

5 Conclusion

In this talk, I have shown almost nothing to do with the property of the system governed by the long range forces, but the possible method to approach the study in terms of the effective field theory. For nearly BPS particles, the inclusion of the velocity-dependent interactions is essential. The field theoretic study of this exotic matter system is under investigation, which is very parallel to [1].

We note that since we deal with the grand canonical formalism, we can take the coalescence of particles (i.e. the change and transfer in the particle number) into consideration. This consideration is regarded as a model of the system of interacting black holes.

As we treat isothermal system, we wish to take the temperature as the Hawking temperature in the presence of a black hole. To incorporate such a background effect, we must extend the formulation to the case with fully curved spacetime and inclusion of radiation (and other possible fields). The work is in preparation.

Note added

We have developed semi-general-relativistic formalism for the gravitating particles since after the workshop. We consider the Hamiltonian of Einstein-Hilbert gravity coupled to particles. Then we integrate out the momenta of particle using gaussian approximation. If the transfer energy between particles is negligible, the partition function can be represented by the path integral of the variables.

Here, for simplicity, the shift vector is assumed to vanish. Then the grand canonical partition function in $(1 + d)$ -dimensional spacetime is written as:

$$Z_G = \int [DN] [D\gamma_{ij}] \exp \left\{ \int \left[\frac{\beta}{16\pi G} N {}^{(d)}R + z \left(\frac{m}{2\pi\beta N} \right)^{d/2} e^{-\beta Nm} \right] \sqrt{\gamma} d^d x \right\}, \quad (20)$$

where N is the lapse function, γ_{ij} is the d -dimensional metric and ${}^{(d)}R$ is the scalar curvature constructed from γ_{ij} .

The "classical" equation for N is

$$\nabla^2 N = 4\pi G \left(2 \frac{d-2}{d-1} Nm + \frac{d^2}{d-1} \frac{1}{\beta} \right) z \left(\frac{m}{2\pi\beta N} \right)^{d/2} e^{-\beta Nm}. \quad (21)$$

We emphasize that the $O(v^2)$ velocity-dependent interaction is *automatically* incorporated.

The gravitational potential is involved in N . The coefficient of the $1/\beta$ seemingly differs from the previous approach. But this is not the real defect because there is $N^{d/2}$ in the right hand side and the gravitational potential is also included in the laplacian in the left hand side through γ_{ij} . The correct flat-space limit yields the agreement with the previous result.²(The interpretation is possible that the half of the averaged energy of a particle is shared with the gravitational potential, which cannot be neglected in this case.)

Taking other forces into account is straightforward.

One more comment is in order: even in the $(1 + 2)$ -dimensions, isothermal spheres can exist! (but global topology of space seems to be curious.)

Further development and field theoretic study in this line will be reported elsewhere.

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²To see this, substitute the metric (for $d \neq 2$) as $\gamma_{ij} \rightarrow N^{-\frac{2}{d-2}} \gamma_{ij}$ in eq. (20).

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Singularity and Blackhole Solutions in (2+1)-Dimensional Gravity

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Abstract

The classical vacuum equation of (2+1)-dimensional gravity with cosmological term are considered. We show mathematical relationship between the two spinless solutions, the naked singularity solution by Deser and Jackiw and the blackhole solution by Bañados, Teitelboim and Zanelli (BTZ) with $J = 0$. As a generalization of this spinless case, we consider a vacuum solution around a spinning object, including BTZ solution with nonzero J . Constructing method of multi-source solutions is also considered.

1 Introduction

In $(2 + 1)$ -dimensional cosmological ($\lambda \neq 0$) gravity, since the Riemann tensor is written by the Ricci tensor, any solution of the classical field equation has constant curvature. Because of this fact, the classical theory looks like trivial at first sight. In another word, we might think that the manifolds in our consideration are restricted to the de-Sitter (dS) and/or the anti de-Sitter (adS) space-time. They are characterized by positive and/or negative cosmological constant, respectively. However we have already known two types of non-trivial vacuum solutions. Here, we use the term non-trivial solution to mean that gravitational field around some material source.

One of them is the Deser and Jackiw [1] naked singularity (NS) solutions without spin ($J = 0$). Their simplest solution represents the field around a delta-function source at the origin like a Coulomb field in electromagnetism. Moreover, they have given a systematic method to construct multi-source solutions.

The other type of the non-trivial solution is given by Bañados, Teitelboim and Zanelli [2], which looks like spinning ($J \neq 0$) blackhole (BH) solution with $\lambda < 0$. Although this describes a constant curvature manifold, *i.e.* adS, global aspects of the manifold give a kind of singularity [3].

In the case without spin, since both type of the solution satisfies the same vacuum field equation, we expect that they will be describing a same manifold with constant curvature. In fact, we can find single BH and NS are equivalent to each other by coordinate transformation, and only differ by sign of the parameters, as we will see. We can accordingly derive both of them by assuming the “conformally flat” ansatz for the 2-space metric.

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In this talk we consider a potential for generalization of this fact. Is it possible to generalize Deser and Jackiw's systematic construction method of the static multi-NS solutions including spin? If it is, we will get the exact metric of the static multi-BH via an analogy of the single spinless object case as mentioned above. We will show later that the analogy is valid exactly for the single spinning case.

Although it is pointed out [4] that the static multi-BHs geometry is inconsistent, the global geometry of time symmetric initial condition, *i.e.* an unstable equilibrium state, is investigated in [5] [6]. This situation is similar to the spinless cases of multi-NS, in which the physical (reality) conditions on the metric strongly restrict positions of the sources. Hence it is interesting to investigate reflections of the global aspects of the manifolds to form of the metrics.

In section 2, we consider the spinless cases and find mathematical relationship between BH and NS solutions. In section 3, we examine the gravitational field around a single spinning source, which includes BTZ solution. In section 4, generalization to the multi-source system is considered.

2 Interrelation between spinless BH and NS

In this section we derive the spinless NS and BH solutions exactly and clarify their mutual relation. First of all, we solve field equation for metric of the following form,

$$ds^2 = -e^{2V(x,y)} dt^2 + e^{2U(x,y)} (dx^2 + dy^2), \quad (1)$$

where x and y are 2-space and t is time coordinates, respectively. We refer the form of (1) as “conformally flat” by its spatial style. Note that we treat both of the spatial coordinate equivalently. Their defining ranges are $(-\infty, \infty)$ commonly. The (μ, ν) components of the field equation is,

$$R_{\mu\nu} = 2\lambda g_{\mu\nu} + 8\pi G T_{\mu\nu}, \quad (2)$$

where $T_{\mu\nu}$ is some combination of energy-momentum tensor of matter. When we consider the vacuum solutions, *i.e.* $T_{\mu\nu} \rightarrow 0$, each component of (2) is,

$$(tt) : \Delta_2 U(x, y) = \lambda e^{2U(x,y)} \quad (3)$$

$$(xx) + (yy) : \Delta_2 e^{V(x,y)} = \lambda e^{2U+V} \quad (4)$$

where $(xx) + (yy)$ means a linear combination of (xx) and (yy) components, and Δ_2 is laplacian with respect to the 2-space. Notice the form of (3) is well known Liouville equation. We can sum up the rest of them, *i.e.* $(xx) - (yy)$, (xy) and (yx) components, into Cauchy-Riemann equation for a complex function,

$$\zeta(x, y) := \zeta_x + i\zeta_y, \quad (5)$$

where

$$\zeta_j = \lambda^{-1} e^{-2U(x,y)} \partial_j e^{2V(x,y)}, \quad (6)$$

and $j = x$ and y .

This system of differential equations have been solved by [1]. We can interpret the solutions as gravitational field around point particle sources, *i.e.* delta-functions. In the derivation, the positions and numbers of poles and zeros in the analytic function $\zeta(x, y)$ play a crucial role. We can construct solutions describing a field around any number of point particle sources through their method. The simplest one of them is a single particle solution, of course. We now consider it as a preparation to compare with the BTZ solution. Assuming y -independence in (3) the solution is,

$$ds^2 = -c \left(\frac{e^{\sqrt{c}x} - e^{-\sqrt{c}x}}{e^{\sqrt{c}x} + e^{-\sqrt{c}x}} \right)^2 dt^2 + \frac{4c}{\lambda} (e^{\sqrt{c}x} + e^{-\sqrt{c}x})^{-2} (dx^2 + dy^2), \quad (7)$$

where c is an integration constant. Depending on the sign of c and λ , we will find that there are different “physical interpretations” of this solution.

For $c > 0$ and $\lambda > 0$ case, if we introduce new coordinate $r := e^x$ and spatial identification $y = 0$ and 2π , the spatial part of the solution (7) becomes,

$$g_{ij} dx^i dx^j = \frac{4c}{\lambda} r^{-2} (r^{\sqrt{c}} + r^{-\sqrt{c}})^{-2} (dr^2 + r^2 dy^2) \quad (8)$$

This is Deser and Jackiw’s NS solution with $\lambda > 0$. Having performed the identification, the y coordinate plays a role of “angle”. The constant c can be interpreted as “deficit angle” around the singularity or particle at the origine. We can analyze the singularity behavior ($r \rightarrow 0, \infty$) by Green function equation in two dimension. That is,

$$\Delta_2 U(x, y) \sim \Delta_2 \log r \sim \delta^{(2)}(r) \sim T_u \quad (9)$$

On the other hand, we can obtain the BTZ solution for $c < 0$ and $\lambda < 0$ case. In this case, we find the solution (7) is written by trigonometric functions as follows,

$$ds^2 = -|c| \tan^2 \sqrt{|c|x} dt^2 + c\lambda^{-1} \sec^2(\sqrt{|c|x}) (dx^2 + dy^2) \quad (10)$$

Performing the coordinate transformation $\rho := \sqrt{|c|} \sec \sqrt{|c|x}$, (where we restrict $0 \leq \sqrt{|c|x} < \pi/2$) and the similar identification $y = 0$ and 2π , we find (7) becomes,

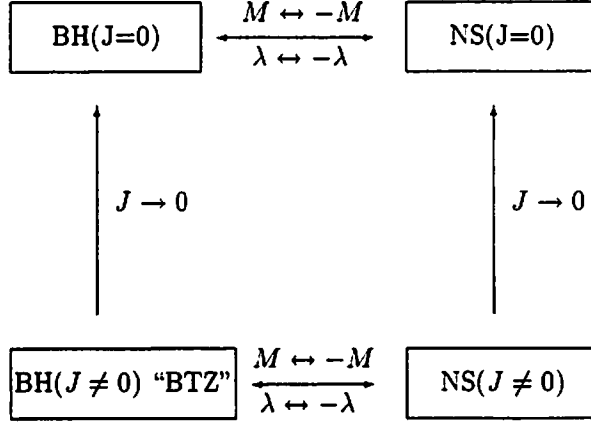
$$ds^2 = -(|\lambda|\rho^2 - |c|) dt^2 + \frac{d\rho^2}{|\lambda|\rho^2 - |c|} + \rho^2 dy^2 \quad (11)$$

This is the well known BTZ BH solution with $J = 0$. The constant $|c|$ is now interpreted as “mass” of the BH. The horizon radius is determined by the pole of $g_{\rho\rho}$.

Having seen the single object case, we observe the mathematical interrelation between BH and NS solutions. We point out two points. One thing is that they have opposite sign of constant curvature, *i.e.* λ . The other is that their difference is only the sign of “mass”, aside from the curvature. We are possible to generalize this argument into the multi-object cases through Deser and Jackiw’s construction. This point is discussed in section 4.

3 Single spinning object

In this section we shall introduce spin in the conformally flat construction given in the last section. Through the exact derivation we saw that the mathematical interrelation holds at least for the single object. This is shown on the upper side in the following diagram. Note that we replaced the constant $|c|$ by M following convention.



We now consider a relation in the lower side of the diagram. Note that the spinning NS solutions have never been known in the cosmological gravity.

Since our formalism treat the coordinates x and y equivalently, to include “spin” we should accordingly introduce non-zero g_{xt} and g_{tx} components in addition to g_{yt} and g_{ty} . However, it is too complicated to investigate such a general case firstly. To simplify the problem, we mainly consider a field with non-zero off-diagonal components; g_{yt} and g_{ty} . We thus take the following generalized conformally flat ansatz,

$$ds^2 = -e^{2V(x,y)} dt^2 + e^{2U(x,y)} (dx^2 + dy^2) + J dt dy. \quad (12)$$

As in the spinless case we impose y -independence on the field to find a solution around single source. We explicitly give some of the component of the field equation,

$$(tt) : \quad \begin{aligned} & V'' + V'(2V' - U') \\ & - V'(V' - U') W e^{2V+2U} = 2\lambda e^{2U} \end{aligned} \quad (13)$$

$$(xx) + (yy) : \quad \begin{aligned} & U'' + \frac{J^2}{4} U' (U' + V') W \\ & = \lambda (e^{2U} + \frac{J^2}{4} e^{-2V}) \end{aligned} \quad (14)$$

$$(yt) : U' V' W e^{2V} = \lambda. \quad (15)$$

Here the function W is defined as,

$$W := (e^{2V+2U} + \frac{J^2}{4})^{-1}. \quad (16)$$

We solve this system through the following assumption

$$e^{2V} = e^{2U} - M, \quad (17)$$

where M is a constant. Then we find a part of the field equation (15) becomes a separation form after integration once,

$$(U')^2 e^{2U} = \lambda(e^{4U} - M e^{2U} + \frac{J^2}{4}) \quad (18)$$

This can be reduced to elliptic integral, and we obtain the solution by making use of Jacobian elliptic functions,

$$ds^2 = (r_+^2 \frac{\text{sn}^2 \sqrt{\alpha} x}{\text{cn}^2 \sqrt{\alpha} x} - r_-^2) dt^2 + \frac{r_+^2}{\text{cn}^2 \sqrt{\alpha} x} (dx^2 + dy^2) + J dy dt, \quad (19)$$

where $\alpha = M \sqrt{1 + \lambda(J/M)^2}$, and $r_{\pm}^2 = -(1 \pm \sqrt{1 + \lambda(J/M)^2})M/2\lambda$. Of course this solves the other components of the field equation.

We can easily find that (19) reduces to (7) by taking $J \rightarrow 0$ limit via properties of Jacobian elliptic functions. This fact is represented by the upgoing arrow in the previous diagram at least single object case.

Now we examine the solution (19) as in the previous spinless object. Does this have a horizon and/or NS according to the sign of M and λ ? For the case with $M > 0$ and $\lambda < 0$, we can show a BH (BTZ) horizon appears by introducing $\rho := r_+/\text{cn}\sqrt{\alpha}x$ and identifying $y = 0$ and 2π , similarly in the spinless case.

$$ds^2 = -(|\lambda|r^2 - M)dt^2 + (|\lambda|\rho^2 - M + \frac{J^2}{4r^2})^{-1} d\rho^2 + \rho^2 dy^2 + J dy dt \quad (20)$$

The constants M and J are interpreted as mass and spin, respectively. There appears two horizons $r_{\pm}$, outer and inner, like the Kerr BH.

On the other hand, Jacobian elliptic functions are written by infinite product of trigonometric functions [7], for example,

$$\text{cn}\sqrt{\alpha}x = \frac{2\sqrt{k'}\sqrt{q}}{\sqrt{k}} \cos \frac{\pi\sqrt{\alpha}x}{2K} \prod_{n=1}^{\infty} \frac{1 + 2q^{2n} \cos \frac{\pi\sqrt{\alpha}x}{K} + q^{4n}}{1 - 2q^{2n-1} \cos \frac{\pi\sqrt{\alpha}x}{K} + q^{4n-2}}. \quad (21)$$

In this expression, k (k') is the (complementary) modulus, $q = \exp(-\pi K'/K)$ and K and K' are complete elliptic integrals. For the case with $M < 0$ and $\lambda > 0$ we find negative valued α . Hence the conformal factor e^{2U} of the 2-space is represented by infinite product of hyperbolic functions through,

$$\text{cn}(i\sqrt{|\alpha|x}) = \frac{2\sqrt{k'}\sqrt{q}}{\sqrt{k}} \cosh \frac{\pi\sqrt{|\alpha|x}}{2K} \prod(\cdots). \quad (22)$$

Performing a coordinate transformation $r := \exp(\pi x/2K)$ like Deser and Jackiw, we can similarly find the NS appears at the origin.

4 Toward construction of spinning multi-object solutions

So far we have observed the mathematical relationship between single object solutions. Precisely speaking, once we had a NS solution with positive λ , we could transform it into BH by changing the sign of “mass” and λ . The situation holds even in the case of spinning object as seen in the last section. The left side category in the following diagram represents our previous observations. We consider potential constructing the right side category. We have the Deser and Jackiw (DJ) method in the case of spinless objects represented by the rightgoing arrow at the top. However, for spinning field there does not exist general method to obtain exact metric of multi-source systems, indicated by ??.

$$\begin{array}{ccc}
 \text{single BH} \sim \text{NS}(J = 0) & \xrightarrow{\text{DJ-construction}} & \text{multi-BH} \sim \text{NS}(J = 0) \\
 \uparrow J \rightarrow 0 & & \uparrow J \rightarrow 0 \\
 \text{single BH} \sim \text{NS}(J \neq 0) & \xrightarrow{??} & \text{multi-BH} \sim \text{NS}(J \neq 0)
 \end{array}$$

For the purpose of future development, we briefly see the DJ method to solve (3). First of all, we change the spatial variables x and y to the complex coordinates $z := x + iy$ and *c.c.*. Having got $\zeta(z)$ ($\bar{\zeta}(\bar{z})$) is an analytic function of z , we have the general solution of (3) in the following form,

$$e^{2u} = \frac{c}{\lambda \zeta(z) \bar{\zeta}(\bar{z}) \cosh^2 \sqrt{c} \eta(z, \bar{z})}, \quad (23)$$

$$\eta(z, \bar{z}) = \frac{1}{2} \left(\int^z \frac{dw}{\zeta(w)} + \int^{\bar{z}} \frac{d\bar{w}}{\bar{\zeta}(\bar{w})} \right). \quad (24)$$

We need to determine $\zeta(z)$ through physical conditions on the metric. For example, if we choose $\zeta(z) = 1/z$, we obtain the single object solution (7). For solutions with more singularities, the positions of them are uniquely determined by reality conditions on $g_{\mu\nu}$. In general, as the number of poles of $\zeta(z)$ increases the singularity number does. Finally we must give an appropriate identification of the spatial coordinates such as $y = 0$ and 2π .

When we generalize this method to nonzero J , we need an answer of the question “Is there any kind of such a complex analysis?”. From mathematical perspective, when we introduce spin the functional space of metric gets out of the space of trigonometric functions into that of Jacobian elliptic functions. Hence we suppose that an analysis on general Riemann surfaces will be relevant in the consideration of multi-object system.

Acknowledgements

I would like to thank Takuya Maki and Takao Koikawa for valuable discussion and active collaboration on this field.

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Exact Solution of Two-Body Motion in (1+1) Dimensional Dilaton Gravity

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Abstract

We present the exact solution of two-body motion in (1+1) dimensional dilaton gravity by solving the constraint equations in the canonical formalism. The determining equation of the Hamiltonian is derived in a transcendental form and the Hamiltonian is expressed for the system of two identical particles in terms of the Lambert W function which has two real branches. On the principal branch the Hamiltonian reduces to the Newtonian limit for small coupling constant. On the other branch the Hamiltonian yields a new set of motions which can not be understood as relativistically correcting the Newtonian ones. The explicit trajectories in the phase space (r, p) are illustrated for various values of the energy.

1 Introduction

Lower dimensional gravity [1, 2] has in this decade attracted many researchers in the prospect that it may give some essential clues to the problems which are not yet exactly solved in the real (3+1) dimensions due to the complexity of the space-time structure. Several types of the theories have been proposed and shown to yield interesting features corresponding to those in (3+1) theory, such as gravitational collapse, black holes, cosmological solutions and quantization.

However, one of the fundamental features of these theories, namely the characteristics as the relativistic extension of Newtonian gravity, is not fully investigated. While Einstein's general relativity is surely the relativistic version of Newtonian gravity, its (2+1) dimensional version does not include Newtonian gravity and in (1+1) dimensions Einstein's gravity is trivial [3]. In this respect the dilaton-gravity theory in (1+1) dimensions [4] was contrived as the relativistic extension of Newtonian gravity, though of course it was inspired also from the string theory [5]. Up to now the post-Newtonian analysis of

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the equations of motion has been presented [6, 7], but the full analysis of the motion is not yet reported.

In this paper we shall present the exact solution of the motion of two-body in the (1+1)-dim $R = T$ dilaton gravity, which contains many analogous characteristics with (3+1) - gravity [4, 6, 8, 9]. It is recognized that a formal beauty of this theory is the decoupling of the dilaton from both the gravitational field equation and the geodesic equations of the bodies. However in the Lagrangian it is easily read that the dilaton field plays an important role in the dynamics through the mixing with the graviton. To investigate its dynamical role, we formulated the theory in the canonical form [10], where the Hamiltonian of the system is given by

$$H = -\frac{1}{\kappa} \int dx \Delta \Psi . \quad (1)$$

Here Ψ is the dilaton field, $\kappa = \frac{8\pi G}{c^4}$ and $\Delta = \frac{\partial^2}{\partial x^2}$. This expression is quite analogous to the ADM Hamiltonian in (3+1) dimensions [11, 12]

$$H = -\frac{1}{\kappa} \int dx^3 \Delta h \quad (2)$$

where h is one component of the metric g_{ij} given by $g_{ij} = \delta_{ij}(1+h) + h_{ij}^{TT}$, h_{ij}^{TT} being the transverse-traceless part and Δ being the three- dimensional Laplacian. The correspondence between (1) and (2) reveals that Ψ has a role like a component of the metric in (3+1) dimensions. The dilaton field Ψ and then the Hamiltonian is determined by solving the constraint equations.

In (3+1) dimensions the exact solution of the two-body motion can not be obtained due to the dissipation of the energy as gravitational radiation. Actually only the static sector of the Hamiltonian was determined exactly [13]. On the contrary in (1+1) dimensions there is no transverse direction and then no gravitational radiation. This restriction simplifies the dynamics and we can derive the determining equation of the Hamiltonian - a kind of transcendental equation. For the system of two identical particles the exact solution of the Hamiltonian is given in terms of the Lambert W function [14]. The canonical equations are also expressed in terms of W . While the W function has many branches with complex values in general, there are two real branches- the principal branch W_0 and W_{-1} , which join smoothly onto each other. The Hamiltonian on the principal branch reduces to the Newtonian limit in the small κ . The explicit trajectory of the motion in the phase space (r, p) is given by setting $H = H_0$. For H_0 sufficiently small, the trajectory can be regarded as the relativistic perturbation of the Newtonian case. As H_0 increases, the trajectory becomes more 'S'-shaped. For larger H_0 , the trajectory is described by the Hamiltonian on the W_{-1} branch, which presents a new set solutions in the relativistic motion at high energy.

2 Matching conditions and the determining equation for the Hamiltonian

According to the canonical formalism we developed in [10], the constraint equations $R^\mu = 0$ lead to

$$\Delta\Psi - \frac{1}{4}(\Psi')^2 + \kappa^2\pi^2 + \kappa\sum_a\sqrt{p_a^2 + m_a^2}\delta(x - z_a) = 0 \quad (3)$$

$$2\pi' + \sum_a p_a\delta(x - z_a) = 0 \quad (4)$$

where π is the conjugate momentum to $\gamma \equiv g_{11}$. The coordinate conditions we imposed are analogous to the ADM conditions in (3+1) dimensions and the differential form of them are

$$\gamma = 1 \quad \Pi = 0, \quad (5)$$

Π being the conjugate momentum to the dilaton field Ψ .

Defining ϕ and χ by

$$\Psi = -4\log|\phi| \quad \pi = \chi' \quad (6)$$

the constraints (3) and (4) for a two-particle system become

$$\Delta\phi - \frac{\kappa^2}{4}(\chi')^2\phi = \frac{\kappa}{4}\left\{\sqrt{p_1^2 + m_1^2}\phi(z_1)\delta(x - z_1) + \sqrt{p_2^2 + m_2^2}\phi(z_2)\delta(x - z_2)\right\} \quad (7)$$

$$\Delta\chi = -\frac{1}{2}\{p_1\delta(x - z_1) + p_2\delta(x - z_2)\}. \quad (8)$$

The general solution to (8) is

$$\chi = -\frac{1}{4}\{p_1|x - z_1| + p_2|x - z_2|\} - \epsilon Xx + \epsilon C_\chi. \quad (9)$$

The factor ϵ ($\epsilon^2 = 1$) has been introduced in the constants X and C_χ so that the T-inversion (time reversal) properties of χ are explicitly manifest. By definition, ϵ changes sign under time reversal and so, therefore, does χ .

Consider first the case $z_2 < z_1$, for which we may divide spacetime into three regions:

$$z_1 < x \quad (+) \text{ region}$$

$$z_2 < x < z_1 \quad (0) \text{ region}$$

$$x < z_2 \quad (-) \text{ region}$$

In each regions, χ' is constant:

$$\chi' = \begin{cases} -\epsilon X - \frac{1}{4}(p_1 + p_2) & (+) \text{ region} \\ -\epsilon X + \frac{1}{4}(p_1 - p_2) & (0) \text{ region} \\ -\epsilon X + \frac{1}{4}(p_1 + p_2) & (-) \text{ region} \end{cases} \quad (10)$$

General solutions to the homogeneous equation $\Delta\phi - (\kappa^2/4)(\chi')^2\phi = 0$ in each region are

$$\begin{cases} \phi_+(x) = A_+e^{\frac{\epsilon}{2}\{X + \frac{1}{4}(p_1 + p_2)\}x} + B_+e^{-\frac{\epsilon}{2}\{X + \frac{1}{4}(p_1 + p_2)\}x} \\ \phi_0(x) = A_0e^{\frac{\epsilon}{2}\{X - \frac{1}{4}(p_1 - p_2)\}x} + B_0e^{-\frac{\epsilon}{2}\{X - \frac{1}{4}(p_1 - p_2)\}x} \\ \phi_-(x) = A_-e^{\frac{\epsilon}{2}\{X - \frac{1}{4}(p_1 + p_2)\}x} + B_-e^{-\frac{\epsilon}{2}\{X - \frac{1}{4}(p_1 + p_2)\}x} \end{cases}. \quad (11)$$

For these solutions to be the actual solutions to (7) with delta function source terms, they must satisfy the following matching conditions at $x = z_1, z_2$:

$$\phi_+(z_1) = \phi_0(z_1) = \phi(z_1) \quad (12a)$$

$$\phi_-(z_2) = \phi_0(z_2) = \phi(z_2) \quad (12b)$$

$$\phi'_+(z_1) - \phi'_0(z_1) = \frac{\kappa}{4} \sqrt{p_1^2 + m_1^2} \phi(z_1) \quad (12c)$$

$$\phi'_0(z_2) - \phi'_-(z_2) = \frac{\kappa}{4} \sqrt{p_2^2 + m_2^2} \phi(z_2) . \quad (12d)$$

These matching conditions lead to

$$A_+ = \frac{\sqrt{p_1^2 + m_1^2} + 4X + \epsilon p_2}{4X + \epsilon(p_1 + p_2)} e^{-\frac{\kappa}{4} p_1 z_1} A_0 + \frac{\sqrt{p_1^2 + m_1^2} + \epsilon p_1}{4X + \epsilon(p_1 + p_2)} e^{-\kappa(X + \frac{1}{4} p_2) z_1} B_0 \quad (13a)$$

$$B_+ = -\frac{\sqrt{p_1^2 + m_1^2} - \epsilon p_1}{4X + \epsilon(p_1 + p_2)} e^{\kappa(X + \frac{1}{4} p_2) z_1} A_0 - \frac{\sqrt{p_1^2 + m_1^2} - 4X - \epsilon p_2}{4X + \epsilon(p_1 + p_2)} e^{\frac{\kappa}{4} p_1 z_1} B_0 \quad (13b)$$

$$A_- = -\frac{\sqrt{p_2^2 + m_2^2} - 4X + \epsilon p_1}{4X - \epsilon(p_1 + p_2)} e^{\frac{\kappa}{4} p_2 z_2} A_0 - \frac{\sqrt{p_2^2 + m_2^2} + \epsilon p_2}{4X - \epsilon(p_1 + p_2)} e^{-\kappa(X - \frac{1}{4} p_1) z_2} B_0 \quad (13c)$$

$$B_- = \frac{\sqrt{p_2^2 + m_2^2} - \epsilon p_2}{4X - \epsilon(p_1 + p_2)} e^{\kappa(X - \frac{1}{4} p_1) z_2} A_0 + \frac{\sqrt{p_2^2 + m_2^2} + 4X - \epsilon p_1}{4X - \epsilon(p_1 + p_2)} e^{-\frac{\kappa}{4} p_2 z_2} B_0 . \quad (13d)$$

Since the magnitudes of both ϕ and χ increase with increasing $|x|$, it is necessary to impose a boundary condition which guarantees that the integration (1) is definite. In the iterative analysis in [10] this condition has been shown to be

$$\Psi^2 - 4\kappa^2 \chi^2 = 0 \quad \text{in the region } z_1 < x \text{ and } x < z_2 \quad (14)$$

which implies

$$A_- = B_+ = 0 \quad (15)$$

$$-\log A_+ - \frac{\kappa \epsilon}{8} (p_1 z_1 + p_2 z_2) = \log B_- + \frac{\kappa \epsilon}{8} (p_1 z_1 + p_2 z_2) = \frac{\kappa}{2} C_x . \quad (16)$$

The condition (15) leads to

$$\frac{A_0}{B_0} = -\frac{\sqrt{p_2^2 + m_2^2} + \epsilon p_2}{\sqrt{p_2^2 + m_2^2} - 4X + \epsilon p_1} e^{-\kappa\{X - \frac{1}{4}(p_1 - p_2)\} z_2} \quad (17)$$

and

$$\frac{A_0}{B_0} = -\frac{\sqrt{p_1^2 + m_1^2} - 4X - \epsilon p_2}{\sqrt{p_1^2 + m_1^2} - \epsilon p_1} e^{-\kappa\{X - \frac{1}{4}(p_1 - p_2)\} z_1} . \quad (18)$$

From (17) and (18) we have

$$\begin{aligned} & (\sqrt{p_1^2 + m_1^2} - \epsilon p_2 - 4X)(\sqrt{p_2^2 + m_2^2} + \epsilon p_1 - 4X) \\ & = (\sqrt{p_1^2 + m_1^2} - \epsilon p_1)(\sqrt{p_2^2 + m_2^2} + \epsilon p_2) e^{\kappa\{X - \frac{1}{4}(p_1 - p_2)\}(z_1 - z_2)} . \end{aligned} \quad (19)$$

Inserting the explicit solutions $\Psi'_\pm$ into (1) we find that $H = 4X$. Including the analysis for $z_1 < z_2$, we have the general equation for the Hamiltonian

$$\begin{aligned} & (\sqrt{p_1^2 + m_1^2} - \epsilon \tilde{p}_2 - H)(\sqrt{p_2^2 + m_2^2} + \epsilon \tilde{p}_1 - H) \\ & = (\sqrt{p_1^2 + m_1^2} - \epsilon \tilde{p}_1)(\sqrt{p_2^2 + m_2^2} + \epsilon \tilde{p}_2) e^{\frac{\kappa}{4}\{H - \epsilon(\tilde{p}_1 - \tilde{p}_2)\}|r|} \end{aligned} \quad (20)$$

where $r \equiv z_1 - z_2$ and $\tilde{p}_a \equiv p_a \text{sgn}(z_1 - z_2)$.

3 Canonical equations of motion and the trajectories in the phase space

From the determining equation (20) for the Hamiltonian the canonical equations of motion are derived. For the case of $z_2 < z_1$ they are

$$\dot{p}_1 = -\frac{\partial H}{\partial z_1} = \frac{-\frac{\kappa}{4}L_0L_1L_2}{L_1 + L_2 - \frac{\kappa r}{4}L_1L_2} \quad (21)$$

$$\dot{z}_1 = \frac{\partial H}{\partial p_1} = \epsilon - \frac{\epsilon L_0L_1}{L_1 + L_2 - \frac{\kappa r}{4}L_1L_2} \cdot \frac{1}{\sqrt{p_1^2 + m_1^2}} \quad (22)$$

and similarly

$$\dot{p}_2 = \frac{\frac{\kappa}{4}L_0L_1L_2}{L_1 + L_2 - \frac{\kappa r}{4}L_1L_2} \quad (23)$$

$$\dot{z}_2 = -\epsilon + \frac{\epsilon L_0L_2}{L_1 + L_2 - \frac{\kappa r}{4}L_1L_2} \cdot \frac{1}{\sqrt{p_2^2 + m_2^2}} \quad (24)$$

where

$$L_0 \equiv 4X - \epsilon p_1 + \epsilon p_2, \quad L_1 \equiv 4X - \epsilon p_1 - \sqrt{p_2^2 + m_2^2}, \quad L_2 \equiv 4X + \epsilon p_2 - \sqrt{p_1^2 + m_1^2}.$$

It is evident that the Hamiltonian and the total momentum $P = p_1 + p_2$ are the constants of motion.

We shall try to solve Eq.(20) for a system of two identical particles. We may choose the center of inertia frame with $p_1 = -p_2 = p$. Then Eq.(20) becomes

$$(H - \sqrt{p^2 + m^2} - \epsilon \tilde{p})^2 = (\sqrt{p^2 + m^2} - \epsilon \tilde{p})^2 e^{\frac{\kappa}{4}(H - 2\epsilon \tilde{p})|r|}. \quad (25)$$

The solution is

$$H = \sqrt{p^2 + m^2} + \epsilon p \text{sgn}(r) - 8 \frac{W\left(-\frac{\kappa}{8}\left(|r|\sqrt{p^2 + m^2} - \epsilon pr\right) \exp\left[\frac{\kappa}{8}\left(|r|\sqrt{p^2 + m^2} - \epsilon pr\right)\right]\right)}{\kappa|r|} \quad (26)$$

where $W(x)$ is the Lambert W -function [14] defined via

$$y \cdot e^y = x \quad \implies \quad y = W(x). \quad (27)$$

In general W is complex and multivalued. When x is real, the function has two real branches shown in Fig.1. The branch satisfying $-1 \leq W(x)$ (a solid line) is denoted by $W_0(x)$ and is referred to as the principal branch. The branch satisfying $W(x) \leq -1$ (a dashed line) is denoted by $W_{-1}(x)$, and is real-valued only for $-1/e \leq x < 0$. Then for $-1/e \leq x < 0$ the function is double valued. The principal branch is analytic at $x = 0$ and has a derivative singularity at $x = -1/e$ beyond which $W(x)$ becomes complex. The series expansion of the principal branch is given by

$$W_0(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n. \quad (28)$$

The Hamiltonian (26) describes the surface in (τ, p, H) space of all allowed phase-space trajectories. Since W and then H are double valued, as the initial condition we need to specify the branch in addition to the starting point (τ_0, p_0) . This condition is equivalent to fixing the value of H_0 , by which a trajectory in the (τ, p) plane is uniquely determined on setting $H = H_0$ in (26).

For H_0 sufficiently small, the Hamiltonian (26) reduces to the Newtonian limit for small κ and the trajectory can be regarded as a relativistic perturbation of the Newtonian case. However once H_0 is sufficiently large there exist a qualitatively new set of trajectories described by the branch W_{-1} which cannot be understood as relativistically 'correcting' the Newtonian ones. For H_0 in the intermediate range the trajectory in (τ, p) plane moves over both branches as shown in fig. 2.

Some characteristic phase-space plots for the equal mass case are given in figs. 3-4. In each of these we have included the corresponding trajectory from the Newtonian theory for comparison. We see that as H_0 increases, the trajectory becomes more 'S'-shaped, with the particles reaching their maximum separation for some positive value of p , where the velocity $\dot{r} = 0$. This occurs because p depends upon the momenta of the particles and of the metric. Under time-reversal, the trajectory for a given value of H_0 is obtained by reflection in the $p = 0$ axis.

For the unequal masses the Hamiltonian is

$$H = \frac{\sqrt{p_1^2 + m_1^2} + \sqrt{p_2^2 + m_2^2}}{2} + \frac{\epsilon}{2}(\tilde{p}_1 - \tilde{p}_2) - 8 \frac{\kappa}{|r|} \mathcal{W} \left[\frac{\kappa |r|}{16} (\tilde{M}_1 + \tilde{M}_2); \frac{\kappa |r|}{16} (\tilde{M}_1 - \tilde{M}_2) \right]. \quad (29)$$

where $\mathcal{W}(x; a)$ is a solution of the equation

$$(y^2 - a^2)e^{2y} = (x^2 - a^2)e^{2x} \quad a > 0 \quad (30)$$

and is related with the Lambert W -function as $\lim_{a \rightarrow 0} \mathcal{W}(x; a) = W(-xe^x)$. In general $\mathcal{W}$ is also complex and multivalued, and when x is real, the function has two real branches shown in Fig. 5. The principal branch is analytic at $x = 0$ and has a derivative singularity at $x = \mathcal{W}^{-1}(-\frac{1}{2}(1 + \sqrt{1 + 4a^2}))$ beyond which it becomes complex. The other branch satisfies $\mathcal{W} < -\frac{1}{2}(1 + \sqrt{1 + 4a^2})$ and joins smoothly onto the first branch. The full function is double valued for $a < x < \mathcal{W}^{-1}(-\frac{1}{2}(1 + \sqrt{1 + 4a^2}))$.

Fig. 6 shows three phase space plots for the unequal mass case, with the mass of particle 2 being $1/10$, equal to and twice the mass of particle 1.

4 Discussion

We have obtained an exact self-consistent solution to the 2-body problem in a $(1 + 1)$ dimensional dilaton theory of gravity. To our knowledge, this is the only exact relativistic 2-body solution of this type. We are able to explore all possible limits of this solution, including large and small gravitational coupling and/or mass and/or momenta [15].

The Hamiltonian (26) shows that the physical region of a phase space is restricted by

$$\frac{\kappa}{8} \left(|r| \sqrt{p^2 + m^2} - \epsilon p r \right) \exp \left[\frac{\kappa}{8} \left(|r| \sqrt{p^2 + m^2} - \epsilon p r \right) \right] < \frac{1}{e} \quad (31)$$

Beyond this region, in other words, for the strong gravity of the large coupling constant κ , the W -function and H become complex as $\lim_{r \rightarrow -\infty} \text{Imaginary part of } W(x) = \pm\pi$. What is perhaps most interesting, is to quantize the degrees of freedom of the 2 body system we consider based on the Hamiltonian given in (26). The quantum theory based on (26) is a quantum theory of gravity coupled to matter whose slow-motion weak field limits should be straightforwardly comparable to that of the non-relativistic mechanics of 2 particles in a linear confining potential. As such it should offer interesting insights into the behaviour of quantum gravity.

Acknowledgements

We would like to thank all participants of the JGRG workshop. This workshop was supported in part by the Monbuscho Grant-in-Aid for Scientific Research.

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Figure Captions

- Fig. 1** The Lambert W -function. A solid line represents W_0 and a dashed line is W_{-1} .
- Fig. 2** A slice of both branches of H at $r = 1$. The horizontal line corresponds to $H_0 = 15$.
- Fig. 3** A comparison of relativistic (s-shaped) and non-relativistic (oval) phase-space trajectories for $H = 2.2m$. All axes are in units of m .
- Fig. 4** A comparison of relativistic (s-shaped) and non-relativistic (oval) phase-space trajectories for $H = 4m$. All axes are in units of m .
- Fig. 5** The function $\mathcal{W}(x; a)$ at $a = 0.1$.
- Fig. 6** Phase-space trajectories for $H = 4m_1$, with $m_2 = 2m_1$ (innermost curve), $m_2 = m_1$ and $m_2 = .1m_1$ (outermost curve). All axes are in units of m_1 .

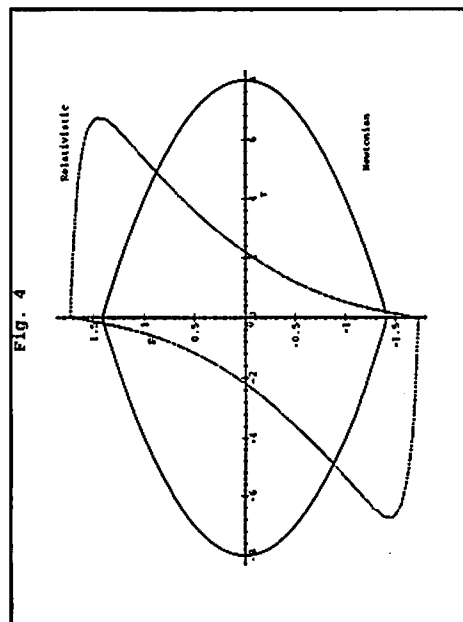
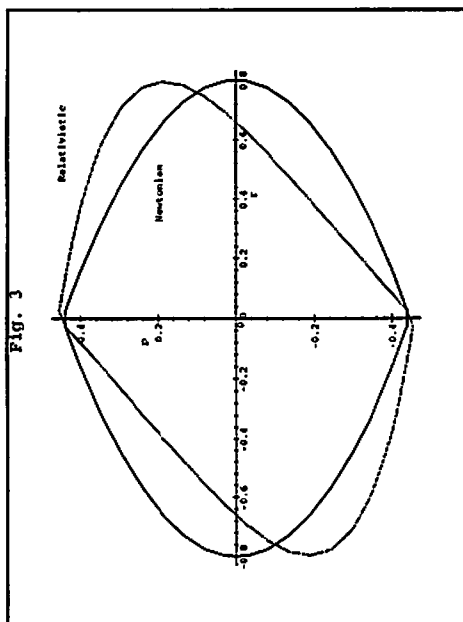
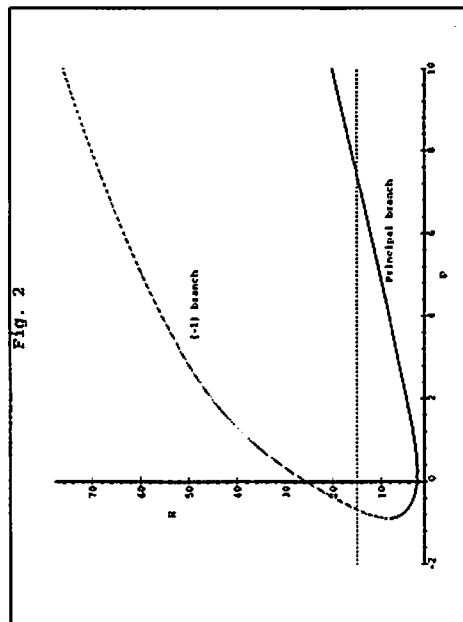
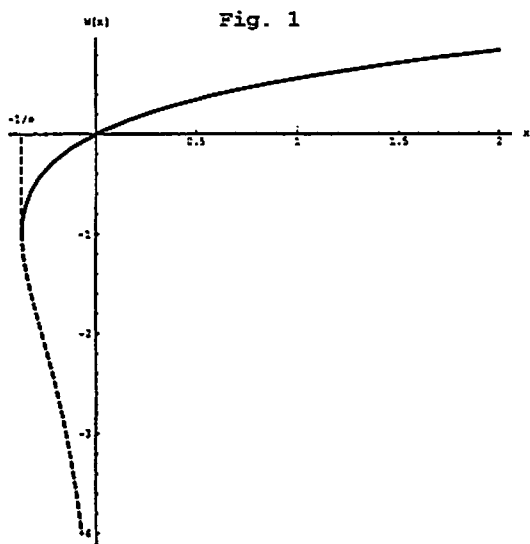


Fig.3

Fig.4

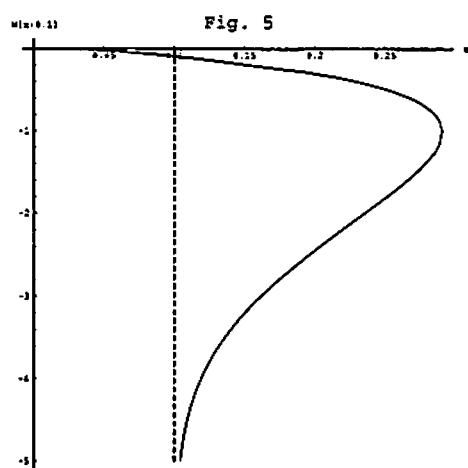


Fig.5

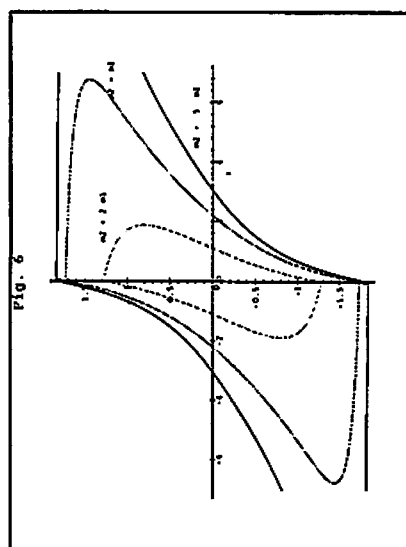


Fig.6

Quasi-Local Energy in Scalar-Tensor Theory of Gravity

– Case of a Spherical Dust Shell –

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Abstract

We investigate about the quasi-local energy (Misner-Sharp energy) of the spherically symmetric space-time with a dust shell in the scalar-tensor theory of gravity. In contrast with general relativity, there is a degree of freedom of gravitational radiation even in the spherically symmetric space-time under this theory. Reflecting this fact, it is difficult to obtain exact solutions of this dust shell system even if it is spherically symmetric. Hence we adopt the post-Newtonian approximation scheme to obtain the solutions.

1 Introduction

As general relativity (GR), the scalar-tensor (ST) theory of gravity is one of viable theories against present experimental tests[1]. But almost all tests with very high precision have been performed in the weak gravitational field. Hence projects to detect gravitational waves by large interferometric detectors (LIGO, VIRGO, GEO, TAMA) is very important since the gravitational wave generation might be a phenomenon under the strong gravitational field, and there is a possibility to check the theory of gravity in such strong field regions by the direct detection of gravitational waves[2].

Here we consider a spherically symmetric dust shell in the frame work of the ST theory of gravity. The spherically symmetric dust shell is a very simple system since like as a particle, its motion is specified by the position as a function of time. However the spherically symmetric dust shell in the ST theory has following interesting contents;

1. binding energy (“internal structure”) is naturally introduced.
2. there is a degree of freedom of gravitational radiation.
3. the quasi-local energy is less ambiguous due to the spherical symmetry of this space-time.

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One of the important differences between GR and the ST theory is that the disagreement of the gravitational and inertial masses of a binding object, the so-called Nordtvedt effect appears in the ST theory but does not appear in GR up to the first post-Newtonian (1PN) order[1]. Because of 1, that is, since the dust shell can be bound, it undergoes the Nordtvedt effect. Next even in the spherical symmetric space-time, gravitational waves can exist in the ST theory. So we can observe that the radiation reaction in the motion of the spherically symmetric dust shell. In GR, the most famous definition of the quasi-local energy of the spherically symmetric space-time is the so-called Misner-Sharp energy[3]. This expression has been re-discovered by many researchers from both physical and geometrical points of view[4] and one of its reasons seems to come from the lack of degrees of freedom of the gravitational radiation. Hence it is important to investigate the Misner-Sharp energy in the ST theory since as mentioned, there is a degree of freedom of spherical gravitational radiation.

However, although the dust shell is one of the simplest objects, it is difficult to obtain exact solutions analytically in the ST theory. Hence here we adopt the post-Newtonian approximation scheme and obtain solutions up to 2.5PN order to see the radiation reaction due to the emission of the scalar gravitational waves. We find that the radiation reaction force strongly depends on the parameter included in the theory. Then using this approximate solution, we obtain the quasi-local energy and investigate the relation between the motion of the dust shell and the quasi-local energy. However, in the ST theory, there is an ambiguity in the choice of a frame where the quasi-local energy is defined. In this article, we investigate the quasi-local energy defined in the Einstein frame only. Although we shall not show, the quasi-local energy defined in the physical frame is ill behaved at future null infinity.

Here our convention of Riemann tensor and signature of the metric follows MTW[5]. We adopt the unit $c = 1$.

2 Basic Equations of the Scalar-Tensor Theory of Gravity

First we show the basic equations of the ST theory of gravity. The equation for the space-time geometry, i.e., the metric tensor $g_{\mu\nu}$ is given by

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi}{\phi}T_{\mu\nu} + \frac{\omega(\phi)}{\phi^2} \left[(\nabla_\mu\phi)\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla^\rho\phi)\nabla_\rho\phi \right] + \frac{1}{\phi}(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^\rho\nabla_\rho\phi), \quad (1)$$

where $\omega(\phi)$ is the coupling function which is determined by experiments. For the scalar field, ϕ , and the matter, $T_{\mu\nu}$, the equations are given by

$$\nabla^\rho\nabla_\rho\phi = \frac{1}{2\omega+3} \left[8\pi T^\rho_\rho - \frac{d\omega}{d\phi}(\nabla^\rho\phi)\nabla_\rho\phi \right], \quad (2)$$

and

$$\nabla_\mu T^{\mu\nu} = 0. \quad (3)$$

The theory of $\omega = \text{constant}$ is known as the Brans-Dicke theory while that of $\omega \rightarrow \infty$ is just GR. The solar experiments now give the most stringent limit on the value of this

coupling function as $\omega > 500[1]$. However, as will be mentioned later, for the derivative of the coupling function, $d\omega/d\phi$, there is not a stringent limit.

In the ST theory, there is a freedom of the frame choice. This means that we can introduce a conformally transformed metric,

$$g_{\mu\nu}^* = D(\phi)g_{\mu\nu}, \quad (4)$$

where $D(\phi)$ is an arbitrary function of the scalar field ϕ , and we may define the physical (observable) quantities in the space-time with the metric, $g_{\mu\nu}^*$, rather than $g_{\mu\nu}$. Then a question arises; what frame should be chosen? The answer is given by experiment only. Here we should note that in the frame with the metric, $g_{\mu\nu}$, the matter couples gravity through the metric tensor only (See Eq.(3). This coupling is called the universal coupling) while in the other frame it does not. This means that the ST theory satisfies the Einstein's equivalence principle[1] in the frame with $g_{\mu\nu}$. Since the Einstein's equivalence principle has passed many experimental tests with very high precision, the frame with $g_{\mu\nu}$ is called the *physical frame*. However, here we shall introduce another important frame, the so-called *Einstein frame* which metric is defined by Eq.(4) with the following $D(\phi)$,

$$D(\phi) = \frac{\phi}{\phi_0}, \quad (5)$$

where ϕ_0 is some positive constant. Then redefining the scalar field by

$$\frac{d\varphi}{d\phi} = -\frac{1}{2\phi}\sqrt{2\omega+3}, \quad (6)$$

the basic equations in this frame become as follows. The equation for the metric, $g_{\mu\nu}^*$, is written in the form,

$$G_{\mu\nu}^* = 8\pi\phi_0^{-1}T_{\mu\nu}^* + 2\left[(\nabla_\mu^*\varphi)\nabla_\nu^*\varphi - \frac{1}{2}g_{\mu\nu}^*(\nabla^{\rho*}\varphi)\nabla_\rho^*\varphi\right], \quad (7)$$

where

$$T_{\mu\nu}^* = \frac{\phi_0}{\phi}T_{\mu\nu}. \quad (8)$$

The equations for the scalar field, φ , and the matter, $T_{\mu\nu}^*$, are given by,

$$\nabla^{\mu*}\nabla_\mu^*\varphi = -4\pi\phi_0^{-1}CT_{\mu}^{*\mu}, \quad (9)$$

and

$$\nabla_\beta^*T_\alpha^{*\beta} = CT_\beta^{*\beta}\partial_\alpha\varphi, \quad (10)$$

where

$$C = \frac{1}{\sqrt{2\omega+3}}. \quad (11)$$

The reason why the Einstein frame is important is that the contribution of the scalar field to the right hand side of the equation for the metric (7) satisfies the weak, dominant and strong energy conditions while in the physical frame it may not (See Eq.(1)).

3 The Spherically Symmetric Dust Shell

The dust shell is regarded as a timelike singular hypersurface on which the density of the dust is infinite. In GR, the formalism to deal with a general timelike and spacelike singular hypersurface has been given by Israel and then Suffern presented the formalism in the Brans-Dike theory[6] while for general class of the ST theory, Sakai and Maeda gave the formalism[7].

In this section, we show basic equations for a spherically symmetric dust shell. A spherical shell divides the space-time into two regions; one is the inside of the shell, $V_{(-)}$, and the other is the outside of that, $V_{(+)}$. Hereafter the subscript (+) means a quantity in $V_{(+)}$ while (-) means that in $V_{(-)}$. In this article, we assume the “vacuum”, i.e., no matter $T_{\mu\nu} = 0$ both in $V_{(+)}$ and $V_{(-)}$.

We adopt the coordinate system in which the metric is expressed in the form,

$$ds_{(\pm)}^2 = -A_{(\pm)}^2 dt_{(\pm)}^2 + B_{(\pm)}^2 dR^2 + R^2 d\Omega^2. \quad (12)$$

Then following the formalism by Sakai and Maeda[7], we derive the equation of motion for the dust shell and the jump conditions on the metric variables and the scalar field across the dust shell as follows. The surface stress-energy tensor of the dust shell is,

$$S_{\mu\nu} = \sigma v_\mu v_\nu, \quad (13)$$

where σ is the surface density of the dust and v^μ is the 4-velocity of that. Denoting the radial coordinate and the proper time of the dust shell by R_s and τ , respectively, v^μ is given by

$$v^\mu = \left(\frac{dt}{d\tau}, v_p, 0, 0 \right), \quad (14)$$

where

$$v_p = \frac{dR_s}{d\tau}. \quad (15)$$

From the normalization condition, we obtain,

$$\frac{dt}{d\tau} = \frac{1}{A} \Gamma \equiv \frac{1}{A} \sqrt{1 + B^2 v_p^2}. \quad (16)$$

Then the equation of motion for the dust shell is given by

$$\frac{dv_p}{d\tau} + 2 \left(\frac{\dot{B}}{AB} \Gamma \right)_{(\pm)} v_p + \left(\frac{A'}{A} + \frac{B'}{B} \right)_{(\pm)} v_p^2 + \left(\frac{A'}{AB^2} \right)_{(\pm)} = \pm \frac{1}{2} \frac{G_s M_s}{R_s^2} \left(\frac{\Gamma}{B} \right)_{(\pm)}, \quad (17)$$

where $M_s = 4\pi\sigma R_s^2$ is the conserved mass and

$$G_s = G(\phi_s) = \left\{ \phi^{-1} \frac{2\omega + 4}{2\omega + 3} \right\}_{\phi=\phi_s}. \quad (18)$$

ϕ_s is the value of the scalar field ϕ on the dust shell while prime and dot mean the radial and time derivatives, respectively. Here note that if the right hand side of Eq.(17) vanishes, this equation agrees with the geodesic equation in the space-time with the metric (12).

Hence the right hand side of Eq.(17) is regarded as gravitational acceleration produced by the dust shell itself and G_s plays a role of the Newton's gravitational constant on the shell.

The jump conditions on the metric variables are given by

$$[R]^\pm = 0, \quad \left[\frac{dR}{d\tau}\right]^\pm = 0, \quad \left[\frac{\Gamma}{B}\right]^\pm = -\frac{\gamma_s G_s M_s}{R_s} \quad \text{and} \quad \left[\frac{dt}{d\tau}\right]^\pm = \left[\frac{\Gamma}{A}\right]^\pm = 0, \quad (19)$$

where

$$\gamma_s = \gamma(\phi_s) \equiv \left\{ \frac{\omega + 1}{\omega + 2} \right\}_{\phi=\phi_s}, \quad (20)$$

and we have denoted a jump of a quantity F across the dust shell by

$$[F]^\pm \equiv \{F_{(+)} - F_{(-)}\}_{R=R_s}. \quad (21)$$

Here note that since the ST theory agrees with GR in the limit of $\omega \rightarrow \infty$, γ is unity in GR, and $0 < \gamma < 1$ for positive and finite ω . The first and second equations of Eq.(19) mean the continuity of the circumference radius across the dust shell. The third equation leads a jump condition on B while the fourth equation leads that on the lapse function A . Note that this fourth equation corresponds to the continuity of the time coordinate function t across the dust shell.

Using Eq.(19) and introducing the coordinate velocity v by

$$v = \frac{dR_s}{dt}, \quad (22)$$

the jump condition on the scalar field is written in the form

$$[\phi]^\pm = 0, \quad \frac{\Gamma}{A} v \left[\frac{B}{A} \dot{\phi} \right]^\pm + \left[\frac{\Gamma}{B} \phi' \right]^\pm = -(1 - \gamma_s) \phi_s \frac{G_s M_s}{R_s^2} \quad \text{and} \quad [\dot{\phi}]^\pm + v[\phi']^\pm = 0. \quad (23)$$

In contrast to GR, the spherically symmetric and vacuum space-time is not necessarily the Schwarzschild solution since even if there is no matter, the scalar field can exist in the vacuum region and the scalar field generates non-trivial configurations of the metric variables through Eq.(1). Hence to obtain a solution for the spherically symmetric dust shell system, we need to solve the equations of motion for the scalar field and of metric variables in vacuum regions $V_{(\pm)}$.

4 Quasi-Local Energy

In the spherically symmetric space-time, the quasi-local energy is rather well defined in GR. The most famous expression has been given first by Misner and Sharp[3]. This expression was re-discovered by many researchers[4]. If we define this quasi-local energy in the physical frame, it is given by

$$M_{QL}(t, R) = \frac{1}{2} R \left(1 + \frac{1}{2} R^2 \theta_+ \theta_- \right), \quad (24)$$

where $\theta_{\pm}$ mean the expansion of outgoing and ingoing null. As Hayward pointed out from a geometrical point of view, the above definition is the simplest one constructed from *invariant quantities* in the spherically symmetric space-time. Further the relation between its value and the circumference radius R is deeply related to the existence of trapped region; the region of $R < 2M_{QL}$ is trapped, that of $R = 2M_{QL}$ is marginally trapped and $R > 2M_{QL}$ is untrapped[4]. However even if we adopt the definition given by Misner and Sharp, there is an ambiguity with respect to the choice of a frame. From experimental tests, it is likely that the physical quantities is defined in the physical frame. However the quasi-local energy need not be done so from a theoretical consideration.

In the calculation of energy flow carried by gravitational waves at the future null infinity, the Nutku-Dykla (ND) pseudo-tensor is usually adopted and this is just the Landau-Lifshitz (LL) pseudo-tensor defined in the Einstein frame[8]. As Lee has shown, the ND pseudo-tensor is well behaved but the LL pseudo-tensor defined in the physical frame is not at the future null infinity[8]. Hence, we expect that the quasi-local energy defined in the Einstein frame is well behaved. In reality, as we shall show in part, the asymptotic behavior of the quasi-local energy defined in the Einstein frame is reasonable from a point of view of the radiation reaction on the motion of the dust shell and agrees with the energy defined by the ND pseudo-tensor at the spacelike and null infinity. Further since the energy conditions are satisfied in the Einstein frame, trapped and marginally trapped surfaces are necessarily placed inside the event horizon in this frame. Here note that the causal structure is invariant under the regular conformal transformation and hence the event horizon is invariant notion in all frames but trapped and marginally trapped surfaces are frame dependent notion. In this sense, the trapped and marginally trapped surfaces defined in the Einstein frame has a meaning in the discussion of causal structure and hence the quasi-local energy defined in the Einstein frame is also related to the causal structure. Hence here we shall investigate the quasi-local energy defined in the Einstein frame only.

The metric in the Einstein frame are written as

$$ds^2 = -A_*^2 dt^2 + B_*^2 dR^2 + R_*^2 d\Omega^2. \quad (25)$$

The above metric variables are related to those in the physical frame by,

$$A_* = \sqrt{\phi/\phi_0} A, \quad B_* = \sqrt{\phi/\phi_0} B \quad \text{and} \quad R_* = \sqrt{\phi/\phi_0} R, \quad (26)$$

and the new scalar field φ is defined by Eq.(6). Then the quasi-local energy in this frame is given by

$$\begin{aligned} M_{QL}^*(t, R) &= \frac{1}{2} R_* \left(1 + \frac{1}{2} R_*^2 \theta_+^* \theta_-^* \right) \\ &= \left(\frac{\phi}{\phi_0} \right)^{\frac{1}{2}} \left[M_{QL} - \frac{R^2}{2B^2} \frac{\phi'}{\phi} + \frac{R^3}{8\phi^2} \left(\frac{\dot{\phi}^2}{A^2} - \frac{\phi'^2}{B^2} \right) \right], \end{aligned} \quad (27)$$

where θ_+^* is the expansion of outgoing null and θ_-^* is that of ingoing null, both which are defined in the Einstein frame, and M_{QL} is the quasi-local energy defined in the physical frame. $\theta_{\pm}^*$ are related to the quantities in the physical frame by,

$$\theta_{\pm}^* = \sqrt{\frac{\phi_0}{\phi}} \left[\theta_{\pm} + \frac{1}{\sqrt{2}} (A^{-1} \partial_t \pm B^{-1} \partial_R) \ln \left(\frac{\phi}{\phi_0} \right) \right]. \quad (28)$$

Then, the region of $R_* < 2M_{QL}^*$ is trapped, that of $R_* = 2M_{QL}^*$ is marginally trapped and that of $R_* > 2M_{QL}^*$ is untrapped, and the trapped region is necessarily placed inside the event horizon if the cosmic censorship holds.

5 Post-Newtonian Approximation

In general, it is difficult to solve the equations given in Sec.3 analytically. Hence here we shall use the post-Newtonian approximation scheme to obtain the solutions. In this scheme, we assume the speed of the dust shell, $|v|$ is much smaller than the speed of light c (note that in our unit $c = 1$) and the potential energy Ω_s is same order as v^2 , i.e.,

$$|v| \ll 1 \quad \text{and} \quad \Omega_s(t) \equiv \frac{G_s M_s}{R_s} = O(v^2). \quad (29)$$

We write the scalar field in the form,

$$\phi = \phi_0(1 + \psi), \quad (30)$$

where ϕ_0 is chosen to be the asymptotic value of the scalar field, ϕ , for $R \rightarrow \infty$. Then we expand the quantities in the form,

$$A = 1 + {}^{(2)}A + {}^{(4)}A + {}^{(6)}A + {}^{(7)}A + \dots, \quad (31)$$

$$B = {}^{(2)}B + {}^{(4)}B + {}^{(6)}B + {}^{(7)}B + \dots, \quad (32)$$

$$\psi = {}^{(2)}\psi + {}^{(4)}\psi + {}^{(5)}\psi + \dots, \quad (33)$$

where suffix (n) denotes the order of v^n .

To solve the equation for the scalar field, ϕ , we impose the outgoing wave boundary condition.

5.1 Quasi-Local energy

First, we show the quasi-local energy in the form,

$$M_{QL}^* = {}^{(2)}M_{QL}^* + {}^{(4)}M_{QL}^* + {}^{(6)}M_{QL}^* + {}^{(7)}M_{QL}^* + \dots \quad (34)$$

In the near zone which scale is much smaller than the wave length of the scalar field, we obtain for $V_{(+)}$,

$${}^{(2)}M_{QL(+)}^* = \phi_0^{-1} M_s, \quad (35)$$

$${}^{(4)}M_{QL(+)}^* = \frac{1}{2} \phi_0^{-1} \left[v^2 - \Omega_s - \frac{1}{2} (1 - \gamma_0) \frac{G_0 M_s}{R} \right] M_s, \quad (36)$$

$$\begin{aligned} {}^{(6)}M_{QL(+)}^* &= \frac{1}{24} \phi_0^{-1} [9v^4 + 8(1 + 2\gamma_0)\Omega_s v^2 + 6(1 - \gamma_0)(2 + \alpha_0)\Omega_s^2] M_s \\ &\quad + \frac{1}{12} \phi_0^{-1} (1 - \gamma_0) \{v^2 + (7 + 6\alpha_0)\Omega_s\} \Omega_s M_s \left(\frac{R_s}{R}\right) \\ &\quad - \frac{1}{4} \phi_0^{-1} (1 - \gamma_0) \gamma_0 \Omega_s^2 M_s \left(\frac{R_s}{R}\right)^2, \end{aligned} \quad (37)$$

$${}^{(7)}M_{QL(+)}^* = \frac{1}{6} \phi_0^{-1} (1 - \gamma_0) (4 + 3\alpha_0) \Omega_s^2 v M_s, \quad (38)$$

for $V_{(-)}$,

$$^{(2)}M_{QL(-)}^* = 0, \quad (39)$$

$$^{(4)}M_{QL(-)}^* = 0, \quad (40)$$

$$^{(6)}M_{QL(-)}^* = \frac{1}{12}\phi_0^{-1}(1-\gamma_0)\Omega_s v^2 M_s \left(\frac{R}{R_s}\right)^3, \quad (41)$$

$$^{(7)}M_{QL(-)}^* = 0, \quad (42)$$

where $\gamma_0 = \gamma(\phi_0)$, $G_0 = G(\phi_0)$ and

$$\alpha_0 = \left\{ (\omega + 2)^{-1} (2\omega + 3)^{-1} \phi \left(\frac{d\omega}{d\phi} \right) \right\}_{\phi=\phi_0}. \quad (43)$$

In the far zone which is far from the dust shell, we easily estimate the quasi-local energy flux as

$$\partial_t M_{QL}^* = -4\pi S_-^R \equiv -\frac{1}{18}\phi_0^{-1}(1-\gamma_0)(4+3\alpha_0)^2 \left(\frac{\Omega_s^3(\bar{t})v^2(\bar{t})}{R_s(\bar{t})} \right)_{i=t-R} M_s, \quad \text{for } R \longrightarrow \infty. \quad (44)$$

The above equation means that the quasi-local energy flow at the future null infinity is $O(v^{10})$ and this corresponds to the time variation of the quasi-local energy of $O(v^9)$, i.e., the 2.5PN order. Here note that we regard $\phi_0^{-1}M_s$ to be $O(v^2)$ but M_s to be $O(v^0)$ because ϕ_0^{-1} has same dimension as the Newton's gravitational constant. In general, the conservation law of the quasi-local energy up to $O(v^n)$ leads the equation of motion up to $O(v^{n-2})$, i.e., $(n/2 - 2)$ PN order, regarding the equation of motion up to $O(v^2)$ as the Newtonian order. Hence the quasi-local energy up to $O(v^9)$ leads the equation of motion up to 2.5PN order and therefore the radiation reaction on the dust shell appears first at 2.5PN order.

5.2 The equation of motion for the dust shell

The equation of motion for the dust shell is written in the form,

$$\frac{dv_p}{d\tau} = a_N + a_{1\text{PN}} + a_{2\text{PN}} + a_{2.5\text{PN}}, \quad (45)$$

where

$$a_N = -\frac{\Omega_s}{2R_s}, \quad (46)$$

$$a_{1\text{PN}} = -\frac{\Omega_s}{4R_s} \left[v^2 - 2(1-\gamma_0)(1+\alpha_0)\Omega_s \right] - \frac{\Omega_s}{6R_s} (1-\gamma_0)(4v^2 + \Omega_s) \quad (47)$$

$$\begin{aligned} a_{2\text{PN}} = & -\frac{1}{16}(3v^2 + 8\Omega_s)\frac{\Omega_s}{R_s}v^2 \\ & - \frac{1}{120}(1-\gamma_0)[72v^4 + 4(33 - 10\gamma_0 - 20\alpha_0)\Omega_s v^2 \\ & + \{59 - 70\gamma_0 + 10(19 - 9\gamma_0)\alpha_0 + 30(7\alpha_0^2 - 4\delta_0)\}\Omega_s^2]\frac{\Omega_s}{R_s}, \end{aligned} \quad (48)$$

$$a_{2.5\text{PN}} = \frac{1}{18}(1-\gamma_0)(4+3\alpha_0)[12v^2 + (5-3\alpha_0)\Omega_s]\frac{\Omega_s^2 v}{R_s}, \quad (49)$$

and

$$\delta_0 = \left\{ (2\omega + 3)^{-1} (2\omega + 4)^{-2} \phi^2 \left(\frac{d^2\omega}{d\phi^2} \right) \right\}_{\phi=\phi_0}. \quad (50)$$

The power due to radiation reaction is given by

$$\begin{aligned} M_s v a_{2.5\text{PN}} &= {}^{(8)} \left\{ M_s v \frac{dv}{dt} \right\} = \frac{1}{18} (1 - \gamma_0) (4 + 3\alpha_0) \{ 12v^2 + (5 - 3\alpha_0) \Omega_s \} \frac{\Omega_s^2 v^2}{R_s} M_s \\ &= -4\pi\phi_0 S_-^R - 2\phi_0 \frac{d}{dt} \left(v^2 [{}^{(7)} M_{QL}^*]^\pm \right). \end{aligned} \quad (51)$$

From the above equation, we find that the power due to the radiation reaction force is written by the energy flux measured at the null infinity and the time derivative term. If the motion is periodic or if there are same initial and final asymptotic states in its motion of the Newtonian order (a kind of the scattering problem), the time derivative term does not contribute of the energy gain or loss because of the time averaging discards it. However, here the dust shell motion is not such a case and hence the time derivative term contributes the energy gain or loss due to the radiation reaction. The same situation exists in GR or Maxwell theory, although the radiation reaction appears first in the quadrupole motion or dipole motion in these cases.

We should note that the acceleration due to the radiation reaction does not have a definite direction. To see this fact, we rewrite $a_{2.5\text{PN}}$ in the form,

$$a_{2.5\text{PN}} = \frac{1}{18} (1 - \gamma_0) (4 + 3\alpha_0) [24M_s^{-1} E_N + (17 - 3\alpha_0) \Omega_s], \quad (52)$$

where $E_N = M_s(v^2 - \Omega_s)/2$ is the energy of the Newtonian order and we assume $E_N < 0$, i.e., the bound motion of the dust shell. Then, if $\alpha_0 \geq 17/3$, the acceleration due to the radiation reaction is always ingoing. On the other hand, for $\alpha_0 < 17/3$ except for the case of $\alpha_0 = -4/3$, we define,

$$R_c \equiv \frac{G_0 M_s^2}{24|E_N|} (17 - 3\alpha_0), \quad (53)$$

and when the radius of the dust shell, R_s , is larger than R_c , the acceleration due to the radiation reaction is ingoing while that is outgoing for $R_s < R_c$ and vanishes at $R_s = R_c$. Damour and Esposito-Farese have shown that α_0 should be larger than about -5 from the binary pulsar observation[9]. So yet there are both possibilities that $4 + 3\alpha_0$ is non-negative and that it is negative. When α_0 is equal to $-4/3$, there is no radiation reaction up to the 2.5PN order. On the other hand, the energy flow in the far zone is always outgoing and hence the total energy of this system is non-increasing with respect to time. However, if $\alpha_0 = -4/3$ there is no energy flux in the far zone and this is consistent to the no radiation reaction on the dust shell.

6 Conclusion

We have investigated the spherically symmetric dust shell and the quasi-local energy of this system under the ST theory of gravity by the post-Newtonian approximation scheme. In order to solve the equation for the scalar field, we have imposed the outgoing

wave condition, and the radiation reaction on the dust shell is related to this boundary condition. Although here we have not shown the solution of the scalar field, we have found the effect of the outgoing wave boundary condition on the scalar field appears at 1.5PN order. Reflecting this fact, there appears the quasi-local mass of $O(v^7)$, i.e., $^{(7)}M_{QL}^*$, which corresponds to the correction in the equation of motion at 1.5PN order. However, there is no correction term of 1.5PN order in the equation of motion of the dust shell. So $^{(7)}M_{QL}^*$ expresses the “energy” of the scalar field and the scalar gravitational waves only and this appears in the radiation reaction on the motion of the dust shell at 2.5PN order. We should note that there are contributions of the scalar field on the quasi-local energy up to 2PN order which corresponds to the equation of motion for the dust shell up to 1PN order. The energy which is obtained by integration of the equation of motion for the dust shell (45) up to 1PN order, does not agree with the quasi-local energy up to 2PN except for its asymptotic value for $R \rightarrow \infty$. Hence the value of the quasi-local energy in the intermediate region comes from the scalar field but this does not relate the motion of the dust shell.

Here we should comment on the terminology of “energy”. It might to be usually recognized in the physics that in order to define the energy of waves (electromagnetic or gravitational waves), the averaging procedure over the region larger than the wavelength of those is necessary, but in our case the quasi-local energy is calculated in the near zone which scale is smaller than the wavelength of the scalar field and the scalar gravitational radiation by its definition. In this sense, it does not seem to be appropriate that the quasi-local energy deserves the title of the “energy” of the scalar *wave*. But as mentioned, the quasi-local energy calculated here is an invariant quantity associated to the spherical symmetry and is a measure of the expansion of a spherical wavefront of light ray. Hence the quasi-local energy expresses a kind of energy of the scalar *field* even though it is not the energy of *waves*. The more detailed discussion will appear elsewhere[10].

Acknowledgements

We would like to thank H. Sato, T. Nakamura, M. Sasaki, M. Shibata, T. Tanaka and M. Siino for their helpful discussion. This work was supported in part by the Grant-in-Aid for Encouragement of Young Scientists (No.08740341) of Ministry of Education, Culture, Science and Sports.

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An attempt to construct well-behaved Quasi-local energy

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Abstract

A new definition of the quasi-local energy is presented. It is a modification of Hayward's definition and based on the $2 + 2$ formalism of the General Relativity. The new point is that it incorporates the total Hamiltonian in the definition. It agrees with the ADM and Bondi-Sachs energy at the asymptotic region. The form of the new quasi-local energy for massless free fields reduces to the vacuum one regardless of the type of the fields unless the field is spinless. It is revealed that the new-quasi local energy can be interpreted as the gravitational mass for a freely infalling 2-surface by physical consideration.

1 Introduction

It is well known that, due to the equivalence principle, the gravitational field does not have local (pointwise) energy density in General Relativity. On the contrary, one can define the local (pointwise) energy density of the other matter fields through the energy-momentum tensor T_{ab} . Hence it is impossible to construct the combined energy density of gravity and matter in a purely local manner.

For asymptotically flat spacetimes, however, it was shown that the notion of total energy of whole 3-space is valid and that one can consistently define the total energy at spatial and null infinity: the ADM energy E_{ADM} [1] and the Bondi-Sachs energy E_{BS} [2, 3]. They have several nice features which are considered to contribute to the most remarkable successes of the theory of General Relativity. They are defined in entirely covariant way[4] and thus have mathematically beautiful structure. The two facts that they are positive under reasonable, fairly weak assumption[5, 6] and that the energy loss rate is given by the integral of the energy flux measured at infinity which was also shown to be positive[2, 7], show that these definitions for asymptotic energy behave quite well in a physical viewpoint also.

The great success of these total energies indicates that the notion of gravitational energy is not meaningless. Since they are defined in spacetime with fixed asymptotic structure and represent the total energy of the entire 3-space, one is motivated to seek definition of energy which is not restricted by the asymptotic structure and is valid for finite region of spacetime. Such definitions cannot be found locally, so we are led to

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consider quasi-local energy. The word “quasi-local” was introduced by Penrose[8] and refers to a compact orientable spatial 2-surface.

The well-defined quasi-local energy would be useful to investigate the local evolution of the spacetime. Consider, for example, the evolution of the gravitational collapse of a massive object. The quasi-local energy of a surface surrounding the source object would be a good measure to tell whether the object collapses to a black hole, or the naked singularity appears, or all the initial material is radiated away to give regular spacetime. Such a measure is valuable in numerical general relativity. Also, the quasi-local energy is expected to give an unified expression independent of the asymptotic structure for the total energy.

Hawking[9] derived an expression of quasi-local energy that is not restricted to spherically symmetric cases. This has several nice properties such as converging to the Bondi-Sachs energy with appropriate limit in asymptotically flat spacetimes and increasing monotonically outward[10], while it also has a disadvantage that it is nonzero in the Minkowski spacetime.

Hayward[11] gave another definition of quasi-local energy in general spacetime based on 2+2 formalism[12]. It is basically a value of the Hamiltonian of the system. Its primary advantage is that it is determined directly through the dynamically preferred quantity, Hamiltonian. Also, the Hayward energy has many favorable features: it agrees with the Misner-Sharp energy when spherical symmetry holds, it vanishes in the Minkowski spacetime, and it converges to the ADM and Bondi-Sachs energy in asymptotically flat spacetime if appropriate limits are taken. However, he did not use the full Hamiltonian in the definition but used the gravitational part of the Hamiltonian only.

We modify Hayward’s definition to employ the full Hamiltonian and investigate the various properties of that definition. In addition to the fact that this new quasi-local energy shares all the good properties of the Hayward energy in vacuum spacetime, it is shown to have the common form in vacuum, Einstein-Maxwell and Einstein-spinor model. This common form is expressed in purely geometrical quantities associated naturally with the 2-surface, and converges to the ADM and Bondi-Sachs energy in asymptotically flat spacetime with appropriate limits. Moreover, the common form is shown to be a well-defined gravitational mass of the 2-surface in the sense that it represents correctly the inward acceleration of the freely infalling 2-surface.

2 Definition of Quasi-local energy based on the 2+2 formalism

2.1 Brief overview on the 2 + 2 formalism

We take two commuting vector fields u^a and v^a , and regard them as the evolution vectors. Taking the evolutional direction to be null does not lose generalities, so this “double null foliation” is assumed hereafter. Taking the parameters of u^a and v^a to be ξ and η , the two-surfaces $\{S_{\xi,\eta}\}$, generated by Lie-propagating a fixed two-dimensional spacelike surface S along u^a and v^a , serve as a foliation of spacetime in neighborhood of S . We take the origins of the parameters ξ and η so that $S_{0,0}$ coincides with S . Since we assumed double

null foliation, we restrict ourselves to the cases such that u^a and v^a give null three-surfaces $\mathcal{L}_u S$ and $\mathcal{L}_v S$. Define h_{ab} , the induced metric on $S_{\xi,\eta}$, and $r_a \equiv h_{ab}u^b$, $s_a \equiv h_{ab}v^b$ and $m \equiv -\log(-(u-r)_a(v-s)^a)$. It is easy to see that $u-r$ and $v-s$ are the null normal vectors to the foliation. Thus these quantities satisfy

$$g_{ab} = h_{ab} - e^m((u-r)_a(v-s)_b + (v-s)_a(u-r)_b). \quad (1)$$

The dynamical equations for the system are derived from the variational principle. The Lagrangian for 2 + 2 formalism, L , is obtained by expressing the four dimensional Lagrangian ${}^{(4)}L$ in terms of the quantities defined on $S_{\xi,\eta}$:

$$\int {}^{(4)}L = \int_{S_{\xi,\eta}} d\xi d\eta \int L, \quad (2)$$

where ${}^{(4)}L$ is sum of the Einstein-Hilbert Lagrangian $\frac{{}^{(4)}\epsilon}{16\pi}R$ and the matter Lagrangian ${}^{(4)}L_m$, where ${}^{(4)}\epsilon$ is four-dimensional volume form and R is the Ricci scalar of the spacetime metric.

It is easily verified that the dynamical equations obtained by extremizing the action integral (2) are written as the Euler-Lagrange equations

$$\mathcal{L}_u \frac{\delta L}{\delta \mathcal{L}_u q} + \mathcal{L}_v \frac{\delta L}{\delta \mathcal{L}_v q} - \frac{\delta L}{\delta q} = 0, \quad (3)$$

where q denotes the dynamical degrees of freedom such as h_{ab}, r^a, s^a, m . Of course q includes the degrees of freedom of matter fields, too.

It is also easy to verify that the Euler-Lagrange equations (3) are equivalent to a Hamiltonian system where the Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H} = \frac{\delta L}{\delta \mathcal{L}_u q} \mathcal{L}_u q + \frac{\delta L}{\delta \mathcal{L}_v q} \mathcal{L}_v q - L. \quad (4)$$

The gravitational part of the Lagrangian which follows from the Einstein-Hilbert Lagrangian, after removing the total divergence, is

$$\begin{aligned} L_G = & \mu \frac{e^{-m}}{16\pi} [\mathcal{R} + \frac{e^m}{2} ({}^{Tr}(\mathcal{L}_{u-r} h \mathcal{L}_{v-s} h) - {}^{Tr} \mathcal{L}_{u-r} h {}^{Tr} \mathcal{L}_{v-s} h \\ & + {}^{Tr} \mathcal{L}_{u-r} h \mathcal{L}_{v-s} m + {}^{Tr} \mathcal{L}_{v-s} h \mathcal{L}_{u-r} m) + \frac{1}{2} D_a m D^a m + 2\omega_a \omega^a - 2\Lambda], \end{aligned}$$

where μ , $\mathcal{R}$ and D_a are the area two-form, the scalar curvature and the covariant derivative of the 2-surface $S_{\xi,\eta}$, respectively, and

$$\begin{aligned} {}^{Tr} \mathcal{L}_u h & \equiv h^{ab} \mathcal{L}_u h_{ab} \\ {}^{Tr} \mathcal{L}_v h & \equiv h^{ab} \mathcal{L}_v h_{ab} \\ {}^{Tr}(\mathcal{L}_u h \mathcal{L}_v h) & \equiv h^{ab} h^{cd} \mathcal{L}_u h_{ac} \mathcal{L}_v h_{bd} \\ \omega_a & \equiv \frac{h_{ab} [u-r, v-s]^b}{2 (u-r)^c (v-s)_c}. \end{aligned} \quad (5)$$

2.2 Hayward's definition of the quasi-local energy

In the standard 3 + 1 formalism, the total energy associated with an asymptotically flat spacelike hypersurface is defined as the integral of the Hamiltonian over that hypersurface. The analogous quantity for 2 + 2 formalism is the integral of the Hamiltonian $\mathcal{H}$ over the surface S . We would like to relate the integral $\int_S \mathcal{H}$ to the quasi-local energy associated to S . However, it is not satisfactory to simply regard it as the quasi-local energy for S due to the following problems:

- The integral does not have the right dimension for energy. It is a dimensionless quantity (in geometrical units $c = G_N = 1$).
- The value of the 2 + 2 Hamiltonian $\mathcal{H}$ is not a geometrically invariant quantity. Rather, it depends on the choice of the evolution vectors u^a and v^a .

The first problem is not difficult to overcome. The area radius $r = (A/(4\pi))^{1/2}$ where A is the area of S is a natural geometrical quantity with dimension and $r \int_S \mathcal{H}$ has the dimension of energy.

In order to overcome the second difficulty, we must somehow fix the coordinate freedom, i.e., the gauge freedom. This can be done in a covariant way as follows.

Of the geometrical quantities introduced above, r^a , s^a and m actually represent the coordinate freedom and can be set to zero on any particular surface S . More precisely, for any given spacelike 2-surface S , one can always find double null foliation around S such that $r^a = s^a = 0, m = 0, \nabla_a m = 0$ on S . (Of course they cannot be set to zero throughout the foliation.) Under this gauge, the gravitational part of the Hamiltonian $\mathcal{H}_G$, which is derived from the gravitational part of the Lagrangian L_G , becomes simple:

$$\begin{aligned} \mathcal{H}_G &= \frac{\mu}{16\pi} (-\mathcal{R} - \frac{1}{2} ({}^{Tr} \mathcal{L}_u h {}^{Tr} \mathcal{L}_v h - {}^{Tr} (\mathcal{L}_u h \mathcal{L}_v h))) + 2\omega_a \omega^a + 2\Lambda \\ &= \frac{\mu}{16\pi} (-R_{abcd} h^{ac} h^{bd} + 2\omega_a \omega^a + 2\Lambda). \end{aligned} \quad (6)$$

There still remains a gauge freedom even after imposing the above gauge condition on S . The gauge condition $r^a = s^a = 0$ only fixes the internal coordinate of $S_{\xi, \eta}$. The condition $m = 0, \nabla_a m = 0$ fixes the foliation around S along the direction of the null normals while the foliation along the surface S is not determined by these conditions. The remaining freedom is manifested on S as a rescaling freedom of the null normals $u - r$ and $v - s$. It should be noted that this freedom is not expressed as such a simple rescaling of null normals on the foliation surfaces other than S because the surfaces do not coincide between the different gauges in general.

This freedom will be employed in the next section to fix the gauge dependence which comes from the matter part of the Hamiltonian $\mathcal{H}_m$. The explicit procedure of fixing this gauge freedom is described later.

Now we are in a position to define the quasi-local energy based on the 2 + 2 formalism. Hayward[11] defined his quasi-local energy as

$$\begin{aligned} E_{\text{Hayward}}(S) &\equiv -\left(\frac{A}{4\pi}\right)^{\frac{1}{2}} \int_S \mathcal{H}_G \\ &= \left(\frac{A}{4\pi}\right)^{\frac{1}{2}} \int_S \frac{\mu}{16\pi} (R_{abcd} h^{ac} h^{bd} - 2\omega_a \omega^a). \end{aligned} \quad (7)$$

The gauge condition on S is understood implicitly in the above expression, and the minus sign on the right hand side of the first line should be there so as that Schwarzschild spacetime gives $E_{\text{Hayward}} = M$. It should be noted that he chose to take the gravitational part of the Hamiltonian $\mathcal{H}_G$, not the total Hamiltonian in the definition. Also, he did not take the cosmological constant into consideration, so the cosmological constant is simply zero in $\mathcal{H}_G$ in his definition.

The Hayward energy $E_{\text{Hayward}}(S)$ has several nice features as summarized below. As is clear from the above derivation, Hayward energy can be defined for *any* spatial 2-surface in *any* spacetime. It requires neither of symmetry about the spacetime, special condition about the 2-surface, nor the special local structure of the spacetime. In fact, among many other quasi-local energies proposed by various authors[13], the one that can be defined completely general like Hayward's is the Hawking energy[9] only.

Another good feature of the Hayward energy is that it vanishes for general spatial 2-surface in the Minkowski spacetime. This is not the case for the Hawking energy. This feature is discussed further in the next section together with the new quasi-local energy proposed below.

It also follows that the Hayward energy converges to the ADM energy and the Bondi-Sachs energy with appropriate limits in asymptotically flat spacetime. Both of the ADM energy and the Bondi-Sachs energy can be expressed as an asymptotic limit of the integral $(\frac{A}{4\pi})^{\frac{1}{2}} \int_S \frac{\mu}{16\pi} C_{abcd} h^{ac} h^{bd}$. [11, 4, 14] Since $\omega_a \omega^a \sim \mathcal{O}(r^{-6})$ [11], E_{ADM} and E_{BS} are given as the appropriate limits of the Hayward energy in the vacuum case $R_{ab} = 0$.

Actually the fall-off condition of the energy-momentum tensor T_{ab} , the projection tensor h^{ab} and the spacetime metric g^{ab} in an asymptotically flat spacetime[4, 14] assures that the Ricci part $R_{ab}h^{ab} - R/3$ in the Hayward energy (7) does not contribute to the limit in the asymptotic limits even the spacetime is not vacuum. Hence the E_{ADM} and E_{BS} are represented as the limit of E_{Hayward} in the general asymptotically flat spacetimes.

2.3 New definition using the full Hamiltonian

Although Hayward's definition (7) has various nice features, it is apparently unnatural because it uses only the gravitational part of the Hamiltonian. Therefore, we take the total Hamiltonian $\mathcal{H}$ and propose another definition

$$E(S) \equiv -\left(\frac{A}{4\pi}\right)^{\frac{1}{2}} \int_S \mathcal{H}. \quad (8)$$

Of course the gauge condition is assumed in this definition, too. The new definition does not agree with $E_{\text{Hayward}}(S)$ when there is non-vacuum matter field. However, it shares some of the desirable features with $E_{\text{Hayward}}(S)$, as shown in the next section, such as approaching to E_{ADM} and E_{BS} if appropriate limits are taken in asymptotically flat spacetimes.

Of course this new definition can also be defined for any spatial 2-surface S in any spacetime as the Hayward energy and the Hawking energy. This feature, as well as the fact that $E(S)$, $E_{\text{Hayward}}(S)$ and $E_{\text{Hawking}}(S)$ are written in terms of the local geometric quantity on S , is well adapted for numerical relativity.

3 Various features of the new Quasi-local energy

In this section, we give the explicit expressions of the quasi-local energy $E(S)$ for many models and show that $E(S)$ has a remarkably simple common term for any massless free fields with non-zero spin. It is only when the total Hamiltonian is used in the definition of quasi-local energy that this common term arises.

3.1 Vacuum

In the model where there is only cosmological constant Λ and no matter fields exist, the quasi-local energy $E(S)$ is

$$E(S) = \left(\frac{A}{4\pi}\right)^{\frac{1}{2}} \int_S \frac{\mu}{16\pi} (C_{abcd} h^{ac} h^{bd} - 2\omega_a \omega^a - \frac{4}{3}\Lambda), \quad (9)$$

where C_{abcd} is the Weyl tensor of the spacetime metric g_{ab} . Since no matter field is present, $E(S)$ and $E_{Hayward}(S)$ give the same result when $\Lambda = 0$.

The first term on the right hand side of eq.(9) is in good concordance with the expressions for E_{ADM} and E_{BS} . Actually, as pointed out by Hayward[11], in an asymptotically flat spacetime ($\Lambda = 0$), the twist term in eq.(9) damps fast enough at asymptotic region so as that both $E(S)$ and $E_{Hayward}(S)$ converge to E_{ADM} and E_{BS} at spatial and null infinity, respectively.

In the Minkowski spacetime, C_{abcd} and Λ vanish. Take S and foliation 2-surfaces $S_{\xi,\eta}$ to be embedded in $t = \text{constant}$ hypersurfaces in a fixed global Lorentz frame. Then the twist ω_a vanishes, too.

Therefore, $E(S)$ is zero for any spacelike 2-surface S embedded in global $t = \text{constant}$ hypersurface. This is one of the nice features of $E(S)$ and $E_{Hayward}(S)$ which is not seen for Hawking's energy[11].

In this sense, $E(S)$ and $E_{Hayward}(S)$ can be regarded as improvements of $E_{Hawking}(S)$ which provide the right value for the Minkowski spacetime.

3.2 Einstein-Maxwell

Next consider a model in which there is electromagnetic field $F_{ab} = \nabla_a A_b - \nabla_b A_a$. The matter part of the four-dimensional Lagrangian of this model is

$${}^{(4)}L_m = -\frac{{}^{(4)}\epsilon}{16\pi} F_{ab} F^{ab}. \quad (10)$$

In order to recast this model into $2 + 2$ formalism, it is necessary to project the tensor indexes onto S so that they can be regarded as the dynamical variables of 2-surface S . We decompose the four-dimensional vector potential A_a into the "parallel" part $A_a^{\parallel} \equiv h_a^b A_b$ and the "perpendicular" parts $Au \equiv u^a A_a$, $Av \equiv v^a A_a$. Parallel part $A_a^{\parallel}$ is a vector field and perpendicular part Au , Av are two scalar fields on 2-surfaces $S_{\xi,\eta}$.

An interesting phenomenon occurs when combining the gravitational part and the matter part of the Hamiltonian. It turns out that the Ricci part in $\mathcal{H}_G|_g$ and $E_{\parallel}^2 + B_{\parallel}^2$ in

$\mathcal{H}_m|_g$ cancel each other leaving only $\frac{2}{3}\Lambda$ because of the Einstein equations $R_{ab} - \frac{R}{2}g_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$.

Hence the quasi-local energy $E(S)$ for this model is just

$$E(S) = \left(\frac{A}{4\pi}\right)^{\frac{1}{2}} \int_S \frac{\mu}{16\pi} \left[C_{abcd} h^{ac} h^{bd} - 2\omega_a \omega^a - \frac{4}{3}\Lambda - 4h^{ab} (\mathcal{L}_u A_a^\parallel \mathcal{L}_v A_b^\parallel - D_a A_u D_b A_v) \right].$$

Note that the form of the first line of the right hand side is the same as that of vacuum spacetime.

It turns out that the remaining term $-4h^{ab} (\mathcal{L}_u A_a^\parallel \mathcal{L}_v A_b^\parallel - D_a A_u D_b A_v)$ is not invariant under the remaining gauge freedom. In other words, this term is dependent on the foliation of 2-surfaces around S , i.e., the coordinate transformation around S . It means that this term is unphysical.

In order to make $E(S)$ physical, it is necessary to remove this unphysical gauge dependence. The most natural way to achieve this is to fix the gauge so that this term vanishes throughout S .

Another way to see the fictitious nature of this term is to note that it is not invariant under the $U(1)$ gauge transformation $A_a \rightarrow A_a + \nabla_a \chi$. Hence it is reasonable to fix the coordinate gauge just to eliminate this term.

In this way, we have fully fixed the gauge freedom and obtained the quasi-local energy:

$$E(S) = \left(\frac{A}{4\pi}\right)^{\frac{1}{2}} \int_S \frac{\mu}{16\pi} (C_{abcd} h^{ac} h^{bd} - 2\omega_a \omega^a - \frac{4}{3}\Lambda). \quad (11)$$

This is completely physical and has the exactly same form as the vacuum case, eq.(9). Thus, in accord with the vacuum case, $E(S)$ approaches to E_{ADM} or E_{BS} in asymptotically flat spacetime when appropriate limit is taken.

3.3 Einstein-spinor

Here we consider a model which contains one massless 2-spinor field ψ^A as a matter field. Such spinor is also known as the (left handed) Weyl spinor.

The readers should refer to the standard literature[15, 16] for the notation and the handling of the spinor fields on curved spacetimes. Be carefull about the difference of the metric sign convention when comparing the expressions found there with those of us. They take the signature of the spacetime metric as $(+---)$ there but we adopt $(-+++)$ here.

We adopt a spin-frame such that the preferred spatial subspaces spanned by m^a and $\bar{m}^a$ are tangent to S . Then the total Hamiltonian for this model is

$$\begin{aligned} \mathcal{H}|_g = & \frac{\mu}{16\pi} (C_{abcd} h^{ac} h^{bd} - 2\omega_a \omega^a - \frac{4}{3}\Lambda) \\ & + \frac{\mu}{2\sqrt{2}} i (\bar{\psi}^{0'} \bar{m}^a D_a \psi^1 + \bar{\psi}^{1'} m^a D_a \psi^0 - m^a \omega_a \bar{\psi}^{1'} \psi^0 - \text{c.c.}) \end{aligned} \quad (12)$$

It is remarkable that we have the same common term $C_{abcd} h^{ac} h^{bd} - 2\omega_a \omega^a - \frac{4}{3}\Lambda$ again. In addition, the remaining term

$$\bar{\psi}^{0'} \bar{m}^a D_a \psi^1 + \bar{\psi}^{1'} m^a D_a \psi^0 - m^a \omega_a \bar{\psi}^{1'} \psi^0 - \text{c.c.} \quad (13)$$

is not invariant under the remaining gauge freedom.

By investigating the behaviour of this term under that gauge transformation, it is shown that it can be eliminated by fixing the gauge and the quasi-local energy for Einstein-spinor system takes the form

$$E(S) = \left(\frac{A}{4\pi}\right)^{\frac{1}{2}} \int_S \frac{\mu}{16\pi} \left[C_{abcd} h^{ac} h^{bd} - 2\omega_a \omega^a - \frac{4}{3}\Lambda \right], \quad (14)$$

the same as the cases of the vacuum and Einstein-Maxwell system. Hence $E(S)$ coincides with E_{ADM} and E_{BS} at spatial and null infinity in asymptotically flat spacetimes.

4 Physical Interpretation of the new quasi-local energy

One apparent drawback of 2+2 formalism compared to 3+1 formalism is that its physical meaning is less clear than the other. Nevertheless, it can be shown that our quasi-local energy $E(S)$ has a concrete physical interpretation: the gravitational mass attracting the 2-surface S . The aim of this section is to validate this interpretation and provide a firm physical position to stand on with $E(S)$.

In Newtonian theory of gravitation, the motion of a test particle with vanishing angular momentum is determined by

$$\ddot{r} = -\frac{M}{r^2}. \quad (15)$$

This equation relates the gravitational mass M with $\ddot{r}$, the time evolution of a geometrical quantity. We seek for a similar relation in the context of General Relativity. Note that the relation we are seeking for should be more complex, because the nature of General Relativity tells us that what we are seeking for is (and what we can expect as much as possible is) the *effective* gravitational mass including the contribution of the centrifugal force. In the following, we first establish the mass-geometry relation in terms of the quantities associated with freely falling 2-surface in Newtonian theory and then extend that relation to the case of General Relativity, and compare the result with $E(S)$.

4.1 Effective mass in Newtonian theory

Consider a 2-surface S which consists of freely infalling test particles in the Newtonian absolute spacetime. Denote the tangent vector of each test particle by t^a and define the 3-velocity v^a by $t^a = (\partial/\partial t)^a + v^a$, where t is the absolute time. Denote the flat metric of the absolute space by g_{ab} and the natural derivative operator of the spacetime by ∇_a . Then the equation of motion takes the form

$$t^a \nabla_a v^b = -g^{ab} \nabla_b \Phi, \quad (16)$$

where Φ is the gravitational potential which satisfies

$$g^{ab} \nabla_a \nabla_b \Phi = 4\pi\rho \quad (17)$$

with ρ being the density of mass.

Denoting the projection tensor onto S by h_a^b , $h_{ab} \equiv h_a^c g_{cb}$ is the metric of S . These metrics satisfy $g^{ab} = h^{ab} + n^a n^b$ where n^a is the unit outward normal vector of S and $h^{ab} \equiv g^{ac} h_c^b$. Define $\hat{\theta}$ by $h_a^c h_b^d \mathcal{L}_t \mu_{cd} = \hat{\theta} \mu_{ab}$ where μ_{ab} is the area 2-form of S . Then $\hat{\theta} = h_b^a \nabla_a t^b = h_b^a \nabla_a v^b$ gives the evolution rate of the infinitesimal area element δA which consists of some fixed members of the test particles: $\hat{\theta} = (\delta A) / \delta A$, and $\hat{\theta}$ is also the expansion of the congruence of the world lines of test particles.

Introduce the shear $\hat{\sigma}_{ab} \equiv 2h_a^c h_b^d \nabla_{(c} v_{d)} - \hat{\theta} h_{ab}$ and the rotation $\hat{\omega}_{ab} \equiv 2h_a^c h_b^d \nabla_{[c} v_{d]}$ of the congruence of the world lines of the test particles³. Then we have, by the relation $t^c \nabla_c n^a = -n_b h^{ac} \nabla_c v^b$,

$$t^c \nabla_c \hat{\theta} + \frac{1}{2} \hat{\theta}^2 + \frac{1}{4} \hat{\sigma}_{ab} \hat{\sigma}^{ab} - \frac{1}{4} \hat{\omega}_{ab} \hat{\omega}^{ab} = -\underline{\theta} n^c \nabla_c \Phi - D_a (h^{ab} D_b \Phi) + h^{ab} (n_c \nabla_a v^c) (n_d \nabla_b v^d), \quad (18)$$

where $\underline{\theta} \equiv h^{ab} \nabla_a n_b$ is the trace of the extrinsic curvature of S .

Integrating the above equation over the closed 2-surface S , we have

$$\int_S \mu \left(t^c \nabla_c \hat{\theta} + \frac{1}{2} \hat{\theta}^2 + \frac{1}{4} \hat{\sigma}_{ab} \hat{\sigma}^{ab} - \frac{1}{4} \hat{\omega}_{ab} \hat{\omega}^{ab} \right) = \int_S \mu \left(-\underline{\theta} n^c \nabla_c \Phi + h^{ab} (n_c \nabla_a v^c) (n_d \nabla_b v^d) \right). \quad (19)$$

The first term on the right hand side basically gives the material mass inside S and the second term corresponds to the contribution of centrifugal force. This is most easily seen for the initial configuration in which S is sphere and v^a is tangent to S . For such surface, $\underline{\theta} = 2/r$ and $n_c h^{ab} \nabla_a v^c = -v^c h^{ab} \nabla_a n_c = -v^a / r$. Thus the value of eq.(19) is

$$\int_S \mu \left(-\frac{2}{r} n^c \nabla_c \Phi + \frac{v^a v_a}{r^2} \right) = 8\pi \left(-\frac{M}{r} + \frac{l^2}{2r^2} \right),$$

where $M \equiv \int_V dV \rho$ with V being the region inside S and $l \equiv r(v^a v_a)^{1/2}$ is the angular momentum of test particles of unit mass. This is (up to a numerical factor 8π) an effective potential of test particles with the angular momentum l .

From these consideration, we call

$$M_{eff}(S) \equiv -\left(\frac{A}{4\pi}\right)^{\frac{1}{2}} \int_S \frac{\mu}{8\pi} \left(\hat{\theta} + \frac{1}{2} \hat{\theta}^2 + \frac{1}{4} \hat{\sigma}_{ab} \hat{\sigma}^{ab} - \frac{1}{4} \hat{\omega}_{ab} \hat{\omega}^{ab} \right) \quad (20)$$

the effective mass for general closed surface S .

4.2 Effective mass in General Relativity

Next we consider a freely falling 2-surface in General Relativity. Take a fixed spatial 2-surface S and a unit timelike vector field t^a orthogonal to S over S . Extend each t^a along its direction to be geodesic. As a result we have a two-dimensional geodesic congruence which stems from S . Denote the metric of the 2-surface which is Lie-propagated along t^a by h_{ab} . Then the expansion $\hat{\theta} \equiv h^{ab} \nabla_a t_b$ of the congruence again satisfies $h_a^c h_b^d \mathcal{L}_t \mu_{cd} = \hat{\theta} \mu_{ab}$ and equals to the evolution rate of the infinitesimal area element which is spanned by some

³The rotation $\hat{\omega}_{ab}$ and the twist ω_a is different quantity.

fixed members of the congruence: $\hat{\theta} = (\delta A)/\delta A$. In this case the rotation $\hat{\omega}_{ab} \equiv 2h_a^c h_b^d \nabla_{[c} t_{d]}$ of the congruence vanishes since t^a is taken to be orthogonal to S , so the shear is given by $\hat{\sigma}_{ab} = 2h_a^c h_b^d \nabla_c t_d - \hat{\theta} h_{ab}$.

Then, by the relation $t^c \nabla_c n^a = -n_b h^{ac} \nabla_c t^b$ where n^a is unit spacelike vector orthogonal to both h_{ab} and t^a , we have

$$t^c \nabla_c \hat{\theta} + \frac{1}{2} \hat{\theta}^2 + \frac{1}{4} \hat{\sigma}_{ab} \hat{\sigma}^{ab} = -\frac{1}{2} C_{abcd} h^{ac} h^{bd} - R_{ab} t^a t^b + \frac{1}{2} R_{ab} h^{ab} - \frac{R}{3} + \omega^a \omega_a. \quad (21)$$

This equation should be compared to eq.(18).

Therefore the effective mass (20) is

$$M_{eff}(S) = \left(\frac{A}{4\pi}\right)^{\frac{1}{2}} \int_S \mu \left[\frac{1}{16\pi} (C_{abcd} h^{ac} h^{bd} - 2\omega_a \omega^a - \frac{4}{3} \Lambda) \right. \\ \left. T_{ab} t^a t^b - \frac{1}{2} T_{ab} h^{ab} + \frac{2}{3} T_a^a \right]. \quad (22)$$

where we have used the Einstein equations $R_{ab} - Rg_{ab}/2 = 8\pi T_{ab} - \Lambda g_{ab}$.

This is very similar to the common form (9) of our quasi-local energy $E(S)$. We encounter a familiar form $C_{abcd} h^{ac} h^{bd} - 2\omega_a \omega^a - \frac{4}{3} \Lambda$ again here. The essential difference between $E(S)$ and $M_{eff}(S)$ is only the term $T_{ab} t^a t^b - \frac{1}{2} T_{ab} h^{ab} + \frac{2}{3} T_a^a$. This remaining term should be considered as manifestation of the naiveness of the definition of $M_{eff}(S)$ relying the analogue of the Newtonian theory rather than a fault of $E(S)$. Actually, $E(S)$ should contain full information of the dynamics of the system since it is derived from the total Hamiltonian of the system, while $M_{eff}(S)$ does not. In addition, this remaining term is confirmed to vanish in the Reissner-Nordström and Schwarzschild-de Sitter spacetime and thus $E(S) = M_{eff}(S)$ for those spacetimes. This also lends support for regarding the remaining term as not so important.

For these reasons, we conclude that $E(S)$ has a firm physical meaning although 2+2 formalism itself may be hard to interpret physically. The meaning of $E(S)$ is the gravitational mass attracting the 2-surface S and accelerating it inward. This interpretation is in good accord with General Relativity and the full dynamics of the system is taken into it.

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On the null surface formalism

– Formulation in three dimensions and gauge freedom –

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Abstract

The null surface formalism of GR in three dimensions is presented, and the gauge freedom thereof, which is not just diffeomorphism, is discussed briefly.

1 Introduction

Recently, Frittelli, Kozameh, and Newman [1, 2, 3] presented an unconventional version of GR, where the variable is not any field such as the metric or connection, but 2-parameters family of foliations

$$Z(x^a; \zeta, \bar{\zeta}) = \text{const.} \quad (1)$$

of a manifold, where x^a is an arbitrary coordinate system, and ζ and $\bar{\zeta}$ are parameters. In this formalism, called the *null surface formalism (NSF)*, part of main equations, called the *metricity condition*, is obtained from the requirement that the hypersurfaces $Z(x^a; \zeta, \bar{\zeta}) = \text{const.}$ be *null* with respect to a metric. This condition not only says the existence of a corresponding metric on the manifold, but also contains information of null geodesics. (For an attempt toward quantization, see Refs.[4, 5].)

We shall in this article present the three-dimensional version of NSF. An establishment of NSF in dimensions other than four was nontrivial at the point the original formulation was presented, since the NSF is not a conventional “field” theory. Our success in three dimensions may suggest that NSF can establish in any dimensions equal to or higher than three. Moreover, the simplicity in three dimensions makes the algebraic structure of the NSF transparent. We use our formalism to discuss “gauge” freedom of NSF, which is also rather unconventional.

2 Null foliations and the intrinsic coordinates

Consider a Lorentzian metric $g^{ab}(x^a)$ on a three dimensional manifold M , and 1-parameter family of null foliations of M ;

$$Z(x^a, \zeta) = \text{const.}, \quad (2)$$

where x^a is an arbitrary coordinate system, and ζ is a real parameter.

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As is emphasized in Refs.[1, 2], a family of null foliations defines a particular coordinate system θ^i , called the *intrinsic coordinates*, which is given by successive derivatives of Z with respect to the parameter ζ ;

$$\begin{aligned}\theta^0 &\equiv u \equiv Z(x^a, \zeta), \\ \theta^1 &\equiv \omega \equiv \partial Z(x^a, \zeta), \\ \theta^2 &\equiv R \equiv \partial^2 Z(x^a, \zeta),\end{aligned}\tag{3}$$

where

$$\partial \equiv \frac{\partial}{\partial \zeta}.\tag{4}$$

We may write as

$$\theta^i = \partial^i Z(x^a, \zeta).\tag{5}$$

For later convenience, we define the “Jacobian”

$$\theta^i_a \equiv \theta^i_{,a}, \quad \theta^i_a \theta^j_a \equiv \delta^i_j.\tag{6}$$

We also define

$$\Lambda \equiv \partial^3 Z(\theta^i, \zeta).\tag{7}$$

We can find the components of the metric with respect to θ^i

$$g^{ij} = g^{ab} \partial^i Z_{,a} \partial^j Z_{,b},\tag{8}$$

by successively operating ∂ on the null condition of $Z(x^a, \zeta)$

$$g^{00} = g^{ab} (x^a) Z_{,a} Z_{,b} = 0.\tag{9}$$

For example, the first derivative of Eq.(9), $g^{ab} \partial Z_{,a} Z_{,b} = 0$, immediately implies $g^{01} = 0$. If we define

$$\Omega^2 \equiv g^{02},\tag{10}$$

which is independent of $Z(x^a, \zeta)$, the final result is

$$g^{ij} = \Omega^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -(1/3)\Lambda_{,2} \\ 1 & -(1/3)\Lambda_{,2} & -(1/3)\partial(\Lambda_{,2}) + (1/9)(\Lambda_{,2})^2 + \Lambda_{,1} \end{pmatrix}.\tag{11}$$

Here, the operation of ∂ on a function $f(\theta^i, \zeta)$ is well-defined;

$$\partial f(\theta^i, \zeta) = \partial' f + (\partial \theta^i) f_{,i} = \partial' f + \omega f_{,0} + R f_{,1} + \Lambda f_{,2},\tag{12}$$

where ∂' is the differential operator for fixed θ^i .

With Eq.(11), we can check that $l^a \equiv (\partial/\partial R)^a$ is a null geodesic generator;

$$l^a l_a = 0, \quad l^b \nabla_b l^a = -2 \frac{\Omega_{,2}}{\Omega} l^a.\tag{13}$$

3 The metricity conditions and Einstein's equation

As in [2], the metricity conditions can be expressed as

$$\begin{aligned} \partial g^{ab}(x^a) &= 0 \\ \Leftrightarrow \partial(g^{ij}(\theta^i, \zeta)\theta_i^a\theta_j^b) &= 0 \\ \Leftrightarrow \partial g^{ij}(\theta^i, \zeta) + g^{km}(\theta^i, \zeta)\partial(\theta_k^a\theta_m^b)\theta_i^a\theta_j^b &= 0. \end{aligned} \quad (14)$$

That is, we demand that the ζ -dependence of $g^{ij}(\theta^i, \zeta)$ can be “absorbed” into a coordinate transformation of the form (3). It is important to note that θ^i themselves are regarded as ζ -dependent, so $g^{ij}(\theta^i, \zeta) = g^{ij}(\theta^i(\zeta), \zeta)$. This implies that even though g^{ij} has no explicit dependence of ζ , the metricity conditions are nontrivial.

To proceed, we define matrix T^i_j by

$$\partial\theta^i_a \equiv T^i_j\theta^j_a. \quad (15)$$

With this, Eq.(14) can be written as

$$\partial g^{ij} - T^i_k g^{kj} - T^j_k g^{ki} = 0. \quad (16)$$

We can easily have the explicit form of T^i_j . For example, noting $\theta^i_a = \theta^i_{,a}$, we have for $i = 0$ for Eq.(15)

$$\begin{aligned} \partial Z_{,a} &= T^0_j\theta^j_{,a} \\ \Leftrightarrow \theta^1_{,a} &= T^0_j\theta^j_{,a}, \end{aligned} \quad (17)$$

so that

$$T^0_j = \delta_j^1. \quad (18)$$

Similarly, we have for other components

$$T^1_j = \delta_j^2, \quad T^2_j = \Lambda_j. \quad (19)$$

We have six conditions Eq.(16) (under Eqs.(18) and (19)), and moreover we impose the null condition $g^{00} = 0$ and define Eq.(10). We therefore have eight inputs, so eight outputs must follow. Six of them are given by the components of g^{ij} , Eq.(11). The rest is the metricity conditions we searched for, which is the following two consistency conditions;

$$\Lambda_{,2} = 3\Omega^{-1}\partial\Omega, \quad (20)$$

and

$$-\frac{4}{9}(\Lambda_{,2})^3 + 2\Lambda_{,2}\partial(\Lambda_{,2}) - 2\Lambda_{,1}\Lambda_{,2} - \partial^2(\Lambda_{,2}) + 3\partial(\Lambda_{,1}) - 6\Lambda_{,0} = 0. \quad (21)$$

If we find a solution of Eqs.(20) and (21) for Λ and Ω , we obtain a metric (11), and simultaneously, have essentially all information of null geodesics thereof (cf. Eq.(13)).

Next, we consider Einstein's equation with cosmological constant λ ,

$$G^{ab} = \lambda g^{ab}, \quad (22)$$

where G^{ab} is the Einstein tensor.

We cast the gradient of Z to Eq.(22);

$$\begin{aligned} G^{ab}Z_{,a}Z_{,b} &= \lambda g^{ab}Z_{,a}Z_{,b} = 0 \\ \Leftrightarrow G^{00} &= 0 \\ \Leftrightarrow G_{22} &= 0. \end{aligned} \quad (23)$$

From a straightforward calculation, $G_{22} = \Omega^{-1}\Omega_{,22}$. We thus have

$$\Omega^{-1}\Omega_{,22} = 0. \quad (24)$$

If we operate ∂ on (the first row of) Eq.(23), we obtain

$$G^{11} + G^{02} = 0, \quad (25)$$

but this does *not* mean the two components of Einstein's equation for $(ij) = (11)$ and (02) , given by

$$G^{11} = -\lambda\Omega^2 \quad (26)$$

and

$$G^{02} = \lambda\Omega^2. \quad (27)$$

Therefore, we need impose one of the two equations independently of Eq.(23) (or Eq.(24)).² We take Eq.(27), which is, after some calculations, found to be

$$\begin{aligned} & -\frac{1}{36}\Omega^2((\Lambda_{,22})^2 + 6\Lambda_{,221}) + \frac{1}{9}(\Omega_{,2})^2(-3\partial(\Lambda_{,2}) + (\Lambda_{,2})^2 + 9\Lambda_{,1}) \\ & + \frac{1}{18}\Omega\Omega_{,2}(3\partial(\Lambda_{,22}) + \Lambda_{,2}\Lambda_{,22}) + \frac{2}{3}\Omega_{,2}(3\Omega_{,0} - \Lambda_{,2}\Omega_{,1}) \\ & + \frac{1}{6}\Omega(\Lambda_{,22}\Omega_{,1} + 2\Lambda_{,2}\Omega_{,12} + 6\Omega_{,11} - 6\Omega_{,02}) - (\Omega_{,1})^2 = \lambda. \end{aligned} \quad (28)$$

We have used, as well as Eq.(24), identity

$$(\partial(\Lambda_{,2}))_{,2} = \partial(\Lambda_{,22}) + \Lambda_{,12} + \Lambda_{,2}\Lambda_{,22}, \quad (29)$$

obtained from more general identity for an arbitrary function ϕ

$$\partial(\phi_{,2}) - (\partial\phi)_{,2} = -\phi_{,1} - \Lambda_{,2}\phi_{,2}. \quad (30)$$

We can check that any other components of Einstein's equation follow from the successive differentiations with respect to ζ of Eq.(25). Thus, we need only two component equation, Eqs.(24) and (28), rather than six components.

Our fundamental equations are therefore the four coupled nonlinear partial differential equations (20), (21), (24), and (28). The following fact may be noteworthy at this point. First, we can immediately solve Eq.(24), and find that Ω is a polynomial of variable R at most of first order. Then, it is also easy to see from Eq.(20) that Λ is a polynomial of variable R at most of third order. Thus, both Λ and Ω are polynomials of R in *any* "gauge", discussed in the next section.

²Ref.[2] lacks the corresponding equation.

4 “Gauge” degrees of freedom

For a fixed spacetime, we can consider many varieties of 1-parameter family of null foliations, and which, in general, correspond to different Λ and Ω . Therefore, even in the present formalism there exists a sort of “gauge” freedom.

More precisely, we can categorize this “gauge” into two parts, one is essential *deformations* of way of taking the 1-parameter family of null foliations, and the other is just *reparametrizations* of the 1-parameter family of null foliations.

We, in this section, discuss two simple cases of reparametrizations.

4.1 Reparametrization of ζ

Consider a 1-parameter family of foliations $u = \text{const.} = Z(x^a, \zeta)$. Reparametrization of the form

$$\zeta \rightarrow f(\zeta), \quad (31)$$

where f is a real function, apparently preserves the original foliation. Hence, this is a sort of gauge.

Under the transformation (31), we have

$$Z(x^a, \zeta) \rightarrow \tilde{Z}(x^a, \zeta) = Z(x^a, f(\zeta)). \quad (32)$$

We denote the new intrinsic coordinates and Λ obtained from $\tilde{Z}$ as $(\tilde{u}, \tilde{\omega}, \tilde{R})$ and $\tilde{\Lambda}$, respectively. Operating ∂ successively on $\tilde{Z}$, we can easily have

$$\begin{aligned} \tilde{u} &= u, \quad \tilde{\omega} = \omega f'(\zeta), \\ \tilde{R} &= R f'(\zeta)^2 + \omega f''(\zeta), \\ \tilde{\Lambda} &= \Lambda f'(\zeta)^3 + 3R f'(\zeta) f''(\zeta) + \omega f'''(\zeta). \end{aligned} \quad (33)$$

Thus, we have obtained the transformation rule for Λ . The rule for Ω can also be obtained, if we think of Eq.(33) as a coordinate transformation $\theta^i \rightarrow \tilde{\theta}^i$.

From the Jacobian

$$\tilde{\theta}^i{}_{,k} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f'(\zeta) & 0 \\ 0 & f''(\zeta) & f'(\zeta)^2 \end{pmatrix}, \quad (34)$$

we have

$$\begin{aligned} \tilde{g}^{ij} &= g^{kl} \tilde{\theta}^i{}_{,k} \tilde{\theta}^j{}_{,l} \\ &= f'(\zeta)^2 \Omega^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & * \\ 1 & * & ** \end{pmatrix}, \end{aligned} \quad (35)$$

where

$$* \equiv -\frac{f''(\zeta)}{f'(\zeta)} - \frac{1}{3} f'(\zeta) \Lambda_{,2} \quad (36)$$

and

$$** \equiv -\left(\frac{f''(\zeta)}{f'(\zeta)}\right)^2 - \frac{2}{3} f''(\zeta) \Lambda_{,2} + f'(\zeta)^2 \left(-\frac{1}{3} \partial \Lambda_{,2} + \Lambda_{,1} + \frac{1}{9} (\Lambda_{,2})^2\right). \quad (37)$$

Comparing with Eq.(11), we find

$$\Omega \rightarrow \tilde{\Omega} = f'(\zeta)\Omega. \quad (38)$$

That is, Ω so transforms as ω . (We can check that $*$ and $**$ are certainly of the form of Eq.(11), if written in terms of the tilded variables.)

4.2 Reparametrization of u

Consider a foliation

$$\text{const.} = Z(x^a). \quad (39)$$

Then, the foliation

$$\text{const.} = \phi(Z(x^a)), \quad (40)$$

where ϕ is a real function, is geometrically the same foliation as the original.

We thus consider the transformation

$$Z(x^a, \zeta) \rightarrow \tilde{Z}(x^a, \zeta) = \phi(Z(x^a, \zeta)). \quad (41)$$

The following procedure to obtain the transformation rule for Λ and Ω is exactly the same as before. We denote the new intrinsic coordinates and Λ obtained from $\tilde{Z}$ as $(\tilde{u}, \tilde{\omega}, \tilde{R})$ and $\tilde{\Lambda}$, respectively. Operating ∂ successively on $\tilde{Z}$, we have

$$\begin{aligned} \tilde{u} &= \phi(u), \quad \tilde{\omega} = \phi'(u)\omega, \\ \tilde{R} &= \phi''(u)\omega^2 + \phi'(u)R, \\ \tilde{\Lambda} &= \phi'''(u)\omega^3 + 3\phi''(u)\omega R + \phi'(u)\Lambda. \end{aligned} \quad (42)$$

The rule for Ω can also be obtained by observing the coordinate transformation $\theta^i \rightarrow \tilde{\theta}^i$. We find

$$\Omega \rightarrow \tilde{\Omega} = \phi'(u)\Omega. \quad (43)$$

Again, Ω transforms like ω .

5 An example

As an example of solution, we consider the Minkowski space, which is the only (local) vacuum solution of Einstein's equation in three dimensions.

The metric is, in the standard coordinates, given by

$$ds^2 = dt^2 - dx^2 - dy^2. \quad (44)$$

From the 1-parameter family of (or " S^1 's worth" of) null vector fields given by

$$l^a = \left(\frac{\partial}{\partial t}\right)^a - \cos \zeta \left(\frac{\partial}{\partial x}\right)^a - \sin \zeta \left(\frac{\partial}{\partial y}\right)^a, \quad (45)$$

we can easily have a 1-parameter family of null foliations;

$$u = Z(x^a, \zeta) = l_a x^a = t + x \cos \zeta + y \sin \zeta. \quad (46)$$

Operating ∂^n , we have

$$\begin{aligned}\omega &= \partial Z = -x \sin \zeta + y \cos \zeta, \\ R &= \partial^2 Z = -x \cos \zeta - y \sin \zeta, \\ \Lambda &= \partial^3 Z = x \sin \zeta - y \cos \zeta.\end{aligned}\tag{47}$$

We, thus, have

$$\Lambda = -\omega.\tag{48}$$

As for Ω , like in the previous section, thinking of Eqs.(46) and (47) as a coordinate transformation, we have

$$\Omega = 1,\tag{49}$$

since the metric in terms of the intrinsic coordinates is found to be

$$ds^2 = du^2 + 2dudR - d\omega^2.\tag{50}$$

We can easily check that Eqs.(48) and (49) satisfy the fundamental equations (20), (21), (24), and (28). Also, we can check $l^a = (\partial/\partial R)^a$.

6 Conclusion

We have formulated the three dimensional version of NSF. It inherits all the properties of the original one, but is, of course, much simpler. We have explicitly discussed two simple classes of gauge with our simple version of NSF. As we have seen, gauge in NSFs is not just diffeomorphism of the manifold — it presumably corresponds to a subclass of the diffeomorphisms, but is unclear up to now.

Acknowledgments

I would like to thank Professor H. Kodama for helpful comments, T. Koike for an enlightening conversation, and M. Siino for discussions at an early stage of this work.

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Topology of Event Horizon

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Abstract

The topologies of event horizons are investigated. The relation between the topology of the event horizon and the endpoint of the event horizon is revealed. A torus event horizon is caused by two-dimensional endpoints. One-dimensional endpoints provide the coalescence of spherical event horizons. These aspects can be removed by an appropriate timeslicing. In a sense, the result of Shapiro, Teukolsky et. al.[8] is generalized in the present article.

1 Introduction

The existence of an event horizon is one of the most characteristic aspects of general relativity. So, many authors have been studied the properties of the event horizon.

Mathematically, an event horizon is defined as the boundary of the causal past of the future null infinity[1]. Since a natural asymptotic structure is supposed to be asymptotic flat (The topology of the future null infinity is $S^2 \times R$), we naively think that the (spatial) topology of the event horizon will be S^2 .

The simplest situation arises in the general stationary spacetime, for which it can be shown that any event horizon must have the spherical topology[2][3]. The first work regarding the topology of non-stationary black holes is due to Gannon[4]. Assuming a physically reasonable condition of asymptotic flatness, he proved that the topology of a smooth event horizon must be either a sphere or a torus (when the dominant energy condition is satisfied). Gannon's approach has recently been extended and generalized to yield stronger theorems, with the assumptions of asymptotic flatness, global hyperbolicity, and a suitable energy condition. Friedmann, Schleich, and Witt proved the "topological censorship" theorem that any two causal curves extending from past to the future null infinity is homotopy equivalent to each other[5]. Jacobson and Venkataramani[6] have established a theorem that strengthens a recent result due to Browdy and Galloway that the topology of an event horizon with a timeslicing is a sphere if no new null generators enter the horizon at later times[7]. The theorem of Jacobson and Venkataramani limits the time for which a torus event horizon can persist.

On the contrary, event horizons whose topology is not a single S^2 are reported in the numerical simulations of gravitational collapses. Shapiro, Teukolsky et. al.[8] numerically observed a torus event horizon in the collapse of a toroidal matter. Seidel et. al. numerically shown the coalescence of two spherical event horizons[9]. On first sight,

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their numerical results contradict some of the above mentioned theorems, though evidently their simulations are physically reasonable. As suggested by Shapiro, Teukolsky, et. al., an important aspect is indifferenciability of the event horizons in their numerically generated spacetime. In a physically reasonable collapse, one cannot expect the differentiability of the event horizon. In the present article, such indifferenciability which occurs in physically reasonable spacetimes are distinguished from indifferenciability caused by the pathological structure of a spacetime and treated as the endpoint of the null geodesics generating the event horizon[10].

In a physically realistic gravitational collapse, it is believed that a spacetime is stationary far in the future. Therefore it is natural to assume that the topology of an event horizon is a sphere for a single asymptotic region. So, the problem of the topology of the event horizon is regarded as topology changes in three-dimensional manifolds from non-spherical surface to a sphere. We put the theories of the topology change[11][12] into this case.

In the next section, we review the theories of the topology change of spacetime which is applied to the event horizon in the section 3. Final section is devoted to the summary and discussions.

2 The topology change of (2+1)-spacetime

Many works have been concerned in the topology change of a spacetime. Some of these are useful to discuss the topology of an event horizon (EH) which is a three-dimensional null surface embedded in a four-dimensional spacetime. Now we briefly present several theorems about the topology change of the spacetime.

2.1 Poincaré Hopf Theorem

Our investigation is based on the well known theorem about the relation between the topology of a manifold and a vector field on it. The following Poincaré-Hopf theorem (Milnor 1965) is essential for our analysis.

Theorem 2.1 Poincaré-Hopf *Let M^n be a compact n -dimensional ($n \geq 2$) C^∞ manifold. X is any C^∞ vector field with at most a finite number of zeros, satisfying following two conditions. (a) The zeros of X are contained in $\text{Int}M^n$. (b) X has outward directions at ∂M^n . Then the sum of the indices of X at all its zeros is equal to the Euler number χ of M^n ,*

$$\chi(M^n) = \text{ind}(X). \quad (1)$$

The index of the vector field X at a zero p is defined as follows. Let $X_a(x)$ be the components of X with respect to local coordinates $\{x^a\}$ in a neighborhood of p . Set $v_a(x) = X_a(x)/|X|$. If we evaluate v on a small sphere centered at $x(p)$, we can regard $v_a(S^{n-1})$ as a C^∞ mapping from S^{n-1} into S^{n-1} . The mapping degree of this map is called the index of X at the zero p . For example, if the map is homeomorphic, the mapping degree of the orientation preserving (reversing) map is $+1$ (-1). In the present article, we treat three-dimensional manifold embedded into a four-dimensional spacetime manifold as

an EH. The three-dimensional manifold has two two-dimensional boundaries as an initial boundary and a final boundary (this is assumed to be a sphere in the next section). For such a manifold, we use the following modification of the Poincaré-Hopf theorem. Now we consider odd-dimensional manifold with two boundaries $\Sigma_{1,2}$.

Theorem 2.2 Sorkin 1986 *Let M^n be a compact n -dimensional ($n \geq 2$ is an odd number) C^∞ manifold with $\Sigma_1 \cup \Sigma_2 = \partial M^n$ and $\Sigma_1 \cap \Sigma_2 = \emptyset$. X is any C^∞ vector field with at most a finite number of zeros, satisfying following two conditions (a) The zeros of X is contained in $\text{Int} M^n$. (b) X has inward directions at Σ_1 and outward directions at Σ_2 . Then the sum of the indices of X at all its zero is related to the Euler number of Σ_1 and Σ_2 .*

$$\chi(\Sigma_2) - \chi(\Sigma_1) = 2\text{ind}(X). \quad (2)$$

Its proof is given in Sorkin's work[11].

2.2 Geroch's Theorem

Geroch stressed that no closed timelike curve in a spacetime (M, g) needs diffeomorphic initial and final hypersurfaces[12].

Theorem 2.3 Geroch 1967 *Let M be a smooth n -dimensional compact geometry whose boundary is the disjoint union of two compact spacelike $(n-1)$ -manifolds, Σ_1 and Σ_2 . Suppose M is isochronous, and has no closed timelike curve. Then Σ_1 and Σ_2 are diffeomorphic, and further M is topologically $\Sigma_1 \times [0, 1]$.*

This theorem is not directly applicable to a null surface H where a chronology is determined by null geodesics generated by a null vector field K . In this case, isochronous means that there is no zero of K in the interior of H . On the other hand, the closed timelike curve does not rigorously correspond to a closed null curve, since on a null surface an imprisoned null geodesic cannot be distorted so that, while remaining null, it becomes closed curve as in the case of the theorem 2.3[12]. Then we require the strongly causal condition[10] to a spacetime rather than no closed causal curve condition. The following modified version of Geroch's theorem arises.

Theorem 2.4 *Let H be a smooth n -dimensional compact null surface whose boundary is the disjoint union of two compact spacelike $(n-1)$ -manifolds, Σ_1 and Σ_2 . Suppose that there exists a null vector field K which is nowhere zero in the interior of H and has inward and outward direction at Σ_1 and Σ_2 , respectively, and H is embedded into a strongly causal spacetime (M, g) . Then Σ_1 and Σ_2 are diffeomorphic, and further H is topologically $\Sigma_1 \times [0, 1]$.*

3 The topology of event horizon

Now we apply the topology change theories given in the previous section to EHs. Let (M, g) be a four-dimensional C^∞ spacetime. In the rest of this article, the spacetime

(M, g) is supposed to be strongly causal. Furthermore, for simplicity, the topology of the EH (TOEH)² is assumed to be a smooth S^2 far in the future. This assumption is valid in the case we consider only one regular ($\sim R \times S^2$) asymptotic region to define the EH, namely the future null infinity $\mathcal{J}^+$. The following result, however, is easily extended to the case of the different final TOEH.

In the present article, the most important concept is the existence of the endpoints of null geodesics λ which completely lies in the EH and generates it. We call them the endpoints of the EH. Here we regard the null geodesics λ are on the EH and the endpoint is considered as the endpoint of the null geodesics belonging to the EH, though the null geodesic can continue to the outside or the inside of the EH through the endpoint. We consider a null vector field K on the EH which is tangent to the null geodesics λ . The zeros of K are the endpoint of these null geodesics and it can only be past endpoints since λ generates the EH[10].

First we pay attention to the relation between the endpoint and the differentiability of the EH. We see that the EH is not smooth at the past endpoint.

Lemma 3.1 *Suppose that H is a three-dimensional null surface embedded into the space-time (M, g) by a function f as*

$$H : f(x) = 0. \quad (3)$$

When H is generated by the set of null geodesics whose tangent vector field is K , the embedding function f is indifferentiable at the endpoint of the null geodesic.

From now we require EHs can only have the above mentioned indifferentiability.

Condition 3.2 *An EH is indifferentiable only on compact subsets and they become the endpoints of null geodesics generating the EH.*

In the case in which the future null infinity possesses pathological structure, its EH could be nowhere differentiable[13]. Moreover, there might be the case where indifferentiable point is not the endpoint of the EH. However, we admit the condition 3.2 as a physically valid condition. We have no concrete example of a physically reasonable spacetime with the non-compact indifferentiability. On the other side, the reason why we consider the indifferentiability only caused by the endpoints, is that every EH possesses at least one endpoint except for an eternal EH. Most of the indifferentiability, which we can presume, is concerned by the endpoint.

When we suppose there is no past endpoint of null generator between Σ_1 and Σ_2 on an EH, Geroch's theorem stresses the topology of the smooth EH does not change.

Proposition 3.3 *Let H be the subset of an EH, whose boundaries are an initial spatial section Σ_1 and a final spatial section Σ_2 . Σ_2 is assumed to be far in the future and a smooth sphere. Suppose that H is smooth. Then the topology of Σ_1 is S^2 .*

From Sorkin's theorem there should be any zero of null generator K in the interior of H provided that the Euler number of $\Sigma_2 \sim S^2$ is different from that of Σ_1 . Such a

²The TOEH means the topology of the spatial section of the EH. Of course, it depends on a timeslicing.

zero can only be a past endpoint since the generator of the EH, K , cannot have a future endpoint in the interior of H [10]. For such past endpoints we can prove following two propositions.

Proposition 3.4 *Suppose that the EH of strongly causal spacetime (M, g) satisfies the condition 3.2. The set of the past endpoints of the EH is a spacelike set.*

Proposition 3.5 *Suppose that the EH of the strongly causal spacetime (M, g) satisfying the condition 3.2 and the spatial section of it is a smooth sphere far in the future. The SOEP of the EH is arc-wise connected. Moreover, the collared SOEP is topologically D^3 .*

Now we give theorems and corollaries about the topology of spatial sections of the EH on a timeslicing. First we consider the case where the EH is dynamically formed with a simple structure.

Theorem 3.6 *Let S_H be the section of an EH by a spacelike hypersurface. Suppose the EH, which is embedded into the strongly causal spacetime (M, g) and satisfies the condition 3.2, is not eternal one. If S_H is smooth, it is topologically $\emptyset$ or S^2 .*

On the other hand, we get the following theorem about the change of the TOEH with the aid of Sorkin's theorem.

Theorem 3.7 *Consider a timeslicing T to which spatial sections of an EH, Σ_1 and Σ_2 are tangent. Let H be the subset of an EH in a strongly causal spacetime (M, g) , whose boundaries are the initial spatial section Σ_1 and the final spatial section Σ_2 , and K be the null generator of the EH. Suppose that Σ_2 is a sphere and H satisfies the condition 3.2. If, in the timeslicing T , the TOEH changes (Σ_1 is not homeomorphic to Σ_2) then*

- *the one-dimensional segment of the SOEP causes coalescence of two spherical EHs.*
- *the two-dimensional segment of the SOEP causes the change of the TOEH from a torus to a sphere.*

Proof: First we carry out the regularization of H and K . Consider a homeomorphism $\phi : M \rightarrow M$. It maps

$$\phi : H \longrightarrow \widetilde{H}, \quad \lambda \longrightarrow \tilde{\lambda} \quad (4)$$

$$\Sigma_{1,2} \longrightarrow \widetilde{\Sigma}_{1,2}, \quad T \longrightarrow \widetilde{T}, \quad (5)$$

$$\phi^* : K \longrightarrow \widetilde{K}, \quad (6)$$

so that $\widetilde{K}$ is the tangent field of curves $\tilde{\lambda}$ generating $\widetilde{H}$. We call ϕ a smoothing map when $\widetilde{H}$ is embedded into M by a smooth function f as

$$\widetilde{H} : f(x) = 0. \quad (7)$$

Such ϕ always exists, when H satisfies the condition 3.2.

Now $\widetilde{K}$ is not always null. Furthermore, using the mapped timeslicing $\widetilde{T}$, we modify $\widetilde{K}$ so that the SOEP of $\tilde{\lambda}$ becomes zero-dimensional set (the SOEP is no longer arc-wise

connected). To make the SOEP zero-dimensional, $\bar{K}$ should be given on the SOEP of $\tilde{K}$ so as to generate it. On the SOEP of $\tilde{K}$, $\bar{K}$ is determined by the timeslice $\tilde{T}$ so that $\bar{K}$ is tangent to the SOEP of $\tilde{K}$ and directed to the future in the sense of the timeslicing $\tilde{T}$. Especially, at the boundary of the SOEP of $\tilde{K}$, we should be careful about a non-zero-dimensional boundary so that the boundary is also generated by $\bar{K}$. Here it is noted that the case in which the boundary is tangent to the timeslicing $\tilde{T}$ is possible. Since such a situation is unstable under the small deformation of the timeslicing, however, we omit this degeneration as mentioned in the following remark. Hence $\bar{K}$ is determined on the SOEP of $\tilde{K}$ (see, for example, Fig.1) and has some zeros there. Smoothing $\bar{K}$ along $\bar{K}$ on the SOEP (There appears no additional zero of $\bar{K}$ by this smoothing since future directed null vector field K is also future directed in the sense of the timeslicing $\mathcal{T}$), we get $\bar{K}$ on the whole of $\tilde{H}$ and its integral curves $\bar{\lambda}$. From the construction of $\bar{K}$, there are some zeros of $\bar{K}$ only on the SOEP and $\bar{K}$ is everywhere future directed in the sense of the timeslicing $\tilde{T}$ (though they will be spacelike somewhere).

Now we apply the theorem 2.2 to $\tilde{H}$ whose boundaries are $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2 \sim S^2$, with the modified vector field $\bar{K}$. Since $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are tangent to $\tilde{T}$, $\bar{K}$ has inward directions at $\tilde{\Sigma}_1$ and outward directions at $\tilde{\Sigma}_2$.

From the construction, the type of the zero of $\bar{K}$ depends on the dimensions of the SOEP. Especially for the zero most in the future, the one-dimensional SOEP provides a zero corresponding to index = -1 and the two-dimensional SOEP gives a zero with index = +1. By the theorem 2.2, the Euler number changes at the zero by $2 \times \text{index}$. Therefore if there is the one- (two)-dimensional SOEP, the timeslicing $\mathcal{T}$ gives the topology change of the EH from two spheres (a torus) to a sphere. When $\tilde{H}$ contains the whole of the SOEP, it will present all changes of the TOEH from the formation of the EH as shown in Fig.1. To complete discussions we consider dull cases provided by an appropriate timeslicing. When the edge of the SOEP is hit by a timeslice from the future, it gives a zero with its index being zero from the above construction (Fig.1(c)). $\square$

This result is suggested in Shapiro, Teukolsky et. al.[8].

Incidentally a certain timeslicing gives the further changes of the Euler number.

Corollary 3.8 *The topology change processes of an EH with $(n \times S^2 \rightarrow S^2, n = 1, 2, 3, \dots)$ can change each other and with (a surface with genus= $n \rightarrow S^2, n = 1, 2, 3, \dots$) can also change, under the appropriate deformation of a timeslicing.*

As shown in the above corollary 3.8, the TOEH highly depends on a timeslicing. Nevertheless the theorem 3.7 tells that there is a determined difference between the coalescence of n spheres where the Euler number decreases by the one-dimensional SOEP and the EH of a surface with genus= n where the Euler number increases by the two-dimensional SOEP. Finally we see that, in a sense, the TOEH is transient term.

Corollary 3.9 *All the changes of the TOEH are reduced to the trivial creation of an EH which is topologically S^2 .*

Thus we see that the change of the TOEH is determined by the topology of the SOEP and the slicing way of it. We can draw the graph of the SOEP as Fig.3. To determine the topology of the SOEP we must only give the order to each vertex of the graph by a

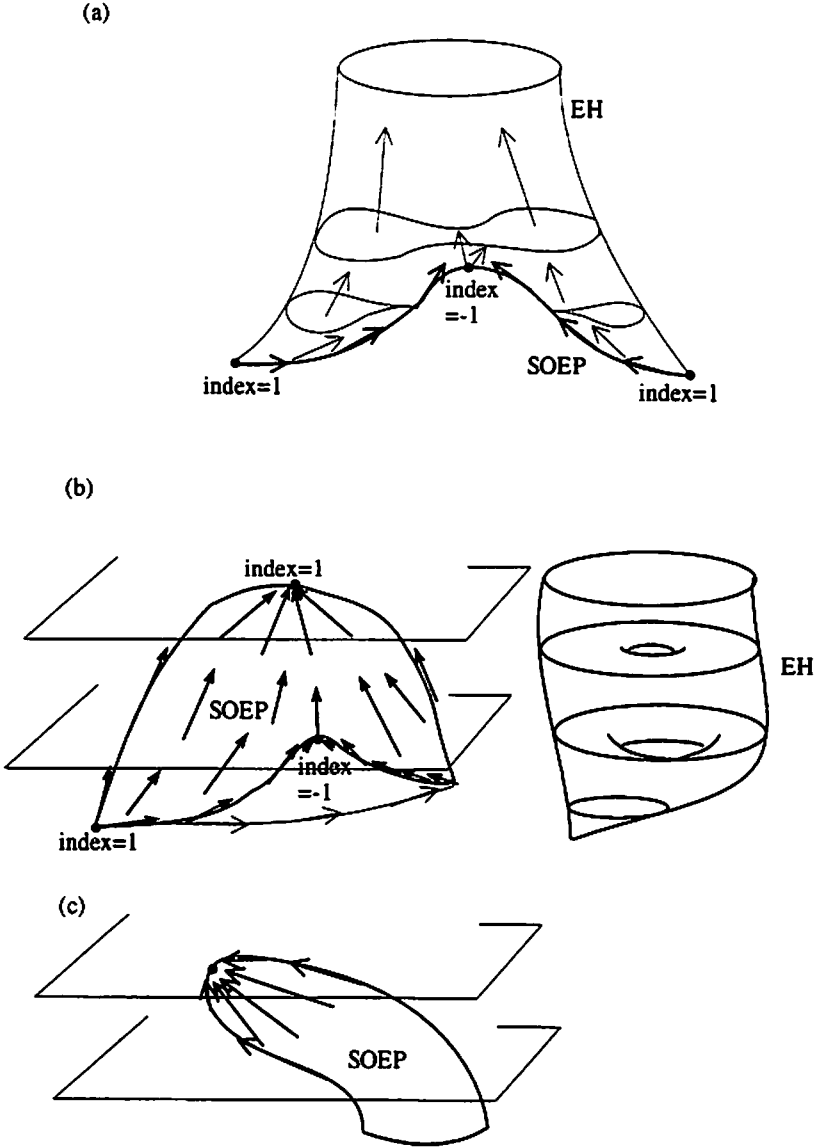


Figure 1: (a) and (b) are the one-dimensional and two-dimensional SOEP, respectively (In (b) we draw the whole of the EH separately). (c) is the case in which the edge of the SOEP is hit from the future. By these vector fields $\vec{K}$, the SOEPs are generated. The zeros of $\vec{K}$ and their indices are indicated.

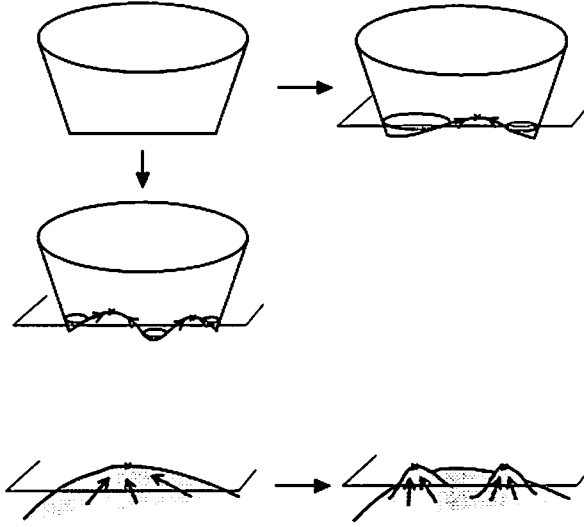


Figure 2: By the appropriate changes of the timeslicing, the number of zeros of the vector field $\overline{K}$. (a) and (b) are the one-dimensional and two-dimensional SOEP, respectively.

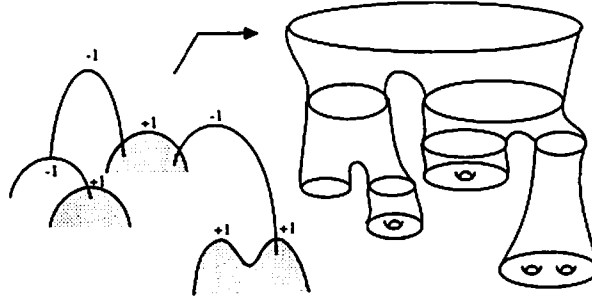


Figure 3: A graph is given. Determining the order to the vertex, we see the topology of the EH from the index of each zero.

slicing. The graph in Fig.3 might be rather complex. Nevertheless, considering a small scale inhomogeneity, for example a single particle scale, the EH may admit such a complex SOEP. It will be smoothed out in our macroscopic physics.

4 Summary and Discussion

We have studied the topology of the EH (TOEH), partially considering the indifferentiability of the EH. We have found that the coalescence of EHs is caused by the one-dimensional past endpoints and a torus EH two-dimensional. In a sense, this is the generalization of the result of Shapiro, Teukolsky et. al.[8]. Furthermore such changes of the TOEH can be removed by an appropriate timeslicing since the set of endpoints (SOEP) is a connected spacelike set. We see that the TOEH strongly depends on the timeslicing. The dimen-

sions of the SOEP, however, play an important role and are invariant under the change of the timeslicing. Hence it arises the question what controls the dimensions of the SOEP. One may expect that something like an energy condition restricts the possibilities of the SOEP. Nevertheless it is doubtful since, in fact, both cases (the coalescence of EHs—the one-dimensional SOEP and a torus EH—the two-dimensional SOEP) are reported in the numerical simulations [8][9]. Are these generic in real gravitational collapses? It is probable that the gravitational collapse in which the EH is a single sphere for all the timeslicing is not generic, since the zero-dimensional SOEP reflects the higher symmetry of system than that of the one- or two-dimensional SOEP. On balance, the symmetry of matter configurations will control it. It should be cleared in future.

In the present article, we have assumed some conditions about the structure of spacetime. Can other weaker conditions take the place of them? First, the differentiability of (M, g) can be weakened to a certain level. On the other side, the strongly causal condition may be a little strong. For, this condition is needed only on the EH. For example, the global hyperbolicity implies the strong causality on the EH, because the global hyperbolicity exclude a closed causal curve and a past imprisoned causal curve and there is no future imprisoned null curve on the EH. On the topology of the spacetime, we required that $\mathcal{I}^+$ is topologically $R \times S^2$, which implies that the TOEH is S^2 far in the future. This, however, is not crucial. Since the present investigation is based on the topology change theory, the same discussion is possible for other final TOEH. The differentiability of the EH is supposed by the condition 3.2 while it might be able to be violated in the realistic situations. The case in which the EH is not indifferntiable only on compact subsets will be excluded by the realistic requirement about the asymptotic structure, as a nowhere differentiable spacetime[13] is excluded by asymptotic flatness. On the contrary, the condition, that an indifferntiable point is an endpoint, is not clear whether it can be implied by other physically reasonable conditions. The indifferntiable point which is the endpoint, however, is overwhelmingly easy to occur. Every non-eternal EH possesses such an indifferntiable point while we do not have any simple example where the indifferntiable point is not the endpoint.

Incidentally, some of the statement appeared in this article, may overlap the result in the former works[2]~[7]. Nevertheless the condition required here is pretty different from that of them (Especially the energy condition is not assumed). They might be the extension of the former work.

Finally we remind the essential question. How can we see the topology of the EH? Some of the former works, for example “cosmic censorship”, oppose. We expect the phenomena highly depending on the existence of the EH, for instance the quasi-normal mode of gravitational wave[14] or Hawking radiation[15], reflect the TOEH. It is our future problem.

Acknowledgements

We would like to thank Professor H. Kodama, Dr. K. Nakao, Dr. T. Chiba and Dr. A. Ishibashi for helpful discussions. We are grateful to Professor H. Sato and Professor N. Sugiyama for their continuous encouragement. The author thanks the Japan Society for

the Promotion of Science for financial support. This work was supported in part by the Japanese Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Science, Culture and Sports.

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Geodesical Completeness and Globally Hyperbolicity : Rigid Singularity Theorem

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Abstract

We show the rigid singularity theorem, that is, a globally hyperbolic spacetime satisfying the strong energy condition and containing trapped sets is incomplete or splits isometrically as space $\times$ time. Further, we discuss Geroch's theorem containing an assumption which is the "no observer horizon" condition. These results are related to Yau's Lorentzian splitting conjecture. Thus, in other words, we establish one version of Lorentzian splitting theorem.

1 Introduction

30 years ago, Penrose-Hawking have shown that spacetimes are geodesically incomplete under some physically reasonable conditions [P][H][HP][HE]. The generic condition is the key assumption to induce singularities *rigidly*. Geroch improved these theorems without the generic condition and with "no observer horizon" condition for the spatially closed universe [Ge]. Here, "no observer horizon" condition is that for some $p \in M$, the set $M - [I^+(p) \cap I^-(p)]$ is compact (This definition is what Bartnik improved the original Geroch's definition[Ba]). He proved that a globally hyperbolic spacetime which has compact acausal hypersurfaces without edge (i.e. compact Cauchy surfaces) is geodesically incomplete or Ricci flat. In this statement flatness implies that the spacetime is static, i.e. a timelike Killing vector field is orthogonal to a family of spacelike hypersurface, (of course, there exists a static and non-flat spacetime, e.g. Einstein static universe). From the standpoint the *rigidity*, we would like to show that spacetimes are singular except under very special circumstances (e.g. static) even if the strict curvature inequalities do not hold, i.e. no generic condition. A question arises whether or not spacetimes are static if spacetimes are timelike geodesically complete.

In 1988, Bartnik conjectured that if spacetimes which are timelike geodesically complete and globally hyperbolic with compact Cauchy surfaces satisfy the timelike convergence condition (strong energy condition), the spacetime splits isometrically as space $\times$ time, i.e. static[?]. Eschenburg and Galloway proved the Bartnik's conjecture without global hyperbolicity and with modified "no observer horizon" condition which is $I^-(\gamma) \cap S \neq \emptyset$ where S is a compact partial Cauchy surface and γ is an S -ray[EG].

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This result is closely related Yau's Lorentzian splitting conjecture, that is, a spacetime (M, g) which is timelike geodesically complete, obeys the timelike convergence condition (strong energy condition), splits as an isometric product $(\mathbf{R} \times V, -dt^2 \oplus h)[Y]$. This conjecture is proved by Eschenburg[E], Galloway[?], Newman[N] and lastly Galloway and Horta[GHo] with the assumption of the existence of a complete timelike line.

The Lorentzian splitting theorem is slightly different from Bartnik conjecture since the former requires the assumption of the existence of a timelike line and the latter requires the assumption of the existence of a compact spacelike hypersurface. The advantage of Eschenburg and Galloway's theorem[EG] is that if there exists a compact spacelike hypersurface, there *always* exists a line under "no observer horizon" condition. Our aim is to show this theorem without the modified "no observer horizon" condition. This is surely Geroch's conjecture[Ge].

The purpose of this paper is to show the rigid singularity theorem in globally hyperbolic spacetimes. To do this we show Eschenburg and Galloway's theorem[EG] without the modified "no observer horizon" condition and with global hyperbolicity. Also, we show the existence of compact Cauchy surfaces in globally hyperbolic and geodesically complete spacetimes with trapped sets.

This fact, regardless of the assumption of the existence of a compact spacelike hypersurface without edge, suggests that we can show the rigid singularity theorem.

Further, we should show that a static, globally hyperbolic, timelike geodesically complete spacetime satisfying the strong energy condition and containing a compact Cauchy surface satisfies $R_{ab}K^aK^b \equiv 0$. This result is obtained directly from our theorems and the theorem of Garfinkle and Harris[GHa]. This fact is closely related to Geroch's theorem[Ge].

Our notation and fundamental definitions (such as $I^\pm, J^\pm, E^\pm, D^\pm, H^\pm$,) are as those of Hawking and Ellis[HE] and Beem et al.[?].

2 Preliminaries

Let M be a timelike geodesically complete time oriented Lorentzian manifold.

A spacetime M is said to be *globally hyperbolic* if the strong causality condition holds on M and if for any two points $p, q \in M$, $J^+(p) \cap J^-(q)$ is compact. Globally hyperbolicity of M implies the existence of a Cauchy surface for M . A *Cauchy surface* is a subset of M which every inextendible nonspacelike curve intersects exactly once. To prove the rigid singularity theorem there exists a necessary concept, that is, a trapped set. We call an achronal set S *future (or past) trapped* such that $E^+(S)$ (or $E^-(S)$) is compact. In geodesically complete spacetimes satisfying the strong energy condition, the existence of a trapped set implies the existence of (1) a compact achronal set without edge, (2) a closed trapped surface or (3) a point p such that on every past (or future) null geodesic from p the expansion of the null geodesics from p become negative. This means that in the universe the attraction by the matter surpasses the repulsion.

Next, we will represent definitions and lemmas about curves. The following tools is very strong arms to construct a timelike line. We denote the Lorentzian arc length of causal curves γ as $L(\gamma)$.

We fix a complete Riemannian metric h for the spacetime (M, g) throughout this paper.

Given a inextendible nonspacelike curve γ in (M, g) , γ is assumed to be reparametrized to be an h -unit speed curve. The following formulation of limit curve is very useful.

Lemma 1 (lemma 14.2 of [BEE]) *Let $\gamma_n : (-\infty, \infty) \rightarrow M$ be a sequence of inextendible nonspacelike curves (parametrized by arc length in h which is a complete Riemannian metric). Suppose that $p \in M$ is an accumulation point of the sequence $(\gamma_n(0))$. Then there exists an inextendible nonspacelike curve $\gamma : (-\infty, \infty) \rightarrow M$ such that $\gamma(0) = p$ and a subsequence (γ_{n_k}) which converges uniformly (with respect to h) to γ on compact subsets of $\mathbf{R}$. γ is called a limit curve of (γ_n) . $\square$*

A sequence $\gamma_n : [a_n, b_n] \rightarrow (M, g)$ of h -arc length parametrized causal curves is said to be *limit maximizing* if

$$L(\gamma_n) \geq d(\gamma_n(a_n), \gamma_n(b_n)) - \epsilon_n \quad (1)$$

where $\epsilon_n \rightarrow 0$ as $n \rightarrow +\infty$ and d denotes the Lorentzian distance function such that

$$d(p, q) = \sup\{L(\gamma); \gamma \in C(p, q)\} \leq \infty, \quad (2)$$

where $C(p, q)$ denotes the set of causal curves from p to q .

A *ray* is a maximal future inextendible causal geodesic $\gamma : [0, \infty) \rightarrow M$. We can get rays from limit constructions as follows.

Lemma 2 (Lemma 14.8 of [BEE]) *Let $\{z_n\}$ be a sequence in (M, g) with $z_n \rightarrow z$. Let $p_n \in I^+(z_n)$ with $d(z_n, p_n) < +\infty$. Let $\gamma_n : [0, a_n] \rightarrow (M, g)$ be a limit maximizing sequence of future causal curves with $\gamma_n(0) = z_n$ and $\gamma_n(a_n) = p_n$. Let $\tilde{\gamma}_n : [0, +\infty) \rightarrow (M, g)$ be any future inextendible extension of γ_n . Suppose either $p_n \rightarrow +\infty$ or $d(z_n, p_n) \rightarrow +\infty$. Then any limit curve $\gamma : [0, +\infty) \rightarrow (M, g)$ of the sequence $\{\tilde{\gamma}_n\}$ is a nonspacelike geodesic ray starting at z . $\square$*

We extend the notion of a ray as follows. Let $S \subset M$ be a subset containing $\gamma(0)$ such that γ maximizes distance to S , i.e. for any $t \in [0, \infty)$,

$$L(\gamma|[0, t]) = d(S, \gamma(t)), \quad (3)$$

where $L(\gamma)$ is the Lorentzian arc length and $d(S, x) = \sup\{d(q, x); q \in S\}$. Then γ is called an *S-ray*.

Let $\gamma : [0, \infty) \rightarrow M$ be a future inextendible S -ray and let $z \in I^-(\gamma) \cap J^+(S)$. Let $z_n \rightarrow z$ in $J^+(S)$ and put $p_n = \gamma(r_n)$ for some sequence $\gamma_n \rightarrow \infty$. Then $z_n \in I^-(p_n)$ for sufficiently large n , and $d(z_n, p_n) < \infty$. Assume either

- a. $p_n \rightarrow \infty$, or
- b. $d(z_n, p_n) \rightarrow \infty$.

Consider a limit maximizing sequence μ_n of causal curves from z_n to p_n . By lemma 2, any limit curve $\mu : [0, \infty) \rightarrow M$ of the μ_n is a ray starting at z . Such a ray is called a *co-ray* of γ . Note that μ is contained $\overline{I^-(\gamma)}$.

Let $\gamma : [0, \infty) \rightarrow M$ be a timelike S -ray. The Busemann function $b : M \rightarrow (-\infty, \infty)$ associated to γ is defined as follows,

$$b(x) = \lim_{t \rightarrow \infty} b_t(x), \quad (4)$$

where

$$b_t(x) = d(\gamma(0), \gamma(t)) - d(x, \gamma(t)). \quad (5)$$

$b_t(x)$ decreases monotonically with t since, for $s > t$, $b_t(x) \geq b_s(x)$. This inequality is given by

$$d(x, \gamma(s)) \geq d(x, \gamma(t)) + d(\gamma(t), \gamma(s)), \quad (\text{Reverse Triangle Inequality}) \quad (6)$$

$$d(\gamma(0), \gamma(s)) = d(\gamma(0), \gamma(t)) + d(\gamma(t), \gamma(s)). \quad (7)$$

Further, for $x \in I^-(\gamma) \cap J^+(S)$,

$$b(x) \geq d(S, x) \geq 0 \quad (8)$$

since

$$d(S, x) + d(x, \gamma(t)) \leq d(\gamma(0), \gamma(t)) < \infty \quad (9)$$

implies

$$b_t(x) \geq d(S, x) \quad (10)$$

for any t . We should recall that b_t is upper semicontinuous since d is lower semicontinuous and b is also upper semicontinuous since b is the decreasing limit of the b_t .

Lemma 3 *Let $\gamma : [0, \infty) \rightarrow M$ be a timelike S -ray and $\mu : [0, \infty) \rightarrow M$ be a co-ray with $\mu(0) = z \in I^-(\gamma) \cap J^+(S)$. If μ is a null ray, then $b(x) \leq b(z)$ for any $x \in I^-(\mu)$.*

Proof: Let μ_n is a limit maximizing sequence from z_n to $\gamma(r_n)$ where $z_n \rightarrow z \in I^-(\gamma) \cap J^+(S)$ and $r_n \rightarrow \infty$. Then

$$b_{r_n}(x) = d(\gamma(0), \gamma(r_n)) - d(x, \gamma(r_n)), \quad (11)$$

$$b_{r_n}(\mu_n(t)) = d(\gamma(0), \gamma(r_n)) - d(\mu_n(t), \gamma(r_n)), \quad (12)$$

and, by reverse triangle inequality,

$$d(x, \gamma(r_n)) \geq d(x, \mu_n(t)) + d(\mu_n(t), \gamma(r_n)). \quad (13)$$

($x \in I^-(\mu_n(t))$ since μ is limit curve of $\{\mu_n\}$.) Thus,

$$b_{r_n}(x) \leq b_{r_n}(\mu_n(t)). \quad (14)$$

Also,

$$b_{r_n}(\mu_n(t)) - b_{r_n}(z_n) = d(z_n, \mu_n(t)). \quad (15)$$

By (14) – (15),

$$b_{r_n}(x) - d(z_n, \mu_n(t)) \leq b_{r_n}(z_n). \quad (16)$$

Taking the limit as $n \rightarrow \infty$, we have $d(z_n, \mu_n(t)) \rightarrow d(z, \mu(t)) = 0$ since μ is null. Further, by Busemann function is upper semicontinuous, we get $b(x) \leq b(z)$. $\square$

Remark. From the previous discussion and the above lemma we have $0 \leq d(S, x) \leq b(x) \leq b(z)$.

To show that co-ray is timelike we need Lorentzian Busemann function[BEE]. From this, we get the following lemma.

Lemma 4 *Let (M, g) be a timelike geodesically complete manifold containing a compact Cauchy surface S . Then there exists a timelike S -ray and any co-ray of γ is timelike.*

Proof: Let $p \in S$ and $\mu : [0, \infty) \rightarrow M$ be a future inextendible timelike geodesic with $\mu(0) = p$. Let $r_n \rightarrow \infty$ be monotone sequence and $p_n = \mu(r_n)$. Then $p_n \rightarrow \infty$, otherwise the strong causality violates. Since S is a Cauchy surface, there exists maximal curve γ_n from S to p_n . Let $q_n = \gamma_n(0) \in S$. By compactness of S , $q_n \rightarrow q \in S$. From lemma 2, there exists an S -ray γ as a limit curve of $\{\gamma_n\}$. If γ is null $d(\gamma(0), \gamma(t)) = 0$, where $\gamma : [0, \infty) \rightarrow M$, $\gamma(0) \in S$ and $0 < t < \infty$. This means $\gamma(t) \in J^+(S)$. But, by compactness of S , $J^+(S) = S$. Therefore causality violates. Thus, S -ray γ is timelike.

Now let η be a co-ray of γ with $\eta(0) = u \in J^+(S)$. Similarly as above discussion, $\eta(s) \in I^+(S)$ for any $s > 0$. Let $s_n \rightarrow \infty$ be a monotone sequence and $v_n = \eta(s_n)$. We can construct a sequence $\{u_n\} \subset I^-(\gamma) \cap J^+(S)$ with $u_N \in I^-(v_n)$ for all n such that $d(S, u_n) \rightarrow \infty$. If η is null, for Busemann function of γ , $b(u_n) \leq b(u) < \infty$. On the other hand, $b(u_n) \leq d(S, u_n) \rightarrow \infty$. This is a contradiction. Thus, η is timelike.

Let $\zeta : [0, w_n] \rightarrow M$ be a limit maximizing sequence of curves from S to v_n . Let $\zeta_{\eta_n}(0) = z_n \in S$ and $z_n \rightarrow z \in S$ by compactness of S . We have $v_n \rightarrow \infty$ otherwise the strong causality violates. Then, by lemma 2, $w_n \rightarrow \infty$ and we get S -ray $\zeta : [0, \infty) \rightarrow M$ as a limit curve of ζ_n . By compactness of S , ζ is timelike. Further, ζ has infinite length by completeness. Thus, for any $w > 0$ and for large enough n ,

$$d(S, v_n) \geq L(\zeta_n|[0, w_n]) \geq L(\zeta_n|[0, w]) \rightarrow L(\zeta|[0, w]) \quad (17)$$

(See proposition 14.7 of [BEE]). Since $L(\zeta|[0, w]) \rightarrow \infty$ as $w \rightarrow \infty$, we get $d(S, v_n) \rightarrow \infty$. Therefore any co-ray of S -ray γ is timelike. $\square$

Remark. The last two lemmas are a special case of lemma 2.4 and 5 of [EG].

3 Theorems

Lemma 5 *A globally hyperbolic spacetime (M, g) contains a compact achronal hypersurface without edge (i.e. a compact Cauchy surface) if the spacetime (M, g) contains a future (or past) trapped set T .*

Proof: $E^+(T)$ is a compact since T is a trapped set. Also $E^+(T) = I^+(T)$ because of the spacetime is globally hyperbolic,

i.e. necessarily causally simple. Furthermore $I^+(T)$ is edgeless since it is the full boundary of the future of T and, by proposition 6.3.1 of [HE], $I^+(T)$ is a closed, imbedded, achronal three-dimensional C^{1-} sub manifold. Thus, $E^+(T)$ is a compact achronal hypersurface without edge. Let S be a Cauchy surface in $I^-(E^+(T))$. (M, g) admits a past directed C^1 timelike vector field. Each integral curve of the field intersects S and $E^+(T)$ at most once. Therefore, they define a continuous one-to-one map $\phi : E^+(T) \rightarrow S$. Since $E^+(T)$ is compact, its image $\phi(E^+(T))$ also is compact and is homeomorphic to $E^+(T)$. Thus, $\phi(E^+(T))$ is a three-dimensional manifold without boundary. If S is non-compact, $\phi(E^+(T))$ must have a boundary in S . This is contradiction. Then S is a compact Cauchy surface. $\square$

We recall that a timelike line is an inextendible maximal timelike geodesic.

Lemma 6 *Let (M, g) be a spacetime which is timelike geodesically complete and contains a compact Cauchy surface S . Then (M, g) contains a timelike line.*

Proof: By lemma 4 every compact Cauchy surface S admits a future timelike S -ray $\gamma : [0, \infty) \rightarrow M$ and a past timelike S -ray $\beta : (-\infty, 0] \rightarrow M$. Let $r_n, r'_n \rightarrow \infty$ be monotone sequence. γ is a limit curve of the sequence $\{\gamma_n\}$ which are maximal curves with past endpoint $\gamma_n(0) \in S$ and with future endpoint $v_n = \mu(r_n)$, where μ is a future inextendible timelike geodesic $\mu : [0, \infty) \rightarrow M$ with $\mu(0) = r \in S$. Similarly, β is a limit curve of the sequence $\{\beta_n\}$ which are maximal curves with future endpoint $\beta_n(0) \in S$ and with past endpoint $v'_n = \nu(r'_n)$, where ν is a past inextendible timelike geodesic $\nu : (-\infty, 0] \rightarrow M$ with $\nu(0) = r \in S$ by construction(cf. lemma 2).

We shall show $I^-(\mu) \subseteq I^-(\gamma)$. By construction, since μ and γ are relative causally, $I^-(\mu) \subset I^-(\gamma)$, $I^-(\mu) \supset I^-(\gamma)$ or $I^-(\mu) = I^-(\gamma)$. If $I^-(\mu) \supset I^-(\gamma)$, there exist points $\{v_n\}$ on μ in the future of γ such that $n > n_0$ where n_0 is some integer. But this is a contradiction because of γ is a limit curve of γ_n at $v_n \rightarrow \infty$ as $n \rightarrow \infty$ (cf. lemma 1). Thus, $I^-(\mu) \subseteq I^-(\gamma)$. Similarly, $I^+(\nu) \subseteq I^+(\beta)$.

Since $r \in I^-(\mu)$ and $r \in I^+(\nu)$, from the above discussion, $r \in I^-(\gamma)$ and $r \in I^+(\beta)$. Therefore, $I^-(\gamma) \cap I^+(\beta) \neq \emptyset$.

t

Let $t_n, s_n \rightarrow \infty$ be monotone sequence and set $q_n = \gamma(t_n)$ and $p_n = \beta(-s_n)$. Let $\lambda_n : [a_n, b_n] \rightarrow M$ be a limit maximizing causal curve from p_n to q_n . We can construct such a sequence since $I^-(\gamma) \cap I^+(\beta) \neq \emptyset$ for sufficiently large n . λ_n must intersect with S at z_n since $p_n \in D^-(S)$ and $q_n \in D^+(S)$. We can choose the parameter such that $z_n = \lambda_n(0)$. Since S is compact, $z_n \rightarrow z \in S$. Thus, there exists a limit curve λ of the sequence $\{\lambda_n\}$ through S (cf. lemma 1).

By lemma 4, $\lambda^+ \equiv \lambda : [0, \infty)$ is a timelike ray since λ^+ is a co-ray of γ . Furthermore, there exists $0 < \varepsilon$ such that $\lambda : [-\varepsilon, \infty)$ is maximizing. Therefore $\lambda : [-\varepsilon, \infty)$ is also a timelike ray.

Let $\lambda^- \equiv \lambda : (-\infty, 0] \rightarrow M$ be a co-ray of β . Since $\lambda_n : [-\varepsilon, 0] \rightarrow \lambda : [-\varepsilon, 0]$ which is a timelike geodesic, $\lambda_n(s) \in I^-(z)$ for sufficiently large n . For appropriate $s \in [-\varepsilon, 0]$, $z \in I^+(\beta(s_n)) \subset I^+(\beta)$. Furthermore, λ^- is a co-ray of β and $a_n \rightarrow -\infty$. Thus, λ is a line and timelike since λ^+ is timelike. $\square$

The next assertion is our main result which is the rigid singularity theorem in globally hyperbolic spacetimes.

Theorem 1 *Spacetime (M, g) is timelike geodesically incomplete or splits, i.e. (M, g) is isometric to $(\mathbf{R} \times S, -dt^2 \oplus h)$, where (S, h) is a complete Riemannian manifold if the following conditions hold:*

1. $R_{ab}K^aK^b \geq 0$ for all timelike vector K (strong energy condition),
4. there exists a future (or past) trapped set $T \subset M$.
5. (M, g) is globally hyperbolic,

Proof: Suppose that the spacetime is geodesically complete. By lemma 5, there exists a compact Cauchy surface S in the past of $E^+(T)$ (or in the future of $E^-(T)$). Therefore,

from lemma 6 there exists a line. Since the condition 1 hold, by Lorentzian splitting theorem [E][G89][N][GHo], the spacetime splits as an isometric product $(\mathbf{R} \times S, -dt^2 \oplus h)$. Thus, the proof is complete. $\square$

Finally, we comment on an application for the above result. To do this we need the following proposition which is the contradiction of corollary 7 of Garfinkle and Harris[GHa].

Proposition 1 *Let (M, g) be a static, globally hyperbolic, timelike or null geodesically complete spacetime containing a compact Cauchy surface. Then $R_{ab}K^aK^b \leq 0$ for all timelike vectors K^a .* $\square$

From theorem 2 and proposition 1, we get the following theorem.

Theorem 2 *Let (M, g) be a static, globally hyperbolic, timelike or null geodesically complete spacetime containing a compact Cauchy surface. Then $R_{ab}K^aK^b \equiv 0$ for all timelike vectors K^a .* $\square$

Remark. Geroch's result is not " $R_{ab}K^aK^b \equiv 0$ " but " $R_{ab} \equiv 0$ ", i.e. Ricci flat if spacetimes are geodesically complete[Ge]. The energy condition of Geroch's theorem is that $R_{ab}K^aK^b \equiv 0$, equality holding only if $R_{ab} = 0$. This is the stronger condition. From this fact, Einstein static universe is excluded. However, the spacetime satisfying the conditions of theorem 3 includes Einstein static universe. Indeed, Einstein static universe satisfies the conditions of theorem 3 but this universe is not Ricci flat.

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Gravitational Wave by a Spinning Particle Plunging into a Kerr Black Hole

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Abstract

We analyze a spinning particle plunging into a Kerr black hole, and study a spin effect on emitted gravitational radiation using a black hole perturbation approach. We consider a spinning test particle moving on an equatorial plane and calculate the total emitted energy of gravitational radiation. We find that it largely depends on the spin-spin coupling terms and it becomes larger as the rotation parameter a and the particle spin s gets larger. That is, when the particle spin is parallel to the black hole spin, the emitted energy is enhanced, while for the anti-parallel case it is suppressed.

1 Introduction

Ever since the discovery of the binary pulsar PSR1916+13 by Hulse and Taylor [1], we believe in the existence of gravitational waves emitted from compact objects. Their continual observations over 20 years show that both the observed rate of change of the orbital period and that predicted by general relativity agree to within 0.3% accuracy [2].

Currently, a number of worldwide projects for detecting gravitational radiation directly using a km-scale laser interferometer such as LIGO, VIRGO, GEO600, and TAMA are progressing. In particular, the LIGO project [3] will begin operating within a few years. Among the many possible sources of gravitational waves, one of the most plausible sources is a coalescing binary, composed of neutron star-neutron star (NS-NS), neutron star-black hole (NS-BH), or black hole-black hole (BH-BH). These direct detections of gravitational waves will enable us to see strong gravitational phenomena such as coalescence of compact stars or black holes. In order to extract signals from noise, we need to prepare in advance a list of expected templates of gravitational waves, which depend on many parameters of sources (masses and spins of compact objects, orbital angular momentum, eccentricity of their orbits, ...). Therefore one of the current most important subjects in general relativity is to predict a precise waveform of gravitational waves for a set of given parameters.

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The black hole perturbation approach is one of the ways to study such a waveform, which is based on the linearized Einstein equation. For the Kerr black hole, Teukolsky [4] derived the basic equations. The advantage of this method is that it enables us to treat full relativistic effects and it may apply to a generic motion of a particle. In fact, there are many papers which have studied gravitational radiation by a test particle around a black hole using this approach. Nakamura and Sasaki [5] discuss a particle falling along the rotation axis of a Kerr black hole, while Kojima and Nakamura [6] studied a particle on the equatorial plane. Shibata calculated gravitational radiation from a circular orbit [7], an eccentric orbit [8] on the equatorial plane, and a precessional orbit [9] of a test particle. If we will extrapolate these results to a realistic close binary system, we should also consider the spin of a test particle, which may correspond to spin for a companion star or a black hole.

There are, so far, two papers which treat a spinning test particle using a black hole perturbation approach. Mino, Shibata and Tanaka (MST) [10] calculated gravitational waveforms and total emitted energy from a spinning particle moving along the rotation axis of a Kerr black hole. Tanaka, Mino, Sasaki and Shibata [11] calculate the total emitted energy of gravitational radiation from a spinning particle with a circular orbit on the equatorial plane of a Kerr black hole using the post-Newtonian expansion method.

In this paper, we extend these two studies to a spinning particle moving on the equatorial plane. Just for simplicity, we assume an additional condition, i.e., $u^{(3)} = 0$, which means that the angular momentum observed by a ZAMO (zero angular momentum local observer) vanishes. This condition is also adopted by MST. Then, one of the main purposes of our paper is to discuss how the orbit of a spinning particle affects the gravitational radiation, comparing with the results of MST.

This paper is organized as follows. In §2, we write down our basic equation for a spinning particle. We present our black hole perturbation approach in §3. We then analyze the total emitted energy and total angular momentum of gravitational radiation from a spinning particle plunging into a Kerr black hole in §4.

Throughout this paper, we use the units of $c = G = 1$ and the metric signature $(-, +, +, +)$. The round and square brackets on the indices denote symmetrization and anti-symmetrization, respectively.

2 Spinning Particle

In this section, we start by considering the motion of the spinning particle in a Kerr background spacetime. Its equation of motion was first derived by Papapetrou [12]. It was then reformulated by Dixon [13], in which he discussed a generic motion of extended bodies, which means that in a point-particle limit a particle has arbitrary higher multipole moments. Here we follow Dixon's approach neglecting higher multipole moments other than a mass monopole and a spin dipole moment. Then, the equations of motion are given as

$$\frac{D}{d\tau}p^\mu = -\frac{1}{2}R^\mu_{\nu\rho\sigma}(z)v^\nu S^{\rho\sigma},$$

$$\frac{D}{d\tau} S^{\mu\nu} = 2p^{[\mu}v^{\nu]}, \quad (1)$$

where the vector p^μ and the antisymmetric tensor $S^{\mu\nu}$ represent the linear momentum and spin angular momentum of the particle, respectively, and $v^\mu = dz^\mu/d\tau$ is the 4-velocity. The quantity $\mu^2 = -p_\mu p^\mu$ is conserved and then μ is regarded as the mass of a test particle. We also include a normalized momentum u^μ ($u^\mu u_\mu = -1$) as $p^\mu = \mu u^\mu$. $D/d\tau$ denotes the covariant derivative along the particle trajectory (v^μ).

In order to determine a relation between v^μ and u^μ (or p^μ), we need to impose a supplementary condition, which fixes the center of mass of a spinning particle as

$$S^{\mu\nu} p_\nu = 0. \quad (2)$$

We then set

$$S^{\mu\nu} S_{\mu\nu} = 2\mu^2 s^2. \quad (3)$$

Noting that the above equations of motion are reparametrization invariant for a free parameter τ of a trajectory, we can fix an orbital parameter τ to satisfy

$$u^\mu(\tau)v_\mu(\tau) = -1. \quad (4)$$

Using Eqs. (1), (2) and (4) the relation between u^μ and v^μ becomes

$$v^\mu - u^\mu = \frac{S^{\mu\nu} R_{\nu\gamma\sigma\lambda} u^\gamma S^{\sigma\lambda}}{2[\mu^2 + \frac{1}{4} R_{\alpha\beta\delta\epsilon}(z) S^{\alpha\beta} S^{\delta\epsilon}]}. \quad (5)$$

The energy-momentum tensor of a spinning particle $T^{\mu\nu}$ is given [10] as

$$T^{\mu\nu}(x) = \frac{\mu}{\sqrt{-g}} \int d\tau \left\{ u^{(\mu}(x, \tau) v^{\nu)}(x, \tau) \delta^{(4)}(x - z(\tau)) - \nabla_\lambda \left[S^{\lambda(\mu}(x, \tau) v^{\nu)}(x, \tau) \delta^{(4)}(x - z(\tau)) \right] \right\}, \quad (6)$$

where the definition of the bitensors $v^\mu(x, \tau)$, $u^\mu(x, \tau)$, $S^{\mu\nu}(x, \tau)$ are given in [10].

Here just for simplicity, we restrict our present analysis with two conditions for a motion of a spinning particle. First, a spinning particle moves only on the equatorial plane of Kerr black hole. Secondly, we choose $u^{(3)} = 0$, where $u^{(a)}$ denotes the tetrad components of a normalized momentum observed by ZAMO (zero angular momentum local observer).

These conditions lead to $J_z = (a + s)E$, where E and J_z represents the energy and the z -component of total angular momentum of a spinning particle, respectively. Then the equation of motion of the spinning particle becomes

$$\begin{aligned} \frac{dt}{d\tau} &= \frac{r^2 + a^2}{\Delta}, \\ \frac{dr}{d\tau} &= -\frac{\sqrt{r^2 - \Delta}}{r}, \\ \frac{d\varphi}{d\tau} &= \frac{a}{\Delta}, \end{aligned}$$

with $\theta = \frac{\pi}{2}$, where a is the rotation parameter of a Kerr black hole and $\Delta = r^2 - 2Mr + a^2$. From these equations, we can easily see that the motion of the spinning particle never depends on its spin.

3 Generalized Regge-Wheeler Equation

Now, we consider a black hole perturbation approach of a Kerr background spacetime by a spinning particle and study the gravitational waves in such a system.

In order to calculate metric perturbations of a Kerr spacetime, we adopt the Sasaki-Nakamura formalism [14], in which the Regge-Wheeler equation is generalized by a transformation from the Teukolsky equation [4]. The radial wave equation becomes

$$\left[\frac{d^2}{dr^{*2}} - F(r) \frac{d}{dr^*} - U(r) \right] X_{lm\omega}(r) = S_{lm\omega}(r), \quad (7)$$

where the tortoise coordinate r^* is defined as

$$dr^* = \frac{r^2 + a^2}{\Delta} dr. \quad (8)$$

The potential functions $F(r)$ and $U(r)$ are given as

$$\begin{aligned} F(r) &= \frac{\Delta}{r^2 + a^2} \frac{1}{\gamma} \frac{d\gamma}{dr}, \\ U(r) &= G^2 - FG + \frac{\Delta}{(r^2 + a^2)} \frac{dG}{dr} + \frac{\Delta}{(r^2 + a^2)^2} U_1, \end{aligned}$$

with

$$\begin{aligned} \gamma(r) &= c_0 + \frac{c_1}{r} + \frac{c_2}{r^2} + \frac{c_3}{r^3} + \frac{c_4}{r^4}, \\ c_0 &= -12iM\omega + \lambda(\lambda + 2) - 12a\omega(a\omega - m), \\ c_1 &= 8ia[3a\omega - \lambda(a\omega - m)], \\ c_2 &= 24iaM(a\omega - m) + 12a^2[1 - 2(a\omega - m)^2], \\ c_3 &= 24ia^3(a\omega - m) - 24Ma^2, \\ c_4 &= 12a^4, \\ G(r) &= -\frac{1}{r^2 + a^2} \frac{d\Delta}{dr} + \frac{r\Delta}{(r^2 + a^2)^2}, \\ U_1(r) &= V_T + \frac{\Delta^2}{\beta} \left[\frac{d}{dr} \left(2\alpha + \frac{d\beta/dr}{\Delta} \right) - \frac{d\gamma/dr}{\gamma} \left(\alpha + \frac{d\beta/dr}{\Delta} \right) \right], \\ \alpha &= -i \frac{K\beta}{\Delta^2} + 3i \frac{dK}{dr} + \lambda + \frac{6\Delta}{r^2}, \\ \beta &= 2\Delta \left(-iK + r - M - \frac{2\Delta}{r} \right). \end{aligned}$$

V_T is the potential in the Teukolsky equation, which is given as

$$V_T(r) = -\frac{K^2 + 4i(r - M)K}{\Delta} + 8i\omega r + \lambda, \quad (9)$$

where $K = (r^2 + a^2)\omega - ma$ and where ${}_sS_{lm}^{\omega}(\theta)$ is a spin-weighted spheroidal function, which obeys

$$\left[\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d}{d\theta} \right) - \left(a^2 \omega^2 \sin^2 \theta + \frac{(m + s \cos \theta)^2}{\sin^2 \theta} + 2a\omega s \cos \theta - s - 2am\omega - \lambda \right) \right] {}_sS_{lm}^{\omega}(\theta) = 0, \quad (10)$$

with $s = -2, -1, 0, 1$ or 2 and λ being the eigenvalue. ${}_sS_{lm}^{\omega}(\theta)$ is normalized as

$$\int_0^\pi d\theta \sin \theta |{}_sS_{lm}^{\omega}(\theta)|^2 = 1.$$

Here we shall present the explicit form of the source term $S_{lm\omega}(r)$ for a spinning particle with the condition $u^{(3)} = 0$. The source term $S_{lm\omega}$ is given by

$$S_{lm\omega} = \frac{\gamma \Delta}{(r^2 + a^2)^{3/2} r^2} \mathcal{W} \exp(-i \int_r^r \frac{K}{\Delta} dr). \quad (11)$$

Here $\mathcal{W}$ is divided into three parts as

$$\mathcal{W} = \mathcal{W}_{nn} + \mathcal{W}_{\bar{m}n} + \mathcal{W}_{\bar{m}\bar{m}}, \quad (12)$$

$$\mathcal{W}_{nn} = f_0(r) e^{i\chi(r)} + \int_r^\infty dr' f_1(r') e^{i\chi(r')} + \int_r^\infty dr' \int_{r'}^\infty dr'' f_2(r'') e^{i\chi(r'')}, \quad (13)$$

$$\mathcal{W}_{\bar{m}n} = g_0(r) e^{i\chi(r)} + \int_r^\infty dr' g_1(r') e^{i\chi(r')}, \quad (14)$$

$$\mathcal{W}_{\bar{m}\bar{m}} = h_0(r) e^{i\chi(r)} + \int_r^\infty dr' h_1(r') e^{i\chi(r')} + \int_r^\infty dr' \int_{r'}^\infty dr'' h_2(r'') e^{i\chi(r'')}, \quad (15)$$

where

$$\chi = \omega(t + r^*) - m\tilde{\varphi}, \quad (16)$$

$$f_0(r) = -\frac{1}{\omega^2} w_{nn}, \quad (17)$$

$$f_1(r) = -\frac{2}{\omega^2} \left(\frac{dw_{nn}}{dr} + i(2a\omega - m) \frac{d\tilde{\varphi}}{dr} w_{nn} \right), \quad (18)$$

$$f_2(r) = -\frac{1}{\omega^2} \left[\left(\frac{d^2 w_{nn}}{dr^2} - im \frac{d^2 \tilde{\varphi}}{dr^2} w_{nn} - 2im \frac{d\tilde{\varphi}}{dr} \frac{dw_{nn}}{dr} - m^2 \left(\frac{d\tilde{\varphi}}{dr} \right)^2 w_{nn} \right) + 2ia\omega \left(\frac{dw_{nn}}{dr} \frac{d\tilde{\varphi}}{dr} + w_{nn} \frac{d^2 \tilde{\varphi}}{dr^2} - im \left(\frac{d\tilde{\varphi}}{dr} \right)^2 w_{nn} \right) - a^2 \omega^2 \left(\frac{d\tilde{\varphi}}{dr} \right)^2 w_{nn} \right], \quad (19)$$

$$g_0(r) = w_{\bar{m}n}^{(1)} - \frac{i}{\omega} w_{\bar{m}n}^{(2)}, \quad (20)$$

$$g_1(r) = -\frac{i}{\omega} \frac{dw_{\bar{m}n}^{(2)}}{dr} + (a\omega - m) \frac{w_{\bar{m}n}^{(2)}}{\omega} \frac{d\tilde{\varphi}}{dr}, \quad (21)$$

$$h_0(r) = w_{\bar{m}\bar{m}}^{(1)} - \frac{i}{\omega} w_{\bar{m}\bar{m}}^{(2)} - \frac{1}{\omega^2} w_{\bar{m}\bar{m}}^{(3)}, \quad (22)$$

$$h_1(r) = -\frac{2}{\omega^2} \left(\frac{dw_{\tilde{m}\tilde{m}}^{(3)}}{dr} + i(2a\omega - m) \frac{d\tilde{\varphi}}{dr} w_{\tilde{m}\tilde{m}}^{(3)} \right) + \frac{i}{\omega} w_{\tilde{m}\tilde{m}}^{(4)} - \frac{i}{\omega} \frac{dw_{\tilde{m}\tilde{m}}^{(4)}}{dr} + (a\omega - m) \frac{w_{\tilde{m}\tilde{m}}^{(4)}}{\omega} \frac{d\tilde{\varphi}}{dr}, \quad (23)$$

$$h_2(r) = -\frac{1}{\omega^2} \left[\left(\frac{d^2 w_{\tilde{m}\tilde{m}}^{(3)}}{dr^2} - im \frac{d^2 \tilde{\varphi}}{dr^2} w_{\tilde{m}\tilde{m}}^{(3)} - 2im \frac{d\tilde{\varphi}}{dr} \frac{dw_{\tilde{m}\tilde{m}}^{(3)}}{dr} - m^2 \left(\frac{d\tilde{\varphi}}{dr} \right)^2 w_{\tilde{m}\tilde{m}}^{(3)} \right) + 2ia\omega \left(\frac{dw_{\tilde{m}\tilde{m}}^{(3)}}{dr} \frac{d\tilde{\varphi}}{dr} + w_{\tilde{m}\tilde{m}}^{(3)} \frac{d^2 \tilde{\varphi}}{dr^2} - im \left(\frac{d\tilde{\varphi}}{dr} \right)^2 w_{\tilde{m}\tilde{m}}^{(3)} \right) - a^2 \omega^2 \left(\frac{d\tilde{\varphi}}{dr} \right)^2 w_{\tilde{m}\tilde{m}}^{(3)} \right] + \frac{i}{\omega} \frac{dw_{\tilde{m}\tilde{m}}^{(4)}}{dr} - \frac{1}{\omega} (a\omega - m) w_{\tilde{m}\tilde{m}}^{(4)} \frac{d\tilde{\varphi}}{dr}, \quad (24)$$

with

$$\tilde{\varphi} = \varphi + \int \frac{a}{\Delta} dr, \quad (25)$$

$$\hat{S} = \left(a\omega - m - i\frac{s}{r} \right) \left(\frac{d_{-2} S_{lm}^{\omega\omega}}{d\theta} \Big|_{\theta=\pi/2} + (a\omega - m)_{-2} S_{lm}^{\omega\omega} \left(\frac{\pi}{2} \right) \right) - \frac{\lambda}{2} {}_{-2} S_{lm}^{\omega\omega} \left(\frac{\pi}{2} \right), \quad (26)$$

$$w_{nn} = -\frac{dr}{d\tau} \hat{S} \left[1 - \frac{as}{r^2} - i\frac{s}{r} (a\omega - m) \right], \quad (27)$$

$$w_{\tilde{m}\tilde{m}}^{(1)} = i\frac{s}{2} \left| \frac{dr}{d\tau} \right|^{-1} \frac{r^3}{\Delta} \left(1 + \frac{dr}{d\tau} \right) \left(\frac{d_{-2} S_{lm}^{\omega\omega}}{d\theta} \Big|_{\theta=\pi/2} + (a\omega - m)_{-2} S_{lm}^{\omega\omega} \left(\frac{\pi}{2} \right) \right), \quad (28)$$

$$w_{\tilde{m}\tilde{m}}^{(2)} = i\frac{s}{2} \left[1 + \left(2 - \frac{1}{2\Delta} \frac{d\Delta}{d\tau} + ir \frac{K}{\Delta} \right) \left(1 + \frac{dr}{d\tau} \right) \right] \left(\frac{d_{-2} S_{lm}^{\omega\omega}}{d\theta} \Big|_{\theta=\pi/2} + (a\omega - m)_{-2} S_{lm}^{\omega\omega} \left(\frac{\pi}{2} \right) \right), \quad (29)$$

$$w_{\tilde{m}\tilde{m}}^{(1)} = \frac{1}{2} \left(\frac{dr}{d\tau} \right)^{-1} a s {}_{-2} S_{lm}^{\omega\omega} \left(\frac{\pi}{2} \right), \quad (30)$$

$$w_{\tilde{m}\tilde{m}}^{(2)} = \frac{as}{r} \left(1 - \frac{dr}{d\tau} \right) {}_{-2} S_{lm}^{\omega\omega} \left(\frac{\pi}{2} \right), \quad (31)$$

$$w_{\tilde{m}\tilde{m}}^{(3)} = -\frac{dr}{d\tau} \frac{as}{r^4} \Delta {}_{-2} S_{lm}^{\omega\omega} \left(\frac{\pi}{2} \right), \quad (32)$$

$$w_{\tilde{m}\tilde{m}}^{(4)} = 2 \frac{as}{r^2} \frac{dr}{d\tau} {}_{-2} S_{lm}^{\omega\omega} \left(\frac{\pi}{2} \right), \quad (33)$$

where an overbar denotes the complex conjugate.

To solve the wave function $X_{lm\omega}(r)$ in Eq. (7) using the Green function method, we need two independent homogeneous solutions of the Sasaki-Nakamura equation, i.e.,

$$X_{lm\omega}^{\text{in}(0)}(r) = \begin{cases} e^{-ikr^*} \\ A_{lm\omega}^{\text{in}} e^{-i\omega r^*} + A_{lm\omega}^{\text{out}} e^{i\omega r^*} \end{cases} \quad r^* \rightarrow -\infty, \\ X_{lm\omega}^{\text{out}(0)}(r) = \begin{cases} B_{lm\omega}^{\text{in}} e^{-ikr^*} + B_{lm\omega}^{\text{out}} e^{ikr^*} \\ e^{i\omega r^*} \end{cases} \quad \begin{matrix} r^* \rightarrow \infty \\ r^* \rightarrow -\infty \end{matrix},$$

where $k = \omega - ma/2r_+$ with $r_+ = M + \sqrt{M^2 - a^2}$ being the radius of the event horizon, and the Wronskian W

$$W \equiv X_{lm\omega}^{in(0)} \frac{dX_{lm\omega}^{out(0)}}{dr^*} - X_{lm\omega}^{out(0)} \frac{dX_{lm\omega}^{in(0)}}{dr^*} = 2i\omega A_{lm\omega}^{in} \frac{\gamma}{c_0}.$$

Then the inhomogeneous solution of Eq. (7) is given as

$$X_{lm\omega}(r) = X_{lm\omega}^{in(0)} \int_{r^*}^{\infty} \frac{S_{lm\omega}(r)}{W} X_{lm\omega}^{out(0)} dr^* + X_{lm\omega}^{out(0)} \int_{-\infty}^{r^*} \frac{S_{lm\omega}(r)}{W} X_{lm\omega}^{in(0)} dr^*.$$

For observation, we only have to know the asymptotic behavior of the wave function $X_{lm\omega}(r)$ at infinity, which is

$$X_{lm\omega}^{(\infty)}(r) = A_{lm\omega}^{(\infty)} e^{i\omega r^*}, \quad (34)$$

where

$$A_{lm\omega}^{(\infty)} = \frac{c_0}{2i\omega A_{lm\omega}^{in}} \int_{-\infty}^{\infty} \frac{S_{lm\omega}(r)}{\gamma} X_{lm\omega}^{in(0)}(r) dr^*. \quad (35)$$

Then the waveform at infinity is

$$h_+ - ih_\times = \frac{8}{r} \int_{-\infty}^{\infty} d\omega e^{i\omega(r^* - t)} \sum_{l,m} \frac{A_{lm\omega}^{(\infty)}}{c_0} {}_{-2}S_{lm}^{\alpha\omega}(\theta) \frac{e^{im\varphi}}{\sqrt{2\pi}}. \quad (36)$$

From Eq. (36), the energy and total angular momentum fluxes of gravitational waves at infinity are obtained by

$$\left(\frac{dE}{dt} \right)_{GW} = \frac{4}{\pi} \sum_{l,m} \left| \int_{-\infty}^{\infty} d\omega \omega \frac{A_{lm\omega}^{(\infty)}}{c_0} e^{i\omega(r^* - t)} \right|^2, \quad (37)$$

$$\left(\frac{dJ_z}{dt} \right)_{GW} = \frac{4}{\pi} \sum_{l,m} \int_{-\infty}^{\infty} d\omega \omega \frac{A_{lm\omega}^{(\infty)}}{c_0} e^{i\omega(r^* - t)} \int_{-\infty}^{\infty} d\omega' m \frac{\bar{A}_{lm\omega'}^{(\infty)}}{\bar{c}_0} e^{-i\omega'(r^* - t)}. \quad (38)$$

Finally, we find the total energy, total angular momentum, energy spectrum, and angular momentum spectrum of the emitted gravitational waves are given as

$$\Delta E = \int_{-\infty}^{\infty} d\omega \sum_{l,m} \left(\frac{dE}{d\omega} \right)_{lm\omega}, \quad \Delta J_z = \int_{-\infty}^{\infty} d\omega \sum_{l,m} \left(\frac{dJ_z}{d\omega} \right)_{lm\omega}, \quad (39)$$

and

$$\left(\frac{dE}{d\omega} \right)_{lm\omega} = 8\omega^2 \left| \frac{A_{lm\omega}^{(\infty)}}{c_0} \right|^2, \quad \left(\frac{dJ_z}{d\omega} \right)_{lm\omega} = 8m\omega \left| \frac{A_{lm\omega}^{(\infty)}}{c_0} \right|^2, \quad (40)$$

respectively.

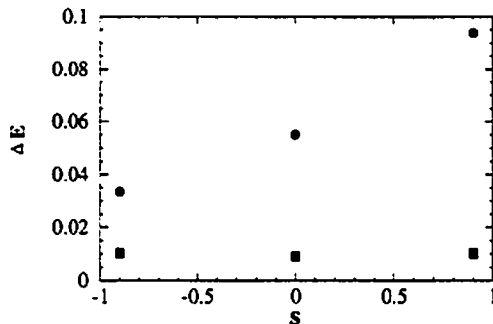


Fig. 1: The total emitted energy of gravitational radiation from a spinning particle plunging into a Kerr black hole.

We show only the $l = 2$ mode. The circle and the square correspond to the cases of $a = 0.9$ and $a = 0$, respectively. The emitted energy of gravitational radiation gets larger as the spin of a test particle is larger, if its direction is parallel to the black hole rotation.

4 Results

We show our numerical results in Figs. 1 and 2. Although the condition $u^{(3)} = 0$ may be very limited, we find that the emitted energy of gravitational radiation gets larger as the spin of a test particle is larger, if its direction is parallel to the black hole rotation. On the other hand, if the spin direction is anti-parallel the spin effect suppress emission of gravitational waves. This result is exactly opposite to that of MST (spinning particle moves along the rotation axis). Therefore, the total energy from gravitational radiation largely depends on the choice of orbital plane. From Fig. 2, the total emitted angular momentum of gravitational radiation shows a similar dependence on the spin to the total energy, but when the spin-spin coupling term appears, it becomes larger. This is because the condition $u^{(3)} = 0$ implies that the "orbital angular momentum" is equal to a Kerr rotation parameter a and then a spin-orbit coupling effect may enhance the emitted energy and the angular momentum of gravitational waves.

In order to give a more concrete picture for the effect of the spin of a test particle, we are planning to study the gravitational radiation for more generic cases.

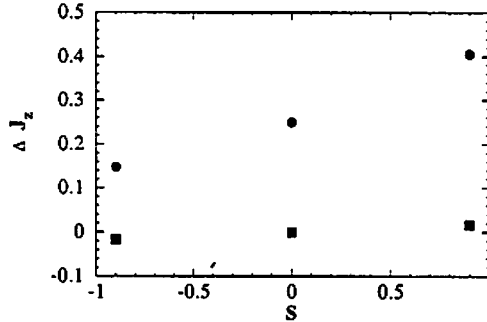


Fig. 2: The total angular momentum of gravitational radiation from a spinning particle plunging into a Kerr black hole. We show only the $l = 2$ mode. The marks denote the same as those in Fig. 1. The total emitted angular momentum of gravitational radiation shows a similar dependence on the spin to the total energy, but when the spin-spin coupling term appears, it becomes larger.

Acknowledgments

We would like to thank Y. Mino, T. Nakamura, M. Sasaki, S. Suzuki and T. Tanaka for many useful discussions. we also would like to thank P. Haines for critical reading of our manuscript.

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Gravitational Radiation Reaction to a Particle Motion II

- Spinning Particle -

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Abstract

A particle traveling in a curved spacetime is known to trace a geodesic of the background spacetime if the mass of the particle is negligibly small. In this paper, we discuss the leading order correction to the equation of motion of a particle with spin. Previous work has discovered the leading correction due to coupling of the spin of the particle and the Riemann tensor of the surrounding spacetime. Recently we found the equation of motion of a non-spinning particle which presumably describes the effect of gravitational radiation reaction, but its derivation seems difficult to apply to a spinning particle, such as a Kerr black hole. This paper is devoted to discussion of the consistent derivation of the equation of motion which is corrected both by the spin-Riemann coupling and the gravitational radiation reaction. The result of this paper implies that it is possible to treat a spinning particle in the linear perturbation theory, which has been conjectured in previous work. It is thought useful in constructing the theoretical template of the gravitational waves, say, coming from rotating black hole binary systems.

1 Introduction

The gravitational waves from an inspiralling binary is one of the most promising sources expected to be detected by the near-future interferometric gravitational wave detectors such as LIGO, VIRGO and LISA[1, 2]. In order to extract the information of the binaries from the last inspiralling stage, we need accurate theoretical templates of the gravitational wave forms[3]. As an approach, perturbations of a black hole by an orbiting point particle have been studied[4].

Most of the previous papers considered the case that the orbiting point particle is a structureless one and is characterized only by a mass parameter. In those papers, the point particle is defined by using a Dirac's delta function, therefore the induced perturbation diverges at the particle. This leads to some conceptual problems in the perturbation study; 1) how we can prescribe the strong non-linearity of the perturbation around the

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particle, and 2) how we define the motion of the particle though the metric diverges at the particle.

Our previous paper[5] considered these problems, extensively using the technique of the matched asymptotic expansion that has been already developed by many authors (e.g., D'Eath [6] and Thorne and Hartle [7]). The matched asymptotic expansion is a technique to match physical quantities in different perturbation schemes, which we think useful in order to take into account the non-linearity of the perturbation around the particle. We first construct the metric in two different regions. The metric around the point particle (body's neighborhood[7]) is assumed to be approximated by a Schwarzschild black hole and its perturbation due to the tidal distortion given by the background curvature (internal scheme[5] or body expansion[7]). Here the metric is expanded in powers of r/L , where r is a circumference radius measured in the local inertial frame of the background and L is the characteristic curvature scale of the background spacetime. The metric far from the point particle (external universe[7]) is approximated by the linear perturbations on the background spacetime generated by a point source (external scheme[5] or external-universe expansion[7]). The metric in this region is expanded in powers of Gm/r , where m is the mass scale of the particle. The particle is presumed to be compact and isolated in the sense that Gm is far smaller than L , then the metrics constructed in different expansion schemes are matched in the overlapping region (buffer region[7]) where both expansion schemes are valid, taking the gauge degree of freedom into account. In practice, the metric is expanded in powers of both r/L and Gm/r .

In [5], the equation of motion was derived from the condition that the asymptotic matching of the metric can be consistently done. In the internal scheme, the background spacetime is a Schwarzschild black hole, and has spherical symmetry. We decompose the metric perturbation in the internal scheme by spherical tensor harmonics. The important step is to require that the gauge mode of translation should vanish in the internal scheme, which corresponds to fixing the center of mass of the particle. Equating the each component of the metrics derived in the internal region and in the external region at the overlapping region, we obtain the information how the center of mass moves in the background spacetime in the external scheme and we obtain the equation of motion of the particle with the effect of gravitational radiation reaction.

Showing that the asymptotic matching can be consistently done with the external metric generated by the point particle, we clarify that the point particle used in computing the gravitational radiation reaction to the orbit so far[4], approximates the Schwarzschild black hole.

Since we use the metric at the overlapping region, we do not encounter the divergence of the particle and the equation of motion is understood as a well-defined manner, that means, we derive the equation of motion of the local asymptotic frame of the particle with respect to the background spacetime.

On the other hand, the post-Newtonian calculation indicates the spin of the particle could modify the gravitational wave forms of the realistic binary system[8], which motivates the problem how we can consider a spinning particle in the perturbation study. Recently we discuss the perturbation due to a spinning point particle[9] and derive the post-Newtonian expansion of the gravitational radiation reaction to the energy and the angular momentum of the orbit. Though these results are the same as those derived

by the standard post-Newtonian calculation[8], we do not know what the spinning point particle in [9] represents.

The effect of the spin to the equation of motion was discussed by many authors in various ways. Beginning from the work of Papapetrou[10], Dixon[14] have found the covariant description of the energy momentum tensor of an extended body and the equation of motion of it. The resulted equation of motion indicates that the coupling of the spin and the Riemann curvature affect the orbit of the body. However the derivation in [14] does not take into account the self-gravity of the body, therefore the equation of motion does not have the effect of gravitational radiation reaction.

The effect of the spin is also discussed, for example, in D'Eath[6] and Thorne and Hartle[7], in which the strong self-gravity of the body is taken into account by using the technique of matched asymptotic expansion. Especially Thorne and Hartle[7] establish the frame work of matched asymptotic expansion of the metric around the particle. Using the metric thus derived, they integrate non-covariant conservation laws in the inertial frame of the background spacetime at the particle (body's local asymptotic rest frame in [7]). They find that these integrations are useful in extracing the information of the motion of the particle and the evolution of the spin, and they call them 'laws of motion and precession' which can be converted into the equations of motion with an approximate solution of the Einstein equations of the surrounding spacetime. The laws of motion indicate that the geodesic law of motion is corrected by the coupling of the spin of the body and the Riemann curvature of the surrounding spacetime. As a specific example of a system, they consider the case that two Kerr black holes are orbiting each other. However they assume that the separation of these black holes is enough large and derive the equation of motion and precession accurately through post^{1.5}-Newtonian order. Therefore the radiation reaction force derived in [5] does not appear in [7].

The purpose of this paper is 1) to clarify the use of a spinning point particle in the perturbation study[9], and 2) to derive the correction to the equation of motion due to the radiation reaction and due to the coupling of the spin and the Riemann curvature in a unified manner.

Again we use the technique of matched asymptotic expansion in order to construct the metric. Since the spin-zero limit in [9] is found to approximate a Schwarzschild black hole[5], we suspect that the spinning point particle in [9] approximate a Kerr black hole, therefore we assume that the metric around the particle is approximated by that of a Kerr black hole with perturbation. The metric far from the particle is again approximated by a linear perturbation of a general background spacetime.

The next step is to extract the information of the motion of the particle from the metric thus constructed. At this point we have two methods to obtain the equation of motion; the consistency condition of the matching with appropriate gauge fixing in the internal scheme[5], and the use of non-covariant conservation laws in the inertial frame of the background spacetime[7]. The former method developed by Mino et. al.[5] seems difficult to apply to the present case. This is because we do not know the general method to distinguish the translation gauge mode of metric perturbation in a Kerr background, hence we cannot fix the center of mass condition in the internal scheme when matching the metrics in the internal and external scheme at the overlapping region. Thus we consider to integrate the conservation laws as in [7], which is equivalent to convert the

laws of motion derived in [7] into the equation of motion with a linear perturbed metric of the general background spacetime at the overlapping region.

This paper is organized as follows. In section 2, we construct the metric in the overlapping region (buffer region), using the covariant expansion method of the tensor Green function[5]. Here we suppose the reader is familiar with the concept of ‘bi-tensors’[11] and we skip the computation of the tensor Green function[5]. In section 3, we derive the equation of motion of the spinning point particle. Section 4 summarizes the result.

2 Matched Asymptotic Expansion

Following the procedure given in [5], we first discuss the external scheme of the metric. We have a general background metric $g_{\mu\nu}(x)$ which satisfies the vacuum Einstein equations around the particle in the external scheme. We compute its linear perturbation with the Green function method. The perturbation in the external scheme is constructed with an assumption on the perturbation source. (See Eq. (3) below.) We will discuss the consistency of this assumption when we compute the metric in the overlapping region later.

We put the metric $g_{\mu\nu}(x) + h_{\mu\nu}(x)$ into the Einstein equations and consider to solve the linearized equations of $h_{\mu\nu}$ assuming that the background metric $g_{\mu\nu}$ satisfies the vacuum Einstein equations around the particle of interest². We introduce the trace-reversed metric perturbation $\psi_{\mu\nu}(x)$, and put the covariant harmonic gauge condition on it.

$$\psi_{\mu\nu}(x) = h_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)g^{\xi\pi}(x)h_{\xi\pi}(x) \quad (1)$$

$$\psi^{\mu\nu}{}_{;\nu} = 0 \quad (2)$$

We can make use of the tensor Green function of the linearized Einstein equations with retarded boundary condition derived in Sec. 2 of [5]. Here we assume the metric perturbation is generated by the following source term[9, 14],

$$T^{\mu\nu}(x) = m \int dT \left\{ v^\mu(x, T) v^\nu(x, T) \frac{\delta^{(4)}(x - z(T))}{\sqrt{-g}} - \nabla_\xi \left(S^{\xi(\mu}(x, T) v^{\nu)}(x, T) \frac{\delta^{(4)}(x - z(T))}{\sqrt{-g}} \right) \right\}, \quad (3)$$

where $v^\mu(x, T) = \bar{g}^\mu{}_\alpha(x, z(T))\dot{z}^\alpha(T)$ and $S^{\mu\nu}(x, T) = \bar{g}^\mu{}_\alpha(x, z(T))\bar{g}^\nu{}_\beta(x, z(T))S^{\alpha\beta}(T)$. $\bar{g}^m u_\alpha(x, z(T))$ is a bivector of parallel displacement defined, for example, in Appendix A of [5], and $S^{\alpha\beta}(T)$ is an anti-symmetric tensor which is called spin tensor of the particle and is assumed to satisfy the center of mass condition, $S_{\alpha\beta}(T)v^\beta(T) = 0$. (A dot over a function means T -derivative of it, such as $\dot{z}^\alpha(T) = dz^\alpha/dT$.) The amplitude of the spin tensor $S_{\alpha\beta}(T)$ is usually assumed to be at most of order m in the case of Kerr black holes, which seems to indicate the 2nd order perturbation theory is necessary when we want to discuss the spin effect of the particle in the exact manner. But one will find it not so if we assume that the construction of the metric by the matched asymptotic expansion is

²In this derivation, we follow the notation used in [11]. We assign the indices $\alpha, \beta, \gamma, \delta$ for the point on the particle trajectory $z(T)$, and the indices μ, ν, ξ, ρ for the field point x

correct. The reason why we include the spin term in (3) is our interest in its contribution to the motion via gravitational radiation reaction, which is important for constructing the gravitational wave template. The exact form of the metric perturbation becomes

$$\begin{aligned}
\psi^{\mu\nu}(x) = & 2Gm \left(\left[\frac{1}{\dot{\sigma}(x, z(T))} u^{\mu\nu}{}_{\alpha\beta}(x, z(T)) \dot{z}^\alpha(T) \dot{z}^\beta(T) \right. \right. \\
& + \frac{\ddot{\sigma}(x, z(T))}{\dot{\sigma}^3(x, z(T))} u^{\mu\nu}{}_{\alpha\beta}(x, z(T)) \sigma_{;\gamma}(x, z(T)) S^{\gamma\alpha}(T) \dot{z}^\beta(T) \\
& + \frac{1}{\dot{\sigma}(x, z(T))} u^{\mu\nu}{}_{\alpha\beta;\gamma}(x, z(T)) S^{\gamma\alpha}(T) \dot{z}^\beta(T) \\
& - \frac{1}{\dot{\sigma}^2(x, z(T))} \frac{d}{dT} \left(u^{\mu\nu}{}_{\alpha\beta}(x, z(T)) \sigma_{;\gamma}(x, z(T)) S^{\gamma\alpha}(T) \dot{z}^\beta(T) \right) \\
& + \left. \frac{1}{\dot{\sigma}(x, z(T))} v^{\mu\nu}{}_{\alpha\beta}(x, z(T)) \sigma_{;\gamma}(x, z(T)) S^{\gamma\alpha}(T) \dot{z}^\beta(T) \right]_{T=T_{Ret}(x)} \\
& - \int_{-\infty}^{T_{Ret}(x)} dT \left(v^{\mu\nu}{}_{\alpha\beta}(x, z(T)) \dot{z}^\alpha(T) \dot{z}^\beta(T) + v^{\mu\nu}{}_{\alpha\beta;\gamma}(x, z(T)) S^{\gamma\alpha}(T) \dot{z}^\beta(T) \right) \Big). \tag{4}
\end{aligned}$$

$\sigma(x, z)$ is a half the squared geodetic interval between x and z and is defined, for example, in Appendix A of [5]. $u^{\mu\nu\alpha\beta}(x, z)$, $v^{\mu\nu\alpha\beta}(x, z)$ appear when computing various tensor Green functions, and are defined in Sec. 2 of [5]. $T_{Ret}(x)$ is the retarded time of the particle and is a scalar function determined by $\sigma(x, z(T_{Ret})) = 0$ with the condition that $z(T_{Ret})$ is inside the past light cone of x .

We next consider the metric in the overlapping region, in which the metric is expanded in powers of Gm/r and r/L . Since m and L are the only dimensional quantities of the system, it is also the expansion in powers of r . We first introduce coordinates $\{T, X^i\} = \{X^a\}$, ($i = 1, 2, 3; a = 0, 1, 2, 3$)³, such that they become those introduced in [11] when taking the test particle limit $m \rightarrow 0$

$$\lim_{m \rightarrow 0} (\sigma_{;\alpha}(x, z(T)) + e_{\alpha i}(T) X^i) = 0, \tag{5}$$

$$\lim_{m \rightarrow 0} \dot{z}^2(T) = -1, \tag{6}$$

$$\lim_{m \rightarrow 0} \dot{z}^\alpha(T) e_{\alpha i}(T) = 0, \tag{7}$$

$$\lim_{m \rightarrow 0} e^\alpha{}_i(T) e_{\alpha j}(T) = \delta_{ij}. \tag{8}$$

(In the overlapping region, $X^i X^j / L^2$ is so small that the coordinate transformation $x^\alpha = x^\alpha(X^a)$ becomes single valued[11].) (7) and (8) shows that $e_{\alpha i}(T)$ are the spatial triad at $z(T)$ on the 3-hypersurface normal to $\dot{z}(T)$. The coordinate system for finite m shall be determined later.

We note the components of the metric whose order of magnitude are $O((Gm)^n / L^m)$ by ${}^{(m)}_{(n)} h_{ab}$.

$$g_{ab}^{full} = \sum_{m,n=0}^{\infty} {}^{(m)}_{(n)} h_{ab} \tag{9}$$

³We assign the indices, a, b, c, d , for the spacetime coordinates i.e. 0, 1, 2, 3, while the indices, i, j, k, l , for the spatial coordinates i.e. 1, 2, 3. Minkowski and Kronecker summation conventions are taken.

We call these terms the $\binom{m}{n}$ -components of the metric[5]. Simple dimensional analysis shows

$$\binom{m}{n}h_{ab} \sim |X|^{(m-n)}. \quad (10)$$

(Here we use $|X| = \sqrt{X^i X^i}$.) In the next section, we use the Landau-Lifshitz pseudotensor in order to compute the equation of motion, and the metric is expanded by taking the flat Minkowski metric η_{ab} as a background. For its purpose, we raise the indices of the $\binom{m}{n}$ -components of the metric by η_{ab} , and we define the trace-reverse $\binom{m}{n}$ -components of the metric with respect to the flat Minkowski.

$$\binom{m}{n}\bar{h}_{ab} = \binom{m}{n}h_{ab} - \frac{1}{2}\eta_{ab}\eta^{cd}\binom{m}{n}h_{cd} \quad (11)$$

Because of the limited length of this manuscript, we omit the derivation of each component of the metric and the coordinate condition for finite m . We note that the $\binom{m}{0}$ -components of the metric are obtained from the background metric $g_{\mu\nu}(x)$ and that $\binom{m}{1}$ -components of the metric are obtained from the background metric $g_{\mu\nu}(x)$ and the linear perturbation (4).

$$\binom{0}{0}h_{ab} = \eta_{ab} \quad (12)$$

$$\binom{1}{0}\bar{h}_{ab} = 0 \quad (13)$$

$$\begin{aligned} \binom{2}{0}\bar{h}_{00} &= \frac{2}{3}R_{\alpha\beta\gamma\delta}(z(T))\dot{z}^\alpha(T)X^\beta(T)\dot{z}^\gamma(T)X^\delta(T) \\ &\quad + O(Gm|X|^2/L^3) \end{aligned} \quad (14)$$

$$\begin{aligned} \binom{2}{0}\bar{h}_{0i} &= \frac{2}{3}R_{\alpha\beta\gamma\delta}(z(T))e^\alpha{}_i(T)X^\beta(T)\dot{z}^\gamma(T)X^\delta(T) \\ &\quad + O(Gm|X|^2/L^3) \end{aligned} \quad (15)$$

$$\begin{aligned} \binom{2}{0}\bar{h}_{ij} &= \frac{1}{3}R_{\alpha\beta\gamma\delta}(z(T))X^\beta(T)X^\delta(T)(e^\alpha{}_i(T)e^\gamma{}_j(T) + \delta_{ij}\dot{z}^\alpha(T)\dot{z}^\gamma(T)) \\ &\quad + O(Gm|X|^2/L^3) \end{aligned} \quad (16)$$

$$\binom{0}{1}\bar{h}_{00} = \frac{4Gm}{|X|} \quad (17)$$

$$\binom{0}{1}\bar{h}_{0i} = 0 \quad (18)$$

$$\binom{0}{1}\bar{h}_{ij} = 0 \quad (19)$$

$$\binom{1}{1}\bar{h}_{ab} = 0 \quad (20)$$

$$\begin{aligned} \binom{2}{1}\bar{h}_{00} &= \dot{z}^\alpha(T)\frac{D}{dT}e_{\alpha i}(T)X^i(T) \\ &\quad + \frac{10Gm}{3|X|}R_{\alpha\beta\gamma\delta}(z(T))\dot{z}^\alpha(T)X^\beta(T)\dot{z}^\gamma(T)X^\delta(T) \\ &\quad - 2V_{\alpha\beta\gamma}(T)\dot{z}^\alpha(T)\dot{z}^\beta(T)X^\gamma(T) + O((Gm)^2|X|/L^3) \end{aligned} \quad (21)$$

$$\binom{2}{1}\bar{h}_{0i} = e^\alpha{}_i(T)\frac{D}{dT}e_{\alpha j}(T)X^j$$

$$+2GmR_{\alpha\beta\gamma\delta}(z(T))X^\beta(T)\dot{z}^\gamma(T)\left(-\dot{z}^\alpha(T)e^\delta_i(T)+\frac{2}{3|X|}e^\alpha_i(T)X^\delta(T)\right) \\ -2V_{\alpha\beta\gamma}(T)\dot{z}^\alpha(T)e^\beta_i(T)X^\gamma(T)+O((Gm)^2|X|/L^3) \quad (22)$$

$$\begin{aligned} {}^{(2)}_{(1)}\bar{h}_{ij} &= \delta_{ij}\dot{z}^\alpha(T)\frac{D}{dT}e_{\alpha k}(T)X^k \\ &+2GmR_{\alpha\beta\gamma\delta}(z(T))\left(\frac{1}{3|X|}X^\beta(T)X^\delta(T)(e^\alpha_i(T)e^\gamma_j(T)-\delta_{ij}\dot{z}^\alpha(T)\dot{z}^\gamma(T)) \right. \\ &\quad \left.-2|X|\dot{z}^\alpha(T)e^\beta_i(T)\dot{z}^\gamma(T)e^\delta_j(T)\right) \\ &-2V_{\alpha\beta\gamma}(T)e^\alpha_i(T)e^\beta_j(T)X^\gamma(T)+O((Gm)^2|X|/L^3) \end{aligned} \quad (23)$$

$${}^{(0)}_{(2)}\bar{h}_{00} = \frac{(Gm)^2}{|X|^2} \quad (24)$$

$${}^{(0)}_{(2)}\bar{h}_{0i} = -\frac{2Gm}{|X|^3}S_{\alpha\beta}(T)X^\alpha(T)e^\beta_i(T)+O((Gm)^3/L|X|^2) \quad (25)$$

$${}^{(0)}_{(2)}\bar{h}_{ij} = \frac{(Gm)^2}{|X|^2}\left(-2\delta_{ij}+\frac{X^iX^j}{|X|^2}\right) \quad (26)$$

$${}^{(1)}_{(2)}\bar{h}_{ab} = 0 \quad (27)$$

Here the coordinate conditions are determined such that $(\frac{1}{0})$ -components of the metric vanish.

$$\sigma_{;a}(x, z(T)) + e_{\alpha i}(T)X^i = 0 \quad (28)$$

$$\dot{z}^2(T) = -1 + V_{\alpha\beta}(T)\left(g^{\alpha\beta}(z(T)) + 2\dot{z}^\alpha(T)\dot{z}^\beta(T)\right) + O((Gm)^2/L^2) \quad (29)$$

$$\dot{z}^\alpha(T)e_{\alpha i}(T) = -2V_{\alpha\beta}(T)\dot{z}^\alpha(T)e^\beta_i(T) + O((Gm)^2/L^2) \quad (30)$$

$$e^\alpha_i(T)e_{\alpha j}(T) = \delta_{ij} + V_{\alpha\beta}(T)\left(-\delta_{ij}g^{\alpha\beta}(z(T)) + 2e^\alpha_i(T)e^\beta_j(T)\right) + O((Gm)^2/L^2) \quad (31)$$

One may see that $(\frac{0}{n})$ -components of the metric realizes the asymptotic gravitational field of a Kerr black hole, thus we find it consistent to use (3) to generate the external metric.

3 Equation of Motion

We apply (2.2a), (2.2b), (2.3a) and (2.3b) of [7] in order to derive the equation of motion.

$$P^a(T, r) = \frac{1}{16\pi G} \int_{|X|=r} d^2S_j H^{ab0j}_{,b} \quad (32)$$

$$J^{ij}(T, r) = \frac{1}{16\pi G} \int_{|X|=r} d^2S_k \left(X^i H^{ja0k}_{,a} - X^j H^{ia0k}_{,a} + H^{ik0j} - H^{jk0i} \right) \quad (33)$$

$$\frac{d}{dT}P^a(T, r) = -\frac{1}{G} \int_{|X|=r} d^2S_j \dot{t}^{aj} \quad (34)$$

$$\frac{d}{dT}J^{ij}(T, r) = -\frac{1}{G} \int_{|X|=r} d^2S_k \left(X^i \dot{t}^{jk} - X^j \dot{t}^{ik} \right) \quad (35)$$

As in [7], we adopt the choice of Landau-Lifshitz for H^{abcd} , t^{ab} .

$$H^{abcd} = H_{L-L}^{abcd} \quad (36)$$

$$t^{ab} = (-g)t_{L-L}^{ab} \quad (37)$$

(The explicit expressions are given in standard texts, such as, Sec.20 of Misner-Thorne-Wheeler[12] or Sec.96 of Landau-Lifshitz[13].) The integrals are taken over a 2-surface of constant $|X|$ in the overlapping region for definiteness, but we focus on r -independent terms in the integrals since it is enough to only consider them in order to derive the leading correction to the orbit $z(T)$.

Before considering (32) and (34), we first show that the spin tensor $S_{\alpha\beta}(T)$ is geodetic transported in the background geometry $g_{\alpha\beta}(z(T))$ in the test particle limit $m \rightarrow 0$. Eq. (33) has a dimension of $(length)^2$ and we consider the terms of order $(Gm)^2$. We obtain

$$J^{ij}(T, r) = mS_{\alpha\beta}(T)e^{\alpha i}e^{\beta j} + O(Gm^3/L) + (r \text{ dependent terms}). \quad (38)$$

(35) has a dimension of $(mass)^2/(length)$ and we consider the terms of order m^2/L in the same way. Then we find that (35) vanishes.

$$\frac{d}{dT}J^{ij}(T, r) = O(Gm^3/L^2) + (r \text{ dependent terms}) \quad (39)$$

Since the spatial triad are geodetic transported in the background geometry to the leading order, (38) and (39) result in

$$\frac{DS^{\alpha\beta}}{dT}(T) = 0. \quad (40)$$

We next consider (32) and (34). (32) has a dimension of $(mass)^1$ and we consider the terms of order m and Gm^2/L . We obtain

$$P^0(T, r) = m + O(G^2m^3/L^2) + (r \text{ dependent terms}), \quad (41)$$

$$P^i(T, r) = O(G^2m^3/L^2) + (r \text{ dependent terms}). \quad (42)$$

(34) has a dimension of $(mass)/(length)$ and we consider the terms of order m/L and Gm^2/L^2 . A straightforward computation results in

$$\frac{d}{dT}P^0(T, r) = O(G^2m^3/L^3) + (r \text{ dependent terms}), \quad (43)$$

$$\begin{aligned} \frac{d}{dT}P^i(T, r) &= \frac{m}{2}R_{\alpha\beta\gamma\delta}(z(T))e^{\alpha i}(T)\dot{z}^{\beta}(T)S^{\gamma\delta}(T) \\ &\quad - \frac{m}{2}V_{\alpha\beta\gamma}(T)e^{\gamma i}(T)\left(2\dot{z}^{\alpha}(T)\dot{z}^{\beta}(T) + g^{\alpha\beta}(z(T))\right) \\ &\quad + m\dot{z}^{\alpha}(T)\frac{D}{dT}e_{\alpha i}(T) + O(G^2m^3/L^3) + (r \text{ dependent terms}). \end{aligned} \quad (44)$$

With T -derivative of (29) and (30), we finally get the equation of motion,

$$\begin{aligned} \frac{D}{dT}\dot{z}^{\alpha}(T) &= \bar{V}_{\beta\gamma\delta}(T)\left(2\dot{z}^{\beta}(T)g^{\alpha\gamma}(z(T))\dot{z}^{\delta}(T) - \dot{z}^{\beta}(T)\dot{z}^{\gamma}(T)g^{\alpha\delta}(z(T))\right) \\ &\quad + \frac{1}{2}R^{\alpha}{}_{\beta\gamma\delta}(z(T))\dot{z}^{\beta}(T)S^{\gamma\delta}(T) + O(G^2m^2/L^3), \end{aligned} \quad (45)$$

where we define $\bar{V}_{\alpha\beta\gamma} = V_{\alpha\beta\gamma} - 1/2g_{\alpha\beta}V^{\delta}_{\gamma}$. We introduce the proper time τ of the orbit,

$$\frac{d\tau}{dT} = 1 - \bar{V}_{\alpha\beta}(T)\dot{z}^{\alpha}(T)\dot{z}^{\beta}(T), \quad (46)$$

and we finally get

$$\begin{aligned} \frac{D}{d\tau} \frac{dz^{\alpha}}{d\tau}(\tau) = & \bar{V}_{\beta\gamma\epsilon}(\tau) \left(\frac{dz^{\alpha}}{d\tau} \frac{dz^{\beta}}{d\tau} \frac{dz^{\gamma}}{d\tau} \frac{dz^{\epsilon}}{d\tau} + 2 \frac{dz^{\beta}}{d\tau} g^{\alpha\gamma}(z(\tau)) \frac{dz^{\epsilon}}{d\tau} - \frac{dz^{\beta}}{d\tau} \frac{dz^{\gamma}}{d\tau} g^{\alpha\epsilon}(z(\tau)) \right) \\ & + \frac{1}{2} R^{\alpha}_{\beta\gamma\epsilon}(z(\tau)) \frac{dz^{\beta}}{d\tau} S^{\gamma\epsilon}(\tau) + O(G^2 m^2 / L^3). \end{aligned} \quad (47)$$

4 Conclusion

In this paper, we have derived the equation of motion of a spinning particle on a given background spacetime. We have constructed the metric using the technique of matched asymptotic expansion, and computed conservation laws introduced in [7]. The conservation laws are used to extract the information of the motion.

The conclusion of our previous paper[5] is that the non-spinning particle moves along a geodesic of the regularized perturbed space,

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + h_{(v)\mu\nu}(x), \quad (48)$$

where $h_{(v)\mu\nu}(x)$ is the tail part of the metric perturbation (4),

$$h_{(v)\mu\nu}(x) = \psi_{(v)\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) g^{\xi\rho}(x) \psi_{(v)\xi\rho}(x), \quad (49)$$

$$\psi_{(v)\mu\nu}(x) = -2Gm \int_{-\infty}^{T_{Ret}(x)} dT \left(v^{\mu\nu}_{\alpha\beta}(x, z(T)) \dot{z}^{\alpha}(T) \dot{z}^{\beta}(T) + v^{\mu\nu}_{\alpha\beta;\gamma}(x, z(T)) S^{\gamma\alpha}(T) \dot{z}^{\beta}(T) \right). \quad (50)$$

We find that the spin of the particle gives rise to an effective force in addition to this, which is the same as discussed in [10, 14, 7].

$$\frac{\tilde{D}}{dT} \dot{z}(T) = \frac{1}{2} R^{\alpha}_{\beta\gamma\delta}(z(T)) \dot{z}^{\beta}(T) S^{\gamma\delta}(T), \quad (51)$$

As was commented in [5], it is still under investigation to compute $h_{(v)\mu\nu}(x)$ even for well-known background geometries, such as a Kerr geometry.

The important progress made in this paper is the justification of the use of a spinning point particle in the perturbation study. This fact is important both numerically and analytically[9]. Numerical study of a spinning particle falling into a rotating black hole will not only give us a quite accurate estimate of emitted gravitational waves, but also become an efficient method to check the future-coming study done by numerical relativity. Analytical study of a spinning particle rotating around a black hole is important in constructing the theoretical template of gravitational wave form coming from rotating black hole binary system.

We have considered the spinning particle in the present paper. The application of the present derivation to a particle with higher multipole structure is straightforward. As an example, the computation of the innermost stable circular orbit of a spheroidal star rotating around a Kerr black hole is under investigation[15].

Acknowledgements

We thank all the members of Japan gravitational-wave theoretical group, especially M. Sasaki, T. Nakamura, M. Shibata, and T. Tanaka for fruitful conversations. We thank Prof. H. Sato and Prof. S. Ikeuchi for their continuous encouragements. This work was supported in part by Monbusho Grant-in-Aid for Scientific Research No. 5427.

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Analytic Solutions of the Teukolsky Equation and their Properties

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Abstract

We explain the analytic solutions of Teukolsky equation in the Kerr geometry which Mano, Suzuki and Takasugi found recently. The properties of these solutions are examined. Solutions are quite useful to construct theoretical templates of gravitational wave from binaries for LIGO, VIRGO, TAMA and GEO600.

1 Introduction

In this report, we will discuss the analytic solutions of Teukolsky equation in Kerr geometry. This talk is based on the works by Mano, Suzuki and Takasugi[1] and Mano and Takasugi[2]. The solutions consists of hypergeometric type which is convergent except $r = \infty$ and the Coulomb type which is convergent $r > r_+$ where r_+ is the outer horizon. These two solutions are analytically matched in the intermediate region where both series are convergent[1],[2].

These solutions are essentially given in the expansion of $\epsilon = 2M\omega$, M being the mass of black hole and ω an angular frequency. Thus, the solutions are quite useful to obtain the Post-Newtonian expansion of the gravitational wave for example from a particle in circular orbits around the Kerr black hole.

We present the expressions without full explanations and derivations which we refer to Refs.1 and 2.

2 Analytic solutions

Teukolsky equation for radial function without source term is

$$\Delta R'' + 2(r - M)(s + 1)R' + [(K^2 - 2is(r - M)K)/\Delta + 4is\omega r - \lambda]R = 0, \quad (1)$$

with M being the mass of the black hole, aM its angular momentum, $\Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-)$ where r_+ and r_- are positions of outer and inner horizons, respectively, $K = (r^2 + a^2)\omega - am$, $\lambda = E - s(s + 1) - 2ma\omega + a^2\omega^2$. s is the spin weight and l is an angular momentum which satisfies $l \geq \max(|m|, |s|)$.

Hypergeometric type solution of Teukolsky equation is expressed as

$$R_{in;s}^\nu = A_s e^{i\epsilon\kappa x} (-x)^{-s-i\epsilon_-} (1-x)^{i\epsilon_-} \sum_{n=-\infty}^{\infty} a_n^\nu(s) \\ \times F(n + \nu + 1 - i\tau, -n - \nu - i\tau; 1 - s - 2i\epsilon_+; x), \quad (2)$$

where $x = -\frac{\omega}{\epsilon\kappa}(r-r_+)$, $\kappa = \sqrt{1-q^2}$ with $q = \frac{a}{M}$, $\tau = \frac{\epsilon-mq}{\kappa}$ and $\epsilon_{\pm} = \frac{(\epsilon \pm \tau)}{2}$. This solution satisfies the incoming boundary condition on the horizon.

Coefficients $a_n^\nu(s)$ satisfy the three term recurrence relation. The solution is characterized by ν which is determined so that the series of $a_n^\nu(s)$ is convergent. By examining the three term recurrence relation, we find that ν is a renormalized (shifted) angular momentum and it behaves as $\nu = l + O(\epsilon^2)$ for small ϵ . In addition, we find that (1) if ν is a solution, then $-\nu - 1$ is, and (2) $\nu(s)$ is a real and an even function of s . By using large $|n|$ behaviors of coefficients and hypergeometric functions, we can prove that the solution is convergent for the entire complex plane except infinity for all finite ϵ , i.e., $r < \infty$. Another independent solution is the outgoing solution on the outer horizon and is given by $R_{\text{out};s}^\nu = \Delta^{-s}(R_{\text{in};-s}^\nu)^*$. These two solutions form a pair of solutions.

The hypergeometric type solution is expressed as $R_{\text{in};s}^\nu = A_s e^{i\epsilon\kappa} [R_{0;s}^\nu + R_{0;s}^{-\nu-1}]$. Here,

$$R_{0;s}^\nu = e^{-i\epsilon\kappa\tilde{x}} (\tilde{x})^{\nu+i\epsilon_+} (\tilde{x}-1)^{-s-i\epsilon_+} \sum_{n=-\infty}^{\infty} \frac{\Gamma(1-s-2i\epsilon_+)\Gamma(2n+2\nu+1)}{\Gamma(n+\nu+1-i\tau)\Gamma(n+\nu+1-s-i\epsilon)} \times a_n^\nu(s) \tilde{x}^n F(-n-\nu-i\tau, -n-\nu-s-i\epsilon; -2n-2\nu; \frac{1}{\tilde{x}}). \quad (3)$$

Then, $R_{0;s}^\nu$ and $R_{0;s}^{-\nu-1}$ form a pair of independent solutions.

Coulomb type solution is expressed as

$$R_{C;s}^\nu = z^{-1-s} \left(1 - \frac{\epsilon\kappa}{z}\right)^{-s-i\epsilon_+} \sum_{n=-\infty}^{\infty} (-i)^n \frac{(\nu+1+s-i\epsilon)_n}{(\nu+1-s+i\epsilon)_n} a_n^\nu(s) e^{-iz} \times (2z)^{n+\nu} z \frac{\Gamma(n+\nu+1-s+i\epsilon)}{\Gamma(2n+2\nu+2)} \Phi(n+\nu+1-s+i\epsilon, 2n+2\nu+2; 2iz), \quad (4)$$

where $(a)_n = \Gamma(n+a)/\Gamma(a)$ and $z = \omega(r-r_-) = \epsilon\kappa(1-x) = \epsilon\kappa\tilde{x}$. Another independent solution is obtained by changing ν to $-\nu-1$, $R_{C;s}^{-\nu-1}$. They form a pair of independent solution. By using the large $|n|$ behaviors of coefficients and Coulomb wave functions, we can prove that this solution is convergent for $r > r_+$.

Coulomb type solutions contain both the incoming $R_{+;s}^\nu$ and the outgoing solutions $R_{-;s}^\nu$ at infinity. Thus, solutions are decomposed into another pair of solutions, $R_{C;s}^\nu = R_{+;s}^\nu + R_{-;s}^\nu$.

Matching of two types of solutions is performed by comparing $R_{0;s}^\nu$ and $R_{C;s}^\nu$ which are solutions of Teukolsky equation. Both behaves like $\tilde{x}^\nu$ multiplied by a single-valued function for large $\tilde{x}$ ($z = \epsilon\kappa\tilde{x}$). Thus, $R_{0;s}^\nu$ should be proportional to $R_{C;s}^\nu$. The same is true for $R_{0;s}^{-\nu-1}$ and $R_{C;s}^{-\nu-1}$. It should be noted that $\nu - (-\nu-1) = 2\nu+1$ is not an integer so that $R_{0;s}^\nu$ and $R_{0;s}^{-\nu-1}$ are solutions with different characteristic exponents. Defining

$$R_{0;s}^\nu = K_\nu(s) R_{C;s}^\nu. \quad (5)$$

Then, by comparing each power of $\tilde{x}$ in the region $1 \ll \tilde{x} < \infty$ where both solutions converge, we find

$$K_\nu(s) = \frac{(2\epsilon\kappa)^{-\nu+s-\tilde{r}} 2^{-s} i^{\tilde{r}} \Gamma(1-s-2i\epsilon_+) \Gamma(\tilde{r}+2\nu+1) \Gamma(\tilde{r}+2\nu+2)}{\Gamma(\nu+1-i\tau) \Gamma(\nu+1-s-i\epsilon) \Gamma(\tilde{r}+\nu+1-s+i\epsilon)}$$

$$\begin{aligned}
& \times \frac{\Gamma(\nu+1+i\tau)\Gamma(\nu+1+s+i\epsilon)}{\Gamma(\tilde{r}+\nu+1+i\tau)\Gamma(\tilde{r}+\nu+1+s+i\epsilon)} \\
& \times \left(\sum_{n=\tilde{r}}^{\infty} \frac{(\tilde{r}+2\nu+1)_n}{(n-\tilde{r})!} \frac{(\nu+1+s-i\epsilon)_n}{(\nu+1-s+i\epsilon)_n} a_n^\nu(s) \right)^* \\
& \times \left(\sum_{n=-\infty}^{\tilde{r}} \frac{(-1)^n}{(\tilde{r}-n)!(\tilde{r}+2\nu+2)_n} \frac{(\nu+1+s-i\epsilon)_n}{(\nu+1-s+i\epsilon)_n} a_n^\nu(s) \right)^{-1}, \quad (6)
\end{aligned}$$

where $\tilde{r}$ can be any integer and $K_\nu(s)$ is independent of the choice of $\tilde{r}$.

Property of coefficients is summarized as follows: With the initial condition $a_0^\nu(s) = a_0^{-\nu-1}(s) = 1$, we find $a_{-n}^{-\nu-1}(s) = a_n^\nu(s)$ and

$$a_n^\nu(-s) = \left| \frac{(\nu+1+s+i\epsilon)_n}{(\nu+1-s+i\epsilon)_n} \right|^2 a_n^\nu(s), \quad a_n^\nu(s)^* = (-1)^n \frac{(\nu+1+i\tau)_n}{(\nu+1-i\tau)_n} a_n^\nu(s). \quad (7)$$

These relations are useful to prove analytically the Teukolsky-Starobinsky identities.

Solutions valid in the entire plane of r are now given by taking the hypergeometric type solution around the horizon and the Coulomb type solution around infinity.

Asymptotic forms of Teukolsky functions are given in compact forms following the definition by Chandrasekhar[5]:

$$\begin{aligned}
R_s^{(\text{trans})} &= A_s \left(\frac{\epsilon\kappa}{\omega} \right)^{2s} \sum_{n=-\infty}^{\infty} a_n^\nu(s), \\
R_s^{(\text{inc})} &= \frac{A_s e^{i\epsilon\kappa}}{\omega} \left[K_\nu(s) - i e^{-i\pi\nu} \frac{\sin \pi(\nu-s+i\epsilon)}{\sin \pi(\nu+s-i\epsilon)} K_{-\nu-1}(s) \right] A_{+,s}^\nu, \\
R_s^{(\text{ref})} &= \frac{A_s e^{i\epsilon\kappa}}{\omega^{1+2s}} \left[K_\nu(s) + i e^{i\pi\nu} K_{-\nu-1}(s) \right] A_{-,s}^\nu, \quad (8)
\end{aligned}$$

where

$$\begin{aligned}
A_{+,s}^\nu &= 2^{-1+s-i\epsilon} e^{i(\pi/2)(\nu+1-s)} e^{-\pi\epsilon/2} \frac{\Gamma(\nu+1-s+i\epsilon)}{\Gamma(\nu+1+s-i\epsilon)} \sum_{n=-\infty}^{\infty} a_n^\nu(s), \\
A_{-,s}^\nu &= 2^{-1-s+i\epsilon} e^{-i(\pi/2)(\nu+1+s)} e^{-\pi\epsilon/2} \sum_{n=-\infty}^{\infty} (-1)^n \frac{(\nu+1+s-i\epsilon)_n}{(\nu+1-s+i\epsilon)_n} a_n^\nu(s). \quad (9)
\end{aligned}$$

3 The Teukolsky-Starobinsky identities

In below, we confine $s = 1/2, 1, 3/2$ and 2 . The Teukolsky-Starobinsky (T-S) identities[3],[4] are expressed as

$$\Delta^s (D^\dagger)^{2s} \Delta^s R_s = C_s^* R_{-s}, \quad (\text{T-S identity (A)}) \quad (10)$$

$$(D)^{2s} R_{-s} = C_s R_s, \quad (\text{T-S identity (B)}) \quad (11)$$

where D and $D^\dagger$ are differential operators

$$D = \frac{\partial}{\partial r} - i \frac{K}{\Delta}, \quad D^\dagger = \frac{\partial}{\partial r} + i \frac{K}{\Delta}, \quad (12)$$

where $(D^\dagger)^{2s}$ means that the differential operation $D^\dagger$ applies $2s$ times and the same for $(D)^{2s}$. The Starobinsky constants[4] C_s are given explicitly in Ref.2. Solutions of Teukolsky equation should satisfy the T-S identities.

Fortunately, we can show analytically that $R_{in;\pm s}^\nu$ for both (hypergeometric and Coulomb) types satisfy the T-S identity (A). From this, we can fix the relative normalization between solutions with spin weight s and $-s$ analytically.

The normalization constants A_s are determined for with the choice $A_{-s} = 1$

$$A_s = C_s^* \left(\frac{\omega}{\epsilon \kappa} \right)^{2s} \frac{\Gamma(1+s-2i\epsilon_+)}{\Gamma(1-s-2i\epsilon_+)} \left| \frac{\Gamma(\nu+1-s+i\epsilon)}{\Gamma(\nu+1+s+i\epsilon)} \right|^2. \quad (13)$$

Also, with the choice $B_{-s} = 1$

$$B_s = C_s^* \omega^{2s}. \quad (14)$$

Identities among sums of coefficients are derived by considering the T-S identity (B). It is not possible to prove analytically, we consider it by using the asymptotic forms. We find a nontrivial identity:

$$\sum_{n=-\infty}^{\infty} a_n^\nu(-s) = |C_s|^2 \left| \frac{\Gamma(\nu+1-s+i\epsilon)}{\Gamma(\nu+1+s+i\epsilon)} \right|^2 \sum_{n=-\infty}^{\infty} a_n^\nu(s). \quad (I-1) \quad (15)$$

No more identity arises from the T-S identities. By considering the conserved current from the Wronskian and we find an identity:

$$\frac{\left| \sum_{n=-\infty}^{\infty} (-1)^n \frac{(\nu+1+s-i\epsilon)_n}{(\nu+1-s+i\epsilon)_n} a_n^\nu(s) \right|^2}{\left| \sum_{n=-\infty}^{\infty} a_n^\nu(s) \right|^2} = |C_s|^2 \left| \frac{\Gamma(\nu+1-s+i\epsilon)}{\Gamma(\nu+1+s+i\epsilon)} \right|^2. \quad (I-2) \quad (16)$$

These identities will be useful to express the various physical quantities given in the exact forms in simpler forms and also to test the accuracy of the computation.

4 The absorption rate and the emission rate of black hole

By using our solutions, we find the absorption rate without any approximation as

$$\begin{aligned} \Gamma_s = & \delta_s \left| \frac{R_s^{(\text{trans})}}{R_s^{(\text{inc})}} \right|^2 = 1 - \frac{(2\omega)^{4s}}{|C_s|^2} \left| \frac{R_s^{(\text{ref})}}{R_s^{(\text{inc})}} \right|^2 = (2\epsilon\kappa)^{2\nu+1} \frac{e^{\pi\epsilon}}{\pi} \left\{ \frac{\sinh 2\pi\epsilon_+}{\cosh 2\pi\epsilon_+} \right\} \\ & \times \frac{D_s^\nu}{\left| 1 + i \frac{(2\epsilon\kappa)^{2\nu+1} e^{i\pi\nu} (-1)^{2s} \sin \pi(\nu-i\tau)}{\pi} \left(\frac{\sin \pi(\nu-s-i\epsilon)}{\sin 2\pi\nu} \right)^2 D_s^\nu \right|^2}, \end{aligned} \quad (17)$$

where the upper column ($\sinh 2\pi\epsilon_+$) is taken for bosons ($s = 0, 1, 2$) and the lower column ($\cosh 2\pi\epsilon_+$) is taken for fermions ($s = 1/2, 3/2$).

The emission rate from the black hole is given by

$$\langle N \rangle \equiv \frac{\Gamma_s}{e^{\frac{2\pi}{\kappa'}(\omega - m\Omega_+) \mp 1}} = 2 \left\{ \frac{(\sinh 2\pi\epsilon_+)^{-1}}{(\cosh 2\pi\epsilon_+)^{-1}} \right\} e^{-\frac{\pi}{\kappa'}(\omega - m\Omega_+)} \Gamma_s. \quad (18)$$

Here, the upper column for bosons and the lower column for fermions, κ' is surface gravity, Ω_+ is angular velocity on horizon and we used $2\epsilon_+ = \frac{\omega - m\Omega_+}{\kappa'}$. By substituting Γ_s into the evaporation rate formula, we find

$$\langle N \rangle = \frac{2(2\epsilon\kappa)^{2\nu+1}}{\pi} \frac{e^{-\frac{\pi}{\kappa'}(\frac{\omega}{1+\kappa} - m\Omega_+)} D_s^\nu}{\left| 1 + i \frac{(2\epsilon\kappa)^{2\nu+1} e^{i\pi\nu} (-1)^{2s} \sin \pi(\nu - i\tau)}{\pi} \left(\frac{\sin \pi(\nu - s - i\epsilon)}{\sin 2\pi\nu} \right)^2 D_s^\nu \right|^2}. \quad (19)$$

It is interesting to observe that the denominator in the emission rate $\langle N \rangle$ which shows the difference between the boson and the fermion emissions is canceled by the $\sinh 2\epsilon_+$ or $\cosh 2\epsilon_+$ factor in Γ_s . As a result, the emission rate shows no characteristic difference for the emissions of bosons and fermions.

5 Summary

We showed the exact solutions of Teukolsky equation. By using them, we can examine various physical quantities analytically. Furthermore, these solutions are useful for investigating analytically various gravitational wave emission processes such as from a binary. The emission of gravitational wave from a particle in circular orbits can be attacked in the Post-Newtonian approximation or the numerical computation by just solving the three term recurrence relation rather than solving a differential equation.

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Time symmetric initial conditions of gravitational waves for 3D numerical relativity

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Abstract

We study time symmetric initial data sets of pure gravitational waves for 3D numerical relativity. We adopt two types of spatial line elements; one is the Brill wave type metric, $\gamma_{rr} = \gamma_{\theta\theta}/r^2$, and the other is that the line element of the unit sphere is conformal flat, i.e., $\gamma_{\theta\theta} = \gamma_{\varphi\varphi}/\sin^2 \theta$. In both cases, the Hamiltonian constraint reduces to the simple 3D elliptic-type equation for the conformal factor which can be accurately solved. We numerically solve the elliptic-type equations for gravitational waves of 2^2 and 2^4 poles. For solutions of high amplitude gravitational waves, we determine the apparent horizon encompassing them to clarify a criterion of formation of black hole from gravitational collapse of gravitational waves as well as to know the required grid resolution for simulation of the collapse of gravitational waves to be a black hole. It is found that a fine grid of its size $< 0.01M$ for each direction of 3D Cartesian coordinates will be required in numerical simulation.

1 Introduction

In this paper, we investigate 3D time symmetric initial conditions of gravitational waves such as the Brill wave[1] which is often used in axisymmetric computations[2][3][4]. The purpose is as follows; (1) to prepare simple 3D initial data sets of gravitational waves in which the amplitude is so high that the AH encompassing gravitational waves appears; (2) to clarify the criteria of formation of black hole in the collapse of gravitational waves; (3) to know the resolution required to simulate dynamical collapse of gravitational waves to be a black hole. In section 2, we describe the formulation to prepare the 3D initial data of pure gravitational waves; in section 2.1, we present two simple geometries where the initial data sets can be easily computed; in section 2.2, we show equations to determine the AH. In section 3, numerical results for various initial data sets are shown. Using the 3D AH finder, we determine the AH for high amplitude gravitational waves, and investigate a criterion for formation of the AH. Then, we clarify the grid resolution required to compute dynamical collapse of gravitational waves to be a black hole. Section 4 is devoted to summary. Throughout the paper, we use the units $c = 1$, where c is the speed of light.

2 Basic equations

2.1 Geometry

To prepare time symmetric initial conditions of pure gravitational waves, we must solve the Hamiltonian constraint

$$R = 0, \quad (1)$$

where R is the scalar curvature of a 3D hypersurface. Following York[6], we define $\tilde{\gamma}_{ij} = \Phi^{-4}\gamma_{ij}$. Then, the Hamiltonian constraint can be written as

$$\tilde{\Delta}\Phi = \frac{\Phi}{8}\tilde{R}, \quad (2)$$

where $\tilde{\Delta}$ and $\tilde{R}$ are the covariant derivative and the scalar curvature with respect to $\tilde{\gamma}_{ij}$.

As the line element, we consider the following two types. One is just the 3D version of the Brill wave[1]

$$dl^2 = \Phi^4 [e^\eta (dr^2 + r^2 d\theta^2) + r^2 \sin^2 \theta d\varphi^2]. \quad (3)$$

Even in the 3D case, the Hamiltonian constraint of the Brill wave becomes a very simple form as

$$\begin{aligned} & \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \left[\frac{\partial^2 \Phi}{\partial \theta^2} + \cot \theta \frac{\partial \Phi}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial}{\partial \varphi} \left(e^\eta \frac{\partial \Phi}{\partial \varphi} \right) \right] = -\frac{\Phi}{8} S_B(\eta) \\ & = -\frac{\Phi}{8} \left[\frac{\partial^2 \eta}{\partial r^2} + \frac{1}{r} \frac{\partial \eta}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \eta}{\partial \theta^2} + \frac{e^\eta}{r^2 \sin^2 \theta} \left\{ 2 \frac{\partial^2 \eta}{\partial \varphi^2} + \frac{3}{2} \left(\frac{\partial \eta}{\partial \varphi} \right)^2 \right\} \right]. \end{aligned} \quad (4)$$

The other is that the line element of the unit sphere is conformal flat (i.e., $\gamma_{\theta\theta} = \gamma_{\varphi\varphi} / \sin^2 \theta$)

$$dl^2 = \Phi^4 [dr^2 + r^2 e^\eta (d\theta^2 + \sin^2 \theta d\varphi^2)]. \quad (5)$$

In this case, the Hamiltonian constraint also becomes a simple form,

$$\begin{aligned} & \frac{1}{r^2} \frac{\partial}{\partial r} \left(e^\eta r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \left[\frac{\partial^2 \Phi}{\partial \theta^2} + \cot \theta \frac{\partial \Phi}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 \Phi}{\partial \varphi^2} \right] = -\frac{\Phi}{8} S_S(\eta) \\ & = -\frac{\Phi}{8} \left[e^\eta \left(2 \frac{\partial^2 \eta}{\partial r^2} + \frac{6}{r} \frac{\partial \eta}{\partial r} + \frac{3}{2} \left(\frac{\partial \eta}{\partial r} \right)^2 + \frac{2}{r^2} \right) - \frac{2}{r^2} + \frac{1}{r^2} \left(\frac{\partial^2 \eta}{\partial \theta^2} + \cot \theta \frac{\partial \eta}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 \eta}{\partial \varphi^2} \right) \right] \quad (6) \end{aligned}$$

Hereafter, we call the former the B type metric, and the latter the S type metric for simplicity. We notice that from the regularity conditions at the origin and z axis, η must behaves as $\rightarrow r^2 \sin^2 \theta f(x, y, z)$ at z axis for the B type metric, and as $\rightarrow r^2 f(x, y, z)$ at the origin for the S type metric, where f is a regular function.

For each geometry, the gravitational mass of the system is calculated by

$$M = \frac{1}{2\pi} \int \rho d^3x, \quad (7)$$

where ρ denotes $\Phi S_B/8$ or $\Phi S_S/8$. It is possible to calculate M by other methods in order to check the accuracy of numerical calculation of M [7].

In the case of the S type metric, there is an additional advantage for numerical computation: For the linear perturbation, in which $|\eta| \ll 1$ and $|\Phi - 1| \ll 1$, Eq.(6) may be rewritten as

$$\Delta_{\text{flat}}\phi = -\frac{1}{8}\left[2\frac{\partial^2\eta}{\partial r^2} + \frac{6}{r}\frac{\partial\eta}{\partial r} + \frac{2\eta}{r^2} + \frac{1}{r^2}\left(\frac{\partial^2\eta}{\partial\theta^2} + \cot\theta\frac{\partial\eta}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\eta}{\partial\varphi^2}\right)\right], \quad (8)$$

where $\phi = \Phi - 1$ and Δ_{flat} is the Laplacian for the flat space. Then, we can construct exact solutions for Eq.(8) as follows. In the linear perturbation theory, Eq.(2) is rewritten as

$$\Delta_{\text{flat}}\hat{\phi} = \frac{1}{8}\left[-\eta^{kl}D_k^{(0)}D_l^{(0)}(h_{ij}\eta^{ij}) + \eta^{kl}D_k^{(0)}(\eta^{ij}D_i^{(0)}h_{lj})\right], \quad (9)$$

where η^{ij} is the flat metric, $D_k^{(0)}$ is the covariant derivative with respect to η_{ij} , and h_{ij} denotes the linear perturbation of $\tilde{\gamma}_{ij}$ (i.e., $\gamma_{ij} = (1 + \hat{\phi})^4(\eta_{ij} + h_{ij})$). Hence, if h_{ij} satisfies the transverse and tracefree conditions, $\eta^{ij}D_i^{(0)}h_{lj} = 0$ and $h_{ij}\eta^{ij} = 0$, $\hat{\phi}$ is equal to zero. Solutions of h_{ij} which satisfy such conditions are given by previous papers. Since we only consider the even parity gravitational wave, such solutions become[8][5]

$$h_{ij} = \begin{pmatrix} a_{\ell m}\hat{Y}_{\ell m} & b_{\ell m}\partial\hat{Y}_{\ell m}/\partial\theta & b_{\ell m}\partial\hat{Y}_{\ell m}/\partial\varphi \\ * & g_{\ell m}\hat{Y}_{\ell m} + f_{\ell m}W_{\ell m} & f_{\ell m}X_{\ell m} \\ * & * & \sin^2\theta(g_{\ell m}\hat{Y}_{\ell m} - f_{\ell m}W_{\ell m}) \end{pmatrix}, \quad (10)$$

where $\hat{Y}_{\ell m}$ is the real part of the spherical harmonic function defined as $\hat{Y}_{\ell m} = (Y_{\ell m} + Y_{\ell m}^*)/\sqrt{2}$ ($Y_{\ell m}^*$ denotes the complex conjugate of $Y_{\ell m}$), and $X_{\ell m}$ and $W_{\ell m}$ are given by

$$X_{\ell m} = 2\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\theta} - \cot\theta\right)\hat{Y}_{\ell m}, \quad W_{\ell m} = \left(\frac{\partial^2}{\partial\theta^2} - \cot\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)\hat{Y}_{\ell m}. \quad (11)$$

$a_{\ell m}$, $b_{\ell m}$, $g_{\ell m}$ and $f_{\ell m}$ are functions of r , and satisfy the following equations;

$$g_{\ell m} = -\frac{r^2}{2}a_{\ell m}, \quad (12)$$

$$b_{\ell m} = \frac{1}{\ell(\ell+1)r}\frac{d}{dr}(r^3a_{\ell m}), \quad (13)$$

$$f_{\ell m} = \frac{1}{(\ell-1)(\ell+2)}\left[-\frac{r^2}{2}a_{\ell m} + \frac{d}{dr}\left\{\frac{r}{\ell(\ell+1)}\frac{d}{dr}(r^3a_{\ell m})\right\}\right]. \quad (14)$$

Thus, for an arbitrary choice of $a_{\ell m}(r)$, we can obtain a solution for h_{ij} .

Then, using the gauge transformation, we transform h_{ij} to the S type metric as

$$\hat{h}_{ij} = h_{ij} + D_i^{(0)}\xi_j + D_j^{(0)}\xi_i, \quad (15)$$

where

$$\xi_i = (A_{\ell m}(r)\hat{Y}_{\ell m}, B_{\ell m}(r)\partial\hat{Y}_{\ell m}/\partial\theta, B_{\ell m}(r)\partial\hat{Y}_{\ell m}/\partial\varphi). \quad (16)$$

Now, to guarantee $\hat{h}_{r\theta} = \hat{h}_{r\varphi} = \hat{h}_{\theta\varphi} = 0$, $A_{\ell m}$ and $B_{\ell m}$ must satisfy the following equations;

$$A_{\ell m} + \frac{dB_{\ell m}}{dr} - \frac{2}{r}B_{\ell m} + b_{\ell m} = 0, \quad (17)$$

$$B_{\ell m} + f_{\ell m} = 0. \quad (18)$$

Also, η and ϕ can be related to h_{ij} and ξ_i from $\hat{h}_{rr} = 4\phi$ and $\hat{h}_{\theta\theta} + \hat{h}_{\varphi\varphi}/\sin^2\theta = 2r^2(\eta + 4\phi)$ as

$$\phi = \frac{1}{4} \left(a_{\ell m} + 2 \frac{dA_{\ell m}}{dr} \right) \hat{Y}_{\ell m}, \quad (19)$$

$$\eta = -4\phi + \frac{1}{r^2} \left(g_{\ell m} - \ell(\ell+1)B_{\ell m} + 2rA_{\ell m} \right) \hat{Y}_{\ell m}. \quad (20)$$

Hence, if we give $a_{\ell m}$, gauge variables $A_{\ell m}$ and $B_{\ell m}$ are calculated from Eqs.(17) and (18), and as a result, we can get solutions for ϕ and η as

$$\begin{aligned} \phi = \phi_{\ell} \hat{Y}_{\ell m} &= \frac{1}{8} \left[(\ell^2 - 4)(\ell - 1)(\ell + 3)Q - 12(\ell^2 + \ell - 7) \frac{dQ}{dr} r - 3(\ell + 6)(\ell - 5) \frac{d^2Q}{dr^2} r^2 \right. \\ &\quad \left. + 26 \frac{d^3Q}{dr^3} r^3 + 2 \frac{d^4Q}{dr^4} r^4 \right] \hat{Y}_{\ell m}, \\ \eta = \eta_{\ell} \hat{Y}_{\ell m} &= \left[-\ell(\ell - 2)(\ell + 1)(\ell + 3)Q + 8(\ell^2 + \ell - 3) \frac{dQ}{dr} r + 2(\ell^2 + \ell - 18) \frac{d^2Q}{dr^2} r^2 \right. \\ &\quad \left. - 12 \frac{d^3Q}{dr^3} r^3 - \frac{d^4Q}{dr^4} r^4 \right] \hat{Y}_{\ell m}, \end{aligned} \quad (21)$$

where $Q(r) = 2a_{\ell m}(r)/\{(\ell - 1)\ell(\ell + 1)(\ell + 2)\}$. Once we find the solution of the linear perturbation equation, we can calculate the gravitational mass of system from the second order perturbation equation as

$$M = \frac{1}{16\pi} \int_0^\infty dr \left[8r^2 \left(\frac{d\phi_{\ell}}{dr} \right)^2 + 8\ell(\ell + 1)\phi_{\ell}^2 - \frac{r^2}{2} \left(\frac{d\eta_{\ell}}{dr} \right)^2 \right]. \quad (22)$$

These solutions are useful to check the accuracy of numerical solution for Φ in the Hamiltonian constraint and M . On the other hand, calibration of numerical code for the B type metric can be performed only for axisymmetric configurations by comparing 3D results with those computed in axisymmetric code.

Before closing this section, we define the following two quantities which are convenient for numerical analysis. One is the Riemann invariant defined as

$$R_{\text{inv}} = {}^{(4)}R_{\mu\nu\sigma\lambda} {}^{(4)}R^{\mu\nu\sigma\lambda} = 8R_{ij}R^{ij}, \quad (23)$$

where ${}^{(4)}R_{\mu\nu\sigma\lambda}$ and R_{ij} are the 4D Riemann tensor and 3D Ricci tensor, respectively. We expect that the Riemann invariant will indicate us region where the gravitational tidal field is strong. Then, we define

$$M_r(r) = \frac{1}{2\pi} \int_0^r r^2 dr \int d(\cos\theta) \int d\varphi \rho. \quad (24)$$

Here, $M_r(r)$ cannot be regarded as the gravitational mass inside r , and we do not know how to estimate the gravitational mass except for that at spatial infinity. However, it may be useful to see size of extension for mass density of gravitational waves; hereafter, we define a outermost radius r_w which satisfies the relation $|M_r(r_w) - M| = 10^{-2}M$. Then, it will be regarded as a rough measure of size of extension for gravitational waves.

2.2 Apparent horizon finder

In the case of time symmetric space, the equation to determine the AH is

$$D_i s^i = 0, \quad (25)$$

where D_i is the covariant derivative with respect to γ_{ij} and s^i is the unit normal of two surface of the AH. Denoting the location of the AH as $r = h(\theta, \varphi)$, s^i is expressed as

$$s_i = C\Phi^2 \left(1, -\frac{\partial h}{\partial \theta}, -\frac{\partial h}{\partial \varphi} \right), \quad (26)$$

where C is a normalization factor determined from $s^i s_i = 1$. Substituting Eq.(26) into Eq.(25), the following equation is derived;

$$\begin{aligned} \frac{\partial^2 h}{\partial \theta^2} + \cot \theta \frac{\partial h}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 h}{\partial \varphi^2} - 2h &= \frac{1}{C} \left(\frac{\partial C}{\partial r} h^2 - \frac{\partial C}{\partial \theta} \frac{\partial h}{\partial \theta} - \frac{e^\eta}{\sin^2 \theta} \frac{\partial C}{\partial \varphi} \frac{\partial h}{\partial \varphi} \right) \\ &+ \frac{4}{\Phi} \left(\frac{\partial \Phi}{\partial r} h^2 - \frac{\partial \Phi}{\partial \theta} \frac{\partial h}{\partial \theta} - \frac{e^\eta}{\sin^2 \theta} \frac{\partial \Phi}{\partial \varphi} \frac{\partial h}{\partial \varphi} \right) - \frac{e^\eta - 1}{\sin^2 \theta} \frac{\partial^2 h}{\partial \varphi^2} - \frac{e^\eta}{\sin^2 \theta} \frac{\partial \eta}{\partial \varphi} \frac{\partial h}{\partial \varphi} \end{aligned} \quad (27)$$

for the B type metric, and

$$\begin{aligned} \frac{\partial^2 h}{\partial \theta^2} + \cot \theta \frac{\partial h}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 h}{\partial \varphi^2} &= \frac{1}{C} \left(h^2 e^\eta \frac{\partial C}{\partial r} - \frac{\partial C}{\partial \theta} \frac{\partial h}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial C}{\partial \varphi} \frac{\partial h}{\partial \varphi} \right) \\ &+ \frac{4}{\Phi} \left(e^\eta h^2 \frac{\partial \Phi}{\partial r} - \frac{\partial \Phi}{\partial \theta} \frac{\partial h}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial \Phi}{\partial \varphi} \frac{\partial h}{\partial \varphi} \right) + e^\eta \left(\frac{\partial \eta}{\partial r} h^2 + 2h \right), \end{aligned} \quad (28)$$

for the S type metric. In each case, the area of the AH is calculated as

$$A_B = 8 \int_0^{\pi/2} d\varphi \int_0^{\pi/2} d\theta \Phi^4 h e^{\eta/2} \sqrt{\left\{ h^2 + \left(\frac{\partial h}{\partial \theta} \right)^2 \right\} \sin^2 \theta + e^\eta \left(\frac{\partial h}{\partial \varphi} \right)^2}, \quad (29)$$

for the B type metric case, and

$$A_S = 8 \int_0^{\pi/2} d\varphi \int_0^{\pi/2} d\theta \Phi^4 h e^{\eta/2} \sqrt{\left\{ e^\eta h^2 + \left(\frac{\partial h}{\partial \theta} \right)^2 \right\} \sin^2 \theta + \left(\frac{\partial h}{\partial \varphi} \right)^2}, \quad (30)$$

for the S type metric case. In the following section, we will determine the AH by using technique developed in a previous paper[9].

3 Numerical results

We have calculated Φ for several configurations of η both for the B and S type metrics. For the B type metric case, we choose η as

$$\eta = B_m e^{-r^2/2} (r \sin \theta)^{m+2} \cos(m\varphi) \quad m = 0, 2, \text{ and } 4. \quad (31)$$

Thus, in the case $m = 0$, it is the same form as that adopted by Miyama[3] in his axisymmetric calculation. We compare the results obtained in this paper with his to check the present calculation.

For the S type metric case, we first determine η by setting $Q = -Q_0 e^{-r^2/2}/2$ ($Q_0 \ll 1$) and $\ell = 2$ in Eq.(21) to check the numerical accuracy. In this case, ϕ_2 and η_2 become

$$\phi_2 = \frac{Q_0}{4} r^2 (-r^6 + 19r^4 - 78r^2 + 42) e^{-r^2/2}, \quad (32)$$

$$\eta_2 = Q_0 r^4 (r^4 - 18r^2 + 63) e^{-r^2/2}. \quad (33)$$

In fig.1, we show the numerical and analytical solutions of ϕ for the case $Q_0 = 10^{-3}$ in the equatorial plane (for $\varphi = 0$ and $\pi/2$). The solid and dotted lines are numerical and analytic solutions, respectively. We see that two solutions agree very well.

After checking the numerical code, we simply choose η as

$$\eta = S_m r^2 e^{-r^2/2} (r \sin \theta)^m \cos(m\varphi) \quad m = 2 \text{ and } 4. \quad (34)$$

In fig. 2, we show the relation between the amplitude of gravitational waves (B_m or S_m) and the gravitational mass of the system. The horizontal axis denotes amplitude of gravitational waves and the vertical axis denotes the gravitational mass. For the low amplitude cases, the mass is proportional to the square of the amplitude exactly. As Eppley pointed out in the axisymmetric case[2], for sufficiently high amplitude, regular solutions do not exist because a point where $\Phi = 0$ appears. Fig. 2 shows that except for the scaling of the amplitude of η , the relation between the amplitude and gravitational mass is very similar each other irrespective of the metric form (see B_2 and S_2 or B_4 and S_4).

For the high amplitude cases, we apply the AH finder to investigate whether the AH exists or not. Since the configuration of gravitational waves adopted in this paper is not so nonspherical[10], we can find the AH for all cases: We find that the AH appears when $B_0 \geq 12$ for the B type metric of $m = 0$, $B_2 \geq 3.0$ for the B type metric of $m = 2$, $B_4 \geq 0.57$ for the B type metric of $m = 4$, and $S_2 \geq 3.2$ for the S type metric of $m = 2$, and $S_4 \geq 0.45$ for the S type metric of $m = 4$. Note that as for the B type metric of $m = 0$, Miyama also determined the critical amplitude for the appearance of the AH in his axisymmetric calculation, and the present result agrees with his.

To investigate a criterion of the appearance of the AH, we plot M_r as a function of the coordinate radius r for high amplitude gravitational waves in figs.3(a-e). In figs.3(a-c), we show not only M_r (dotted-dashed line), but also dM_r/dr (solid line) for the B type metric of $m = 0$ (a), $m = 2$ (b) and $m = 4$ (c), respectively. In figs.3(d) and (e), we show M_r for the S type metric of $m = 2$ (d) and $m = 4$ (e), respectively. In all figures, dotted line denotes $M_r = 2r$. These figures show that the size of extension for gravitational waves r_w is ~ 4 irrespective of the configuration and amplitude of gravitational waves. For the case when the AH appears, r_w satisfies the condition $M/2r_w > 0.9 - 1.2$. Since the radius of the event horizon of the Schwarzschild black hole in the isotropic coordinate is $M/2$, the present results seem quite reasonable.

Finally, we discuss the resolution required to simulate the dynamical collapse of gravitational waves to be a black hole. The typical scale of variation of the metric is

$$\sim \left(\exp(-\eta) \frac{\partial \exp(\eta)}{\partial r} \right)^{-1} = \left(\frac{\partial \eta}{\partial r} \right)^{-1} \equiv L. \quad (35)$$

Present numerical results show that the minimum value of L is $\sim 0.01 - 0.03M$ when the AH encompasses the wave packet of gravitational waves. From these results, we may

conclude that the typical scale of variation of the metric is $\sim$ a few $\times 0.01M$. In 3D numerical relativity, we will use the Cartesian coordinate to evolve the geometry, so that the grid size must be less than $\sim 0.01M$ in each Cartesian coordinate in order to simulate the collapse of gravitational waves to be a black hole accurately.

4 Summary

In this paper, we have computed time symmetric initial data sets of pure gravitational waves. We adopt two types of simple 3D line elements to describe such initial data. One is the Brill wave type metric and the other is that the line element of the unit sphere is conformal flat ($\gamma_{\theta\theta} = \gamma_{\varphi\varphi} / \sin^2 \theta$). As a result, the equations for the conformal factor reduce to simple elliptic-type equations, and by solving these equations, we can prepare arbitrary initial data sets of gravitational waves even when their amplitude is high enough to appear the AH encompassing them.

From numerical analysis, it is found that for initial data considered in this paper, a criterion of the formation of the AH is $M/2r_w > 0.9 - 1.2$, where r_w is the size of extension for gravitational waves defined in section II-A. For the initial data in which the AH appears, the minimum scale of variation of the metric is a few $\times 0.01M$ irrespective of configuration of gravitational waves. This means that we will have to adopt a grid size less than $\sim 0.01M$ in numerical simulation of collapse of gravitational waves to be a black hole.

Acknowledgments

Numerical computations was in part performed on FACOM VPP-300 in data processing center of National Astronomical Observatory in Japan. This work was in part supported by a Grant-in-Aid for Creative Basic Research of Ministry of Education, Culture, Science and Sports, No. 08NP0801.

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Figure captions

Fig.1 Comparison of numerical(solid lines) and analytical(dotted lines) solutions of $\Phi - 1$ in the case of linear perturbation for the S type metric.

Fig.2 Gravitational mass as a function of the amplitude of gravitational waves.

Figs.3 M_r as a function of r In figs.3(a), (b) and (c), we show M_r (dotted-dashed line) as well as dM_r/dr for B type metric of $m = 0, 2$ and 4 . In figs.3(d) and (e), we show M_r for S type metric of $m = 2$ and 4 . Dotted line denotes $M_r = 2r$ in all figures.

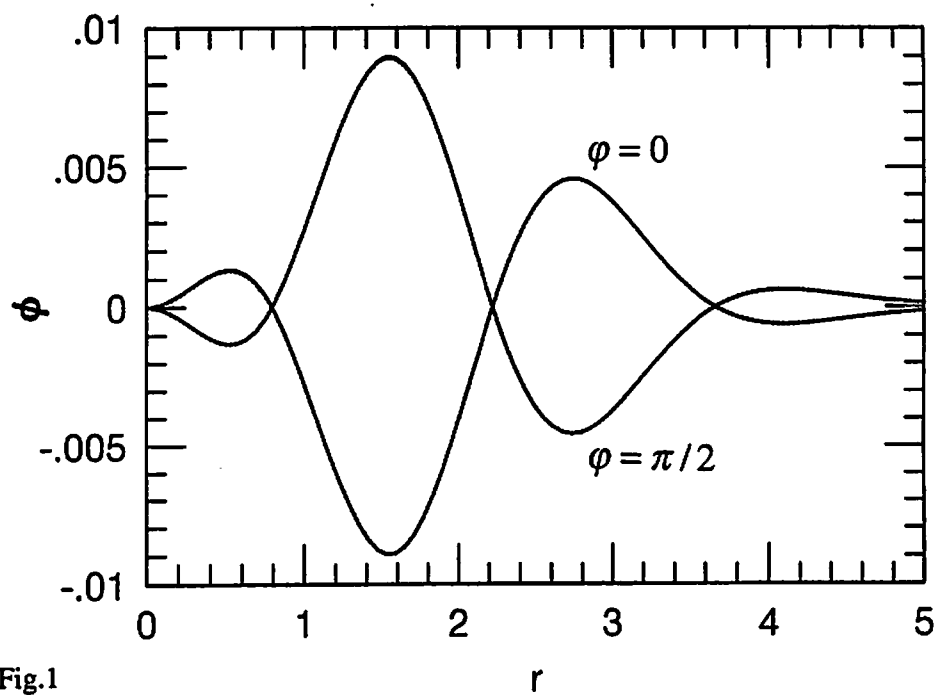


Fig.1

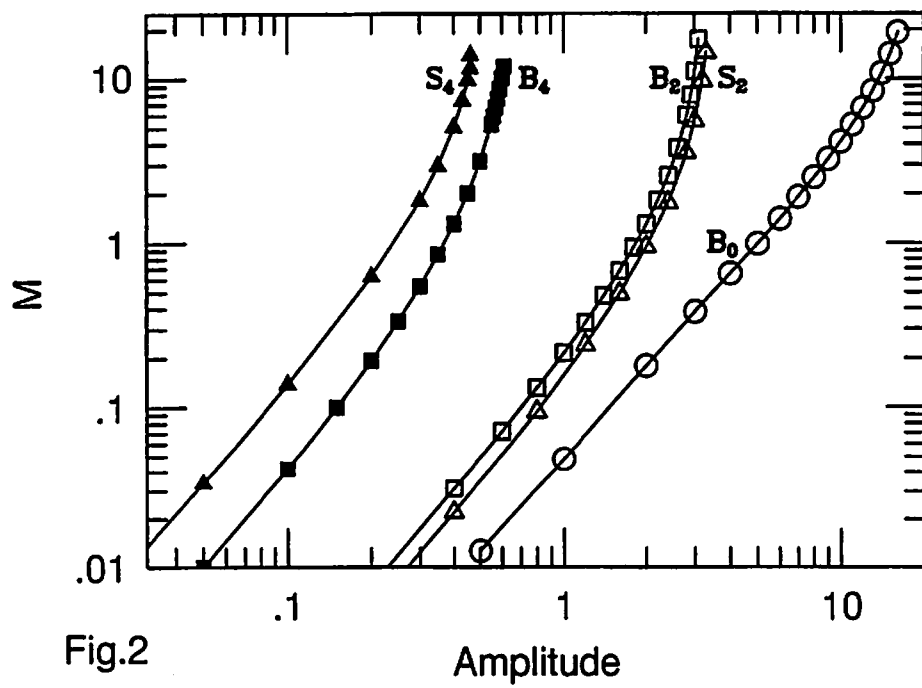


Fig.2

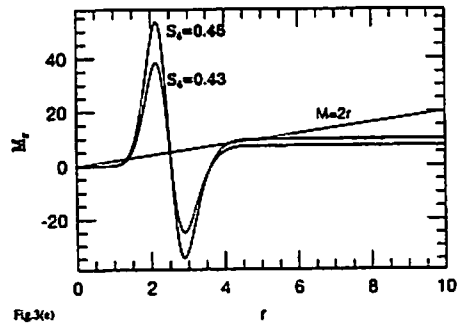
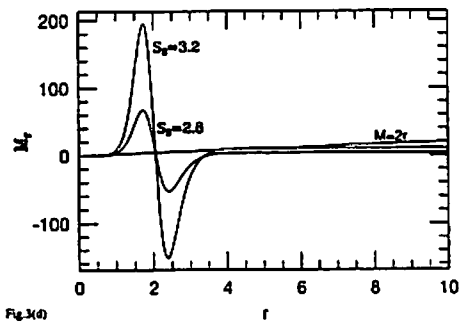
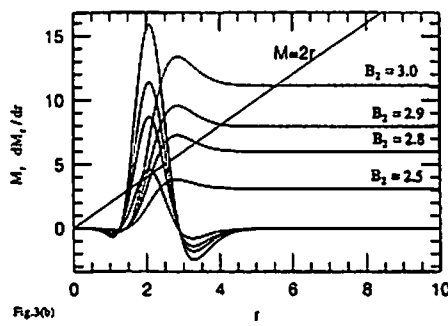
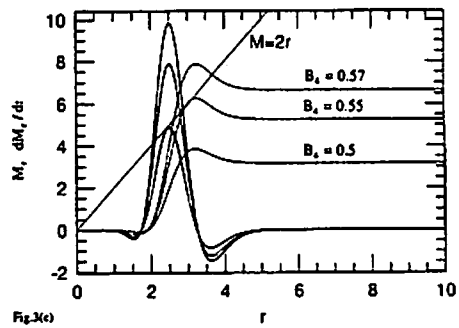
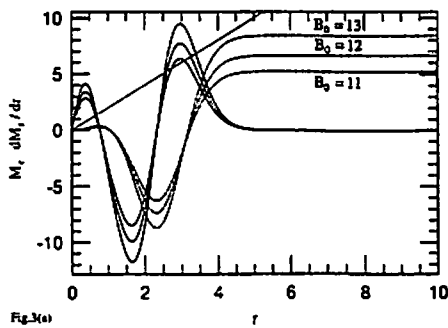


Fig.3

Post-Newtonian expansion of gravitational waves from a particle in circular orbits around a rotating black hole ¹

– Effect of black hole absorption –

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Abstract

When a particle moves around a Kerr black hole, it radiates gravitational waves. Some of those waves are absorbed by the black hole. We calculate such absorption of gravitational waves induced by a particle of mass μ in circular orbit on a equatorial plane around a Kerr black hole of Mass M . We assume that the velocity of the particle is much smaller than the speed of light and calculate the gravitational waves using a post-Newtonian expansion technique of the Teukolsky equation. We adopt a analytic technique for the Teukolsky equation found by Mano, Suzuki, and Takasugi. Using the formulas for the energy flux into a black hole, investigate the effect of the absorption on the orbital evolution of coalescing compact binaries.

1 Introduction

Among the possible sources of gravitational waves, coalescing compact binaries are the most promising candidates for detection by near-future, ground based laser interferometric detectors such as LIGO, VIRGO, GEO600, and TAMA. When a signal of gravitational waves is detected, we will try to extract parameters for binaries, such as masses and spins etc., from inspiral wave forms using the matched filtering technique. In this method, parameters for binaries are determined by cross-correlating the noisy signal from detectors with theoretical templates. If the signal and the templates lose phase with each other by one cycle over $\sim 10^3 - 10^4$ cycles as the waves sweep through the interferometer band, their cross correlation will be significantly reduced. This means that, in order to extract the information optimally, we need to make theoretical templates which are accurate to better than one cycle during an entire sweep through the interferometer's band. [1]

The standard method to calculate inspiraling wave forms from coalescing binaries is the post-Newtonian expansion of the Einstein equations, in which the orbital velocity v of the binaries are assumed to be small compared to the speed of light. Since, for coalescing binaries, the orbital velocity is not so small when the frequency of gravitational waves is in LIGO/VIRGO band, it is necessary to carry the post-Newtonian expansion up to

¹This work was based on a collaboration with S. Mano, and E. Takasugi.

extremely high order in v . A post-Newtonian wave generation formalism which can handle the high order calculation has been developed by Blanchet, Damour and Iyer[2, 3]. Based on this formalism, calculations have been carried out up to post^{5/2}-Newtonian order, or $O(v^5)$ beyond the leading order quadrupole formula[4]. Another formalism is also developed up to $O(v^4)$ by Will and Wiseman[5] which is based on Epstein-Wagoner's formalism[6].

Another post-Newtonian expansion techniques based on black hole perturbation formalism is developed. In this analysis, one considers gravitational waves from a particle of mass μ orbiting a black hole of mass M when $\mu \ll M$. Although this method is restricted to the case when $\mu \ll M$, one can calculate very high order post-Newtonian corrections to gravitational waves using a relatively simple algorithm in contrast with the standard post-Newtonian analysis. The gravitational radiation at infinity were calculated in these method to $O(v^8)$ by Tagoshi, Shibata, Tanaka, and Sasaki [7] in a Kerr black hole case, and to $O(v^{11})$ by Tanaka, Tagoshi, and Sasaki [8] in a Schwarzschild black hole case.

On the other hand, a study of gravitational waves which are absorbed by a black hole is developed very little in the context of the post-Newtonian approximation. The absorption of gravitational waves into the black hole horizon was first calculated by Poisson and Sasaki in the case when a test particle is in a circular orbit around a Schwarzschild black hole[9]. In this case, since the effect of the black hole absorption appears at $O(v^8)$, this effect is negligible for the orbital evolution of coalescing compact binaries in the above laser interferometer's band. However, as we shall see in this paper, the black hole absorption appears at $O(v^5)$ when a black hole is rotating. Then it is important to investigate the effect of black hole absorption to the orbital evolution of coalescing compact binaries when at least one of the stars is a rotating black hole.

In order to calculate the post-Newtonian expansion of ingoing gravitational waves into a Schwarzschild black hole, Poisson and Sasaki[9] used two type of representations of solutions of the homogeneous Teukolsky equation. One is expressed in terms of the spherical Bessel functions which can be used at large radius, and the other is expressed in terms of a hypergeometric function which can be used near the horizon. Then two type of expressions are matched at some region where both formulas can be applied. They got formulas of a solution of the Teukolsky equation which can be used to calculate ingoing gravitational waves to $O(v^{13})$, although they gave formulas for ingoing waves only to $O(v^8)$.

However, it is difficult to extend their method to higher order. It is also difficult to apply their method to a rotating black hole case. On the other hand, another analytic techniques for the Teukolsky equation was found by Mano, Suzuki, and Takasugi[10]. Since this method is very powerful for the calculation of the post-Newtonian expansion of Teukolsky equation, we adopt this method in this paper.

2 The Teukolsky formalism

In the Teukolsky formalism, gravitational perturbations of the Kerr black hole are described by a function $\psi_4 = C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$. $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor, $n^\alpha = ((r^2 + a^2), -\Delta, 0, a)/(2\Sigma)$, $m^\alpha = (ia \sin \theta, 0, 1, i/\sin \theta)/(\sqrt{2}(r + ia \cos \theta))$. We use the Boyer-

Lindquist coordinate (t, r, θ, ϕ) in this paper and $\Sigma = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 - 2Mr + a^2$, and a is a spin parameter of a Kerr black hole. We decompose ψ_4 into Fourier-harmonic components according to

$$(r - ia \cos \theta)^4 \psi_4 = \sum_{\ell m} e^{-i\omega t + im\varphi} {}_{-2}S_{\ell m}(\theta) R_{\ell m \omega}(r), \quad (1)$$

where ${}_2S_{\ell m}(\theta)$ are spin-weighted spheroidal harmonics. The radial function $R_{\ell m \omega}$ satisfies the Teukolsky equation

$$\Delta^2 \frac{d}{dr} \left(\frac{1}{\Delta} \frac{dR_{\ell m \omega}}{dr} \right) - V(r) R_{\ell m \omega} = T_{\ell m \omega}, \quad (2)$$

The potential $V(r)$ is given by

$$V(r) = -\frac{K^2 + 4i(r - M)K}{\Delta} + 8i\omega r + \lambda, \quad (3)$$

where $K = (r^2 + a^2)\omega - ma$ and λ is the eigenvalue of ${}_2S_{\ell m}^{\omega}$. The source term $T_{\ell m \omega}$ is given in refs. [7].

We define two kind of homogeneous solutions of the Teukolsky equation.

$$R_{\ell m \omega}^{\text{in}} \rightarrow \begin{cases} C_{\ell m \omega} \Delta^2 e^{-ikr^*} & \text{for } r \rightarrow r_+ \\ r^3 A_{\ell m \omega}^{\text{out}} e^{i\omega r^*} + r^{-1} A_{\ell m \omega}^{\text{in}} e^{-i\omega r^*} & \text{for } r \rightarrow +\infty, \end{cases} \quad (4)$$

$$R_{\ell m \omega}^{\text{up}} \rightarrow \begin{cases} B_{\ell m \omega}^{\text{out}} e^{ikr^*} + \Delta^2 B_{\ell m \omega}^{\text{in}} e^{-ikr^*} & \text{for } r \rightarrow r_+, \\ D_{\ell m \omega} r^3 e^{i\omega r^*} & \text{for } r \rightarrow +\infty \end{cases} \quad (5)$$

where $k = \omega - ma/2Mr_+$ and r^* is the tortoise coordinate defined by

$$\begin{aligned} r^* &= \int \frac{dr^*}{dr} dr \\ &= r + \frac{2Mr_+}{r_+ - r_-} \ln \frac{r - r_+}{2M} - \frac{2Mr_-}{r_+ - r_-} \ln \frac{r - r_-}{2M}, \end{aligned} \quad (6)$$

where $r_{\pm} = M \pm \sqrt{M^2 - a^2}$.

We solve the Teukolsky equation by using the Green function method. A solution of the Teukolsky equation which has purely outgoing property at infinity and has purely ingoing property at the horizon is given by

$$R_{\ell m \omega} = \frac{1}{W} \left\{ R^{\text{up}} \int_{r_+}^r dr' R^{\text{in}} T_{\ell m \omega} \Delta^{-2} + R^{\text{in}} \int_r^{\infty} dr' R^{\text{up}} T_{\ell m \omega} \Delta^{-2} \right\}, \quad (7)$$

where the Wronskian W is given by

$$W = 2i\omega D_{\ell m \omega} A_{\ell m \omega}^{\text{in}}. \quad (8)$$

Then, ψ_4 has an asymptotic property at the horizon as

$$R_{\ell m \omega}(r \rightarrow r_+) \rightarrow Z_{\text{hole}} \Delta^2 e^{-ikr^*}, \quad (9)$$

where

$$Z_{\text{hole}} = \frac{C_{\ell m \omega}}{2i\omega D_{\ell m \omega} A_{\ell m \omega}^{\text{in}}} \int_{r_+}^{\infty} dr' R^{\text{up}} T_{\ell m \omega} \Delta^{-2}. \quad (10)$$

The energy flux formulas of ingoing wave into a Kerr black hole is given by (Teukolsky and Press, *Astrophys. J.* **193**, 443 (1974))

$$\left(\frac{dE_{\text{hole}}}{dt d\Omega} \right) = \sum_{\ell m} \int d\omega \frac{2S_{\ell m}^2}{2\pi} \frac{128\omega k(k^2 + 4\varepsilon^2)(k^2 + 16\varepsilon^2)(2Mr_+)^5}{|C|^2} |Z_{\text{hole}}|^2, \quad (11)$$

where

$$|C|^2 = (Q + 4a\omega m - 4a^2\omega^2)[(Q - 2)^2 + 36a\omega m - 36a^2\omega^2] + (2Q - 1)(96a^2\omega^2 - 48a\omega m) + 144\omega^2(M^2 - a^2). \quad (12)$$

The post-Newtonian expansion of these formulas can be done by solving the Teukolsky equation by setting $\varepsilon \equiv 2M\omega \sim O(v^3)$, $z = \omega r \sim O(v)$ and by assuming $\varepsilon \ll z \ll 1$. Here, we define $v \equiv (M/r_0)^{1/2}$ and r_0 is the orbital radius of the test particle. In the next section, we explain how to perform the post-Newtonian expansion.

3 Analytic solutions of the homogeneous Teukolsky equation and the post-Newtonian expansion

New analytic representations of the homogeneous Teukolsky equation were found by Mano, Suzuki and Takasugi [10]. Since all the details are written in that paper, we only describe the outline of the method. In this method, two kind of expansion are used. One is a series of hypergeometric functions which is defined as

$$R_0^\nu = e^{i\varepsilon\kappa z} (-x)^{\nu-s-\frac{1}{2}(\varepsilon-\tau)} (1-x)^{\frac{1}{2}(\varepsilon-\tau)} \sum_{n=-\infty}^{\infty} a_n^\nu \frac{\Gamma(1-s-i\varepsilon-i\tau)\Gamma(2n+2\nu+1)}{\Gamma(n+\nu+1-i\tau)\Gamma(n+\nu+1-s-i\varepsilon)} \\ \times (-x)^n F(-n-\nu-i\tau, -n-\nu+s+i\varepsilon; -2n-2\nu; \frac{1}{x}), \quad (13)$$

where

$$x = -\frac{\omega}{\varepsilon\kappa}(r-r_+), \quad \kappa = \sqrt{1-q^2}, \quad q = \frac{a}{M}, \quad \tau = \frac{\varepsilon-mq}{\kappa}. \quad (14)$$

The coefficient a_n^ν obey a three terms recurrence relation

$$\alpha_n^\nu a_{n+1}^\nu + \beta_n^\nu a_n^\nu + \gamma_n^\nu a_{n-1}^\nu = 0. \quad (15)$$

The expansion Eq.(13) converge if $r < \infty$.

The homogeneous solution R^{in} is given by

$$R^{\text{in}} = R_0^\nu + R_0^{-\nu-1}.$$

The other solution is a series of Coulomb wave functions. defined as [12]

$$R_C^\nu = z^{-1-s} \left(1 + \frac{\varepsilon\kappa}{z}\right)^{\frac{1}{2}(\varepsilon-\tau)} \sum_{n=-\infty}^{\infty} b_n^\nu F_{n+\nu}(z), \quad (16)$$

where $F_{n+\nu}$ is the Coulomb wave function,

$$F_{n+\nu} = e^{-iz} (2z)^{n+\nu} z \frac{\Gamma(n+\nu+1-s+i\varepsilon)}{\Gamma(2n+2\nu+2)} \Phi(n+\nu+1-s+i\varepsilon, 2n+2\nu+2; 2iz),$$

and $z = \omega(r - r_+)$. The coefficients b_n^ν obey the same three terms recurrence relation as Eq.(15) if we redefine b_n^ν as

$$b_n^\nu = i^n \frac{\Gamma(\nu+1-s+i\varepsilon)\Gamma(\nu+1-s-i\varepsilon)}{\Gamma(\nu+1+s+i\varepsilon)\Gamma(\nu+1+s-i\varepsilon)} \times \frac{\Gamma(n+\nu+1+s+i\varepsilon)\Gamma(n+\nu+1+s-i\varepsilon)}{\Gamma(n+\nu+1-s+i\varepsilon)\Gamma(n+\nu+1-s-i\varepsilon)} a_n^\nu. \quad (17)$$

The expansion Eq.(16) converges if $z > \varepsilon\kappa$. The upgoing solution $R_{\ell m \omega}^{\text{up}}$ is given by

$$R_{\text{up}}^\nu = \left[\frac{\sin \pi(\nu-s+i\varepsilon)}{\sin \pi(\nu+s-i\varepsilon)} R_C^\nu - i e^{i\pi\nu} R_C^{\nu-1} \right] \times \left[e^{2i\pi\nu} + \frac{\sin \pi(\nu-s+i\varepsilon)}{\sin \pi(\nu+s-i\varepsilon)} \right]^{-1}. \quad (18)$$

The relation of above two solutions are given by

$$R_0^\nu = K_\nu R_C^\nu,$$

where

$$K_\nu = \frac{(\varepsilon\kappa)^{-\nu-r+s} 2^{-\nu-r} (-i)^r \Gamma(1-s-i\varepsilon-i\tau)}{\Gamma(1+r+\nu+i\tau)\Gamma(1+r+\nu-s-i\varepsilon)\Gamma(1+r+\nu-s+i\varepsilon)} \times \sum_{n=r}^{\infty} \frac{\Gamma(n+\nu+1+i\tau)\Gamma(n+r+2\nu+1)}{(n-r)!\Gamma(n+\nu+1-i\tau)} a_n^\nu \times \left[\sum_{n=-\infty}^r (-i)^n \frac{b_n^\nu}{(r-n)!\Gamma(n+r+2\nu+2)} \right]^{-1}, \quad (19)$$

where r is an arbitrary integer. This relation holds where both expansion Eq.(13) and Eq.(16) converge.

Using this relation, it's possible to obtain the amplitude of $R_{\ell m \omega}$ at infinity. They are given by

$$A^{\text{out}} = \omega^3 e^{i\varepsilon \ln \varepsilon} e^{-\frac{1}{2}\pi(\nu+1+s-i\varepsilon)} 2^{-1-s+i\varepsilon} (K_\nu + i e^{i\pi\nu} K_{-\nu-1}) \sum_{n=-\infty}^{\infty} b_n^\nu (-i)^n, \quad (20)$$

and

$$A^{\text{in}} = \omega^{-1} e^{-\varepsilon \ln \varepsilon} e^{-\frac{1}{2}\pi(-\nu-1+s-i\varepsilon)} 2^{-1+s-i\varepsilon} \left(K_\nu - i e^{-i\pi\nu} \frac{\sin \pi(\nu-s+i\varepsilon)}{\sin \pi(\nu+s-i\varepsilon)} K_{-\nu-1} \right) \times \sum_{n=-\infty}^{\infty} b_n^\nu i^n \frac{\Gamma(n+\nu+1-s+i\varepsilon)}{\Gamma(n+\nu+1+s-i\varepsilon)}. \quad (21)$$

Finally, we explain the relation of the method above to the post-Newtonian expansion. It is easy to see that if we set $a_0'' = 1$ at $n = 0$, the order of a_n in ε increase as $|n|$ increase. Further, from the behavior of each term of the expansion Eq.(16) as a function of z , we see that the post-Newtonian order of the expansion of Eq.(16) increase. This can be checked by setting $\varepsilon \sim v$ and $z \sim v$ where $v = (M/r_0)^{1/2}$ and r_0 is the orbital radius of the test particle. Then, the above expansion is very useful in the post-Newtonian expansion. (Note, however, that those increase of the post-Newtonian order are not monotonous for $n < 0$. But this fact doesn't change the usefulness of the method for the post-Newtonian expansion.)

4 Results and Conclusion

We evaluate Eq.(11) using a method explained in the last section

and assuming $\varepsilon \ll z \ll 1$. All the detailed calculation are presented elsewhere [13]. In order to express the post-Newtonian corrections to the luminosity, we define $\eta_{\ell m}^H$ as

$$\left(\frac{dE}{dt}\right)_{\ell m} \equiv \frac{1}{2} \left(\frac{dE}{dt}\right)_N \eta_{\ell m}^H, \quad (22)$$

where $(dE/dt)_N$ is the Newtonian quadrupole luminosity at infinity:

$$\left(\frac{dE}{dt}\right)_N = \frac{32\mu^2 M^3}{5r_0^5} = \frac{32}{5} \left(\frac{\mu}{M}\right)^2 v^{10},$$

and $(dE/dt)_{\ell m}$ is the luminosity of each mode. We only show $\eta_{\ell m}$ for $m > 0$ mode since $\eta_{\ell, m} = \eta_{\ell, -m}$. The luminosity for each mode to the horizon become to $O(v^9)$ as

$$\begin{aligned} \eta_{2,2}^H &= \left(\frac{-q}{4} - \frac{3q^3}{4}\right) v^5 + \left(\frac{-3q}{4} - \frac{9q^3}{4}\right) v^7 \\ &+ \left(\frac{1}{2} + 2\alpha(q)q + \frac{43q^2}{36} + 6\alpha(q)q^3 - \frac{155q^4}{12} + \frac{\sqrt{1-q^2}}{2} + \frac{13q^2\sqrt{1-q^2}}{2} + 3q^4\sqrt{1-q^2}\right) v^8 \\ &+ \left(\frac{-199q}{42} - \frac{593q^3}{42} + \frac{2q^5}{7}\right) v^9, \\ \eta_{2,1}^H &= \frac{-q(4-3q^2)v^7}{16} + \frac{q^2(4-3q^2)v^8}{12} \\ &- \frac{q(4-3q^2)(12+q^2)v^9}{36}, \\ \eta_{3,3}^H &= \frac{-15q(4+5q^2)(1+8q^2)v^9}{896}, \\ \eta_{3,1}^H &= \frac{-q(4-3q^2)(9-8q^2)v^9}{8064}, \end{aligned}$$

where $\alpha(q) = \text{Im} \left[\psi^{(0)} \left(3 + \frac{2iq}{\sqrt{1-q^2}} \right) \right]$ and $\psi^{(n)}(z)$ is polygamma functions.

q	$(1.4M_{\odot}, 10M_{\odot})$	$(1.4M_{\odot}, 40M_{\odot})$	$(1.4M_{\odot}, 70M_{\odot})$
0	$< 10^{-2}$	0.02	0.1
0.9	0.7	1.7	2.4
-0.9	0.3	0.45	0.39

Table 1: The difference of the total cycle N caused by the formulas for dE/dt with and without the black hole absorption effect for typical neutron star(NS)-black hole(BH) binaries with mass $(M_{\text{NS}}, M_{\text{BH}})$.

The total luminosity is given by

$$\begin{aligned} \left\langle \frac{dE}{dt} \right\rangle = & \left(\frac{dE}{dt} \right)_N \left[\left(\frac{-q}{4} - \frac{3q^3}{4} \right) v^5 + \left(-q - \frac{33q^3}{16} \right) v^7 \right. \\ & + \left(\frac{1}{2} + 2\alpha(q)q + \frac{55q^2}{36} + 6\alpha(q)q^3 - \frac{79q^4}{6} + \right. \\ & \left. \frac{\sqrt{1-q^2}}{2} + \frac{13q^2\sqrt{1-q^2}}{2} + 3q^4\sqrt{1-q^2} \right) v^8 \\ & \left. + \left(\frac{-43q}{7} - \frac{4651q^3}{336} - \frac{17q^5}{56} \right) v^9 \right], \end{aligned}$$

If we set $q = 0$ above formula reduced to

$$\left\langle \frac{dE}{dt} \right\rangle = \left(\frac{dE}{dt} \right)_N v^8, \quad (23)$$

which was derived by Poisson and Sasaki[9]. We see that the absorption effect is more important in the case $q \neq 0$.

Using the above results, we estimate the effect of the black hole absorption to the orbital evolution of the coalescing compact binaries. We ignore the finite mass effect in the post-Newtonian formulas and interpret M as the total mass and μ as the reduced mass of the system.

The total cycle N of gravitational waves from an inspiraling binary is given by

$$N = \int_{v_f}^{v_i} dv \frac{\Omega}{\pi} \frac{dE/dv}{|dE/dt|}, \quad (24)$$

where $v_i = (M/r_i)^{1/2}$, $v_f = (M/r_f)^{1/2}$, and r_i and r_f are the initial and final orbital separation of the binary. We adopt $r_f = 6M$ and r_i is the one at which the frequency of wave is 10Hz. We evaluate the difference of the total cycle N caused by the formulas for dE/dt with and without the black hole absorption effect. The results for the typical NS-BH systems are given in Table 1.

We find that the black hole absorption is more important in cases $q > 0$. We also find that, in cases $q > 0$, the black hole absorption is more important when the mass of the black hole is large.

All the details of this paper will be presented elsewhere [13].

Acknowledgments

The author thanks S.Mano, E.Takasugi for the collaboration, and thanks M.Sasaki for discussion. The author was supported by Research Fellowships of the Japan Society for the Promotion of the Science for Young Scientists.

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Many-body system of spinning particles

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Abstract

We propose a new approach to the effect of spin on motion of many-body system and we calculate the post-Newtonian spin effect of two-body system by using the energy-momentum tensor of spinning particle in context of post-Newtonian scheme.

1 Introduction

At present, in order to detect and observe gravitational waves it starts to construct laser interferometers at various place. The most promising candidates of the targets of the observation of these laser interferometer are consider to be coalescing compact binaries. For observation of the gravitational waves from these system, in advance we must know templates of gravitational waves that would be detected. For this purpose, many author calculate the wave form and the energy transportation by the gravitational waves from binary compact objects by each method. These gravitational waves from the binaries bring us many information of the system, for example, the equation of states of the neutron stars, structure, spin, and mass ratio *etc.* Therefore, the study of the effect by them is a important problem not only theoretical but experimentally.

By the way, it is also a important problem how extend bodies move in general theory of gravitation as well as in the Newton mechanics. In other words, the problem is how neutron stars or black holes move on the background space-time. In general relativity, motion of objects with strong self gravity is not clear because of non-linearity of gravity itself. This influence is more important in the observation of the gravitational waves from compact binaries for the motion may be affected by self-gravity and the released gravitational waves also may be affected by them. In this article, we consider only spin effect among the constructure of extend bodies we study the effect to the motion of many-body system consisted of spinning particles.

In this article, the method that we use is to solve the Einstein equation which has the energy-momentum tensor as its source by the post-Newtonian scheme. This method is the extension of so-called EIH scheme which was used to solve motion of many body system of point particles in general relativity. Bu using this method, it becomes easier to study the influence on motion of the binaries by their spin. Here, we derive the spin-spin coupling force and the spin-orbit coupling force at leading order by this scheme. These force has been derived already by many authors, however the derivation of them by this method shows the advantage of our method.

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2 Post-Newtonian approximation

We use the post-Newtonian approximation to derive the spin-spin interaction and the spin-orbit interaction. In this approximation, we expand the Einstein equation in term of $1/c$, and we assume that the gravitational field is enough weak. In the Einstein equation.

$$R_{\mu\nu} = \frac{8\pi G}{c^4} \left(g_{\mu\rho} g_{\nu\sigma} - \frac{1}{2} g_{\mu\nu} g_{\rho\sigma} \right) T^{\rho\sigma}, \quad (1)$$

we put that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (2)$$

where $\eta_{\mu\nu}$ is Lorentz metric, and we get the following equations,

$$2^{(1)}R_{00} = -h_{00,ii} - h_{ii,00} + 2h_{0i,0i} \quad (3)$$

$$\begin{aligned} 2^{(2)}R_{00} = & h_{ij}(h_{00,ij} + h_{ij,00} - 2h_{0i,0j}) \\ & - \frac{1}{2}h_{00,i}(h_{00,i} + h_{jj,i} - 2h_{ij,j}) - \frac{1}{2}h_{00,0}(h_{ii,0} - 2h_{0i,i}) \\ & - h_{0i,0}(2h_{ij,i} - h_{jj,i}) + \frac{1}{2}h_{ij,0}h_{ij,0} + \frac{1}{2}(h_{0i,j} - h_{0j,i})(h_{0i,j} - h_{0j,i}) \end{aligned} \quad (4)$$

$$2^{(1)}R_{0i} = h_{0j,ij} + h_{ij,0j} - h_{0i,jj} - h_{jj,0i} \quad (5)$$

$$\begin{aligned} 2^{(2)}R_{0i} = & h_{0j,j}h_{00,i} - h_{jk,j}(h_{0k,i} + h_{ik,0} - h_{0i,k}) - \frac{1}{2}h_{jj,0}h_{00,j} \\ & + \frac{1}{2}h_{jj,k}(h_{0k,i} + h_{ik,0} - h_{0i,k}) + h_{0j}(h_{ij,00} - h_{0i,0j} - h_{0j,0i} + h_{00,ij}) \\ & - h_{jk}(h_{0j,ik} + h_{ij,0k} - h_{0i,jk} - h_{jk,0i}) + \frac{1}{2}h_{jk,0}h_{jk,i} \\ & + \frac{1}{2}h_{00,j}(h_{ij,0} - h_{0j,i} - h_{0i,j}) + \frac{1}{2}(h_{0k,j} - h_{0j,k})(h_{0k,j} - h_{0j,k}) \end{aligned} \quad (6)$$

$$2^{(1)}R_{ij} = -h_{0i,0j} - h_{0j,0i} + h_{00,ij} + h_{ij,00} + h_{ik,jk} + h_{jk,ik} - h_{ij,kk} - h_{kk,ij} \quad (7)$$

$$\begin{aligned} 2^{(2)}R_{ij} = & \frac{1}{2}(h_{0i,j} + h_{0j,i} - h_{ij,0})(2h_{0k,k} - h_{00,0} - h_{kk,0}) \\ & + \frac{1}{2}(h_{ik,j} + h_{jk,i} - h_{ij,k})(2h_{0k,0} - 2h_{kl,l} - h_{00,k} + h_{ll,k}) \\ & + h_{00}(h_{00,ij} - h_{0i,0j} - h_{0j,0i} + h_{ij,00}) \\ & + h_{0k}(h_{ik,0j} + h_{jk,0i} + h_{0i,jk} + h_{0j,ik} - 2h_{ij,0k} - 2h_{0k,ij}) \\ & + h_{kl}(h_{ij,kl} + h_{kl,ij} - h_{ik,jl} - h_{jk,il}) - 2h_{0k,i}h_{0k,j} + \frac{1}{2}h_{kl,i}h_{kl,j} \\ & + \frac{1}{2}(h_{0k,i} + h_{ik,0} - h_{0i,k})(h_{0k,j} + h_{0j,k} - h_{jk,0}) \\ & + \frac{1}{2}(h_{0i,k} + h_{0k,i} - h_{ik,0})(h_{0k,j} + h_{jk,0} - h_{0j,k}) - h_{ik,l}(h_{jl,k} - h_{jk,l}), \end{aligned} \quad (8)$$

where $^{(1)}R_{\mu\nu}$ and $^{(2)}R_{\mu\nu}$ are the linear part and the quadratic part of $R_{\mu\nu}$ in term of $h_{\mu\nu}$, and any pair of repeated indices are meant to be summed over $i = 1, 2, 3$ etc. We also expand the source term in the field equation, and we have the following,

$$T^{\alpha\beta} = T_{\text{mono}}^{\alpha\beta} + T_{\text{spin}}^{\alpha\beta} \quad (9)$$

$$T_{\text{mono}}^{\alpha\beta} = \int c d\lambda p^{(\alpha} v^{\beta)} \frac{\delta^4(x - z(\lambda))}{\sqrt{-g}} \quad (10)$$

$$T_{\text{spin}}^{\alpha\beta} = - \int c d\lambda \nabla_\gamma \left(S^{\gamma(\alpha} v^{\beta)} \frac{\delta^4(x - z(\lambda))}{\sqrt{-g}} \right) \quad (11)$$

$$T_{\mu\nu} = \left(g_{\mu\alpha} g_{\nu\beta} - \frac{1}{2} g_{\mu\nu} g_{\alpha\beta} \right) T^{\alpha\beta} \quad (12)$$

$${}^{(-2)}T_{\text{mono}}^{00} = \frac{1}{2} m c^2 \delta^3(x - z(z^0)) \quad (13)$$

$${}^{(0)}T_{\text{mono}}^{00} = \frac{1}{2} m c^2 \delta^3(x - z(z^0)) \left\{ \frac{3}{2} \left(\frac{\vec{V}}{c} \right)^2 - h_{00}^{(2)} - \frac{1}{2} h_{ii}^{(2)} \right\} \quad (14)$$

$${}^{(1)}T_{\text{mono}}^{00} = \frac{1}{2} m c^2 \delta^3(x - z(z^0)) \left\{ -h_{00}^{(3)} - \frac{1}{2} h_{ii}^{(3)} - h_{0i}^{(2)} \frac{V^i}{c} \right\} \quad (15)$$

$${}^{(-1)}T_{\text{mono}}^{0i} = - m c \delta^3(x - z(z^0)) V^i \quad (16)$$

$${}^{(0)}T_{\text{mono}}^{0i} = - \frac{1}{2} m c \delta^3(x - z(z^0)) h_{0i}^{(2)} \quad (17)$$

$${}^{(2)}T_{\text{mono}}^{0i} = - \frac{1}{2} m c \delta^3(x - z(z^0)) \left\{ h_{0i}^{(3)} + \left(\frac{\vec{V}}{c} \right)^2 - h_{kk}^{(2)} \frac{V^i}{c} + 2 h_{ij}^{(2)} \frac{V^j}{c} \right\} \quad (18)$$

$${}^{(-2)}T_{\text{mono}}^{ij} = \frac{1}{2} m c^2 \delta^3(x - z(z^0)) \delta_{ij} \quad (19)$$

$${}^{(0)}T_{\text{mono}}^{ij} = \frac{1}{2} m c^2 \delta^3(x - z(z^0)) \delta_{ij} \left\{ -\frac{1}{2} \left(\frac{\vec{V}}{c} \right)^2 \delta_{ij} + 2 \frac{V^i V^j}{c^2} h_{ij}^{(2)} - \frac{1}{2} h_{kk}^{(2)} \delta_{ij} \right\} \quad (20)$$

$${}^{(1)}T_{\text{mono}}^{ij} = \frac{1}{2} m c^2 \delta^3(x - z(z^0)) \left\{ h_{ij}^{(3)} - \frac{1}{2} h_{kk}^{(3)} + 2 h_{0i}^{(2)} \frac{V^j}{c} + 2 h_{0j}^{(2)} \frac{V^i}{c} + \frac{V^i}{c} - h_{0k}^{(2)} \frac{V^k}{c} \delta_{ij} \right\} \quad (21)$$

$${}^{(-1)}T_{\text{spin}}^{00} = m c^2 \left(\frac{\vec{V}}{c} \times \vec{S} \right)^k \frac{\partial}{\partial x^k} \delta^3(x - z(x^0)) \quad (22)$$

$${}^{(0)}T_{\text{spin}}^{00} = m c^2 \epsilon_{ijk} h_{0j}^{(2)} S^k \frac{\partial}{\partial x^i} \delta^3(x - z(x^0)) \quad (23)$$

$${}^{(-2)}T_{\text{spin}}^{0i} = \frac{1}{2} m c^2 \left(\vec{S} \times \vec{\nabla} \delta^3 \right) \quad (24)$$

$${}^{(-1)}T_{\text{spin}}^{ij} = \frac{1}{2} m c^2 \left\{ \epsilon_{kil} S^l V^j + (i \leftrightarrow j) \right\} \frac{\partial}{\partial x^k} \delta^3, \quad (25)$$

where indices (-2) etc. show the power of $1/c$. Here we show only the lowest order terms as to spin part. We can get the general solutions of the field equation from the above equations. In order to derive the spin-spin interaction and the spin-orbit interaction, we must put the solutions of the field equation into the equation of motion. In post-Newtonian approximation, to be able to solve the $N + 1$ -th post-Newtonian field equations, it is necessary to satisfy the N -th post-Newtonian equation of motion. This is equivalent to satisfy the Bianchi identity. Therefore, we use the equation derived from the Bianchi identity as the equation of motion. By changing the time parameter to z^0 from the proper time τ , the equation have the following form,

$$\begin{aligned} \frac{d^2 x^i}{(d x^0)^2} = & - \left(\Gamma^i_{\alpha\beta} - V^i \Gamma^0_{\alpha\beta} \right) V^\alpha V^\beta \\ & - \frac{1}{2} \left(R^i_{\lambda\rho\sigma} - V^i R^0_{\lambda\rho\sigma} \right) V^\lambda \epsilon^{\rho\sigma}_{\alpha\beta} V^\alpha S^\beta. \end{aligned} \quad (26)$$

The S in this equation represents the particle own spin.

From above equations, we obtain the spin-spin coupling force and the spin-orbit coupling force. In two body system, the spin-spin coupling force is

$$F^i_{(ss)} = \frac{G}{c^2} \frac{\partial}{\partial x^i} \left[2 \frac{\vec{S}_A \cdot \vec{S}_B}{r^3} - 3 \frac{(\vec{S}_A \times \vec{r}) \cdot (\vec{S}_B \times \vec{r})}{r^5} \right] \quad (27)$$

where $\vec{r}$ is the vector from the object A to the object B. And the spin-orbit coupling force is

$$\begin{aligned} F^i_{(so)} = & \frac{3Gm}{c^2} \left[\frac{(\vec{V}_A \times \vec{S}_A)^i}{r^3} + 2 \frac{\vec{r} \cdot (\vec{V}_A \times \vec{S}_A)}{r^5} r^i - \frac{(\vec{r} \cdot \vec{V}_A)(\vec{r} \times \vec{S}_A)^i}{r^5} \right] \\ & - \frac{2Gm}{c^2} \left[\frac{(\vec{V}_A \times \vec{S}_B)^i}{r^3} - \frac{(\vec{r} \cdot \vec{V}_A)(\vec{r} \times \vec{S}_B)^i}{r^5} \right] \end{aligned} \quad (28)$$

3 Validity of the point-like source

We use the delta function as the source of the field equations. In general, the real stars (neutron stars, black holes, ordinal stars etc.) must be deformed by the tidal force of the companion objects. This seems to mean that it is not right to treat the objects as point-like particles. However we can show that point-like particles are able to represent the external gravity field of any object.

In the post-Newtonian gravity, the field equations have the following form.

$$\Delta \phi^{(n)} = N(\phi^{(n-1)}) + T, \quad (29)$$

where $N(\phi^{(n-1)})$ represents the nonlinear term of the lower order field, and T is the source term of the objects. Here splitting the $\phi^{(n)}$ into $\phi_Q^{(n)} + \phi_T^{(n)}$ such as

$$\Delta \phi_Q^{(n)} = N(\phi^{(n-1)}), \quad \Delta \phi_T^{(n)} = T. \quad (30)$$

Hereafter we consider the only $\phi_T^{(n)}$, therefore we omit the subscript 'T'.

If T has a compact support, its external field is a vacuum solution, and it can be expanded with spherical harmonics. On the other hand, if T is proportional to the delta function $\frac{\partial^n}{\partial x^{i_1 \dots i_n}} \delta^3$, the solution is

$$\phi = \frac{\partial^n}{\partial x^{i_1 \dots i_n}} \frac{1}{r}. \quad (31)$$

This also is able to be expanded with spherical harmonics. If we could show that the free degrees of spherical harmonics coincide with that of the independent derivative of the delta functions, we could show our insistence. Indeed, we can show it easily by using the STF (Symmetric Trace Free) parts of the 'delta' functions such as

$$\frac{\partial^n}{\partial x^{<i_1 \dots i_n>}} \frac{1}{r}. \quad (32)$$

where $< \dots >$ shows STF part and the definition of the STF part is the perfectly symmetric and that it vanishes by any contraction of two indices. This lemma suggests us that the effect of any deformation of the objects can be introduced by the change of the coefficients of the 'delta' functions, for instance, that by the tidal force is by those of quadrupoles. Thus we know that we can use the point-like particle as a source instead of any extended object. On the other side, Einstein, Infeld & Hoffman showed that the motion of particles is determined by only the gravity on a 2-surface (in the vacuum). This means that we don't need to know the internal structure of the objects, that is, even if the object is a black hole, neutron star, or other object, the motion is same. Therefore to use a delta function seems to be valid in the post-Newtonian scheme.

4 Discussion and summary

We calculate the spin-spin coupling force and the spin-orbit coupling force in post-Newtonian approximation. at lowest order. These result is confirmed with the before calculation by some authors. We can also calculate higher corrections by this scheme. However some problem is lying and we consider them here.

First, in our treatment, we consider the objects to be point-like particles. When calculating higher correction in this scheme, there appears so-called self-gravity divergence. Ohta *et. al.* neglected those divergence by the empirical rule, however it must be showed. Damour showed that up to 2.5PN order such empirical rule is valid in the case of non-spinning particles. This is naturally conclusion for simple delta function describes a particle with 'spherical' symmetry and the divergence does not directly contribute the motion of particle. Any contribution of the divergence may be introduced in corrections to the mass (inertial) of the particle. We cannot apply to the case of spinning particles because of non-spherical symmetry of spinning particle from the derivative of the delta functions. In order to avoid this ambiguity, we seem to need to treat regularly the delta functions as the distribution(hyperfunction).

Second, we have discussed the validity of using delta function for describing the distribution of the objects, but we have not yet know the connection of the coefficient of the derivative of the delta functions and the equation of states of the objects. It is not clear

that at each order of post-Newtonian approximation we can simply consider the coefficient to be the multipole moment of the objects. Conversely, this point is a virtue of this scheme for we introduce the effect of tidal deformation by a phenomenal rule like the Hook's rule.

Because of these problems, it is difficult to give a physical interpretation to this scheme. Therefore, to settle these problems and to give a physical interpretation are future subjects.

5 Acknowledgements

We would like to thank T. Nakamura for useful comments and encouragement and T. Tanaka for useful comments.

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Post-Newtonian Darwin Problem

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Abstract

Using the equation of motion for the first post-Newtonian hydrodynamics, we calculate the orbital angular velocity and the total energy of the binary neutron stars constructed by incompressible, homogeneous stars. In the present article we derive the lowest and the second lowest order contribution of the finite-size stars in the first post-Newtonian (PN) order. It is found that the most effective finite-size term in the orbital angular velocity contributes as 3PN order.

1 Introduction

The laser interferometers such as LIGO[1], VIRGO[2], GEO600[3] and TAMA300[4] are currently being constructed and the detection of gravitational waves is expected at the end of this century. One of the most important astrophysical sources of gravitational waves for these detectors is the coalescing binary neutron stars. The gravitational waves emitted in the inspiral phase so-called the last three minutes[5] have frequencies in the sensitive region of these detectors, i.e., from 10Hz to 1000Hz. We can know each mass, each spin and radius of the binary neutron star by comparing the detected wave forms with the theoretical template[6].

The radius of the neutron stars which construct a binary system, from which we have the constraint on the equation of state of neutron star, is obtained from the relation between the frequency of gravitational waves at the innermost stable circular orbit (ISCO)[7] and the radius of the neutron star[8]. Since it is difficult to solve fully general relativistic equations, some approximations are required in these papers[8].

In this article, we solve the Darwin problem using the hydrodynamic equation of motion in the post-Newtonian approximation. Here, Darwin problem is concerned with the equilibrium and the stability of a homogeneous body rotating about another body considering the mutual tidal interactions between the star 1 and the star 2. We assume the neutron stars in the binary system as incompressible, homogeneous ellipsoids. The reward

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is that all the calculations can be done analytically so that our results will contribute to the understanding how the finite-size contributions in PN influence to the orbital angular velocity.

This article is organized as follows. In section 2, the first virial equation is derived. We show the main results of this paper in section 3. In section 4, it is discussed on the result derived in section 3. We use the unit of $G = 1$ through this article.

2 Formulation

The equation of hydrodynamics governing an ideal fluid in the post-Newtonian approximation are derived in many papers[9]. Here, we use Shibata's formulation.

In this article, we consider the equilibrium configuration of binary neutron stars of equal masses whose separation is R . We assume that the center of mass of a star (star 1) locates at the origin and the other one (star 2) locates at $(x_1, x_2, x_3) = (-R, 0, 0)$.

The integrated form of the Euler equation for incompressible fluid in the first post-Newtonian approximation is written as

$$\frac{P}{\rho} - \frac{1}{2c^2} \left(\frac{P}{\rho} \right)^2 = U - \frac{X_0}{c^2} + \left\{ \frac{\varpi^2}{2} + \frac{1}{c^2} (2\varpi^2 U - X_\Omega + \hat{\beta}_\varphi) \right\} \Omega^2 + \frac{\varpi^4}{4c^2} \Omega^4 + \text{const.}, \quad (1)$$

where $\varpi^2 = (x_1 + R/2)^2 + x_2^2$.

We will take into account the coefficients of the terms such as $x_1^{m_1} x_2^{m_2} x_3^{m_3}$ in X_0 , $\Omega^2 X_\Omega$, $\Omega^2 \hat{\beta}_\varphi$, and so on, where $0 \leq m_1, m_2, m_3 \leq 5$ and $0 \leq m_1 + m_2 + m_3 (\equiv m_t) \leq 5$, up to $O(R^{-5})$ for the case m_t is odd and up to $O(R^{-3})$ for the case m_t is even.

Next, we calculate Eq.(1) and express it in terms of the coordinates x^i and the separation R .

The equation for the Newtonian potential is

$$\Delta U = -4\pi\rho. \quad (2)$$

There are two kinds of contribution for U at star 1: one is the contribution from star 1 and the other is that from star 2. Hence, we write U as $U = U^{1-1} + U^{2-1}$ where U^{i-j} denotes the potential generated by star i at star j . U^{1-1} and U^{2-1} are

$$U^{1-1} = \pi\rho \left(A_0 - \sum_l A_l x_l^2 \right), \quad (3)$$

$$\begin{aligned} U^{2-1} = \frac{M}{R} \left\{ 1 - \frac{x_1}{R} + \frac{2x_1^2 - x_2^2 - x_3^2}{2R^2} + \frac{-2x_1^3 + 3x_1(x_2^2 + x_3^2)}{2R^3} \right. \\ \left. + \frac{8x_1^4 + 3x_2^4 + 3x_3^4 - 24x_1^2(x_2^2 + x_3^2) + 6x_2^2x_3^2}{8R^4} \right\} \\ + \frac{3\mathcal{I}_{11}}{2R^3} \left(1 - \frac{3x_1}{R} + \frac{12x_1^2 - 5x_2^2 - 5x_3^2}{2R^2} \right) + \frac{3}{2R^5} (\mathcal{I}_{22}x_2^2 + \mathcal{I}_{33}x_3^2), \quad (4) \end{aligned}$$

where

$$I_{ijkl\dots} = \int d^3x \rho x_i x_j x_k x_l \dots, \quad \mathcal{I}_{ii} = I_{ii} - \frac{1}{3} \sum_l I_{ll}. \quad (5)$$

$A_{ijk\dots}$ are index symbols defined by Chandrasekhar[10], and $A_0 = \sum_l A_l a_l^2$, where a_l are axes of the ellipsoid. Note that U^{2-2} and U^{1-2} are obtained by changing x_1 into $-(x_1 + R)$ in U^{1-1} and U^{2-1} , respectively.

In Newtonian order, the pressure is written as

$$P = P_0 \left(1 - \sum_l \frac{x_l^2}{a_l^2} \right), \quad (6)$$

and a_l is determined from

$$-\frac{P_0}{\rho a_1^2} = -\pi \rho A_1 + \frac{\Omega_N^2}{2} + \frac{M}{R^3}, \quad (7)$$

$$-\frac{P_0}{\rho a_2^2} = -\pi \rho A_2 + \frac{\Omega_N^2}{2} - \frac{M}{2R^3}, \quad (8)$$

$$-\frac{P_0}{\rho a_3^2} = -\pi \rho A_3 - \frac{M}{2R^3}, \quad (9)$$

where Ω_N is the orbital angular velocity at the Newtonian order. From Eqs.(7)-(9), the pressure at origin of star 1 is calculated as

$$P_0 = \frac{\rho}{3} \left[\pi \rho A_0 - \frac{\Omega_N^2}{2} (a_1^2 + a_2^2) - \frac{M}{2R^3} (2a_1^2 - a_2^2 - a_3^2) \right] + O(R^{-5}). \quad (10)$$

As in the case of U , X_0 is divided into two parts as $X_0 = X_0^{1-1} + X_0^{2-1}$. The equation for X_0^{1-1} is

$$\Delta X_0^{1-1} = 4\pi\rho \left[2\pi\rho \left(A_0 - \sum_l A_l x_l^2 \right) + \frac{3P_0}{\rho} \left(1 - \sum_l \frac{x_l^2}{a_l^2} \right) + 2U^{2-1} \right], \quad (11)$$

and the solution becomes

$$\begin{aligned} X_0^{1-1} = & -\alpha_0 U^{1-1} + \alpha_1 D_1 + \sum_l \eta_l D_{ll} - \frac{M}{R^3} (2D_{11} - D_{22} - D_{33}) \\ & - \frac{M}{R^4} (-2D_{111} + 3D_{122} + 3D_{133}), \end{aligned} \quad (12)$$

where

$$\alpha_0 = 2\pi\rho A_0 + \frac{3P_0}{\rho} + \frac{2M}{R} + \frac{3\mathcal{I}_{11}}{R^3}, \quad (13)$$

$$\alpha_1 = \frac{2M}{R^2} + \frac{9\mathcal{I}_{11}}{R^4}, \quad (14)$$

$$\eta_l = 2\pi\rho A_l + \frac{3P_0}{\rho a_l^2}. \quad (15)$$

D_i , D_{ii} and D_{1ii} are expressed for star 1 as

$$D_i = \pi \rho a_i^2 \left(A_i - \sum_l A_{il} x_l^2 \right) x_i, \quad (16)$$

$$D_{ii} = \pi \rho \left[a_i^4 \left(A_{ii} - \sum_l A_{iil} x_l^2 \right) x_i^2 + \frac{1}{4} a_i^2 \left(B_i - 2 \sum_l B_{il} x_l^2 + \sum_{l,m} B_{ilm} x_l^2 x_m^2 \right) \right], \quad (17)$$

$$D_{1ii} = \pi \rho \left[a_i^6 \left(A_{1ii} - \sum_l A_{1iil} x_l^2 \right) x_1^3 + \frac{3}{4} a_i^4 \left(B_{1i} - 2 \sum_l B_{1il} x_l^2 + \sum_{l,m} B_{1ilm} x_l^2 x_m^2 \right) x_1 \right], \quad (18)$$

$$D_{1ii} = \pi \rho \left[a_1^2 a_i^4 \left(A_{1ii} - \sum_l A_{1iil} x_l^2 \right) x_1 x_i^2 + \frac{1}{4} a_1^2 a_i^2 \left(B_{1i} - 2 \sum_l B_{1il} x_l^2 + \sum_{l,m} B_{1ilm} x_l^2 x_m^2 \right) x_1 \right] \quad (19)$$

where $B_{ijk\dots}$ are shown in Chandrasekhar's textbook[10]. On the other hand, the contributions from star 2 at star 1 we need later are

$$D_1^{2-1} = \frac{I_{11}}{R^2} \left(1 - \frac{2x_1}{R} + \frac{6x_1^2 - 3x_2^2 - 3x_3^2}{2R^2} + \dots \right), \quad (20)$$

$$D_2^{2-1} = \frac{I_{22}}{R^2} \left(\frac{x_2}{R} - \frac{3x_1 x_2}{R^2} + \dots \right), \quad (21)$$

$$D_{ii}^{2-1} = \frac{I_{ii}}{R} \left(1 - \frac{x_1}{R} + \frac{2x_1^2 - x_2^2 - x_3^2}{2R^2} + \dots \right) + \frac{3I_{iil}}{2R^3} \left(1 - \frac{3x_1}{R} \right). \quad (22)$$

The equation for X_0^{2-1} is written as

$$\Delta X_0^{2-1} = 4\pi \rho \left[2\pi \rho \left(A_0 - \sum_l A_{il} y_l^2 \right) + \frac{3P_0}{\rho} \left(1 - \sum_l \frac{y_l^2}{a_l^2} \right) + 2U^{1-2} \right], \quad (23)$$

where $y_1 = -(x_1 + R)$, $y_2 = x_2$ and $y_3 = x_3$. Then the solution is

$$\begin{aligned} X_0^{2-1} &= -\alpha_0 U^{2-1} + \sum_l \eta_l \left\{ \frac{I_{ll}}{M} U_{\text{point}}^{2-1} + \frac{3I_{lil}}{2R^3} \left(1 - \frac{3x_1}{R} \right) \right\} \\ &\quad - \frac{M}{R^3} (2D_{11}^{2-1} - D_{22}^{2-1} - D_{33}^{2-1}) - \alpha_1 D_1^{2-1}, \end{aligned} \quad (24)$$

where

$$I_{iil} = I_{iil} - \frac{1}{3} \sum_l I_{iil}, \quad (25)$$

$$U_{\text{point}}^{2-1} = \frac{M}{R} \left(1 - \frac{x_1}{R} + \frac{2x_1^2 - x_2^2 - x_3^2}{2R^2} + \frac{-2x_1^3 + 3x_1(x_2^2 + x_3^2)}{2R^3} \right). \quad (26)$$

The equation for X_Ω is

$$\Delta X_\Omega = 8\pi \rho \left(x_1^2 + x_2^2 + R x_1 + \frac{R^2}{4} \right). \quad (27)$$

Then the solution is written as $X_{\Omega}^{1-1} + X_{\Omega}^{2-1}$, where

$$X_{\Omega}^{1-1} = -2 \left(D_{11} + D_{22} + R D_1 + \frac{R^2}{4} U^{1-1} \right), \quad (28)$$

$$X_{\Omega}^{2-1} = -2 \left(D_{11}^{2-1} + D_{22}^{2-1} - R D_1^{2-1} + \frac{R^2}{4} U^{2-1} \right). \quad (29)$$

The definition of $\hat{\beta}_{\varphi}$ is

$$\begin{aligned} \hat{\beta}_{\varphi} = & -\frac{7}{2} \left(x_1 P_1 + x_2 P_2 + \frac{R}{2} P_1 \right) \\ & -\frac{1}{2} \left\{ \left(x_1 + \frac{R}{2} \right)^2 P_{2,2} + x_2^2 P_{1,1} - \left(x_1 + \frac{R}{2} \right) x_2 (P_{1,2} + P_{2,1}) \right\}. \end{aligned} \quad (30)$$

where

$$\Delta P_1 = -4\pi\rho \left(x_1 + \frac{R}{2} \right), \quad (31)$$

$$\Delta P_2 = -4\pi\rho x_2. \quad (32)$$

P_i is also written as $P_i^{1-1} + P_i^{2-1}$, where

$$P_1^{1-1} = D_1 + \frac{R}{2} U^{1-1}, \quad P_2^{1-1} = D_2, \quad (33)$$

and

$$P_1^{2-1} = D_1^{2-1} - \frac{R}{2} U^{2-1}, \quad P_2^{2-1} = D_2^{2-1}. \quad (34)$$

Substituting these expressions into Eq.(1), the integrated form of the Euler equation can be written as

$$\begin{aligned} \frac{P}{\rho} = & U_N + \delta U + \frac{1}{2} \varpi^2 \Omega^2 + \frac{1}{\varrho^2} \left[\gamma_0 + \sum_i \gamma_i x_i^2 \right. \\ & \left. + \sum_{i \geq m} \gamma_{im} x_i^2 x_m^2 + x_1 \left(\beta_0 + \sum_i \beta_i x_i^2 + \sum_{i \geq m} \beta_{im} x_i^2 x_m^2 \right) \right] + \text{const.}, \end{aligned} \quad (35)$$

where δU is the PN correction of U which we will mention in the next section, and

$$\begin{aligned} \beta_0 = & -\frac{M\pi\rho}{R^2} \left(\frac{6P_0}{5\pi\rho^2} + \frac{11A_0}{10} + \frac{3}{2} a_1^2 A_1 + a_2^2 A_2 \right) + \frac{9M^2}{4R^3} \\ & + \frac{1}{R^4} \left[\frac{9}{2} \pi \rho f_{11} (-a_1^2 A_1 - 2a_2^2 A_2 + A_0) + \frac{9}{2} \sum_i \left(2\pi \rho A_i + \frac{3P_0}{\rho a_i^2} \right) f_{i111} \right. \\ & \left. - \frac{9}{2} \pi \rho f_{11} \left(2A_0 + \frac{3P_0}{\pi\rho^2} \right) + \frac{3}{4} \pi \rho M a_1^2 (-2a_1^2 B_{11} + a_2^2 B_{12} + a_3^2 B_{13}) \right] \\ & + \frac{M}{8R^5} (118I_{11} - 93I_{22} - 59I_{33}), \end{aligned} \quad (36)$$

$$\begin{aligned}\beta_1 = & \frac{M\pi\rho}{R^2} \left(\frac{3}{2}a_1^2 A_{11} + a_2^2 A_{12} - \frac{1}{2}A_1 \right) \\ & + \frac{1}{R^4} \left[\frac{9}{2}\pi\rho \mathcal{I}_{11}(a_1^2 A_{11} + 2a_2^2 A_{12} - A_1) + \sum_l \eta_l I_{ll} - \left(2\pi\rho A_0 + \frac{3P_0}{\rho} \right) M \right. \\ & \left. + \frac{1}{2}\pi\rho M a_1^2 (-4a_1^4 A_{111} + 6a_1^2 B_{111} - 3a_2^2 B_{112} - 3a_3^2 B_{113}) \right] - \frac{M^2}{4R^5}, \quad (37)\end{aligned}$$

$$\begin{aligned}\beta_2 = & \frac{M\pi\rho}{R^2} \left(\frac{1}{2}a_1^2 A_{12} - a_2^2 A_{12} + 3a_2^2 A_{22} + A_1 - \frac{3}{2}A_2 \right) \\ & + \frac{1}{R^4} \left[\frac{9}{2}\pi\rho \mathcal{I}_{11}(-a_1^2 A_{12} - 2a_2^2 A_{12} + 6a_2^2 A_{22} - 3A_2 + 2A_1) \right. \\ & \left. - \frac{3}{2}\sum_l \eta_l I_{ll} + \frac{3}{2} \left(2\pi\rho A_0 + \frac{3P_0}{\rho} \right) M \right. \\ & \left. - \frac{3}{2}\pi\rho M a_1^2 (-2a_1^2 B_{112} - 2a_2^4 A_{122} + a_2^2 B_{122} + a_3^2 B_{123}) \right] + \frac{29M^2}{8R^5}, \quad (38)\end{aligned}$$

$$\begin{aligned}\beta_3 = & \frac{M\pi\rho}{R^2} \left(\frac{3}{2}a_1^2 A_{13} + a_2^2 A_{23} - \frac{1}{2}A_3 \right) \\ & + \frac{1}{R^4} \left[\frac{9}{2}\pi\rho \mathcal{I}_{11}(a_1^2 A_{13} + 2a_2^2 A_{23} - A_3) - \frac{3}{2}\sum_l \eta_l I_{ll} + \frac{3}{2} \left(2\pi\rho A_0 + \frac{3P_0}{\rho} \right) M \right. \\ & \left. - \frac{3}{2}\pi\rho M a_1^2 (-2a_1^2 B_{113} - 2a_3^4 A_{133} + a_2^2 B_{123} + a_3^2 B_{133}) \right] + \frac{39M^2}{8R^5}, \quad (39)\end{aligned}$$

$$\beta_{11} = \frac{M\pi\rho}{R^4} \left[2a_1^6 A_{1111} + \frac{3}{4}a_1^2 (-2a_1^2 B_{1111} + a_2^2 B_{1112} + a_3^2 B_{1113}) \right], \quad (40)$$

$$\beta_{22} = \frac{M\pi\rho}{R^4} \left[-3a_1^2 a_2^4 A_{1222} + \frac{3}{4}a_1^2 (-2a_1^2 B_{1122} + a_2^2 B_{1222} + a_3^2 B_{1223}) \right], \quad (41)$$

$$\beta_{33} = \frac{M\pi\rho}{R^4} \left[-3a_1^2 a_3^4 A_{1333} + \frac{3}{4}a_1^2 (-2a_1^2 B_{1133} + a_2^2 B_{1233} + a_3^2 B_{1333}) \right], \quad (42)$$

$$\beta_{12} = \frac{M\pi\rho}{R^4} \left[2a_1^6 A_{1112} - 3a_1^2 a_2^4 A_{1122} + \frac{3}{2}a_1^2 (-2a_1^2 B_{1112} + a_2^2 B_{1122} + a_3^2 B_{1123}) \right], \quad (43)$$

$$\beta_{13} = \frac{M\pi\rho}{R^4} \left[2a_1^6 A_{1113} - 3a_1^2 a_3^4 A_{1133} + \frac{3}{2}a_1^2 (-2a_1^2 B_{1113} + a_2^2 B_{1123} + a_3^2 B_{1133}) \right], \quad (44)$$

$$\beta_{23} = \frac{M\pi\rho}{R^4} \left[-3a_1^2 (a_2^4 A_{1223} + a_3^4 A_{1233}) + \frac{3}{2}a_1^2 (-2a_1^2 B_{1123} + a_2^2 B_{1223} + a_3^2 B_{1233}) \right]. \quad (45)$$

Note that since we would like to get the orbital angular velocity in this article, it is not necessary to derive γ_0 , γ_l and γ_{lm} which have no relation with the orbital angular velocity.

3 The Finite-Size Contributions

In this section, the orbital angular velocity in the post-Newtonian order is derived using the equations we calculate the above section. We show how the finite-size terms contribute to the orbital angular velocity.

Ω^2 is determined from the first tensor virial (TV) equation. In the Newtonian order,

the angular velocity Ω_N^2 becomes

$$\Omega_N^2 = \frac{2M}{R^3} + \frac{18I_{11}}{R^5}. \quad (46)$$

Since the following relation holds even in the PN order,

$$\int d^3x \frac{\partial P}{\partial x_1} = 0, \quad (47)$$

the first TV equation in the PN order becomes

$$\begin{aligned} 0 = & \frac{R}{2} M \delta \Omega^2 + \int d^3x \rho \xi_1 \Omega_N^2 + M \beta_0 + 3\beta_1 I_{11} + \beta_2 I_{22} + \beta_3 I_{33} + 5\beta_{11} I_{1111} \\ & + 3\beta_{12} I_{1122} + 3\beta_{13} I_{1133} + \beta_{22} I_{2222} + \beta_{33} I_{3333} + \beta_{23} I_{2233} + \delta \int d^3x \rho \frac{\partial U}{\partial x_1}, \end{aligned} \quad (48)$$

where ξ_1 is the x_1 component of Lagrangian displacement ξ_i , and $\delta \Omega^2$ denotes the PN correction of the angular velocity.

3.1 The Post-Newtonian Deformation

Chandrasekhar and his coraborators considered the method to calculate the post-Newtonian effects on the uniformly rotating isolated bodies[11]. First they construct the ellipsoidal figures in Newtonian order, and second, consider the displacement in post-Newtonian order. Using such a method, they have the equilibrium figures in post-Newtonian order. In this article we accept their method.

In order to express Eq.(35) using only the coordinate and the index symbols, we have to calculate δU . δU is defined as

$$\delta U = - \sum_i \frac{\partial}{\partial x_i} \int d^3x' \frac{\rho \xi_i}{|\mathbf{x} - \mathbf{x}'|}. \quad (49)$$

Here, ξ_i denotes the Lagrangian displacement of the fluid element induced by the PN correction, and is used to guarantee that the pressure of the deformed Darwin ellipsoid at the boundary becomes zero. This means that it is determined from γ_0 , γ_i , γ_{ij} , β_0 , β_i and β_{ij} in Eq.(35).

In choosing ξ_i , we require it to be divergent free because $\delta \rho = - \sum_i \partial_i (\rho \xi_i)$. To obtain $\delta \Omega^2$ up to $O(R^{-3}) \times \Omega_N^2$, we only need the following:

$$\xi_i^{(0)} = \left(\frac{1}{2}, 0, 0 \right), \quad (50)$$

$$\xi_i^{(21)} = \left(\frac{1}{2} x_1^2, 0, -x_1 x_3 \right), \quad (51)$$

$$\xi_i^{(22)} = (0, x_1 x_2, -x_1 x_3). \quad (52)$$

Thus, we set

$$\xi_i = S_0 \xi_i^{(0)} + S_{21} \xi_i^{(21)} + S_{22} \xi_i^{(22)}. \quad (53)$$

There are no space to explain more detail about the Lagrangian displacement here, however we can show S_0 is indeterminant, and S_{21} and S_{22} are $O(R^{-4})$ [12]. This means that the PN deformation of the figure has no effect on the Eq.(48).

3.2 The Post-Newtonian Orbital Angular Velocity

Since S_0 is arbitrary, we set $S_0 = 0$. Hence, from Eq.(48) we have $\delta\Omega^2$. Adding it to Ω_N^2 , Ω^2 is expressed as

$$\begin{aligned}\Omega^2 = & \frac{2M}{R^3} \left[1 + \frac{1}{c^2} \left\{ 2\pi\rho A_0 - \frac{9M}{4R} - \frac{M}{10R^3} (28a_1^2 - 14a_2^2 - 9a_3^2) + O(R^{-4}) \right\} \right] \\ & + \frac{18\mathcal{I}_{11}}{R^5} \left\{ 1 + \frac{28}{15c^2} \pi\rho A_0 + O(R^{-2}) \right\}.\end{aligned}\quad (54)$$

This is the orbital angular velocity we would like to derive. However, there are M in Eq.(54). This is Newtonian order mass, and is not an appropriate value to represent “mass” in the post-Newtonian approximation.

We consider the PPN mass[13] defined as

$$\begin{aligned}M_{\text{PPN}} &= \int d^3x \rho \left[1 + \frac{1}{c^2} \left(\frac{v^2}{2} + 3U - \frac{1}{2}U_{\text{self}} + \frac{v_{\text{self}}^2}{2} \right) \right] \\ &= M \left[1 + \frac{1}{c^2} \left(\frac{13M}{4R} + 2\pi\rho A_0 + \frac{M}{20R^3} (38a_1^2 - 7a_2^2 - 15a_3^2) + O(R^{-5}) \right) \right].\end{aligned}\quad (55)$$

In the PPN formalism, the center of mass of each star is also defined as

$$x_{\text{PPN}}^i = \frac{1}{M_{\text{PPN}}} \int d^3x \rho \left[1 + \frac{1}{c^2} \left(\frac{v^2}{2} + 3U - \frac{1}{2}U_{\text{self}} + \frac{v_{\text{self}}^2}{2} \right) \right].\quad (56)$$

Thus, we should change the orbital separation as

$$R_{\text{PPN}} = R \left[1 + \frac{1}{c^2} \left\{ -\frac{4Ma_1^2}{5R^3} + O(R^{-5}) \right\} \right].\quad (57)$$

When we rewrite Eq.(54) using the PPN mass and the PPN separation, the orbital angular velocity does not depend on the internal structure of the star only up to $O(R^{-1})$ in the post-Newtonian term as

$$\begin{aligned}\Omega^2 = & \frac{2M_{\text{PPN}}}{R_{\text{PPN}}^3} \left[1 + \frac{1}{c^2} \left\{ -\frac{11M_{\text{PPN}}}{2R_{\text{PPN}}} - \frac{M_{\text{PPN}}}{20R_{\text{PPN}}^3} (142a_1^2 - 35a_2^2 - 33a_3^2) + O(R_{\text{PPN}}^{-4}) \right\} \right] \\ & + \frac{18(\mathcal{I}_{11})_{\text{PPN}}}{R_{\text{PPN}}^5} \left[1 + \frac{1}{c^2} \left\{ -\frac{2}{15} \pi\rho A_0 - \frac{13M_{\text{PPN}}}{4R_{\text{PPN}}} + O(R_{\text{PPN}}^{-2}) \right\} \right].\end{aligned}\quad (58)$$

3.3 The Total Energy

The total energy is calculated from $E = E_N^{\text{def}} + E_{\text{PN}}^{\text{def}}/c^2$, where

$$E_N^{\text{def}} = \int \rho \left(\frac{1}{2}v^2 - \frac{1}{2}U \right) d^3x,\quad (59)$$

$$E_{\text{PN}}^{\text{def}} = \int \rho \left(\frac{5}{8}v^4 + \frac{5}{2}v^2U + \frac{P}{\rho}v^2 - \frac{5}{2}U^2 + \frac{1}{2}\hat{\beta}_\varphi\Omega^2 \right) d^3x.\quad (60)$$

If we substitute Ω^2 into the above formulas, the final form of E becomes $E_N + E_{\text{PPN}}/c^2$ where

$$E_N = M \left[-\frac{4\pi\rho A_0}{5} - \frac{M}{2R} + \frac{3I_{11}}{2R^3} + \frac{2M}{5R^3}(a_1^2 + a_2^2) + O(R^{-5}) \right], \quad (61)$$

$$E_{\text{PPN}} = 2M \left[-\frac{34}{21}(\pi\rho A_0)^2 - \frac{19M\pi\rho A_0}{6R} - \frac{25M^2}{32R^2} + \frac{M\pi\rho A_0}{R^3} \left\{ \frac{22}{21}(a_1^2 + a_2^2) - \frac{158}{35} \frac{I_{11}}{M} \right\} \right. \\ \left. + \frac{M^2}{240R^4}(26a_1^2 + 35a_2^2 - 155a_3^2) + O(R^{-5}) \right]. \quad (62)$$

Now, we define $a_{\text{PPN}} = (3M_{\text{PPN}}/4\pi\rho)^{1/3}$. If the total energy is expressed by M_{PPN} and a_{PPN} instead of M and a_i in the limiting case $a_i/R \rightarrow 0$, then

$$E = M_{\text{PPN}} \left[-\frac{8\pi\rho a_{\text{PPN}}^2}{5} - \frac{M_{\text{PPN}}}{2R_{\text{PPN}}} + \frac{1}{c^2} \left\{ -\frac{16}{7}(\pi\rho a_{\text{PPN}}^2)^2 + \frac{27M_{\text{PPN}}^2}{16R_{\text{PPN}}^2} \right\} \right], \quad (63)$$

Except for terms depending on a_{PPN} , this equation agrees with the energy for the two point particles in circular orbits[15].

4 Discussion

The energy emission rate or the orbital angular velocity of the binary neutron stars in the post-Newtonian approximation is usually expressed using the PPN mass. In such papers, even if the effects of the finite-mass and finite-spin contribution are discussed, the finite-size effect is neglected. Even if the finite-size effect in gravitational radiation is discussed, it is considered only up to Newtonian quadrupole moment[14]. In this article it is found from Eq.(58) that the most effective term of the finite-size contribution in the post-Newtonian order contributes as large as the finite-mass contribution of "3PN" order. Therefore, the finite size contributions in the post-Newtonian order are effective to the gravitational wave forms when we extract the information using the matched filter technique.

We can determine the ISCO of this system finding the minimum point of the total energy, and this work is in progress[12].

Acknowledgements

We thank H. Asada and T. Tanaka for useful discussions and also K.T. would like to thank H. Sato for continuous encouragement. This work was in part supported by a Grant-in-Aid of Ministry of Education, Culture, Science and Sports(Nos. 08237210 and 08NP0801).

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Scalar Gravitational Wave from Oppenheimer-Snyder Collapse in Scalar-Tensor Theories of Gravity – An Effective Mass of Scalar Field –

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Abstract

We investigate scalar gravitational wave from Oppenheimer-Snyder collapse in scalar-tensor theories of gravity. We solve numerically the generation and propagation of the scalar gravitational wave from a spherically symmetric and homogeneous dust collapse under the approximation that we can neglect the back reaction of the scalar wave on the space-time. In the scalar-tensor theories, it would be possible to get the information of the first derivative of the coupling function contained in the theory because the wave form of the scalar gravitational wave greatly depends on it. We also discuss an effective mass of the scalar field and its implication to the wave form of the scalar gravitational wave.

1 Introduction

General relativity is not the only gravitational theory that passes all experimental tests that have been carried out until now. There are alternative theories of gravity that pass the present weak-field gravitational tests. In strong-field regime, these theories may deviate very widely from general relativity. Therefore experiments in strong-field regimes are necessary in order to determine the correct theory of gravity.

Scalar-tensor theories of gravity are among those viable theories and naturally arise as the low-energy limit of string theory [1, 2] or other unified theories. They play an important role in some inflation model [3]. Interestingly, laser interferometric gravitational wave observatories (LIGO [4]) project will be in practical use next decade [5] and other projects (VIRGO [6], GEO [7] and TAMA [8]) of gravitational wave observations are being carried forward so that high-accuracy tests of the scalar-tensor theories may be expected [9, 10].

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In the scalar-tensor theories of gravity we consider scalar fields as well as a metric tensor as fields that are related to gravity. The scalar-tensor theories must agree with general relativity within an accuracy of at least $\sim 0.1\%$ in the weak-field region from the results of solar-system experiments and pulsar tests [11]. But a class of scalar-tensor theories that satisfies the above constraints may predict order-of-magnitude deviations from general relativity in the regime of strong gravitational field [12]. One example of such phenomena in the strong-field regime may be the gravitational collapse of a star.

Since in general relativity no gravitational wave is radiated from a spherically symmetric collapse, we cannot obtain any information of collapse from observations of gravitational waves if realistic gravitational collapses to black hole are almost spherically symmetric. However in scalar-tensor theories scalar gravitational waves are radiated even from spherically symmetric collapse and these scalar gravitational waves can be detected by laser interferometric gravitational wave detectors [10].

Here we take a spherically symmetric, homogeneous dust collapse as a model of the collapse of the stellar core and examine the scalar gravitational wave radiation and its wave form. As a first step to understand the nature of the scalar gravitational waves in general, we neglect the back reaction of the scalar field on the space-time. The method for numerical calculations is similar to that of Cunningham, Price and Moncrief [13]. We use null coordinates in order to see the propagation of the scalar gravitational wave. We use units of $c = 1$. The Greek indices $\mu, \nu, \dots$ run over 0,1,2,3. A comma represents a partial derivative. We follow Misner, Thorne and Wheeler's [14] sign conventions of the metric tensor, Riemann tensor and Einstein tensor.

2 Basic Equations

2.1 Field Equations of Scalar-Tensor Theory

We consider a class of scalar-tensor theories of gravity in which gravitation is mediated by a mass-coupled, long-range scalar field in addition to space-time curvature. The action is given by [11]

$$I = \frac{1}{16\pi} \int \sqrt{-g} \left(\dot{\phi} R - \frac{\omega(\phi)}{\phi} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \right) d^4x + I_m[\Psi_m, g_{\mu\nu}], \quad (1)$$

where $g_{\mu\nu}$ is the metric tensor, R is the scalar curvature, and ϕ is the scalar field that couples to gravity. ϕ approaches the cosmological value ϕ_0 in the asymptotic region. $\omega(\phi)$ is a dimensionless arbitrary function of ϕ . Ψ_m represents matter fields, and I_m is the action of them.

Varying the action by $g_{\mu\nu}$ and ϕ we have

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi}{\phi} T_{\mu\nu} + \frac{\omega(\phi)}{\phi^2} \left(\phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \right) + \frac{1}{\phi} (\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \square \phi), \quad (2)$$

$$\square \phi = \frac{1}{3 + 2\omega(\phi)} \left(8\pi T - \frac{d\omega}{d\phi} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \right), \quad (3)$$

where ∇_ν and $\square$ are a covariant derivative and the d'Alembertian of $g_{\mu\nu}$ respectively. The equations of motion for matter fields are given by

$$\nabla_\nu T^{\mu\nu} = 0, \quad (4)$$

where the energy-momentum tensor $T^{\mu\nu}$ is defined by

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta I_m[\Psi_m, g_{\mu\nu}]}{\delta g_{\mu\nu}}. \quad (5)$$

In the asymptotic region, the gravitational "constant" G_0 , which is measured in time dilation experiment or by observation of the Keplerian motion of a test particle, is given by [16]

$$G_0 = \phi_0^{-1} \frac{4 + 2\omega(\phi_0)}{3 + 2\omega(\phi_0)}. \quad (6)$$

Taking the units of $G_0 = 1$, we have

$$\phi_0 = \frac{4 + 2\omega(\phi_0)}{3 + 2\omega(\phi_0)}. \quad (7)$$

If $\omega(\phi)$ is a constant, the theory reduces to the Brans-Dicke theory [15]. In the limit of $\omega \rightarrow \infty$, the theory approaches general relativity.

We can take another representation of the theory by the conformal transformation from the "physical" Brans-Dicke frame to the "unphysical" frame in which the Einstein-Hilbert action is separated from the scalar field action [17]. This new frame is often called the "Einstein" frame. In this frame, the action is [18]

$$I = \frac{1}{16\pi G_*} \int \sqrt{-g_*} (R_* - 2g_*^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu}) d^4x + I_m[\Psi_m, A^2(\varphi) g_{*\mu\nu}], \quad (8)$$

where G_* is the "bare" gravitational constant and φ is the scalar field. The metric tensor $g_{*\mu\nu}$ in the Einstein frame is related to the metric tensor $g_{\mu\nu}$ in the Brans-Dicke frame by the conformal transformation as

$$g_{\mu\nu} = A^2(\varphi) g_{*\mu\nu}. \quad (9)$$

R_* is the scalar curvature of $g_{*\mu\nu}$. φ and $A(\varphi)$ are determined by

$$G_* A^2(\varphi) = \frac{1}{\phi}, \quad (10)$$

$$\alpha^2(\varphi) = \frac{1}{3 + 2\omega(\phi)}, \quad (11)$$

where $\alpha(\varphi)$ is defined by

$$\alpha(\varphi) \equiv \frac{d \ln A(\varphi)}{d\varphi}. \quad (12)$$

The field equations are given by

$$G_{*\mu\nu} = R_{*\mu\nu} - \frac{1}{2} R_* g_{*\mu\nu} = 8\pi G_* T_{*\mu\nu} + 2 \left(\varphi_{,\mu} \varphi_{,\nu} - \frac{1}{2} g_{*\mu\nu} g_*^{\alpha\beta} \varphi_{,\alpha} \varphi_{,\beta} \right), \quad (13)$$

$$\square_* \varphi = -4\pi G_* \alpha(\varphi) T_*, \quad (14)$$

with the equations of motion given by

$$\nabla_{\mu} T^{\mu}_{\nu} = \alpha(\varphi) T_{\mu} \nabla^{\mu} \varphi, \quad (15)$$

where ∇_{μ} , $\square$, and $R_{\mu\nu}$ are a covariant derivative, the d'Alembertian and the Ricci tensor of $g_{\mu\nu}$, respectively. $T^{\mu\nu}$ is defined by

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta I_m[\Psi_m, A^2(\varphi)g_{\mu\nu}]}{\delta g_{\mu\nu}}, \quad (16)$$

and is related to the energy-momentum tensor $T_{\mu\nu}$ in the physical Brans-Dicke frame by

$$T_{\mu\nu} = A^2(\varphi) T_{\mu\nu}. \quad (17)$$

φ approaches the cosmological value φ_0 in the asymptotic region. Hereafter we take the units $G_* = 1$.

Generally, $\alpha(\varphi)$ is an arbitrary function that characterizes the theory. If $\alpha(\varphi)$ is constant, the theory reduces to the Brans-Dicke theory. In the limit $\alpha \rightarrow 0$, the theory approaches general relativity. We denote

$$\alpha_0 \equiv \alpha(\varphi_0), \quad (18)$$

$$\beta_0 \equiv \left. \frac{d\alpha(\varphi)}{d\varphi} \right|_{\varphi=\varphi_0}. \quad (19)$$

In the post-Newtonian approximation, the PPN parameters γ_{Edd} and β_{Edd} are expressed as [18]

$$1 - \gamma_{Edd} = \frac{2\alpha_0^2}{1 + \alpha_0^2}, \quad (20)$$

$$\beta_{Edd} - 1 = \frac{\beta_0 \alpha_0^2}{2(1 + \alpha_0^2)^2}. \quad (21)$$

Results of solar-system test experiments (the time-delay and deflection of light) and pulsar-timing tests constrain γ_{Edd} as [19]

$$\gamma_{Edd} = 0.9996 \pm 0.0017. \quad (22)$$

This inequality is rewritten in terms of ω and α_0 as

$$\omega > 500, \quad \alpha_0^2 < 10^{-3}. \quad (23)$$

The deviation of β_{Edd} from unity implies the breakdown of the gravitational weak equivalence principle. Bounds on this effect from the results of the lunar-laser-ranging experiment constrain β_{Edd} as [20]

$$|\beta_{Edd} - 1| < 7 \times 10^{-4}. \quad (24)$$

This constraint on β_{Edd} comes only through the combination of α_0 and β_0 in Eq. (21). Therefore the smaller the α_0 is, the looser it constrains β_0 . Very recently, Damour and Esposito-Farèse [21] found that binary-pulsar experiments constrain β_0 to

$$\beta_0 > -5, \quad (25)$$

independently of the value of α_0 for a particular choice of the functional form $\alpha(\varphi)$.

We can easily show that the form of the metric perturbation from the Minkowski space-time is given in the appropriate gauge by [22]

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \frac{\delta\phi}{\phi_0} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (26)$$

where h_+ , $h_\times$ represent the plus and cross mode, respectively. These modes propagate independently under the background space-time. We can detect the scalar mode by a laser interferometric gravitational wave detector as shown by Shibata, Nakao and Nakamura [10]. If four detectors are available, we can distinguish the plus, cross and the scalar modes and determine the direction of the source in principle [10].

From Eqs. (10)-(12), we can relate the scalar field perturbation $\delta\phi = \phi - \phi_0$ in the Brans-Dicke frame to the scalar field perturbation $\delta\varphi = \varphi - \varphi_0$ in the Einstein frame in the linear approximation as

$$\delta\phi = -2\alpha_0(1 + \alpha_0^2)\delta\varphi. \quad (27)$$

2.2 Wave Equation

We assume that $|\alpha_0| \ll 1$. Although it may exclude interesting non-perturbative effects, such as a *spontaneous scalarization* [12, 21], we neglect the back reaction of the scalar field on the space-time. In other words, we solve the wave equation

$$\square^{(E)}\varphi = -4\pi\alpha(\varphi)T^{(E)}, \quad (28)$$

to determine the time evolution of the scalar field. This approximation is assured if the energy-momentum tensor of the scalar field is sufficiently smaller than that of the dust [22]. We rewrite Eq. (28) under the background of the Oppenheimer-Snyder collapse. The Oppenheimer-Snyder space-time is given as follows. In the interior of the dust one can write the line element in the closed Friedmann form

$$ds^2 = -d\tau^2 + a(\tau)^2(d\chi^2 + \sin^2\chi d\Omega^2) \quad (29)$$

$$= a(\eta)^2(-d\eta^2 + d\chi^2 + \sin^2\chi d\Omega^2), \quad (30)$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2, \quad (31)$$

where

$$a(\eta) = \frac{1}{2}a_0(1 + \cos\eta), \quad (32)$$

$$\tau(\eta) = \frac{1}{2}a_0(\eta + \sin\eta). \quad (33)$$

The density of the dust is given by

$$\rho(\eta) = \frac{3a_0}{8\pi}a^{-3} = \frac{3}{8\pi a_0^2} \left\{ \frac{1}{2}(1 + \cos\eta) \right\}^{-3}. \quad (34)$$

The ranges of η and χ are

$$0 \leq \eta < \pi, \quad (35)$$

and

$$0 \leq \chi \leq \chi_0 < \frac{\pi}{2}. \quad (36)$$

Let $r_s(t)$ be the circumferential radius of the stellar surface. Then in the exterior of the dust ($r > r_s(t)$), the space-time is expressed by the Schwarzschild metric as

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \quad (37)$$

The matching conditions are expressed as

$$r_s = a(\eta) \sin \chi_0, \quad (38)$$

$$M = \frac{1}{2} a_0 \sin^3 \chi_0, \quad (39)$$

$$t = 2M \ln \left| \frac{\left(\frac{r_{s0}}{2M} - 1\right)^{\frac{1}{2}} + \tan \frac{\eta}{2}}{\left(\frac{r_{s0}}{2M} - 1\right)^{\frac{1}{2}} - \tan \frac{\eta}{2}} \right| + 2M \left(\frac{r_{s0}}{2M} - 1\right)^{\frac{1}{2}} \left[\eta + \left(\frac{r_{s0}}{4M}\right) (\eta + \sin \eta) \right], \quad (40)$$

where $r_{s0} \equiv r_s(t=0)$. The wave equation in the interior ($0 \leq \chi \leq \chi_0$) is given by

$$\frac{1}{a^2} \left\{ -\frac{1}{a^2} \frac{\partial}{\partial \eta} \left(a^2 \frac{\partial \delta \varphi}{\partial \eta} \right) + \frac{1}{\sin^2 \chi} \frac{\partial}{\partial \chi} \left(\sin^2 \chi \frac{\partial \delta \varphi}{\partial \chi} \right) \right\} = 4\pi \alpha(\varphi) \rho, \quad (41)$$

while in the exterior ($r > r_s(t)$) it is given by

$$- \left(1 - \frac{2M}{r}\right)^{-1} \frac{\partial^2 \delta \varphi}{\partial t^2} + \frac{1}{r^2} \frac{\partial}{\partial r} \left\{ r^2 \left(1 - \frac{2M}{r}\right) \frac{\partial \delta \varphi}{\partial r} \right\} = 0. \quad (42)$$

For simplicity we take the initial condition that $\varphi = \varphi_0$ and the time derivative of φ vanishes on the initial hypersurface $\eta = 0$ and $t = 0$. Namely in the interior of the dust we set

$$\delta \varphi = 0, \quad (43)$$

$$\frac{\partial \delta \varphi}{\partial \eta} = 0, \quad (44)$$

at $\eta = 0$, and in the exterior of the dust

$$\delta \varphi = 0, \quad (45)$$

$$\frac{\partial \delta \varphi}{\partial t} = 0, \quad (46)$$

on the first ray $\bar{u} = \bar{u}_0 \equiv -r_{*,s}(t=0)$, because $\varphi = \varphi_0$ is the static solution in the exterior region that satisfies the boundary condition in the asymptotic region.

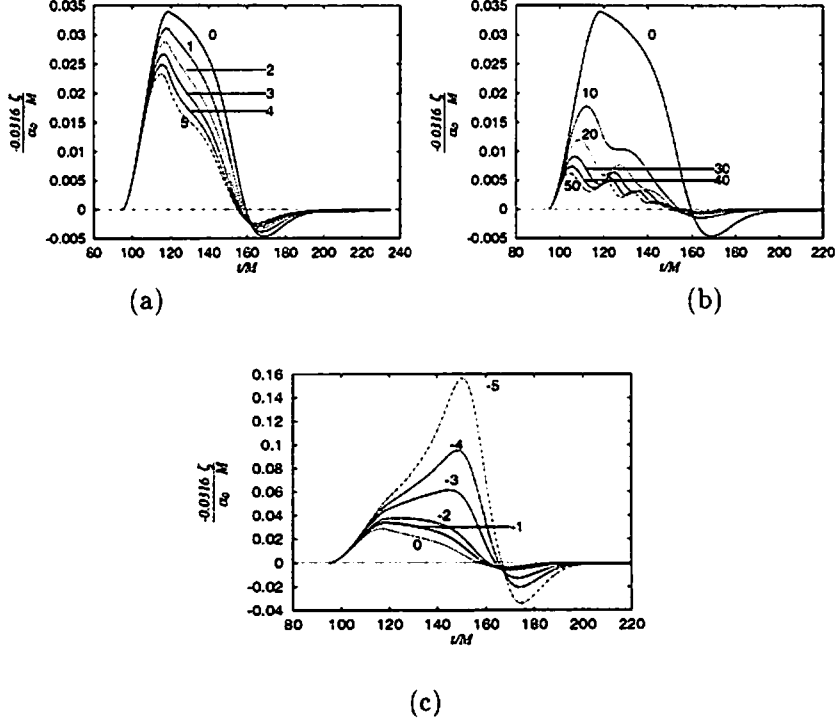


Figure 1: The wave form of the scalar gravitational wave for various β at $r = 100M$. The numbers attached to the curves represent the value of β respectively. The initial dust radius r_{s0} is fixed to $10M$.

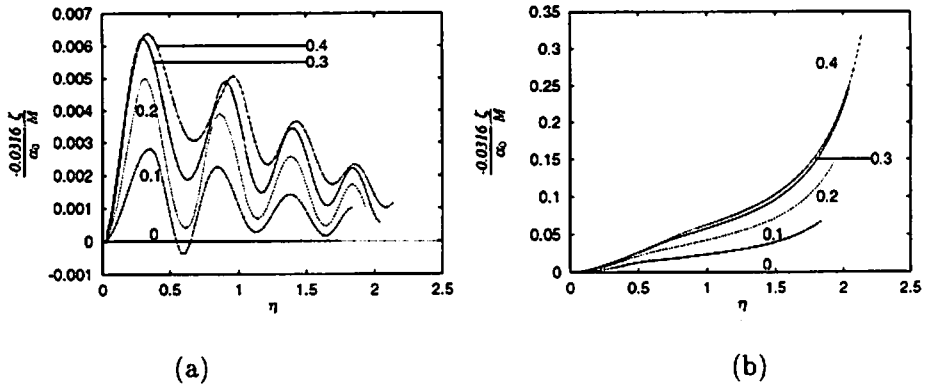


Figure 2: The time evolution of the scalar field seen by comoving observers in the interior of the dust for $\beta = 50$ (a) and $\beta = -5$ (b). The abscissa is the conformal time η and the numbers attached to the curves are the values of the fixed radial coordinates χ of the comoving observers.

3 Numerical Results

We integrate numerically the wave equation for the scalar field. We use comoving coordinate for the stellar interior, its appropriate extension for the inner vacuum region and Schwarzschild coordinate for the outer vacuum region. For the purpose of accurate calculation of the wave form in the asymptotic region, we transform them to null coordinates. The details of those coordinate transformations and the numerical code are described in [22]. As for the coupling function, for simplicity, we assume that

$$\alpha(\varphi) = \alpha_0 + \beta(\varphi - \varphi_0). \quad (47)$$

Hereafter in place of $\delta\varphi$ we define ζ by

$$\zeta \equiv \begin{cases} a \sin \chi \delta\varphi & (\text{interior}) \\ r \delta\varphi & (\text{exterior}). \end{cases} \quad (48)$$

In Fig. 1 the wave forms of the scalar gravitational waves seen by an observer at $r = 100M$ are shown, for various β , from the collapse of the dust whose initial radius is fixed to $10M$. In each figure, the scalar field approaches the asymptotic value after the peak, the quasi-normal mode of the scalar wave having passed the point of the observer, which is consistent with the black hole no hair conjecture.

First we consider the case of positive β . The amplitude of the scalar gravitational wave decreases as β increases. As seen from Figs. 1(a) and (b) there appears another oscillation mode whose frequency is characteristic of the relevant gravitational theory and the dust density. The larger β is, the shorter is the period of this oscillation mode and the more the scalar field oscillates before it approaches the asymptotic value. This oscillation mode reflects an oscillation mode in the interior of the dust seen in Fig. 2(a), which shows the time evolution of the scalar field observed by comoving observers in the interior of the dust for $\beta = 50$. The ratio of the energy density of the scalar field to that of the dust remains within $\sim O(\alpha_0^2)$. So, if $|\alpha_0| \ll 1$, the back reaction of the scalar field on the space-time is negligible.

Second we consider the case of negative β . The amplitude of the scalar wave increases as $|\beta|$ increases, as seen in Fig. 1(c). This reflects an exponential growth of the scalar field in the interior of the dust seen in Fig. 2(b), which shows the time evolution of the scalar field observed by comoving observers in the interior of the dust for $\beta = -5$. The ratio of the energy of the scalar field to that of the dust is $\sim 10 \times O(\alpha_0^2)$ for $\beta = -1$, $\sim 100 \times O(\alpha_0^2)$ for $\beta = -5$ and $\sim 10^4 \times O(\alpha_0^2)$ for $\beta = -10$. Therefore, if we consider a finite value of α_0 , there is a critical value of β for which the back reaction of the scalar field cannot be neglected.

As we have seen in Sec. 2.1, in fact, the solar-system test experiments for the post-Newtonian order deviation from general relativity do not constrain directly the value of β because β is constrained only through the combination with α_0 in Eq. (21) (however, see [21]). So the wave form for β , the value of which is allowed by the solar-system experiments, may be quite different from that in the Brans-Dicke theory.

4 Discussions and Summary

Here we discuss an effective mass of the gravitational scalar field. The wave equation (28) becomes the Klein-Gordon like equation and the gravitational scalar field gets an effective mass m given by

$$m^2 = -4\pi\beta T. \quad (49)$$

Here T denotes the trace of the energy momentum tensor and is given for perfect fluid by

$$T = -\rho + 3P. \quad (50)$$

So for an ordinary matter T will be negative. Therefore we conclude that the scalar field gets a real effective mass for $\beta > 0$ while an imaginary effective mass for $\beta < 0$ if $T < 0$. But we must be reminded that the scalar field remains massless in vacuum or in radiation fluid. The oscillations for $\beta > 0$ and monotonic growth for $\beta < 0$ of the scalar field in the interior of the dust are explained from its effective mass. If we consider the homogeneous scalar field, the real mass results in the harmonic oscillations and the imaginary the exponential growth. The time scale of the oscillations and growth is approximately $|m|^{-1} \sim 1/\sqrt{\beta\rho}$. The reasons why the considerable growth of the wave form does not occur for β a little smaller than zero are the finiteness of the time for the formation of the event horizon and the finiteness of the region where the scalar field gets an imaginary mass.

We summarize the scalar gravitational wave in the scalar-tensor theories of gravity. We calculate the scalar gravitational wave from the dust collapse in the scalar-tensor theory given by Eq. (47). The wave form greatly depends on the parameter β . If $\beta > 0$, the amplitude of the scalar gravitational wave is suppressed and a new oscillation mode whose time scale is $\sim 1/\sqrt{\beta\rho}$ appears. The back reaction of the scalar field on the space-time is within $\sim O(\alpha_0^2)$. If $\beta < 0$, the amplitude of the scalar gravitational wave is enhanced and its back reaction would not be negligible. Hence we have to solve the whole field equations fully numerically, which is the subject of our next paper [23].

Acknowledgements

We would like to thank T. Damour for suggesting to investigate this problem. We would like to thank M. Siino, M. Shibata, S. Hayward, Y. Fujii and H. Sato for useful discussions. T.H. and T.C. are also grateful to H. Sato for continuous encouragement. This work was in part supported by the Grant-in-Aid for Basic Research (No.08NP0801) and for Encouragement of Young Scientists (No.08740341) of Ministry of Education, Culture, Science and Sports.

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Odd Parity Perturbation of the Lemaître-Tolman-Bondi Spacetime

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Abstract

We consider odd-parity perturbation of the Lemaître-Tolman-Bondi (LTB) spacetime which expresses spherically symmetric inhomogeneous dust collapse model. Near the origin, LTB spacetime is similar to Friedmann-Robertson-Walker (FRW) spacetime due to the regularity condition at the origin. To investigate the approximate evolution of perturbation of LTB spacetime, we consider a propagation equation on the FRW spacetime. The solution of this equation shows that the perturbation has a growing mode and can affect to the formation of singularity.

1 Introduction

The cosmic censorship conjecture is one of the most important unsolved problems on the classical general relativity. Many researchers have made a lot of efforts to prove this conjecture. However their trials have resulted in failure and many counterexamples have produced. One of them is the spherically inhomogeneous dust collapse model described by Lemaître-Tolman-Bondi spacetime. We know that naked singularity does occur in the LTB spacetime from the generic initial condition[1, 2, 3, 4]. For example, in the case of marginally bound collapse of LTB spacetime, if the initial density profile $\rho(r)$ has a negative second derivative at the origin $\rho''(0) < 0$, then the shell focusing naked singularity appears at the origin. This genericity is the most important issue of the naked singularity examples. Here we consider stability of the solution against nonspherical metric perturbations to investigate the genericity of the solution.

2 Gauge invariant formulation

Here we introduce a gauge invariant formalism of the most general spherically symmetric spacetime[5]. We consider any spherically symmetric spacetime with a metric

$$g_{\mu\nu}^{(0)} dx^\mu dx^\nu = g_{ab} dx^a dx^b + R^2(t, r) d\Omega^2, \quad (1)$$

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and a energy-momentum tensor

$$T_{\mu\nu}dx^\mu dx^\nu = T_{ab}dx^a dx^b + \frac{1}{2}T_B{}^B R^2(t, r)d\Omega^2, \quad (2)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Latin indices $a, b, \dots$ refer to t and r and capital latin indices $A, B, \dots$ refer to θ and ϕ . We use the geometrized unit $G = c = 1$ in this article.

Odd parity perturbation which is away from eq.(1),(2) is

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & h_0\Phi_{mB}^l \\ & 0 & h_1\Phi_{mB}^l \\ & & h_2\chi_{mAB}^l \end{pmatrix}, \quad (3)$$

$$\delta T_{\mu\nu} = \begin{pmatrix} 0 & 0 & t_0\Phi_{mB}^l \\ & 0 & t_1\Phi_{mB}^l \\ & & t_2\chi_{mAB}^l \end{pmatrix}, \quad (4)$$

where $h_{0,1,2}$ and $t_{0,1,2}$ are the function of t and r , and Φ_{mA}^l and χ_{mAB}^l are the odd-parity vector and tensor spherical harmonics, respectively.

The set of odd-parity gauge invariant variables is defined as

$$k_a = h_a - \frac{1}{2}R^2\partial_a\left(\frac{h_2}{R^2}\right) \quad (\text{metric}), \quad (5)$$

$$L_a = t_a - \frac{1}{2}T_B{}^B h_a \quad (\text{matter}), \quad (6)$$

$$L = t_2 - \frac{1}{2}T_B{}^B h_2 \quad (\text{matter}). \quad (7)$$

Introduce these variables into linearized Einstein equations, then we get the odd-parity equations

$$k^a{}_{|a} = 16\pi L \quad (l \geq 2), \quad (8)$$

$$(R^4 W^{ab})_{|b} + (l-1)(l+2)k^a = 16\pi R^2 L^a \quad (l \geq 1), \quad (9)$$

where

$$W_{ab} = \left(\frac{hb}{R^2}\right)_{|a} - \left(\frac{h_a}{R^2}\right)_{|b} \quad (10)$$

$$= \left(\frac{k_b}{R^2}\right)_{|a} - \left(\frac{k_a}{R^2}\right)_{|b}. \quad (11)$$

The vertical bar, $|$, denotes covariant derivative with respect to t, r . The linearized conservation equations reduce to the single scalar equation

$$(R^2 L^a)_{|a} = (l-1)(l+2)L \quad (l \geq 1). \quad (12)$$

3 Basic equations

Here we apply gauge invariant formulation of section 2 into the LTB space-time.

3.1 Odd-parity perturbation on the LTB space-time

The LTB spacetime is expressed by the use of the synchronous and comoving coordinate system in which a metric is

$$ds^2 = -dt^2 + A^2(t, r)dr^2 + R^2(t, r)(d\theta^2 + \sin^2 \theta d\phi^2). \quad (13)$$

The matter is assumed to be the pressureless dust fluid whose energy-momentum tensor is given by

$$T_{\mu\nu} = \rho(t, r)u_\mu u_\nu, \quad (14)$$

where ρ is the energy-momentum density and u_μ is the 4-velocity of the fluid element, both of which are functions of t and r .

On the LTB space-time, gauge invariant matter perturbations become

$$\begin{cases} L_0 = \rho U \\ L_1 = L = 0, \end{cases} \quad (15)$$

for density perturbation vanishes and only velocity perturbation contributes to the perturbation of energy-momentum tensor. So, eqs.(8),(9) are

$$\partial_t (A k_0) = \partial_r \left(\frac{k_1}{A} \right), \quad (16)$$

$$\partial_r \left(\frac{R^2}{A} K_s \right) + A(l-1)(l+2)k_0 = 16\pi A R^2 L_0, \quad (17)$$

$$\partial_t \left(\frac{R^2}{A} K_s \right) + \frac{1}{A}(l-1)(l+2)k_1 = 0, \quad (18)$$

respectively. Here we introduce another gauge invariant variable

$$K_s = R^2 \partial_t \left(\frac{h_1}{R^2} \right) - R^2 \partial_r \left(\frac{h_0}{R^2} \right) = R^2 \partial_t \left(\frac{k_1}{R^2} \right) - R^2 \partial_r \left(\frac{k_0}{R^2} \right). \quad (19)$$

The conservation equation(12) becomes

$$\partial_t (A R^2 L_0) = 0. \quad (20)$$

This equation can be easily integrated

$$A R^2 L_0 = \partial_r J(r), \quad (21)$$

where $J(r)$ is arbitrary function of r . From this and eqs.(17),(18), we get the wave equation

$$\begin{aligned} \partial_t \left(\frac{A}{R^2} \partial_t (R^4 \psi_s) \right) - \partial_r \left(\frac{1}{AR^2} \partial_r (R^4 \psi_s) \right) + (l-1)(l+2) A \psi_s \\ = -16\pi \partial_r \left(\frac{\partial_r J(r)}{AR^2} \right), \end{aligned} \quad (22)$$

where

$$\psi_s = \frac{K_s}{AR^2}. \quad (23)$$

3.2 Analysis of $l=1$ mode

When $l = 1$, eqs.(18),(17) are

$$\partial_t \left(\frac{R^2}{A} K_s \right) = 0, \quad (24)$$

$$\partial_r \left(\frac{R^2}{A} K_s - 16\pi J(r) \right) = 0, \quad (25)$$

respectively. Eq.(25) can be easily integrated into

$$\frac{R^2}{A} K_s - 16\pi J(r) = c(t), \quad (26)$$

where $c(t)$ is any function of t . We substitute it into eq.(24). Then

$$\partial_t c(t) = 0 \quad (27)$$

$$\therefore c(t) = \text{const}. \quad (28)$$

From the regularity condition at the origin, this *const* is zero. Therefore, from eq.(26), we get the solutions of the differential equations (24),(25) as follows,

$$\frac{R^2}{A} K_s - 16\pi J(r) = 0 \quad (29)$$

$$\therefore K_s = 16\pi \frac{A}{R^2} J(r). \quad (30)$$

For $l \geq 2$ mode, we have to solve the wave equation (22). But it is very difficult to do it algebraically and we should calculate it numerically.

4 Riemann tensor

In order to see the change of the geometry, we investigate the perturbation of the Riemann tensor $R_{\mu\nu\alpha\beta}$. the Riemann tensor is decomposed into the Ricci tensor $R_{\mu\nu}$ (trace part)

and the Weyl tensor $C_{\mu\nu\alpha\beta}$ (traceless part). Further the Weyl tensor is decomposed into electric part $E_{\alpha\beta}$ and magnetic part $B_{\alpha\beta}$ which are defined by,

$$E_{\alpha\beta} = C_{\alpha\mu\beta\nu} n^\mu n^\nu, \quad (31)$$

$$B_{\alpha\beta} = \frac{1}{2} \epsilon_\alpha^{\sigma\mu\nu} C_{\mu\nu\beta\lambda} n_\sigma n^\lambda, \quad (32)$$

where $\epsilon_{\mu\nu\alpha\beta}$ is 4-dim skew tensor and n_μ is unit normal to the spacelike hypersurface,

$$n_\mu = (1/\sqrt{-g_{00}}, 0, 0, 0). \quad (33)$$

In the LTB spacetime, the Ricci tensor $R_{\mu\nu}$ have non-zero value in the region of $\rho \neq 0$ through the Einstein equation and also the electric part $E_{\alpha\beta}$ have non-zero value in the synchronous and comoving coordinate $n_\mu = u_\mu$. On the other hand the magnetic part $B_{\alpha\beta}$ with $n_\mu = u_\mu$ is identically zero. The magnetic part is deeply related to the irrotational motion of the dust and hence we should pay a special attention to its perturbation.

4.1 Perturbation of Riemann tensor

We calculate the axisymmetric perturbation ($m = 0$) of tetrad components of the Ricci and Weyl tensors. Here the tetrad of the perturbed LTB spacetime is chosen as

$$e_{(t)}^\mu = \left(1, 0, 0, \frac{h_0 P_{l,\theta}}{R^2 \sin \theta} \right), \quad (34)$$

$$e_{(r)}^\mu = \left(0, \frac{1}{A}, 0, -\frac{h_1 P_{l,\theta}}{AR^2 \sin \theta} \right), \quad (35)$$

$$e_{(\theta)}^\mu = \left(0, 0, \frac{1}{R}, -\frac{h_2}{2R^3 \sin^2 \theta} [\sin \theta P_{l,\theta,\theta} - \cos \theta P_{l,\theta}] \right), \quad (36)$$

$$e_{(\phi)}^\mu = \left(0, 0, 0, \frac{1}{R \sin \theta} \right), \quad (37)$$

where $P_l(\cos \theta)$ is the Legendre polynomial with angular momentum l and comma denotes partial derivative with respect to θ .

The perturbation of the Ricci tensor is expressed by the matter perturbation through the Einstein equation as

$$\delta(R_{(t)(\phi)}) = \frac{8\pi}{R} L_0 P_{l,\theta} = \frac{8\pi}{AR^3} \frac{dJ}{dr} P_{l,\theta}, \quad (38)$$

and other components vanish, where we use eq.(21) in the last equality. The perturbation of the tetrad components of the electric part $E_{\alpha\beta}$ are given as follows;

$$\delta(E_{(r)(\phi)}) = \left[\frac{1}{2AR^2} (l-1)(l+2)k_1 + \frac{\dot{R}}{2AR} K_s \right] \sin \theta P_{l,\theta}, \quad (39)$$

$$\delta(E_{(\theta)(\phi)}) = \frac{1}{2AR^2} \left[\left(\frac{k_1}{A} \right)' - \dot{A}k_0 \right] (\sin \theta P_{l,\theta,\theta} - \cos \theta P_{l,\theta}), \quad (40)$$

and the other components vanish. The perturbation of the tetrad components of magnetic part $B_{\alpha\beta}$ are obtained by

$$\delta(B_{(r)(r)}) = \frac{1}{2AR^2}l(l+1)K_sP_l, \quad (41)$$

$$\delta(B_{(r)(\theta)}) = \frac{1}{4AR^2} \left[R^2 \left(\frac{K_s}{A} \right)' - A(l-1)(l+2)k_0 \right] P_{l,\theta}, \quad (42)$$

$$\delta(B_{(\theta)(\theta)}) = -\frac{1}{AR^2} \left[R \left(\frac{k_0}{R} \right)' + \left(\frac{\dot{R}}{R} - \frac{\dot{A}}{A} \right) k_1 + \frac{1}{2}K_s \right] P_{l,\theta,\theta} \quad (43)$$

$$-\frac{1}{2AR^2} \left[R \left(\frac{k_0}{R} \right)' + \left(\frac{\dot{R}}{R} - \frac{\dot{A}}{A} \right) k_1 + K_s \right] l(l+1)P_l, \quad (44)$$

$$\delta(B_{(\phi)(\phi)}) = -\frac{1}{AR^2} \left[R \left(\frac{k_0}{R} \right)' + \left(\frac{\dot{R}}{R} - \frac{\dot{A}}{A} \right) k_1 + \frac{1}{2}K_s \right] \cot \theta P_{l,\theta} \quad (45)$$

$$-\frac{1}{2AR^2} \left[R \left(\frac{k_0}{R} \right)' + \left(\frac{\dot{R}}{R} - \frac{\dot{A}}{A} \right) k_1 + K_s \right] l(l+1)P_l, \quad (46)$$

and the other components vanish.

4.2 Behavior near the origin

Now we will investigate the behavior of the Ricci and Weyl tensors near the origin where the naked singularity appears. From the regularity conditions[6], we can see that the perturbations of the Ricci and Weyl tensors obtained above behave near the origin as follows;

for the Ricci tensor,

$$\delta(R_{(t)(\phi)}) \rightarrow \frac{8\pi}{R_c} L_c r^l; \quad (47)$$

for the Weyl tensor of $l \geq 2$,

$$\delta(E_{(r)(\phi)}) \rightarrow \frac{1}{2R_c^4}(l-1) \left[(l+2)k_{1c} - R_c \dot{R}_c k_{0c} \right] (\partial_\theta P_l) r^{l-1}, \quad (48)$$

$$\delta(E_{(\theta)(\phi)}) \rightarrow \frac{1}{2R_c^4} \left[(l+2)k_{1c} - R_c \dot{R}_c k_{0c} \right] (\partial_\theta^2 P_l - \cot \theta \partial_\theta P_l) r^{l-1}, \quad (49)$$

$$\delta(B_{(r)(r)}) \rightarrow -\frac{1}{2R_c^3}(l-1)l(l+1)k_{0c}P_l r^{l-2}, \quad (50)$$

$$\delta(B_{(r)(\theta)}) \rightarrow -\frac{1}{2R_c^3}(l-1)l(l+1)k_{0c}(\partial_\theta P_l) r^{l-2}, \quad (51)$$

$$\delta(B_{(\theta)(\theta)}) \rightarrow -\frac{1}{2R_c^3}(l+1)k_{0c}(\partial_\theta^2 P_l + lP_l) r^{l-2}, \quad (52)$$

$$\delta(B_{(\phi)(\phi)}) \rightarrow -\frac{1}{2R_c^3}(l+1)k_{0c}(\cot \theta \partial_\theta P_l + lP_l) r^{l-2}, \quad (53)$$

and the other components vanish;
for the Weyl tensor of $l = 1$,

$$\delta(E_{(r)(\phi)}) \rightarrow -\frac{1}{2R_c^3}\dot{R}_c K_{sc} r^2 \sin \theta, \quad (54)$$

$$\delta(B_{(r)(r)}) \rightarrow \frac{1}{R_c^3} K_{sc} r \cos \theta, \quad (55)$$

$$\delta(B_{(r)(\theta)}) \rightarrow -\frac{1}{4R_c^3} K_{sc} r \sin \theta, \quad (56)$$

$$\delta(B_{(\theta)(\theta)}) \rightarrow -\frac{1}{2R_c^3} K_{sc} r \cos \theta, \quad (57)$$

$$\delta(B_{(\phi)(\phi)}) = \delta(B_{(\theta)(\theta)}), \quad (58)$$

and the other components vanish.

From the above equations we see that only the quadrapole mode $l = 2$ of the magnetic part $B_{\alpha\beta}$ does not vanish at the origin and hence the naked singularity formation may be mostly affected by the quadrapole mode.

5 Approximate analysis

Here we consider the time evolution of ψ_s with $J(r) = 0$, i.e., the dynamics which obeys the homogeneous propagation equation. Near the origin, the background spacetime is well approximated by the flat Friedmann-Robertson-Walker (FRW) spacetime because of the regularity condition.

5.1 Propagation in the FRW spacetime

When the background is the FRW spacetime, metric functions are

$$R(t, r) = a(t)r, \quad A(t, r) = a(t), \quad (59)$$

where

$$a(t) = a_0(-t/t_c)^{2/3}. \quad (60)$$

We choose the time coordinate $t \leq 0$ as singularity appears at $t = 0$. Here we introduce dimensionless variables

$$\tau \equiv \frac{t}{t_c}, \quad z \equiv \frac{a_0 r}{t_c}. \quad (61)$$

Then, the homogeneous propagation equation for ψ_s becomes

$$\psi_{s,\tau,\tau} = (-\tau) \left(\psi_{s,z,z} + \frac{6}{z} \psi_{s,z} \right) - \frac{14}{3} \frac{1}{\tau} \psi_{s,\tau} - \frac{8}{3} \frac{1}{\tau^2} \psi_s, \quad (62)$$

where commas followed by τ and by z denote $\partial/\partial\tau$ and $\partial/\partial z$, respectively. Further we shall introduce the conformal time η defined by

$$\eta \equiv -3(-\tau)^{1/3}, \quad (63)$$

and a new variable ϕ_s by

$$\phi_s \equiv z^3 \eta^6 \psi_s. \quad (64)$$

then eq.(62) becomes as

$$\phi_{s,\eta,\eta} - \frac{6}{\eta} \phi_s = \phi_{s,z,z} - \frac{6}{z} \phi_s. \quad (65)$$

The solution of the above equation is easily obtained by the method of separation of variables. Denoting $\phi_s = T(\eta)V(z)$, we obtain the equations for T and V as

$$\frac{d^2 T}{d\eta^2} + \left(\sigma^2 - \frac{6}{\eta^2} \right) T = 0, \quad (66)$$

$$\frac{d^2 V}{dz^2} + \left(\sigma^2 - \frac{6}{z^2} \right) V = 0, \quad (67)$$

where σ is an arbitrary constant. The above equations are completely same and the solutions are expressed by the $\pm 5/2$ th Bessel functions. Imposing the regularity on V at the origin, then we obtain a solution for ϕ_s as

$$\phi_s = \sqrt{-z\eta} \left[C_+ J_{\frac{5}{2}}(-\sigma\eta) + C_- J_{-\frac{5}{2}}(-\sigma\eta) \right] J_{\frac{5}{2}}(\sigma z) \quad (68)$$

where $C_{\pm}$ are arbitrary constants. To construct a general solution, we use the Bessel transformation and the result is given by

$$\psi_s = \frac{1}{z^{5/2}(-\eta)^{11/2}} \int_0^\infty \left\{ f_+(\sigma) J_{\frac{5}{2}}(-\sigma\eta) + f_-(\sigma) J_{-\frac{5}{2}}(-\sigma\eta) \right\} J_{\frac{5}{2}}(\sigma z) \sigma d\sigma, \quad (69)$$

where $f_{\pm}(\sigma)$ are arbitrary functions. Then the asymptotic behavior for $\eta \rightarrow 0$ is given by

$$\psi_s \rightarrow \text{const} \times \eta^{-8}. \quad (70)$$

Therefore if the gravitational wave has enough time to grow at the origin, the naked singularity can be considerably affected.

5.2 The region of approximation

In the marginally bound collapse solution of LTB space time, we can express $R(t, r)$ as follows,

$$R(t, r) = \left(\frac{9}{2} m(r) \right)^{1/3} (t_s(r) - t)^{2/3}, \quad (71)$$

where $m(r)$ is mass function and $t_s(r)$ is chosen with some arbitrary constants t_c and r_c as

$$t_s(r) = t_c \left(\frac{r}{r_c} \right)^2. \quad (72)$$

We can approximate this solution to the one of FRW spacetime if the condition,

$$t_s(r) \ll |t|, \quad (73)$$

is satisfied. Here we define the region which can be approximated by FRW spacetime is

$$t_s(r) = t_c \left(\frac{r}{r_c} \right)^2 < -t. \quad (74)$$

If we use the conformal time η instead of t , this condition can be expressed as

$$\eta < -3 \left(\frac{r}{r_c} \right)^{2/3}. \quad (75)$$

The conformal time η when the origin is determined only by the FRW region has upper bound η_f . For $\eta > \eta_f$, we should consider the effect of inhomogeneity of background spacetime.

6 Conclusion

In this article, we have investigated the behavior of the odd parity perturbation of the LTB spacetime. Near the origin, we can approximate the background spacetime to the FRW spacetime. When the back ground is the flat FRW spacetime, the homogeneous propagation equation can be solved analytically and we obtain the solution which will grow toward the singularity at the origin. However the region which can be approximated to FRW spacetime is restricted. Just before the occurrence of the singularity, we need information of inhomogeneous region. To investigate the effect of inhomogeneity on the evolution of the perturbation, we must solve the wave equation(22) numerically.

Acknowledgments

We would like to thank H.Sato for useful discussions and continuous encouragement.

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Cylindrical Domain Walls and Gravitational Waves

– Scattering Problem of Einstein-Rosen Waves –

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Abstract

A cylindrical spacetime with a domain wall is considered in order to set up the scattering problem of gravitational wave by a domain wall to clarify the relation between spontaneous emission of gravitational wave by domain walls and local curvature scales of its world volume. In general, to consider a cylindrical domain wall including its gravity corresponds to the radiation reaction problem. Furthermore, it is possible to set the scattering problem if one admit singular spacetime as a background spacetime of scattering problem.

1 Introduction

Domain walls are the simplest type of topological defects associated with the symmetry breaking in unified theories. In the early universe, the existence of topologically stable domain walls can easily be ruled out because they overdominate the radiation energy density[1]. On the other hand, unstable ones such as walls bounded by strings and domain wall bubbles might be harmless to cosmological models[2]. The estimation by Vachaspati et.al[3], provided that the energy loss rate which is given by the quadrupole formula, tells us the dominant decay process of domain walls is the emission of gravitational waves. Although this is an interesting conclusion, it is unclear whether or not it can be taken seriously. Because there is no guarantee that the quadrupole formula does give a correct estimate of the gravitational wave emission rate by topological defects which has a large negative internal pressure.

As a first step in order to give a complete answer to this question, an exactly solvable case is considered within the general-relativistic perturbation framework[4]. In Ref.[4], the exact solution which is spherically symmetric and locally Minkowskian outside a domain wall[5] as the background spacetime is considered. The domain wall is treated as an infinitesimally thin singular hypersurface with a surface energy-momentum density specific to the domain wall (this treatment is called *thin wall approximation*) and the whole spacetime consists of two identical Minkowskian regions matched at the domain wall. On this special background, the spacetime perturbation can be solved explicitly, and the global solutions can be obtained by matching via the junction condition of Israel[6]. This consideration corresponds to the gravitational wave scattering problem by a domain wall.

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and then the fact that no resonance pole exists in this problem is obtained. This fact can be interpreted that the domain wall does not emit gravitational waves spontaneously by its free oscillation. The same result comes from another solvable model with the background spacetime which consists of deSitter and Minkowski regions matched at a domain wall[7]. This result is quite embarrassing because, in the treatment neglecting gravity, a domain wall has its own dynamical freedom and its oscillation is expected to become a source of gravitational waves even in the first order with respect to its oscillation amplitude. Then, different qualitative behavior of the wall is obtained whether one neglects the gravity or not.

One of the reasons for this difference is in the estimation of the order of the back reaction effect to the motion of the domain wall due to the gravitational wave emission[7]. In the above two solvable model, the equation for the perturbative motion of the wall reduced from the junction formula coincide with the equation derived from Nambu action together with the perturbative term of geometries. This shows that the perturbative motion of a domain wall, gravitational emission and its back reaction to the wall motion are all same order. It is quite different from the picture which is considered when one derive the quadrupole formula for gravitational wave emission. However, it might not necessarily leads to the above conclusion that a domain wall does not emit gravitational wave spontaneously by its free oscillation, though it tells us that the effect of the back reaction is quite important when one considers the motion of domain walls and the emission of gravitational wave from them.

Another possible reason is the topology or intrinsic curvature of the world volume of the domain wall. In above analyses, the world volume of the domain wall is three dimensional deSitter space. On the other hand, one of typical examples which has resonance poles for the perturbative gravitational wave is Schwarzschild spacetime with mass scale M . This resonance poles are called quasi-normal modes and their frequencies are proportional to $1/M$. It might be natural to consider that there are some relations between resonance poles or spontaneous emission of gravitational wave and local curvature scales of background spacetime. So, in this talk, we pay our attention to the scale in the intrinsic curvature of the world volume of the domain wall, and propose a model to clarify the essential reason for the no spontaneous emission of gravitational wave due to the motion of domain wall, which has a local curvature scale in the intrinsic curvature of the world volume of the domain wall. Our model proposed here is a cylindrically symmetric domain wall.

A cylindrically symmetric domain wall in $3 + 1$ dimensional spacetime is essentially different from a spherically symmetric one. A spherically symmetric domain wall can move without emission of gravitational wave, since there is no local modes of gravity in a spherically symmetric spacetime. Intuitively, one might consider that a cylindrically symmetric domain wall in $3 + 1$ dimensional spacetime is equivalent to a cosmic string in $2 + 1$ dimensional spacetime[8], in which there is no local modes. However, to consider a cylindrical domain wall including its gravity corresponds to the radiation reaction problem like the motion of an accelerated electron in classical electromagnetism, in general. There is local modes, which is called *Einstein-Rosen wave*, in cylindrically symmetric spacetime unlike spherically symmetric one, and a cylindrical domain wall excites this Einstein-Rosen wave. So, it seems to be difficult to set up scattering problem of gravitational wave

by a cylindrical domain wall.

However, it is possible to set up scattering problem if we admit singular spacetime as a background which is an exact solution discussed by Ipser and Sikivie[5]. Though this spacetime has planer symmetry and locally Kasner spacetime outside a domain wall, this symmetry may be regard as a cylindrical one. The domain wall is treated by thin wall approximation like above analyses. The whole spacetime consists of two identical Kasner region matched by the domain wall. On this background solution, we consider only cylindrically symmetric mode *i.e.* the perturbative Einstein-Rosen wave as gravitational wave, for simplicity. Although our background spacetime has naked singularities, we adopt an appropriate boundary condition for the gravitational wave at naked singularity in the hope that the existence of these singularities is not essential to the relation between a local curvature scale and spontaneous emission of gravitational wave due to the domain wall motion. Unfortunately, there is no conclusion in this proceeding. It will be obtained in near future.

Throughout this proceedings, we use the units such that the light velocity $c = 1$ and $\kappa^2 = 8\pi G$ where G is Newton's constant of gravity. The signature of Lorentzian metrics is chosen to be $(-, +, +, \dots)$ and indices $a, b, \dots$ runs from 0 to 3.

2 Cylindrical Spacetime and Domain Wall

A cylindrical spacetime can be characterized by the existence of two commutable spacelike Killing vector fields, which are both hypersurface orthogonal. The Killing orbit for one of these Killing vector fields, which is denoted by z^a , is R^1 , and the other one, which is denoted by ϕ^a , is S^1 . We introduce coordinate functions z and ϕ by $z^a \nabla_a z = 1$ and $\phi^a \nabla_a \phi = 1$, respectively, and introduce a time coordinate t and a radial coordinate r so that the basis $\{\nabla_a t, \nabla_a r, z_a, \phi_a\}$ are orthogonal to each other. When the momentum densities along these Killing directions vanish and energy momentum tensor of matter fields satisfy the condition $T_t^t + T_r^r = 0$, spacetime metric can be written by

$$ds^2 = e^{2(\gamma-\psi)}(-dt^2 + dr^2) + e^{2\psi}dz^2 + r^2e^{-2\psi}d\phi^2, \quad (1)$$

which is called *Weyl canonical form*, ψ and γ are functions of t and r . From this metric (1), in the vacuum region of spacetime Einstein equations can be reduced to the following three equations,

$$\partial_t^2 \psi - \frac{1}{r} \partial_r (\partial_r r \psi) = 0, \quad (2)$$

$$\partial_r \gamma = r \left((\partial_t \psi)^2 + (\partial_r \psi)^2 \right), \quad (3)$$

$$\partial_t \gamma = 2r(\partial_t \psi)(\partial_r \psi). \quad (4)$$

Note that (2) is a wave equation for cylindrically symmetric scalar wave in a flat spacetime. These equations can be solved explicitly and the solution is well-known as Einstein-Rosen wave.

Next, we consider Israel's Junction formula to construct cylindrically symmetric spacetime with a domain wall within the thin wall approximation[9]. Our whole spacetime

$(\mathcal{M}, g_{ab})$ contains two vacuum regions $(\mathcal{M}_+, g_{ab+})$ and $(\mathcal{M}_-, g_{ab-})$ with boundaries $\partial\mathcal{M}_\pm = \Sigma_\pm = \Sigma$. This boundary Σ corresponds to the world volume of the domain wall and (Σ, h_{ab}) is a submanifold of the whole spacetime $\mathcal{M} = \mathcal{M}_- \cup \Sigma \cup \mathcal{M}_+$, where h_{ab} is the induced metric on Σ from g_{ab} and it is also projection operator from the tangent space of $\mathcal{M}$ to that of Σ . Since we consider a cylindrical domain wall, we may assume that $g_{ab\pm}$ can be represented by Weyl canonical form (1).

Now, we denote 4-velocity of domain wall by u^a ($u^a u_a = 1$), assume that $u^a z_a = u^a \phi_a = 0$, and introduce time coordinate τ by $u^a \nabla_a \tau = 1$. Furthermore, we assume that the Killing vectors $z_\pm^a, \phi_\pm^a$ on $\Sigma_\pm$ can be identified with Killing vectors on Σ such that $z^a|_\Sigma = z_\pm^a = h_b^a z_\pm^b$, $\phi^a|_\Sigma = \phi_\pm^a = h_b^a \phi_\pm^b$, respectively, and extend functions z and ϕ on $\mathcal{M}_\pm$ as smooth functions on $\mathcal{M}$. By assuming the function r on $\mathcal{M}_\pm$ to be a C^1 -function of $\mathcal{M}$, these above assumptions become to be consistent and the world volume of our domain wall Σ is characterized by a function $R = R(\tau)$ where R is the value of the function r on Σ with a fixed value of τ . In terms of this coordinate system $\{\tau, z, \phi\}$, the metric h_{ab} on Σ can be given by

$$ds^2|_\Sigma = h_{ab} dx^a dx^b = -d\tau^2 + e^{2\psi_\pm} dz^2 + R^2 e^{-2\psi_\pm} d\phi^2, \quad (5)$$

where $\psi_\pm = \psi_\pm(\tau) = \psi_\pm$ and $\psi_\pm$ is a projected values of metric function ψ on $\Sigma_\pm$, respectively.

Equation (5) gives intrinsic geometry of Σ . On the other hand, junction formula also gives the condition for extrinsic curvature $K_{\pm}^a = h^{ac} h_b^d \nabla_c n_{d\pm}$ of $\Sigma_\pm$ in $\mathcal{M}_\pm$ where $n_{a\pm}$ are unit normal vector of $\Sigma_\pm$, respectively. For a domain wall which is created by the symmetry breaking of scalar field, the junction condition is reduced to

$$[K_b^a] = -\lambda h_b^a, \quad (6)$$

where λ must be positive constant on Σ due to the momentum constraint on Σ and it corresponds to the energy density of the domain wall. In (6),

$$[A] = A_+ - A_-, \quad (7)$$

for arbitrary functions $A_\pm$ on $\Sigma_\pm$. The coordinate system $\{\tau, z, \phi\}$ is characterized by u^a , z^a and ϕ^a and the components of u^a and $n_\pm^a$ in terms of Weyl canonical form on $\mathcal{M}_\pm$ are given by

$$n_\pm^a = \frac{dR}{d\tau} \left(\frac{\partial}{\partial t_\pm} \right)^a + X_\pm \left(\frac{\partial}{\partial r} \right)^a, \quad u^a = \left(\frac{\partial}{\partial \tau} \right)^a = X_\pm \left(\frac{\partial}{\partial t_\pm} \right)^a + \frac{dR}{d\tau} \left(\frac{\partial}{\partial r} \right)^a. \quad (8)$$

where

$$X_\pm = \frac{dt_\pm}{d\tau} = \sqrt{\left(\frac{dR}{d\tau} \right)^2 + e^{-2(\gamma_\pm - \psi_\pm)}}, \quad (9)$$

$\gamma_\pm = \gamma_\pm(\tau)$ are projected values of metric function $\gamma(t, r)$ in (1) on $\Sigma_\pm$, respectively. In (8), we choose u^a to be future directed and $n_\pm^a \nabla_a R > 0$. (If one impose reflection symmetry on $\mathcal{M}$ with respect to Σ , we should choose $n_-^a \nabla_a R > 0$ and $n_+^a = -n_-^a$.) In

terms of the coordinate system $\{\tau, z, \phi\}$, the components of extrinsic curvature $K_{b\pm}^a$ are given by

$$K_{\tau\pm}^\tau = \frac{1}{X_\pm} \left(\frac{d^2 R}{d\tau^2} - (D_\parallel \psi) \frac{dR}{d\tau} + R((D_\parallel \psi)^2 + (D_\perp \psi)_\pm^2) \right) - (D_\perp \psi)_\pm, \quad (10)$$

$$K_{z\pm}^z = (D_\perp \psi)_\pm, \quad (11)$$

$$K_{\phi\pm}^\phi = \frac{1}{R} X_\pm - (D_\perp \psi)_\pm, \quad (12)$$

where $D_\parallel A = u^a \nabla_a A$, $(D_\perp A)_\pm = n_\pm^a \nabla_a A$ for arbitrary scalar function A on Σ . Hence, (6) can be reduced to

$$\left(-\frac{d^2 R}{d\tau^2} + \frac{dR}{d\tau} D_\parallel \psi - R(D_\parallel \psi)^2 \right) \left[\frac{1}{X} \right] - R \left[\frac{(D_\perp \psi)^2}{X} \right] = 2\lambda, \quad (13)$$

$$[D_\perp \psi] = -\lambda, \quad (14)$$

$$[X] = -2\lambda R. \quad (15)$$

When $\gamma_\pm$ and ψ_s are boundary values of a solution of (2-4) on $\mathcal{M}_\pm$, respectively, the other constraints are automatically satisfied. Furthermore, using (2-4), (8) and (9), (13) can be reduced from (14) and (15). Hence, to construct cylindrical spacetime with a domain wall, we must solve Eq.(2-4) on $\mathcal{M}_+$ and $\mathcal{M}_-$, separately, and then build up $\mathcal{M} = \mathcal{M}_- \cup \Sigma \cup \mathcal{M}_+$ by imposing boundary conditions (14) and (15).

Physical meaning of above procedure becomes to be more clear if one impose reflection symmetry on $\mathcal{M}$ with respect to Σ . In this case, (14) and (15) can be reduced to

$$\left(\frac{dR}{d\tau} (\partial_t \psi) + \lambda R (\partial_r \psi) \right) \Big|_\Sigma = \frac{\lambda}{2}, \quad (16)$$

$$\left(\frac{dR}{d\tau} \right)^2 + e^{-2(\gamma_s - \psi_s)} = \lambda^2 R^2, \quad (17)$$

where $\gamma_s = \gamma(t, r)|_\Sigma$. (16) gives boundary condition for Einstein-Rosen wave ψ which is determined by wave equation (2). Hence, to construct a cylindrical domain wall spacetime with reflection symmetry, one must solve (2) under the boundary condition (16), and then integrate (3), (4) to obtain $\gamma(t, r)$. Then, one can obtain the motion of the wall by (17). This is just the radiation reaction problem, i.e. a cylindrical domain wall becomes to be a source of Einstein-Rosen wave ψ . Note that this radiation is due to the condition for the $z - z$ components of extrinsic curvature (14) which affect to the excitation of the local mode of Einstein-Rosen wave. It also shows that a loop of cosmic string in 2 + 1 dimensional spacetime is not equivalent to a cylindrically symmetric domain wall.

3 Einstein Rosen Wave Scattering

As mentioned in previous section, the problem to build up a cylindrical spacetime with a domain wall corresponds to the radiation reaction problem, in general. On the other hand, it is possible to set the scattering problem of gravitational wave by a domain wall if we

admit singular spacetime. In this section, we propose the model of a cylindrical domain wall as a scattering problem to clarify the relation between the spontaneous gravitational wave emission from domain walls and local curvature scale of its world volume.

3.1 Background spacetime

Now, we consider a special solution of vacuum Einstein equation (2-4)

$$\psi = \frac{1}{2} \ln \left(\frac{r}{\alpha} \right), \quad \gamma = \frac{1}{4} \ln \frac{r}{\alpha} \quad (18)$$

as a background spacetime of our scattering problem and hence spacetime metric is given by

$$ds^2 = \sqrt{\frac{\alpha}{r}} (-dt^2 + dr^2) + \frac{r}{\alpha} dz^2 + \alpha r d\phi^2, \quad (19)$$

where α is a constant which has a dimension of length. This solution is a static solution of (2-4). Since general static solution of (2-4) is well-known as spatially Kasner solution, (18) is also a Kasner solution with a fixed Kasner parameter. We note that this spacetime has a local curvature scale α , but the physical meaning of this scale α is not clear.

We also note that this solution belongs to type D of Petrov classification, in the spacetime (19) all points on the axis $r = 0$ are curvature singularities and the Riemann polynomial diverge as $R^{abcd} R_{abcd} \propto r^{-3}$. These singularities are naked. On the other hand, all components of Riemann curvature R_{abcd} vanish when $r \rightarrow \infty$. In this sense, we may regard our spacetime to be asymptotically flat[10]. Furthermore, since the section $dz = d\phi = 0$ of our spacetime is conformally flat and we consider the region $(t, r) \in (-\infty, \infty) \times (0, \infty)$, it is easy to clarify the global structure of our spacetime by usual procedure[11] and one can write conformal diagram of this section $dz = d\phi = 0$ which has timelike, spatial and null infinities. This conformal diagram is analogous to Schwarzschild spacetime with a negative mass parameter.

Using this spacetime (19), one can easily obtain a domain wall spacetime with naked singularities[5]. We impose reflection symmetry on $\mathcal{M}$ with respect to Σ . In this case, (16) becomes to be trivial and the motion of the domain wall can be obtained by

$$\left(\frac{dR}{d\tau} \right)^2 + \left(\frac{R}{\alpha} \right)^{1/2} - \lambda^2 R^2 = 0 \quad (20)$$

from (17). Here the local energy condition for the domain wall $\lambda > 0$ leads to $X \equiv X_- = -X_+ > 0$ which means that $\mathcal{M}_\pm$ must be the interior of cylinder $r = R(\tau)$ with naked singularities. The last term of the left hand side of (20) caused by the negative pressure of the domain wall itself and this term leads to deSitter expansion. The second term corresponds to the repulsive force which is generated by the naked singularities.

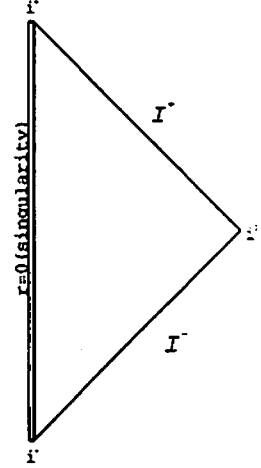


Fig.1: The conformal diagram of the section $dz = d\phi = 0$ of the spacetime (19)

By virtue of $dt/d\tau = X = \lambda R$, this motion of the domain wall in (t, r) -plane can be obtained by the solution of a differential equation

$$\left(\frac{dr}{dt}\right)^2 + \frac{1}{\lambda^2 \alpha^{1/2} r^{3/2}} - 1 = 0. \quad (21)$$

Although a solution of this equation cannot be obtained explicitly, it is easy to see the behavior of the wall motion from (21). Seeing from an observer whose world line is $r = \text{constant}$ in either vacuum region, the domain wall first starts to shrink from the past null infinity at the light velocity, decelerates its speed, and bounces at a finite radius $r = \lambda^{-4/3} \alpha^{-1/3}$. After that it expands to the null infinity. Hence, our resulting spacetime $\mathcal{M}$ by the Israel's formula has future and past null infinities each of which have two connected components. The global structure of this spacetime is analogous to that in Ref.[4]. When $\alpha \sim r$, the solution of (21) is approximately three dimensional deSitter spacetime and Σ is contained in the region $r^2 - t^2 > 0$ of our spacetime. However, our spacetime is singular one, the topology of Σ is $R^2 \times S^1$. This caused by the local curvature scale α . Hence, we may say that our spacetime is constructed by introducing the curvature scale α into the spacetime in Ref.[4] then spacetime has become singular and the topology of the world volume of the domain wall has changed.

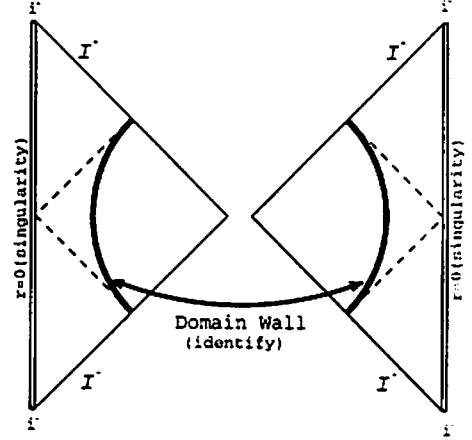


Fig.2: The construction of our background spacetime $\mathcal{M} = \mathcal{M}_- \cup \Sigma \cup \mathcal{M}_+$.

3.2 Scattering Problem

As mentioned above, we have constructed a cylindrically symmetric background spacetime of the scattering problem for gravitational wave, which has a local curvature scale α unlike the background in Ref.[4]. Although Σ cannot be expressed by an explicit form as a function of t and r , one can construct scattering problem of gravitational wave by the domain wall. For simplicity, our consideration is restricted to the only the perturbative Einstein-Rosen wave (2), though it is possible to consider all modes of perturbation of geometry.

To consider the perturbative Einstein-Rosen wave, we add a perturbative term to the first equation of (18),

$$\psi = \frac{1}{2} \ln \left(\frac{r}{\alpha} \right) + \epsilon \varphi, \quad (22)$$

where ϵ is a infinitesimal parameter for the perturbation theory. Since (2) is linear, the equation for φ is the same as (2). Within the accuracy of $O(\epsilon)$, one can easily obtain the

metric function

$$\gamma = \frac{1}{4} \ln \frac{r}{\alpha} + \epsilon\varphi, \quad (23)$$

where we have ignored the constant term of γ which corresponds to the deficit angle around axis $r = 0$. We must note that $\epsilon\varphi$ does not affect to the section $d\phi = dz = 0$ of the spacetime within the accuracy of $O(\epsilon)$, i.e. the $t - r$ part of the metric (19) is unchanged.

To construct the global solution of perturbative Einstein-Rosen wave, we must consider the Israel's junction condition at Σ ,

$$[\varphi] = 0, \quad [D_{\perp}\varphi] = 0, \quad (24)$$

where we have not imposed the reflection symmetry on the perturbative wave φ . Nevertheless the perturbative Einstein-Rosen wave corresponds to the even mode of the perturbation with a background spacetime (19), this boundary conditions have the same as that for the odd mode perturbation in Ref.[4]. So, this boundary condition might lead the same conclusion in Ref.[4]. But it is not trivial because our background world volume of the domain wall Σ is not three dimensional deSitter space.

We must also note that the junction condition (15) leads to the same equation of motion of the domain wall as (20) even if we include the perturbative term $\epsilon\varphi$ within the accuracy of $O(\epsilon)$, because the perturbative term $\epsilon\varphi$ does not affect to the $t - r$ part of the metric (19). However, this does not mean that $\epsilon\varphi$ does not make any fluctuations of geometry. Actually, the circumference function $\mathcal{R} = 2\pi e^{-\psi} R$ of our domain wall along the Killing direction ϕ^a , which has a definite geometrical meaning, is fluctuated as

$$\mathcal{R} = 2\pi\sqrt{\alpha R}(1 - 2\epsilon\varphi_s) \quad (25)$$

within the same accuracy, where φ_s is a value of φ on Σ and a function of τ . Then, the intrinsic geometries of Σ are fluctuated by the perturbative Einstein-Rosen wave φ only when φ_s does not vanish. Thus, there is a direct relation between gravitational waves and fluctuations of Σ . This situation is the same one in Ref.[4].

Now, we set up the scattering problem of the perturbative Einstein-Rosen wave by the domain wall. First, we must consider the boundary condition of the perturbation at the singular axes $r = 0$. Einstein-Rosen wave is determined by the wave equation (2) whose solutions can be represented by the superposition of Bessel function and Neumann function with the order of 0. When one treat regular spacetime, one impose the boundary condition so that there is no Neumann mode which has logarithmic divergence at $r \rightarrow 0$. However, since our background spacetime has naked singularities at axes $r = 0$, there is no guiding principle to impose the boundary conditions at $r = 0$. But we impose the boundary condition such that there is no Neumann mode in the hope that the existence of this singularity is not essential for the relation between local curvature scales and the spontaneous emission of gravitational wave by domain walls. To impose this boundary condition corresponds to the restriction of our consideration of perturbative modes to the modes which is regular at $r = 0$ and belongs to $L^2(-\infty, \infty)$ with respect to t for any fixed value of r and corresponds to giving these singularities a physical character such that these singularities of our background spacetime are neither source nor sink for Einstein-Rosen wave.

In general, if one adopt the above boundary condition for Einstein-Rosen wave, there is no spontaneous emission of Einstein-Rosen wave in a cylindrical spacetime. Because Einstein-Rosen wave propagates on the flat space ($t - r$ coordinate plane) without any effects of gravitational potential. Hence, one may say that if one can obtain a sort of resonance poles of Einstein-Rosen wave in our cylindrical background this is caused by the existence of the domain wall.

As mentioned previously, the world volume Σ of the domain wall is contained in the region $r^2 - t^2 > 0$ of spacetime. So, to construct scattering problem of gravitational wave by our domain wall it is useful and enough to consider only the interior of the Rindler patch (ζ, χ) defined by

$$r = \chi \cosh \zeta, \quad t = \chi \sinh \zeta. \quad (26)$$

If one define the wave variable by $\Phi = \sqrt{r}\varphi$, in terms of the coordinate (ζ, χ) wave equation (2) can be given by

$$-\partial_\zeta^2 \Phi + \chi \partial_\chi (\chi \partial_\chi \Phi) + \frac{1}{4 \cosh^2 \zeta} \Phi = 0. \quad (27)$$

The solution of this equation can be obtained as follows,

$$\Phi \propto P_{-1/2}^{\pm ik}(\tanh \zeta) \chi^{ik}, \quad (28)$$

where k is a separation constant. Here, we consider the modes with real k . This corresponds to our boundary condition for the perturbation. As a solution of (27), we adapt the superposition of (28) such that

$$\sqrt{r}\varphi = \Phi = \int_{-\infty}^{\infty} dk P_{-1/2}^{-ik}(\tanh \zeta) (C_k e^{ik \ln \chi} + D_k e^{-ik \ln \chi}). \quad (29)$$

This is reduced to

$$\sqrt{r}\varphi \rightarrow \int_{-\infty}^{\infty} dk (C_k e^{ik \ln |u|} + D_k e^{-ik \ln |v|}), \quad (30)$$

in the limit $\tau \rightarrow \pm\infty$, where $v = t + r$, $u = t - r$. (30) shows that C_k corresponds to the amplitude of incoming wave from the region $\tau \rightarrow -\infty$ to our Rindler wedge and D_k corresponds to the amplitude of outgoing wave from our Rindler wedge to the region $\tau \rightarrow +\infty$. To obtain the solutions of this scattering problem of Einstein-Rosen wave by our domain wall, we must consider the solution (29) in two region of spacetime ($\mathcal{M}_\pm, g_{ab\pm}$). We denote the amplitudes of the incoming and outgoing wave in these two regions by $C_k^\pm$, $D_k^\pm$, respectively. Then we must determine the relation between $D_k^\pm$ and $C_k^\pm$ by imposing boundary condition (24). Thus, we have been able to set up the scattering problem of gravitational wave by the domain wall with a local curvature scale α in the region of Rindler wedge of our spacetime $\mathcal{M}$. Our formulation is very simple and very analogous to that in Ref.[4].

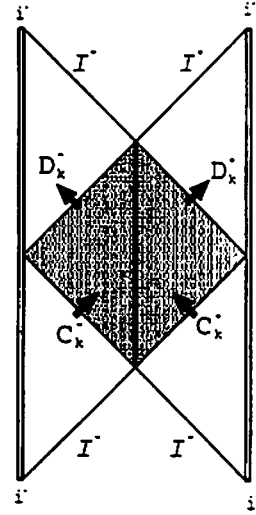


Fig.3: The background spacetime $\mathcal{M}$ of our scattering problem. The shaded region is $r^2 - t^2 > 0$ in which Σ is contained.

4 Summary and Discussion

In summary, we have considered a cylindrical domain wall to clarify the relation between the spontaneous emission of gravitational wave due to the motion of domain walls and a local curvature scale of its world volume. We have considered the formulation to construct a cylindrically symmetric spacetime with a domain wall from more general point of view. In general, to consider a cylindrical domain wall including its gravity corresponds to the radiation reaction problem like the gravitational wave emission by the accelerated point mass. It might be difficult to obtain a regular solution of this system.

On the other hand, we consider a special solution of above system as a background spacetime and perturbative Einstein-Rosen wave. Although our background spacetime is a singular one, we have set a well-defined scattering problem by the restriction of our consideration to the perturbative modes which does not diverge at singularities. Our main purpose is to clarify the relation between the spontaneous emission of gravitational wave via the wall motion and a curvature scale of its world volume α . To discuss this relation, all we have to do is to obtain the relation between the amplitude of incoming wave $C_k^\pm$ and that of outgoing wave $D_k^\pm$ by imposing Israel's junction condition and to seek the resonance poles from this relation. However, it does not seem to be easy to obtain this resonance poles exactly, since one cannot obtain the explicit solution of the equation of background motion of the domain wall. It seems to be better to try to obtain the relation between $C_k^\pm$ and $D_k^\pm$ by some appropriate approximations. This will be done in near future. Of course, there is no guiding principle to impose the boundary conditions at singularities and there is no way to confirm whether our boundary condition is appropriate or not. We hope that this mode restriction is not essential to our purpose.

Since our domain wall might be unstable for the other mode, we might have to consider general modes of perturbation at linear level to complete our consideration. However, since our purpose is to clarify the α dependence in the relation between $C_k^\pm$ and $D_k^\pm$, it will be enough to consider only Einstein-Rosen wave as a first step. If we obtain some resonance poles which depend on α , it might be interpret that the domain wall oscillate with the frequency which is affected by the gravitational potential near the domain wall. Of course we might obtain the same result as in Ref.[4]. Even if so, it is interesting to compare with each other to clarify the interaction between gravitational wave and topological defects.

Finally, we must note that even if we obtain the quasi-normal modes of domain wall it does not means that all problems are made clear. The original main problem is whether the dominant decay process of domain walls is gravitational emission or not. To solve this problem, we must see whether a domain wall does lose its energy by the gravitational wave emission and estimate energy loss rate due to the emission of gravitational wave and compare with the other physical decay process. We might have to consider the correction due to the finite thickness to do this. This will be one of the next future works.

Acknowledgments

We would like to thank T.Okamura, A.Ishibashi and T.Mishima for valuable comments and discussions.

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Predictions of the one-bubble open inflation

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Abstract

We review the one-bubble open inflationary universe scenario. First we discuss the reason why we investigate a model which describes creation of an open universe. And why inflation is necessary is also explained. After a brief summary of the basic idea of the one-bubble open inflationary universe scenario, we classify potential models of the inflaton field, which have been already proposed, into three groups. To discuss the viability of the models, we give a short explanation of the method to evaluate quantum fluctuations of the inflaton field, which evolve to density fluctuations of the universe. For each group of the potential models, we summarize the specific features of the predicted fluctuation spectrum.

1 Introduction

We begin with re-examining the advantage of the inflationary universe scenario. There are many aspects that can be naturally explained by the existence of the period during which the universe experienced the accelerated expansion. Here we focus on the matter production and the homogeneity. In order to explain the amount of the matter density of the universe successfully, say, by considering the heavy particle decay, we need a sufficiently high number density of heavy particles as the initial condition. In other words, there should be large excitations of the degrees of freedom with large wave numbers in comparison with the cosmological scale. On the other hand, the homogeneity of the universe on cosmological scale means that there should be only small excitations of the degrees of freedom with small wave numbers. The only known mechanism to explain this kind of initial condition of the universe, i.e., large excitation on a small scale and small excitation on a large scale, is inflation. So let us accept the inflationary universe scenario as our starting point of discussion.

Now we turn to recent observational constraints on the cosmological parameters. The Hubble parameter is now settling down to $60 - 80 \text{ km/s Mpc}[1]$. One strong constraint comes from the consistency between the ages of the universe and the oldest globular cluster. Now it is believed that the Einstein-de Sitter universe, i.e., a spatially flat model without a cosmological constant, is excluded by this constraint unless there is a serious error in the estimate of either the Hubble parameter or the age of stars in globular clusters. The Einstein-de Sitter model is very attractive for the inflationary universe paradigm

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because a sufficient expansion during the inflation necessarily makes the universe spatially flat.

For the sake of satisfying the spatial flatness, a model with a positive cosmological constant is also a possible choice. This modification solves the age problem. However, the fraction of the energy density of the universe that is attributed to the cosmological constant changes as time goes. Thus in order to make this fraction at present an appropriate value, a fine tuning of the value of the cosmological constant is necessary.

However, if we allow a fine tuning of a certain parameter, there is no reason to reject a spatially open model which can also solve the age problem. Then there arises a question if a spatially open model is really compatible with inflation. For the standard inflationary universe scenario, the answer to this question is No. This fact can be understood as follows.

The physical scale of the spatial curvature changes proportional to the scale factor $a(t)$. We normalize the scale factor so that the spatial curvature scale is given by the scale factor itself. When we consider the evolution of small perturbations on a Friedmann-Robertson-Walker universe, it is convenient to decompose the perturbations by using the spatial harmonics. The mode characterized by comoving wavenumber k has the physical scale $a(t)/k$ as its wavelength. Thus the ratio of this scale to the curvature scale stays constant.

Keeping in mind the above fact, let us look at the Friedmann equation in an open universe:

$$H^2(t) = \frac{1}{a^2(t)} + \Omega(t)H^2(t), \quad (1)$$

where $\Omega(t)$ is the density parameter. This equation can be rewritten as

$$H^{-1}(t) = \sqrt{1 - \Omega(t)} a(t), \quad (2)$$

and it means that

$$(\text{horizon scale}) = \sqrt{1 - \Omega(t)} (\text{curvature scale}). \quad (3)$$

Thus the assumption that the present universe is spatially open, i.e., $1 - \Omega(t_0) \sim O(1)$, means that the present Hubble horizon scale is comparable to the spatial curvature scale. Hence the perturbations on the present Hubble horizon scale are those on the curvature scale. But we have observed that the perturbations on the curvature scale remain to be so at all times. Therefore, the perturbations on the present Hubble horizon scale were always the curvature scale perturbations. It is well known that the amplitude of the curvature scale perturbations approximately stays constant. But these perturbations must be small enough in order not to contradict with the observed isotropy of the cosmic microwave background. Therefore we arrive at the conclusion that the curvature scale perturbations must have been small from the beginning. In other words, to obtain a model which can explain the present-day open universe, a homogeneous and isotropic open universe must be prepared as a pre-inflationary initial condition. This implies that the homogeneity and isotropy problems are not solved by the standard inflation.

Then is there a mechanism to give such an initial condition? The answer to this question is Yes. In the next section, we explain the one-bubble open inflationary universe scenario as an example. Here we said 'as an example' but we do not know any other scenario which can explain the creation of an open universe as successfully as this scenario.

2 One-bubble open inflationary universe scenario

The basic idea of the one-bubble open inflationary universe scenario was proposed by Gott III[2] fifteen years ago. Motivated by the increasing observational data which suggest that our universe may be open, this scenario was revived by several authors[3, 4].

As is explained later, the one-bubble open inflationary universe scenario can be classified into three groups; the single field model, the coupled two field model and the decoupled two field model. For the purpose of explaining the essence of this scenario, here we consider the single field model.

In the single field model, we consider a single scalar field with a potential such as in the new inflation scenario but with a high potential barrier before the slow rolling inflationary phase as shown in Fig. 1. The scenario essentially consists of three parts.

1) (First inflation) The scalar field is initially supposed to be in the false vacuum. The vacuum energy causes the exponential expansion of the cosmic scale and the universe becomes homogeneous and isotropic. Then the spacetime will be approximated by the de Sitter space.

2) (Bubble nucleation) After the first inflation, nucleation of a vacuum bubble occurs through quantum tunneling. In general, the bubble nucleation process is described by the bounce solution with $O(4)$ -symmetry, which is a non-trivial classical solution of the field equation in Euclidean spacetime[5, 6]. Then the expanding bubble after nucleation is described by the classical solution obtained by analytic continuation of the bounce solution to Lorentzian spacetime. Owing to the $O(4)$ -symmetry of the bounce solution, the expanding bubble has the $O(3,1)$ -symmetry. This implies that the system is homogeneous and isotropic on the hyperbolic time slicing inside the bubble and that the nucleation of a bubble can be regarded as the creation of an open universe[2, 7].

3) (Second inflation) To obtain a realistic model of an open universe, a large amount of matter must be produced. But the interior of the bubble just after nucleation is almost empty and the cosmic expansion is dominantly determined not by the matter energy density but by the contribution from the curvature term. If only the standard matter fields are considered, the matter energy density decreases as $1/a^3(t)$ or $1/a^4(t)$ while the curvature term decreases as $1/a^2(t)$. Therefore the initially curvature dominated universe stays so forever. Thus it is necessary to consider a component of the energy density which decays more slowly than the curvature term. This means that the second inflation inside the bubble is required. In the single field model, the inflaton field of this second inflation is the same scalar field that mediates the false vacuum decay.

As mentioned above, we classify the models of one-bubble open inflationary universe scenario into three groups.

i) (the single field model) This model is explained above and is the original model of the one-bubble open inflationary universe scenario[3, 4].

ii) (the coupled two field model) A model categorized in this group was first proposed by Linde and Mezhlumian[8, 9]. Then several variants of this model have been proposed recently[10, 11]. In the two field model, we consider two different inflaton fields for the first and second stages of inflation. The first inflaton field, σ , has a double well potential as shown in Fig 2(a). When the σ -field is in the false vacuum denoted by σ_F , the energy density of the universe is dominated by the vacuum energy of the σ -field. At this stage,

the second inflaton field, ϕ , has the potential shown by the dashed line in Fig. 2(b). After a sufficiently long era of inflation the σ -field makes a phase transition to the true vacuum through bubble nucleation. Here it is assumed that the potential of the ϕ -field changes into the one shown by the solid line in the Fig. 2(b) due to the coupling to the σ -field. Then as the ϕ -field slow rolls, the second inflation occurs in the nucleated bubble.

iii) (the decoupled two field model) Linde and Mezhlumian has proposed another kind of two scalar field model[8, 9]. This model looks more natural. We do not suppose any coupling between two fields, σ and ϕ . So the potential of the second inflaton field, $U(\phi)$, is always like the one drawn by the solid line in Fig. 2(b). When the first inflaton field, σ , is in the false vacuum, the characteristic expansion rate of the cosmic scale, H_F , is determined by the vacuum energy of the σ -field as

$$H_F = \frac{\sqrt{V(\sigma_F)}}{m_{pl}}, \quad (4)$$

where $m_{pl}^2 = 3/8\pi G$. Then from the field equation in the slow rolling approximation, the gradient of the ϕ -field is estimated as

$$|\nabla\phi|_F \sim \frac{1}{3H_F} \frac{dU(\phi)}{d\phi}. \quad (5)$$

On the other hand, after the phase transition, the expansion rate of the cosmic scale is determined by the vacuum energy of the σ -field as

$$H_T = \frac{\sqrt{U(\phi)}}{m_{pl}}, \quad (6)$$

and the gradient of the ϕ -field is estimated as

$$|\nabla\phi|_T \sim \frac{1}{3H_T} \frac{dU(\phi)}{d\phi}. \quad (7)$$

Therefore it is expected that

$$|\nabla\phi|_F \ll |\nabla\phi|_T. \quad (8)$$

This means that the ϕ -field can be considered as a constant field in the false vacuum if its gradient is compared with that in the true vacuum. Thus the initial condition for the evolution of the ϕ -field in the bubble is approximately $O(3, 1)$ -symmetric. Hence we expect the second inflaton ϕ also has the $O(3, 1)$ -symmetry approximately. The larger the value of H_F/H_T is, the more accurate is the $O(3, 1)$ -symmetry of the ϕ -field in the bubble. Thus to explain the homogeneity and isotropy in this model, a large ratio of H_F to H_T is required.

In the above, we have observed that the one-bubble inflation scenario can explain an open universe, solving the homogeneity, isotropy and entropy problems. So the first step is accomplished and now we move to the second step. In order to construct a consistent cosmological model, we must show that the expected spectrum of the primordial fluctuation is consistent with the observations. In the standard inflation, quantum fluctuations

of the inflaton give rise to primordial density perturbations. Of course, one may consider a model in which an alternative source of the primordial fluctuations dominates but such a model significantly loses the predictability. Thus, in the next section, we focus our attention on the case in which the primordial density perturbation is totally determined by

$$\frac{\delta\rho}{\rho} \sim \frac{H\delta\phi}{\dot{\phi}}, \quad (9)$$

where $\delta\phi$ is the quantum fluctuation of the second inflaton field.

3 Quantum tunneling in general relativity

In this section, we give a brief sketch of the method to evaluate the quantum fluctuation after bubble nucleation. Here we construct the unperturbed background geometry taking account of general relativity but we neglect the quantum fluctuation of the metric. A fully general relativistic formulation in which the metric perturbation is also taken into account is now under investigation. (As for recent developments in this direction, see the reference[12].)

We consider the quantum tunneling in the WKB approximation. In the lowest order WKB approximation, the tunneling process is described by the bounce solution, which is a Euclidean solution connecting the initial and final configurations. The general relativistic bounce solution for the bubble nucleation is known as the Coleman-DeLuccia (CD) instanton[6]. A schematic picture which represents the geometry of the CD instanton for the two dimensional case, embedded in the three dimensional Euclidean space, is presented in Fig. 3. It is natural to consider that the Euclidean time in the present case is an angular coordinate. the vertically sliced hypersurface represents the spatial geometry in the initial state and the horizontally sliced hypersurface represents that in the final state. As time goes on the initial spatial geometry is metamorphosed into the final spatial geometry[13]. In the thin wall approximation, the whole geometry is constructed just by a junction of two pieces of sphere with different radius. The true vacuum side (inside the bubble wall) consists of a portion of a sphere with radius H_T^{-1} and the false vacuum side consists of that with radius H_F^{-1} . In the bubble wall, the σ -field changes from σ_F to σ_T .

To understand the meaning of the bounce solution, the above choice of the time coordinate is appropriate. But to write down the metric and the configuration of the tunneling field, it is better to make use of another coordinate which respects the symmetry of the bounce solution. The expression that shows the $O(4)$ -symmetry most manifestly is given by

$$\begin{aligned} \sigma &= \sigma(\tau_E), \\ ds_E^2 &= d\tau_E^2 + a_E^2(\tau_E) \left(dr_E^2 + \sin^2 r_E d\Omega^2 \right) \\ &= a_E^2 \left(d\eta_E^2 + dr_E^2 + \sin^2 r_E d\Omega^2 \right). \end{aligned} \quad (10)$$

The metric and the field configuration depend only on the coordinate τ_E (or η_E) and can be considered as a Euclidean version of the FRW universe.

The analytic continuation of this bounce solution gives the geometry and the field configuration after the bubble nucleation in the lowest WKB order. Here the analytic

continuation requires a little care but we give the detailed discussion on this point later. The analytic continuation to the open universe that is realized in the nucleated bubble is done by

$$\eta_R = -\eta_E - \frac{\pi}{2}i, \quad r_R = ir_E, \quad (11)$$

and

$$a = -ia_E. \quad (12)$$

Then we find that the metric and the field configuration become those of an open FRW universe as

$$\sigma = \sigma(\eta_R), \quad ds_R^2 = a^2(\eta_R) \left(-d\eta_R^2 + dr_R^2 + \sinh^2 r_R d\Omega^2 \right). \quad (13)$$

This is the lowest WKB order description.

In order to extract the information of the quantum fluctuations, it is necessary to develop the WKB approximation to the next order in the multi-dimensional configuration space (in fact it is infinite-dimensional in the case of field theory). The formalism to obtain the multi-dimensional WKB wave function to the next leading order was proposed by Gervais and Sakita[14]. By using a slightly different approach, an important development was achieved by Vachaspati and Vilenkin[15]. Then, based on these pioneering works, we developed a formalism to investigate the quantum state of a scalar field after false vacuum decay[16, 17]. The formalism was applied to the bubble nucleation which occurs on the Minkowski background, and the spectrum of field fluctuations after the decay was studied [18, 19]. The basic formalism was extended to the case in which the bubble nucleation occurs on the de Sitter spacetime[13]. There the effect of the non-trivial geometry of the instanton with gravity, i.e., the CD instanton[6] was taken into account. At that time, however, an appropriate set of the mode functions to describe the initial vacuum state, which is expected to be the Bunch-Davies vacuum due to the sufficient inflation before the tunneling, had not been known. Thus there was a technical difficulty in applying our formalism. Recently, we have succeeded in describing the Bunch-Davies vacuum state in the spatially open chart[20]. Thus this problem has been overcome. Combining the results in these two papers[13, 20], we now have a tool to deal with the quantum fluctuations after tunneling which include the effect of the geometry of the CD instanton similar to the case in the Minkowski background[18].

We consider the perturbation of the second inflaton field. We divide the degrees of freedom of the second inflaton field into two parts as $\phi = \phi_B + \varphi$, where ϕ_B is the semi-classical, spatially homogeneous part of the field and φ represents the quantum fluctuations around this background. We denote the $O(4)$ -symmetric background of the σ -field by σ_B . Thus σ_B is a function of only one coordinate, η . In addition, we assume that ϕ_B also respects the $O(4)$ -symmetry. Thus a model we have in mind is a two-field one, with ϕ_B being constant at $\sigma_B = \sigma_F$. Note that we can consider a single field model by identifying φ with the quantum fluctuations of σ and letting $U(\sigma, \phi) = V''(\sigma)\phi^2/2$.

With these assumptions, the action for φ is given by

$$S_\varphi = \int \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} \mathcal{M}^2(\eta) \varphi^2 \right) \sqrt{-g} d^4x, \quad (14)$$

where

$$\mathcal{M}^2(\eta) = \frac{\partial^2 U}{\partial \phi^2}(\sigma_B(\eta), \phi_B(\eta)). \quad (15)$$

As noted above, in the case of a single field model, ϕ_B and $\mathcal{M}^2(\tau)$ in the following discussion should be replaced by σ_B and $\frac{\partial^2 V}{\partial \sigma^2}(\sigma_B(\tau))$, respectively.

The tunneling wave functional is written in terms of the bounce solution, σ_B , which gives the semi-classical picture of the nucleation, and a set of mode functions which describe the quantum fluctuations φ on the bounce solution[16]. The boundary condition for these mode functions is determined from the fact that the wave function describes the false vacuum state at $\sigma_B = \sigma_F$. It then turns out that the procedure to obtain an appropriate set of mode functions is equivalent to finding out a complete set of them which are regular in one hemisphere of the Euclideanized spacetime[13, 20]. That is, to find a complete set of mode functions v_k which obey

$$[\nabla^\mu \nabla_\mu - \mathcal{M}^2]v_k(\eta, \mathbf{x}) = 0, \quad (16)$$

in the Lorentzean region, and which are regular on the lower hemisphere as shown in Fig. 4. Here, it is important to note that “a complete set” means a set of all modes which are properly normalized by the Klein-Gordon norms on a Cauchy surface Σ of spacetime,

$$\begin{aligned} \langle v_k, v_{k'} \rangle &:= -i \int_\Sigma d\Sigma_\mu g^{\mu\nu} \{v_k \partial_\nu \bar{v}_{k'} - (\partial_\nu v_k) \bar{v}_{k'}\} \\ &= \delta_{kk'}. \end{aligned} \quad (17)$$

Once they are obtained, the quantum fluctuations of the field are described by the “vacuum state”, $|\Psi\rangle$, such that $\hat{a}_k|\Psi\rangle = 0$ for any k where the fluctuating field is expressed in the Heisenberg representation as

$$\hat{\varphi}_H = \sum_k (v_k \hat{a}_k + \bar{v}_k \hat{a}_k^\dagger), \quad (18)$$

with $\hat{a}_k$ and $\hat{a}_k^\dagger$ being the annihilation and creation operators, respectively. Thus the mode functions v_k play the role of the positive frequency functions.

Here we note that the spacetime shown in Fig. 4 can be also understood as a spacetime which represents the creation of the universe from nothing[21, 22]. The only difference from the familiar scenario of the creation of the universe is that the true vacuum bubble is simultaneously created. So it is also possible to consider a scenario which creates the universe from nothing with a true vacuum bubble instead of the nucleation of the bubble after the first inflation.

Now we return to the issue of the analytic continuation. If we continue the bounce solution to the Lorentzian region by the relation $t = i\tau$, thus obtained universe becomes a closed FRW universe and the hypersurface of the continuation ($t = 0$) becomes the one indicated as the initial state in Fig. 3. The hypersurface of the continuation must be the one indicated as the final state in Fig. 3. This hypersurface is specified by $r_E = \pi/2$. Thus the analytic continuation is done by changing the signature in r_E direction as

$$\begin{cases} r_C = i(r_E - \pi/2), & (-\infty, \infty) \\ \eta_C = \eta_E, & (-\infty, \infty). \end{cases} \quad (19)$$

Then the continued metric becomes

$$ds_C^2 = a_E^2 \left(d\eta_C^2 - dr_C^2 + \cosh^2 r_C d\Omega \right). \quad (20)$$

This coordinate does not represent the open universe but covers the region C in Fig. 5, which is a conformal diagram of the idealized one-bubble open universe. Here in this diagram, it is assumed that the spacetime before the quantum tunneling is given by the de Sitter space. In order to obtain the coordinate to cover the regions L and R in Fig. 5, further continuations given by

$$\begin{cases} \eta_R = -\eta_C \mp \frac{\pi}{2}i, & (-\infty, \infty) \\ r_R = r_C \pm \frac{\pi}{2}i, & (0, \infty), \end{cases} \quad \begin{cases} \eta_L = \eta_C \mp \frac{\pi}{2}i, & (-\infty, 0) \\ r_L = r_C \pm \frac{\pi}{2}i, & (0, \infty), \end{cases} \quad (21)$$

are necessary. Then the metric in region $J = L$ or R becomes

$$ds_J^2 = a^2 \left(-d\eta_J^2 + dr_J^2 + \sinh^2 r_J d\Omega \right). \quad (22)$$

Here we should note that two different choices of signature are possible. If we take the upper signature, the $r_E = 0$ line in the Euclidean region is smoothly connected to the spatial origin of the left and right open universes, $r_R = r_L = 0$. If we take the lower signature, the $r_E = \pi$ line is so. So the regularity of the positive frequency functions in the lower hemisphere requires that we use the upper signature in the continuation of those functions to the regions L and R .

In many of the previous works on the quantum field on an open inflationary universe, only the R -region was taken into consideration. However, it was pointed out in the reference[20] that the $\eta_R = \text{constant}$ hypersurface is not a Cauchy surface. This fact indicates that the normal modes defined only in the R -region do not give a complete set of the functions that are normalizable with respect to the Klein-Gordon norm. Thus the quantization restricted to the R -region does not work even if we restrict our observation to the operators which have support only in the R -region. In some case, in fact, we will find later that there appear modes which cannot be expressed by the language of the quantization restricted to the R -region.

It is clear that the quantization is most conveniently performed in the C -region because the $r_C = \text{constant}$ hypersurface is manifestly a Cauchy surface. Once we determine the normalized positive frequency functions v_k in the C -region, we can analytically continue them into the R -region.

Setting $v_k = a_E(\tau)^{-1} \chi_p(\tau) f_{pl}(\rho) Y_{lm}(\Omega)$, the equations for χ_p and f_{pl} in the C -region become, respectively,

$$\left[-\frac{d^2}{d\eta^2} + \mathcal{U} - p^2 \right] \chi_p = 0, \quad (23)$$

with

$$\mathcal{U}(\eta_C) = a_E^2 \left(\mathcal{M}^2 - \frac{R}{6} \right), \quad (24)$$

where R is the scalar curvature, and

$$\left[-\frac{1}{\cosh^2 r_C} \frac{\partial}{\partial r_C} \left(\cosh^2 r_C \frac{\partial}{\partial r_C} \right) - \frac{l(l+1)}{\cosh^2 r_C} - (p^2 + 1) \right] f_{pl} = 0. \quad (25)$$

The first equation determines the spatial harmonics, $\chi_p(\eta_C)Y_{lm}(\Omega)$, while the second one is the equation that determines the time evolution.

The equation for $\chi_p(\eta_C)$ looks like the Schrödinger equation in quantum mechanics with a potential $\mathcal{U}$, which goes to zero at $\eta_C \rightarrow \pm\infty$. Thus for each positive p^2 , there are two independent modes that correspond to the left and right moving waves. We label them by $\sigma = \pm$. Then those modes are normalized as

$$\int_{-\infty}^{\infty} d\eta_C \chi_p^\sigma \overline{\chi_{p'}^{\sigma'}} = \delta_{\sigma\sigma'} \delta(p - p'). \quad (26)$$

When p^2 is negative, the boundary condition at $\eta_C \rightarrow \pm\infty$ is fixed by the requirement of the regularity. Hence the normalizable modes exist only for discrete values of p^2 . As the valley of the potential $\mathcal{U}$ becomes wider and deeper, the number of discrete modes increases. Labeling the discrete eigen values by $\Lambda = ip$, we normalize χ_Λ as

$$\int_{-\infty}^{\infty} d\eta_C \chi_\Lambda \overline{\chi_{\Lambda'}} = \delta_{\Lambda\Lambda'}. \quad (27)$$

Now we investigate the equation for the time evolution. The equation for $f_{pl}(r_C)$ is independent of the details of the model. It is readily solved as

$$f_{pl} = \frac{N_{pl}}{\sqrt{\sinh r}} P_{ip-1/2}^{-l-1/2}(\cosh r), \quad (28)$$

where N_{pl} is a normalization constant and

$$r = r_R = r_L = r_C \pm \frac{\pi}{2}i = \begin{cases} ir_E, \\ ir_E - \pi i. \end{cases} \quad (29)$$

Here the two different signatures correspond to the difference in the way of analytic continuation. In the above, the associated Legendre function of the first kind was chosen because f_{pl} should be regular at r_R or $r_L = 0$ when it is analytically continued. Now we find the choice of the upper signature satisfies the requirement that the positive frequency function is regular on the lower hemisphere ($0 \leq \tau_E \leq \pi/2$). If the lower signature is chosen, $\cosh r$ becomes -1 at $\tau_E = 0$ and $P_{ip-1/2}^{-l-1/2}(\cosh r)$ becomes singular there. On the other hand, if the upper signature is chosen, $\cosh r$ becomes $+1$ at $\tau_E = 0$ and $P_{ip-1/2}^{-l-1/2}(\cosh r)$ is regular there. In order to normalize the mode functions with respect to the Klein-Gordon norm, the normalization constant must be determined to satisfy

$$i \cosh^2 r_C \left\{ \frac{\partial f_{pl}}{\partial r_C} \overline{f_{pl}} - f_{pl} \frac{\partial \overline{f_{pl}}}{\partial r_C} \right\} = 1. \quad (30)$$

Then f_{pl} is fixed except for the phase factor as

$$f_{pl} = \sqrt{\frac{\pi}{2p \sinh \pi p}} F_{pl}, \quad (31)$$

where

$$F_{pl} = \sqrt{\frac{\Gamma(ip+l+1)\Gamma(-ip+l+1)}{\Gamma(ip+1)\Gamma(-ip+1)}} \frac{p}{\sqrt{\sinh r}} P_{ip-1/2}^{-l-1/2}(\cosh r). \quad (32)$$

The function f_{pl} determines the time evolution in the C -region. But after the analytic continuation to the L - or R -region, it becomes a part of the spatial harmonics. The spatial harmonics in the L - or R -region are given by $Y_{plm} = F_{pl}(r_R)Y_{lm}(\Omega)$. For $p^2 > 0$, they are normalized as

$$\int_0^\infty dr_R \sinh^2 r_R \int d\Omega Y_{plm} \overline{Y_{p'l'm'}} = \delta(p - p') \delta_{ll'} \delta_{mm'}. \quad (33)$$

For $p^2 < 0$, however, Y_{plm} are not normalizable in the L - or R -region. Since these modes have coherence length larger than the curvature scale in the L - or R -region, we call them the supercurvature modes[20, 23].

Once we obtain these normalized mode functions v_{plm} in the C -region, they are analytically continued to the R -region. Then the expectation value of the quantum fluctuation of the second inflaton field is evaluated as

$$\langle \varphi(x) \varphi(x') \rangle = \sum_{\mathbf{k}} v_{\mathbf{k}}(x) \overline{v_{\mathbf{k}}(x')}. \quad (34)$$

4 Constraint on models

In the preceding section, we have outlined the method to evaluate the quantum fluctuation after false vacuum decay with the effect of curved spacetime. Now we apply it to study the spectrum of the primordial fluctuations in the one-bubble open inflationary universe scenario.

The meaningful constraint to the model is obtained from large angle anisotropies in the cosmic microwave background. This is because the typical scale of the excitation spectrum is determined by the curvature scale, $p^2 \sim 1$. For the fluctuation with $p^2 \gg 1$, the quantum excitation due to tunneling is significantly suppressed and thus the state does not differ much from the vacuum without tunneling. So the spectrum on the small scale is independent of the detail of a model.

The fluctuation of the cosmic microwave background is characterized by the temperature autocorrelation function, $C(\Theta) := \langle \delta T / T(\vec{\gamma}) \delta T / T(\vec{\gamma}') \rangle_{\vec{\gamma} \cdot \vec{\gamma}' = \cos \Theta}$. Writing them in the form,

$$C(\Theta) = \frac{1}{4\pi} \sum_l (2l+1) C_l P_l(\cos \Theta), \quad (35)$$

the multipole moment C_l is usually used to represent $C(\Theta)$.

The multipole moments C_l can be calculated once the spectrum of the fluctuations of the second inflaton, $\langle (\varphi_p)^2 \rangle$, is obtained. But the relation between them is not so trivial. So it is useful to introduce two reference states that are called the Bunch-Davies vacuum and the conformal vacuum.

The former is defined for a massive field in a pure de Sitter space. Since in the R -region the second inflaton is approximated by a massless scalar field, by the Bunch-Davies vacuum here we refer to the massless limit of it on the de Sitter space with the same expansion rate H as in the R -region. This state respects the full de Sitter invariance and the behavior of the two point function in the coincidence limit is the same as the one for the Minkowski vacuum in the Minkowski space. In this case, it is known that there is

one discrete spectrum at $p^2 = -1$ and the spectrum for $p^2 > 0$ at $\eta_R \rightarrow 0$ is given by[20]

$$\langle |\varphi_p| \rangle_{BD}^2 = \frac{\coth \pi p}{2p(1 + p^2)}. \quad (36)$$

The physical meaning of the conformal vacuum seems hard to explain. It is defined by considering the quantization under the restriction to the R -region. Although this is not a well-defined concept, it can be done practically as follows. In order to normalize the mode functions, $v_{plm}^{(R)} := a^{-1} \chi_p^{(R)}(\eta_R) Y_{plm}(r_R, \Omega)$, one defines the pseudo-Klein-Gordon inner product by

$$\begin{aligned} (v_{plm}^{(R)}, v_{p'l'm'}^{(R)})^{(R)} &:= ia^2 \int_0^\infty dr_R \sinh^2 r_R \int d\Omega \left\{ \frac{dv_{plm}^{(R)}}{d\eta_R} \bar{v}_{p'l'm'}^{(R)} - v_{plm}^{(R)} \frac{d\bar{v}_{p'l'm'}^{(R)}}{d\eta_R} \right\} \\ &= i \left\{ \frac{d\chi_p^{(R)}}{d\eta_R} \bar{\chi}_{p'}^{(R)} - \chi_p^{(R)} \frac{d\bar{\chi}_{p'}^{(R)}}{d\eta_R} \right\} \delta(p - p') \delta_{ll'} \delta_{mm'}. \end{aligned} \quad (37)$$

Since this norm diverges for negative p^2 , the discrete spectrum cannot exist in this approach. The conformal vacuum state is defined by choosing the ‘positive frequency function’, as[24]

$$\chi_p^{(R)}(\eta_R) := \frac{1}{\sqrt{2p(p^2 + 1)}} e^{-ip\eta_R} (ip + \coth \eta_R). \quad (38)$$

Then the spectrum for the fluctuation of the second inflaton field at $\eta_R \rightarrow 0$ becomes

$$\langle |\varphi_p| \rangle_C^2 = \frac{1}{2p(1 + p^2)}. \quad (39)$$

For these two reference states, we show the plots of C_l for $\Omega_0 = 0.1$ and $\Omega_0 = 0.2$ in Fig. 6. In both plots, the solid lines correspond to the Bunch-Davies vacuum and the dotted lines correspond to the conformal vacuum. The dashed lines are also plots for the Bunch-Davies vacuum but excluding the contribution from the discrete mode. From these plots, we can see that the difference in the continuum spectrum of the quantum fluctuations of the second inflaton field does not affect much the resulting temperature fluctuations. The difference between these two reference states mainly comes from the contribution from the discrete mode.

4.1 single field model

First we consider the single field model. A schematic picture of the potential of the inflaton field is shown in Fig. 1. Two tunneling processes for the false vacuum decay with gravity are known. One is that described by the Coleman-DeLuccia instanton, which is explained in Sec. 2. The other is that described by the Hawking-Moss instanton[25]. In the latter case, the false vacuum decay does not give rise to the $O(3,1)$ -symmetry. So the false vacuum decay must be proceeded by the former process. This condition requires that the curvature of the potential at the top of the barrier is much greater than the expansion rate in the false vacuum, $H_F^2 = 8\pi G V(\sigma_F)/3$, i.e.,

$$-\mathcal{M}_{top}^2 := -\frac{d^2 V}{d\sigma^2} \Big|_{top} \gg H_F^2. \quad (40)$$

So it is natural to suppose that the mass squared in the false vacuum, $\mathcal{M}_F^2$, is also much greater than the expansion rate. For simplicity, we assume that the energy density is approximately a constant, $3H^2/8\pi G$. Then the scale factor is approximated by

$$a_E \sim \frac{1}{H \cosh \eta_C}, \quad (41)$$

and the scalar curvature, R , is given by $12H^2$. With these assumptions, the potential in Eq. (23) becomes

$$\mathcal{U} = \frac{1}{\cosh^2 \eta_C} \left(\frac{\mathcal{M}^2}{H^2} - 2 \right). \quad (42)$$

From the original potential shown in Fig. 1, its second derivative, $\mathcal{M}^2$, will have the form as shown in Fig. 7(a). And it is translated into the potential, $\mathcal{U}$, as shown in Fig. 7(b). By the above mentioned reason, $\mathcal{U}$ becomes much larger than unity around $\eta_C = 0$.

Again, we make use of analogy to the quantum mechanics. This is just the problem to find the wave function in an energy eigen state. As mentioned before, there are two independent modes for each positive p^2 . Since there is a high barrier in the potential, we can approximately construct wave functions which have their amplitude only in the right or left region for not so large p . We call them the right and left modes, respectively. The coherence between them is exponentially suppressed. Therefore these two modes can be treated as a set of two independent modes. Only the right mode contributes to the quantum fluctuation in the R -region. Under this condition, we can show that the quantum fluctuation in the R -region is expressed in the following form,[26]

$$\langle \varphi_p^2 \rangle = \frac{\cosh \pi p + \cos \delta_p}{2p(1 + p^2) \sinh \pi p}, \quad (43)$$

where δ_p is a real number. Comparing this result with the reference spectrum, we find that

$$\frac{\langle \varphi_p^2 \rangle - \langle \varphi_p^2 \rangle_{BD}}{\langle \varphi_p^2 \rangle_{BD} - \langle \varphi_p^2 \rangle_C} = e^{-\pi p} \cos \delta_p. \quad (44)$$

Therefore, as for the continuous spectrum with not so large p , the deviation of the present spectrum from the spectrum for the Bunch-Davies vacuum cannot become significantly larger than the difference between the spectra for the Bunch-Davies and conformal vacua. When $p \gg 1$ the above ratio can be large. However, for such p , we have $\langle \varphi_p^2 \rangle \sim \langle \varphi_p^2 \rangle_{BD}$ because $\cosh \pi p \gg \cos \delta_p$. Hence there is no chance that the present spectrum gives a significantly different estimate for the multipole moments C_l .

For the single field model, there is an interesting feature in the discrete mode. To find a discrete mode is the same problem to find a bound state in analogy with the quantum mechanics. In the single field model, assuming $\mathcal{M}_F^2 \gg H^2$, we can show that there exists one discrete mode, which is called the wall fluctuation mode[19, 26, 27, 28] as follows.³

³Here, we are restricting our attention to the fluctuation of the scalar field. However, since the metric perturbation is neglected, the present analysis is not complete. By recent investigations[29, 12], it is suggested that the wall fluctuation mode as a discrete spectrum may disappear when the effect of gravity is fully taken into account.

We write the equation that the homogeneous background solution (bounce solution) satisfies as

$$\frac{1}{a_E^2} \frac{d}{d\eta_C} a_E^2 \frac{d}{d\eta_C} \sigma_B = a_E^2 \frac{dV}{d\sigma}(\sigma_B). \quad (45)$$

This equation comes from the field equation of the σ -field. Then taking a time-derivative of this equation, we obtain

$$\left[-\frac{d^2}{d\eta_C^2} + \mathcal{U} + 4 \right] \frac{d\sigma_B}{d\eta_C} = 0, \quad (46)$$

which is just Eq. (23) with $p^2 = -4$. We can also show that the regularity condition at $\eta_C \rightarrow \pm\infty$ is satisfied for $d\sigma_B/d\eta_C$ from the regularity of the bounce solution. Thus the derivative of the background solution gives the mode function corresponding to the discrete spectrum with $p^2 = -4$. By construction, the existence of this mode is independent of the detailed form of the model of the inflaton potential.

The contribution to C_l from the wall fluctuation mode is plotted in Fig. 8. As is expected, the amplitude is large for small l . Thus if the wall fluctuation mode dominates over the continuous spectrum, the anticipated C_l will contradict with the spectrum of the temperature fluctuation in the microwave background observed by COBE[30]. By a detailed analysis, one can derive the condition on the model parameters from this constraint[26]. Roughly speaking, it is concluded that the condition is

$$\Delta V \gtrsim \frac{\mathcal{M}}{H} V_F, \quad (47)$$

where $\Delta V := V|_{top} - V_F$ is the height of the potential barrier and $\mathcal{M}$ is the typical mass scale around the potential barrier. As is explained above, $\mathcal{M}/H$ must be large enough for the $O(4)$ -symmetric bounce solution to dominate the process of the false vacuum decay. Thus we can conclude that the height of the potential barrier, ΔV , must be sufficiently higher than the height of the potential in the false vacuum, V_F .

4.2 coupled two field model

Here we investigate the coupled two field model. We begin with discussing the continuous spectrum. For simplicity, again, we approximate the background geometry by a pure de Sitter space, although this approximation is not essential. In this model, the mass of the second inflaton, $\mathcal{M}$, changes due to the coupling to the σ -field. Thus it will be natural to assume that $\mathcal{M}^2$ in the false vacuum and in the nucleated bubble can be approximated by M^2 and 0, respectively. Then the potential for Eq. (23) will take the form as shown in Fig. 9. If $M^2/H^2 \gg 1$, the same discussion given in the preceding subsection holds, and the contribution to C_l from the continuous spectrum is not much different from the case of the Bunch-Davies or conformal vacuum. On the other hand, if $M^2/H^2 \ll 1$, we have approximately $\mathcal{M} \equiv 0$, which implies the state is approximately the Bunch-Davies vacuum. So in both extreme cases, apart from the contribution of the discrete mode, C_l is approximated by that for the Bunch-Davies vacuum. Hence one expects that the contribution of the continuous spectrum is always approximated in the same manner. This is true. A detailed analysis can be found in the reference[26].

Here we have one remark. If $M^2/H^2 \ll 1$, the model is not the coupled two field model any longer but becomes the decoupled two field model, which is discussed in the next subsection. As mentioned before, since $H_F \gg H_T$ is required in the decoupled model, the result for this limiting case obtained in the present approximation does not have any direct realistic implication.

Now we turn to the discrete spectrum. As is explained in the subsequent subsection, there is one discrete mode with $p^2 = -1$ for $\mathcal{M} \equiv 0$. Once we know this fact, it will be easy to understand that a discrete mode exists for small M^2/H^2 and η_W , where η_W is the value of η_C on the bubble wall. For larger M^2/H^2 and η_W , it becomes more difficult to make a bound state. Thus for sufficiently large M^2/H^2 and η_W , the discrete mode ceases to exist. As is anticipated from this fact, the contribution to C_l from the discrete mode becomes most effective for the limiting case with small M^2/H^2 and η_W . This is nothing but the Bunch Davies vacuum. Thus the spectrum is bounded from the above by the estimate for the Bunch-Davies vacuum. Denoting C_l for the Bunch-Davies vacuum and that for the conformal vacuum by C_{lBD} and C_{lC} , respectively, we find by a more detailed analysis the relation,

$$C_{lBD} > C_l > C_{lC}, \quad (48)$$

in the case when the background geometry is approximated by a pure de Sitter space[26]. Thus we conclude that the prediction of the coupled two field model is very weakly model-dependent and it is approximately evaluated by using the reference models. The detailed comparison of these reference models with the observed spectrum was done by Yamamoto and Bunn[31]. They found that the small value of Ω_0 , such as 0.2 is mildly rejected.

4.3 decoupled two field model

Finally, we investigate the decoupled two field model. In this case a sufficiently large difference between the energy densities in the false and true vacua is necessary. By definition of the decoupled model, the mass $\mathcal{M}$ is expected to be almost constant. Since $\mathcal{M} \ll H_T^2 \ll H_F^2$, we can approximate $\mathcal{M}$ by zero. Then Eq. (23) becomes

$$\left[-\frac{d^2}{d\eta_C^2} + \frac{1}{a_E} \left(\frac{d^2 a_E}{d\eta_C^2} \right) - (p^2 + 1) \right] \chi_p = 0. \quad (49)$$

Now it is easy to see that the discrete mode for $p^2 = -1$ always exists. The eigen function is given by $\chi_i = N a_E$, where N is the normalization constant. The equation for the normalization (27),

$$1 = \int |\chi_p|^2 d\eta_C = |N|^2 \int a_E^2 d\eta_C, \quad (50)$$

determines N . The integral, $\int a_E^2 d\eta_C$, is totally determined by the energy density in the false vacuum, $3H_F^2/8\pi G$, as $\sim H_F^{-2}$ because the maximum value of a_E is H_F^{-1} . Thus the normalization constant is evaluated as $|N|^2 = O(H_F^2)$. Next, to compare with the Bunch-Davies vacuum, we assume that $a_E = 1/H_T \cosh \eta_C$. Then the integral is evaluated as $\sim H_T^{-2}$. Thus the normalization for the discrete mode of the Bunch-Davies vacuum is given by $|N|^2 = O(H_T^2)$. This difference in the normalization results in the amplification in the amplitude of the discrete mode. Further analysis can show that this kind of

drastic amplification does not occur for the continuous spectrum[32]. As was shown in Fig. 6, the contribution to C_l from the discrete mode is not so significant for the Bunch-Davies vacuum. However if it is amplified by the factor $H_F^2/H_T^2(\gg 1)$ it will cause a serious conflict with the observation. Hence we conclude that the decoupled two field model is very natural in construction but it seems very difficult to reconcile the predicted fluctuation in the cosmic microwave background with the observed one.

5 Conclusion

We reviewed the current status of the study of the one-bubble open inflationary universe scenario. We divided the models into three groups: i) the single field model, ii) the coupled two field model and iii) the decoupled two field model. In the sense of the naturalness of the inflaton potential, iii) is most attractive and i) is rather awkward. We investigated the temperature fluctuation on large angle scales based on the ansatz that the temperature fluctuation is caused by the quantum fluctuation of the inflaton field. In this estimate, the excitation through the tunneling process is estimated by using the WKB approximation. In the lowest WKB order, the background FRW universe is described. In the next leading WKB order, the quantum fluctuation around this background FRW universe is described. In the lowest WKB order, the effect of gravity is taken into account while the metric perturbation which belongs to the next leading order was neglected. For ii), any model predicts a similar temperature fluctuation spectrum and there arises no serious constraint on the model parameters. For i), there is a constraint to the model parameters that the potential barrier must be sufficiently high. For iii), there arises a large amplification of the discrete (supercurvature) mode. Hence no model of this class seems viable.

Acknowledgments

We would like to thank K. Yamamoto for fruitful collaboration on which this work is mainly based. This work was supported in part by Monbusho Grant-in-Aid for Scientific Research No. 07304033.

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Figure Captions

- Fig. 1 : A schematic picture of the potential of the inflaton field for the single field model. The meaning of the stages 1, 2, 3 is explained in the text.
- Fig. 2 : A schematic picture of the potential of the inflaton fields for the two field model. The panel (a) shows the potential of the first inflaton field and the panel (b) shows that of the second inflaton field. For the coupled two field model, the dashed and solid lines in the panel (b) represent the potentials in the false and true vacua, respectively. For the decoupled two field model, the potential for the second inflaton field is always given by that drawn by the solid line.
- Fig. 3 : The CD instanton for the case that the spacetime dimension is two, embedded in the three dimensional Euclidean space.
- Fig. 4 : The analytic continuation of the CD instanton to the Lorentzean region. The lower half is the same that is shown in Fig. 3 and represents the Euclidean region. The upper half represents the continuation to the Lorentzean region.
- Fig. 5 : A partial conformal diagram of the spacetime one-bubble open inflationary universe. Only the region after quantum tunneling is shown.
- Fig. 6 : The multipole moments for the temperature fluctuation, C_l , for the cases of the two reference states. The upper and the lower panels show the cases for $\Omega = 0.1$ and 0.2 , respectively. The solid lines are the plots for the Bunch-Davies vacuum and the dotted lines are the plots for the conformal vacuum. The dashed lines are also the plots for the Bunch-Davies vacuum but excluding the contribution from the discrete mode. This figure is taken from the reference[3].
- Fig. 7 : Schematic pictures for (a) the mass squared and (b) the potential $\mathcal{U}$ in the single field model.
- Fig. 8 : The contribution from the wall fluctuation mode to the multipole moment of the temperature fluctuation, C_l . This figure is taken from the reference[26].
- Fig. 9 : A schematic picture for the potential $\mathcal{U}$ in the coupled two field model.

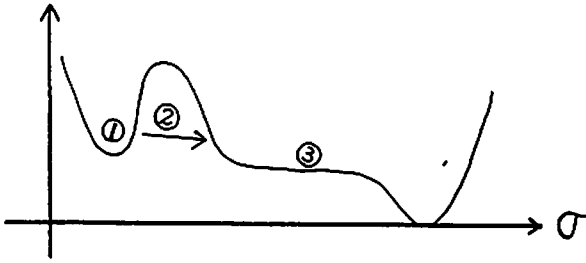


Fig. 1

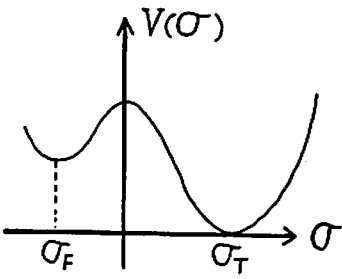


Fig. 2(a)

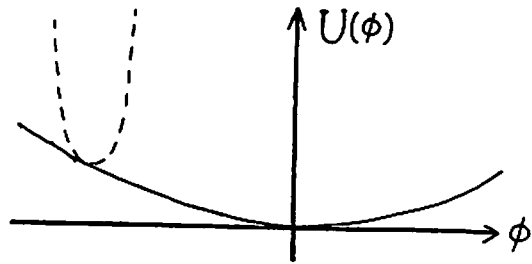


Fig. 2(b)

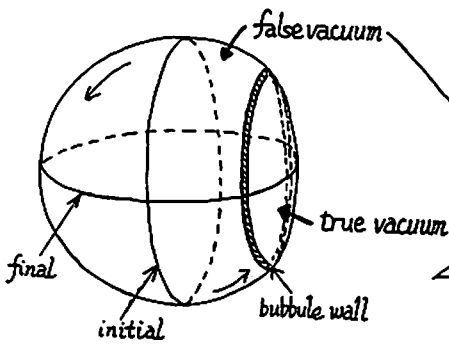


Fig. 3

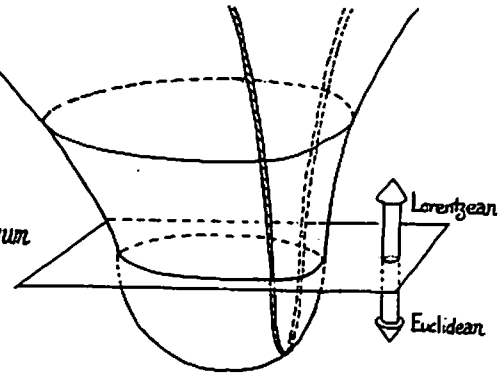


Fig. 4

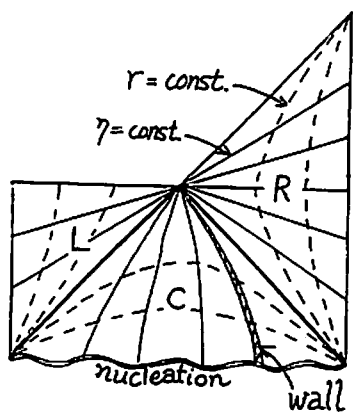


Fig. 5

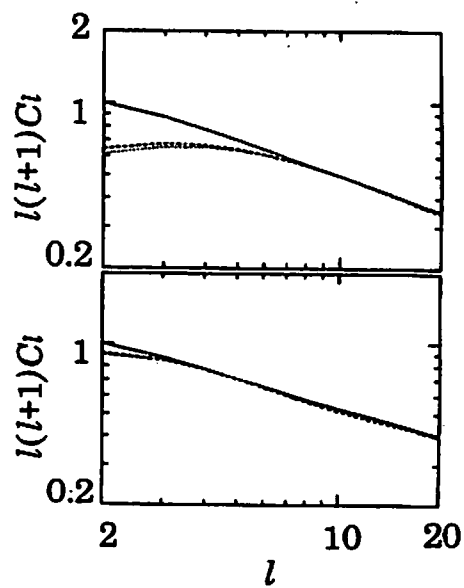


Fig. 6

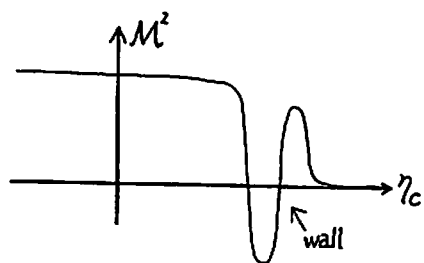


Fig. 7(a)

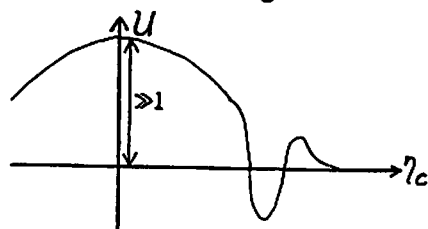


Fig. 7(b)

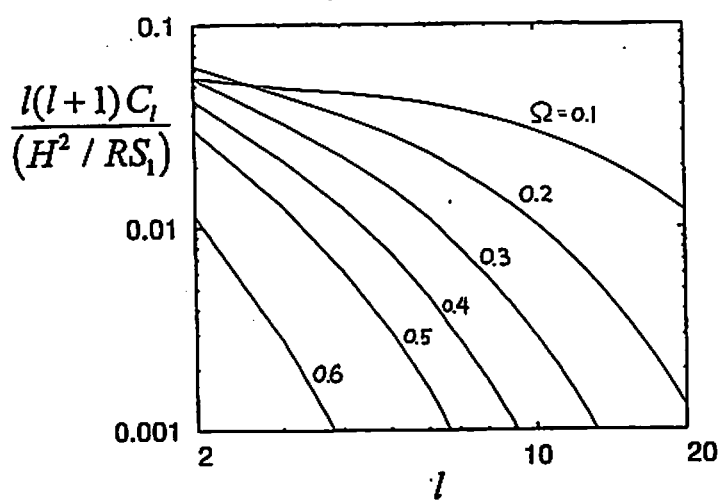


Fig. 8

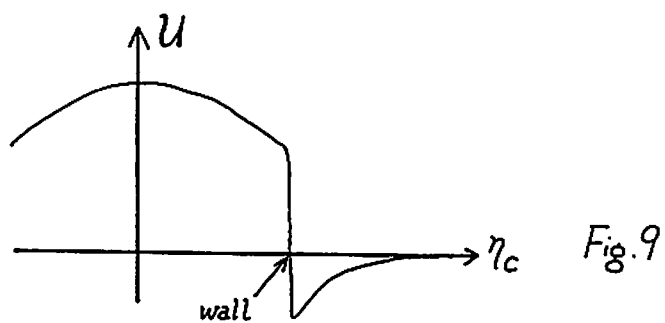


Fig. 9

Evolution of Cosmological Perturbation in Reheating Phase of the Universe

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Abstract

The evolution of the cosmological perturbation during the oscillatory stage of the scalar field is investigated. For the power law potential of the inflaton field, the evolution equation of the Mukhanov's gauge invariant variable is reduced to the Mathieu equation and the density perturbation grows by the parametric resonance.

1 introduction

Recently, the importance of the parametric resonance is recognized in the reheating phase of the inflationary model[2, 3]. Due to the coherent oscillation of the background inflaton field, the fluctuation of the non-linearly coupled boson field or the fluctuation of self interacting inflaton field itself are amplified and the catastrophic particle creation occurs. This effect drastically changes the scenario of the reheating considered so far. When one pay attention to the evolution of the metric perturbation during the coherent oscillatory phase of the scalar field dominated universe, a naive question arises: does the metric perturbation undergo a influence of the parametric resonance by the background oscillation of the scalar field?

It is well known that the cosmological perturbation of the scalar field in the oscillatory stage has the problematic aspect[4]. If one try to write down the evolution equation for the density contrast or the Newtonian potential, the coefficient of the equation becomes singular periodically in time due to the background oscillation of the scalar field. This behavior is not real. It appears through the reduction of the constrained system to the second order differential equation. But it makes difficult to understand the behavior of the metric perturbation in the oscillatory phase.

In this article, we consider the evolution of the cosmological perturbation using the gauge invariant variable introduced by Mukhanov. The evolution equation for this variable has no singular behavior and is suitable to apply the oscillatory phase of the scalar field.

2 single scalar field case

For simplicity, we first consider the system with only the inflaton field.

2.1 basic equation

We treat a spatially flat FRW universe with a minimally coupled scalar field with a potential

$$V(\phi) = \frac{\lambda}{4} \phi_0^4 \left(\frac{\phi}{\phi_0} \right)^{2n}, \quad (n = 1, 2, \dots) \quad (1)$$

The background equations are

$$H^2 = \frac{\kappa}{3} \left(\frac{1}{2} \dot{\phi}^2 + V(\phi) \right), \quad (2)$$

$$\dot{H} = -\frac{\kappa}{2} \dot{\phi}^2, \quad (3)$$

$$\ddot{\phi} + 3H\dot{\phi} + V_{\phi} = 0, \quad (4)$$

where $\kappa = 8\pi G$. The scalar field oscillates if the condition $\phi \ll m_{pl}$ is satisfied. In such a situation, we cannot have the exact solution, but using the time averaged potential energy and the kinetic energy have the relation

$$\langle \dot{\phi}^2 \rangle = 2n \langle V(\phi) \rangle, \quad (5)$$

the scale factor and the Hubble parameter can be approximately expressed as

$$a(t) \approx \left(\frac{t}{t_0} \right)^{\frac{n+1}{3n}}, \quad H \approx \frac{n+1}{3n} \frac{1}{t}. \quad (6)$$

We define new variables for the scalar field and the time:

$$\eta = nt_0 a^{\frac{3}{n+1}} = nt_0 \left(\frac{t}{t_0} \right)^{1/n}, \quad (7)$$

$$\phi(t) = \phi_0 a^{-\frac{3}{n+1}} \tilde{\phi}, \quad (8)$$

where $\tilde{\phi} = 1$ at $t = t_0$ and ϕ_0 is a initial value of the scalar field ($\phi_0 \ll m_{pl}$). Using the new variables, the evolution equation of the scalar field becomes

$$\tilde{\phi}_{\eta\eta} + m^2 n \tilde{\phi}^{2n-1} = 0, \quad m^2 = \frac{\lambda}{2} \phi_0^2. \quad (9)$$

For $n = 1$ (massive scalar field), $\tilde{\phi} = \cos(m\eta)$. For $n = 2$, $\tilde{\phi} = cn(\sqrt{\lambda}\phi_0\eta; \frac{1}{\sqrt{2}})$ where cn is a elliptic function.

2.2 perturbation equation

We use the gauge invariant variables to treat the perturbation whose wavelength is larger than the horizon scale. We found that the most convenient variable is

$$Q = \delta\phi - \frac{\dot{\phi}}{H} \psi = \delta\phi^{(g)} - \frac{\dot{\phi}}{H} \Psi, \quad (10)$$

where ψ is the perturbation of the three curvature, $\delta\phi^{(g)}$ is the gauge invariant variable for the scalar field perturbation and Ψ is the gauge invariant Newtonian potential. For the

zero curvature slice, Q represents the fluctuation of the scalar field. This variable Q was first introduced by Mukhanov[5]. As already mentioned, the coefficient of the evolution equation for the Newtonian potential or the gauge invariant density contrast becomes singular because of the background oscillation of the scalar field. But the evolution equation for Q does not have such a singular behavior:

$$\ddot{Q} + 3H\dot{Q} + \left[V_{\phi\phi} + \left(\frac{k}{a} \right)^2 + 2 \left(\frac{\dot{H}}{H} + 3H \right) \right] Q = 0. \quad (11)$$

The Newtonian potential and the variable Q is connected by the relation

$$-\frac{k^2}{a^2}\Psi = \frac{\kappa}{2} \frac{\dot{\phi}^2}{H} \left(\frac{H}{\dot{\phi}} Q \right), \quad (12)$$

and the gauge invariant density contrast Δ which is equal to $(\delta\rho/\rho)$ on the co-moving time slice is

$$\Delta = \frac{\kappa}{3} \frac{\dot{\phi}^2}{H^3} \left(\frac{H}{\dot{\phi}} Q \right). \quad (13)$$

2.3 behavior of gauge invariant variable

To make the eq.(11) more tractable, we change the variable

$$Q = a^{-\frac{1}{n+1}} \tilde{Q}, \quad \eta = nt_0 \left(\frac{t}{t_0} \right)^{1/n}, \quad (14)$$

Then

$$\tilde{Q}_{\eta\eta} + a^{\frac{6(n-1)}{n+1}} \left[V_{\phi\phi} + \left(\frac{k}{a} \right)^2 + 2 \left(\frac{\dot{H}}{H} + 3H \right) - \frac{9n}{(n+1)^2} H^2 - \frac{3}{n+1} \dot{H} \right] \tilde{Q} = 0. \quad (15)$$

Using the background equation, we can estimate the time dependence of the each terms in brackets:

$$\begin{aligned} V_{\phi\phi} &= n(2n-1)m^2 a^{-\frac{6(n-1)}{n+1}} \tilde{\phi}^{2n-2} \sim O \left(\left(\frac{\eta}{t_0} \right)^{2-2n} \right), \\ \left(\frac{\dot{H}}{H} + 3H \right) &= 6Hn^n \left(\frac{\eta}{t_0} \right)^{1-n} \tilde{\phi}^{2n-1} \tilde{\phi}_\eta \sim O \left(\left(\frac{\eta}{t_0} \right)^{1-2n} \right), \\ H^2, \dot{H} &\sim O \left(\left(\frac{\eta}{t_0} \right)^{-2n} \right). \end{aligned}$$

As we are considering the situation $\eta \geq t_0$, we can neglect the $H^2, \dot{H}$ terms in eq.(15) because they are higher order in powers of $(1/\eta)$ compared to other terms. Our basic equation for the gauge invariant variable becomes

$$\tilde{Q}_{\eta\eta} + \left[n(2n-1)m^2 \tilde{\phi}^{2n-2} + k^2 a^{\frac{4(n-2)}{n+1}} + 4n(n+1) \frac{1}{\eta} \tilde{\phi}^{2n-1} \tilde{\phi}_\eta \right] \tilde{Q} = 0. \quad (16)$$

$\tilde{\phi}$ is the solution of eq.(9).

2.3.1 $n = 1$ case

We first consider $n = 1$ case(massive scalar). The background scalar field is sinusoidal and given by

$$\tilde{\phi} = \cos(m\eta). \quad (17)$$

Eq.(16) becomes

$$\tilde{Q}_{\eta\eta} + \left[m^2 + \left(\frac{k}{a} \right)^2 - \frac{8}{\eta} \sin(m\eta) \cos(m\eta) \right] \tilde{Q} = 0. \quad (18)$$

We introduce a dimensionless time variable $\tau = m\eta$,

$$\tilde{Q}_{\tau\tau} + \left[1 + \left(\frac{k}{ma} \right)^2 - \frac{4}{\tau} \sin(2\tau) \right] \tilde{Q} = 0, \quad (19)$$

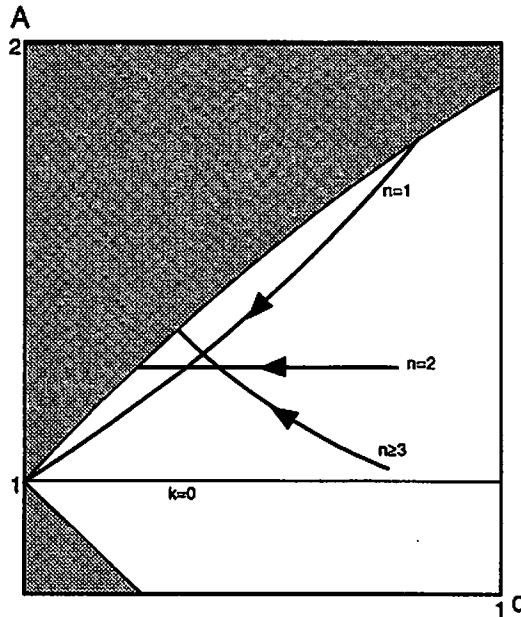
where $a \propto \tau^{2/3}$. This equation has the same form as the Mathieu equation:

$$Y_{\tau\tau} + [A - 2q \sin(2\tau)] Y = 0. \quad (20)$$

In our case the coefficient A, q are time dependent functions and the relation between A and q is

$$A = 1 + \left(\frac{mt_0}{2} \right)^{4/3} \left(\frac{k}{m} \right)^2 q^{4/3}. \quad (21)$$

Using the stability/instability chart of the Mathieu equation, we can know that the perturbation will have the effect of parametric resonance of the first unstable band of the Mathieu function and grows in time(see figure). The grey is stable and the white is unstable region. The curves show the time evolution of the parameter A, q . $A = 1$ line corresponds to $k = 0$.



We can derive its time evolution by solving eq.(19) using multiple time scale method[6]. We introduce a parameter

$$\epsilon = \frac{4}{\tau_0}. \quad (22)$$

As the condition $\tau_0 \gg 1$ is equivalent to the condition of coherent oscillation, ϵ is small parameter. Rewrite the eq.(19) as follows:

$$\tilde{Q}_{\tau\tau} + \left[1 + 2\epsilon\omega_1 - \epsilon\frac{\tau_0}{\tau}\sin(2\tau)\right]\tilde{Q} = 0, \quad (\tau \geq \tau_0) \quad (23)$$

where $\omega_1 = \frac{1}{2\epsilon}\left(\frac{k}{ma}\right)^2$. We assume the condition $\left(\frac{k}{ma}\right)^2 < 1(2\epsilon\omega_1 < 2)$ which means the wavelength of the perturbation is larger than the Compton length. We define slow time scale $\tau_n = \epsilon^n\tau$. The time derivative with respect to τ is replaced by

$$\frac{d}{d\tau} = D_0 + \epsilon D_1 + \dots, \quad (24)$$

where $D_n = \frac{\partial}{\partial \tau_n}$. We expand

$$\tilde{Q} = Q^{(0)} + \epsilon Q^{(1)} + \dots. \quad (25)$$

By substituting these expression to eq.(19) and collect the terms of each order of ϵ . From the $O(\epsilon^0)$ and $O(\epsilon^1)$, we have

$$O(\epsilon^0) : D_0^2 Q^{(0)} + Q^{(0)} = 0, \quad (26)$$

$$O(\epsilon^1) : D_0^2 Q^{(1)} + Q^{(1)} = -\left(2D_0 D_1 Q^{(0)} + 2\omega_1 Q^{(0)} - \frac{\tau_0}{\tau}\sin(2\tau)Q^{(0)}\right). \quad (27)$$

The solution of eq.(26) is

$$Q^{(0)} = \mathcal{A}(\tau_1)e^{i\tau} + \mathcal{A}^*(\tau_1)e^{-i\tau}. \quad (28)$$

We substitute this to the right hand side of eq.(27) and demand that the secular term which is proportional to $e^{i\tau}$ vanishes. This gives the equation for the amplitude $\mathcal{A}$:

$$i\frac{\partial \mathcal{A}}{\partial \tau_1} + \omega_1 \mathcal{A} + i\frac{\tau_0}{4\tau}\mathcal{A}^* = 0. \quad (29)$$

Using the definition of ϵ and τ_1 , we have

$$i\frac{\partial \mathcal{A}}{\partial \tau} + \frac{1}{2}\left(\frac{k}{ma}\right)^2 \mathcal{A} + \frac{i}{\tau}\mathcal{A}^* = 0. \quad (30)$$

Writing $\mathcal{A} = u + iv$ (u, v are real), u satisfies the following second order differential equation:

$$u_{\tau\tau} + \frac{4}{3\tau}u_\tau + \left[\frac{1}{4}\left(\frac{k}{ma}\right)^4 - \frac{2}{3\tau}\right]u = 0. \quad (31)$$

Using the eq.(13) and (28),

$$\Delta \propto \tilde{Q}_\eta \tilde{\phi}_\eta - \tilde{Q} \tilde{\phi}_{\eta\eta} - \frac{1}{\eta} \tilde{Q}_\eta \tilde{\phi} = u + O\left(\frac{u}{\eta}\right). \quad (32)$$

Therefore the function u is equal to the gauge invariant density contrast Δ within the approximation we are using here. The solution of eq.(31) is

$$u = \tau^{-1/6} Z_{\pm 5/2} \left(\left(\frac{k}{a} \right)^2 \frac{1}{mH} \right), \quad (33)$$

where Z_ν is a Bessel function of order ν . We have the critical wavelength $\lambda_J = (mH)^{-1/2}$. The mode whose wavelength is larger than λ_J can grow. If the wavelength is shorter than λ_J initially, the wavelength is stretched by the cosmic expansion and it exceeds the critical length. We can see this behavior by using the chart of Mathieu function. The trajectory which started from the stable region moves to the unstable region. For $k \rightarrow 0$ limit,

$$\begin{aligned} \Delta &\propto \tau^{2/3} = a, \quad \tau^{-1} = a^{-3/2}, \\ \Psi &\propto \text{constant}, \quad \tau^{-5/3} = \frac{H}{a}. \end{aligned} \quad (34)$$

This behavior is same as the perturbation in the dust dominated universe.

2.3.2 $n \geq 2$ case

For $n \geq 2$, the scalar field oscillation is not sinusoidal. We start searching the solution of the equation for $y = \tilde{\phi}_\eta$:

$$y_{\eta\eta} + n(2n-1)m^2 \tilde{\phi}^{2n-2} y = 0. \quad (35)$$

Eq.(16) reduces to this equation if the third and the forth term do not exist. We approximate the solution of eq.(35) by sinusoidal function: $y = m \sin(cm\eta)$. c is a some numerical constant which defines the period of scalar field oscillation. Using the equation for $\tilde{\phi}$, we have

$$\begin{aligned} n(2n-1)m^2 \tilde{\phi}^{2n-2} &= -\frac{y_{\eta\eta}}{y}, \\ 2n(n+1)m^2 \tilde{\phi}^{2n-1} \tilde{\phi}_\eta &= (\tilde{\phi}_\eta \tilde{\phi})_{\eta\eta}, \end{aligned}$$

and using $y = \tilde{\phi}_\eta = m \sin(cm\eta)$, the equation of $\tilde{Q}$ becomes

$$\tilde{Q}_{\eta\eta} + \left[c^2 m^2 - \frac{4cm}{\eta} \sin(2cm\eta) + k^2 a^{\frac{4(n-2)}{n+1}} \right] \tilde{Q} = 0. \quad (36)$$

By introducing the dimensionless time variable $\tau = cm\eta$,

$$\tilde{Q}_{\tau\tau} + \left[1 - \frac{4}{\tau} \sin(2\tau) + \left(\frac{k}{cm} \right)^2 a^{\frac{4(n-2)}{n+1}} \right] \tilde{Q} = 0. \quad (37)$$

This is also Mathieu equation. It is surprising that this equation contains the $n = 1$ case if we set $c = 1$. The function A and q are

$$A = 1 + \alpha \tau^{\frac{4(2-n)}{3}}, \quad q = \frac{2}{\tau}, \quad (38)$$

where $\alpha = \left(\frac{k}{cm}\right)^2 (cnt_0 m)^{4(n-2)/3}$ and we have the relation

$$A(q) = 1 + \alpha \left(\frac{q}{2}\right)^{\frac{4(2-n)}{3}}. \quad (39)$$

Using the chart of Mathieu function, we find that the perturbation also get the effect of the parametric resonance of the first unstable band and grows. But as the time goes on, the trajectory moves from the unstable region to the stable region and the perturbation will oscillate with a constant amplitude. To investigate these behavior, we introduce slow time variable and derive the equation for slowly changing amplitude of $\tilde{Q}$. The procedure is completely same as the $n = 1$ case. The result is

$$\tilde{Q} = \mathcal{A}e^{i\tau} + \mathcal{A}^*e^{-i\tau}, \quad (40)$$

$$i\frac{\partial \mathcal{A}}{\partial \tau} + \frac{1}{2} \left(\frac{k}{cm}\right)^2 a^{\frac{4(n-2)}{n+1}} \mathcal{A} + \frac{i}{\tau} \mathcal{A}^* = 0. \quad (41)$$

The real part of $\mathcal{A}$ obeys

$$u_{\tau\tau} - \frac{4}{3}(n-2)\frac{1}{\tau}u_{\tau} + \left[\frac{\alpha^2}{4}\tau^{\frac{8}{3}(n-2)} - \left(\frac{4}{3}n - \frac{2}{3}\right)\frac{1}{\tau^2}\right]u = 0. \quad (42)$$

The solution of this equation is

$$u = \tau^{\frac{1}{6}(4n-5)} Z_{\nu} \left(\frac{3\alpha}{2(4n-5)} \tau^{\frac{1}{3}(4n-5)} \right), \quad \nu = \pm \frac{4n+1}{2(4n-5)}. \quad (43)$$

For $k \rightarrow 0$ limit, we have

$$\Delta \propto \tau^{\frac{2}{3}(2n-1)} = a^{\frac{2(2n-1)}{n+1}}, \quad \tau^{-1} = a^{-\frac{3}{n+1}}, \quad (44)$$

$$\Psi \propto \text{constant}, \quad \tau^{-\frac{1}{3}(4n+1)} = \frac{H}{a}. \quad (45)$$

3 two scalar field case

We must treat the non-linearly interacting two scalar field system to make the model of reheating more realistic. We consider the system with the massive inflaton field interacting with massless field. This model describes the decay of the inflaton field to the massless field. The background equations are

$$H^2 = \frac{\kappa}{3} \left(\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \dot{\chi}^2 + V(\phi, \chi) \right), \quad (46)$$

$$\dot{H} = -\frac{\kappa}{2} (\dot{\phi}^2 + \dot{\chi}^2), \quad (47)$$

$$\ddot{\phi} + 3H\dot{\phi} + V_{\phi} = 0, \quad (48)$$

$$\ddot{\chi} + 3H\dot{\chi} + V_{\chi} = 0, \quad (49)$$

where $V(\phi, \chi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}\lambda\phi^2\chi^2$. The gauge invariant perturbation variables are

$$Q_\phi = \delta\phi - \frac{\dot{\phi}}{H}\Psi, \quad (50)$$

$$Q_\chi = \delta\chi - \frac{\dot{\chi}}{H}\Psi. \quad (51)$$

The evolution equations of these variables become

$$\begin{aligned} \ddot{Q}_\phi + 3H\dot{Q}_\phi + \left(\frac{k}{a}\right)^2 Q_\phi \\ + \left[V_{\phi\phi} - \frac{\kappa}{a^3} \left(\frac{a^3}{H} \dot{\phi}^2 \right) \right] Q_\phi + \left[V_{\phi\chi} - \frac{\kappa}{a^3} \left(\frac{a^3}{H} \dot{\phi}\dot{\chi} \right) \right] Q_\chi = 0, \end{aligned} \quad (52)$$

$$\begin{aligned} \ddot{Q}_\chi + 3H\dot{Q}_\chi + \left(\frac{k}{a}\right)^2 Q_\chi \\ + \left[V_{\phi\chi} - \frac{\kappa}{a^3} \left(\frac{a^3}{H} \dot{\phi}\dot{\chi} \right) \right] Q_\phi + \left[V_{\chi\chi} - \frac{\kappa}{a^3} \left(\frac{a^3}{H} \dot{\chi}^2 \right) \right] Q_\chi = 0. \end{aligned} \quad (53)$$

These equations decouple if the conditions

$$|\dot{\chi}| \ll |\dot{\phi}|, \quad \lambda\chi^2 \ll m^2$$

are satisfied and become

$$(a^{3/2}Q_\phi)_{''} + \left[m^2 + \left(\frac{k}{a}\right)^2 - \frac{\kappa}{a^3} \left(\frac{a^3}{H} \dot{\phi}^2 \right) - \frac{9}{4}H^2 - \frac{3}{2}\dot{H} \right] (a^{3/2}Q_\phi) = 0, \quad (54)$$

$$(a^{3/2}Q_\chi)_{''} + \left[\lambda\phi^2 + \left(\frac{k}{a}\right)^2 - \frac{9}{4}H^2 - \frac{3}{2}\dot{H} \right] (a^{3/2}Q_\chi) = 0. \quad (55)$$

Using the background solution of the inflaton field $\phi(t) \approx \phi_0 a^{-3/2} \cos(mt)$ (ϕ_0 is a constant), we can rewrite the perturbation equations as the form of the Mathieu equation:

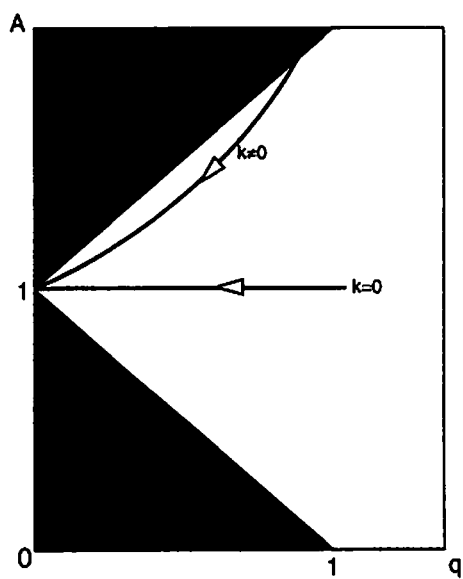
$$(a^{3/2}Q_\phi)_{\tau\tau} + \left[1 + \left(\frac{k}{ma}\right)^2 - \frac{4}{\tau} \sin(2\tau) \right] (a^{3/2}Q_\phi) = 0, \quad (56)$$

$$(a^{3/2}Q_\chi)_{\tau\tau} + \left[\frac{\lambda\phi_0^2}{2m^2} \frac{1}{a^3} + \left(\frac{k}{ma}\right)^2 + \left(\frac{\lambda\phi_0^2}{2m^2} - \frac{3\kappa}{8\pi}\phi_0^2 \right) \frac{1}{a^3} \cos(2\tau) \right] (a^{3/2}Q_\chi) = 0, \quad (57)$$

where $\tau = mt$. The chart of the Mathieu function becomes

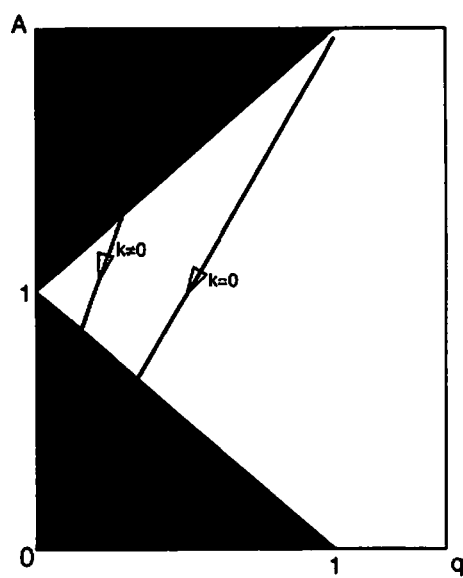
$a^{3/2}Q_\phi$:

$$A = 1 + \left(\frac{k}{ma}\right)^2, \quad 2q = \frac{4}{\tau}$$



$a^{3/2}Q_\chi$:

$$A = \frac{\lambda\phi_0^2}{2m^2} \frac{1}{a^3} + \left(\frac{k}{ma}\right)^2, \quad 2q = \left(\frac{\lambda\phi_0^2}{2m^2} - \frac{3\kappa}{8\pi}\phi_0^2\right) \frac{1}{a^3}$$



From the condition of the coherent oscillation, the parameter q of the Mathieu equation for the variable $a^{3/2}Q_\phi$ is smaller than unity and the perturbation of ϕ field get the effect of parametric resonance of the first unstable band(narrow resonance). On the other hand, the parameter q for the $a^{3/2}Q_\chi$ becomes greater than unity if we choose the value of the coupling constant λ large. Therefore the perturbation of χ field will have the effect of parametric resonance of the broad band whose growth rate of the perturbation is larger than the narrow case.

We can obtain the behavior of the $a^{3/2}Q_\chi$ using the multiple time-scale method. For the long wave limit($k = 0$) and the center of the first unstable band, we have

$$a^{3/2}Q_\chi \approx \tau^{1/2} [u \cos(\alpha_0 \ln \tau + \theta_0) - v \sin(\alpha_0 \ln \tau + \theta_0)], \quad (58)$$

$$u, v \sim \left(\frac{\tau}{\tau_*}\right)^{C\alpha_0}, \quad \alpha_0 = \left(\frac{\lambda\phi_0^2\tau_0^2}{2m^2}\right)^{1/2}, \quad (59)$$

where C is a numerical factor. The perturbation of χ field grows in powers of time and square of the power index is proportional to the coupling λ . When the parameter (A, q) goes out of the unstable band, $a^{3/2}Q_\chi$ oscillates with a constant amplitude.

4 summary

In summary, we found that the evolution equation of the Mukhanov's gauge invariant variable in the oscillatory phase of the scalar field can be reduced to the Mathieu equation and the evolution of this variable undergoes the effect of the parametric resonance. We can interpret the growth of the density perturbation in this phase is caused by the parametric resonance.

For the model of two scalar fields, we found that the perturbation of χ field can get the effect of the parametric resonance of the broad band if the coupling constant is sufficiently large. So there is a possibility that the spectrum of the cosmological perturbation which was generated by the quantum fluctuation during the inflation is modified by the evolution through the stage of the reheating. To check this, we must investigate the behavior of the adiabatic(curvature) and the isocurvature perturbation and this is our next problem to be solved.

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Dynamics of Electroweak Phase Transition : Interpretation of Numerical Results

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Abstract

In a weakly first-order phase transition such as the electroweak phase transition, we have a double-well potential, whose barrier is quite shallow. At a glance of it, it seems that after some period, the universe is distributed into the symmetric minimum and the asymmetric one with the same ratio even if the universe is localized around a minimum initially. However, unless fluctuations around one minimum are so large, the universe keeps a memory of the initial condition and can have a fraction of one minimum different from 0.5.

1 Introduction

Recently, the role of large amplitude fluctuations (called subcritical bubbles) came to draw attention in the dynamics of the first-order phase transition. In particular, their roles on the dynamics of the electroweak phase transition have been investigated by many authors [1]-[8] in connection with baryogenesis.

Borrill and Gleiser [4] examined numerically whether the universe stays homogeneously at the symmetric minimum above the critical temperature. The master equation is the following phenomenological Langevin equation,

$$\square\phi(\mathbf{x},t) + \eta\dot{\phi}(\mathbf{x},t) + V'_{\text{eff}}(\phi, T_c) = \xi(\mathbf{x},t) . \quad (1)$$

Here an overdot denotes time derivation and $\xi(\mathbf{x},t)$ is a random Gaussian noise with the correlation function

$$\langle \xi(\mathbf{x}_1, t_1) \xi(\mathbf{x}_2, t_2) \rangle = \mathcal{D} \delta(\mathbf{x}_1 - \mathbf{x}_2) \delta(t_1 - t_2) , \quad (2)$$

satisfying the fluctuation-dissipation relation $\mathcal{D} = 2\eta T_c$. Also, $V_{\text{eff}}(\phi, T_c)$ is a one-loop finite temperature effective potential of the Higgs field, ϕ , in the electroweak theory and given in [1][9][10] as

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$$V_{\text{eff}}(\phi, T) = D(T^2 - T_2^2)\phi^2 - ET\phi^3 + \frac{1}{4}\lambda_T\phi^4, \quad (3)$$

where $D = 0.17$, $E = 0.0097$, $T_2 = 92.64\text{GeV}$, and $\lambda = (2D)^{-1/2}\lambda_T = 0.06$ for the Higgs mass $M_H = 60\text{GeV}$ which is the experimental lower bound [11]. The potential has two degenerate minima, $\phi = 0$ and $\phi = \phi_+$, at the critical temperature $T = T_c$ (See Fig. ??). They divided the universe into two phases. One is the symmetric phase, corresponding to the field, $\phi \leq \phi_-$, and they called its fraction $f_0(t)$. The other is the asymmetric phase, corresponding to the rest of the field, and they called its fraction $f_+(t)$. According to the above Langevin equation (1), they followed the evolution of $f_0(t)$, starting from $\phi = \phi_- = 0$ for all lattices. The result is depicted in Fig. ??. Thus they concluded that phase mixing is manifest for the Higgs mass permitted from the experiment.

Note that at some λ 's, $f_0(t)$ seems to take an "equilibrium" value between 0.5 and 1.0. Since the minima of the potential are degenerate and the potential is perfectly symmetric along the maximum of the potential, ϕ_- , we cannot prefer one minimum. Therefore, one may wonder that this is because our simulation time is so short that the system still keeps the memory of the particular initial condition, and that if we could trace the evolution for a long enough period the system would relax into a state with $f_0 = 0.5$.

However, in fact, if we examine the time variation of the field configuration by observing its snapshot at different times, we can easily convince ourselves that the system is in a stationary state with a constant $f_0(> 0.5)$ repeating creation and annihilation of a number of small domains of the asymmetric state whose typical radius is at most a few times the lattice spacing. In next section we would like to present an analytic argument to support that the state they(we) followed in the numerical simulation [4] [8] is a thermal equilibrium state. Similar analysis has already been done by Gleiser, Heckler, and Kolb [12] in a slightly different situation. See also Gelmini and Gleiser [13] and Shiromizu and Morikawa [14].

2 Analytic interpretation of numerical results.

Let $d_+(r, \phi, t)$ be the number density of the asymmetric-state bubbles with radius $r(> \delta x)$ and amplitude $\phi(> \phi_-)$ at t . In order to obtain the Boltzmann equation for $d_+(r, \phi, t)$, we count the processes which change it: i) Thermal nucleation of asymmetric-state bubbles in an almost homogeneous symmetric-state sea. ii) Annihilation of these asymmetric-state bubbles into the symmetric-state sea. This is to be distinguished from the process of nucleation of a symmetric-state bubble in a homogeneous background of the asymmetric-state whose rate would be identical to that of i) in a degenerate potential. We expect the process i) has smaller rate than the process ii), since the former requires more energy. We do not take into account the process that nucleated bubbles dynamically shrink ($|v|\partial d_+/\partial r$ term in [12]) because the typical radius of nucleated bubbles is comparable to the lattice spacing and shrinking bubbles and annihilating bubbles are hardly distinguishable in our simulations. Then the Boltzmann equation for $d_+(r, \phi, t)$ becomes

$$\frac{\partial d_+(r, \phi, t)}{\partial t} = (1 - f_+(t)) G_{(s\text{-phase} \Rightarrow r, \phi)} - \frac{4\pi r^3}{3} d_+(r, \phi, t) G_{(s\text{-phase} \Leftarrow r, \phi)}, \quad (4)$$

where $G_{(s\text{-phase}\Rightarrow r, \phi)}$ is the nucleation rate for the process i), $G_{(s\text{-phase}\Leftarrow r, \phi)}$ is that for the process ii). We also assume that nucleation rates can be obtained from the Gibbs distribution, namely $G = A/T_2^4 \exp(-F/T_c)$, where A is a constant. For $G_{(s\text{-phase}\Rightarrow r, \phi)}$ we put $F = b\phi^2 r$ taking surface tension of the created bubbles into account with b a constant. Since the inverse process is not Boltzmann suppressed, we assume its rate is a constant: $G_{(s\text{-phase}\Leftarrow r, \phi)} = B/T_2^4$.

In an equilibrium state, $\partial d_+/\partial t$ equals zero for all ϕ 's and r 's,

$$(1 - f_+^{eq}) \frac{G_{(s\text{-phase}\Rightarrow r, \phi)}}{G_{(s\text{-phase}\Leftarrow r, \phi)}} = \frac{4\pi r^3}{3} d_+(r, \phi, t). \quad (5)$$

Summing these equations for all ϕ 's and r 's leads to

$$\int_{\delta x}^{\infty} \int_{\phi_-}^{\infty} (1 - f_+^{eq}) \frac{G_{(s\text{-phase}\Rightarrow r, \phi)}}{G_{(s\text{-phase}\Leftarrow r, \phi)}} d\phi dr = \int_{\delta x}^{\infty} \int_{\phi_-}^{\infty} \frac{4\pi r^3}{3} d_+(r, \phi, t) d\phi dr = f_+^{eq}. \quad (6)$$

Then f_+^{eq} becomes

$$f_+^{eq} = \frac{I}{I + 1}, \quad (7)$$

where

$$\begin{aligned} I &= \int_{\delta x}^{\infty} \int_{\phi_-}^{\infty} \frac{G_{(s\text{-phase}\Rightarrow r, \phi)}}{G_{(s\text{-phase}\Leftarrow r, \phi)}} d\phi dr \\ &= \frac{A}{B} \int_{\delta x}^{\infty} \int_{\phi_-}^{\infty} \exp(-F/T_c) d\phi dr. \end{aligned} \quad (8)$$

f_+^{eq} is depicted in Fig. ?? as a function of λ . Here the lattice spacing $\delta x = 1.0/(\sqrt{2DT_2})$. Then we can fit the analytic solution with the numerical simulation very well with $A/B = 65$ and $b = 2.77$.

Note that even if $|v|$ equals zero, f_+^{eq} can become values different from 0.5. The essence lies in the fact that creation rate of an asymmetric-state domain in the symmetric phase and that of its reverse can be different due to the surface tension even at the critical temperature if the background is sufficiently homogeneous. Of course, in the case the average amplitude of fluctuations is large enough, percolation occurs quickly and the two processes will have the same rate, resulting in $f_+^{eq} = 0.5$.

3 Summary and Discussion

For the degenerate potential, it seems to be plausible that ultimately two phases should have the same fraction, that is, 0.5 irrespective of the initial condition. However, the system starting from the particular initial condition can bear the trace of the initial state in case thermal fluctuations are not too large. The reason comes from the fact that if the system stays homogeneously at one minimum initially, the transition rate from the minimum to the other is smaller than the contrary because of the surface tension.

In the previous section, we argued only the case when ϕ is localized around the origin initially. If it is localized around $\phi = \phi_+$ initially, on the contrary, we expect that the system will settle into a state with $f_+^{eq} = 1 - I/(I + 1)$ or $f_0^{eq} = I/(I + 1)$, since the potential we are using is symmetric. In order to see what happens in the case ϕ is not localized around either minima, we have run five simulations starting from a checkerboard made of $\phi = 0$ and $\phi = \phi_+$ using different realization of random numbers. The result is depicted in Fig. ?? for $\lambda = 0.06$ and $\delta x = 7.0/(\sqrt{2DT_2})$, which shows that the system approaches to either equilibrium state with f_+^{eq} or $f_0^{eq} = I/(I + 1)$, although it takes much longer time to relax than in the cases with $\phi = 0$ or $\phi = \phi_+$ initially.

This result implies that the configuration with $f_+ = f_0 = 0.5$ is unstable in this case even if it contains maximum number of microscopic states. Note that this configuration also costs more energy of domain boundaries than any other configuration. This is why the system may relax to a configuration with $f_+ \neq f_0$.

Acknowledgements

We would like to thank S. Mukohyama for discussion. MY is grateful to Professor K. Sato for his continuous encouragement and to M. Morikawa, T. Shiromizu, and, M. Kawasaki for their useful comments. This work was partially supported by the Japanese Grant in Aid for Scientific Research Fund of the Ministry of Education, Science, Sports and Culture Nos. 07304033(JY) and 08740202(JY).

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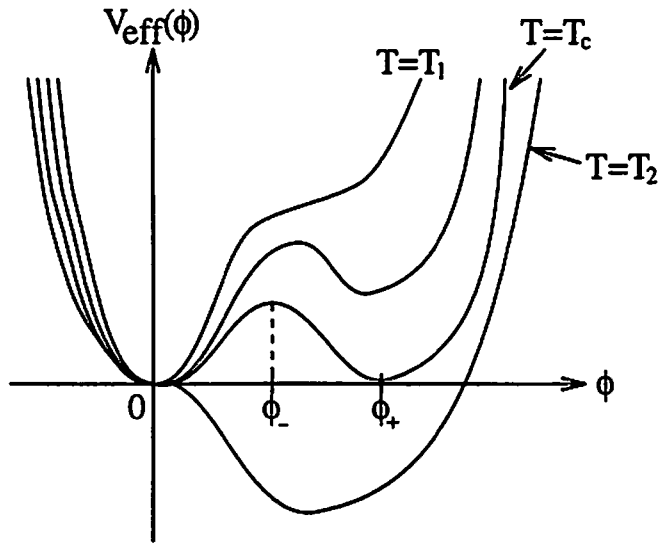


Figure 1: One-loop finite temperature effective potential of the Higgs field.

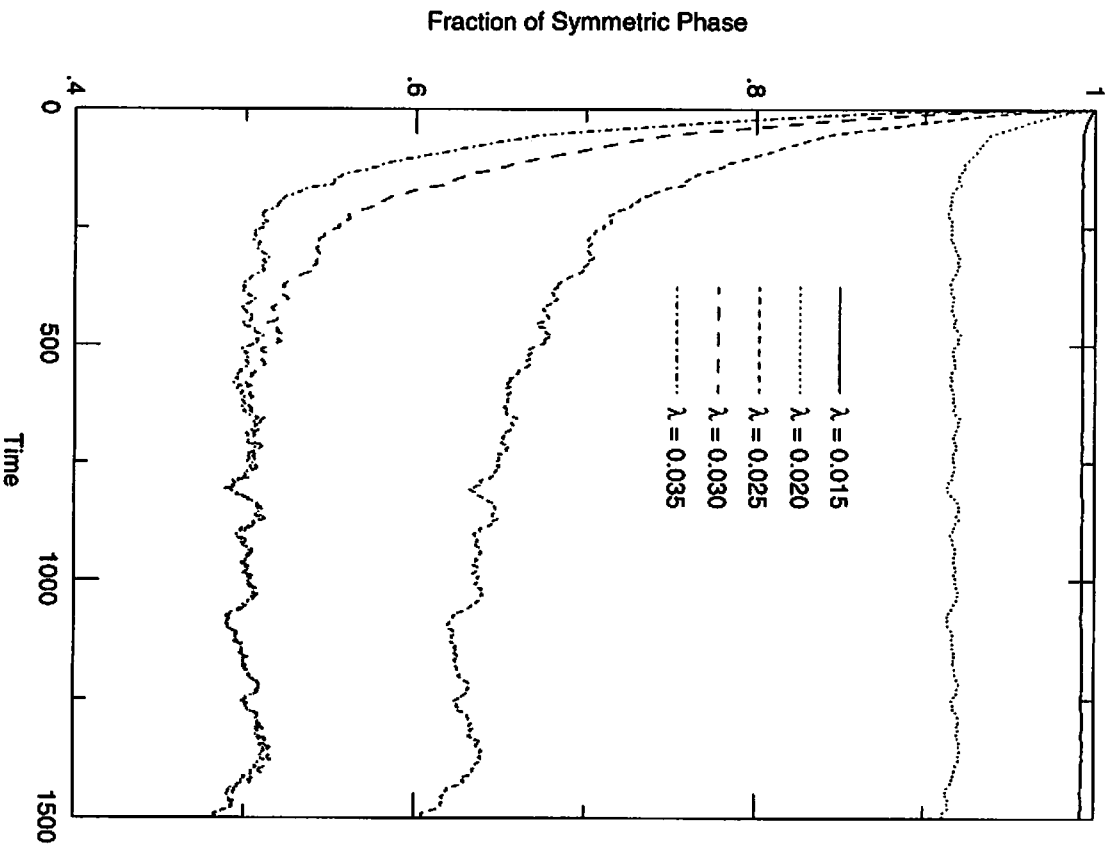


Figure 2: Fraction of the symmetric phase, f_0 , for $\delta x = 1.0/((2D)^{1/2}T_2)$. The vertical axis is fraction, and the horizontal one is the normalized time $\tilde{t} \equiv (2D)^{1/2}T_2t$.

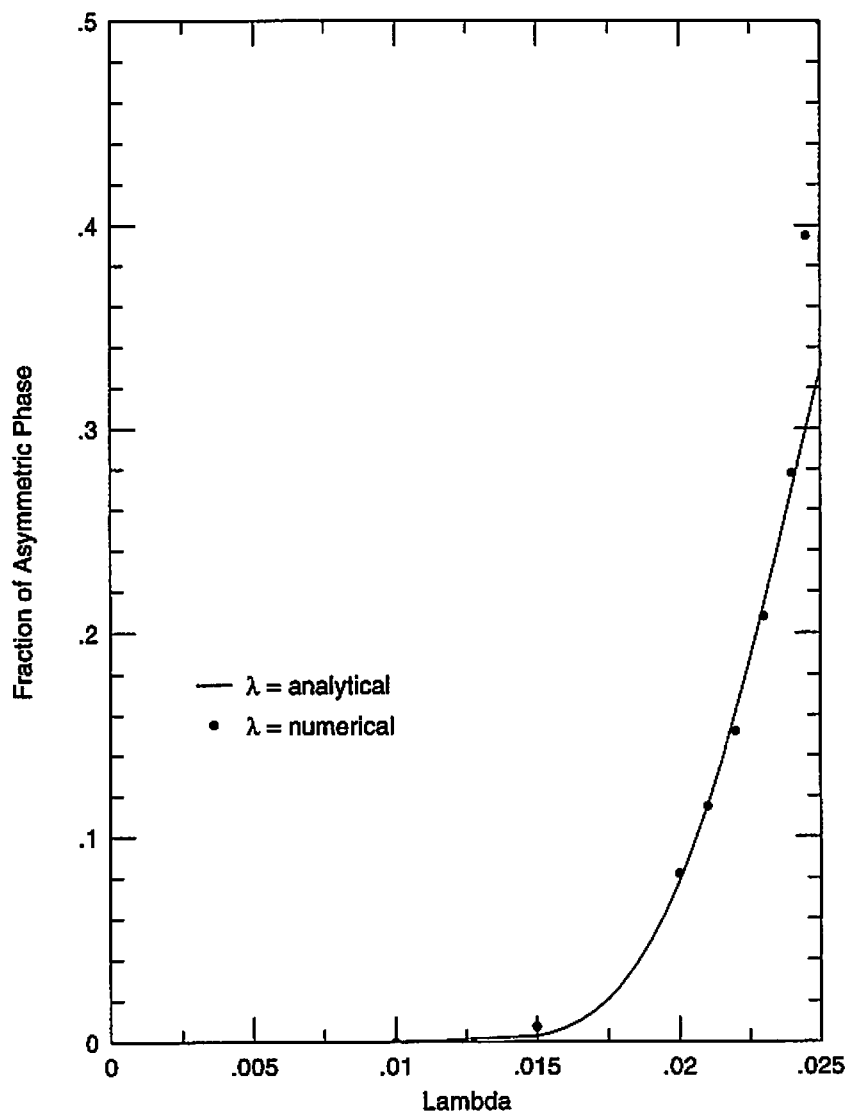


Figure 3: Comparison of analytical and numerical results.

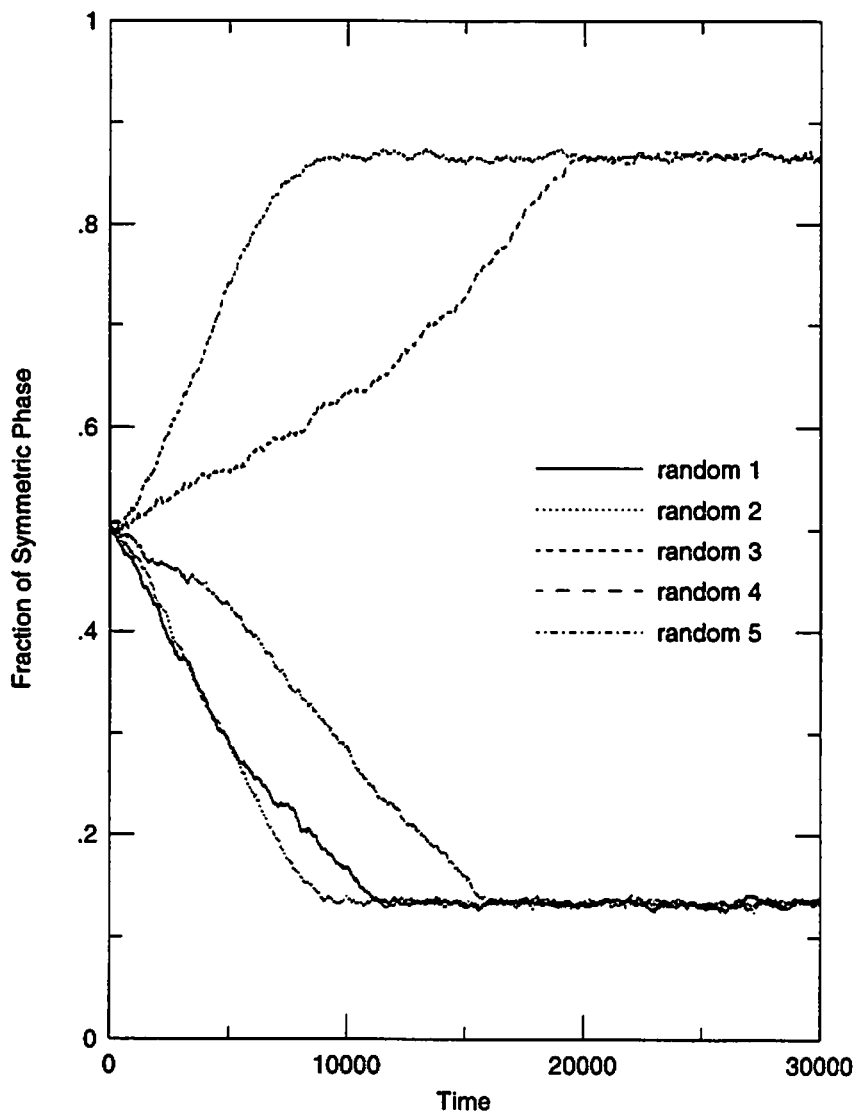


Figure 4: Fraction of the symmetric, f_0 starting from a checkerboard.

Perturbed equation of motion of a domain wall

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Abstract

We explicitly derive the perturbed equation of motion of domain wall coupled with gravitational waves from the Nambu-Goto action. We see that the equation actually coincides with that we obtained by the metric junction formalism.

As a possible evidence of cosmological phase transitions in the early universe, topological defects, such as cosmic strings, domain walls, may remain somewhere in our universe (See Ref. [1]). So far, much of interest has focused on cosmic strings as viable candidates for the seed fluctuations leading to the large scale structure formation. On the other hand, domain wall is the simplest example of the topological defects in accordance with the spontaneous breaking of discrete symmetries. Though stable domain walls cause serious cosmological problem; the energy density of the universe is overdominated, unstable domain walls are probably harmless because they may radiate away their energy by emitting gravitational waves, vector, or scalar waves and disappear quickly. Such decaying domain walls with radiating energies may play important roles in the history of evolution of the universe. So, it has been of interest to investigate the domain walls, especially, accompanied with radiation.

The decaying processes of domain walls with radiation have been investigated by Vachaspati, et.al [2]. They estimated the order of magnitude of the emission rate and showed that the dominant decay process of domain walls is the emission of gravitational waves. Naively one might expect that deformed domain wall oscillates freely and the oscillation yields gravitational waves and that the emission rate of gravitational waves can be estimated by using quadrupole formula. However, as is well known, domain wall is a relativistic object with large negative internal pressure [3]. There is no guarantee that the quadrupole formula give a correct estimate of gravitational wave emission rate by relativistic objects. Thus more precise investigations of the behavior of domain walls with emission process of gravitational waves are necessary. This motivates us to investigate the motion of domain walls in general relativistic framework.

Recently, there is renewed interest in domain walls in the context of open inflation [4] that predicts a low density, negative curvature universe and has been studied widely (See Ref. [5] and references cited therein). The essential feature of this model is that the observable open universe is contained in a single vacuum bubble nucleated during an inflationary phase. Such the universe is surrounded by a domain wall (called a bubble

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wall) and hence the conclusive verification of the model is to detect the existence of the bubble wall. Indeed, there appear specific modes of cosmological perturbations associated with a bubble wall fluctuations and their influence for the cosmic microwave background anisotropies are investigated to constrain the model [6]. Thus a bubble wall perturbations are especially relevant in cosmological perturbations of this model. This also motivates us to investigate the perturbative motion of domain wall.

The motion of domain wall is regarded as a three-dimensional timelike hypersurface Σ in the four-dimensional space-times in the thin wall approximation. Hereafter we call such a hypersurface the worldhypersheet. Wall perturbations are represented by a single scalar field ϕ on the background worldhypersheet which describes the background motion of the wall [7]. The equation of the scalar field takes the form of the Klein-Gordon wave equation

$$(-\square + m^2)\phi = 0, \quad (1)$$

on the background worldhypersheet. The mass m^2 is expressed in terms of the energy density deficit ρ in the vacuum bubble and the surface energy density ζ of the bubble wall. This equation has oscillatory solutions and they might generate gravitational waves estimated by the quadrupole formula. In addition, such gravitational waves might have some informations of inherent properties of the domain wall. It should be noted that, in these analyses, the gravitational backreactions of domain walls have been ignored.

On the other hand, the perturbations of domain walls interacting with the gravitational waves have been investigated, by Kodama, et.al. [8] in flat background case, and by us in open inflation background case [9]. In these analyses, the domain walls whose background worldhypersheets are three-dimensional de Sitter space-times have been treated in the thin wall approximation. In the metric junction formalism [10] the gravitational backreactions of the walls are taken into account. The results are different from those of the analyses neglecting the gravitational backreactions. It has turned out that the deformations of domain wall do not propagate on its background worldhypersheet and accordingly the deformed domain wall does not emit gravitational waves by its free oscillations. Such a domain wall does not emit its inherent modes. This suggests that the backreactions of gravity play a significant role in the behavior of the domain walls; the gravitational perturbations induced by the perturbative motion of domain wall affect to the domain wall motion in the same order in its oscillation amplitude.

In these works [8, 9], the perturbed equations of wall motions coupled with gravity also have been derived from the perturbed metric junction condition (see Eq. (7) below quoted from Ref. [9], see also Eq.(5.2) in Ref. [8]). Comparing them with Eq. (1), we notice that there are extra terms described by metric perturbations. We naturally guess that these additional terms express the influence of gravitational backreactions and induce the difference of the results. If these equations actually express the linear perturbations of the wall coupled with gravitational waves, they also should be derived from the Nambu-Goto action.

In this paper, we will derive the perturbed equation of domain wall motion coupled with gravitational perturbation from the Nambu-Goto action and compare it with the equation derived by the metric junction formalism in the previous paper [9]. We will demonstrate it explicitly on the one-bubble open inflation background.

First, we review the result in Ref. [9]. Second, we derive Eq. (1) from the Nambu-Goto action without taking account of the gravitational backreactions according to Garriga and Vilenkin, and Guven [7] on the one-bubble open inflation background. Then, we demonstrate how to derive the perturbative equation of motion coupled with gravitational perturbations.

Throughout the paper, all of the definitions of geometrical quantities and the notation are the same of those in Ref. [9].

Let us consider the geometry containing a $SO(3,1)$ symmetric infinitely thin bubble wall Σ such that inside the bubble is the Minkowski space-time (M_-, g_-) and outside the de Sitter space-time (M_+, g_+) as the background geometry for perturbative analysis of the bubble wall. We construct the space-time manifold $M = M_+ \cup \Sigma \cup M_-$ by matching the boundaries $\Sigma_{\pm} := \partial M_{\pm}$ at Σ . Hereafter, we describe quantities inside the bubble with subscript “-” and outside “+”. The geometries of the boundaries $\Sigma_{\pm}$ are characterized by the induced intrinsic metrics $(q_{ij}d\xi^i d\xi^j)_{\pm}$ and the extrinsic curvatures $(K_{ij}d\xi^i d\xi^j)_{\pm}$, where $\xi^i_{\pm}$ (the indices $i, j, \dots$ run over 0,1,2) are three-dimensional intrinsic coordinates on $\Sigma_{\pm}$. These quantities are specified by the junction conditions

$$[q_{ij}] = 0, \quad (2)$$

$$[K_j^i] = \kappa \left(S_j^i - \frac{1}{2} \delta_j^i S \right), \quad (3)$$

$$\bar{K}_{ij} S^{ij} = -\rho, \quad (4)$$

$$^{(3)}D_j S^{ij} = 0, \quad (5)$$

where $S_{ij}(\xi^i)$ is the wall stress-energy tensor. In the present case $S_{ij}(\xi^i) = -\zeta q_{ij}$ and Eq. (5) yields $\zeta = \text{const.}$ For quantities $Q_{\pm}$ on $\Sigma_{\pm}$, $[Q] := Q_+ - Q_-$, and $\bar{Q} := \frac{1}{2}\{Q_+ + Q_-\}$.

First, we mention the equation of bubble wall motion coupled with gravitational waves in the framework of the metric junction formalism. Rewriting the perturbation of the junction condition Eq. (4):

$$\delta \bar{K} = 0, \quad (6)$$

in terms of the wall displacement variable $X_{\perp}$, we can obtain the equation

$$\left\{ -D_{\parallel}^2 - 2 \left(\frac{D_{\parallel} r}{r} \right) D_{\parallel} - \frac{l(l+1)}{r^2} + \frac{3}{2} H^2 + 3 \left(\frac{H^2}{4\lambda} \right)^2 + 3\lambda^2 \right\} X_{\perp} - \frac{3}{2} \overline{(K_a^a)} \overline{F_{\perp\perp}} - \frac{1}{2r^4} \overline{D_{\parallel}(r^4 F_{\parallel\perp})} - \frac{1}{4} \overline{D_{\perp} F} = 0, \quad (7)$$

where the subscripts $\parallel$ and $\perp$ denote the tangential and transverse components to the wall, respectively. Here r is an area radius of the bubble, the gauge-invariant metric perturbation variables $F_{\perp\perp}, F_{\parallel\perp}, F$ are constructed by combinations of the metric perturbations (in detail, see Ref. [9]), H is the Hubble radius of the de Sitter space-time, and $\lambda := \kappa\zeta/4$. We will see that the first line of this equation equals to Eq. (1).

Next, in order to compare Eq. (1) with the equation derived above, let us review the derivation of the perturbed wall equation Eq. (1) from action principle following Refs. [7] ignoring the gravitational backreaction. We should comment on that the space-time metric is not smooth because the whole space-time contains a singular hypersurface:

the bubble wall. So, in the formulation of action principle, we should consider suitably approximated smooth space-time metric. We should evaluate the extrinsic curvature K_{ij} , on such a smooth space-time. Here we simply evaluate the extrinsic quantities by the mean value of both sides.

Let $x^\mu = x^\mu(\xi^i)$ be wall extrinsic coordinates which describe the position of the wall in the four-dimensional space-time. The intrinsic coordinates ξ^i of the $SO(3, 1)$ symmetric domain wall are naturally taken by

$$\xi^i = (\tilde{\tau}, \theta, \phi), \quad (8)$$

where θ, ϕ are the standard angular coordinates on the unit two-sphere and $\tilde{\tau}$ is the wall proper time so that the vector $\tau^\mu := (\partial_{\tilde{\tau}})^\mu$ become a normalized future directed timelike vector. The tangential vectors of Σ are defined as

$$e_i^\mu := \frac{\partial x^\mu}{\partial \xi^i}. \quad (9)$$

For example, we can choose e_i^μ such as $e_0^\mu \equiv \tau^\mu$. Let $(n_\pm)^\mu$ be unit normal vectors to the boundaries $\Sigma_\pm$, respectively, such that both $(n_\pm)^\mu$ direct from M_- to M_+ , i.e.,

$$(g_{\mu\nu} n^\mu n^\nu)_\pm = 1, \quad (g_{\mu\nu} e_i^\mu n^\nu)_\pm = 0. \quad (10)$$

In terms of these quantities, the wall intrinsic metric q_{ij} and the extrinsic curvature K_{ij} are defined by

$$q_{ij} := g_{\mu\nu} e_i^\mu e_j^\nu, \quad K_{ij} := -e_i^\mu e_j^\nu \nabla_\nu n_\mu. \quad (11)$$

Here, we also note that

$$q^{ij} e_i^\mu e_j^\nu = g^{\mu\nu} - n^\mu n^\nu. \quad (12)$$

Let us take the Nambu-Goto action for the bubble wall system

$$S = -\zeta \int_\Sigma d^3\xi \sqrt{-q} + \rho \int_{M_-} d^4x \sqrt{-g}, \quad (13)$$

where $g := \det(g_{\mu\nu})$ and $q := \det(q_{ij})$, respectively. The first term is integrated over the three-volume of the worldhypersheet and the second term over the four-volume inside the bubble. This is a generalization of the Nambu-Goto action for strings. By the variation of the action Eq. (13) with respect to small change of $x^\mu(\xi^i)$ in the worldhypersheet, we obtain the equation of the bubble wall motion,

$$\Delta x^\mu + \Gamma_{\nu\lambda}^\mu q^{ij} e_i^\nu e_j^\lambda + \frac{\rho}{\zeta} = 0, \quad (14)$$

where

$$\Delta := \frac{1}{\sqrt{-q}} (\partial_i \sqrt{-q} q^{ij} \partial_j), \quad (15)$$

and $\Gamma_{\lambda\nu}^\mu$ are the space-time Christoffel symbols on Σ . By using the Gauss-Weingarten equation

$${}^{(3)}D_i e_j^\mu + \Gamma_{\nu\lambda}^\mu e_i^\nu e_j^\lambda = n^\mu K_{ij}, \quad (16)$$

($\Delta x^\mu \equiv {}^{(3)}D^j e_j^\mu$), we can further simplify Eq. (14) in the form,

$$K = \frac{\rho}{\zeta}. \quad (17)$$

Because ρ/ζ is constant, the perturbed equation of the bubble wall is obtained as

$$\delta K = 0, \quad (18)$$

by the simple linearization of the background motion Eq. (17).

Let us express this perturbed equation in terms of wall displacement variable ϕ , which should be compared with Eq. (7), by taking a perturbation of Eq. (14). Let $\tilde{\delta}$ be a physical perturbation of the wall variable x^μ such as

$$\tilde{\delta} : x^\mu \longrightarrow \tilde{x}^\mu := x^\mu + \tilde{\delta} x^\mu. \quad (19)$$

Since only motion transverse to the wall is physically observable, the wall displacement $\tilde{\delta} x^\mu$ has only one degree of freedom. Thus physical displacement of the wall is described with a single scalar field $\phi(\xi)$ on Σ such as $\tilde{\delta} x^\mu \equiv \phi n^\mu$. The tangential directions of $\tilde{\delta} x^\mu$ to Σ correspond to diffeomorphisms on Σ :

$$\tilde{\delta} : \xi^i \rightarrow \xi^i + \tilde{\delta} \xi^i. \quad (20)$$

For convenience of calculations, we choose the Gaussian normal coordinates in the neighborhood of considering bubble wall. In the Gaussian normal coordinates, we can express the whole space-time metric as

$$g_{\mu\nu} dx^\mu dx^\nu = d\chi^2 + q_{ij} d\xi^i d\xi^j, \quad (21)$$

and the normal vector as $n^\mu = (\partial_\chi)^\mu$. In this coordinates, the extrinsic curvature is represented by

$$K_{ij} = -\frac{1}{2} n^\lambda \partial_\lambda g_{ij}(x) = -\frac{1}{2} \partial_\chi g_{ij}. \quad (22)$$

The change of the space-time metric is described as

$$\begin{aligned} g_{\mu\nu}(x + \tilde{\delta} x) &\simeq g_{\mu\nu}(x) + \tilde{\delta} x^\lambda \partial_\lambda g_{\mu\nu}(x), \\ &\simeq g_{\mu\nu}(x) - 2\phi K_{ij} e_\mu^i e_\nu^j. \end{aligned} \quad (23)$$

Under this perturbation $\tilde{\delta}$, the induced metric q_{ij} varies as

$$\begin{aligned} \tilde{\delta} : q_{ij}(x) \longrightarrow q_{ij}(\tilde{x}) &= g_{\mu\nu}(x + \tilde{\delta} x) \tilde{e}_i^\mu \tilde{e}_j^\nu \\ &\simeq q_{ij}(x) - 2\phi K_{ij}, \end{aligned} \quad (24)$$

where $\tilde{e}_i^\mu := \partial_i \tilde{x}^\mu$ and we use the orthogonarities $e_i^\mu n_\mu = 0$. The perturbations of the wall intrinsic metric q_{ij} are

$$\tilde{\delta} q_{ij}(x) = -2\phi K_{ij}, \quad (25)$$

$$\tilde{\delta} q^{ij}(x) := -q^{il} q^{jk} \tilde{\delta} q_{lk}(x) = 2\phi K^{ij}. \quad (26)$$

By taking the perturbation $\tilde{\delta}$ of Eq. (14) with the help of Eqs. (23), (25), (26), and the Gauss-Codazzi equation, we can obtain the equation for ϕ on the background worldhypersheet of the wall as the form of Eq. (1) with mass

$$m^2 = -R_{ij}q^{ij} + {}^{(3)}R - \left(\frac{\rho}{\zeta}\right)^2, \quad (27)$$

where $R_{\mu\nu}$ is the space-time Ricci tensor. In detail of this derivation, see Ref. [7]. We naturally evaluate $R_{ij}q^{ij} = \overline{R_{ij}q^{ij}}$: the mean value. Assuming that $\phi(\xi)$ is expressed as $\phi(\tau)Y_m^l(\theta, \varphi)$, we can reduce the perturbed equation of the wall Eq. (1) to

$$\left\{ -D_{\parallel}^2 - 2\left(\frac{D_{\parallel}\tau}{r}\right)D_{\parallel} - \frac{l(l+1)}{r^2} + \frac{3}{2}H^2 + 3\left(\frac{H^2}{4\lambda}\right)^2 + 3\lambda^2 \right\} \phi(\tau) = 0. \quad (28)$$

This equation describes the wall displacement on the background worldhypersheet without treating the bubble wall as a source of gravity.

In usual, as examined in Ref. [7], the perturbations of the wall intrinsic metric are induced by $\tilde{\delta}x^\mu$ through the relation $q_{ij}(x) = g_{\mu\nu}(x)e_i^\mu e_j^\nu$. However to study a coupled system of the domain wall and gravitational waves, we should introduce an additional perturbation δ' of the space-time metric $g_{\mu\nu}$, i.e. metric perturbations:

$$\delta' : g_{\mu\nu}(x) \longrightarrow g'_{\mu\nu}(x) := g_{\mu\nu}(x) + h_{\mu\nu}(x). \quad (29)$$

Now, let us consider full perturbation $\delta := \tilde{\delta} + \delta'$. The space-time metric changes as

$$\begin{aligned} \delta : g_{\mu\nu}(x) \longrightarrow g'_{\mu\nu}(\tilde{x}) &= (g_{\mu\nu} + h_{\mu\nu})(x + \tilde{\delta}x) \\ &\simeq g_{\mu\nu}(\tilde{x}) + h_{\mu\nu}(x) \\ &\simeq g_{\mu\nu}(x) - 2\phi K_{\mu\nu} + h_{\mu\nu}(x). \end{aligned} \quad (30)$$

Therefore

$$\delta g_{\mu\nu}(x) = -2\phi K_{\mu\nu} + h_{\mu\nu}(x). \quad (31)$$

Under these perturbations, the wall intrinsic metric q_{ij} changes

$$\begin{aligned} \delta : q_{ij}(x) \longrightarrow q'_{ij}(\tilde{x}) &= (g_{\mu\nu} + h_{\mu\nu})(x + \delta x) \tilde{e}_i^\mu \tilde{e}_j^\nu \\ &\simeq g_{\mu\nu}(x + \delta x) \tilde{e}_i^\mu \tilde{e}_j^\nu + h_{\mu\nu} e_i^\mu e_j^\nu \\ &\simeq q_{ij}(x) - 2\phi K_{ij} + h_{ij}(x). \end{aligned} \quad (32)$$

where $\tilde{e}_i^\mu := \partial_i \tilde{x}^\mu$ and we use the orthogonarities $e_i^\mu n_\mu = 0$. Thus we obtain the perturbations of q_{ij}

$$\delta q_{ij}(x) = -2\phi K_{ij} + h_{ij}(x), \quad (33)$$

$$\delta q^{ij}(x) := -q^{il}q^{jk}\delta q_{lk}(x) = 2\phi K^{ij} - h^{ij}(x). \quad (34)$$

As mentioned above, the only diffeomorphism invariant wall displacement is normal to the wall. Thus we are interested in the normal direction of the perturbation of Eq. (14), i.e.

$$n_\mu \delta \left(\Delta x^\mu + \Gamma_{\nu\lambda}^\mu e_i^\nu e_j^\lambda + \frac{\rho}{\zeta} n^\mu \right) = 0. \quad (35)$$

The second term of Eq. (35) is decomposed as

$$n_\mu \delta(\Gamma_{\nu\lambda}^\mu q^{ij} e_i^\nu e_j^\lambda) = n_\mu \delta\Gamma_{\nu\lambda}^\mu q^{ij} e_i^\nu e_j^\lambda + n_\mu \Gamma_{\nu\lambda}^\mu \delta q^{ij} e_i^\nu e_j^\lambda + 2n_\mu \Gamma_{\nu\lambda}^\mu q^{ij} e_i^\nu \delta e_j^\lambda. \quad (36)$$

The first term of Eq. (36) becomes

$$\begin{aligned} n_\mu \delta\Gamma_{\nu\lambda}^\mu q^{ij} e_i^\nu e_j^\lambda &= \frac{1}{2} q^{ij} n^\mu (\nabla_i \delta g_{\mu j} + \nabla_j \delta g_{\mu i} - \nabla_\mu \delta g_{ij}) \\ &= \frac{1}{2} n^\mu q^{ij} (\nabla_i h_{\mu j} + \nabla_j h_{\mu i} - \nabla_\mu h_{ij}) \\ &\quad + \frac{1}{2} n^\mu q^{ij} \{-2\nabla_i (\phi K_{\mu j}) - 2\nabla_j (\phi K_{\mu i}) + 2\nabla_\mu (\phi K_{ij})\}, \end{aligned} \quad (37)$$

because of $\delta g_{\mu\nu} = h_{\mu\nu} - 2\phi K_{\mu\nu}$. Expanding $h_{\mu\nu}$ by the tensor harmonics in the way of Ref. [9], we can express Eq. (37) as

$$\begin{aligned} n_\mu \delta\Gamma_{\nu\lambda}^\mu q^{ij} e_i^\nu e_j^\lambda &= -D_\parallel f_{\parallel\perp} Y - (K_a^\alpha) f_{\parallel\parallel} Y - 3(K_a^\alpha) f_{\perp\perp} Y \\ &\quad + \frac{1}{2} D_\perp f_{\parallel\parallel} Y - 2 \left(\frac{D_\parallel r}{r} \right) f_{\parallel\perp} Y + (K_a^\alpha) f^{(e0)} Y \\ &\quad - \frac{1}{2} D_\perp f^{(e0)} Y - \frac{l(l+1)}{r} f_\perp^{(e1)} Y \\ &\quad - 6(K_a^\alpha)^2 \phi + 3\phi D_\perp (K_a^\alpha), \end{aligned} \quad (38)$$

where $f_{\parallel\perp}$ etc are the expansion coefficients of the metric perturbations. The second term of Eq. (36) becomes

$$n_\mu \Gamma_{\nu\lambda}^\mu \delta q^{ij} e_i^\nu e_j^\lambda = 6(K_a^\alpha)^2 \phi + (K_a^\alpha) f_{\parallel\parallel} Y - (K_a^\alpha) f^{(e0)} Y, \quad (39)$$

and the last term of Eq. (36) vanishes. Similarly, for the first and third terms of Eq. (35), we obtain the expressions by the tensor harmonics expansion coefficients as

$$n_\mu \delta(\Delta x^\mu) = \left\{ -D_\parallel^2 - 2 \left(\frac{D_\parallel r}{r} \right) D_\parallel - \frac{l(l+1)}{r^2} \right\} \phi(\tau) Y, \quad (40)$$

$$n_\mu \delta \left(\frac{\rho}{\zeta} n^\mu \right) = n_\mu \frac{\rho}{\zeta} \delta n^\mu = -\frac{3H^2}{8\lambda} f_{\perp\perp} Y. \quad (41)$$

Here we omit the angular indices l, m and the summation $\sum_{l,m}$ for simplicity. Rewriting the metric perturbation expansion coefficients f in Eqs. (38), (39), (40), and (41) by corresponding gauge-invariant variables F , we consequently obtain the first order equation of wall perturbation coupled with gravitational waves in the form

$$\begin{aligned} &\left\{ -D_\parallel^2 - 2 \left(\frac{D_\parallel r}{r} \right) D_\parallel - \frac{l(l+1)}{r^2} + \frac{3}{2} H^2 + 3 \left(\frac{H^2}{4\lambda} \right)^2 + 3\lambda^2 \right\} \phi(\tau) \\ &+ \frac{3H^2}{8\lambda} \overline{F_{\perp\perp}} - \frac{1}{2r^4} \overline{D_\parallel(r^4 F_{\parallel\perp})} - \frac{1}{4} \overline{D_\perp F} = 0. \end{aligned} \quad (42)$$

This is the same as that derived by the metric junction formalism. (Here, $\overline{K_a^\alpha} = H^2/4\lambda$).

When we analyze the whole system, we must solve Eq. (42) and the perturbed Einstein equations

$$\delta G_{\mu\nu} = \zeta h_{\mu\nu} \delta(\chi - \chi|_{\Sigma}), \quad (43)$$

simultaneously. In general, it is hard for us to solve such a system of the Einstein equation and equation of motion of some matter objects, even in the cases that the system has some symmetries. However, as the results in Refs. [8, 9] show, if the gravitational backreactions are taken into account, the behaviors of domain walls drastically change. Thus, we cannot ignore the extra additional terms in Eq. (42) even in the first order of perturbations of domain wall system. Fortunately, in the cases considered in Refs. [8, 9], we can obtain the results which are equivalent to solve simultaneously the system of domain walls and gravitational waves, thanks to high symmetry $SO(3, 1)$ of the background geometries.

In order to clarify the feature of the system in more detail, we should analyze more generic cases. As one of such trials, a cylindrically symmetric case is under consideration by Nakamura and Ishihara [11]. Generalization of the study for the systems of strings and gravity is one of the next steps.

Acknowledgements

We would like to thank Professor T.Tanaka for informative discussion and useful comments. We are also grateful to Dr.K.Nakamura for fruitful discussion.

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Two scalar fields in the reheating stage

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Abstract

Recently, new scenario of the reheating after inflation has been developed. In this scenario, the decay process due to parametric resonance is considered. It can be understood from the fact that the mode equation for the massless boson particles which is produced from the coherently oscillating inflaton becomes the same form as the Mathieu equation. We investigate how the coherent inflaton works on the parametric resonance under the power-law inflaton potential $V(\phi) \propto \phi^{2n}$. We apply the Floquet analysis to obtain the growth-rate for the boson field in long-wave mode. We find that the analysis based on the Mathieu equation can provide the maximal growth-rate correctly even when the oscillation of the inflaton is not sinusoidal.

1 Introduction

Reheating is one of the most significant phenomena in the inflationary universe. It represents the inflaton decay process into the other light boson fields after the slow-rolling regime. Recently, a new scenario of the reheating is developed, in which the decay process occurs due to parametric resonance induced by the coherent oscillating inflaton. According to Kofman *et al.* [1], there are typically three stages in the reheating phase and the first stage (preheating) is specifically “broad” resonance, which is expected that the decay process proceeds extremely rapidly (explosively).

The parametric resonance can be understood from the fact that the mode equation for the boson field denoted by χ can be reduced to the same form as the Mathieu equation

$$\ddot{x} + [A + 2q \cos(2t)]x = 0, \quad (1)$$

where the oscillating term is appeared due to the presence of the coherent inflaton. It is the well-known that eq.(1) has the exponentially growing solutions $x \propto \exp(\mu t)$, where μ is the growth rate and determined by the parameters (A, q) . The broad resonance appears at $q \gg 1$, at which the growth rate can become larger than at $q \leq 1$. Therefore it is expected that the energy density of the inflaton is transferred into the χ -field rapidly for $q \gg 1$.

However we should keep in mind that the mode equation cannot become the Mathieu equation exactly. Due to the cosmic expansion, the parameters (A, q) have time-dependence. Another important point is that the oscillating term does not always become sinusoidal. Behavior of the oscillating term depends on the shape of the inflaton

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potential. This means that the growth-rate μ might be changed by the inflation model and the analysis using the Mathieu equation can not use. For the self-interacting inflaton with the potential $\lambda\phi^4/4$, in which case the fluctuation of the inflaton also leads to the parametric resonance, the analysis based on the Mathieu equation yields the maximal growth-rate of the fluctuation incorrectly [2]. The maximal growth-rate is an important value to estimate the efficiency of the decay process in the preheating stage, which is related with the reheating temperature.

In this article, in order to investigate the effect of the oscillating inflaton on the growth-rate μ for the massless boson field, we analyze the mode equation under the power-law type inflaton potential $V(\phi) \propto \phi^{2n}$. Observing the n dependence of the growth-rate, we show that the Mathieu equation analysis can provide the correct maximal value of the growth-rate even when the oscillation of the inflaton is not sinusoidal. We organize this article as follows: In Sec.2, we describe the model and write down the basic equations in the preheating stage. Substituting the background solutions into the mode equation of χ -field, we evaluate the growth-rate with the help of the Floquet theory in Sec.3. We also consider the validity of the Floquet analysis to take into account the effect of the cosmic expansion. The last section 4 is devoted as the summary and the discussions.

2 Model and Basic equations

We consider the two scalar fields (ϕ, χ) in the flat FRW background. One is the coherent inflaton (ϕ) with the power-law type potential

$$V(\phi) = \frac{\lambda}{4}\phi_0^4 \left(\frac{\phi}{\phi_0}\right)^{2n} = \frac{1}{2}m^2\phi_0^2 \left(\frac{\phi}{\phi_0}\right)^{2n} ; \quad m^2 \equiv \frac{\lambda}{2}\phi_0^2, \quad (2)$$

where $n = 1, 2, \dots$. The other is the light boson (χ) and we assume χ is a massless field. The reheating process can be achieved in the presence of the interaction. In this article, we shall treat the case with $V_{int} = g^2\phi^2\chi^2/2$. By solving the mode equation of χ -field, we examine how the oscillating inflaton affects the growth rate for χ -field.

We wish to investigate the decay process in the preheating stage, which is initially dominated by the coherent oscillating inflaton and the backreaction due to the created χ -particle is neglected. In this situation, $V(\phi) \gg V_{int}$ and the energy density of the inflaton is much larger than that of χ -field. Further we require that the leading process is the decay from the coherent inflaton into χ -field. Then the total system can be reduced to the background part and the fluctuating part. The background part is the homogeneous Einstein equations:

$$H = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3m_{pl}^2} \left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right),$$

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0, \quad (3)$$

which determine the cosmic expansion and oscillating behavior of the coherent inflaton. As for the fluctuation, the mode equation of χ -field is written by

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left[\left(\frac{k}{a}\right)^2 + g^2\phi^2(t)\right]\chi_k = 0, \quad (4)$$

where χ_k denotes the Fourier component of χ . We will focus on the mode equation (4) from now on.

3 Floquet analysis and growth-rate formula

3.1 $n = 1$ case

We firstly analyze $n = 1$ case. Oscillating behavior of the coherent inflation appears when $\phi \ll m_{pl}$. We can solve the background solution approximately:

$$\begin{aligned}\phi(t) &\approx \phi_0 a^{-3/2} \cos m(t - t_0), \\ a(t) &\approx \left(\frac{t}{t_0}\right)^{2/3} ; \quad (t \geq t_0),\end{aligned}\tag{5}$$

where ϕ_0 is the initial value of the inflaton field at $t = t_0$. When we introduce the new variables

$$\tau = m(t - t_0), \quad X_k = a^{3/2} \chi_k,\tag{6}$$

the lowest order evaluation for $g^2 m_{pl}^2 \gg m^2$, $t \geq t_0$ yields the following mode equation:

$$X_k'' + [A_k + 2q \cos(2\tau)] X_k \approx 0,\tag{7}$$

where $(') = \frac{d}{d\tau}$. Eq.(7) is the same expression as the Mathieu equation (1). Different from the exact Mathieu equation, the parameters (A_k, q) in (7) are the time-dependent functions given by

$$A_k = \left(\frac{k}{m}\right)^2 a^{-2} + 2q, \quad q = \frac{1}{4} \left(\frac{g\phi_0}{m}\right)^2 a^{-3}.\tag{8}$$

We find from the instability chart of the Mathieu equation that the maximal growth-rate in this model can be obtained along the line $A_k = 2q$, i.e. $k \rightarrow 0$. We concentrate the analysis on the long-wavelength limit hereafter.

We solve (7) approximately and evaluate the growth-rate μ_k given by $X_k \propto \exp(\mu_k \tau)$ in the preheating stage. In $q \ll 1$ case, the perturbation method expanded by the small parameter q such as multiple-scale method is applicable [3]. Because $q \gg 1$, we cannot apply such the perturbation. We shall analyze with the help of the Floquet theory described below.

Consider the equation with the periodically changing mass

$$x'' + \omega(\tau)x = 0 ; \quad \omega(\tau + \pi) = \omega(\tau).\tag{9}$$

This equation has the Floquet solution represented by

$$x_F(\tau) = e^{i\nu\tau} P(\tau) ; \quad P(\tau + \pi) = P(\tau).\tag{10}$$

The growth-rate can be obtained by $\mu = \text{Im}|\nu|$. Using the property of the Floquet solution, we can derive the following formula: Taking the two linearly independent solutions x_1, x_2

satisfying the initial conditions $x_1(0) = 1, x_1'(0) = 0, x_2(0) = 0, x_2'(0) = 1$, the growth-rate can be given by

$$\cosh(\pi\mu) = 1 + 2x_1' \left(\frac{\pi}{2} \right) x_2 \left(\frac{\pi}{2} \right). \quad (11)$$

Thus we can estimate the growth-rate if we get the solution accurately during a half period of the oscillating function $\omega(\tau)$. Note that because of time-dependence of the parameters, eq.(7) does not become the same form as (9) and we cannot apply the Floquet analysis in a rigorous sense. However, assuming that the parameters varies enough slowly, we proceed to analyze the growth-rate by ignoring their time-dependence. Validity of the analysis including the cosmic expansion will be discussed later.

We now construct the solution (7) which is accurate during a half period of the oscillating term. To construct the solution, we apply the Langer's method [3]. Perform the transformation $(\tau, X_k) \rightarrow (z, v)$ such that

$$z = Q(\tau), \quad v = \sqrt{Q'(\tau)} X_k. \quad (12)$$

Eq.(7) in $k \rightarrow 0$ limit becomes

$$v_{,zz} + \left[\left\{ \frac{2\sqrt{q} \cos \tau}{Q'(\tau)} \right\}^2 + \delta \right] v = 0 ; \quad \delta = \frac{\sqrt{Q'}}{(Q')^2} \left(\frac{1}{\sqrt{Q'}} \right)''. \quad (13)$$

As the function $Q(\tau)$ is not given yet, we have a freedom of imposing the condition on Q . We shall take

$$z^2 = \left\{ \frac{2\sqrt{q} \cos \tau}{Q'(\tau)} \right\}^2, \quad (14)$$

then δ becomes $\mathcal{O}(q^{-1})$ and $Q(\tau)$ is determined. Neglecting δ , the solution v can be represented by the parabolic cylinder function. Thus the mode function X_k for $q \gg 1$ is written by

$$X_k \approx \left[\frac{z^2}{4q \cos^2 \tau} \right]^{1/4} \left(C_0 D_{-\frac{1}{2}} \left(\sqrt{2} e^{i\pi/4} z \right) + c.c \right) ; \quad z^2 = 4\sqrt{q} \sin \tau, \quad (15)$$

where C_0 is arbitrary constant and $D_{-\frac{1}{2}}$ denotes the parabolic cylinder function. We observe that the above solution is correct during a period $\tilde{\tau} \leq \tau \leq \tilde{\tau} + \pi/2$, where $\tilde{\tau}$ is a time satisfying $\cos \tilde{\tau} = 0$.

Using (15), we obtain the growth-rate for the mode function X_k . We set the initial condition $\tau = \tilde{\tau}$ and construct the two solutions. As a bonus, we can use the asymptotic form of the parabolic cylinder function at $\tau = \tilde{\tau} + \frac{\pi}{2}$ because of the broad band analysis $q \gg 1$. Substituting them into the formula (11), the growth-rate is described as follows:

$$\mu = \frac{1}{\pi} \log \left[\sqrt{F^2} + \sqrt{F^2 - 1} \right] ; \quad F = \sqrt{2} \cos(4\sqrt{q}). \quad (16)$$

This shows that the instability regions ($\mu > 0$) appear as a function of q and the maximal value becomes $\mu_{\max} = \log(1 + \sqrt{2})/\pi \approx 0.28$, which coincides with the result analyzed by Kofman *et al.* [1]. The similar expression is obtained by Fujisaki *et al.* [4], although the function F is different from ours. We plotted the growth-rate (16) and compare with numerical result (see Fig.1). We see that the analytic result (16) is valid when $q \gg 1$.

3.2 $n \geq 2$ case

We proceed to investigate the case with the potential index $n \geq 2$. Different from $n = 1$, the oscillations of the inflaton ϕ are not sinusoidal. For $n = 2$, it is given by the Jacobi's elliptic function. Taking into account this thing, we carefully evaluate the growth-rate. The background equations in the coherent oscillating stage are approximately solved as

$$\begin{aligned}\phi(t) &\approx \phi_0 a^{-\frac{1}{n+1}} \tilde{\phi}(t), \\ a(t) &\approx \left(\frac{t}{t_0}\right)^{\frac{n+1}{3n}} : (t \geq t_0),\end{aligned}\quad (17)$$

where $\tilde{\phi}$ represents the non-linear oscillation with $\tilde{\phi}(t_0) = 1, \dot{\tilde{\phi}}(t_0) = 0$ as an initial value. Similar to $n = 1$, we introduce the following variables:

$$\begin{aligned}\tau &= nc_n(mt_0)a^{\frac{1}{n+1}} ; \quad c_n = \sqrt{\pi} \frac{n\Gamma(\frac{1}{2n} + \frac{1}{2})}{\Gamma(\frac{1}{2n})}, \\ X_k &= a^{\frac{1}{n+1}} \chi_k.\end{aligned}\quad (18)$$

Note that $1 \leq c_n \leq \frac{\pi}{2}$. The lowest order evaluation for $g^2 m_{pl}^2 \gg m^2, t \geq t_0$ leads to the reduced mode equation and $\tilde{\phi}$ given by

$$X_k'' + [A_k + 2q(2\tilde{\phi}^2 - 1)] X_k \approx 0 \quad (19)$$

and

$$\tilde{\phi}'' + \left(\frac{n}{c_n^2}\right) \tilde{\phi}^{2n-1} = 0. \quad (20)$$

$\tilde{\phi}$ becomes the periodic function of the new variable τ , i.e, $\tilde{\phi}(\tau + 2\pi) = \tilde{\phi}(\tau)$. We see that eq.(19) contains the $n = 1$ case and becomes the Mathieu-type equation. The parameters (A_k, q) are

$$A_k = \left(\frac{k}{c_n m}\right)^2 a^{4(\frac{n-2}{n+1})} + 2q, \quad q = \frac{1}{4} \left(\frac{g\phi_0}{c_n m}\right)^2 a^{6(\frac{n-2}{n+1})}. \quad (21)$$

In this case, the maximal growth-rate can also be obtained along the line $A_k = 2q$. Treating the parameters (A_k, q) as constant values, we analyze the growth-rate in the long-wavelength limit. The analysis is the same as $n = 1$ case and straightforward. Applying the Langer's method, we get the the regular solution during a period $\tilde{\tau} \leq \tau \leq \tilde{\tau} + \frac{\pi}{2}$ such that $\tilde{\phi}(\tilde{\tau}) = 0$:

$$X_k \approx \left[\frac{z^2}{4q \cos^2 \tau}\right]^{1/4} \left(C_0 D_{-\frac{1}{2}}(\sqrt{2}e^{i\pi/4}z) + c.c\right) ; \quad z^2 = 4\sqrt{q} \int_{\tilde{\tau}}^{\tau} |\tilde{\phi}| d\tau. \quad (22)$$

Substituting the asymptotic form of (22) into the formula (11), the growth-rate in the limit $k \rightarrow 0$ becomes

$$\mu_n = \frac{1}{\pi} \log \left[\sqrt{F_n^2 + \sqrt{F_n^2 - 1}} \right] ; \quad F_n = \sqrt{2} \cos(4\alpha_n \sqrt{q}). \quad (23)$$

We see that the maximal growth-rate becomes the same value as obtained at $n = 1$, $\mu_{\max} \approx 0.28$. The only difference between the distinct potential indices appears in α_n which causes the position shift and the width change of the instability band. α_n is defined by

$$\alpha_n = \int_{\tilde{\tau}}^{\tilde{\tau} + \frac{\pi}{2}} |\dot{\phi}(\tau)| d\tau. \quad (24)$$

We find $\frac{\pi}{4} \leq \alpha_n \leq 1$ from the property of the non-linear oscillation (20). Thus except for slightly changes of the instability band on the width and the location, the growth-rate formula (23) can be approximated by the formula constructed from the Mathieu equation ($n = 1$). As a check, we plot the growth-rate of X_k in $n = 5$ case and compare with the numerical computation (Fig.2).

3.3 Validity of the Floquet analysis

So far we have analyzed the growth-rate for the broad resonance ($q \gg 1$) by neglecting the time-dependence of the parameter (A_k, q). Before closing section, we comment on the validity of the Floquet analysis to take account of the cosmic expansion. According to Shtanov *et al.*[5], we can use the growth-rate even in the case when the value of μ is not constant if the adiabatic condition

$$|\mu'| \ll \mu^2. \quad (25)$$

is satisfied. Within the condition we can evaluate behavior of the mode function X_k as $X_k \propto \exp(\int \mu d\tau)$, where the integral is taken over the time such that $\text{Re}\mu \neq 0$. For our purpose, it is sufficient to consider the growth-rate close to $\mu_{\max}$. In this case, we see from (23) that the condition (25) is always satisfied. The time scale for changing the parameter q is given by the inverse of

$$\left| \frac{q'}{q} \right| = \left| \frac{2(n-2)}{nC_n} \right| \frac{a^{-\frac{3}{n+1}}}{mt_0} = \frac{4\sqrt{3}\pi}{c_n} \left| \frac{n-2}{n+1} \right| \frac{\phi_0 a^{-\frac{3}{n+1}}}{m_{pl}}. \quad (26)$$

We understand that the parameter does not change in $n = 2$ case and the Floquet analysis is always valid in the preheating stage. For $n \neq 2$, as long as $\phi(t) \ll m_{pl}$, the variation of q with time is negligible slow. Thus we can say that the Floquet analysis near $\mu \approx \mu_{\max}$ is valid and can evaluate the mode function by using the integral expression $X_k \propto \exp(\int \mu d\tau)$.

4 Summary and Discussions

We have investigated the mode equation for the massless field in the preheating stage. Assuming the power-law inflaton potential with $\phi^2\chi^2$ -coupling, we estimated the growth-rate due to the parametric resonance with the help of the Floquet analysis. The resulting growth-rate formula is well-approximated by the one constructed from the Mathieu equation.

The reason why the maximal growth-rate for $n \geq 2$ coincides with that for the Mathieu equation ($n = 1$) is that we analyze the broad resonance ($q \gg 1$). The solutions of the

mode equation (22) obtained by the Langer's method become the same forms as (15) except for the weak dependence of the potential index α_n . Thus as long as $q \gg 1$, it is expected that the analysis based on the Mathieu equation can also provide the correct maximal growth-rate. To confirm this thing, we must extend the analysis to $A_k \geq 2q$ ($k \neq 0$), which is the important region to evaluate the decay rate of the energy density of the inflaton. This is our next task.

We finally comment on the decay process into the fluctuation of the inflaton as described in Sec.1. Authors[1] [2] considered the production ($\delta\phi$) of the ϕ -particle with $\lambda\phi^4/4$ potential. The mode equation for the ϕ -particle can be reduced to (19) with $n = 2$. By contrast with the χ -particle decay, the parameter q is specified as

$$q = \frac{3}{2c_2^2} \approx 1.04 \quad . \quad (27)$$

For the above parameter, our analysis at $q \gg 1$ is not applicable. Therefore we cannot say anything about the validity of the analysis using Mathieu equation in this model.

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Figure Captions

Fig.1 The growth-rate of the mode function along the $A_k = 2q$ line ($n = 1$ case): The growth-rate μ for the mode equation (7) is plotted as a function of q . (a) numerical result by using the formula (11). (b) analytic result (16).

Fig.2 The growth-rate of the mode function along the $A_k = 2q$ line ($n = 5$ case): (a) numerical result by using the formula (11). (b) analytic result (16), where we have used the constant $\alpha_5 = 0.870$ evaluated numerically.

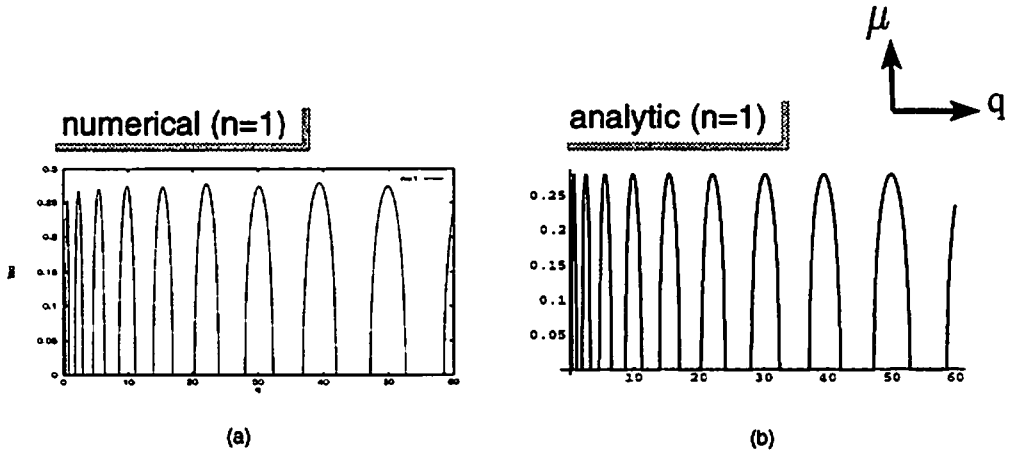


Figure 1:

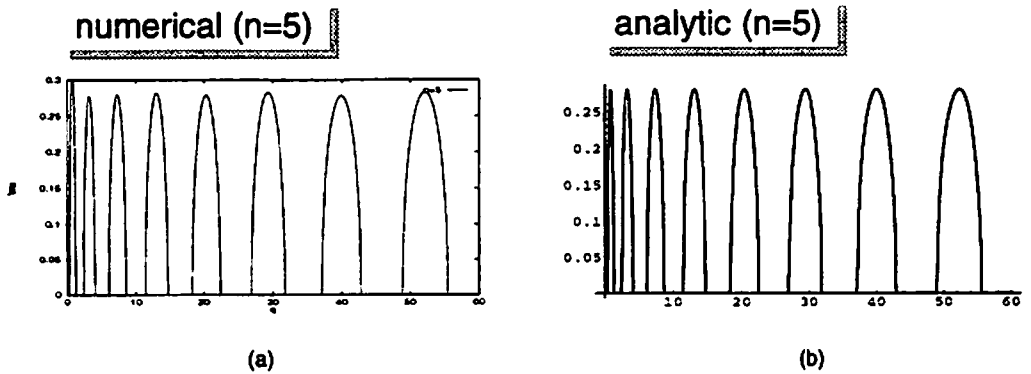


Figure 2:

Perturbations in a Self-Similar Cosmological Model

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Abstract

To clarify the evolution of inhomogeneous substructures in large-scale structures such as voids, we study the perturbations in a self-similar model with pressureless matter, first based on Gerlach and Sengupta's general-relativistic formulation for perturbations in spherically symmetric inhomogeneous models. Next we take the approximate treatment due to the local homogeneous model with the average density parameter which is considered at each point, in order to study the behaviors of perturbations on small scales.

1 Introduction

There are various large-scale dynamical structures such as superclusters and voids which are evolving with the cosmic expansion. In these structures, substructures such as galaxies appear as a result of growth of perturbations. The behaviors of perturbations in general inhomogeneous structures are very complicated, but they may be rather simplified in spherically symmetric structures.

The general-relativistic theory of perturbations in a spherically symmetric inhomogeneous model was derived by Gerlach and Sengupta [1]. They gave the elegant gauge-invariant formulation, but the solutions in cosmological problems have not been derived so far. This may be due to the situations that their perturbation equations are partial differential equations inconvenient for analytical treatments and intuitive insight, and that three kinds of perturbations are not decoupled and their coupling is complicated, as in anisotropic cosmological models ([2] - [4]).

In this note we show the perturbations in a self-similar background model which is a simplest spherically symmetric inhomogeneous model with pressureless matter ([5] - [8]). In this background, dynamics is described using a single coordinate $\xi \equiv ct/r$, where t and r are the cosmic time and a radial coordinate, and the perturbation equations are also reduced to ordinary differential equations. Moreover the background is isotropic and homogeneous in the limit $\xi \rightarrow 0$, so that the behaviors of perturbations are similar to those in the Friedmann model in this limit. Because of these situations, the perturbations can be most easily in a self-similar model. The perturbations are studied first due to Gerlach and Sengupta's formalism, and next in the approximate treatment due to local homogeneous models.

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2 Background model

The Lemaître-Tolman-Bondi solution for spherically symmetric inhomogeneous cosmological models with pressureless matter (Tolman [9], Landau and Lifshitz [10]) is represented by the line-element

$$ds^2 = \bar{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = -c^2 dt^2 + S^2(t, r) \left\{ \frac{(1 + rS'/S)^2}{1 - k\alpha(r)r^2} dr^2 + r^2 d\Omega^2 \right\}, \quad (1)$$

where $(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = (ct, r, \theta, \psi)$, $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, $\alpha(r)$ is an arbitrary function of coordinate radius r , and $S' = \partial S/\partial r$.

For the convenience's sake we assume that the spacetime consists of the inner homogeneous region ($r < r_1$) and the outer self-similar inhomogeneous region ($r > r_1$), as in Ref.[6]. In the inner homogeneous region we have $\alpha(r)/\alpha_0 = \beta(r)/\beta_0 = 1/(r_1)^2$, where $\beta_0 = H_0^{-1}(r_1)^2$ and $H_0 \equiv (S/S)_{t=t_0}$. The Hubble constant H_0 corresponds to the present epoch t_0 . Then the Einstein equations give us

$$\Omega_0 \equiv \rho_{\text{in}}(t_0) / \left(\frac{3H_0^2}{8\pi G} \right) = 1 / (H_0 S_0)^3 \quad (2)$$

and

$$\alpha_0 / (r_1)^2 = (1 - \Omega_0) / (\Omega_0)^{2/3}, \quad (3)$$

for $S_0 \equiv S(t_0, 0)$. The solutions are expressed as

$$S/S_0 = \frac{\Omega_0}{2(1 - \Omega_0)} (\cosh \eta - 1), \quad (4)$$

$$H_0 t = \frac{\Omega_0}{2(1 - \Omega_0)^{3/2}} (\sinh \eta - \eta), \quad (5)$$

$$\rho_{\text{in}} = \left(\frac{3H_0^2 \Omega_0}{8\pi G} \right) \left(\frac{S}{S_0} \right)^{-3}. \quad (6)$$

If an observer is at the center and we consider a light ray which is emitted at the redshift z_1 from the boundary and reaches an observer at the center at present epoch [6], the light-ray equation leads to

$$\sqrt{\alpha_0} = \frac{4\sqrt{1 - \Omega_0}}{\Omega_0^2(1 + z_1)} \left[1 - \Omega_0 + \frac{\Omega_0}{2}(1 + z_1) + \left(\frac{\Omega_0}{2} - 1 \right) \sqrt{1 + \Omega_0 z_1} \right], \quad (7)$$

by which r_1 is related to z_1 using Eq. (3). For $z_1 < 1$, we have $\alpha_0 \simeq (1 - \Omega_0)z_1^2$, $r_1 \simeq \Omega_0^{1/3}z_1$, and, for $z_1 = 1$ and 2 ($\Omega_0 = 0.2$), we have $\alpha_0 = 0.398, 1.095$, $r_1 = 0.412, 0.684$, respectively.

In the self-similar region, $S(t, r)$ is a function of only $\xi (\equiv t/r)$, and $\alpha(r)$ and $\beta(r)$ are taken to be

$$\alpha(r)/\alpha_0 = \beta(r)/\beta_0 = 1/r^2, \quad (8)$$

so that $\alpha(r)r^2$ and $\beta(r)r^2$ may be constant and continuous at the boundary $r = r_1$. The solutions lead to Eq. (4) and

$$\xi \equiv \frac{t}{r} = \frac{H_0^{-1}(r_1)^{-1}\Omega_0}{2(1-\Omega_0)^{3/2}}(\sinh \eta - \eta), \quad (9)$$

$$\rho = \frac{3H_0^2\Omega_0}{8\pi G}\left(\frac{r_1}{r}\right)^2\left(\frac{S}{S_0}\right)^{-3}\left(1 - \xi S_{,\xi}/S\right)^{-1}, \quad (10)$$

where ξ denotes the partial derivative $\partial/\partial\xi$.

The line-element (1) in the self-similar region is expressed as

$$ds^2 = -dt^2 + S^2(\xi)\left\{\frac{(1 - \xi S_{,\xi}/S)^2}{1 - k\alpha_0}dr^2 + r^2d\Omega^2\right\}. \quad (11)$$

If we use for r a radial coordinate x defined by $r/r_1 \equiv \exp(x)$, the line-element is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = r^2 d\bar{s}^2, \quad (12)$$

$$d\bar{s}^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = -d\xi^2 - 2\xi d\xi dx + M^2(\xi)dx^2 + S^2(\xi)d\Omega^2, \quad (13)$$

where $(x^0, x^1, x^2, x^3) = (\xi, x, \theta, \varphi)$,

$$M^2 \equiv N^2(\xi) - \xi^2, \quad (14)$$

and

$$N^2(\xi) \equiv S^2(1 - \xi S_{,\xi}/S)^2/(1 + \alpha_0). \quad (15)$$

The metric components are

$$g_{\mu\nu} = r^2 \bar{g}_{\mu\nu}, \quad g^{\mu\nu} = r^{-2} \bar{g}^{\mu\nu} \quad (16)$$

with

$$(\bar{g}_{00}, \bar{g}_{01}, \bar{g}_{11}) = (-1, -\xi, M^2), \quad (\bar{g}^{00}, \bar{g}^{01}, \bar{g}^{11}) = (-M^2, -\xi, 1)/N^2. \quad (17)$$

The conformal spacetime with $d\bar{s}^2$ is homogeneous in the x direction in the sense that it is invariant for the spatial transformation $x \rightarrow x + \text{const}$.

3 Perturbations in the self-similar region

The angular dependence of all perturbations can be expressed using spherical harmonics such as $Y_{lm}(\theta, \varphi)$ with angular integers l and m , and they have odd parity $((-1)^{l+1})$ or even parity $((-1)^l)$, corresponding to the rotation on the two-dimensional submanifold with $x^0 = \text{const}$ and $x^1 = \text{const}$ (Regge and Wheeler [11], Gerlach and Sengupta [12]).

Odd-parity perturbations include rotational and gravitational-wave perturbations, and even-parity perturbations include density perturbations, rotational and gravitational-wave perturbations.

The perturbation equations are derived in the following steps:

- 1) The (gauge-invariant) perturbed quantities are Fourier-expanded in terms of $\exp[ipx]$ in the hypersurface $\xi = \text{const}$, where p is a wave number and $x \equiv \ln(r/r_1)$.
- 2) Based on the Gerlach and Sengupta formalism, the equations for perturbed quantities such as metric components and matter density and velocities are derived.

The solutions are obtained in the following steps:

- 1) First the solutions are classified in the neighborhood of $\xi = 0$, because their three modes can be decoupled there. The classified three modes are called the *primordial* scalar, vector, and tensor modes of perturbations.
- 2) Next the solutions are derived in series expansion with respect to $\xi^{2/3}$.
- 3) Finally the perturbation equations are numerically solved by using the solutions in series expansion as the initial condition.

For the primordial scalar perturbation, It is found as the result that the density contrast ϵ_m is larger than that in the homogeneous model with the inner density parameter Ω_0 by the factor which is 1.23 at the boundary $r = r_1$ at epoch z_1 , and that ϵ_m at the emission epochs with $z > z_1$ is always larger than that in the ordinary homogeneous model. Here the radial coordinate r is connected with z and η (at the emission epoch) by Eqs. (3.6) and (3.9) in Ref. [6], and the initial value $(\epsilon_m)_i$ is given for arbitrary r at the initial hypersurface $t = t_i$, so that the initial value η_i in the self-similar region is related to $(\eta_{in})_i$ in the inner region by

$$\frac{2(1 - \Omega_0)^{3/2}}{\Omega_0} H_0 t_i = \sinh(\eta_{in})_i - (\eta_{in})_i = \frac{r}{r_i} (\sinh \eta_i - \eta_i), \quad (18)$$

where Eqs.(5) and (9) were used. The behavior of ϵ_m at the emission epochs are shown in Fig. 1 in an example in which $\Omega_0 = 0.2$ and $z_1 = 1.5$ and 2.0.

Fig. 1

4 Approximate treatment due to the local homogeneous model

Now we consider an approximate treatment in which the background inhomogeneous model is replaced by a stepwise series of virtual local homogeneous models and the perturbations are assumed to be caused in these homogeneous models. Because the homogeneous models with $S = S(t)$ must be isotropic at the same time, the perturbations are those in the Friedman model. To determine the model parameter of the local homogeneous models, we define the expansion rates H_t, H_r , and Θ in the transverse and radial directions, and in the average as

$$H_t \equiv \xi(\ln S)_{,\xi}, \quad H_r \equiv \xi[\ln(S - \xi S_{,\xi})]_{,\xi}, \quad (19)$$

and

$$\Theta \equiv (2H_t + H_r)/3, \quad (20)$$

respectively. Their values at the present epoch (i.e., in the $t = t_0$ hypersurface) are the corresponding Hubble constants $(H_t)_0, (H_r)_0$, and Θ_0 . The Hubble constant in the inner

region is H_0 , the present cosmic time is t_0 , and $(H_t)_0$ is equal to Eq. (2.18) in Ref. [6]. The local density parameters can be defined in the two directions, and the average parameter is

$$\bar{\Omega} \equiv \rho(t_0)/\left[\frac{3(\Theta_0)^2}{8\pi G}\right] = \frac{2(\cosh \bar{\eta} - 1)}{3 \sinh^2 \bar{\eta}} \left/ \left[1 - \frac{\sinh \bar{\eta}(\sinh \bar{\eta} - \bar{\eta})}{(\cosh \bar{\eta} - 1)^2}\right] \right., \quad (21)$$

where a mistake in the original expression was corrected (cf [7]) and $\bar{\eta}$ is the present value of η defined by Eqs. (2.19) in Ref. [6]. The radial coordinate r is connected with z and η (at the emission epoch) by Eqs. (3.6) and (3.9) in Ref. [6]. The behaviors of $\bar{\Omega}$ and η (at the emission epoch) are also shown in Fig. 2 as a function of z .

Here we assume that the local homogeneous models have the average density parameter $\bar{\Omega}$, and put attention to the scalar perturbation. The density contrast ϵ_m is derived by solving the perturbation equations in the homogeneous model from an initial epoch t_i to the emission epoch t_e , where the initial value of ϵ_m is assumed to be equal everywhere. The relation between η and the corresponding $\bar{\eta}$ in the local homogeneous model is given so that the cosmic time t may be equal.

As a result of numerical calculations, the behavior of $\epsilon_m(z)/\epsilon_m(0)$ at each emission epoch is shown in Fig. 3 as a function of z at the emission epochs, in an example in which the boundary between the inner homogeneous region and the self-similar region corresponds to the epoch $z_1 = 1.5$ and 2.0. In these two cases, the ratio $\epsilon_m(z)/\epsilon_m(0)$ is by about 30–20% (for $z = 2–3$) larger than that in the ordinary homogeneous model. If we compare this figure with the corresponding one in the previous section, this approximate treatment is found to be good enough and useful.

This consistency may be due to the situation in which the anisotropy is comparatively small and so the coupling between different modes is negligible. In fact the anisotropy is defined by

$$A \equiv H_t/\Theta - 1, \quad (22)$$

and it is shown that, For $\eta \ll 1$, $A \simeq 0.033\eta^2$, and for $\eta = 1.0, 1.5, 2.0$ and 3.0, $A = 0.031, 0.065, 0.104$ and 0.177, respectively. On the other hand, the values of η at the emission and present epochs in the self-similar region are nearly equal to or smaller than 2.0 and 3.0, respectively.

Fig. 2 Fig. 3

5 Conclusion

The gravitational instability in a self-similar inhomogeneous cosmological model was studied by paying attention to the self-similar hypersurface with $\xi = \text{const}$. The perturbation equations reduced to the ordinary differential equations with respect to ξ by expanding perturbations in terms of $\exp[ip \ln(r/r_i)]$. However the formalism was found to be useless for $\xi > \xi_{cr}$, except for those on large scales such that $p = 0 \sim 1, l \approx 1$, because the above hypersurface becomes time-like in the radial direction. Next it was shown that the perturbations on small scales can be regarded as those in local homogeneous (isotropic) models with same density parameter $\Omega(r)$ as in the inhomogeneous model, and that, in

the super-horizon-scale inhomogeneous (void) model consisting of the inner low-density homogeneous ($z \leq z_1$) and the outer self-similar region ($z \geq z_1$), the density contrast in the latter region is by about 20% larger than that in the corresponding one in the ordinary homogeneous model in the case of $\Omega_0 = 0.2$ and $z_1 = 1.5 - 2.0$.

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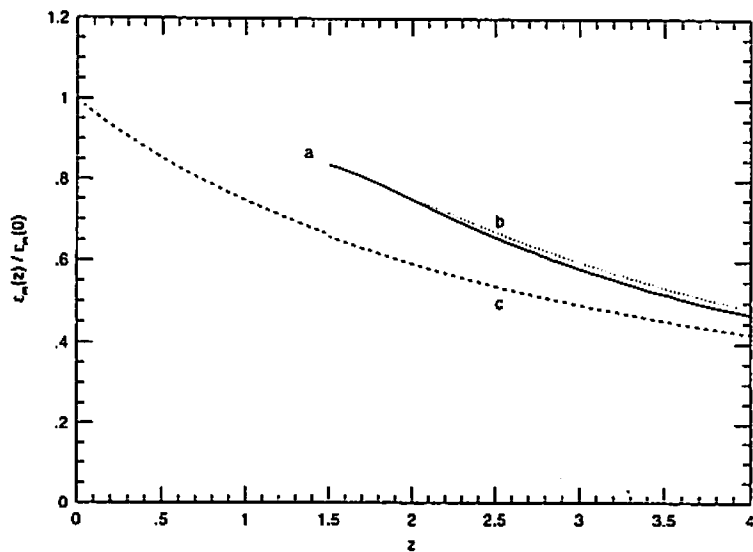


Fig. 1

Figure 1: The ratio $\epsilon_m(z)/\epsilon_m(0)$ in the self-similar region is shown for the inhomogeneous models of (a) $\Omega_0 = 0.2, z_1 = 1.5$, and (b) $\Omega_0 = 0.2, z_1 = 2.0$. For comparison the ratio in (c) the ordinary homogeneous model with $\Omega_0 = 0.2$ is also shown.

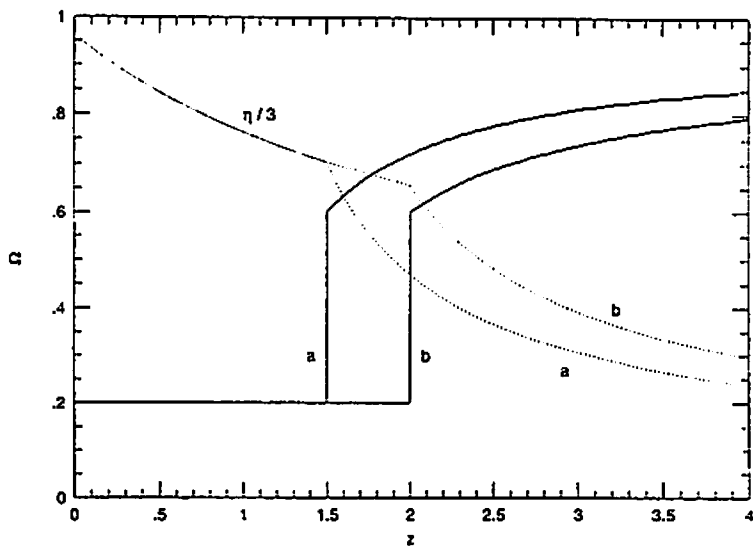


Fig. 2

Figure 2: The local value of the density parameter Ω at present epoch is shown in solid lines. The density parameter Ω_0 in the inner homogeneous region is assumed to be 0.2. The values of $\eta/3$ are also shown in dotted lines. The models with $z_1 = 1.5$ and 2.0 are denoted by a and b, respectively.

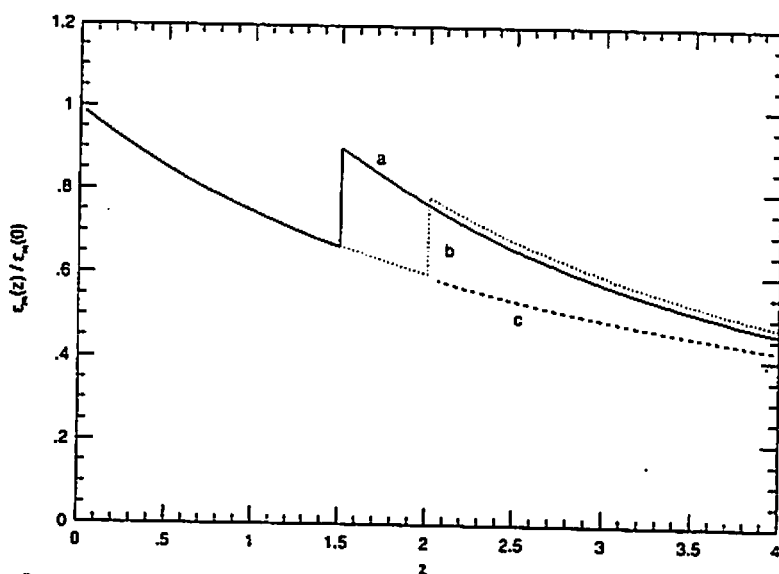


Fig. 3

Figure 3: The ratio $\epsilon_m(z)/\epsilon_m(0)$ is shown for the local homogeneous models corresponding to the cases of (a) $\Omega_0 = 0.2, z_1 = 1.5$, and (b) $\Omega_0 = 0.2, z_1 = 2.0$. For comparison the ratio in (c) the ordinary homogeneous model with $\Omega_0 = 0.2$ is also shown.

Renormalization group for Einstein equation

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Abstract

There is a method for studying long-time asymptotics of nonlinear, partial differential equations. It extracts some dynamical information by using the scale invariance of equations. We apply it to the Einstein equations and study the character of the scale invariant solution by use of the linear perturbation analysis in cosmology.

1 Introduction

Recently the renormalization group (RG) idea is applied to study the long-time asymptotics of the nonlinear, partial differential equations [1]. The RG transformation there is integration of the equation up to a finite time followed by a rescaling of the dependent and independent variables. The RG transformation together with the original differential equation gives a RG equation. Using the RG transformation, the problem on an infinite time is reduced to the problem on a finite time. A fixed point of the RG transformation corresponds to a scale invariant solution of the differential equation. We can obtain the long-time behavior of the equation by studying the behavior around the fixed point.

There are some works on the system of gravity by use of this RG method. Tasaki [2] applied it to the motion of a point particle in the Newton gravity. A similar approach was applied for the Einstein equations to understand the problem of critical behavior in gravitational collapse [3].

Here we apply it to the Einstein equations, especially in the cosmological situation. For simplicity, we consider the case that the universe is filled with dust and has the spherical symmetry.

Observations imply that the present universe has the hierarchical structure such as galaxies, clusters of galaxies, and super clusters, and that the two point correlation function of the galaxies or the clusters of galaxies can be written by a power law [4]. These suggests that the present universe has self-similarity. Thus scale invariant solution should play an important role in cosmology.

In this paper we have applied the renormalization group method to the Einstein equations in the cosmological context. In Sec.2, we explain the RG method for the heat

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equation with the nonlinear term as an example of applications of the RG method to a partial differential equation. In Sec.3, we apply the RG method to the Einstein equations. Sec.4 is devoted to summary and discussions.

2 Renormalization group transformation

In this section, we review the RG method for nonlinear, partial differential equations [1]. First we consider the heat equation with the nonlinear term for a simple example:

$$\frac{\partial u(x, t)}{\partial t} = \frac{1}{2} \left[u''(x, t) + \lambda u^2(x, t) \right], \quad (1)$$

where a prime denotes a spatial derivative and λ is a constant. The equation (1) has scale invariance under the following scaling transformation:

$$\begin{aligned} x &\longrightarrow Lx \\ t &\longrightarrow L^2 t \\ u &\longrightarrow L^2 u, \end{aligned} \quad (2)$$

where L is a parameter of the scaling transformation and is larger than 1. Namely, if $u(x, t)$ is a solution of Eq.(1), the scaled function,

$$^{(L)}u(x, t) = L^2 u(Lx, L^2 t), \quad (3)$$

is also a solution of Eq.(1).

Here we define the RG transformation $\mathcal{R}_L$ of a function of x by

$$\mathcal{R}_L u(x, 1) = ^{(L)}u(x, 1). \quad (4)$$

In short, the $\mathcal{R}_L$ is a map from a set of initial data to another. It is convenient to take the initial time to be $t = 1$. A RG transformation $\mathcal{R}_L$ has a semi-group property:

$$\mathcal{R}_{L^n} = \mathcal{R}_{L^{n-1}} \circ \mathcal{R}_L. \quad (5)$$

Letting $t = 1$ and then $L^2 = t$ in Eqs.(3) and (4), we can express the solution u as the initial data $u(\cdot, 1)$ transformed by RG transformations:

$$u(x, t) = t^{-1} \, ^{(t^{1/2})}u(xt^{-1/2}, 1). \quad (6)$$

Denoting $L = e^\tau$, we have from Eq.(3) that

$$\frac{d^{(L)}u}{d\tau} = L \frac{d^{(L)}u}{dL} = 2^{(L)}u + x u^{(L)'} + 2 \frac{\partial^{(L)}u}{\partial t}. \quad (7)$$

Using the original partial differential equation (1) we have

$$\frac{d^{(L)}u}{d\tau} = 2^{(L)}u + \lambda u^2 + x u^{(L)'} + u^{(L)''}. \quad (8)$$

This is the equation satisfied by the scaled function $^{(L)}u$, which we call the RG equation.

We investigate the fixed point of the RG equation (8). The fixed point u^* is defined by

$$\mathcal{R}_L u^* = u^* \quad (9)$$

for any $L > 1$. This condition means that the field profile is unchanged after a time evolution followed by a suitable rescaling. In general this condition is equivalent to

$$\frac{d^{(L)}u^*}{d\tau} = 0. \quad (10)$$

From Eq.(8), u^* satisfies the following equation,

$$2u^* + \lambda u^{*2} + x u^{*'} + u^{*''} = 0. \quad (11)$$

In the homogeneous case, we can easily obtain the fixed points:

$$u^* = 0 \quad \text{and} \quad -\frac{2}{\lambda}. \quad (12)$$

In order to investigate the character of the fixed point Eq.(12), we consider the linear perturbation around the fixed point Eq.(12). The perturbed quantity $^{(L)}\delta u$ is defined by

$$^{(L)}u = u^* + ^{(L)}\delta u, \quad (13)$$

where $^{(L)}\delta u$ is assumed small. Substituting Eq.(13) into Eq.(8) and neglecting the second order term $^{(L)}\delta u^2$, we obtain the linearized equation for $^{(L)}\delta u$:

$$\frac{d^{(L)}\delta u}{d\tau} = \begin{cases} ^{(L)}2\delta u + x^{(L)}\delta u' + ^{(L)}\delta u'' & (u^* = 0) \\ ^{(L)}-2\delta u + x^{(L)}\delta u' + ^{(L)}\delta u'' & (u^* = -\frac{2}{\lambda}) \end{cases}. \quad (14)$$

We require the boundary condition

$$^{(L)}\delta u \longrightarrow 0 \quad (x \longrightarrow \infty). \quad (15)$$

We are going to find the normal modes with the ansatz:

$$^{(L)}\delta u = f(x)e^{-\frac{\omega}{2}\tau + \omega\tau}, \quad (16)$$

where $f(x)$ is a function to be determined below and ω is a constant. From Eqs.(14) and (16), we have

$$\begin{aligned} f'' - xf' - (\omega - 1)f &= 0 \quad (u^* = 0), \\ f'' - xf' - (\omega + 3)f &= 0 \quad (u^* = -\frac{2}{\lambda}). \end{aligned} \quad (17)$$

The regularity at $x = 0$ and $x = \infty$ implies

$$f(x) = H_n(x), \quad (18)$$

$$\omega = \begin{cases} 1 - n & (u^* = 0) \\ -3 - n & (u^* = -\frac{2}{\lambda}) \end{cases}, \quad (19)$$

where $H_n(x)$ is the Hermite polynomial and $n = 0, 1, 2, \dots$.

From Eq.(19), $u^* = 0$ is an attractor because all ω 's are negative. On the other hand, $u^* = -\frac{2}{\lambda}$ has only one relevant mode ($n = 0$).

3 Renormalization group for Einstein equation

In this section, we apply the RG method which is explained in the previous section to the Einstein equations.

Take a synchronous reference frame where the line element is

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -dt^2 + \gamma_{ij}(x^k, t) dx^i dx^j. \end{aligned} \quad (20)$$

Throughout this paper Latin letters will denote spatial indices and Greek letters spacetime indices.

For example, we consider a spherically symmetric case. In this case, the spatial metric is written by

$$\gamma_{ij}(x^k, t) dx^i dx^j = A^2(r, t) dr^2 + B^2(r, t) (d\theta^2 + \sin^2 \theta d\phi^2). \quad (21)$$

As a simple case, we investigate the universe filled with dust. The Einstein equations are

$$\dot{K}_{ij} = -{}^3R_{ij} + 2K_{il}K_j^l - K K_{ij} + \frac{1}{2} \kappa \rho \gamma_{ij}, \quad (22)$$

$$\kappa \rho = \frac{1}{2} [{}^3R + K^2 - K_l^m K_m^l], \quad (23)$$

$$K_{i;j}^j - K_{;i} = 0, \quad (24)$$

where

$$K_{ij} = \frac{1}{2} \dot{\gamma}_{ij}, \quad (25)$$

${}^3R_{ij}$ is the Ricci tensor associated with γ_{ij} , a dot denotes the derivative with respect to t , a semicolon denotes the covariant derivative with respect to γ_{ij} , ρ is the energy density of dust and $\kappa \equiv 8\pi G$.

From the Bianchi identity $T_{\mu;\nu}^\nu = 0$, we have

$$\kappa \rho = M \gamma^{-1/2}, \quad (26)$$

where M is an arbitrary constant and γ is a determinant of γ_{ij} . Hereafter we consider the RG for the dynamical variables γ_{ij} and K_{ij} .

Here we consider the following scale transformation.

$$r \longrightarrow Lr,$$

$$t \longrightarrow L^3 t,$$

$$\gamma_{rr}(r, t) \longrightarrow \overset{(L)}{\gamma}_{rr}(r, t) \equiv L^{-4} \gamma_{rr}(Lr, L^3 t). \quad (27)$$

$$\gamma_{\theta\theta}(r, t) \longrightarrow \overset{(L)}{\gamma}_{\theta\theta}(r, t) \equiv L^{-6} \gamma_{\theta\theta}(Lr, L^3 t), \quad (28)$$

$$K_{rr}(r, t) \longrightarrow \overset{(L)}{K}_{rr}(r, t) \equiv L^{-1} K_{rr}(Lr, L^3 t), \quad (29)$$

$$K_{\theta\theta}(r, t) \longrightarrow \overset{(L)}{K}_{\theta\theta}(r, t) \equiv L^{-3} K_{\theta\theta}(Lr, L^3 t), \quad (30)$$

$$\rho(r, t) \longrightarrow \overset{(L)}{\rho}(r, t) \equiv L^6 \rho(Lr, L^3 t), \quad (31)$$

where L is a parameter of scale transformation and bigger than 1. Because of the scale invariance of Einstein equations Eqs.(22) - (24), $\overset{(L)}{\gamma}_{rr}$, $\overset{(L)}{\gamma}_{\theta\theta}$, $\overset{(L)}{K}_{rr}$, $\overset{(L)}{K}_{\theta\theta}$ and $\overset{(L)}{\rho}$ satisfy Eqs.(22) - (24).

We derive the RG equation. Letting $L = e^\tau$, the infinitesimal transformation of $\overset{(L)}{\gamma}_{ij}$ and $\overset{(L)}{K}_{ij}$ with respect to τ is

$$\frac{d\overset{(L)}{\gamma}_{rr}}{d\tau} = -4\overset{(L)}{\gamma}_{rr} + r\partial_r \overset{(L)}{\gamma}_{rr} + 6\overset{(L)}{K}_{rr}, \quad (32)$$

$$\frac{d\overset{(L)}{\gamma}_{\theta\theta}}{d\tau} = -6\overset{(L)}{\gamma}_{\theta\theta} + r\partial_r \overset{(L)}{\gamma}_{\theta\theta} + 6\overset{(L)}{K}_{\theta\theta}, \quad (33)$$

$$\begin{aligned} \frac{d\overset{(L)}{K}_{rr}}{d\tau} = & -\overset{(L)}{K}_{rr} + r\partial_r \overset{(L)}{K}_{rr} \\ & + 3 \left[\frac{1}{4} \overset{(L)}{R} \overset{(L)}{\gamma}_{rr} - \overset{(L)}{R}_{rr} + 2\overset{(L)}{K}_{rl} \overset{(L)}{K}_l^r - \overset{(L)}{K} \overset{(L)}{K}_{rr} + \frac{1}{4} \left(\overset{(L)}{K}^2 \overset{(L)}{\gamma}_{rr} - \overset{(L)}{K}_l^m \overset{(L)}{K}_m^l \overset{(L)}{\gamma}_{rr} \right) \right], \end{aligned} \quad (34)$$

$$\begin{aligned} \frac{d\overset{(L)}{K}_{\theta\theta}}{d\tau} = & -3\overset{(L)}{K}_{\theta\theta} + r\partial_r \overset{(L)}{K}_{\theta\theta} \\ & + 3 \left[\frac{1}{4} \overset{(L)}{R} \overset{(L)}{\gamma}_{\theta\theta} - \overset{(L)}{R}_{\theta\theta} + 2\overset{(L)}{K}_{\theta l} \overset{(L)}{K}_\theta^l - \overset{(L)}{K} \overset{(L)}{K}_{\theta\theta} + \frac{1}{4} \left(\overset{(L)}{K}^2 \overset{(L)}{\gamma}_{\theta\theta} - \overset{(L)}{K}_l^m \overset{(L)}{K}_m^l \overset{(L)}{\gamma}_{\theta\theta} \right) \right], \end{aligned} \quad (35)$$

where $\overset{(L)}{R}_{ij}$ is the Ricci tensor associated with $\overset{(L)}{\gamma}_{ij}$. When we derive Eqs.(32) - (35), the equation of motion (22) is used. These equations (32) - (35) are the RG equations.

Letting $t = L^3$, the original variable γ_{ij} is shown by the scaled variable $\overset{(L)}{\gamma}_{ij}$:

$$\gamma_{rr}(r, t) = t^{\frac{4}{3}} \overset{(t^{1/3})}{\gamma}_{rr}(rt^{-1/3}), \quad (36)$$

$$\gamma_{\theta\theta}(r, t) = t^2 \overset{(t^{1/3})}{\gamma}_{\theta\theta}(rt^{-1/3}). \quad (37)$$

Here we investigate the fixed point of the RG equations (32) - (35). The fixed point satisfies the following equations.

$$\frac{r^2 A''}{A} - 3 \frac{r A'}{A} + \frac{r B'}{B} - \frac{1}{2} \left(\frac{r B'}{B} \right)^2 + \frac{r^2 A' B'}{AB} - \frac{9(A^3 - 2A' B B' - A B'^2 + 2A B B'')}{2A^3 B^2} - \frac{1}{2} = 0, \quad (38)$$

$$\frac{r^2 B''}{B} - 5 \frac{r B'}{B} + \frac{1}{2} \left(\frac{r B'}{B} \right)^2 - \frac{9(B'^2 - A^2)}{2A^2 B^2} + \frac{9}{2} = 0. \quad (39)$$

In the spherically symmetric spacetime filled with dust, the exact solution of the Einstein equations is a Tolman-Bondi solution. Thus the Tolman-Bondi solution with the self-similarity is a fixed point. The concrete form of it is

$$A^{*2} = \frac{(1 - 3pr^3)^2}{(1 - pr^3)^{\frac{4}{3}}}, \quad (40)$$

$$B^{*2} = r^2(1 - pr^3)^{\frac{4}{3}}, \quad (41)$$

$$\kappa \rho^* = \frac{4}{3(1 - pr^3)(1 - 3pr^3)}, \quad (42)$$

where p is a constant. When $p = 0$, the above fixed point corresponds to the flat Friedmann universe.

In order to study the behavior of the flow in the RG around the fixed point, we consider the linear perturbation around the fixed point. For simplicity, we concentrate on the spherical modes of linear perturbation. The perturbed quantities δA and δB are defined by

$$\begin{aligned} \gamma_{rr}^{(L)} &= A^{*2} + 2A^* \delta A^{(L)}, \\ \gamma_{\theta\theta}^{(L)} &= B^{*2} + 2B^* \delta B^{(L)}. \end{aligned} \quad (43)$$

Using the comoving gauge, the freedom of the gauge is fixed.

Substituting Eqs.(43) into the momentum constraint (24) and taking the linear order term, we have

$$A^* \left(\frac{\partial \delta A^{(L)}}{\partial \tau} - \frac{\partial \delta B^{(L)}}{\partial \tau} \right) + r A^{*'} (\delta A^{(L)} - \delta B^{(L)}) - r A^* (\delta A'^{(L)} - \delta B''^{(L)}) = 0, \quad (44)$$

where a prime denotes the derivative with respect to r .

We assume that $\delta A^{(L)}$ and $\delta B^{(L)}$ is the following form,

$$\begin{aligned} \delta A^{(L)} &= a(r) e^{\omega \tau}, \\ \delta B^{(L)} &= b(r) e^{\omega \tau}. \end{aligned} \quad (45)$$

Substituting Eqs.(45) into Eq.(44) and integrating, we have

$$a - b' = q(3pr^3 - 1)(pr^3 - 1)^{-1/3}r^\omega, \quad (46)$$

where q is a constant.

Similarly from Eqs.(32) - (35), we get

$$\begin{aligned} r^2 a'' + \left[\frac{2r^2(3pr^3 - 1) + 9(1 - pr^3)^{\frac{2}{3}}}{r(1 - pr^3)(1 - 3pr^3)} - 2r\omega \right] a' \\ + \left[\omega^2 + \frac{3 - pr^3}{1 - pr^3}\omega - \frac{9(1 - pr^3)(1 - 14pr^3 + 15p^2r^6) + 2p^2r^8(1 - 3pr^3)^2}{r^2(1 - 3pr^3)^2(1 - pr^3)^{\frac{2}{3}}} \right] a + \frac{6pr^2(1 - 3pr^3)}{(1 - pr^3)^3} b \\ + 3 \frac{[3(1 - pr^3)^{\frac{2}{3}}(1 - 14pr^3 + 15p^2r^6) - 2pr^5(1 - 3pr^3)^2]}{r^2(1 - pr^3)^2(1 - 3pr^3)^2} b' - \frac{9}{r(1 - 3pr^3)(1 - pr^3)^{1/3}} b'' = 0, \end{aligned} \quad (47)$$

$$r^2 b'' - 2r \left[\omega + \frac{2 - pr^3}{1 - pr^3} \right] b' + \left[\omega^2 + \frac{5 - 3pr^3}{1 - pr^3}\omega + \frac{2(2 - 3pr^3)}{(1 - pr^3)^2} \right] b + \frac{9qr^{\omega-1}}{(1 - 3pr^3)(1 - pr^3)^{1/3}} = 0. \quad (48)$$

From the condition of the regularity at $r = 0$, ω must be positive. Thus this fixed point Eq.(42) is a repeller.

4 Summary and Discussions

We have applied the renormalization group method which is used for the purpose of studying the long time asymptotics of the nonlinear, partial differential equations, to the Einstein equations. This renormalization group is the map between some initial conditions and other initial conditions. The fixed point is a solution which has scale invariance.

With cosmology in mind, we consider the case that the universe is filled with the dust. For simplicity, we imposed the spherically symmetry on the spacetime.

In this case, the Einstein equations have some scale invariance Eqs.(27)-(30). The fixed point of this RG transformation is the Tolman-Bondi solution with self-similarity, which includes the flat Friedmann universe. In order to study the flow of the RG around this fixed point, the linear perturbation analysis is used. We impose the perturbation on the regularity at the origin. The Tolman-Bondi solution with self-similarity is repeler because it has only the relevant mode. This means that the Tolman-Bondi solution with self-similarity is unstable against the spherical mode of linear perturbation. Of course, it is necessary to study the non-spherical mode of perturbation.

When we think the cosmological problem, only the statistical qualities are meaningful. The RG method will be useful to those problems. There the crucial point is how we apply the RG method to the problem of coarse graining of the field quantities. which will be treated in the future work.

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Validity of Zel'dovich-type approximation in general relativity

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Abstract

We compare relativistic approximation methods, which describe gravitational instability in the FLRW universe, in the spherically symmetric model. Approximate solutions from the linear perturbation, the second-order perturbation, Zel'dovich approximation, and post-Zel'dovich approximation are derived and compared with the Lemaitre-Tolman-Bondi solution. We find that the Zel'dovich-type approximations are more accurate than the conventional perturbation theories, especially in the regions $\delta \geq 0.5$ and $\delta < 0$.

1 Introduction

Structure formation in the universe is one of the important subjects of research in cosmology. The most standard scenario of the structure formation is that the density fluctuations with small amplitudes in the early universe have grown to be a variety of cosmic structures due to gravitational instability. The process of the growth can be characterized by the following three stages: 1) linear stage, where the density fluctuations $\delta \equiv (\rho - \rho_b)/\rho_b$ remain in the region $|\delta| \ll 1$ and evolve according to linear perturbation theory, 2) weakly nonlinear stage, where $|\delta|$ becomes comparable to unity and nonlinear effects cannot be ignored, and 3) strongly nonlinear stage, where δ has grown to be much larger than unity.

In the framework of general relativity, linear perturbation theory in the Friedmann-Lemaître-Robertson-Walker (FLRW) universe, pioneered by Lifshitz [1], is now well understood by the Bardeen's gauge-invariant formalism [2]. Most of recent discussions on cosmological perturbation in general relativity have been based on the Bardeen's method. But in the present universe, δ is larger than unity up to the scale of superclusters. Especially at the scale of superclusters, δ is of order 1. Thus it is of interest to consider approximation methods for weakly nonlinear stage of gravitational instability. Tomita [3] developed second-order theory by extending Lifshitz's work to describe nonlinear gravitational instability. His approach, however, still depends on the assumption $|\delta| \ll 1$. An approximation scheme without the assumption was suggested by Zel'dovich [4] in the framework of Newtonian cosmology. This scheme is known as Zel'dovich approximation, which is now widely applied to the problems of the large-scale structure formation. One of remarkably advantageous points of the Zel'dovich approximation is that it includes exact solutions named Zel'dovich's pancake solutions when the collapse is locally one-dimensional. In spite of the success of the Zel'dovich approximation, Newtonian

treatments may be valid only in regions small compared to the Hubble radius. Kasai [5] has recently proposed a relativistic version of the Zel'dovich approximation with clear correspondence to the original one. It is extended to second-order [6].

The Zel'dovich-type approximations are presumably suitable for description of nearly one-dimensional collapse because of the inclusion of the pancake solutions. It is not clear, however, whether they also give high accuracy in the case of the spherically symmetric collapse. In Newtonian framework, it has been investigated with the top-hat model [7] and with more general consideration [8]. In this article, we compare the relativistic perturbative approaches such as the linear perturbation theory, the second-order perturbation theory, Zel'dovich approximation, and post-Zel'dovich approximation with the spherically symmetric exact solution, i.e., the Lemaitre-Tolman-Bondi (LTB) solution in order to show the usefulness of the Zel'dovich-type approximations. Differences between the conventional perturbation theories and the Zel'dovich-type approximations are clarified quantitatively in the spherically symmetric model. Then this work can be regarded as the relativistic version of the works given by Refs. [7, 8].

Throughout this article, units are chosen so that $c = 1$. Indices $\mu, \nu, \dots$ run from 0 to 3 and $i, j, \dots$ run from 1 to 3.

2 Review of relativistic Zel'dovich approximation

In this section, we summarize a general relativistic version of Zel'dovich approximation developed by Kasai [5]. The models we consider contain irrotational dust with density ρ and four-velocity u^μ . Here we neglect effects of the fluid pressure and the vorticity. In comoving synchronous coordinates, the line element can be written in the form

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j \quad (1)$$

with $u^\mu = (1, 0, 0, 0)$. Then the Einstein equations read

$$\frac{1}{2} \left[{}^3R^i_i + (K^i_i)^2 - K^i_j K^j_i \right] = 8\pi G\rho, \quad (2)$$

$$K^i_{j||i} - K^i_{i||j} = 0, \quad (3)$$

$$\dot{K}^i_j + K^k_k K^i_j + {}^3R^i_j = 4\pi G\rho \delta^i_j, \quad (4)$$

where ${}^3R^i_j$ is the three-dimensional Ricci tensor,

$$K^i_j = \frac{1}{2} g^{ik} \dot{g}_{jk} \quad (5)$$

is the extrinsic curvature, $\parallel$ denotes the covariant derivative with respect to the three-metric g_{ij} , and an overdot ($\dot{}$) denotes $\partial/\partial t$. The energy equation $u_\mu T^{\mu\nu}_{;\nu} = 0$ gives

$$\dot{\rho} + \rho K^i_i = 0 \quad (6)$$

with the solution

$$\rho = \rho(t_{in}, \mathbf{x}) \frac{\sqrt{\det[g_{ij}(t_{in}, \mathbf{x})]}}{\sqrt{\det[g_{ij}(t, \mathbf{x})]}}. \quad (7)$$

Let us introduce the following triad

$$g_{ij} = a^2(t) \delta_{(k)(\ell)} e_i^{(k)} e_j^{(\ell)}, \quad (8)$$

where $a(t)$ is the scale factor. Then Eq. (7) is rewritten as

$$\rho = \rho_b \frac{\det[e_i^{(\ell)}(t_{in}, \mathbf{x})]}{\det[e_i^{(\ell)}(t, \mathbf{x})]} \quad (9)$$

with the background density $\rho_b \propto a^{-3}$.

We obtain the following master equation from Eqs. (2) and (4)

$$\frac{\partial}{\partial t} \left[a^3 \left(\ddot{e}_i^{(\ell)} + 2 \frac{\dot{a}}{a} \dot{e}_i^{(\ell)} - 4\pi G \rho_b e_i^{(\ell)} \right) \right] = a \left(P_i^{(\ell)} + Q_i^{(\ell)} + S_i^{(\ell)} \right) \quad (10)$$

with nonlinear source terms

$$P_i^{(\ell)} = \frac{\partial}{\partial t} \left\{ \frac{a^2}{4} \left[(V_k^j)^2 - V_n^k V_n^j \right] e_i^{(\ell)} - a^2 (V_k^k V_i^j - V_k^j V_i^k) e_j^{(\ell)} \right\}, \quad (11)$$

$$Q_i^{(\ell)} = \left(V_k^j \mathcal{R}_i^k + \frac{1}{4} V_i^j \mathcal{R}_k^k - \frac{1}{2} V_n^k \mathcal{R}_k^n \delta_i^j \right) e_j^{(\ell)}, \quad (12)$$

and

$$S_i^{(\ell)} = \left[V_i^j |^k_{|k} + V_k^k |^j_{|i} - V_{k|i}^j |^k - V_i^k |^j_{|k} + k (3V_i^j - V_k^k \delta_i^j) \right] e_j^{(\ell)}, \quad (13)$$

where $V_i^j \equiv K_i^j - (\dot{a}/a) \delta_i^j$ is the deviation of the extrinsic curvature from FLRW universe, $\mathcal{R}_i^j \equiv a^2 {}^3R_i^j - 2k \delta_i^j$ is that of the three-curvature (k is the curvature constant), and $|$ denotes the covariant derivative with respect to $\gamma_{ij} \equiv a^{-2} g_{ij}$. It should be noted that we have not used any approximation methods in deriving Eq. (10) from the Einstein equations.

The master equation (10) is solved perturbatively as follows [6]

$$e_i^{(\ell)} = \left(1 + \frac{10}{9} \Psi(\mathbf{x}) \right) \delta_i^{(\ell)} + \delta_j^{(\ell)} \left(t^{\frac{2}{3}} \Psi_{,i}^j(\mathbf{x}) + t^{\frac{1}{3}} \psi_{,i}^j(\mathbf{x}) + t^{\frac{4}{3}} \varphi_{,i}^j(\mathbf{x}) \right), \quad (14)$$

where $\Psi(\mathbf{x})$ is a spatial arbitrary function of linear order, and $\psi_{,i}^j(\mathbf{x})$ and $\varphi_{,i}^j(\mathbf{x})$ are quadratic quantities with respect to $\Psi(\mathbf{x})$, given by

$$\psi_{,i}^j = \frac{5}{9} \Psi_{,k}^k \Psi_{,i}^j \delta_{,j}^i - \frac{20}{9} (\Psi \Psi_{,i}^j + \Psi_{,i}^j \Psi_j), \quad (15)$$

$$\varphi_{,i}^j = \frac{3}{14} ((\Psi_{,k}^k)^2 - \Psi_{,k}^k \Psi_{,k}^k) \delta_{,j}^i - \frac{6}{7} (\Psi_{,k}^k \Psi_{,i}^j - \Psi_{,i}^k \Psi_{,j}^k). \quad (16)$$

(Here for simplicity, we assume the Einstein-de Sitter background, $k = 0$, and consider only the growing scalar modes up to second-order.)

3 Density fluctuations from approximation methods

In this section, we give expressions of the density fluctuations obtained from various approximation methods. We consider Lifshitz's linear theory (LIN) [1], Tomita's second-order theory (SEC) [3], Zel'dovich approximation (ZA) [5], and post-Zel'dovich approximation (PZA) [6].

The results from LIN and SEC are

$$\delta_{LIN} = -t^{2/3}\Psi_{,k}^k, \quad (17)$$

$$\delta_{SEC} = -t^{2/3}\Psi_{,k}^k + \frac{5}{9}t^{2/3}(\Psi_{,k}^k\Psi_{,k} + 6\Psi\Psi_{,k}^k) + \frac{1}{7}t^{4/3}(5(\Psi_{,k}^k)^2 + 2\Psi_{,l}^k\Psi_{,k}^l). \quad (18)$$

The Zel'dovich-type approximation is obtained by substituting Eq. (14) into Eq. (9). Therefore, δ from ZA and PZA are as follows

$$\delta_{ZA} = \left(\det \left[\delta^i_j + \frac{t^{2/3}\Psi_{,j}^i}{1 + \frac{10}{9}\Psi} \right] \right)^{-1} - 1, \quad (19)$$

$$\delta_{PZA} = \left(\det \left[\delta^i_j + \frac{t^{2/3}\Psi_{,j}^i + t^{2/3}\psi_{,j}^i + t^{4/3}\varphi_{,j}^i}{1 + \frac{10}{9}\Psi} \right] \right)^{-1} - 1. \quad (20)$$

If we expand Eqs. (19) and (20) in the Taylor series, we can reproduce Eqs. (17) and (18), respectively. In this sense, ZA and PZA reproduce LIN and SEC, respectively, in the limit $|\delta| \ll 1$.

4 Comparison with LTB solution

In this section, we compare the relativistic perturbative methods with LTB solution. The line element of the LTB solution reads

$$ds^2 = -dt^2 + \frac{r'^2}{1+f} dR^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (21)$$

where $(') \equiv \partial/\partial R$ and $f = f(R)$ is an arbitrary function which is related to initial velocity of dust. $r = r(t, R)$ satisfies the following differential equation

$$\dot{r}^2 = \frac{F(R)}{r} + f(R) \quad (22)$$

with an arbitrary function $F(R)$, which represents initial distribution of matter. Eq. (22) can be integrated as follows

(1) $f > 0$:

$$r = \frac{F}{2f}(\cosh \eta - 1), \quad t - t_0(R) = \frac{F}{2f^{3/2}}(\sinh \eta - \eta), \quad (23)$$

(2) $f < 0$:

$$r = \frac{F}{-2f}(1 - \cos \eta), \quad t - t_0(R) = \frac{F}{2(-f)^{3/2}}(\eta - \sin \eta), \quad (24)$$

(3) $f = 0$:

$$r = \left(\frac{9F}{4}\right)^{1/3} (t - t_0(R))^{2/3}, \quad (25)$$

where $t_0(R)$ is an integration constant. In three cases, the density reads

$$8\pi G\rho = \frac{F'}{r'r^2}. \quad (26)$$

Considering the correspondence to the linear perturbation theory, we find that the arbitrary functions $f(R)$ and $t_0(R)$ correspond to the growing and the decaying modes, respectively. Here we set $t_0(R) = 0$ because we neglect the decaying modes in this article.

Let us turn our attention to the approximation methods. If we impose $\Psi = \Psi(R)$, Eqs. (17), (18), (19) and (20) become

$$\delta_{LIN} = -t^{2/3}(\alpha + 2\beta), \quad (27)$$

$$\delta_{SEC} = -t^{2/3}(\alpha + 2\beta) + \frac{10}{3}t^{2/3}\Psi(\alpha + 2\beta) + \frac{5}{9}t^{2/3}\Psi'^2 + t^{4/3}\left(\alpha^2 + \frac{20}{7}\alpha\beta + \frac{24}{7}\beta^2\right) \quad (28)$$

$$\delta_{ZA} = \left(1 + \frac{t^{2/3}\beta}{1 + \frac{10}{9}\Psi}\right)^{-2} \left(1 + \frac{t^{2/3}\alpha}{1 + \frac{10}{9}\Psi}\right)^{-1} - 1, \quad (29)$$

$$\begin{aligned} \delta_{PZA} = & \left(1 + \frac{t^{2/3}\beta - \frac{20}{9}t^{2/3}\Psi\beta + \frac{5}{9}t^{2/3}\Psi'^2 - \frac{3}{7}t^{4/3}\beta^2}{1 + \frac{10}{9}\Psi}\right)^{-2} \\ & \times \left(1 + \frac{t^{2/3}\alpha - \frac{20}{9}t^{2/3}\Psi\alpha - \frac{5}{3}t^{2/3}\Psi'^2 - \frac{3}{7}t^{4/3}\beta(2\alpha - \beta)}{1 + \frac{10}{9}\Psi}\right)^{-1} - 1, \end{aligned} \quad (30)$$

where $\alpha = \alpha(R) \equiv \Psi''$ and $\beta = \beta(R) \equiv \Psi'/R$.

For a simple example, we set an initial density profile as

$$\delta(t_{in}, R) = \epsilon \exp \left[- \left(\frac{R}{R_0} \right)^2 \right], \quad (31)$$

where ϵ is a small parameter ($|\epsilon| \ll 1$) and R_0 is a scale of the fluctuations. Then we can evaluate δ at the center of the fluctuations ($R = 0$) from the LTB solution and the approximation methods in the following form

$$\delta_{LTB}(R = 0) = \begin{cases} \frac{9}{2} \frac{(\theta - \sin \theta)^2}{(1 - \cos \theta)^3} - 1, & \text{for } \epsilon > 0, \\ \frac{9}{2} \frac{(\theta - \sinh \theta)^2}{(\cosh \theta - 1)^3} - 1, & \text{for } \epsilon < 0, \end{cases} \quad (32)$$

with

$$t = \begin{cases} \frac{9}{20} \sqrt{\frac{3}{5}} \epsilon^{-\frac{3}{2}} (\theta - \sin \theta), & \text{for } \epsilon > 0, \\ \frac{9}{20} \sqrt{\frac{3}{5}} (-\epsilon)^{-\frac{3}{2}} (\sinh \theta - \theta), & \text{for } \epsilon < 0, \end{cases}$$

$$\delta_{LIN}(R=0) = \epsilon t^{\frac{2}{3}}, \quad (33)$$

$$\delta_{SEC}(R=0) = \epsilon t^{\frac{2}{3}} + \frac{17}{21} \epsilon^2 t^{\frac{4}{3}}, \quad (34)$$

$$\delta_{ZA}(R=0) = \left(1 - \frac{\epsilon}{3} t^{\frac{2}{3}}\right)^{-3} - 1, \quad (35)$$

$$\delta_{PZA}(R=0) = \left(1 - \frac{\epsilon}{3} t^{\frac{2}{3}} - \frac{\epsilon^2}{21} t^{\frac{4}{3}}\right)^{-3} - 1. \quad (36)$$

(In fact, these expressions do not depend on detail of initial density profiles. It is of essence in the calculation that $\epsilon > 0$ and $\epsilon < 0$ correspond to $f < 0$ and $f > 0$ at $R = 0$, respectively.) The above relativistic results are quite similar to Newtonian ones, which are given in Refs. [7, 8]. In Fig. 1, we plotted δ from the approximation methods as functions of δ_{LTB} for $\epsilon > 0$. Fig. 2 is the same as Fig. 1, but for $\epsilon < 0$. Physically speaking, the cases $\epsilon > 0$ and $\epsilon < 0$ correspond to the collapse and the expanding void, respectively.

5 Summary

We have tested the relativistic perturbative approximations to gravitational instability with the LTB solution. The values of δ at $R = 0$ have been estimated from the LTB solution, LIN, SEC, ZA, and PZA for a given initial density profile. We found that the Zel'dovich-type approximations are more accurate than the conventional perturbation theories. In particular, the first-order Zel'dovich approximation works better than the second-order perturbation theory in the weakly nonlinear regime $\delta \geq 0.5$ and the underdense region $\delta < 0$. It should be also noted that the results we have obtained here are quite similar to those obtained from Newtonian treatments [7, 8]. In order to see more detailed behavior, it will be necessary to compare the evolved profiles of δ , not only the values of δ at $R = 0$.

In this article, we have given just a simple example. When we consider more general situations including occurrence of the shell-crossing, the Zel'dovich-type approximations may not be quite accurate. Such consideration is of interest in order to clear the limitation of the Zel'dovich-type approximations. These are now under investigation.

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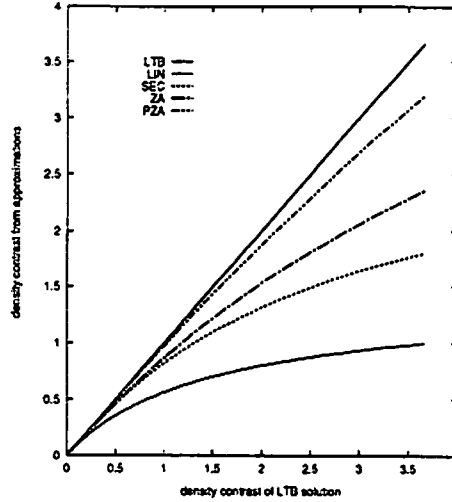


Fig.1 Density contrast δ at $R = 0$ calculated from approximation methods as a function of δ_{LTB} for $\epsilon > 0$. δ_{ZA} catches up with δ_{SEC} at $\delta_{LTB} \sim 0.5$.

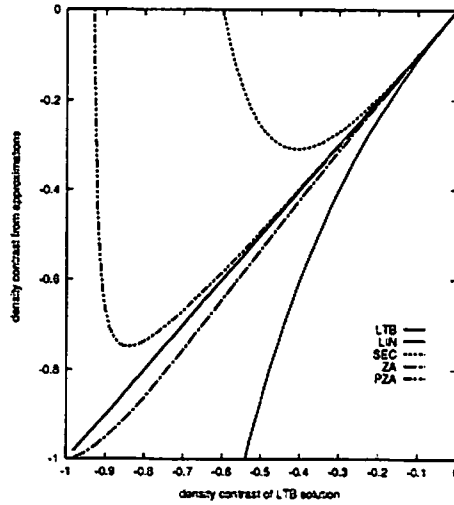


Fig.2 Same as in Fig. 1, but for $\epsilon < 0$.

Constraints on Scalar-Tensor Theories of Gravity from Density Perturbations in Inflationary Cosmology

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Abstract

Spectra of density perturbations produced during chaotic inflation in scalar-tensor theories of gravity are calculated, taking both adiabatic and isocurvature modes into account. It is found that the role of the isocurvature mode is negligible after inflation. Comparing the predicted spectrum of the cosmic microwave background radiation anisotropies with the one observed by the COBE-DMR we calculate constraints on the parameters of these theories, which turn out to be stronger by an order-of-magnitude than those obtained from post-Newtonian experiments.

1 Introduction

Inflationary cosmology gives a natural explanation for the horizon, flatness, homogeneity, and monopole problems[1]. In its simplest form, inflation with a single scalar field predicts an approximately scale-invariant spectrum of Gaussian adiabatic density perturbations[2]. If other fields are present, however, primordially isocurvature perturbations may also be generated to leave an imprint on the universe today [3]. In any case, it may be said that inflation is the unique causal mechanism for generating density perturbations on scales well above the Hubble radius[4].

Scalar-tensor theories of gravitation have recently been received a renewed interest. The main reason is that the low-energy effective action of superstring theories[5] generally involves a dilaton coupled to the Ricci curvature. Moreover, as pointed out by Damour and Esposito-Farese[6] in the study of neutron star models, a wide class of scalar-tensor theories not only pass the present weak-field gravitational tests but also exhibit non-perturbative strong-field deviations away from general relativity. Detectability of scalar gravitational waves by laser interferometric gravitational wave observatories(LIGO) has been explored in [7][8].

Cosmological consequences of such theories, however, have not been fully examined. Recently, one of us has studied the generality of chaotic inflation in scalar-tensor theories in detail and found that the onset of inflation can be greatly affected by the presence of the dilaton in some cases[9]. According to the attractor mechanism of scalar-tensor

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theories[10], which works only for those theories in which the derivative of the Brans-Dicke coupling is positive, even those theories that are quite different from general relativity in the early stage of the universe approach general relativity during the inflationary stage and the matter-dominated stage of the universe.

Even if this is the case, the deviations in the earlier stage could be imprinted in the spectrum of the cosmic microwave background radiation (CMB). We may constrain model parameters using the CMB anisotropy as a probe of the early universe. In fact, because of the presence of two scalar fields, *i.e.*, the inflaton and the dilaton, density perturbations have both adiabatic and isocurvature modes and hence the spectrum might be changed.

In this paper, we study formation and evolution of density perturbations during and after chaotic inflation in scalar-tensor theories paying attention to the role of the isocurvature mode. This paper originates from the analysis by Starobinsky and Yokoyama[11], who calculated the spectrum of density perturbations from inflation in Brans-Dicke theory. The scalar-tensor theories we study include those with negative β (the derivative of the Brans-Dicke coupling) which have not been studied so far and exhibit a significant deviation from general relativity in the strong field region[6].²

The plan of this paper is as follows: In §2, basic equations of background fields are given. Linear perturbation equations are given and solutions to these equations are derived in §3. We study the evolution of isocurvature perturbations after inflation in §4. In §5, the spectrum of density perturbations are calculated and compared with the COBE observation to constrain the parameters of the theory. Finally §6 is devoted to summary.

2 Background field equations

2.1 Scalar-tensor theories of gravitation

We consider the inflaton, σ , coupled to scalar-tensor theories of gravity. The action is

$$S = \int d^4x \sqrt{-\bar{g}} \left[\frac{1}{16\pi} \left(\psi \bar{\mathcal{R}} - \frac{\omega(\psi)}{\psi} \bar{g}^{ab} \psi_{,a} \psi_{,b} \right) - \frac{1}{2} \bar{g}^{ab} \sigma_{,a} \sigma_{,b} - V(\sigma) \right], \quad (1)$$

where ψ is the massless Brans-Dicke dilaton, $\bar{\mathcal{R}}$ is the scalar curvature, $\omega(\psi)$ is a dimensionless coupling parameter and $V(\sigma)$ is the potential of the inflaton. Let us consider the conformal transformation,

$$\bar{g}_{ab} = e^{2a(\varphi)} g_{ab}, \quad (2)$$

such that $e^{-2a(\varphi)} = \psi G$. Then the action can be written as

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} R - \frac{1}{2} g^{ab} \varphi_{,a} \varphi_{,b} - \frac{1}{2} e^{2a} g^{ab} \sigma_{,a} \sigma_{,b} - e^{4a} V(\sigma) \right], \quad (3)$$

where $\kappa^2 = 8\pi G$ and φ is defined by

$$2a(\varphi)^2 \equiv \frac{2}{\kappa^2} \left(\frac{da(\varphi)}{d\varphi} \right)^2 = \frac{1}{2\omega(\psi) + 3}. \quad (4)$$

²The analysis by [12] has been limited to positive β and the isocurvature mode has been treated incorrectly. See below.

The relation to the PPN parameters ($\gamma_{Edd}, \beta_{Edd}$) and the current constraints on them are [13]

$$|\gamma_{Edd} - 1| = \frac{4\alpha(\varphi)^2}{1 + 2\alpha(\varphi)^2} \Big|_0 < 2 \times 10^{-3}, \quad |\beta_{Edd} - 1| = \frac{2\alpha(\varphi)^2\beta(\varphi)}{1 + 2\alpha(\varphi)^2} \Big|_0 < 6 \times 10^{-4}, \quad (5)$$

where $\beta(\varphi) = d\alpha(\varphi)/\kappa d\varphi$ and subscript 0 indicates the present value. Brans-Dicke theory corresponds to $\beta(\varphi) = 0$. In general relativity $\alpha(\varphi) = 0$.

2.2 Background equations

We consider the spatially flat FRW spacetime as a background metric

$$ds^2 = -dt^2 + R(t)^2 d\mathbf{x}^2. \quad (6)$$

The equations of motion derived from the action Eq.(3) are

$$\left(\frac{\dot{R}}{R}\right)^2 \equiv H^2 = \frac{\kappa^2}{6} [\dot{\varphi}^2 + e^{2a}\dot{\sigma}^2 + 2e^{4a}V(\sigma)], \quad (7)$$

$$\ddot{\varphi} + 3H\dot{\varphi} = \kappa\alpha [e^{2a}\dot{\sigma}^2 - 4e^{4a}V(\sigma)], \quad (8)$$

$$\ddot{\sigma} + 3H\dot{\sigma} = -e^{2a}V'(\sigma) - 2\kappa\alpha\dot{\varphi}\dot{\sigma}, \quad (9)$$

$$\dot{H} = -\frac{\kappa^2}{2}(\dot{\varphi}^2 + e^{2a}\dot{\sigma}^2), \quad (10)$$

where $\dot{}$ and $'$ denote time and σ derivatives, respectively.

The conditions for slow-roll inflation ($|\ddot{\sigma}| \lesssim |3H\dot{\sigma}|, |\ddot{\varphi}| \lesssim |3H\dot{\varphi}|$) are written as,

$$\max\left(\frac{1}{2}\dot{\varphi}^2, \frac{1}{2}e^{2a}\dot{\sigma}^2\right) \lesssim e^{4a}V(\sigma), \quad (11)$$

$$e^{-2a}V'(\sigma)^2 \lesssim 6\kappa^2V(\sigma)^2, \quad (12)$$

$$e^{-2a}V''(\sigma) \lesssim 3\kappa^2V(\sigma), \quad (13)$$

$$8\alpha(\varphi)^2 \lesssim 3, \quad (14)$$

$$16\alpha(\varphi)^2 + 4\beta(\varphi) \lesssim 3. \quad (15)$$

When the above inequalities apply, field equations are simplified to

$$3H^2 \simeq \kappa^2 e^{4a}V(\sigma), \quad (16)$$

$$3H\dot{\varphi} \simeq -4\kappa\alpha e^{4a}V(\sigma), \quad (17)$$

$$3H\dot{\sigma} \simeq -e^{2a}V'(\sigma). \quad (18)$$

3 Linear perturbations

3.1 Basic equations

Now we turn to linear perturbations with a perturbed metric in the longitudinal gauge

$$ds^2 = -(1 + 2\Phi)dt^2 + R^2(1 - 2\Psi)d\mathbf{x}^2. \quad (19)$$

From the perturbed Einstein equations, each Fourier mode³ satisfies the following equations of motion to first order,

$$\Phi = \Psi, \quad (20)$$

$$\begin{aligned} \ddot{\delta\varphi} + 3H\dot{\delta\varphi} + \frac{k^2}{R^2}\delta\varphi &+ 4e^{4a}\kappa^2 V(\sigma)(4\alpha^2 + \beta)\delta\varphi - e^{2a}\kappa^2\dot{\sigma}^2(2\alpha^2 + \beta)\delta\varphi \\ &= 4\dot{\Phi}\dot{\varphi} + 2e^{2a}\alpha\kappa\dot{\sigma}\delta\sigma - 4e^{4a}\alpha\kappa(V'(\sigma)\delta\sigma + 2V(\sigma)\Phi), \end{aligned} \quad (21)$$

$$\begin{aligned} \ddot{\delta\sigma} + 3H\dot{\delta\sigma} + \frac{k^2}{R^2}\delta\sigma &+ e^{2a}V''(\sigma)\delta\sigma = -2e^{2a}V'(\sigma)(\Phi + \alpha\kappa\delta\varphi) \\ &+ \dot{\sigma}(4\dot{\Phi} - 2\alpha\kappa\dot{\varphi} - 2\beta\kappa^2\dot{\varphi}\delta\varphi) - 2\alpha\kappa\dot{\varphi}\delta\sigma, \end{aligned} \quad (22)$$

$$\begin{aligned} \ddot{\Phi} + 4H\dot{\Phi} + (\dot{H} + 3H^2)\Phi &= \frac{\kappa^2}{2} \left[\dot{\varphi}\delta\dot{\varphi} + e^{2a}\dot{\sigma}\delta\dot{\sigma} - e^{4a}V'(\sigma)\delta\sigma + (e^{2a}\dot{\sigma}^2 - 4e^{4a}V(\sigma))\alpha\kappa\delta\varphi \right], \end{aligned} \quad (23)$$

together with the Hamiltonian and momentum constraints

$$\begin{aligned} 3H\dot{\Phi} + (\dot{H} + 3H^2)\Phi + \frac{k^2}{R^2}\Phi &= -\frac{\kappa^2}{2} \left[\dot{\varphi}\delta\dot{\varphi} + e^{2a}\dot{\sigma}\delta\dot{\sigma} \right. \\ &\quad \left. + e^{4a}V'(\sigma)\delta\sigma + (e^{2a}\dot{\sigma}^2 + 4e^{4a}V(\sigma))\alpha\kappa\delta\varphi \right], \end{aligned} \quad (24)$$

$$\dot{\Phi} + H\Phi = \frac{\kappa^2}{2}(\dot{\varphi}\delta\varphi + e^{2a}\dot{\sigma}\delta\sigma). \quad (25)$$

3.2 Long wavelength perturbations

Since what we need are the non-decreasing adiabatic and isocurvature modes on large scale $k \ll RH$, which turn out to be weakly time-dependent as will be seen in the final result [14], we may consistently neglect $\ddot{\Phi}$ and those terms containing two time derivatives. Then eqs. (21), (22), and (25) are simplified to

$$\Phi \simeq -2\alpha\kappa\delta\varphi - \frac{1}{2}\frac{V'}{V}\delta\sigma, \quad (26)$$

$$3H\dot{\delta\varphi} \simeq -4\beta\kappa^2 e^{4a}V\delta\varphi, \quad (27)$$

$$3H\dot{\delta\sigma} \simeq -e^{2a}\left(\frac{V'}{V}\right)'V\delta\sigma + 2\alpha\kappa e^{2a}V'\delta\varphi. \quad (28)$$

Following [11], we find the last two equations can be integrated to give

$$\delta\varphi \simeq \frac{4\alpha}{\kappa}Q_1, \quad (29)$$

$$\delta\sigma \simeq \frac{1}{\kappa^2}\frac{V'}{V}(e^{-2a}Q_1 + Q_2), \quad (30)$$

where use has been made of Eqs.(16), (17) and (18). And then

$$\Phi \simeq -8\alpha^2Q_1 - \frac{1}{2\kappa^2}\left(\frac{V'}{V}\right)^2(e^{-2a}Q_1 + Q_2), \quad (31)$$

where Q_1 and Q_2 are constants of integration.

³Our convention of the Fourier transformation is $Y(\mathbf{k}) = (2\pi)^{-3/2} \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} Y(\mathbf{x})$ for each variable Y .

3.3 Adiabatic and isocurvature modes

In order to clarify the physical meaning of the above solutions, we should divide them into adiabatic and isocurvature modes [11][14]. The primordially isocurvature mode is characterized by its vanishingly small contribution to the curvature perturbation initially, while the growing adiabatic mode can be described by the following expressions in the long wavelength limit $k \ll RH$.

$$\Phi_{ad} = C_1 \left[1 - \frac{H}{R} \int R(t') dt' \right] \simeq -C_1 \frac{\dot{H}}{H^2}, \quad (32)$$

$$\frac{\delta\varphi_{ad}}{\dot{\varphi}} = \frac{\delta\sigma_{ad}}{\dot{\sigma}} = \frac{C_1}{R} \int R(t') dt' \simeq \frac{C_1}{H}, \quad (33)$$

where C_1 is a constant and the latter approximate equality in each expression is derived by integration by parts during the inflationary stage.

As discussed by Starobinsky and Yokoyama [11] when the conditions

$$8\alpha^2 \ll \frac{1}{2\kappa^2} \left(\frac{V'}{V} \right)^2 e^{-2a}, \quad (34)$$

$$e^{-2a} \simeq 1 \quad (35)$$

hold, one can discriminate between adiabatic and isocurvature modes in the final results by defining new constants C_1 and C_3 as $C_1 \equiv -Q_1 - Q_2$ and $C_3 \equiv -Q_2$. Using Eqs.(16), (17) and (18) we find

$$\Phi \simeq -C_1 \frac{\dot{H}}{H^2} + C_3 \left[-8\alpha^2 + \frac{1}{2\kappa^2} \left(\frac{V'}{V} \right)^2 (1 - e^{-2a}) \right], \quad (36)$$

$$\frac{\delta\varphi}{\dot{\varphi}} \simeq \frac{C_1}{H} - \frac{C_3}{H}, \quad (37)$$

$$\frac{\delta\sigma}{\dot{\sigma}} \simeq \frac{C_1}{H} - \frac{C_3}{H} (1 - e^{-2a}). \quad (38)$$

In the above expressions, terms in proportion to C_1 and C_3 represent adiabatic and isocurvature modes, respectively if the above conditions hold. However, in more general scalar-tensor theories (especially in positive β theories) these conditions will not hold and consequently terms in proportion to C_3 will contain the adiabatic mode. In any case, as shown above, only the dilaton contributes to the isocurvature mode practically.

3.4 Quantum fluctuations

We shall next determine the constants C_1 and C_3 from amplitudes of quantum fluctuations of the scalar fields generated during the inflationary stage. Due to the inequalities (11),(12) and (13), Eqs.(21) and (22) can be approximated by the equation of motion of a free massless scalar field during inflation for $k \geq RH$. The standard quantization gives the well-known result, that is, the Fourier components of the fields can be represented in the form

$$\delta\varphi(\mathbf{k}) = \frac{H(t_k)}{\sqrt{2k^3}} \epsilon_\varphi(\mathbf{k}), \quad \delta\sigma(\mathbf{k}) = \frac{H(t_k)}{\sqrt{2k^3}} e^{-a} \epsilon_\sigma(\mathbf{k}), \quad (39)$$

where t_k is the time when k -mode leaves the Hubble horizon during inflation. Here $\epsilon_\varphi(k)$ and $\epsilon_\sigma(k)$ are classical random Gaussian variables with the following averages:

$$\langle \epsilon_\varphi(\mathbf{k}) \rangle = \langle \epsilon_\sigma(\mathbf{k}) \rangle = 0, \quad \langle \epsilon_i(\mathbf{k}) \epsilon_j(\mathbf{k}') \rangle = \delta_{ij} \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \quad i, j = \varphi, \sigma. \quad (40)$$

We thus find

$$C_1 = \left[e^{-2a} H \frac{\delta\sigma}{\dot{\sigma}} + (1 - e^{-2a}) H \frac{\delta\varphi}{\dot{\varphi}} \right]_{t_k} = \frac{H^2(t_k)}{\sqrt{2k^3}} \left[\frac{e^{-3a}}{\dot{\sigma}} \epsilon_\sigma(\mathbf{k}) + \frac{1 - e^{-2a}}{\dot{\varphi}} \epsilon_\varphi(\mathbf{k}) \right]_{t_k}, \quad (41)$$

$$C_3 = \left[e^{-2a} H \left(\frac{\delta\sigma}{\dot{\sigma}} - \frac{\delta\varphi}{\dot{\varphi}} \right) \right]_{t_k} = \frac{H^2(t_k)}{\sqrt{2k^3}} \left[\frac{e^{-3a}}{\dot{\sigma}} \epsilon_\sigma(\mathbf{k}) - \frac{e^{-2a}}{\dot{\varphi}} \epsilon_\varphi(\mathbf{k}) \right]_{t_k}. \quad (42)$$

Note that Garcia-Bellido and Wands [12] have incorrectly set $C_3 = 0$ which cannot be justified since both $\delta\sigma$ and $\delta\varphi$ are stochastic quantities.

4 Evolution of isocurvature perturbation after inflation

In order to relate the spectrum of Φ at the end of inflation to that at decoupling, we have to study the evolution of the isocurvature mode together with the adiabatic mode after inflation. Unlike the adiabatic modes we do not have a universal formula for the isocurvature modes, so we have to calculate them explicitly.

4.1 Isocurvature perturbation in the reheating stage with Brans-Dicke dilaton

First let us consider the isocurvature mode during the reheating stage after inflation. In this stage, $\kappa^2 V \ll e^{-2a} V''$ (see Eq.(13)), so that the inflaton oscillates coherently. The cosmic expansion law in this regime is the same as that in the matter-dominated era. However density perturbations might evolve differently because equations of motion of perturbed quantities, (21) through (23), have oscillating coefficients and we must worry about possible parametric resonance effect. This issue has been studied by Kodama and Hamazaki [15] recently in the case only the inflaton is present and oscillating. They have shown that resonance effect is unimportant at least in the long wavelength regime which is of our interest. In the present case with two scalar fields, however, their approach is not directly applicable, in particular, for the isocurvature mode. Hence we have numerically solved the evolution equations in the long wavelength limit. Figure 1 illustrates an example of the results for the Brans-Dicke theory with $\omega = 500$. We find that there is no parametric amplification and the both modes evolve similarly. We also find qualitatively the same results for more generic scalar-tensor theories.

4.2 Isocurvature perturbation in the radiation-dominated stage with Brans-Dicke dilaton

Next let us consider the isocurvature mode during the radiation dominated era after the reheating stage. After the inflaton has decayed into radiation, the evolution equation of

Φ becomes

$$\begin{aligned} \ddot{\Phi} + (4 + 3c_s^2)H\dot{\Phi} + (2\dot{H} + 3(1 + c_s^2)H^2)\Phi + c_s^2 \frac{k^2}{R^2}\Phi \\ = \frac{\kappa^2}{3} \frac{h_r h_d}{h} \left(\frac{\Delta_d}{1 + w_d} - \frac{\Delta_r}{1 + w_r} \right) \end{aligned} \quad (43)$$

where c_s is the sound velocity,

$$c_s^2 = \frac{\dot{p}}{\dot{\rho}} = \frac{3\dot{\varphi}^2 + 4\rho_r/3}{3\dot{\varphi}^2 + 4\rho_r}, \quad (44)$$

with $h_j \equiv \rho_j + p_j$, $w_j \equiv p_j/\rho_j$ ($j = r, d$), and $h \equiv \rho + p$. It is now apparent that terms in the parenthesis in the right hand side of (43) are just the entropy perturbation

$$\frac{\delta S}{S} = \frac{3}{4}\Delta_r - \frac{1}{2}\Delta_d. \quad (45)$$

For the isocurvature perturbation, it is nonvanishing. It is known that if we replace ρ_d with the baryon density ρ_b , the isocurvature mode grows as R [16]. Because the dilaton density ρ_d decays as R^{-6} instead of R^{-3} , it is clear that the special solution originating from the source term decays as R^{-2} . On the other hand, the homogeneous part of the growing mode solution is the same as the adiabatic one. Therefore the isocurvature mode in the radiation dominated stage grows no faster than the adiabatic mode.

To conclude, the isocurvature mode does not evolve faster than the adiabatic counterpart and so is unimportant. The spectrum at the end of inflation is sufficient to compare the observation.

5 Spectra of perturbations and constraints on PPN parameters

Now we calculate the density fluctuation spectra produced by inflation. In order to do so, we have to specify $V(\sigma)$ and $a(\varphi)$. We consider chaotic inflation induced by a mass term $V(\sigma) = m^2\sigma^2/2$, of which we have a sensible particle-physics model in the intermediate scale [17]. Here m is determined from the large-angle microwave anisotropies seen by COBE-DMR[18]. The details of COBE normalization and numerical method are given in [19].

5.1 Brans-Dicke theory

The Brans-Dicke theory corresponds to $\beta \equiv 0$ and thus

$$\alpha(\varphi) = \alpha_0, \quad (46)$$

$$a(\varphi) = \alpha_0 \kappa \varphi. \quad (47)$$

Initial conditions for φ and σ are

$$\kappa\varphi = 4\alpha_0 N + \varphi_e, \quad (48)$$

$$\varphi_e \simeq 10\alpha, \quad (49)$$

$$\kappa^2\sigma^2 = \frac{2}{3} + \frac{1}{2\alpha_0^2}(e^{2\alpha_0\kappa\varphi} - e^{2\alpha_0\kappa\varphi_e}). \quad (50)$$

In Figs. 2, the spectra $k^3\langle|\Phi|^2\rangle$ for each α_0 or ω are shown. In the Brans-Dicke theory, the conditions Eqs.(11) and (12) hold and therefore we are justified to say that terms in proportion to C_3 represent the isocurvature mode. We see that considering the current limit on ω ($\omega > 500$) in Eq.(15), the contribution of the isocurvature mode is totally negligible in the Brans-Dicke theory in accordance with Starobinsky and Yokoyama[11].

The spectral indices on the comoving horizon scale today are $n = 0.970, 0.967, 0.967$ for $\omega = 500, 5000, 50000$, respectively. Note that $n = 0.967$ in general relativity.

5.2 A class of scalar-tensor theory

We take the following simple functional form for $a(\varphi)$ in accord with Damour and Nordtvedt[10]

$$a(\varphi) = \frac{\beta}{2}(\kappa\varphi)^2, \quad \beta \equiv \text{const}, \quad (51)$$

$$\alpha(\varphi) = \beta\kappa\varphi. \quad (52)$$

We choose the origin of the field φ so that $\alpha^2(\varphi) = \alpha_0^2$ (see (5)) at present.⁴ We have to determine the value of φ at the end of inflation, i.e., φ_e , to calculate the spectrum of density perturbation. Since we normalize φ in terms of its present value, we need to go back to the end of inflation to specify φ_e . The evolution of φ in the radiation or matter-dominated universe is analyzed by Damour and Nordtvedt[10]. In the radiation-dominated stage φ is hardly changed. We assume for simplicity that φ_e is equal to φ at the beginning of matter dominated stage. For $|\beta| \ll 1$ (see Eq.(15)), φ_e is determined by

$$\kappa\varphi_e \simeq \frac{\alpha_0}{\beta}e^{10\beta}. \quad (53)$$

Then φ is given in terms of N as

$$\kappa\varphi \simeq \kappa\varphi_e e^{4\beta N} \simeq \frac{\alpha_0}{\beta}e^{4\beta N+10\beta}. \quad (54)$$

For σ with (see Eq.(12))

$$e^{-2a(\varphi_e)} \left(\frac{V'(\sigma_e)}{\kappa V(\sigma_e)} \right)^2 = 6, \quad (55)$$

σ is given by

$$\kappa^2\sigma^2 \simeq \frac{2}{3}e^{-2a(\varphi_e)} + \frac{1}{2\beta} [E_1(2a(\varphi)) - E_1(2a(\varphi_e))], \quad (56)$$

⁴The authors of [12] take $\alpha(\phi) = \alpha_1 + \alpha_2\kappa\phi$ and choose the origin so that $\alpha = 0$ at the end of inflation. However, α deviates from 0 afterwards and may conflict with the limit on α_0 .

where $E_1(x)$ is the exponential integral function.

In Figs. 3, the spectra for several β with $\alpha_0 = 10^{-3}$ are shown. In some cases the mode coming from C_3 part in Eq.(36) is not negligible and the spectrum can be different from flat one. In positive β scalar-tensor theories, the conditions Eqs.(35) and (35) do not hold since initially α can be greatly different from zero (see Eq.(54)), and terms in proportion to C_3 contain the adiabatic mode in addition to the isocurvature mode. For negative β scalar-tensor theories, however, terms in proportion to C_3 represent the isocurvature mode.

In Fig. 4 we show that contour plot of n on $\alpha_0 - \beta$ plane. We see that for $\beta > 0$ n falls down very steeply with increasing β . Thus the observed spectrum by COBE-DMR[18], $n = 1.2 \pm 0.3$, strongly constrains β . Interestingly, the contour level can be fitted by a linear function in $\log \alpha_0$. For $n \geq 0.9$,

$$8.7 \times 10^{-3} \log \left(\frac{\alpha_0}{\sqrt{5} \times 10^{-2}} \right) - 1.5 \times 10^{-2} \leq \beta \leq -8.9 \times 10^{-3} \log \left(\frac{\alpha_0}{\sqrt{5} \times 10^{-2}} \right) + 5.6 \times 10^{-3}. \quad (57)$$

For $n \geq 0.7$,

$$5.0 \times 10^{-3} \log \left(\frac{\alpha_0}{\sqrt{5} \times 10^{-2}} \right) - 3.7 \times 10^{-2} \leq \beta \leq -9.3 \times 10^{-3} \log \left(\frac{\alpha_0}{\sqrt{5} \times 10^{-2}} \right) + 7.6 \times 10^{-3}. \quad (58)$$

If we assume that the tensor mode of perturbations does not contribute to the spectrum significantly, the above inequality gives the constraint on β . An important point to note is that the constraint on β has little dependence on α_0 as well as on the details of normalization of fluctuations, and that the constraint is stronger by an order-of-magnitude than those obtained by post-Newtonian experiments(see Eq.(15)).

6 Summary

We have shown that the spectrum of the density perturbation produced by chaotic inflation in a scalar-tensor theory can be a non-flat one because of the variable Brans-Dicke coupling. The spectrum observed by COBE-DMR thus constrains such a coupling. The constraint is stronger by an order-of-magnitude than those obtained by post-Newtonian experiments. It is interesting to note that the constraint hitherto obtained in the same simple coupling function as ours ($\alpha(\varphi) = \beta\varphi$) by binary pulsar experiment[20] ($\beta > -5$) is 'loose' in a sense (α_0 must be $\alpha_0 < 10^{-1000}(!)$) so that the constraint Eq.(57) or Eq.(58) can be looser than $\beta > -5$. Thus we may conclude that cosmological inflation is suited in the framework of general relativity, or more advocatively, inflation favors Einstein. Although our results have been derived by assuming a simple coupling function, we expect that they hold in general classes of coupling functions. In this case, however, β in the constraints Eq.(57) and Eq.(58) may be referred to that at the end of inflation.

We have assumed that gravitational waves do not make significant contribution to large scale power spectrum. For example, even for power law inflation with $n = 0.8$ gravitational waves contribute 50% to the temperature fluctuation on 10 degree scale[21]. Detailed study of the spectrum of gravitational waves in inflationary models involving multiple scalar fields is necessary and is left as a future important problem.

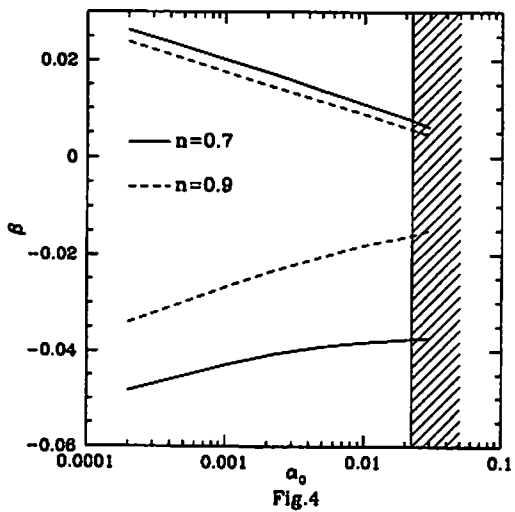
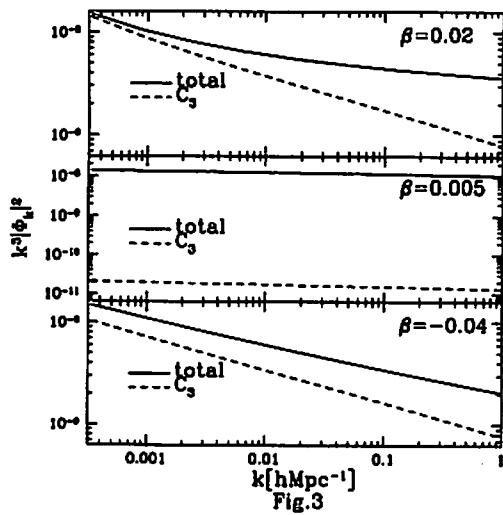
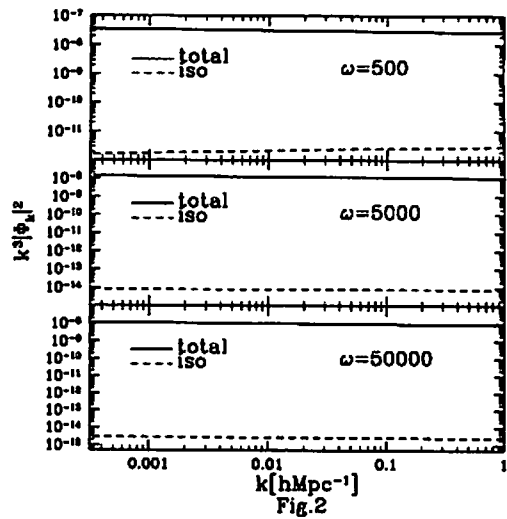
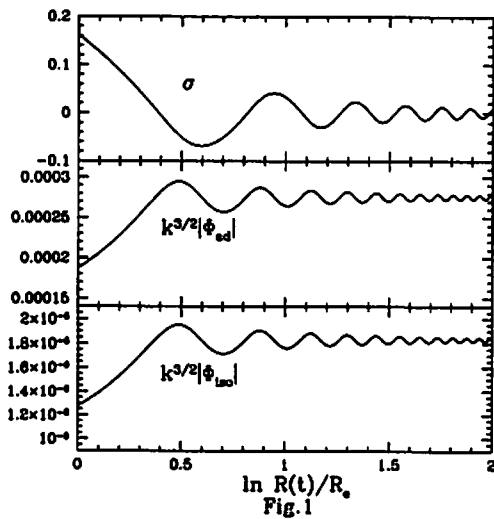
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FIGURE CAPTION

- Fig.1. Behavior of the inflaton and curvature perturbations during field oscillation regime in the Brans-Dicke theory with $\omega = 500$. The abscissa is the e-folding number after the end of inflation.
- Fig.2. The spectra of perturbations in Brans-Dicke theory with $\omega = 500, 5000, 50000$. The abscissa is the wavenumber in unit of $h\text{Mpc}^{-1}$. logarithm is base 10. The isocurvature mode is negligible in this theory.
- Fig.3. The spectra of perturbations in scalar-tensor theories with $\alpha_0 = 1.0 \times 10^{-3}$ and $\beta = 0.02, 0.005, -0.04$. For larger $|\beta|$, the mode coming from C_3 part in Eq.(36) is not negligible and the spectrum has slope. The respective spectral index is $n = 0.724, 0.967, 0.734$.
- Fig.4. The contour plot of spectral index n for scalar-tensor theories. The contour levels $n = 0.9, 0.7$ are shown. We find non-zero β theories are strongly constrained. The shaded region is the constraint by the post-Newtonian experiment ($\alpha_0 < \sqrt{5} \times 10^{-2}$).



Gravitational Lenses in the Clumpy Universe and the Determination of Cosmological Parameters

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Abstract

We investigate the cosmological test recently used by B. Fort, Y. Mellier and M. Dantel-Fort (FMD), where the observed location of the critical line in gravitational lensing is used to determine the cosmological parameters, Ω and λ . Applying this method to the cluster of galaxies Cl0024+1654, FMD put a bound on the cosmological constant, $\lambda > 0.6$, assuming the spatially flat universe. It plays a crucial role in this method that the angular diameter distance-redshift relation depends on the cosmological models through the cosmological parameters. We discuss the effect of the clumpiness of the universe by using the Dyer-Roeder angular diameter distance. It is shown that the effect of the clumpiness can become too large to be ignored, particularly for a high density universe. We also find the important property that the clumpiness changes monotonically the ratio of two distances: the distance from the lens to the source and that from the observer to the source. We discuss the implication of this property to the cosmological test. Moreover, we mention the uncertainty of the determination of the velocity dispersion, which is regarded as one of the most serious problems in this test.

1 Introduction

In understanding the Universe, it has been one of central issues to determine the cosmological parameters; the density parameter Ω and the cosmological constant λ [1, 2]. Although some cosmological tests have been proposed and applied, the conclusive result has not been obtained yet.

The angular diameter distance-redshift relation depends on cosmological parameters. Thus, it is possible to determine cosmological parameters through the dependence, if one can measure the redshift and the apparent size of an object whose physical size is known by another independent method. B. Fort, Y. Mellier and M. Dantel-Fort (FMD) have used the relation between the mass of the gravitational lens (cluster of galaxies) and the location of the critical line [3]. Using the data of cluster of galaxies Cl0024+1654, they obtained a constraint $\lambda > 0.6$, as a preliminary result. Their procedure is as follows (See section 2 for more details): From the observation of a lens object (which is nearly axisymmetric along a line of sight) and background galaxies, one can determine the velocity dispersion, the angular core radius and the maximum angular radius of critical lines. In particular,

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the critical line, defined as the location where large amplification of images occurs [4], can be determined observationally as the location where the number density of background galaxies changes drastically [3]. The location of the critical line depends not only on the mass distribution of the lens but also on the ratio between two distances: the angular diameter distance from the lens to the source and that from the observer to the source. The mass distribution of the lens is usually estimated from the velocity dispersion and the core radius. When the redshift of the source is between 3 and 5, the distance ratio depends strongly on the cosmological parameters but is insensitive to the redshift of the source. Thus the cosmological parameters can be determined by this dependence. FMD have studied the case of the angular diameter distance in the Friedmann-Lemaitre-Robertson-Walker (FLRW) universe. However, the inhomogeneities of the universe may disturb this test by bringing uncertainties to the angular diameter distance-redshift relation. We also investigate this effect by using the Dyer-Roeder (DR) angular diameter distance [5, 6].

This paper is organized as follows. In section 2, the method by FMD is explained. In section 3, the DR angular diameter distance is used in order to take into consideration the effect of clumpiness of the universe. In particular, we prove the property that the clumpiness changes monotonically the ratio of two distances. Section 4 is devoted to discussion and summary.

2 The location of the critical lines and cosmological parameters

In this section, we would like to summarize the method to determine the cosmological parameters by the use of the maximum radius of the critical lines in the gravitational lens. It is assumed that the mass profile of the cluster of galaxies is that of the isothermal sphere with core [3]. Then the mass profile of the cluster is written as

$$\rho(r) = \rho_0 \left(1 + \frac{r^2}{r_c^2}\right)^{-1}, \quad (1)$$

where r_c is a core radius, $\rho_0 = \sigma^2/2\pi G r_c^2$ and σ is a line-of-sight velocity dispersion. The mass within the radius R on the lens plane is expressed as

$$M(R) = \frac{\pi \sigma^2}{G} \left(\sqrt{R^2 + r_c^2} - r_c \right), \quad (2)$$

where R is a radius from the center of the cluster on the lens plane. In the spherically symmetric case such as the isothermal sphere, the critical line appears on the Einstein ring [4];

$$\theta_{cr} = \theta_E = \sqrt{\frac{4GM}{c^2} \frac{D_{LS}}{D_{OL}D_{OS}}}, \quad (3)$$

where θ_{cr} and θ_E are the angular radius of the critical line and that of the Einstein ring, and D_{OL} , D_{OS} and D_{LS} are the angular diameter distances between the observer and the lens, between the observer and the source, and between the lens and the source,

respectively. In the case of isothermal sphere, we may use Eq.(2) to obtain

$$\theta_{cr} = \sqrt{\frac{4\pi\sigma^2}{c^2}((R_{cr}^2 + r_c^2)^{1/2} - r_c)} \frac{D_{LS}}{D_{OL}D_{OS}}, \quad (4)$$

where R_{cr} is the radius of the critical line. It should be noted that the observable is not the radius of the critical line but its angular radius, $\theta_{cr} = R_{cr}/D_{OL}$. From Eq.(4), we obtain the important ratio [3], which depends only on the observables,

$$\frac{D_{LS}}{D_{OS}} = \frac{c^2}{4\pi\sigma^2}((\theta_{cr}^2 + \theta_c^2)^{1/2} + \theta_c), \quad (5)$$

where we defined $\theta_c = r_c/D_{OL}$. Usually, the radius of the critical line (several tens arcsec, several Mpc) is much larger than the core radius (less than Mpc). Then we can use the approximate relation

$$\frac{D_{LS}}{D_{OS}} \approx \frac{c^2}{4\pi\sigma^2}\theta_{cr}. \quad (6)$$

It is noteworthy that the angular diameter distance has not been specified. Since the angular radius of the critical line θ_{cr} , the line-of-sight velocity dispersion in the cluster σ and the angular core radius θ_c can be obtained observationally, the ratio D_{LS}/D_{OS} can be determined from these data. On the other hand, the distance ratio D_{LS}/D_{OS} as well as the angular diameter distances, D_{LS} and D_{OS} , depends on the cosmological models. For instance, in the FLRW universe, the relation between the redshift and the distance ratio depends only on the cosmological parameters, Ω and λ [1, 2]. It should be noted that D_{LS}/D_{OS} is sensitive to Ω and λ , but insensitive to the redshift of the source from 3 to 5. Therefore, if one obtains observationally the maximum of D_{LS}/D_{OS} , the cosmological parameters can be constrained without the precise measurement of redshift of background faint galaxies [3]. In order to constrain the cosmological parameters, it is necessary to obtain D_{LS}/D_{OS} within about ten percent accuracy, for Cl0024+1654 at $z_L = 0.39$.

From Eq.(3), it is clear that the cosmological test discussed here relies on (1) which angular diameter distance to be used and (2) the mass of the cluster as the gravitational lens. We investigate mainly the effect of the first aspect on the cosmological test. In particular, we pay much attention to the distance ratio D_{LS}/D_{OS} . It is worthwhile to mention that the cosmological test discussed here does not depend on the Hubble constant because of our using only the distance ratio D_{LS}/D_{OS} .

3 Cosmological test in the clumpy universe

On a sufficiently large scale, our universe can be described by the FLRW universe. Nevertheless, the light ray propagates from the source to the observer according to the geodesic equation which is determined by the local metric along the light ray [1, 2]. Therefore, it is likely that the light propagation is affected by the inhomogeneities of the universe. Thus, we consider how much the inhomogeneities affect the ratio of angular diameter distances D_{LS}/D_{OS} .

We consider the DR angular diameter distance, for its simplicity, in order to take into account the inhomogeneities [5, 6, 7, 8, 4, 9]. The DR angular diameter distance is determined by

$$\frac{d^2}{dw^2}D + \frac{3}{2}(1+z)^5\alpha\Omega D = 0, \quad (7)$$

where the parameter α represents the clumpiness along the light ray. In the FLRW universe, α is unity, while α vanishes when the light ray propagates through the empty space. In this paper, we consider $0 \leq \alpha \leq 1$, since we have little knowledge of α . Here w is an affine parameter, which is assumed [4] to be given by that in the FLRW universe

$$\frac{dz}{dw} = (1+z)^2 \sqrt{\Omega z(1+z)^2 - \lambda z(2+z) + (1+z)^2}. \quad (8)$$

It is noteworthy that the coefficient of the last term of Eq.(7), $3\alpha\Omega/2$, comes from the Ricci focusing by the clumpy matter and, therefore, the DR angular diameter distance is a decreasing function of α for a fixed redshift [5, 6, 4].

By noticing that α enhances monotonically the Ricci focusing in Eq.(7), one expects that a (nontrivial) relation for $\alpha_1 < \alpha_2$ as

$$\frac{D_{LS}}{D_{OS}}(\alpha_1) < \frac{D_{LS}}{D_{OS}}(\alpha_2). \quad (9)$$

This is proved as follows [10]: Let us fix the redshift of the source and the cosmological parameters, Ω and λ . Then we define a function of z_L parameterized by α as

$$X_\alpha(z_L) = \frac{D_{LS}}{D_{OS}}(\alpha). \quad (10)$$

From Eq.(7), we obtain the equation for $X_\alpha(z_L)$

$$\frac{d^2}{dw_L^2}X_\alpha(z_L) + \frac{3}{2}(1+z_L)^5\alpha\Omega X_\alpha(z_L) = 0, \quad (11)$$

where w_L is an affine parameter at the lens. We define the Wronskian as

$$W(X_{\alpha_1}, X_{\alpha_2}) = \left(X_{\alpha_1} \frac{dX_{\alpha_2}}{dw_L} - X_{\alpha_2} \frac{dX_{\alpha_1}}{dw_L} \right). \quad (12)$$

Using Eq.(11), we obtain

$$\frac{d}{dw_L}W(X_{\alpha_1}, X_{\alpha_2}) < 0, \quad (13)$$

where we used $\alpha_1 < \alpha_2$. Since both X_{α_1} and X_{α_2} vanish at $z_L = z_S$, we obtain

$$W(X_{\alpha_1}(z_S), X_{\alpha_2}(z_S)) = 0. \quad (14)$$

From Eqs.(13) and (14), we find

$$W(X_{\alpha_1}, X_{\alpha_2}) > 0, \quad (15)$$

where we used the fact that the affine parameter w defined by Eq.(8) is an increasing function of z . Eq.(15) is rewritten as

$$\frac{d}{dw_L} \ln \frac{X_{\alpha_2}}{X_{\alpha_1}} > 0. \quad (16)$$

Since X_α always becomes unity at the observer, we find

$$\ln \frac{X_{\alpha_2}(z_L = 0)}{X_{\alpha_1}(z_L = 0)} = 0. \quad (17)$$

From Eqs.(16) and (17), we find

$$\ln \frac{X_{\alpha_2}}{X_{\alpha_1}} > 0. \quad (18)$$

This gives us what we wish to prove.

It should be noted that Eq.(18) holds even if one uses the opposite sign in the definition of the affine parameter in Eq.(8). Eq.(9) means that the distance ratio D_{LS}/D_{OS} for $0 < \alpha < 1$ takes a value between those for $\alpha = 0$ and 1, namely for $0 < \alpha < 1$,

$$\frac{D_{LS}}{D_{OS}}(\alpha = 0; \text{Empty}) < \frac{D_{LS}}{D_{OS}}(\alpha) < \frac{D_{LS}}{D_{OS}}(\alpha = 1; \text{FLRW}). \quad (19)$$

Therefore, all we have to do is to investigate two limiting cases $\alpha = 0$ and 1, in order to know a region where D_{LS}/D_{OS} for $0 \leq \alpha \leq 1$ takes its value for given z_L , Ω and λ . The ratio D_{LS}/D_{OS} for $\alpha = 0$ and 1 has been evaluated numerically, for instance, by Fukugita, Futamase, Kasai and Turner [11]. Eq.(19) shows that, for the light propagation through a lower density region such as a void, the distance ratio D_{LS}/D_{OS} is always less than that in the FLRW universe.

4 Discussion and Summary

We have used the DR angular diameter distance, in order to investigate how much the clumpy universe affects the determination of the cosmological parameters by the method using the location of the critical lines in the gravitational lens. It is shown in section 3 that the distance ratio D_{LS}/D_{OS} in a clumpy universe is necessarily less than that in the FLRW universe. This implies that the lower bound on λ is given by the use of the angular diameter distance in the FLRW universe.

In the above, we have considered only the dependence of D_{LS}/D_{OS} on the cosmological parameters. Here, we briefly mention the dependence of this test on astrophysical models of the cluster of galaxies. As shown in Eq.(5), the precise determination of the velocity dispersion is necessary even if one assumes the isothermal sphere as the lens. If one specifies the redshifts of the lens and the source, the accuracy required for the determination of the velocity dispersion depends on which cosmological models we wish to compare with. If one wishes to distinguish, for instance, the cosmological model with $(\Omega, \lambda) = (0.2, 0)$ from that with $(\Omega, \lambda) = (0.2, 0.8)$, one must distinguish $D_{LS}/D_{OS} = 0.64$ from 0.76 for $z_L = 0.5$ and $z_S = 4$ [10]. This means that one must discriminate the difference of D_{LS}/D_{OS} at

worst by 16 percent. However, the statistical error of the velocity dispersion is at least about ten percent because of the limited number (about a hundred) of galaxies inside the radius of the critical line, even though the velocity of individual galaxies can be measured precisely. This required accuracy is almost unchanged around $z_L = 0.5$. Thus, because of the statistical error of the determination of the velocity dispersion, it is concluded that the current data of Cl0024+1654 does not rule out the cosmological model without λ . However, in the statistical sense, it is possible to put a bound on Ω and λ , if one can apply this method to a lot of clusters of galaxies. For instance, if one uses ten clusters of galaxies, the statistical error of the velocity dispersion becomes less than four percent, so that one may distinguish some cosmological models, say with $(\Omega, \lambda) = (0.2, 0)$ and $(0.2, 0.8)$.

Furthermore, the mass profile used as the lens model must be confirmed in order to make this cosmological test reliable. Thus it is of great importance that the mass or the mass profile of the lens is determined precisely. It is hoped that the mass reconstruction by the weak gravitational lensing [12, 13, 14, 15] as well as the analysis of the velocity field in the cluster of galaxies [16] clarifies the precise mass profile of the cluster as the lens. Then the cosmological test discussed here will work well.

Acknowledgments

I would like to thank M. Sasaki and T. Tanaka for useful discussion. I would like to thank M. Kasai for pointing out a previous work on the distance ratio. I would like to thank B. F. Roukema for pointing out the recent work by FMD. I would like to thank S. Ikeuchi for useful advice and continuous encouragement.

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Gravitational lensing by clusters of galaxies and a new test for the cosmological constant

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Abstract

Highly amplified but less distorted images of distant background galaxies due to gravitational lensing by foreground clusters of galaxies are investigated. Using 16 clusters of galaxies in EMSS sample with $z > 0.2$ and $L_X > 4 \times 10^{44} \text{ erg sec}^{-1}$ for a sample of clusters, which contains six giant luminous arcs, we find that the expected number of highly amplified but less distorted images in the sample is comparable to the one of giant luminous arcs. We also find that the redshift range of source galaxies where such events take place is rather sensitive to the cosmological parameters, to the cosmological constant in particular. Therefore such events can be used as a test for the cosmological constant as well as a tool to explore the nature of distant galaxies.

1 Introduction

After the discovery of the first multiply imaged quasar [1] and the first observations of gravitational giant arc [2, 3] and arclets [4], gravitational lensing becomes one of the most active field in the observational cosmology. Gravitational lensing makes itself evident through the multiplication or elongation of the original distant image. Moreover, the lensing amplification effect, though not directly observable, could make intrinsically small and faint sources observable.

Recently Yee et al. (1996) [5] reported the serendipitous discovery of a protogalaxy candidate, designated cB58, at a high redshift of $z = 2.72$. Ellingson et al. (1996) [6] have analyzed cB58 and concluded that cB58 can be regarded as the most active star-forming galaxy ever discovered.

However, one should consider the possible effect of amplification by gravitational lensing because, in projection, cB58 lies close ($\sim 6''$) to the center of the cluster of galaxies MS1512.4+3647 at $z = 0.373$. Williams and Lewis (1996) [7] claimed that the phenomenal properties of this protogalaxy candidate are in fact due to amplification by gravitational lensing by the foreground cluster, and that the unlensed properties of the source are typical of high-redshift starforming systems.

The lensing amplification is, however, rather sensitive to the assumed cluster mass distribution model. Therefore, detailed knowledge of the cluster mass distribution is very important to quantify the amplification by lensing. Hamana et al. (1997) [8] quantified the lensing amplification of the cB58 based on recent ROSAT and ASCA X-ray observations

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of MS1512.4+3647. They found that the amplification is at most 25, and the lensed image is less distorted.

Motivated by the study of the gravitational lensing effects on cB58, Futamase et al. (1997) [9] pointed out a new class of gravitational lensing events, namely, the highly amplified but less distorted gravitational lensed image. Henceforth we shall call such an event as HALD event. Since the lensing effects act like, so to say, a telescope, such an event may provide us a fruitful information on the distant source object. It was also pointed out that the redshift range of the source galaxies where the events take place is rather sensitive to the cosmological parameters, to the cosmological constant in particular [9]. Therefore such events can be used as a test for the cosmological constant as well as a tool to explore the nature of distant galaxies.

Since such a lensing event may play an important role in probing not only the distant objects but also cosmological parameters, it is worth while to investigate the properties of the lensing event.

We first briefly review the basic theory of the gravitational lensing. Next we discuss the applications of HALD lensing events to the astrophysics. We also discuss a new test for the cosmological parameters.

Throughout this paper, the Hubble constants H_0 is taken to be $50 \text{ km sec}^{-1} \text{ Mpc}^{-1}$.

2 Basic theory of the gravitational lensing

In this section, we briefly review the basic theory of the gravitational lensing (see for detail e.g. Schneider, Ehlers and Falco, 1992, [10]). In the following, the notations are basically followed by [10].

Let us define the appropriately scaled, projected Newtonian potential of the lens $\Psi(\vec{\theta})$ as follows,

$$\Psi(\vec{\theta}) = \frac{D_{ds}}{D_d D_s} \frac{2}{c^2} \int \Phi(D_d \vec{\theta}, z) dz. \quad (1)$$

Using this potential, the lens equation is written as

$$\vec{\beta} = \vec{\theta} - \vec{\nabla} \Psi. \quad (2)$$

If we take the source as the circular disk, the lens equation may be regarded as the mapping from the circular source to elliptical image. The elongations of the image along the major and minor axis are given by

$$\Lambda_+ = |1 - \kappa - \gamma|^{-1}, \quad (3)$$

$$\Lambda_- = |1 - \kappa + \gamma|^{-1}. \quad (4)$$

Here κ and γ are the convergence and shear, respectively, which are given by second partial derivatives of the potential,

$$\kappa = \frac{1}{2}(\Psi_{,11} + \Psi_{,22}), \quad (5)$$

$$\gamma = \sqrt{\gamma_1^2 + \gamma_2^2}, \quad (6)$$

where $\gamma_1 = (\Psi_{,11} - \Psi_{,22})/2$ and $\gamma_2 = \Psi_{,12} = \Psi_{,21}$.

Now we introduce a parameter R which describes a distortion of the image,

$$R \equiv \frac{\Lambda_+}{\Lambda_-}. \quad (7)$$

The lensing amplification is also written in terms of Λ_+ and Λ_- ,

$$A = \Lambda_+ \Lambda_-. \quad (8)$$

It should be noted that the distortion and amplification of the image are determined by fixing redshifts of the lens and source as well as image position θ and the density distribution of the lens. They are, in principle, observable. Therefore, the amplification and distortion of the image may be determined without the information on the unlensed source.

Using A and R , HALD lensed image is expressed by the conditions $R \sim 1$ and $A \gg 1$. For the event with $\kappa \sim 1$, the amplification and distortion of the image become $A \sim 1/\gamma^2$ and $R \sim 1$, respectively. In that case, if the density distribution of the lensing cluster is not strongly eccentric, γ is sufficiently small at close to the center of the lens. This is an essential physical mechanism to creat HALD event.

3 Properties of highly amplified but less distorted lensing events

In this section, we briefly summarize some basic properties of HALD lensing events [9].

3.1 Expected number

The expected number of the highly amplified but less distorted lensing events is examined. As an example of the prediction, we took 16 EMSS distant X-ray luminous clusters of galaxies as the sample of clusters of galaxies in which six giant luminous arcs are discovered [11].

For simplicity, we take the King model as the density distribution of the lensing clusters. The King model is characterized by one-dimensional velocity dispersion and core radius [12]. In order to convert the X-ray luminosity of the clusters in our sample to the velocity dispersion, we employ the quasi Faber-Jackson relation for the X-ray selected clusters [13]. We take a simplifying assumption that the core radius is fixed by 100kpc.

We give criteria for HALD lensing event as; $A > 10$ and $R < 3$. The detection criteria are taken by $m_V < 22$ and the diameter of a lensed image along with the shortest axis has to exceed the seeing FWHM taken by $0''.8$.

For the source galaxies, we use the galaxy luminosity function based on Canada-France redshift survey [14]. We take a simplifying assumption that the luminous area of background galaxy is taken to be a circular disk whose radius is related to its absolute magnitude by the so-called Freeman law [15]. K-corrections are calculated by using SEDs given in [16]. Because of the validity of the redshift range of the galaxy luminosity function, the calculation is restricted up to $z = 1.3$.

Table 1: Expected number of the highly amplified but less distorted lensed images (HALD) and giant luminous arcs (GLA) in the 16 EMSS distant X-ray luminous cluster of galaxies for three universe models.

(Ω_0, λ_0)	(1.0, 0.0)	(0.1, 0.0)	(0.1, 0.9)
HALD	1.5	1.6	2.5
GLA	0.32	0.39	1.9

We take the following three universe models; $(\Omega_0, \lambda_0) = (1.0, 0.0)$, $(0.1, 0.0)$ and $(0.1, 0.9)$. The results are summarized in Table 1. For comparison, we also calculate the expected number of giant luminous arcs in the sample clusters, and list the results in Table 1.

Comparing the expected number of giant luminous arcs with observed one (six events), the zero- λ_0 universe models should be rejected. The expected number of giant luminous arcs is somewhat smaller than observed one even though taking the universe model with large- λ_0 into account. This discrepancy is partly due to the restricted redshift range of our calculation and partly due to the simplifying assumptions we have taken. It has been pointed out that the contribution of the source galaxies beyond $z = 1.3$ is not negligible [17]. Therefore we conclude that in the case of the universe model with large- λ_0 , the discrepancy is not serious.

According to our calculation, it is expected that the expected number of HALD lensing events is comparable with that of giant luminous arcs. We also find that the most of HALD lensing events are expected to take place within the core radius of the cluster of galaxies.

3.2 New test for the cosmological constant

In the case of HALD lensing events, the redshift range of the source galaxies is severely restricted. The ranges are dependent on universe models because such events take place only when κ and γ taken appropriate values and those parameters vary with redshifts of the source galaxies. The dependence of the universe model is mainly due to a combination of angular diameter distances, D_{ds}/D_s , appeared in the lens equation. We should note that the combination is strongly dependent on the cosmological constant but weakly dependent on the density parameter as was pointed out by [18]. Moreover the combination is not dependent on the Hubble constant.

Once the event is discovered, and the redshifts of the lens and source as well as image position and density distribution of the lens cluster of galaxies are fixed by observations, the cosmological parameters, cosmological constant in particular, are constrained by the redshift of the source galaxy.

3.3 Applications to astrophysics

The amplification effect enables us to observe objects which are too distant or intrinsically too faint to be observed without lensing. Therefore lenses allow us to get the information on source objects which are intrinsically far below the resolution limit or sensitivity limit of current observations.

In the most case of giant luminous arcs, the width of the arcs is below the seeing FWHM. Therefore the images can not be resolved. They provide us with poor informations on the distant lensed objects.

On the other hand, HALD lensing events are an ideal for probing distant or faint objects. The lensed objects are not stretched along with one direction but are expanded to the whole direction. Therefore they may provide us fruitful informations on distant source objects.

4 Conclusions

We have examined the basic properties of the HALD lensing events. We emphasize the following three points;

- The number of such events is expected as many as that of giant luminous arcs.
- Such events may be used as a test for the cosmological constant.
- Such events have a potential ability to explore the nature of distant source objects.

To date, cB58 is only candidate for such events. Since the events are hardly identified by their morphology, the redshift survey of galaxies within the clusters should be necessary to discover another candidates of the events.

Acknowledgments

We would like to thank Y. Shioya and T. Yamada for valuable discussions.

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CMB Anisotropies Produced by Voids

– Effect of Gravitational Lensing –

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Abstract

We study the effect of gravitational lensing by a void on CMB anisotropy by using a model of an empty void in an Einstein-de Sitter universe. Using Thompson & Vishniac's results, we make an analytic estimate of the anisotropy. Although only the redshift fluctuations of single photons have been taken into account in the previous works, we find that the effect of gravitational lensing is dominant in CMB anisotropy. For a void with its diameter being $\gtrsim 10\text{Mpc}$, the temperature fluctuations caused by gravitational lensing are of the order $\Delta T/T \gtrsim 10^{-4}$ in the model, which implies the importance of nonlinear effects on the small-scale anisotropy of CMB.

1 Introduction

The gravitational influence of nonlinear voids on the cosmic microwave background (CMB) has been studied in the literature [1]–[4]. Thompson & Vishniac analytically estimated the anisotropy produced by an empty void in an Einstein-de Sitter background using the thin-shell approximation in general relativity [1]. The same results were also obtained by Martinez-Gonzalez, Sanz & Silk in a different scheme using the “potential approximation” [3]. Another method is based on the Lemaitre-Tolman-Bondi solution; the anisotropies in more general models has been investigated by several authors [4].

In all the previous works, temperature fluctuations $\Delta T/T$ have been estimated only from the redshift fluctuations of a photon $\Delta\nu/\nu$. If we take the effect of gravitational lensing into account, however, the number of photons which are received by a detector per solid angle is perturbed. This also contributes on $\Delta T/T$ if we estimate the temperature by flux of light. In this paper we demonstrate that this latter effect dominates the former by using a model of an empty void in an Einstein-de Sitter universe.

The plan of this paper is as follows. In §2, we briefly review the method and results of Thompson & Vishniac [1]. In §3, using their results, we calculate temperature fluctuations caused by the effect of gravitational lensing. §4 is devoted to summary and discussions.

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2 Thompson & Vishniac's Results

Thompson & Vishniac [1] analytically calculated the redshift fluctuations of a photon passing through a void. The model they considered is as follows. The background universe is an Einstein-de Sitter spacetime:

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2) \quad \text{with} \quad a(t) = \left(\frac{t}{t_o}\right)^{\frac{2}{3}}. \quad (1)$$

The void itself is an empty spherical region, and hence a Minkowski spacetime. The matter surrounding a void is assumed to form a thin shell. From momentum conservation and energy conservation, respectively, Maeda & Sato and Bertschinger showed that the shell radius expands asymptotically as [5]

$$r_v(t) \propto t^\beta \quad \text{with} \quad \beta \approx 0.13. \quad (2)$$

In Fig. 1 we schematically depict the trajectory of a photon in terms of the comoving frame. The subscripts 1 and 2 denote quantities at the time the photon enters the void and at the time it leaves, respectively. The subscripts *e* and *o* denote quantities at the last-scattering surface and at an observer, respectively. In this two-dimensional comoving space we denote each position by a vector symbol $\mathbf{r}$. What Thompson & Vishniac did is to derive the expression of the path and the energy of a photon from the time t_1 to t_2 . The momentum vectors just before entering a void and just after leaving it are, respectively, written as

$$k_1^\mu = E_1 \left(1, \frac{\cos \psi_1}{\sqrt{g_{rr,1}}}, \frac{\sin \psi_1}{\sqrt{g_{\theta\theta,1}}}, 0 \right), \quad k_2^\mu = E_2 \left(1, \frac{\cos \psi_2}{\sqrt{g_{rr,2}}}, \frac{\sin \psi_2}{\sqrt{g_{\theta\theta,2}}}, 0 \right). \quad (3)$$

With Lorentz transformation at t_1 and at t_2 , they obtained the redshift fluctuations:

$$\frac{\Delta\nu}{\nu} \equiv \frac{E_2}{E_{2,b}} - 1 = \frac{E_2}{E_1} \frac{E_{1,b}}{E_{2,b}} - 1 = \eta_2^3 \cos \psi_2 \left(\frac{8}{9}\beta - \frac{16}{81} \cos^2 \psi_2 \right) + O(\eta_2^4), \quad (4)$$

where the subscript *b* denotes background unperturbed quantities and we have defined

$$\eta \equiv \frac{a(t)r_v(t)}{t} \propto t^{\beta-\frac{1}{3}}. \quad (5)$$

They also calculated the scattering angle of a photon:

$$\delta \equiv -\psi_1 + \psi'_1 + \psi'_2 - \psi_2 = \frac{4}{9}\eta_2^2 \sin(2\psi_2) + O(\eta_2^3). \quad (6)$$

Equation (4) implies that the redshift fluctuations of a photon are of order η_2^3 ; if we suppose a void with its diameter being $50h^{-1}Mpc$, $\Delta\nu/\nu$ is less than 10^{-6} , which is not comparable to the current observed values. Although (6) has not attracted much attention, it inspires the authors with the idea that the effect of gravitational lensing on temperature fluctuations should be important because δ is of order η_2^2 .

In our calculation in §3, we also use the following relations.

$$\eta_1 = \eta_2 - 2\left(\beta - \frac{1}{3}\right) \cos \psi_2 \eta_2^2 + 2\left(\beta - \frac{1}{3}\right) \left[\left(\beta - \frac{2}{3}\right) \cos^2 \psi_2 + \beta \right] \eta_2^3 + O(\eta_2^4). \quad (7)$$

$$\psi_1 = \psi_2 + 2\beta \sin \psi_2 \eta_2 + O(\eta_2^2). \quad (8)$$

3 Effect of Gravitational Lensing

In this section we estimate temperature fluctuations induced by the effect of gravitational lensing. As mentioned in §1, the number of photons which are received by a detector per solid angle is perturbed by this effect. If photons are distributed homogeneously on the last-scattering surface, the total number of photons received by a detector per solid angle is proportional to the area S of the last-scattering surface the photons leave. This situation is schematically depicted in the comoving space in Fig. 2. ξ and y are defined by reference to Fig. 2. Because S is proportional to ξy , the temperature fluctuations caused by the effect of gravitational lensing are given by

$$\frac{\Delta S}{S} = \frac{\Delta \xi}{\xi} + \frac{\Delta y}{y}. \quad (9)$$

Our main task here is to obtain the position of any photon on the last-scattering surface which eventually reaches an observer: to find the expression of $\mathbf{r}_e - \mathbf{r}_o$. As shown in Fig. 1, we define α as the angle formed between the direction of observation and the direction of the void's center, and d as the comoving distance of the void's center. We introduce two vector bases, $\{\mathbf{e}_x, \mathbf{e}_y\}$ and $\{\mathbf{n}(\alpha), \mathbf{l}(\alpha)\}$, in the two-dimensional comoving space. The trajectory of a photon in a homogeneous region is easily obtained:

$$\mathbf{r}_2 - \mathbf{r}_o = \int_{t_2}^{t_o} \frac{dt}{a} \mathbf{n} = (1 - \sqrt{a_2}) \mathbf{n}, \quad (10)$$

$$\mathbf{r}_e - \mathbf{r}_1 = \int_{t_1}^{t_e} \frac{dt}{a} (\cos \delta \mathbf{n} + \sin \delta \mathbf{l}) = (\sqrt{a_1} - \sqrt{a_e}) (\mathbf{n} + \delta \mathbf{l}) + O(r_2^3). \quad (11)$$

To get the last equality of (11), we have used the facts that δ is of order η_2^2 and that η_2 is of order r_2 with a unit $3t_o = 1$. Explicitly, the geodesic between $\mathbf{r}_o$ and $\mathbf{r}_2$ says

$$\sin \psi_2 = \frac{d}{r_2} \sin \alpha, \quad (12)$$

$$\eta_2 = \frac{3r_2}{1 - d \cos \alpha + r_2 \cos \psi_2}, \quad (13)$$

and hence (6) is rewritten as

$$\delta = \frac{8\lambda\sqrt{1-\lambda^2}}{(1-d\cos\alpha)^2} r_2^2 + O(r_2^3) \quad \text{with} \quad \lambda \equiv \frac{d}{r_2} \sin \alpha. \quad (14)$$

This is just the expression which Martinez-Gonzalez, Sanz & Silk derived [3]. As for $\mathbf{r}_1 - \mathbf{r}_2$, we obtain

$$\mathbf{r}_1 - \mathbf{r}_2 = [r_1 \cos(\delta + \psi_1) + r_2 \cos \psi_2] \mathbf{n} + [r_1 \sin(\delta + \psi_1) - r_2 \sin \psi_2] \mathbf{l}. \quad (15)$$

We use (7) and (8) to expand $\delta + \psi_1$, r_1 and $\sqrt{a_1}$ with r_2 (or η_2), and then we get

$$\mathbf{r}_1 - \mathbf{r}_2 = 2r_2(\cos \psi_2 - \beta \eta_2) \mathbf{n} + O(r_2^3) = (\sqrt{a_2} - \sqrt{a_1}) \mathbf{n} + O(r_2^3). \quad (16)$$

Equation (16) tells us that, although a photon is bent inside the void, the deviation is of order r_2^3 . From (10)-(13), (16), and the relation,

$$\sqrt{a_1} = \sqrt{a_2} + O(r_2) = 1 - d \cos \alpha + O(r_2), \quad (17)$$

we find

$$\begin{aligned} \mathbf{r}_e - \mathbf{r}_o &= (1 - \sqrt{a_e})\mathbf{n} + (1 - d \cos \alpha - \sqrt{a_e})\delta \mathbf{l} + O(r_2^3) \\ &= (\mathbf{r}_e - \mathbf{r}_o)_b + (1 - d \cos \alpha - \sqrt{a_e})\delta \mathbf{l} + O(r_2^3). \end{aligned} \quad (18)$$

This result is quite understandable: as long as we look at the leading terms of r_2 , a void can be approximated to be a point object with a scattering angle δ .

Now it is easy to calculate y and ξ :

$$y = (\mathbf{r}_e - \mathbf{r}_o) \cdot \mathbf{e}_y = (1 - \sqrt{a_e}) \sin \alpha + (1 - d \cos \alpha - \sqrt{a_e})\delta \sin \alpha + O(r_2^3), \quad (19)$$

$$\xi = \left| \frac{d(\mathbf{r}_e - \mathbf{r}_o)}{d\alpha} \right| \Delta \alpha = \left[1 - \sqrt{a_e} + (1 - d \cos \alpha - \sqrt{a_e}) \frac{d\delta}{d\alpha} \right] \Delta \alpha + O(r_2^3). \quad (20)$$

Although r_2 also depends on α , $dr_2/d\alpha$ appears only in higher-order terms. The background unperturbed quantities are simply

$$y_b = (1 - \sqrt{a_e}) \sin \alpha, \quad \xi_b = (1 - \sqrt{a_e}) \Delta \alpha, \quad (21)$$

and hence we find $\Delta y/y$, $\Delta \xi/\xi$, and $\Delta S/S$:

$$\frac{\Delta y}{y} \equiv \frac{y}{y_b} - 1 = \frac{1 - d \cos \alpha - \sqrt{a_e}}{1 - \sqrt{a_e}} \delta \cot \alpha + O(r_2^3), \quad (22)$$

$$\frac{\Delta \xi}{\xi} \equiv \frac{\xi}{\xi_b} - 1 = \frac{1 - d \cos \alpha - \sqrt{a_e}}{1 - \sqrt{a_e}} \frac{d\delta}{d\alpha} + O(r_2^3), \quad (23)$$

$$\frac{\Delta S}{S} = \frac{1 - d \cos \alpha - \sqrt{a_e}}{1 - \sqrt{a_e}} \left(\frac{d\delta}{d\alpha} + \delta \cot \alpha \right) + O(r_2^3). \quad (24)$$

We should note that we have assumed in deriving the formula (6) that all the angles ψ_1 , ψ'_1 , ψ'_2 and ψ_2 are less than $\pi/2$. If one of them exceeds $\pi/2$ —actually ψ'_1 exceeds it first as α increases— Eq. (6) (or (14)) is not valid; neither is our result (24) accordingly. There is some critical value of α above which (24) is not true. Such range of α is, however, very small and does not contribute to CMB anisotropies in the observed angular scale.

The main result here is that the temperature fluctuations caused by the effect of gravitational lensing are of order r_2^2 , which implies that this effect is greater than that of redshift fluctuations of a photon in general. We show an example of the angular dependence of $\Delta y/y$, $\Delta \xi/\xi$, and $\Delta S/S$ in Fig. 3, where we choose $r_2 = 15h^{-1}\text{Mpc}$, $d = 50h^{-1}\text{Mpc}$, and $a_e^{-1} = 1 + z_e = 1000$. Fig. 4 indicates how the anisotropy depends on r_2 and d : $\Delta S/S(\alpha = 0)$ is proportional both to r_2 and to d . Even for not so large voids with $r_2 \sim 5h^{-1}\text{Mpc}$, $\Delta S/S(\alpha = 0)$ becomes greater than 10^{-4} .

4 Summary and Discussions

We have calculated the temperature fluctuations caused by the effect of gravitational lensing for a model of an empty void in an Einstein-de Sitter universe. We have made use of Thompson & Vishniac's results to obtain an analytic expression for the anisotropy.

In the previous works, CMB anisotropies have been estimated only by calculating the redshift fluctuations of a photon $\Delta\nu/\nu$. What we have found is, however, that the effect of gravitational lensing induces much larger fluctuations than $\Delta\nu/\nu$. For a void with its diameter being $\gtrsim 10\text{Mpc}$, the anisotropy caused by gravitational lensing are of the order $\Delta T/T \gtrsim 10^{-4}$, which is one order of magnitude higher than the observed values [6].

Of course, we have to investigate more realistic models to compare theoretical results with observational data. Statistical analysis for a multi-void universe by use of our analytic result is also important and in progress.

Acknowledgments

We would like to thank K. Maeda, N. Sugiyama, and K. Tomita for useful discussions. Thanks are also due to P. Haines for reading the manuscript. This work was supported partially by the Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Science and Culture (No.08740217) and by the Waseda University Grant for Special Research Projects.

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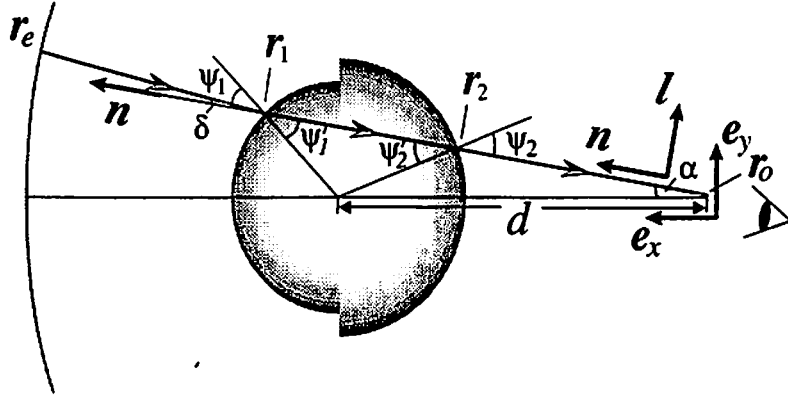


Fig.1 Cross section of a void on the $\varphi = \pi/2$ plane in the comoving frame. We depict the trajectory of a photon from the last-scattering surface to an observer. The subscripts 1 and 2 denote quantities at the time the photon enters the void and at the time it leaves, respectively. The subscripts e and o denote quantities at the last-scattering surface and at present, respectively. We denote each position by a vector symbol $\mathbf{r}$. We define α as the angle formed between the direction of observation and the direction of the void's center, and d as the comoving distance of the void's center. We introduce two vector bases, $\{e_x, e_y\}$ and $\{n(\alpha), l(\alpha)\}$, in the two-dimensional comoving space.

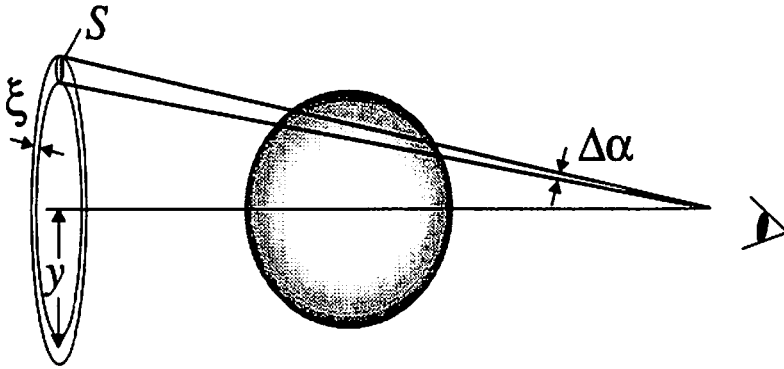


Fig.2 Schematic sketch of a null geodesic congruence. The total number of photons received by a detector per solid angle is proportional to the area S of the last-scattering surface the photons leave. S is proportional to ξy

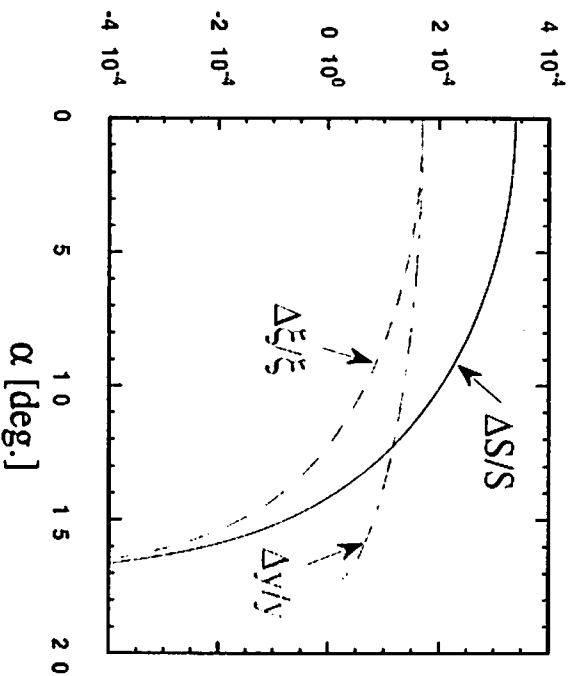


Fig. 3. Example of the angular dependence of CMB anisotropy $\Delta S/S$. For reference, we also plot $\Delta y/y$ and $\Delta \xi/\xi$. We choose $r_2 = 15h^{-1}\text{Mpc}$, $d = 50h^{-1}\text{Mpc}$, and $a_e^{-1} = 1 + z_e = 1000$.

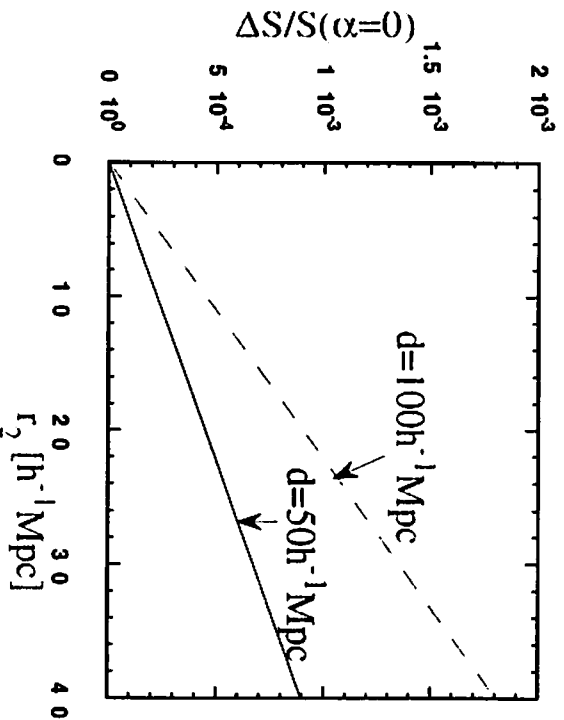


Fig. 4. CMB anisotropy $\Delta S/S$ at $\alpha = 0$; it is proportional both to r_2 and to d . Even for not so large voids with $r_2 \sim 5h^{-1}\text{Mpc}$, $\Delta S/S(\alpha = 0)$ becomes greater than 10^{-4} .

Anisotropies in Luminosity Distance

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Abstract

The fluctuation of luminosity distance caused by the large scale structure (LSS) is studied. We apply the formalism derived by Sasaki to some cosmological models including open models by linear perturbation methods. Numerical calculation is done on flat models. Some implications for determination of cosmological parameters are discussed.

1 Introduction

The largescale structure (LSS) of the universe has been a recent topic of great interest both in observational and theoretical cosmology. For these two decades, we have accumulated a large amount of observational data on the distribution of galaxies in the nearby universe by the redshift surveys. We also have recently succeeded in obtaining the data of cosmic microwave background (CMB) anisotropies by *COBE*. By analyzing these data, our understanding of the structure of the universe has made a significant progress, and we now know that our universe can be described well on large scales by a homogeneous and isotropic Friedman-Robertson-Walker (FRW) model with small inhomogeneities on it. About the parameters necessary to specify the model, however, no definite conclusion has so far been submitted in spite of the great amounts of researches from various approaches. One reason is, at present, the lack of our observational knowledge of the universe at rather high redshifts ($z > 1$), but there are currently two plans of further redshift surveys: the Sloan Digital Sky Survey (SDSS), which is to obtain redshifts for 10^6 galaxies and 10^5 quasars, and the Two-Degree Field Survey (2dF), which will measure 250,000 galaxies at even deeper redshifts over a limited region of the sky. Under these circumstances, it is certain that we will obtain in the near future much observational data which allow us to derive more quantitative information of the present features of the universe.

One important information we can obtain from such observations is the effect of the large-scale structure on the appearance of luminous sources. Sasaki discussed the effect of density inhomogeneities on luminosity distance d_L [1]. He derived a general formula to relate the inhomogeneities of the gravitational potential Ψ with fluctuations of luminosity distance, and applied it to the Einstein-de Sitter universe, giving an analytic expression. Now that we have obtained some information of Ψ from COBE data, it is worthwhile evaluating the amount of the effect in some cosmological models.

Another motivation for investigating the relation between the luminosity distance and the large-scale structure comes from the recent reports on the method to determine the cosmological parameters from the new observational data of supernovae (SNe) which have been systematically searched at high redshifts ($z \simeq 0.35 - 0.45$). By studying the magnitude-redshift relation of type Ia SNe, they give attempts to constrain the mass

parameter and the cosmological constant. In determining the cosmological parameters using the magnitude-redshift relation, one must use data for objects with the same absolute luminosity ("standard candles"). Past attempts to use galaxies as standard candles have failed due to large uncertainties in galaxy luminosity evolution. It is possible to determine peak luminosity of SNe accurately by studying the observed correlation between lightcurve shape and peak luminosity and spectroscopic features correlated with luminosity. The effective dispersion in peak absolute magnitude can be reduced to $\sigma_M \simeq 0.1 - 0.2$, thus the supernovae act as good standard candles.

There is a claim, however, that the large-scale structure of the universe prevents accurate determination of the cosmological parameters by that method[2]. A bundle of light rays from a distant source is focused due to gravitational lensing effects caused by the intervening density inhomogeneities. As a result, the image of the source appears magnified or demagnified compared with the homogeneous case, leading to uncertainties in estimating the luminosity distance.

Motivated by these works, we give in this paper a more detailed investigation of the anisotropies in luminosity distance caused by small perturbations on a FRW model. Following the general formula derived by Sasaki, we rewrite the relation between d_L and Ψ in the form which can also be used for an open universe. Then we made numerical evaluations of the anisotropies in d_L in a spatially flat universe with the cosmological constant, utilizing the information of the gravitational potential inhomogeneities obtained by the 2yr COBE data [3].

The anisotropies in the $d_L - z$ relation bring us two consequences. First, it leads to errors in determining the luminosity distance. This effect is important when an accurate measurement of the distance is needed, such as the case in determining the deceleration parameter.

We also want to point out that such anisotropy patterns give us information about the cosmological parameters because the magnitude and characters of these patterns are model dependent. Thus, if such anisotropy patterns are identified by succeeding deep surveys of the universe in the future, we will have another test for the cosmological model.

This paper is organized as follows. In Section 2, the perturbed $d_L - z$ relation is derived following Sasaki. Application to a spatially flat case is done in Section 3, with results of numerical calculations. In section 4, we give the formula in open models. The last section is devoted for discussions and concluding remarks.

2 Formalism

2.1 Basic equations

First we give basic equations used in the following sections.

We consider the propagation of light rays emitted by some source such as a stellar or a galaxy. It is well described by geometric optics since the curvature scale we are considering is much larger than the wavelength. First, the energy momentum tensor of a congruence of photons with 4-momentum $\hat{k}^\mu$ is given by

$$\hat{T}^{\mu\nu} = \frac{1}{8\pi} \mathcal{A}^2 \hat{k}^\mu \hat{k}^\nu. \quad (1)$$

where A is the scalar amplitude of the wave. We introduce a null vector k^μ conformal to $\hat{k}^\mu$ and an affine parameter with it, defined by

$$k^\mu = a^2 \hat{k}^\mu \equiv \frac{dx^\mu}{d\lambda}. \quad (2)$$

Using the equation of motion and the geodesic equation, one obtains the propagation equation for the scalar amplitude

$$\nabla_\mu (\mathcal{A}^2 a^2 k^\mu) = 2\mathcal{A}a \left[\frac{d}{d\lambda} (\mathcal{A}a) + \frac{1}{2} \mathcal{A}a\theta \right] = 0; \quad \theta = \nabla^\nu k_\nu. \quad (3)$$

where we define θ which is referred as the expansion. This equation is interpreted to represent the conservation of photon number. By differentiating this with respect to λ , one obtains the propagation equation for θ which is called Raychaudhuri equation

$$\frac{d}{d\lambda} \theta = -R_{\mu\nu} k^\mu k^\nu - \frac{1}{2} \theta^2 - 2\sigma^2; \quad \sigma^2 = \frac{1}{2} \left[k^{(\alpha;\beta)} k_{(\alpha;\beta)} - \frac{1}{2} \theta^2 \right], \quad (4)$$

where σ is called the shear of the congruence. Note that the vorticity vanishes in the geometric optic description.

Next, the luminosity distance is derived as follows. Absolute luminosity of a spherically symmetric source with a radius R_s is given by

$$L_s = 4\pi R_s^2 f(\lambda_s) \quad (5)$$

where λ_s is the conformal affine parameter (defined in Eq.(6)) evaluated at the source and f is the amplitude of the energy flux of the source measured by an observer comoving with the source whose 4-velocity is $\hat{u}^\mu$

$$f = \frac{1}{8\pi} \mathcal{A}^2 (-\hat{k}_\nu \hat{u}^\nu)^2 = \frac{1}{8\pi} \mathcal{A}^2 w^2. \quad (6)$$

Here we defined the energy of the photon w .

Luminosity distance to the source measured by an observer at $\lambda = 0$ is defined by

$$d_L^2 = \frac{L_s}{4\pi f(0)} = \frac{f(\lambda_s)}{f(0)} R_s^2 = \frac{\mathcal{A}^2 w^2|_s}{\mathcal{A}^2 w^2|_0} R_s^2 \quad (7)$$

Recalling the definition of redshift $w_s/w_0 = 1 + z$, this is written in terms of z

$$d_L = \frac{\mathcal{A}(\lambda_s)}{\mathcal{A}(0)} (1 + z(\lambda_s)) R_s \quad (8)$$

Thus $d_L - z$ relation is explicitly obtained if the other factors in the right-hand side of Eq.(8) is expressed in terms of z , as we will show in the succeeding sections. Note that this relation holds irrelevantly of inhomogeneities of the universe.

In the next section we express this relation in terms of the variables describing the fluctuations of the universe.

2.2 Anisotropies in d_L caused by LSS

If the universe were completely homogeneous and isotropic, the luminosity distance would be the same at the same redshift regardless of the direction one is looking at, i.e. $d_L - z$ relation has no anisotropies. However, the universe has small fluctuations which have led to the complex structure of the universe. Information of the magnitude of these fluctuations of matter density and gravitational potential has been obtained from the observation of anisotropies of CMB. The fluctuation of the gravitational potential give rise to anisotropies in the $d_L - z$ relation since the photons emitted from the source are reflected by gravitational force.

We utilize the usual perturbation technique; we solve the unperturbed equations first and proceed to the first order thereafter. For notational convenience, we denote a perturbed quantity with a tilde on top and an unperturbed one without the tilde in the following. For example, we express the metric as

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}, \quad \delta g_{0\mu} = (-2A, \beta_i), \quad \delta g_{ij} = H_{ij}. \quad (9)$$

For convenience, we introduce a new null vector $\tilde{K}^\mu$ by

$$\tilde{K}^\mu = -\frac{1}{\tilde{w}(\lambda_s)a[\tilde{\eta}(\lambda_s)]}\tilde{k}^\mu. \quad (10)$$

In the following, λ denotes a new affine parameter associated with $\tilde{K}^\mu$. We also introduce a new velocity variable by $u^\mu = a\tilde{u}^\mu$.

First let us consider the light propagation on the unperturbed background. The spatial part of the background metric is

$$g_{ij}dx^i dx^j = \gamma_{ij}dx^i dx^j = \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2. \quad (11)$$

The wavevector which is normalized correspondingly with (10) is

$$K^\mu = \frac{dx^\mu}{d\lambda} = (-1, \gamma^i). \quad (12)$$

Then inserting these quantities into Eq.(4), and taking $\sigma = 0$ (the shear is not important as long as we consider emissions from a point source), we obtain the following evolution equation of θ ;

$$\frac{d}{d\lambda}\theta = -2K - \frac{1}{2}\theta^2. \quad (13)$$

This is integrated to give

$$\theta_L = 2\sqrt{-K} \coth \sqrt{-K}(\lambda - \lambda_s - \Delta\lambda_s) \quad (14)$$

where the suffix L indicates the appropriate boundary condition for the light rays of interest. $\Delta\lambda_s$ is the infinitesimal affine parameter corresponding to the radius of the source: $R_s = a(\tilde{\eta}[\lambda_s])\Delta\lambda_s$.

Next we consider perturbed quantities. The perturbation of Eq.(13) gives

$$\frac{d}{d\lambda}\delta\theta_L = -\theta_L\delta\theta_L - \delta(R_{\mu\nu}K^\mu K^\nu)_\lambda. \quad (15)$$

Integrating this with the boundary condition $\delta\theta_L(\lambda_s) = 0$ we obtain

$$\delta\theta_L(\lambda) = \frac{1}{\text{sh}^2\sqrt{-K}(\lambda - \lambda_s - \Delta\lambda_s)} \int_\lambda^{\lambda_s} d\bar{\lambda} \text{sh}^2\sqrt{-K}(\bar{\lambda} - \lambda_s - \Delta\lambda_s) \delta(R_{\mu\nu}K^\mu K^\nu)_{\bar{\lambda}} \quad (16)$$

With the aid of Eq.(14), equation (3) is integrated to give

$$\frac{A(\lambda_s)a[\tilde{\eta}(\lambda_s)]}{A(0)a[\tilde{\eta}(0)]} = \frac{\text{sh}\sqrt{-K}(\lambda_s + \Delta\lambda_s)}{\text{sh}\sqrt{-K}\Delta\lambda_s} \exp\left[-\frac{1}{2} \int_0^{\lambda_s} d\lambda \delta\theta_L(\lambda)\right]. \quad (17)$$

Inserting Eq. (17) and the relation $R_s = a[\tilde{\eta}(\lambda_s)]\Delta\lambda_s$ into Eq.(8) and taking the point source limit $\Delta\lambda_s \rightarrow 0$, the perturbed luminosity distance is expressed as a function of λ ,

$$\tilde{d}_L(\lambda_s) = a[\tilde{\eta}(0)] \frac{\text{sh}\sqrt{-K}\lambda_s}{\sqrt{-K}} [1 + \tilde{z}(\lambda_s)] \exp\left[-\frac{1}{2} \int_0^{\lambda_s} d\lambda \delta\theta_L(\lambda)\right]. \quad (18)$$

For comparison with observational data, we should express this as a function of the redshift z . After some manipulations one obtains

$$\tilde{d}_L(z, \gamma^i) = d_L(z) \left[1 + \left(\frac{a'}{a} \delta\eta \right)_0 + \sqrt{-K} \coth\sqrt{-K}\lambda_s \delta\lambda_s - \frac{1}{2} \int_0^{\lambda_s} \delta\theta_L(\lambda) d\lambda \right] \quad (19)$$

where d_L is the luminosity distance on the unperturbed background

$$d_L(\eta) = a(\eta_0) \frac{\text{sh}\sqrt{-K}\lambda_s}{\sqrt{-K}} (1 + z), \quad (20)$$

and $\delta\lambda_s$ is given by

$$\delta\lambda_s = \delta\eta(\lambda_s) - \frac{1}{(a'/a)_s} \left[\frac{a'}{a} \delta\eta + \frac{d}{d\lambda} \delta\eta - A - (\beta_i + v_i) \gamma^i \right]_0. \quad (21)$$

The affine parameter $\lambda_s(z)$ is evaluated at z on the unperturbed universe, namely defined implicitly by the relation

$$1 + z = \frac{a[\eta(0)]}{a[\eta(0) - \lambda_s]}. \quad (22)$$

The variable v_i is the spatial component of perturbation of 4-velocity of a comoving observer which is taken to be

$$u^\mu = (1, 0) \quad \delta u^\mu = (-A, v^i). \quad (23)$$

Then, we define the fluctuation of the luminosity distance to be the difference between the perturbed luminosity distance and the the angular average of that compared at the same value of the redshift, that is,

$$\Delta d_L \equiv \tilde{d}_L(z) - \langle \tilde{d}_L(z) \rangle \quad (24)$$

where

$$\langle \tilde{d}_L(z) \rangle = \frac{1}{4\pi} \int d\Omega \tilde{d}_L(z) \quad . \quad (25)$$

We decompose Eq.(24) into multipole components associated with the conventional spherical hamonic function $Y_{lm}(\Omega)$ as usually done for analysing anisotropies of a physical qunatitiy, e.g. anisotropies of CMB ,

$$\frac{\Delta d_L}{d_L} = \sum_{l,m} C_{lm}(z) Y_{lm}(\Omega) \quad (26)$$

In the following two sections we derive the explicit expression of the perturbed luminosity distance in spatially flat models and open models.

3 Flat universe

3.1 Formalism

We assume the the background universe is spatially flat and matter dominated, and neglect the pressure term.

When $K = 0$ we can write from Eq.(19)

$$\frac{\Delta d_L}{d_L} = \frac{\delta \lambda_s}{\lambda_s} - \frac{1}{2} I_L \quad , \quad (27)$$

$$- \frac{1}{2} I_L = \frac{1}{2\lambda_s} \int_0^{\lambda_s} d\lambda (\lambda - \lambda_s) \lambda \delta(R_{\mu\nu} K^\mu K^\nu)_\lambda. \quad (28)$$

When the spacetime inhomogeneities are due to density perturbations, perturbations of all quantities are described by functions which are scalar with respect to spatial indices. We define such scalars $B, \mathcal{R}, H_T$, and v by

$$\beta_i =: B_{|i}, \quad H_{ij} =: 2\mathcal{R}\gamma_{ij} + 2H_{T|i j}, \quad v_i =: -v_{|i}. \quad (29)$$

(We have assumed that the velocities of the source and the observer are of cosmological origin.) According to the method of gauge-invariant cosmological perturbation theory, the metric perturbations are most conveniently described by the following variables

$$\Psi = A - \frac{1}{a} [a(H_T' - B)]', \quad \Phi = \mathcal{R} - \frac{a'}{a} (H_T' - B), \quad V = v - H_T' \quad . \quad (30)$$

These equations suggest that the longitudinal gauge $H_T' = B = 0$ is a convenient gauge for further calculations. Thus, taking the longitudinal gauge, we rewrite the expression of the fluctuation of luminosity distance in terms of Ψ . First, the perturbation of the curvature term in (28) is

$$\delta(R_{\mu\nu} K^\mu K^\nu)_\lambda = 2\Psi_{|i}^i + 2\Psi'' - 4\Psi'_{|i} \gamma^i \quad . \quad (31)$$

We used the relation $\Psi + \Phi = 0$ which follows from the fact that the pressure fluctuation is negligible.

To evaluate $\delta\lambda_s$, we need $\delta\eta_s$ and $\frac{d}{d\lambda}\delta\eta|_0$. These are obtained from the time component of the geodesic equation

$$\frac{d^2}{d\lambda^2}\delta\eta = 2\frac{d}{d\lambda}\Psi + 2\Psi' \quad (32)$$

Summing up all the terms, we finally obtain the expression for the fluctuation of the luminosity distance in terms of the gravitational potential

$$\begin{aligned} \frac{\Delta d_L}{d_L} = & \frac{1}{\lambda_s} \int_0^{\lambda_s} d\lambda \left[(\lambda - \lambda_s) \lambda (\Psi_{|i}^{|i} + \Psi'' - 2\Psi'_{|i}\gamma^i) \right] + \left(\frac{2}{\lambda_s(a'/a)_s} - 2 \right) \int_0^{\lambda_s} \Psi'(\eta) d\lambda \\ & + \frac{2}{\lambda_s} \left[\int_0^{\lambda_s} d\lambda \Psi + \int_0^{\lambda_s} \int_0^\lambda \Psi'(\bar{\eta}) d\bar{\lambda} \right] - \frac{1}{\lambda_s(a'/a)_s} (\Psi_0 - \Psi_s) - \Psi_s \\ & - \frac{1}{\lambda_s(a'/a)_s} (V_{0|i}\gamma^i - V_{s|i}\gamma^i) - V_{s|i}\gamma^i \end{aligned} \quad (33)$$

where λ_s is defined by Eq.(22) and the velocity V is given in terms of Ψ through the relation

$$V = \frac{2}{3} \left(\frac{a\Psi}{H_0^2 \Omega_0 a_0} \right)' \quad (34)$$

For computation of C_{lm} it is convenient to expand Ψ using $Y_{lm}(\Omega)$ as

$$\Psi(\mathbf{x}) = \frac{2}{\pi} \sum_{lm} Y_{lm}(\Omega) \int k^2 dk \Psi_{lm}(k) j_l(kr) \quad (35)$$

Then one immediately obtains the expression of C_{lm} :

$$C_{lm} = \int d\Omega \frac{\Delta d_L}{d_L} Y_{lm}^*(\Omega) = \frac{2}{\pi} \int k^2 dk G_{lm} \quad (36)$$

$$\begin{aligned} G_{lm} = & \frac{1}{\lambda_s} \int_0^{\lambda_s} d\lambda (\lambda - \lambda_s) \lambda \left[(-k^2 \Psi_{lm}(k, \eta) + \Psi''_{lm}(k, \eta)) j_l(kr) - 2j'_l(kr) \Psi'_{lm}(k, \eta) \right] \\ & + \left(\frac{2}{\lambda_s(a'/a)_s} - 2 \right) \int_0^{\lambda_s} j_l(kr) \Psi'_{lm}(k, \eta) + \frac{2}{\lambda_s} \int_0^{\lambda_s} j_l(kr) \Psi_{lm}(k, \eta) \\ & + \frac{2}{\lambda_s} \int_0^{\lambda_s} d\lambda \int_0^\lambda d\bar{\lambda} j_l(kr) \Psi'_{lm}(k, \bar{\eta}) + \left(\frac{1}{\lambda_s(a'/a)_s} - 1 \right) j_l(kr_s) \Psi_{lm}(k, \eta_s) \end{aligned} \quad (37)$$

$$\begin{aligned} & - \frac{2}{3\Omega_0 \lambda_s(a'/a)_s} \left[j'_l(0) (\Psi_{lm}(k, \eta_0) + \Psi'_{lm}(k, \eta_0)) - j'_l(kr_s) \left(\sqrt{a_s \Omega(a_s)} \Psi_{lm}(k, \eta_s) + a_s \Psi''_{lm}(k, \eta_s) \right) \right] \\ & - \frac{2}{3\Omega_0} j'_l(kr_s) (\Psi_{lm}(k, \eta_s) + \Psi'_{lm}(k, \eta_s)) \end{aligned} \quad (38)$$

where $r = \eta_0 - \eta$, and we have set $a_0 = 1$ and $H_0 = 1$ for simplicity.

z	$3C_1/4\pi$	$\sum_2^{l_c} (2l+1)C_l/4\pi$
0.01	2×10^{-2}	5×10^{-3}
0.1	2×10^{-4}	5×10^{-5}
1.0	2×10^{-6}	5×10^{-5}

Table 1: Multipole moments of anisotropies in d_L for $h = 0.5$, $\Omega_0 = 1.0$ (Einstein-de Sitter model). Dipole terms are seperated. The cutoff l_c is 50, 200, 300 for $z = 0.01, 0.1, 1, 0$, respectively.

3.2 Numerical Calculations

We take as the background model a spatially flat FRW model with the density parameter Ω_0 and the Hubble constant $H_0 = 100 \text{ km/s/Mpc}$. Information of the fluctuations of our universe is often presented in terms of the power spectrum of the density perturbation Δ , which is related with the gravitational potential through the Poisson equation

$$\Psi|_i = \frac{3}{2} \frac{a_0}{a} \Omega_0 H_0^2 \Delta(a(\eta)) \quad . \quad (39)$$

The present matter power spectrum is given by the transfer function T

$$|\Delta_k(\eta_0)|^2 = AkT^2(k) \quad , \quad (40)$$

$$T(k) = \frac{\ln(1 + 2.34q)}{2.34q} [1 + 3.89q + (16.1q)^2 + (5.46q)^3 + (6.71q)^4]^{-1/4} \quad (41)$$

with

$$q = \frac{k(T_0/2.7\text{K})^2}{\Omega_0 h^2 \exp(-\Omega_B - \sqrt{h/0.5\Omega_B/\Omega_0}) \text{Mpc}^{-1}} \quad (42)$$

where Ω_B the baryon density parameter and T_0 is the present temperature of CMB. With these we can calculate the expectation value squared of anisotropies in luminosity distance

$$\left\langle \left| \frac{\Delta d_L}{d_L} \right| \right\rangle^2 := \int \frac{d\Omega}{4\pi} \left| \frac{\Delta d_L}{d_L} \right|^2 =: \sum_l \frac{2l+1}{4\pi} C_l(z) \quad . \quad (43)$$

An example of the results of numercial calculation is given on the table. For the precise values, see the paper to be published in the near future. We will also give the results on other models there.

4 Open universe

We consider the case where the background is an open universe without the cosmological constant. Following the discussion in section 3, we obtain the expression of the fluctuation

of the luminosity distance in an open universe corresponding to Eq.(33)

$$\begin{aligned}
\frac{\Delta d_L}{d_L} = & \frac{\sqrt{-K}}{\text{sh}(\sqrt{-K}\lambda_s)} \int_0^{\lambda_s} d\lambda \frac{\text{sh}\sqrt{-K}(\lambda - \lambda_s)}{-K} (\text{sh}\sqrt{-K}(\lambda - \lambda_s) + \text{sh}\sqrt{-K}\lambda_s) \\
& \times \left(\Psi_{|i}^{|i} + \Psi'' - 2\Psi'_{|i}\gamma^i - 2K \int_0^\lambda \Psi'(\bar{\eta})d\bar{\lambda} \right) \\
& + \frac{2\sqrt{-K}}{\text{sh}\sqrt{-K}\lambda_s} \left(\frac{1}{(a'/a)_s} - \lambda_s \right) \int_0^{\lambda_s} \Psi'(\eta)d\lambda + \frac{2\sqrt{-K}}{\text{sh}\sqrt{-K}\lambda_s} \left[\int_0^{\lambda_s} \Psi(\eta)d\lambda + \int_0^{\lambda_s} d\lambda \int_0^{\lambda_s} d\bar{\lambda} \Psi'(\bar{\eta}) \right] \\
& - (\Psi_0 - \Psi_s) \left(\frac{\sqrt{-K}}{(a'/a)_s \text{sh}(\sqrt{-K}\lambda_s)} \right) - \frac{\Psi_s \sqrt{-K}\lambda_s}{\text{sh}(\sqrt{-K}\lambda_s)} \\
& - (V_{0|i}\gamma^i - V_{s|i}\gamma^i) \frac{\sqrt{-K}}{(a'/a)_s \text{sh}(\sqrt{-K}\lambda_s)} - \frac{V_{s|i}\gamma^i \sqrt{-K}\lambda_s}{\text{sh}(\sqrt{-K}\lambda_s)} .
\end{aligned} \tag{44}$$

5 discussion

First we give some comments on our numerical results. The main origins of fluctuation of luminosity distance are the amplification of luminosity by gravitational lensing and the effect of peculiar velocity on the redshift. The dipole moment is dominated by the latter effect over the whole range of redshift we are considering. The contribution of the effect of peculiar velocity dumps as z becomes large, while the gravitational lensing effect becomes efficient when the redshift is above about 0.5 . The dependence on the cosmological parameters is investigated in the paper we are preparing.

Next, we show what our results imply for observational cosmology. One of the methods to determine the cosmological parameter is studying the redshift-magnitude relation . Especially, we obtain the deceleration parameter q_0 from data at small redshifts. In order to determine the cosmological parameters via the redshifts-magnitude relation, we have to decide the luminosity distance very accurately. To show this, let us remind the relation

$$d_L = \frac{1}{H_0} (z + \frac{1}{2}(1 - q_0)z^2 + \dots), \tag{45}$$

from which follows

$$|\delta q_0| = \frac{1}{2z} \left| \frac{\Delta d_L}{d_L} \right| . \tag{46}$$

Then, if we want to constrain the deceleration parameter q_0 up to 0.1 at $z = 0.1$, we have to keep $\left| \frac{\Delta d_L}{d_L} \right| < 2 \times 10^{-2}$. The value of our calculation is of the same order with this criteria . Of course, however, the present error is dominated by the dispersion in estimating the absolute luminosity $\sigma_M = 0.1 \sim 0.2 \text{ mag}$. Note the relation

$$|\delta q_0| = \frac{1}{z} |\delta M| . \tag{47}$$

As mentioned in the introduction, the value of the dispersion of luminosity distance we calculated here gives the limit on the precision of the measurement of apparent magnitude , or the intrinsic error in determining the luminosity distance, which leads to the error in determining the cosmological parameters. Our calculation shows this effect is negligible at high redshift.

Acknowledgements

The author would like to thank Naoshi Sugiyama and Misao Sasaki for helpful discussions and Humitaka Sato for critical comments and continual encouragement.

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Black Hole Solutions in (2+1)-dimensional Dilaton Gravity¹

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Abstract

We investigate static, spherically symmetric solutions in (2+1)-dimensional dilaton gravity with a finite cosmological constant. By adjusting the integration constant, we find some classes of the exact solutions with no ansatz in the Einstein-Dilaton theory and in the case including the electric fields. We confirm that there exist the solutions with a black hole horizon (BTZ type solutions), a cosmological horizon, and singularities.

1 Introduction

Study of lower dimensional gravity is expected to give some insight into real four-dimensional gravity or into mathematical and conceptual problems of gravity. (2+1)-dimensional gravity as well as (1+1)-dimensional gravity was studied by many authors [1, 2, 3]. It is known that any solution must be locally flat in (2+1)-dimensional Einstein gravity. The single- and multi-centered solution with singularities have been found[2]. Recently black hole-like solutions have been found in the (2+1)-dimensional Einstein-Maxwell theory with a negative cosmological constant by Banados, Teitelboim and Zanelli(BTZ)[4]. The solution is locally flat but has non-trivial global structure. The BTZ solution is found to be a solution of low energy string theory [5]. Also black hole solutions in (2+1)-dimensional dilaton gravity were studied by Chan and Mann[6]. By adopting ansatz for the dilaton field and the metric they found a one-parameter family of BTZ type solutions. They pointed out that there are non-trivial effects of dilaton on the black hole physics.

In the present work we study the static, spherically symmetric solutions in (2+1)-dimensional dilaton gravity. We impose no ansatz to obtain the solutions. The solutions can be obtained analytically for both Einstein-Dilaton and Einstein-Dilaton-Maxwell theory with a finite cosmological constant. The next section sets up the action and the metric in this. In section 3 we first consider the Einstein-Dilaton theory, i.e. without gauge field. In section 4 the case including gauge field is studied in the similar way. Section 5 is devoted to a summary and discussion.

¹This talk is based on the work in collaboration with T. Koikawa and A. Nakamura.

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2 (2+1)-dimensional Dilaton Gravity

We consider the action for the (2+1)-dimensional Einstein-Dilaton-Maxwell theory:

$$S = \int d^3x \frac{\sqrt{\det g}}{16\pi} \left[-R + \lambda(\nabla\phi)^2 + e^{-4a\phi}F^2 + e^{-4b\phi}\Lambda \right], \quad (1)$$

where R is the scalar curvature and the Λ is the cosmological constant. The scalar field ϕ is known as the dilaton field, while F denotes the field strength of $U(1)$ gauge field. This action has two arbitrary dilaton couplings: a is the coupling to the gauge field and b the cosmological constant.

This reduces to the usual Einstein-Maxwell theory if $a = b = 0$, while the action for the low energy string theory is obtained by setting $a=1$, $b=1$, $\lambda=4$ and rescaling the metric. The static, spherically symmetric metric can be generally written as

$$ds^2 = -e^{2V}dt^2 + e^{2U}dr^2 + r^2d\varphi^2, \quad (2)$$

where V and U are functions of only the radius r . Also the dilaton field and the gauge field depend only on the radius r under the static, spherically symmetric condition. Singularities and black hole solutions in the (2+1)-dimensional Einstein gravity have been investigated with finite cosmological constant ($a=0, b=0$) by Deser, Jackiw and 'tHooft, Witten and BTZ. The BTZ solution can be obtained when $\Lambda < 0$. The multi-centered solutions have also found systematically [2] The case including dilaton have been studied by Chan and Mann and black hole solutions have been found for negative cosmological constant and $a = b$. The multi-centered solution of the model is not yet well understood.

3 Spherically symmetric solutions of the Einstein-Dilaton system

We first consider the Einstein-Dilaton system (not including gauge field i.e. $F = 0$ in (1)). Only four independent field equations can be obtained from the action(1) To solve the equations we adopt the method found by Yoshimura and Koikawa[7], which was used for studying Dilaton black holes in four dimensions. Instead of radius r , we introduce the parameter z by

$$re^{V-U} \frac{dz}{dr} = 1. \quad (3)$$

Then the field equations are of (0, 0)-, (1,1)-, and (2,2)- components of the Einstein equation:

$$\ddot{V} = -r^2 e^{2V} e^{4b\phi} \Lambda, \quad (4)$$

$$\dot{V} - \dot{U} = -r^2 e^{V+U} \left\{ \frac{2q^2}{r^2} e^{4a\phi} + e^{4b\phi} \Lambda \right\}, \quad (5)$$

$$\dot{V} + \dot{U} = \lambda e^{-V+U} \dot{\phi}^2, \quad (6)$$

and of the dilaton field equation:

$$\ddot{\phi} = r^2 e^{2V} \left\{ \frac{4a}{\lambda} \frac{q^2}{r^2} e^{4a\phi} + \frac{2b}{\lambda} e^{4b\phi} \Lambda \right\}, \quad (7)$$

where dot denotes derivative with respect to z . While the radius r still remains in the *r.h.s.* of eqs.(4)-(7), we can solve U and V with respect to z as seen later. Then the relation between r and z is obtained.

From eq.(4) and (7) a linear combination of ϕ and V can be integrated:

$$\left(\frac{2b}{\lambda} \right) V + \phi = c_1 z + c_2, \quad (8)$$

where c_1 and c_2 denote integration constants. (Hereafter c with a suffix denotes a constant.) On the other hand other two integrations can be performed from Eqs.(4) and (7) with Eq.(5) :

$$\dot{V} = e^{V-U} + c_3 \quad (9)$$

and

$$\dot{\phi} = -\frac{2b}{\lambda} e^{V-U} + c_4. \quad (10)$$

Note that constant c_1, c_2, c_3, c_4 are not independent on each other:

$$-\frac{2b}{\lambda} c_3 + c_4 = c_1$$

Since the relation V and ϕ is given in Eq.(8), we must solve two equations, for example Eqs.(6) and (10) to obtain the general solution. These are too complicated to be solved analytically. So we make a assumption that $c_4 = 0$. From Eq.(6) using Eq.(10), it is found that a linear combination of V , U and ϕ is constant:

$$V + U + 2b\phi = c_5. \quad (11)$$

Then U and V (thus the exponent in the *r.h.s.* of Eq.(10)) can be rewritten by the dilaton ϕ and z :

$$\dot{\phi} = -\frac{2b}{\lambda} e^{U-V} = -\frac{2b}{\lambda} e^{\alpha\phi + \beta z + \gamma}, \quad (12)$$

where

$$\alpha = \frac{2b^2 - \lambda}{b}, \quad \beta = \frac{\lambda}{b} c_1, \quad \gamma = \frac{\lambda}{b} c_2 - c_5. \quad (13)$$

This equation can be solved as:

$$\phi(z) = -\frac{1}{\alpha} \log \left\{ \alpha \left(\frac{2b}{\lambda} \frac{1}{\beta} e^{\beta z + \gamma} - c_6 \right) \right\}. \quad (14)$$

Substituting this result to Eq.(8) and Eq.(11),

$$V(z) = \frac{\lambda}{2b} \frac{1}{\alpha} \log \left\{ \alpha \left(\frac{2b}{\lambda} \frac{1}{\beta} e^{\beta z + \gamma} - c_6 \right) \right\} + \frac{\lambda}{2b} (c_1 z + c_2). \quad (15)$$

and

$$U(z) = \left(\frac{1}{2} + \frac{b}{\alpha} \right) \log \left\{ \alpha \left(\frac{2b}{\lambda} \frac{1}{\beta} e^{\beta z + \gamma} - c_6 \right) \right\} - \frac{\lambda}{2b} (c_1 z + c_2) + c_3. \quad (16)$$

In order to reexpress these solutions in terms of radius r , it is necessary to know the $r - z$ relation. The relation can be deduced from the definition (3) and the above results (15) and (16):

$$\log(r/r_0) = \int e^{V-U} dz = \frac{\lambda}{2b} \frac{1}{\alpha} \log \left\{ \alpha \left(\frac{2b}{\lambda} \frac{1}{\beta} e^{\beta z} - c_6 e^{-\gamma} \right) \right\}. \quad (17)$$

The dilaton field is written as

$$\phi(r) = -\frac{2b}{\lambda} \log(r/r_0), \quad (18)$$

and the metric as

$$ds^2 = -f(r)dt^2 + \frac{e^{2c_3} (rr_0)^{2b\lambda}}{f(r)} dr^2 + r^2 d\varphi^2, \quad (19)$$

where

$$f(r) = C_0 \left(\frac{r}{r_0} \right) \left\{ \beta c_6 e^{-\gamma} + \frac{\beta}{\alpha} \left(\frac{r}{r_0} \right)^{2b\alpha\lambda} \right\} \quad (20)$$

with the positive definite constant C_0 :

$$C_0 = e^{2c_3 b} \left(\frac{\lambda}{2b} \right) > 0 \quad (21)$$

It is not difficult to verify that the solutions with a horizon exist for certain values of the integration constants. Before the classification, we note the relation between our integration constants and the cosmological constant Λ :

$$\Lambda = \alpha \beta e^{c_3} c_6 \frac{1}{r_0^2} \quad (22)$$

The solutions can be classified as following:

- (1) $\alpha > 0, \beta > 0, c_6 > 0$ ($\Lambda > 0$): NS solution
- (2) $\alpha > 0, \beta > 0, c_6 < 0$ ($\Lambda < 0$): BTZ type solution
- (3) $\alpha > 0, \beta < 0, c_6 > 0$ ($\Lambda < 0$): no solutions

(4) $\alpha > 0, \beta < 0, c_6 < 0, \Lambda > 0$: deSitter type solution

(5) $\alpha < 0, \beta > 0, c_6 > 0, \Lambda < 0$: BTZ type solution

(6) $\alpha < 0, \beta > 0, c_6 < 0, \Lambda > 0, \Lambda > 0$: no solutions

(7) $\alpha < 0, \beta < 0, c_6 > 0, \Lambda > 0$: NS solution

(8) $\alpha < 0, \beta < 0, c_6 < 0, \Lambda < 0$: anti-deSitter type solution

where 'NS solution' represents the solution with a naked singularity and (anti-)deSitter type solution means the solution with a cosmological horizon. Note that there exit BTZ type solutions only for $\Lambda < 0$.

4 Singularities and black hole solutions in Einstein-Maxwell-Dilaton theory

In this section we consider the theory including $U(1)$ gauge field. We study here only the simplest case which the gauge field is pure electric, i.e. only $\partial_r A_0 \neq 0$. In such a case we can solve by an analysis similar to the one in the previous section. Introducing the variable z defined by Eq(3) again, we get similar equations i.e. Eqs(4),(6), the other equations which differ from eqs.(5) and (7) by only terms from the electric field:

$$\ddot{\phi} = r^2 e^{2V} \left\{ \frac{4a}{\lambda} \frac{q^2}{r^2} e^{4a\phi} + \frac{2b}{\lambda} e^{4b\phi} \Lambda \right\}, \quad (23)$$

$$\dot{V} - \dot{U} = -r^2 e^{V+U} \left\{ \frac{2q^2}{r^2} e^{4a\phi} + e^{4b\phi} \Lambda \right\}, \quad (24)$$

and the field equation for the electric field, which can be immediately integrated as

$$F_{tr} = \frac{q}{r} e^{V+U+4a\phi}. \quad (25)$$

Moreover, we concentrate our attention on the low energy string case: $a=1, b=1$ and $\lambda=4$. Then the linear combination of ϕ and V

$$\Phi = \frac{1}{2q^2}(V + 2\phi) \quad (26)$$

satisfy well-known Liouville equation:

$$\ddot{\Phi} = \exp(4q^2\Phi). \quad (27)$$

One of the solution is given by

$$\Phi = -\log \left| \frac{\sinh \left\{ \frac{2\sqrt{2}c_9 q^2(z - z_0)}{2c_9 q} \right\}}{2c_9 q} \right|. \quad (28)$$

On the other hand, from Eqs.(4) and (24)

$$\dot{\phi} = -\frac{1}{2}e^{V-U} + c_7. \quad (29)$$

This is the same equation as Eq.(10). We assume $c_7=0$ as in the previous case. Then a linear combination of V , U and ϕ is to be constant again:

$$V + U + 2\phi = c_8. \quad (30)$$

From Eqs.(28) and(30) with Eq.(26), V and U can be rewritten in terms of ϕ and z . Then Eq.(29) is solved with respect to z under the assumption $c_7=0$. The $r-z$ relation is given by

$$(r/r_0) = \left| \frac{e^{2\sqrt{2}c_9q^2(z-z_0)} - 1}{e^{2\sqrt{2}c_9q^2(z-z_0)} - \frac{A-B}{A+B}} \right|, \quad (31)$$

where

$$A = \frac{1}{2\sqrt{2}q^2c_9} \quad (32)$$

and

$$B = \frac{c_{10}e^{c_8}}{4q^2c_9}. \quad (33)$$

Using this relation we can obtain the solution:

$$\phi(r) = \frac{1}{2} \log \left| \pm 4c_9^2q^2e^{-c_8}(A+B) \left(\frac{r}{r_0} \right) \right|, \quad (34)$$

$$ds^2 = -f(r)dt^2 + \frac{C_2(r/r_0)^2}{f(r)} dr^2 + r^2 d\varphi^2, \quad (35)$$

where

$$f(r) = C_1 \left(1 \mp \left(\frac{r}{r_0} \right) \right) \left\{ 1 \mp \frac{B-A}{B+A} \left(\frac{r}{r_0} \right) \right\}, \quad (36)$$

with

$$C_1 = 2q^2e^{2c_8} > 0 \quad (37)$$

and

$$C_2 = \frac{e^{4c_8}}{4q^2c_9^2} \left(\frac{A}{A+B} \right)^2 > 0. \quad (38)$$

The cosmological constant Λ is related to the constants as

$$\Lambda = \frac{1}{2c_9^2} \left(\sqrt{2}c_9 - c_{10}e^{c_8} \right) e^{-2c_8}$$

The classification of the solutions is summarized as following(details :

(1) $B + A > 0, B - A > 0$ ($\Lambda > 0$): deSitter type

(2) $B + A > 0, B - A > 0$ ($\Lambda < 0$): BTZ type

(3) $B + A > 0, B - A > 0$ ($\Lambda > 0$): no horizon

(4) $B + A > 0, B - A > 0$ ($\Lambda < 0$): anti-deSitter type

In this case there exist solutions with a black hole/cosmological horizon in a certain parameter region. Consequently the solutions agree with that found by Chan and Mann[6], which are imposed some ansatz on the derivation: $\phi \propto \ln(r/r_0)$ and $g_{\varphi\varphi} = \text{const.} \times r^m$ where m is a constant.

5 Concluding Remarks

We investigate static, spherically symmetric solutions of the Einstein-Dilaton theory and the Einstein-Dilaton-Maxwell theory. For the latter, we consider only the case that the dilaton couplings corresponds to the low energy string theory and the magnetic parts vanish. In order to solve analytically, we make a assumption to the integration constant for the each cases. We do not impose any ansatz on the derivation. It is found that there exist the BTZ type solutions if $\Lambda < 0$ for both cases. The results for the case including the electric field agree with Chan and Mann. One of further extensions may be obtained by including the magnetic field. Indeed we can solve analytically by means of our method and get the general solutions in the presence of the magnetic field (in preparation [8]) On the other hand, these physical properties and the mathematical arguments are interesting subjects worth studying.

6 Acknowledgements

I would like to thank Takao Koikawa and Atsushi Nakamura for valuable discussion and active collaboration on the present work.

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String Theory and Quantum Black Holes

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Abstract

In this talk, we present a concise and pedagogical review on the recent progress in understanding black holes within the framework of string theory which is a prime candidate for the unified quantum theory of all interactions including gravity. We emphasize that string theory suggests new perspectives on the question of statistical interpretation of black hole entropy and, in particular, on the nature of event horizon, both of which are radically different from those in classical and semi-classical General Relativity.

1. The Meaning of String Theory
2. Dirichlet Branes and Black Hole Solutions
3. Statistical Interpretation of Black Hole Entropy – Extremal or near extremal limit –
4. Prospect – Small scale structure of space-time implied from string theory –

The following is a compressed version of the pages of the transparencies used in the talk.

1 Introduction: The Meaning of String Theory

What is the fundamental string theory?

In the present stage of development, the string “theory” [1] is actually nothing but the collection of rules for constructing S -matrix elements for which

- (Perturbative) elementary excitations $\Rightarrow$ 1 dim. strings

Namely, the particle spectra are explained from the propagation of 1 dim. strings whose center of mass degrees of freedom correspond to particle excitations in the limit of contracted strings (“zero-slope limit”).

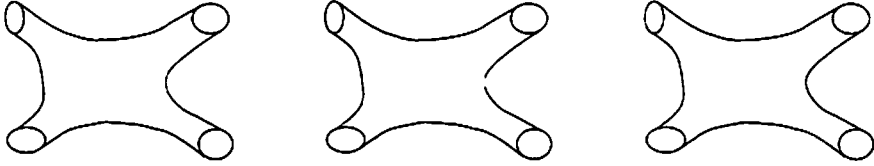
- Feynman rules

$\Rightarrow$ 2 dim. field theory on Riemann surfaces

which is called the conformal field theory and plays the same role as particle quantum mechanics in standard field theories for fundamental interactions.

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Pictorially, an amplitude is given as a sum of diagrams like



Here the overall weights of different string diagrams are determined by unitarity as

$$\sim e^{-N_{\text{euler}}\phi}$$

$$\phi = \text{dilaton}$$

$$N_{\text{euler}} = 2 - 2(\text{handles}) - (\text{holes})$$

One of the most remarkable features of these rules is that we then necessarily have gravitational interaction in a unified manner with other gauge interactions.

Unified theories of all interactions including

Gravity!

In addition to this, string theory exhibits several remarkable properties which are desirable as a candidate for the unified theory.

- The unique fundamental constant

“String Constant” ℓ_s

which roughly characterizes the typical spatial extension of a string and appears as follows.

world-sheet string action
(γ =world-sheet metric)

$$S = \frac{1}{\ell_s^2} \int d^2\xi \sqrt{-\gamma} \frac{1}{2} \gamma^{ab} \partial_a x^\mu \partial_b x_\mu$$

$\Downarrow$

This leads to the Newton constant in flat D-dimensional space-time

$$G_N \sim e^{2\phi} \ell_s^{D-2}$$

$$e^\phi \sim \text{coupling strength of strings} = g$$

- Self-consistency requirement in string theory is so stringent that

▷ Only five different vacuum solutions
in 10 dimensions with flat Minkowski space-time

- Type I (open+closed, $SO(32)$)
- Type IIA (closed)
- Type IIB (closed)
- Heterotic (closed, $SO(32)$)
- Heterotic (closed, $E_8 \times E_8$)

Furthermore, we have

▷ No ultra-violet divergence

▷ No arbitrary parameters other than ℓ_s

e^ϕ (coupling constant) is *in principle* calculable.

▷ Complete “bootstrap”

among elementary excitations and possible external fields

However, the present string theory is limited in that

- Only perturbatively defined
 - Infinite number of perturbative vacua in lower compactified dimensions
 - No definite physical predictions
 - divergence of perturbative series, worse than ordinary renormalizable field theories.
- Only observables are S-matrix elements

⇓

- Problems of string theory : non-perturbative definition of the theory

- Symmetry principle?
- Fundamental degrees of freedom?
- and so on

Now my standpoint in the present talk is that

String theory

||

**A theoretical hypothesis towards
a microscopic theory of gravity and the structure of space-time**

From this point of view,
black hole physics = an ideal arena for testing this hypothesis

It is worthwhile to recall the following analogy:

Blackbody radiation in classical physics

Stefan-Boltzmann law as a relation between the energy density and entropy
($p = \frac{\epsilon}{D-1}$, Kirchhoff's law)

$$s \sim \hbar^{\frac{D-1}{D}} \epsilon^{\frac{D-1}{D}}$$

Classical radiation theory predicts $s = 0$
($\hbar \rightarrow 0, c = 1$, finite ϵ).

$\Downarrow$

Entropy of black hole

Beckenstein-Hawking law
(General Relativity, Hawking radiation) :

$$S (= \frac{\text{Area}}{4G_N}) \sim \frac{1}{4} G_N^{\frac{1}{D-3}} E^{\frac{D-2}{D-3}}$$

Ordinary quantization of gravity predicts $S = 0$
($1/G_N \sim \Lambda_{cutoff}^{D-2} \rightarrow \infty, c = \hbar = 1$, finite mass E)

Any microscopic theory of quantum gravity should provide a statistical explanation for black hole entropy, just like the quantum theory provides the correct statistical interpretation for the entropy of black body radiation.

- First thought : degeneracy of free (SUSY) string states in the large mass limit

$$d(E) = \exp 2\pi \sqrt{\frac{D}{6}} N$$

$$N = (\ell_s E)^2$$

Could this be compatible with BH law?

Ans. Not quite; first of all, the power of the exponential with respect to mass is different, but if we imagine that the string constant ℓ_s might be replaced effectively by the Schwarzschild radius, then

$$\ell_s \rightsquigarrow \ell_{\text{eff}} \sim R_{\text{Schwarzschild}} \sim (G_N E)^{1/\text{ver}D-3}$$

$\Downarrow$

$$\log d(N) \sim G_N^{\frac{1}{D-3}} E^{\frac{D-2}{D-3}} \sim S_{\text{BH}}$$

However, no clear justification of this reasoning so far. To discuss realistic black holes for which

$$\ell_s \ll R_{\text{schwarzschild}},$$

a correct formulation of *non-weak* coupling string theory would be required since for weak coupling we would necessarily have

$$\ell_{\text{Planck}} \sim g^{\frac{1}{D-2}} \ell_s \ll \ell_s \quad \text{for } g \rightarrow 0.$$

It is interesting that the recent explanation of the black hole entropy in string theory for some special extremal black holes is rather reminiscent of this argument (see below ²).

2 Dirichlet Branes and Black Hole Solution

1. Black hole solutions in string theory

There are a variety of black-hole type solutions (in from two to ten space-time dimensions) [2] in the effective low-energy field theory approximation (i.e., “zero-slope” $\ell_s \rightarrow 0$ limit) to string theories.

- A special class among them in (space-time) super symmetric string theories :
BPS saturated solutions which preserve some part of the supersymmetry:
 - no net static force between them
 - the value of mass is protected against various higher order corrections
 - microscopic (stringy) interpretation possible in terms of D-branes

↓

We can hope that

the state counting, using presently known techniques, produce reliable results for BPS saturated solutions, since supersymmetry usually tends to cancel various corrections.

- One typical example : 5 dimensional solution in the semi-classical picture where the flat 10D space-time is compactified into a product of flat 5D space-time and $S^1 \times T^4$.

$$R^{10} \rightarrow R^5 \times S^1 \times T^4$$

- Low energy effective field theory (type IIB)

$$\frac{1}{16\pi G_N} \int d^{10}x \sqrt{-g} \left\{ R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{12}e^{\phi} H_{RR}^2 \right\}$$

²The inverse of the (effective string constant)² is recently referred to as the rescaled string tension.

- H_{RR} = 3-form field strength for the RR gauge field $B_{\mu\nu}^{RR}$
- 6-parameter solution (Q_1, Q_2, n, R, V, E) :
 R = length of S^1
 V = volume of T^4

▷ Charges (S^3 surface integrals in R^5 , 5 dim. Minkowski)

$$Q_1 \sim \int e^\phi * H$$

$$Q_5 \sim \int H$$

• $n \sim$ KK momentum in the direction S^1

▷ Mass E : In the extremal limit,
(6 parameters $\rightarrow$ 5 parameters)

$$E_e = \frac{R|Q_1|}{g} + \frac{RV|Q_5|}{g} + \frac{|n|}{R}$$

▷ Nonzero horizon area $\rightarrow$ BH Entropy (extremal limit)

$$S_e = 2\pi\sqrt{|Q_1 Q_5 n|}$$

▷ String coupling can be made weak everywhere (namely, constant ϕ)
when

$$\frac{R|Q_1|}{g} = \frac{RV|Q_5|}{g} = \frac{n}{R}$$

($\Rightarrow$ 5 dim. Reissner-Nordstøm solution)

More explicitly, the fields and charges take the following forms [6]:

- *Dilaton*

$$e^{-2\phi} = (1 + \frac{r_0^2 \sinh^2 \gamma}{r^2})(1 + \frac{r_0^2 \sinh^2 \alpha}{r^2})^{-1}$$

- *Metric*

$$\begin{aligned} ds^2 = & (1 + \frac{r_0^2 \sinh^2 \alpha}{r^2})^{-3/4} (1 + \frac{r_0^2 \sinh^2 \gamma}{r^2})^{-1/4} [-dt^2 + dx_3^2] \\ & + \frac{r_0^2}{r^2} (\cosh \sigma dt + \sinh \sigma dx_5)^2 + (1 + \frac{r_0^2 \sinh^2 \alpha}{r^2}) dx_i dx^i \\ & + (1 + \frac{r_0^2 \sinh^2 \alpha}{r^2})^{1/4} (1 + \frac{r_0^2 \sinh^2 \gamma}{r^2})^{3/4} [(1 - \frac{r_0^2}{r^2})^{-1} dr^2 + r^2 d\Omega_3^2] \end{aligned}$$

- *Charges*

$$Q_1 = \frac{V}{4\pi^2 g} \int e^\phi * H = \frac{V r_0^2}{2g} \sinh 2\alpha$$

$$Q_5 = \frac{1}{4\pi^2 g} \int H = \frac{r_0^2}{2g} \sinh 2\sigma,$$

$$n = \frac{R^2 V r_0^2}{2g^2} \sinh 2\sigma$$

where the coordinates are

$$x^i \in \{T^4 : x^6, x^7, x^8, x^9\}$$

$$x^5 \in S^1.$$

2. Interpretation in terms of Dirichlet branes

- What are the Dirichlet branes (D-branes)?
 - **Extended dynamical objects** [3] at which strings can end or begin

“Dp-brane”

when the object has p spatial dimensions

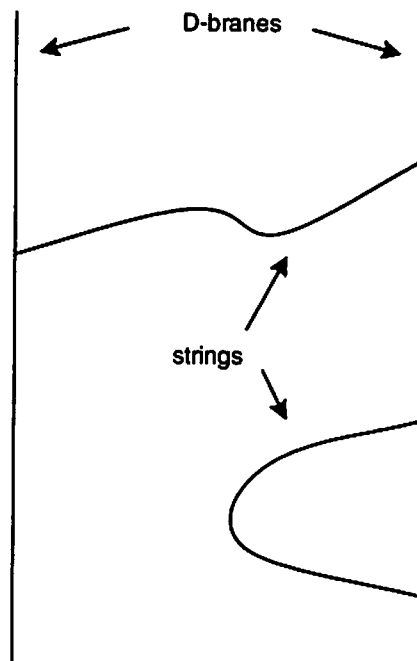
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open strings with Dirichlet boundary condition

↑

“Dirichlet branes”

The Dirichlet branes have finite tension or mass and can move around the space-time, and the open strings coupled with them mediate the interaction among them and also describes their motion as collective coordinates of the Dirichlet branes.



- RR (Ramond-Ramond) charges

Dp-branes have RR charges with respect to $(p + 1)$ -form RR gauge fields. In contrast to this, the strings as the perturbative excitations are neutral with respect to RR charges. The existence of RR charged dynamical objects has been required for the validity of conjectured duality relations of various string models.

$p + 1$ -form fields naturally couple to
p-dimensional extended objects

↓

RR charges corresponding to
 $(p + 2)$ -form field strength

Type IIA $\rightarrow p = \text{even}$

Type IIB $\rightarrow p = \text{odd}$

Examples: Type IIB

D1-branes $\leftrightarrow$ 3-form RR field strength

D5-branes $\leftrightarrow$ 7-form RR field strength

dual to 3-form field strength

- Specific mass (tension) of D-branes $\propto \frac{1}{g}$ ($\leftrightarrow$ no force condition)
- D-branes can be regarded as the solitonic excitation of string theory (plays similar roles [4] as Bogomolnyi-saturated magnetic monopoles, instantons, etc in gauge field theories)

In particular,

BPS saturation $\leftarrow$ partial supersymmetry

$$Q|\psi\rangle = 0$$

for some supergenerators Q and RR (or KK)-charges q

$$0 = \{Q, Q\}|\psi\rangle = (E - cq)|\psi\rangle \rightarrow E - cq = 0$$

All these properties indicate that

D-branes

||

microscopic description of
solitonic classical solutions
with RR (and Kaluza-Klein) charges
in effective low-energy field theories

It is quite remarkable that string theory allows point-like and even -1 dimensional objects as D0(or D-particle) and D -1 branes (D-instantons), respectively. This is perhaps an indication that the fundamental degrees of freedom of string theory are not really the strings!

3 Statistical Interpretation of Black Hole Entropy

[5]

- D-brane description of the extremal 5 dim. black hole solution

The long-distance property of the 5D black hole solution is equivalently described by considering the following D-brane configurations

- Q_5 D5-branes wrapping (singly wound) around

$$T^5 = S^1 \times T^4$$

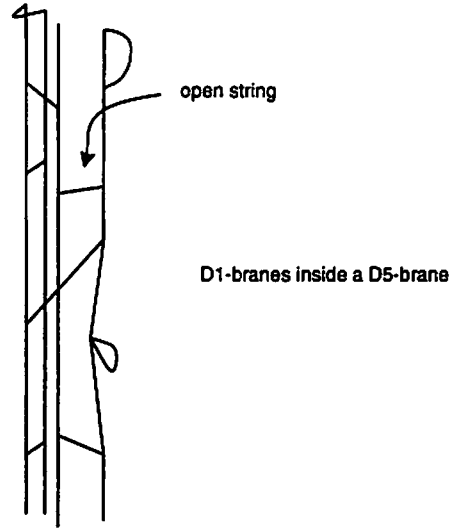
- Q_1 D1-branes wrapping (singly wound) around

$$S^1$$

- n units (*only* right moving or left moving) of the KK momentum along the direction of S^1

$$P = \frac{n}{R}$$

Pictorially, it is something like below. Here the surface of this page is a D5-brane and the vertical lines are D1-branes. There are many open strings whose ends are either on the D5-branes or D1-branes.



- State counting ($Q_5 = 1$ case, for simplicity)

Count the number of right moving massless free open strings (D1-D1) along the S^1 direction whose ends sit on the D1-branes.

(If the D1-branes are dilute inside the D5-brane, this is a valid approximation)

⇓

entropy of one-dimensional gas of $4Q_1$ species of free massless supersymmetric scalar-fermion system in a box of length $2\pi R$

↓

$$E = \frac{n}{R}, \text{ valid for large } n, R, V$$

$$S = \sqrt{\frac{\pi 4Q_1 E \times 2\pi R}{2}} = 2\pi \sqrt{|Q_1|n}$$

coincides with the BH formula ($Q_5 = 1$)

- Other cases (at least if one of the charges are finite) are evaluated similarly and always reduce to the BH formula
- Can be extended to near extremal cases.
The result is consistent with the BH formula

Add anti D-branes and take into account both right and left moving open strings

⇓

Hawking radiation is interpreted from the process :

right moving open strings + left moving open strings

↓

closed strings

- Similar (but slightly more complicated) discussion can be made for 4 dim. black hole solutions to effective low-energy field theories
- If all of the charges are equally large, a similar argument gives the BH entropy by making following replacement,

many singly-wound D-branes

↓

single pair of
multiply-wound D1- and D5-branes
(counting of D1-D5 massless open strings)

In this case, however, the string coupling g cannot be so small. $R, V, g \sim O(1) \leadsto$ non-extremal result is problematical

⇓

State counting reduces to counting the number of massless open string states, attached to the D1-brane, for effective excitation number

$$n_{\text{eff}} \sim |Q_1 Q_5| n$$

This again leads to the above BH formula.

Note: $n \rightarrow n_{\text{eff}}$ is reminiscent of " $\ell_s \rightarrow \ell_{\text{eff}}$ " mentioned at the end of section 1.

4 Prospect

- Realistic (non-extremal) black holes?

All of the known results are reliable only for sufficiently weak string coupling $g \ll 1$ where

$$\ell_{\text{Planck}} \ll g^{2/(D-2)} \ell_s$$

It is not easy to treat macroscopic black holes whose size is

$$r_S \sim (G_N E)^{1/(D-3)} \sim g^{2/(D-3)} E^{1/(D-3)} \ell_s^{(D-2)/(D-3)} \gg \ell_s$$

- Information loss?

If the interpretation in terms of D-branes is correct,
the Hawking radiation is
not different from
the ordinary thermal radiation.

$\Downarrow$

The thermal nature reflects
only our ignorance on the
initial microscopic state of the D-brane configurations.

To prove this statement, however, a completely well defined unitary theory of D-brane-string dynamics would be required.

- Microscopic structure of space-time?

In string theory, it seems that the microscopic structure of space-time near the horizon ceases to be meaningful.

What is the significance of the string constant ℓ_s ?

A proposal

evidence from
D0(particle)-brane dynamics, D-brane-string scattering, string-string scattering,
st-duality, etc

“Space-time uncertainty relation” [7]

$$\Delta X \Delta T > \ell_s^2$$

$\Downarrow$

Observation of a black hole from asymptotic region implies ΔT (uncertainty with respect to time durations) $\rightarrow 0$ near the would-be horizon because of infinite red-shift effect.

$\downarrow$

ΔX (uncertainty with respect to spatial distances) grows indefinitely near the horizon.

This strongly suggests that we cannot talk about the short distance structure of space-time near the horizon and is consistent with other proposals like

“black hole complementarity” [8]

Note:

- 1. However, it is well known that the nature of Hawking radiation crucially depends on arbitrary short distance behavior of the field propagators.
- 2. In string theory, even the global topologies of space-time is not necessarily invariant observables. Symmetry transformations (e.g., T-duality) can change topology.

- Towards non-perturbative definition of string theory?

Is there any model for black hole space-time defined for arbitrary string coupling?

- (Deformed) Matrix models in 1 dimension [9]
 $\rightarrow$ 2 dim. black hole in string theory

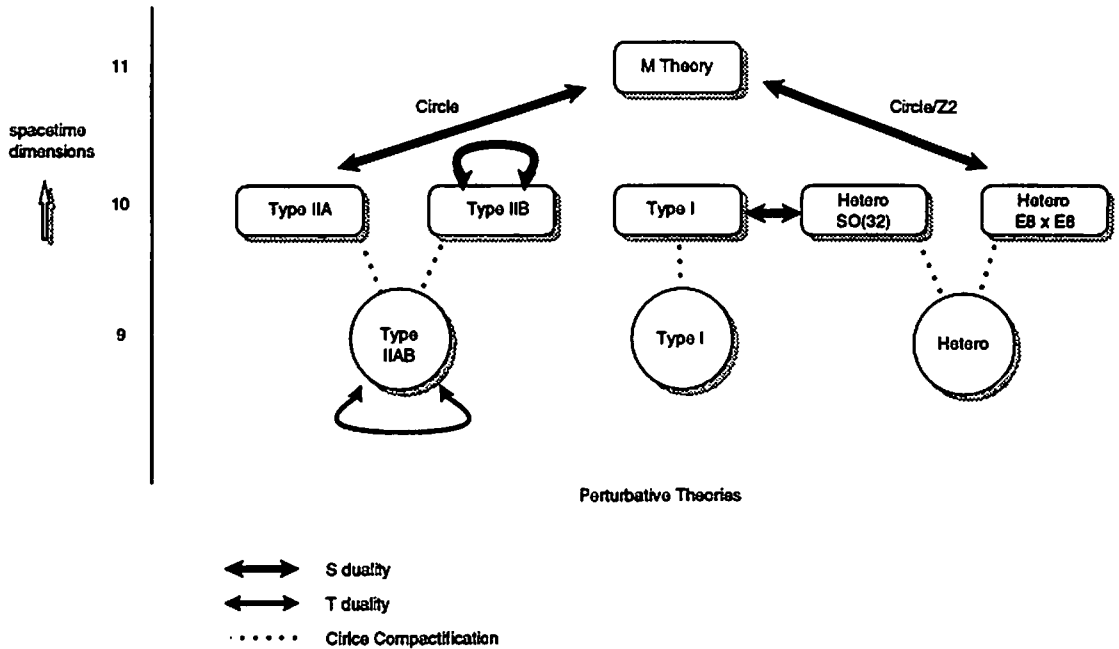
This suggests a possibility that

black hole space-time of string theory in $2(1+1)$ dimensions is mapped into a non-singular quantum mechanical system described by a $U(\infty)$ matrix in 1 dimension.

$\downarrow$

Unitary S-matrix (exactly calculable)

- Recent progress: duality relations among various different “classical” vacua of string theory are made plausible [10].



- Matrix models in (1=time) dimensions
→ 10 + 1 dim. M-theory

“M(atrix)-theory” [11]
(and perhaps other dimensions as well)

is now a plausible candidate for nonperturbative string theory. There are even plausible hints that some zero-dimensional matrix model could be the fundamental formulation we are seeking for. The structure of such possible matrix models seems consistent with the space-time uncertainty relation mentioned above.

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On scattering off the extreme Reissner-Nordström black hole in $N = 2$ supergravity¹

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Abstract

The scattering amplitudes for the perturbed fields of the $N = 2$ supergravity about the extreme Reissner-Nordström black hole is examined. Owing to the fact that the extreme hole is a BPS state of the theory and preserves an unbroken global supersymmetry ($N = 1$), the scattering amplitudes of the component fields should be related to each other. In this paper, we derive the formula of the transformation of the scattering amplitudes.

1 Introduction

Although charged black holes would be less interesting in astrophysical context, since we do not expect the existence of macroscopic charged bodies, in theoretical aspects, charged black holes provide us tools for understanding black holes. Charged black holes are similar to the Kerr holes in some aspects; causal structures and the existence of superradiance phenomena, and so on. The Reissner-Nordström solution is spherically symmetric and so it is more tractable than the Kerr hole. Thus charged black holes provide interesting toy models.

Further as we see later, we can regard the extreme Reissner-Nordström solution as a gravitational soliton [1, 2].

A soliton is a classical solution whose configuration is (i) stationary, (ii) has finite localized energy, (iii) regular and (iv) stable classically and quantum mechanically (in general relativity, we impose (iii) outside horizons). Solitons that are non-perturbative configurations play an important role for studying non-perturbative aspects of quantum field theories. The collective coordinates of a soliton, such as its position and its conjugate momentum, and the fluctuations about it create the “soliton sector” of the particle spectrum which is not described by the conventional perturbation approaches [3]. Further, in some supersymmetric gauge theory, we obtain the duality symmetry between electric and magnetic part, that magnetic monopole solitons behave as weakly coupled elementary fields at strong coupling [4, 5].

Returning to gravitational solitons, we may image black holes as solitons in a gravitational theory [1, 2, 6, 7] and that elementary particles are black holes with very small

¹This work is a collaboration with H.Onozawa, T.Mishima and H.Ishihara.

mass. The Kerr-Newman series without the naked singularity satisfies the conditions, (i), (ii) and (iii). The Schwarzschild solution has non-zero Hawking radiation. Therefore it is unstable in quantum theory. The Kerr series is unstable due to the superradiance. If we consider the theory without charged fields, the Reissner-Nordström solution has no superradiance. Further the extreme one has no Hawking radiation. Hence, the extreme Reissner-Nordström solutions are stable both in classical and quantum theory and we can regard them as the gravitational solitons [1, 2].

Solitons often have some conserved charges. From the stability of solitons in classical and quantum theory, we may think a soliton as the least energy state whose energy is given by the charges. Further we may expect the inequality between the mass and charges of a soliton [8]. In fact, we have the inequality in supersymmetric theory [9]. The states that saturate the inequality is called BPS states. Although BPS states are massive and break the supersymmetry, they still have some unbroken supersymmetry. Therefore the supermultiplet of the BPS states is shortened in comparison with the general massive states. This fact suggests that the mass formula for the BPS state holds exactly even in quantum theory [9]. Further since the mass formula is derived by algebra of supersymmetry, it holds irrespective of the strength of coupling constants in the theory.

Can we expect the same things in gravitational theory? The answer is "Yes". The Einstein and the Einstein-Maxwell system are embedded in the supergravity theories. In the asymptotically flat spacetime, we can obtain the global charges that generate the rigid supersymmetry [10]. Therefore we can follow the argument in the rigid supersymmetry, formally. For the $N = 1$ supergravity the positivity of energy is suggested [10, 11]. Subsequently Witten established the positivity for the general relativity using the trick of "Witten spinor" [12]. Further using the "Witten spinor" motivated by the transformation law of gravitini in the $N = 2$ supergravity, Gibbons and Hull [13] established the inequality between the mass and the electromagnetic charges. Further they showed that the saturated configurations are Majumdar-Papapetrou(MP) solutions, which are assemblages of the extreme Reissner-Nordström holes and that the MP solutions have unbroken supersymmetry. More generalizations of their results are available in Refs. [14, 15].

In addition, the non-renormalization theorem of on-shell effective action for the MP solutions was established [14, 16]. Although supergravity has better ultraviolet behavior than the general relativity, it is known that supergravity is nonrenormalizable at the perturbative level and is not regarded as the final theory. However we may expect that the final theory should include supergravity and may think the non-renormalization for the MP solutions as a guiding principle to the final theory [17, 18].

Thus, it is very significant to investigate extreme black holes in classical and semi-classical framework through the general relativity and supergravity. To more understand extreme holes that are regarded as a kind of "vacuum" state, we need study the fluctuation(excitation) about them.

Originally, the study of the perturbation about the extreme Reissner-Nordström hole was motivated by the interest in the no hair conjecture in supergravity [19] and by interpretation problem of the paradoxical thermal properties of extreme-dilaton black holes [20].

Recently in the study on the method of calculating quasinormal frequencies of the extreme Reissner-Nordström hole, Onozawa, et.al. numerically [21] found that the quasi-

normal frequencies of gravitational waves and electromagnetic waves about it coincide by a suitable shift of the angular momentum indices. Due to the fact that quasinormal frequency is resonance pole of scattering wave, we may expect that gravitational and electromagnetic wave have the same reflection and transmission amplitude. Subsequently, they established coincidence between S-matrices of perturbations of gravitational, electromagnetic and spin-3/2 fields (gravitini) about the extreme Reissner-Nordström background in the $N = 2$ supergravity by finding relation between the Regge-Wheeler potentials of perturbations [22]. From the fact that the extreme Reissner-Nordström hole is a BPS state in extended supergravity, we expect that the coincidence is related to the fact that BPS states preserve unbroken supersymmetries. The purpose of this report is to derive the relation between scattering matrices of graviton, gravitini and photon using supersymmetric transformation.

The report is organized as follows: In the subsequent subsections, we have brief sketches of the representation of supersymmetry and the BPS state. In Sec. II we briefly review the perturbation equations through the Newman-Penrose formalism and the scattering problem. In Sec. III we give the supersymmetric transformation law between the curvatures of perturbed fields. In Sec. IV we seek the correspondence between the radial parts of the perturbations with the suitable total angular momentum and the relations of the reflection and transmission coefficients for them. Sec. V is devoted to summary.

1.1 massless representation

For the extended ($N \geq 2$) supersymmetry algebra with no central charges, the algebra is given by [23]

$$\{Q_A^i, \bar{Q}_{A'}^j\} = \sqrt{2} P_\mu \sigma_{AA'}^\mu \delta^{ij}, \quad (1.1)$$

$$[Q_A^i, J_{\mu\nu}] = \frac{1}{2} (\sigma_{\mu\nu})_A{}^B Q_B^i, \quad (1.2)$$

where where $(Q_A^i, \bar{Q}_{A'}^j)$, P_μ and $J_{\mu\nu}$ are the generators of the supersymmetric transformations, translations and the Lorentz rotations, respectively and we list the algebraic relations which are necessary later. The Eq.(1.2) shows that the generator Q_A^i carries spin-1/2. Since we consider the massless representation of the algebra, we set the Lorentz frame as

$$P_\mu = (E, 0, 0, -E), \quad (1.3)$$

and obtain

$$\{Q_A^i, Q_B^{j\dagger}\} = 2E \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \delta^{ij}, \quad (1.4)$$

where $Q_A^{i\dagger} = \bar{Q}_{A'}^j$. Hereafter we use the notations $A = \{(1), (2)\}$. Thus, in massless representation, $Q_{(1)}^i$ is "dead", $Q_{(1)}^i = 0$, because $Q_{(1)}^i Q_{(1)}^{i\dagger} = 0$. In this frame, the helicity operator is J_{12} and

$$[Q_A^i, J_{12}] = -\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} Q_{(1)}^i \\ Q_{(2)}^i \end{pmatrix}. \quad (1.5)$$

Therefore $Q_{(2)}^i$ and $Q_{(2)}^{i\dagger}$ act as the lowering and raising operator for the helicity, respectively.

We assume that $N = 2$ and the highest helicity is h ,

$$J_{12}|h\rangle = h|h\rangle, \quad (1.6)$$

$$Q_{(2)}^{i\dagger}|h\rangle = 0. \quad (1.7)$$

By acting "alive" $Q_{(2)}^i$ on $|h\rangle$ we acquire the other states

$$|h - \frac{1}{2}\rangle_1 = Q_{(2)}^1|h\rangle, \quad (1.8)$$

$$|h - \frac{1}{2}\rangle_2 = Q_{(2)}^2|h\rangle, \quad (1.9)$$

$$|h - 1\rangle = Q_{(2)}^2|h - \frac{1}{2}\rangle_1 = -Q_{(2)}^1|h - \frac{1}{2}\rangle_2, \quad (1.10)$$

and there is no lower helicity state due to the nilpotency $(Q_A^i)^2 = 0$. Thus the states $\{|h\rangle, |h - 1/2\rangle_i, |h - 1\rangle\} (i = 1, 2)$ form multiplet for $N = 2$ supersymmetry. Further we expect that two fields with helicity $(h - 1/2)$ appear symmetrically and indeed there is $O(2)$ -symmetry between them.

When we need include graviton, we set $h = 2$. If we need a multiplet including graviton and photon, then we should take $N = 2$ supersymmetric theory at minimum.

For general N , the supermultiplet including graviton as the field with the highest helicity have N gravitini among which there is $O(N)$ -symmetry, ${}_N C_2$ vector fields and so on. Thus the massless representation for N -supersymmetry is 2^N -dimensional representation.

For massive representation, $Q_{(1)}^i$ is "alive". Therefore the dimension of representation (number of physical degrees of freedom) for massive is greater than for massless and it is 2^{2N} -dimension.

1.2 BPS state

In supersymmetric theories, we have a general inequality on the mass and charge of all states. The states which saturate the inequality is called "BPS states".

In the extended ($N \geq 2$) supersymmetry algebra, the algebra has the form [23],

$$\{Q_A^i, \bar{Q}_{A'}^j\} = \sqrt{2}P_\mu \sigma_{AA'}^\mu \delta^{ij}, \quad (1.11)$$

$$\{Q_A^i, Q_B^j\} = \epsilon_{AB} Z^{ij}, \quad (1.12)$$

$$\{\bar{Q}_{A'}^i, \bar{Q}_{B'}^j\} = \epsilon_{A'B'} Z^{*ij}, \quad (1.13)$$

where $Z^{ij} (= -Z^{ji})$ is called the central charges that commute with any other generators and are given by something like topological charges [9].

Now let us consider $N = 2$ for the simplicity and massive case (the case of non-vanished central charge implies nonzero mass). Since Z^{ij} is antisymmetric with respect to the indices, $Z^{ij} = \epsilon^{ij}Z$ for $N = 2$. Further we choose the rest frame

$$P_\mu = (M, 0, 0, 0). \quad (1.14)$$

Then by a suitable redefinition of Q s' phases, we obtain the equations

$$M\delta^{ij} = \{Q_{(1)}^i, Q_{(1)}^{j\dagger}\} = \{Q_{(2)}^i, Q_{(2)}^{j\dagger}\}, \quad (1.15)$$

$$\{Q_{(1)}^1, Q_{(2)}^2\} = -\{Q_{(1)}^2, Q_{(2)}^1\} = |Z|, \quad (1.16)$$

Here we define new generators from linear combinations of Q 's

$$\begin{cases} +Q_{(1)} \equiv Q_{(1)}^1 + Q_{(2)}^{2\dagger} \\ +Q_{(2)} \equiv Q_{(2)}^1 - Q_{(1)}^{2\dagger}, \end{cases} \quad (1.17)$$

$$\begin{cases} -Q_{(1)} \equiv Q_{(1)}^1 - Q_{(2)}^{2\dagger} \\ -Q_{(2)} \equiv Q_{(2)}^1 + Q_{(1)}^{2\dagger}, \end{cases} \quad (1.18)$$

and these satisfy the algebra

$$\{\pm Q_{(1)}, \pm Q_{(1)}^\dagger\} = \{\pm Q_{(2)}, \pm Q_{(2)}^\dagger\} = 2(M \pm |Z|) \geq 0. \quad (1.19)$$

The last inequality is followed by the positivity of state space and the equality holds for $M = |Z|$ state(BPS state), then $-Q_{(1)}|\text{BPS}\rangle = -Q_{(2)}|\text{BPS}\rangle = 0$, that is, the BPS state is invariant for a half of $N = 2$ supersymmetry transformations. Thus BPS states preserve unbroken supersymmetry, that is, the states are supersymmetric.

In extended supergravity, if the spacetime is asymptotically flat, then we have the conserved charges, total 4-momentum, supercharges and central charges and they form the rigid supersymmetric algebra [10]. Therefore we can follow the above argument and show a inequality. The corresponding positive energy theorem in the Einstein-Maxwell system is as follows [13]:

Theorem 1 *The ADM or Bondi mass of an asymptotically flat spacetime satisfying*

1. $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = M_p^{-2} \left[T_{\mu\nu}^{\text{matter}} + \frac{1}{4\pi} \left(F_{\mu\lambda}F_{\nu}^{\lambda} - \frac{1}{4}g_{\mu\nu}F^2 \right) \right],$
2. $\nabla_\nu F^{\mu\nu} = 4\pi J^\mu,$
3. $\nabla_\nu {}^*F^{\mu\nu} = 4\pi \tilde{J}^\mu,$
4. $T_{\mu\nu}^{\text{matter}}U^\mu V^\nu \geq \sqrt{8\pi}M_p \left[(J_\mu V^\mu)^2 + (\tilde{J}_\mu V^\mu)^2 \right]^{1/2},$

for all pairs of non-spacelike vectors U^μ, V^ν satisfies inequality

$$M \geq \sqrt{8\pi}M_p(Q^2 + P^2)^{1/2}, \quad (1.20)$$

with equality if and only if there exists two 2-spinor fields satisfying

$$\nabla_\mu \zeta_A^i - \frac{i}{M_p} \epsilon^{ij} F_A^B e_{\mu BA'} \bar{\zeta}^{jA'} = 0. \quad (1.21)$$

Where Q and P are an electric and magnetic charge, respectively and $M_p = (8\pi G)^{-1/2}$ is the Planck mass.

Eq.(1.21) is just transformation law of gravitini on purely bosonic background in $N = 2$ supergravity,

$$\delta\psi_{A\mu}^i = M_p \nabla_\mu \alpha_A^i - i\epsilon^{ij} F_A^B e_{\mu BA'} \alpha^{jA'}, \quad (1.22)$$

and the MP solution satisfies Eq.(1.21), that is, it is a supersymmetric configuration. The general spacetime configurations that admit the existence of solutions of Eq.(1.21) are investigated in detail by Tod [24].

2 The perturbation equations

By linearizing the $N = 2$ supergravity [25] about a purely bosonic background, we have the perturbation equations for linearized Einstein-Maxwell system and for linearized $O(2)$ doublet of spin-3/2 fields, which they are decoupled. Here we follow Chandrasekhar [26] and Torres del Castillo & Silva-Ortigoza [27] for bosonic perturbations and fermionic ones respectively. Hereafter we adopt unit $M = 1$, where M is the mass of background black hole.

On the background of the Reissner-Nordström solution, the bosonic perturbations are described by the Regge-Wheeler equation and the fermionic ones by the similar equation thorough the Newmann-Penrose formalism.

For the perturbations with the helicity- $(+1, +\frac{3}{2}, +2)$, Y_{+s} ($s = 1, \frac{3}{2}, 2$), in phantom gauge [28], their equations of radial parts are

$$\Lambda^2 Y_{+s} + P_s \Lambda_- Y_{+s} - Q_s Y_{+s} = 0, \quad (2.1)$$

$$\frac{d}{dr_*} = \frac{\Delta}{r^2} \frac{d}{dr}, \quad \Delta = r^2 - 2r + Q^2, \quad (2.2)$$

$$\Lambda_{\pm} \equiv \frac{d}{dr_*} \pm i\Omega, \quad \Lambda^2 \equiv \Lambda_+ \Lambda_-, \quad (2.3)$$

where r_* is the tortoise coordinate and we omit the index of distinguishing two gravitini because they follows the same equation as expected from $O(2)$ symmetry between them.

The each P_s and Q_s are given by, for $s = 1, 2$,

$$P_s \equiv \frac{d}{dr_*} \ln\left(\frac{r^8}{D_s}\right), \quad D_s \equiv \Delta^2 \left(1 + \frac{2q_s}{\mu^2 r}\right), \quad (2.4)$$

$$Q_s \equiv \mu^2 \frac{\Delta}{r^4} \left(1 + \frac{2q_s}{\mu^2 r}\right) \left(1 + \frac{q_{s'}}{\mu^2 r}\right) \quad (s, s' = 1, 2; s \neq s'), \quad (2.5)$$

where $q_{1,2}$ are defined by

$$q_1 = 3 + \sqrt{9 + 4Q^2 \mu^2}, \quad q_2 = 3 - \sqrt{9 + 4Q^2 \mu^2}, \quad (2.6)$$

and for $s = \frac{3}{2}$,

$$P_{\frac{3}{2}} \equiv \frac{3}{r^3} (r^2 - 3r + 2Q^2), \quad (2.7)$$

$$Q_{\frac{3}{2}} \equiv \frac{\Delta}{r^6} (\lambda r^2 + 2r - 2Q^2). \quad (2.8)$$

The radial perturbations Y_{+s} are constructed in two ways. One is by the perturbed Weyl scalar Ψ_0 and the perturbed spin connection κ as

$$Y_{+s}(r) = \frac{\Delta^2}{r^3} F_{+s}(r), \quad F_{+s} = R_{+s}(r) + \frac{q_s k(r)}{\mu}, \quad (2.9)$$

$$\Psi_0 = R_{+2}(r) S_{+2}(\theta) e^{i(\Omega t + m\phi)}, \quad \kappa = \sqrt{2} r^2 k(r) S_{+1}(\theta) e^{i(\Omega t + m\phi)}, \quad (2.10)$$

where the constant μ is a eigenvalue of the spin-weighted spherical harmonics,

$$\mathcal{L}_{-1}^{\dagger} \mathcal{L}_2 S_{+2} = -\mu^2 S_{+2} , \quad \mathcal{L}_2 \mathcal{L}_{-1}^{\dagger} S_{+1} = -\mu^2 S_{+1} , \quad (2.11)$$

$$\mu = \sqrt{(J-1)(J+2)} . \quad (2.12)$$

The operators $\mathcal{L}_n$ and $\mathcal{L}_n^{\dagger}$ are defined by

$$\mathcal{L}_n \equiv \partial_{\theta} + \frac{m}{\sin \theta} + n \cot \theta , \quad (2.13)$$

$$\mathcal{L}_n^{\dagger} \equiv \partial_{\theta} - \frac{m}{\sin \theta} + n \cot \theta . \quad (2.14)$$

Besides, the functions S_{+1} and S_{+2} are related in the manner

$$\mathcal{L}_2 S_{+2} = \mu S_{+1} , \quad \mathcal{L}_{-1}^{\dagger} S_{+1} = -\mu S_{+2} . \quad (2.15)$$

Another is by the Weyl scalar Ψ_1 and the spin connection σ as

$$G_{+s}(r) = R_{+s}(r) + \frac{q_s}{\mu} s(r) , \quad (2.16)$$

$$\Psi_1 = \frac{1}{r\sqrt{2}} R_{+1}(r) S_{+1}(\theta) e^{i(\Omega t + m\phi)} , \quad \sigma = r s(r) S_{+2}(\theta) e^{i(\Omega t + m\phi)} , \quad (2.17)$$

and G_{+s} are related to Y_{+s} through the relations,

$$\Delta \left(D_2^{\dagger} - \frac{3}{r} \right) F_{+s} = \mu \left(1 + \frac{2q_s}{\mu^2 r} \right) G_{+s'} \quad (s, s' = 1, 2; s \neq s') , \quad (2.18)$$

where the operator D_n is

$$D_n \equiv \partial_r + \frac{ir^2 \Omega}{\Delta} + 2n \frac{r-1}{\Delta} , \quad D_n^{\dagger} = (D_n)^* . \quad (2.19)$$

For the helicity- $(+3/2)$ perturbations, the supersymmetric gauge invariant quantities are constructed by the supercovariant curvature of the spin- $3/2$ fields ψ_{μ}^i as

$$H_0^i = \Psi_{(ABC)}^i o^A o^B o^C , \quad (2.20)$$

$$\Psi^i{}^A{}_{BC} = \frac{1}{2} \left[\boldsymbol{D}_{(B|A'|} \psi^{iA}{}_{C)}{}^{A'} + \frac{i}{M_p} \epsilon^{ij} \boldsymbol{F}^A{}_{(B} \bar{\psi}^{jA'}{}_{|A'|C)} \right] , \quad (2.21)$$

where $\boldsymbol{D}_{\mu}$ is covariant derivative with respect to the spin connection $\omega_{\mu ab} = \omega_{\mu AB} \epsilon_{A'B'} + \bar{\omega}_{\mu A'B'} \epsilon_{AB}$, for example,

$$\boldsymbol{D}_{\mu} \eta_A = \partial_{\mu} \eta_A + \omega_{\mu A}{}^B \eta_B , \quad (2.22)$$

and the bold-face letters indicate the background quantities, o^A is a principal spinor of $\boldsymbol{F}_{AB}$, and $\boldsymbol{F}_{AB}$ is a 2-spinor representation of the self-dual part of electromagnetic field strength $F_{\mu\nu}$. Hence we obtain the modes of the helicity- $(+3/2)$ perturbations,

$$Y_{+\frac{3}{2}}^j = \frac{\Delta^{\frac{3}{2}}}{r^2} R_{+\frac{3}{2}}^j , \quad (2.23)$$

$$H_0^j = R_{+\frac{3}{2}}^j(r) S_{+\frac{3}{2}}(\theta) e^{i(\Omega t + m\phi)} , \quad (2.24)$$

where m is $+1/2$ or $-1/2$ and the spin-weight $+3/2$ spherical harmonics $S_{+\frac{3}{2}}$ satisfies

$$\mathcal{L}_{-\frac{1}{2}}^\dagger \mathcal{L}_{\frac{1}{2}} S_{+\frac{3}{2}} = -\lambda S_{+\frac{3}{2}}, \quad (2.25)$$

$$\lambda = (J_s - \frac{1}{2})(J_s + \frac{3}{2}) \quad (J_s = \frac{3}{2}, \frac{5}{2}, \dots). \quad (2.26)$$

To set the scattering problem, we need the normalized in(out)-going wave forms for Y_{+s} at the asymptotic regions ($r_- \rightarrow \pm\infty$). At $r_- \rightarrow \infty$, its asymptotic form of each normalized perturbation $Y_{+s}(s = 1, \frac{3}{2}, 2)$ become

$$Y_{+s}^{(+\infty, in)} \sim -4\Omega^2 e^{+i\Omega r_-} \quad \text{and} \quad Y_{+s}^{(+\infty, out)} \sim -\frac{K_s}{4\Omega^2 r_-^{2\|s\|}} e^{-i\Omega r_-}, \quad (2.27)$$

where $\|s\| = 2$ for $s = 1, 2$ and $\|s\| = \frac{3}{2}$ for $s = \frac{3}{2}$. K_s are defined by

$$K_s \equiv \mu^2(\mu^2 + 2) + 2i\Omega\beta_s, \quad \beta_s^2 \equiv q_s^2 \quad (s, s' = 1, 2; s \neq s'), \quad (2.28)$$

$$K_{\frac{3}{2}} \equiv 2i\Omega(\kappa_{\frac{3}{2}} + 2i\Omega\beta_{\frac{3}{2}}), \quad (2.29)$$

$$\kappa_{\frac{3}{2}}^2 \equiv \left[\left(J_s - \frac{1}{2} \right) \left(J_s + \frac{1}{2} \right) \left(J_s + \frac{3}{2} \right) \right]^2, \quad \beta_{\frac{3}{2}}^2 \equiv 4. \quad (2.30)$$

Similarly, at $r_- \rightarrow -\infty$ for $s = 1, 2$

$$\begin{aligned} Y_{+s}^{(-\infty, out)} &\sim 4i\Omega \left(i\Omega - \frac{r_+ - Q^2}{r_+^3} \right) \exp(+i\Omega r_-) \\ Y_{+s}^{(-\infty, in)} &\sim \frac{K_s(1 + \frac{2q_s}{\mu^2 r_+}) \Delta^{\|s\|}}{4r_+^{4\|s\|} \left(i\Omega - 2\frac{r_+ - 1}{r_+^4} \right) \left(i\Omega - \frac{r_+ - Q^2}{r_+^3} \right)} \exp(-i\Omega r_-), \end{aligned} \quad (2.31)$$

and for $s = \frac{3}{2}$

$$\begin{aligned} Y_{+\frac{3}{2}}^{(-\infty, out)} &\sim 4i\Omega \left(i\Omega - \frac{r_+ - 1}{2r_+^2} \right) \exp(+i\Omega r_-) \\ Y_{+\frac{3}{2}}^{(-\infty, in)} &\sim \frac{K_s \Delta^{\|s\|}}{4r_+^{4\|s\|} \left(i\Omega - \frac{r_+ - 1}{2r_+^2} \right) \left(i\Omega - \frac{3}{4} \frac{r_+ - 1}{r_+^2} \right)} \exp(-i\Omega r_-). \end{aligned} \quad (2.32)$$

Using the above basis, the scattering problems of the perturbations are set as

$$\begin{aligned} Y_{+s} &\sim Y_{+s}^{(+\infty, in)} + R_s(\Omega) Y_{+s}^{(+\infty, out)} \quad (r_- \rightarrow \infty), \\ &\sim T_s(\Omega) Y_{+s}^{(-\infty, out)} \quad (r_- \rightarrow -\infty), \end{aligned} \quad (2.33)$$

where R_s and T_s are the reflection and transmission coefficients respectively.

3 The transformation law of the curvatures

In the previous section, we summarize the perturbation equations governing the physical modes. On the extreme Reissner-Nordström background, the quasinormal frequencies of the perturbations with the different helicity coincide by the suitable shift of the total angular momentum [21, 22]. This fact suggests that the reflection and transmission amplitudes are equivalent among the perturbations with the different helicity.

It is well known that the extreme Reissner-Nordström background has an unbroken global supersymmetry in $N = 2$ supergravity [13]. This implies that the perturbations with the different helicity are related to each other.

In this section, we obtain the supersymmetric transformation laws between the curvatures of the perturbed fields through $N = 2$ supergravity. The action of the $N = 2$ supergravity are represented by

$$\begin{aligned} \mathcal{L} = & -\frac{M_p^2 e}{2} R - \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} (\psi_{A\mu}^i e_{\nu}^{AA'} D_{\rho} \bar{\psi}_{A'\sigma}^i - \bar{\psi}_{A'\mu}^i e_{\nu}^{AA'} D_{\rho} \psi_{A\sigma}^i) \\ & - \frac{i}{32 M_p^2} \epsilon^{\mu\nu\rho\sigma} [(\epsilon^{ij} \psi_{\mu}^{iA} \psi_{A\nu}^j) (\epsilon^{kl} \psi_{\rho}^{kB} \psi_{B\sigma}^l) - (\epsilon^{ij} \bar{\psi}_{\mu}^{iA'} \bar{\psi}_{A'\nu}^j) (\epsilon^{kl} \bar{\psi}_{\rho}^{kB'} \bar{\psi}_{B'\sigma}^l)] \\ & - \frac{e}{4} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} + \frac{i}{8 M_p} \epsilon^{\mu\nu\rho\sigma} \hat{F}_{\rho\sigma} \epsilon^{ij} (\psi_{\mu}^{iA} \psi_{A\nu}^j + \bar{\psi}_{\mu}^{iA'} \bar{\psi}_{A'\nu}^j), \end{aligned} \quad (3.1)$$

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \frac{1}{2 M_p} \epsilon^{ij} (\psi_{\mu}^{iA} \psi_{A\nu}^j - \bar{\psi}_{\mu}^{iA'} \bar{\psi}_{A'\nu}^j) \quad (i, j, k, l = 1, 2), \quad (3.2)$$

where $M_p = (8\pi G)^{-1/2}$ is the Planck mass and $e_{\mu}^{AA'} = e_{\mu}^a \sigma_a^{AA'}$, and D_{μ} is covariant derivative with respect to ω_{μ}^{ab} . The connection $\omega_{\mu ab}$ including the torsion is given by tetrad and gravitini through varying the action with respect to $\omega_{\mu ab}$,

$$\omega_{\mu}^{ab} = \omega_{\mu}^{(0)ab} + K_{\mu}^{ab}, \quad (3.3)$$

$$\omega_{\mu}^{(0)ab} = e^{a\nu} \partial_{[\mu} e_{\nu]}^b - \frac{1}{2} e_{c\mu} e^{a\nu} e^{b\lambda} \partial_{\nu} e_{\lambda}^c - (a \leftrightarrow b), \quad (3.4)$$

$$\begin{aligned} K_{\mu}^{ab} = & \frac{i}{2 M_p^2} (e^{a\nu} \bar{\psi}_{A'[\mu}^i \sigma^{bAA'} \psi_{|A|\nu]}^j - \frac{1}{2} e_{c\mu} e^{a\nu} e^{b\lambda} \bar{\psi}_{A'\nu}^i \sigma^{cAA'} \psi_{A\lambda}^j) \\ & - (a \leftrightarrow b), \end{aligned} \quad (3.5)$$

and the curvature is given by

$$R_{\mu\nu}^{ab} = 2 \partial_{[\mu} \omega_{\nu]}^{ab} + 2 \omega_{[\mu}^{ac} \omega_{\nu]}^b{}_c, \quad (3.6)$$

$$R_{ab} = e_{\mu}^a e_{\nu}^b R^{\mu\nu}. \quad (3.7)$$

The action is invariant under the supersymmetric transformations,

$$\delta e_{a\mu} = -\frac{i}{2 M_p} (\alpha_{A'}^i \sigma_a^{AA'} \bar{\psi}_{A'\mu}^i + \bar{\alpha}_{A'}^i \sigma_a^{AA'} \psi_{A\mu}^i), \quad (3.8)$$

$$\delta A_{\mu} = -\frac{1}{2} \epsilon^{ij} (\alpha^{iA} \psi_{A\mu}^j - \bar{\alpha}^{iA'} \bar{\psi}_{A'\mu}^j), \quad (3.9)$$

$$\delta \psi_{A\mu}^i = M_p D_{\mu} \alpha_A^i - i \epsilon^{ij} \hat{F}_A{}^B e_{\mu BA'} \bar{\alpha}^{jA'}, \quad (3.10)$$

where α_A^i are Grassmann odd transformation parameters and $\hat{F}_{AB}$ is 2-spinor representation of the self-dual part of $\hat{F}_{\mu\nu}$.

We can check that $\omega_{\mu ab}$ and $\hat{F}_{\mu\nu}$ are supercovariant, i.e., their transformations have no derivative of transformation parameters. For the spin-3/2 fields, we introduce the supercovariant curvatures of $\psi_{ABA'}^i = e_{BA'}^\mu \psi_{A\mu}^i$ in 2-spinor representation,

$$\Psi^{iA}{}_{BC} = \frac{1}{2} \left[D_{(B|A'|} \psi^{iA}{}_{C)}^{A'} + \frac{i}{M_p} \epsilon^{ij} \hat{F}^A{}_{(B} \tilde{\psi}^{jA'}{}_{|A'|C)} \right], \quad (3.11)$$

$$\Psi^{iA}{}_{B'C'} = \frac{1}{2} \left[D_{B(B'} \psi^{iAB}{}_{C')} - \frac{i}{M_p} \epsilon^{ij} \hat{F}^A{}_{B} \tilde{\psi}^{jA'}{}_{(B'C')} \right]. \quad (3.12)$$

They are transformed according to

$$\delta \Psi^{iA}{}_{BC} = \frac{M_p}{2} R_{BC}{}^{AD} \alpha_D^i + \frac{i}{2} \epsilon^{ij} \left[D_{(B}{}^{A'} \hat{F}_{C)}{}^A \right] \tilde{\alpha}_{A'}^j + O(\psi^2), \quad (3.13)$$

$$\begin{aligned} \delta \Psi^{iA}{}_{B'C'} &= \frac{M_p}{2} R_{B'C'}{}^{AD} \alpha_D^i - \frac{i}{2} \epsilon^{ij} \left[D_{B(B'} \hat{F}^{AB} \right] \tilde{\alpha}_{C')}^j \\ &\quad + \frac{1}{2M_p} \hat{F}^{AD} \tilde{F}_{B'C'} \alpha_D^i + O(\psi^2). \end{aligned} \quad (3.14)$$

Because we will analyze the perturbations about a purely bosonic background, it is sufficient to obtain the transformation laws at linear order of $\psi_{A\mu}^i$.

We introduce an expansion parameter λ and replace fundamental fields, tetrad, connection, gravitini and electromagnetic potential about a background as, for example, $\psi_{A\mu}^i \rightarrow \boldsymbol{\psi}_{A\mu}^i + \lambda \psi_{A\mu}^i$, where we use the bold-face for the background quantities and standard letters for the perturbed quantities, respectively. Various equations and relations for perturbed fields are given by expanding with respect to λ .

The perturbed fields have gauge degree of freedoms originated from arbitrariness of correspondence between the perturbed world and the background world. Due to $\boldsymbol{\psi}_{A\mu}^i = 0$, the bosonic quantities are invariant under the supersymmetric gauge transformations and the fermionic quantities transform, for example, into

$$\delta_g \Psi_{ABC}^i = \frac{M_p}{2} R_{BCA}{}^D \beta_D^i + \frac{i}{2} \epsilon^{ij} \left[D_{(B}{}^{A'} \boldsymbol{F}_{C)A} \right] \tilde{\beta}_{A'}^j, \quad (3.15)$$

where β_A^i are any spinor parameters.

Let us consider supersymmetric transformation laws. Because of $\boldsymbol{\psi}_{A\mu}^i = 0$, the bosonic background quantities are invariant under supersymmetric transformation. On the other hand, fermionic quantities generally change due to non-trivial bosonic background. For example, the background gravitini transform under the supersymmetric transformation into

$$\delta \psi_{A\mu}^i = M_p D_\mu \alpha_A^i - i \epsilon^{ij} F_A{}^B e_{\mu B A'} \tilde{\alpha}^{j A'}, \quad (3.16)$$

and the perturbed supercovariant curvatures of gravitini transform into

$$\begin{aligned} \delta \Psi_{ABC}^i &= \frac{M_p}{2} R_{BCA}{}^D \alpha_D^i + \frac{i}{2} \epsilon^{ij} \left[D_{(B}{}^{A'} F_{C)A} \right] \tilde{\alpha}_{A'}^j \\ &\quad + \frac{i}{2} \epsilon^{ij} \sigma_{a(B}{}^{A'} \left[\omega^a{}_{C)}{}^D \boldsymbol{F}_{DA} + \omega^a{}_{|A|}{}^D \boldsymbol{F}_{C)D} \right] \tilde{\alpha}_{A'}^j. \end{aligned} \quad (3.17)$$

Therefore, if there are some supercovariantly constant spinors (SCCS's),

$$D_\mu \zeta_A^i - \frac{i}{M_p} \epsilon^{ij} F_A{}^B e_{\mu B A'} \bar{\zeta}^{j A'} = 0 , \quad (3.18)$$

the background configurations are invariant under the supersymmetric transformations that are induced by SCCS's. And then, unbroken supersymmetry persists on the system consisting of the perturbed fields.

We introduce the quantities constructed by Ψ_{ABC}^i ,

$$H_0^i \equiv \Psi_{(ABC)}^i o^A o^B o^C , \quad (3.19)$$

$$H_1^i \equiv \Psi_{(ABC)}^i o^A o^B \iota^C , \quad (3.20)$$

$$H_2^i \equiv \Psi_{(ABC)}^i o^A \iota^B \iota^C , \quad (3.21)$$

$$H_3^i \equiv \Psi_{(ABC)}^i \iota^A \iota^B \iota^C , \quad (3.22)$$

where o^A and ι^A are principal spinors of F_{AB} . Here we assume that the background space-time is in the Petrov type D. Then the physical modes are described by H_0^i or H_3^i because they are diffeomorphic, local Lorentz gauge invariant due to the purely bosonic background and supersymmetric gauge invariant due to the type D character and Eq.(3.15).

Therefore we are interested in the transformation laws of H_0^i generated by SCCS's, ζ_A^i ,

$$\begin{aligned} \delta H_0^i &= \frac{M_p}{2} [\zeta_{(0)}^i \Psi_1 - \zeta_{(1)}^i \Psi_0] \\ &+ \frac{i}{2} \epsilon^{ij} (D_a \phi_0) (\bar{\zeta}_{(0')}^j m^a - \bar{\zeta}_{(1')}^j l^a) \\ &+ i \epsilon^{ij} \phi_0 (\beta \bar{\zeta}_{(0')}^j - \epsilon \bar{\zeta}_{(1')}^j) \\ &- i \epsilon^{ij} \phi_1 (\sigma \bar{\zeta}_{(0')}^j - \kappa \bar{\zeta}_{(1')}^j) , \end{aligned} \quad (3.23)$$

where $\zeta_{(0)}^i = o^A \zeta_A^i$ and $\zeta_{(1)}^i = \iota^A \zeta_A^i$ and they have the spin-weight $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. And then, Ψ_0 , Ψ_1 , ϕ_0 , σ and κ are perturbed Weyl scalars, Maxwell scalar and complex spin coefficients, respectively. Further ϵ and β are background spin coefficients.

4 The relations of the reflection and transmittion coefficients

In the previous section, we obtained the transformation law between the perturbed curvatures of gravitini. Using it, we can relate the decoupled modes Y_{+s} on the extreme Reissner-Nordström black hole.

On the extreme hole, there exist the supercovariantly constant spinors,

$$\zeta_{(0)}^i = \sqrt{2} \eta_{(0)}^i(\theta) \exp(im' \phi) , \quad (4.1)$$

$$\zeta_{(1)}^i = \frac{\Delta^{\frac{1}{2}}}{r} \eta_{(1)}^i(\theta) \exp(im' \phi) , \quad (4.2)$$

where m' is $+\frac{1}{2}$ or $-\frac{1}{2}$ and η_A^i satisfy

$$\mathcal{L}_{-\frac{1}{2}}^{m'\dagger} \eta_{(0)}^i = \mathcal{L}_{-\frac{1}{2}}^{m'} \eta_{(1)}^i = 0 , \quad (4.3)$$

$$\mathcal{L}_{+\frac{1}{2}}^{m'} \eta_{(0)}^i = \eta_{(1)}^i , \quad (4.4)$$

$$\mathcal{L}_{+\frac{1}{2}}^{m'\dagger} \eta_{(1)}^i = -\eta_{(0)}^i . \quad (4.5)$$

where the operators $\mathcal{L}_n^m$ and $\mathcal{L}_n^{m\dagger}$ are the same operators as defined in Eqs.(2.13) and (2.14) in the previous section and we manifest azimuthal angular momentum dependence with index m . The supercovariantly constant spinors satisfy the relation,

$$\frac{i}{M_p} \epsilon^{ij} \phi_1 \bar{\zeta}_{A'}^j = -\gamma \left(\frac{2r^2}{\Delta} \bar{\iota}_{A'} \iota^A + \bar{o}_{A'} o^A \right) \zeta_A^i . \quad (4.6)$$

From Eq.(4.6), the transformation of H_0^i , Eq.(3.23) becomes

$$\delta H_0^i = \frac{M_p}{2} \left[\frac{\Delta^{\frac{1}{2}}}{r} \left(\Psi_0 + 4\gamma \frac{r^2}{\Delta} \sigma \right) \eta_{(1)}^i - \sqrt{2} (\Psi_1 - 2\gamma\kappa) \eta_{(0)}^i \right] e^{im'\phi} , \quad (4.7)$$

where we, of course adopt the phantom gauge $\phi_0 = \phi_2 = 0$.

According to Sec. II, we decompose Ψ_0 , Ψ_1 , κ and σ by the spin-weighted spherical harmonics, and we manifest angular momentum dependence. For example,

$$\Psi_0 = R_{+2}^J(r) S_{+2}^J(\theta) e^{i(\Omega t + m\phi)} , \quad (4.8)$$

$$\mathcal{L}_{-1}^{m\dagger} \mathcal{L}_2^m S_{+2}^J = -\mu_J^2 S_{+2}^J , \quad (4.9)$$

$$\mu_J = \sqrt{(J-1)(J+2)} . \quad (4.10)$$

And then

$$\begin{aligned} \delta H_0^i = & \frac{M_p}{2} \left[\frac{\Delta^{\frac{1}{2}}}{r} \left(R_{+2}^J + r \frac{d}{dr} \ln \left(\frac{\Delta}{r^2} \right) s^J \right) (\eta_{(1)}^i S_{+2}^J) \right. \\ & \left. - \frac{1}{r} \left(R_{+1}^J - r^3 \frac{d}{dr} \left(\frac{\Delta}{r^2} \right) k^J \right) (\eta_{(0)}^i S_{+1}^J) \right] e^{i(\Omega t + M\phi)} , \end{aligned} \quad (4.11)$$

where $M = m + m'$.

The each quantity, $\eta_{(1)}^i S_{+2}^J$ and $\eta_{(0)}^i S_{+1}^J$, has spin-weight $+3/2$, but not an eigenstate of total angular momentum respectively. Hence we need decompose them into $(S_{+3/2}^{J+1/2})$ and $(S_{+3/2}^{J-1/2})$ that are eigenstates of the total angular momentum. It is easy to check the equations

$$\mathcal{L}_{-\frac{1}{2}}^{M\dagger} \mathcal{L}_{\frac{3}{2}}^M (\eta_{(1)}^i S_{+2}^J) = -\mu_J (\eta_{(0)}^i S_{+1}^J) - \mu_J^2 (\eta_{(1)}^i S_{+2}^J) , \quad (4.12)$$

$$\mathcal{L}_{-\frac{1}{2}}^{M\dagger} \mathcal{L}_{\frac{3}{2}}^M (\eta_{(0)}^i S_{+1}^J) = -\mu_J (\eta_{(1)}^i S_{+2}^J) - (\mu_J^2 + 3) (\eta_{(0)}^i S_{+1}^J) . \quad (4.13)$$

From these equations, we can decompose $(\eta_{(1)}^i S_{+2}^J)$ and $(\eta_{(0)}^i S_{+1}^J)$ as

$$\eta_{(1)}^i S_{+2}^J = \xi^i \frac{S_{+\frac{3}{2}}^{J+\frac{1}{2}} - S_{+\frac{3}{2}}^{J-\frac{1}{2}}}{q_2 - q_1} , \quad (4.14)$$

$$\eta_{(0)}^i S_{+1}^J = \xi^i \frac{q_1 S_{+\frac{3}{2}}^{J+\frac{1}{2}} - q_2 S_{+\frac{3}{2}}^{J-\frac{1}{2}}}{2\mu_J(q_2 - q_1)} , \quad (4.15)$$

where ξ^i are arbitrary Grassmann odd constants and $q_{1,2}$ are defined by

$$\begin{aligned} q_1 &\equiv 3 + \sqrt{9 + 4\mu_J^2} = 2(J + 2) , \\ q_2 &\equiv 3 - \sqrt{9 + 4\mu_J^2} = -2(J - 1) . \end{aligned} \quad (4.16)$$

The expressions of q_j are the extreme limit of Eq.(2.6).

Because $\Delta = (r - 1)^2$ in the extreme case, Eq.(4.11) is rewritten as

$$\begin{aligned} \delta H_0^i &= \frac{M_p \Delta^{\frac{1}{2}} \xi^i}{2r(q_2 - q_1)} \left[S_{+\frac{3}{2}}^{J+\frac{1}{2}} \left(F_{+1}^J - \frac{q_1}{2\Delta^{\frac{1}{2}}\mu_J} G_{+2}^J \right) \right. \\ &\quad \left. - S_{+\frac{3}{2}}^{J-\frac{1}{2}} \left(F_{+2}^J - \frac{q_2}{2\Delta^{\frac{1}{2}}\mu_J} G_{+1}^J \right) \right] e^{i(\Omega t + M\phi)} , \\ &= \frac{M_p \Delta^{\frac{1}{2}} \xi^i}{2r(q_2 - q_1)} \left[S_{+\frac{3}{2}}^{J+\frac{1}{2}} \left\{ F_{+1}^J - \frac{q_1}{2} \frac{r\Delta^{\frac{1}{2}}}{\mu_J^2 r + 2q_1} \left(D_2^{\dagger} - \frac{3}{r} \right) F_{+1}^J \right\} \right. \\ &\quad \left. - S_{+\frac{3}{2}}^{J-\frac{1}{2}} \left\{ F_{+2}^J - \frac{q_2}{2} \frac{r\Delta^{\frac{1}{2}}}{\mu_J^2 r + 2q_2} \left(D_2^{\dagger} - \frac{3}{r} \right) F_{+2}^J \right\} \right] e^{i(\Omega t + M\phi)} , \end{aligned} \quad (4.17)$$

where we use the relation Eq.(2.18). Eq.(4.17) shows that the helicity- $(+3/2)$ modes with $J + 1/2$ are generated by the unbroken supersymmetry from the helicity- $(+1)$ mode F_{+1}^J with total angular momentum J or the helicity- $(+2)$ mode F_{+2}^{J+1} with $J + 1$.

From Eq.(4.17), radial parts $Y_{+\frac{3}{2}}^{J_s}$ of perturbed curvatures of gravitini are generated from Y_{+s}^J as

$$Y_{+\frac{3}{2}}^{J_s} = \xi^k \left[Y_{+s}^J - C_s^J(r) \Lambda_- Y_{+s}^J \right] , \quad (4.18)$$

or equivalently,

$$\xi^k \left[2i\Omega + \frac{1}{C_s^J} - C_s^J Q_s + \frac{1}{2} P_{\frac{3}{2}} \right] Y_{+s}^J = \left[2i\Omega + \frac{1}{C_s^J} + \frac{1}{2} P_{\frac{3}{2}} \right] Y_{+\frac{3}{2}}^{J_s} + \Lambda_- Y_{+\frac{3}{2}}^{J_s} , \quad (4.19)$$

where

$$C_s^J(r) \equiv \frac{q_s r^2 \Delta^{\frac{3}{2}}}{2\mu_J^2 D_s} , \quad D_s \equiv \Delta^2 \left(1 + \frac{2q_s}{\mu_J^2 r} \right) , \quad (4.20)$$

and J_s is $J + \frac{1}{2}$ for $s = 1$ and $J - \frac{1}{2}$ for $s = 2$. Eqs.(4.18) and (4.19) are our main result and in principle, we can also obtain the relations between potentials of perturbations with different helicities. Hereafter we omit the index k which distinguishes two gravitini.

From Eqs.(4.18) and (4.19), we can obtain the relation between reflection and transmission amplitudes. From them, it follows that $Y_{+\frac{3}{2}}^{J_s}$ derived from $Y_{+s}^{J(+\infty, in)}$ and $Y_{+s}^{J(+\infty, out)}$ ($s = 1, 2$) have, respectively, the asymptotic behaviors at $r_- \rightarrow \infty$

$$Y_{+\frac{3}{2}}^{J_s} \sim Y_{+\frac{3}{2}}^{J_s(+\infty, in)} \quad \text{and} \quad Y_{+\frac{3}{2}}^{J_s} \sim \frac{i\Omega q_s}{\mu_J^2} \frac{K_s}{K_{\frac{3}{2}}} Y_{+\frac{3}{2}}^{J_s(+\infty, out)} . \quad (4.21)$$

Similarly, it follows that $Y_{+\frac{3}{2}}^{J_s}$ derived from $Y_{+s}^{J(-\infty, in)}$ and $Y_{+s}^{J(-\infty, out)}$ ($s = 1, 2$) have, respectively, the asymptotic behaviors at $r_- \rightarrow -\infty$

$$Y_{+\frac{3}{2}}^{J_s} \sim Y_{+\frac{3}{2}}^{J_s(-\infty, out)} \quad \text{and} \quad Y_{+\frac{3}{2}}^{J_s} \sim \frac{i\Omega q_s}{\mu_J^2} \frac{K_s}{K_{\frac{3}{2}}} Y_{+\frac{3}{2}}^{J_s(-\infty, in)} . \quad (4.22)$$

Therefore the asymptotic form of $Y_{+\frac{3}{2}}^{J_s}$ derived from the solution for $Y_{+s}^J (s = 1, 2)$ having the asymptotic behavior

$$\begin{aligned} Y_{+s}^J &\sim Y_{+s}^{J(+\infty, in)} + R_s^J(\Omega) Y_{+s}^{J(+\infty, out)} & (r_+ \rightarrow \infty) , \\ &\sim T_s^J(\Omega) Y_{+s}^{J(-\infty, out)} & (r_+ \rightarrow -\infty) , \end{aligned} \quad (4.23)$$

has the asymptotic behavior

$$\begin{aligned} Y_{+\frac{3}{2}}^{J_s} &\sim Y_{+\frac{3}{2}}^{J_s(+\infty, in)} + R_s^J(\Omega) \frac{i\Omega q_s}{\mu_J^2} \frac{K_s}{K_{\frac{3}{2}}} Y_{+\frac{3}{2}}^{J_s(+\infty, out)} & (r_+ \rightarrow \infty) , \\ &\sim T_s^J(\Omega) Y_{+\frac{3}{2}}^{J_s(-\infty, out)} & (r_+ \rightarrow -\infty) . \end{aligned} \quad (4.24)$$

Accordingly, we obtain the relations of reflection and transmission coefficients,

$$R_{\frac{3}{2}}^{J_s}(\Omega) = \gamma_s R_s^J(\Omega) \quad T_{\frac{3}{2}}^{J_s}(\Omega) = T_s^J(\Omega) \quad (s = 1, 2) , \quad (4.25)$$

$$\gamma_s \equiv \frac{i\Omega q_s}{\mu_J^2} \frac{K_s}{K_{\frac{3}{2}}} , \quad (4.26)$$

where $|\gamma_s| = 1$. Thus, under the suitable shift of angular momentums, while the amplitudes of the transmitted waves are identically the same for three perturbed fields, the reflected amplitudes differ only in their phases.

5 Summary

In the previous section, using the unbroken supersymmetry that the extreme Reissner-Nordström black hole has, we obtain the relation between the reflection and transmission coefficients of decoupled modes with (helicity, total angular momentum) $= (1, J), (\frac{3}{2}, J + \frac{1}{2}), (2, J + 1)$.

These relations are also expected for the perturbations about the superpartners of the extreme Reissner-Nordström black hole [29] and for matter multiplets about them.

In the Ref. [22], we observed that the Regge-Wheeler potential of gravitational perturbation coincides with one of electromagnetic perturbation by inversion of the tortoise coordinate, that is, exchange of the horizon for infinity, vice versa. It is interesting to understand the above correspondence by using the relations of the perturbations obtained in the previous section.

Acknowledgments

The author appreciate the opportunity of talking about this work. He would like to thanks H. Onozawa, T. Mishima and H. Ishihara for valuable comments and stimulating discussions. He also appreciate Professor A. Hosoya for continuous encouragement. The research was supported in part by the Scientific Research Fund of the Ministry of Education.

A

In this Appendix, we summarize 2-spinor notations. At first, we introduce Infeld-van Waerden symbols,

$$\sigma_{AA'}^a = \frac{1}{\sqrt{2}}(1, \Sigma^i), \quad (\text{A.1})$$

where Σ^i are the ordinary Pauli matrices,

$$\Sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.2})$$

The indices A, A' , etc. are raised and lowered by

$$\eta^A = \epsilon^{AB} \eta_B, \quad \zeta_A = \zeta^B \epsilon_{BA}, \quad (\text{A.3})$$

where

$$\epsilon_{AB} = \epsilon^{AB} = \epsilon_{A'B'} = \epsilon^{A'B'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (\text{A.4})$$

We also define the various matrices,

$$(\sigma^a)^{AA'} \equiv \epsilon^{A'B'} \epsilon^{AB} \sigma_{BB'}^a, \quad (\text{A.5})$$

$$(\sigma^{ab})_A{}^B \equiv i \sigma_{AA'}^{[a} \sigma^{b] B A'}, \quad (\text{A.6})$$

$$(\bar{\sigma}^{ab})^{A'}{}_{B'} \equiv i \sigma^{[a|A A'|} \sigma_{A B'}^{b]}. \quad (\text{A.7})$$

The matrices $(\sigma^{ab})_{AB}$ and $(\bar{\sigma}^{ab})_{A'B'}$ are symmetric matrices.

Using these matrices, we obtain the 2-spinor representations of various tensors, for example,

$$v_{AA'} = v_a \sigma_{AA'}^a, \quad v_a = v_{AA'} \sigma_a^{AA'}. \quad (\text{A.8})$$

Further we have the equalities:

$$[(\sigma^{ab})^*]_{A'B'} = (\bar{\sigma}^{ab})_{A'B'} \quad (\text{A.9})$$

$$\sigma_{AA'}^a \sigma^{b B A'} = \frac{\eta^{ab}}{2} \epsilon_{AB} - i (\sigma^{ab})_{AB} \quad (\text{A.10})$$

$$\sigma^{a A}{}_{A'} \sigma^b_{A B'} = -\frac{\eta^{ab}}{2} \epsilon_{A'B'} - i (\bar{\sigma}^{ab})^{A'B'} \quad (\text{A.11})$$

$$\sigma_{AA'}^a \sigma^{b A A'} = \eta^{ab} \quad (\text{A.12})$$

$$\sigma_{AA'}^a \sigma_a^{B B'} = \delta_A^B \delta_{A'}^{B'} \quad (\text{A.13})$$

$$\sigma_{AA'}^a \sigma_{BB'}^b = \frac{\eta^{ab}}{4} \epsilon_{AB} \epsilon_{A'B'} - \frac{i}{2} \epsilon_{A'B'} (\sigma^{ab} \epsilon)_{AB} + \frac{i}{2} \epsilon_{AB} (\epsilon^T \bar{\sigma}^{ab})_{A'B'} - (\sigma^{ac} \epsilon)_{AB} (\epsilon^T \bar{\sigma}^b_c)_{A'B'} \quad (\text{A.14})$$

$$(\sigma^{ab} \epsilon)_{AB} (\epsilon^T \bar{\sigma}_{ab})_{A'B'} = 0 \quad (\text{A.15})$$

$$\sigma^{ab} = \frac{i}{2} \epsilon^{abcd} \sigma_{cd} \quad (\text{A.16})$$

$$\bar{\sigma}^{ab} = -\frac{i}{2} \epsilon^{abcd} \bar{\sigma}_{cd} \quad (\text{A.17})$$

$$\epsilon_{0123} = +1 \quad (\text{A.18})$$

$$\begin{aligned} (\sigma^{ac}\epsilon)_{AB} (\sigma^b_c\epsilon)_{CD} &= \frac{\eta^{ab}}{2} \epsilon_{A(C\epsilon D)B} \\ &- \frac{i}{2} (\sigma^{ab}\epsilon)_{A(C\epsilon D)B} - \frac{i}{2} (\sigma^{ab}\epsilon)_{B(C\epsilon D)A} \end{aligned} \quad (\text{A.19})$$

$$(\sigma^{ab}\epsilon)_{AB} (\sigma_{ab}\epsilon)_{CD} = 2 \epsilon_{A(C\epsilon D)B} \quad (\text{A.20})$$

$$\begin{aligned} \sigma^{ab} \sigma^{cd} &= \frac{i}{4} \epsilon^{abcd} + \frac{1}{2} \eta^a[c \eta^{d]b} \\ &+ \frac{i}{2} (\eta^a[c \sigma^{d]b} - \eta^b[c \sigma^{d]a} - \eta^c[a \sigma^{b]d} + \eta^d[a \sigma^{b]c}) \end{aligned} \quad (\text{A.21})$$

$$\text{Tr}(\sigma^{ab} \sigma_{cd}) = \frac{i}{2} \epsilon^{ab}_{cd} + \delta^{ab}_{cd} \quad (\text{A.22})$$

B

We summarize the notations in Newman-Penrose formalism.

$$l^a = \sigma^a_{AA'} o^A \bar{o}^{A'} \quad (\text{B.1})$$

$$n^a = \sigma^a_{AA'} \iota^A \bar{\iota}^{A'} \quad (\text{B.2})$$

$$m^a = \sigma^a_{AA'} o^A \bar{\iota}^{A'} \quad (\text{B.3})$$

$$\bar{m}^a = \sigma^a_{AA'} \iota^A \bar{o}^{A'} \quad (\text{B.4})$$

where the spinors o^A and ι^A are normalized by $o_A \iota^A = 1$. Thus, the vectors l^a , n^a , m^a and $\bar{m}^a$ are null tetrad basis,

$$l^a n_a = 1, \quad m^a \bar{m}_a = -1. \quad (\text{B.5})$$

Using these basis, we can write the Pauli matrices, and so on:

$$\eta_{ab} = 2 l_{(a} n_{b)} - 2 m_{(a} \bar{m}_{b)} \quad (\text{B.6})$$

$$\epsilon_{AB} = o_A \iota_B - \iota_A o_B \quad (\text{B.7})$$

$$\sigma^a_{AA'} = l^a \iota_A \bar{\iota}_{A'} - m^a \iota_A \bar{o}_{A'} - \bar{m}^a o_A \bar{\iota}_{A'} + n^a o_A \bar{o}_{A'} \quad (\text{B.8})$$

$$\begin{aligned} \sigma^a_{AA'} \sigma^b_{B A'} &= \frac{\eta^{ab}}{2} \epsilon_{AB} + 2 l^{[a} m^{b]} \iota_A \iota_B - 2 n^{[a} \bar{m}^{b]} o_A o_B \\ &- 2(l^{[a} n^{b]} - m^{[a} \bar{m}^{b]}) o_{(A} \iota_{B)} \end{aligned} \quad (\text{B.9})$$

$$\begin{aligned} -\frac{i}{2} (\sigma^{ab}\epsilon)_{AB} &= l^{[a} m^{b]} \iota_A \iota_B - n^{[a} \bar{m}^{b]} o_A o_B \\ &- (l^{[a} n^{b]} - m^{[a} \bar{m}^{b]}) o_{(A} \iota_{B)} \end{aligned} \quad (\text{B.10})$$

Further we define the covariant derivative by the Ricci-rotation coefficients as

$$D_\mu v_a = \partial_\mu v_a + \omega_{\mu a}{}^b v_b, \quad (\text{B.11})$$

$$D_\mu \eta_A = \partial_\mu \eta_A + \omega_{\mu A}{}^B \eta_B, \quad (\text{B.12})$$

$$D_\mu \zeta_{A'} = \partial_\mu \zeta_{A'} + \bar{\omega}_{\mu A'}{}^{B'} \zeta_{B'}, \quad (\text{B.13})$$

$$\omega_{\mu AA' BB'} \equiv \omega_{\mu ab} \sigma_{AA'}^a \sigma_{BB'}^b = \omega_{\mu AB} \epsilon_{A' B'} + \bar{\omega}_{\mu A' B'} \epsilon_{AB}, \quad (\text{B.14})$$

$$\omega_{\mu AB} \equiv -\frac{i}{2} (\sigma^{ab} \epsilon)_{AB} \omega_{\mu ab}, \quad (\text{B.15})$$

$$\bar{\omega}_{\mu A' B'} \equiv \frac{i}{2} (\epsilon^T \bar{\sigma}^{ab})_{A' B'} \omega_{\mu ab}. \quad (\text{B.16})$$

And we define 12 complex spin coefficients as

$$\kappa = l^\mu (m^a D_\mu l_a) = l^\mu (o^A D_\mu o_A), \quad (\text{B.17})$$

$$\sigma = m^\mu (m^a D_\mu l_a) = m^\mu (o^A D_\mu o_A), \quad (\text{B.18})$$

$$\rho = \bar{m}^\mu (m^a D_\mu l_a) = \bar{m}^\mu (o^A D_\mu o_A), \quad (\text{B.19})$$

$$\tau = n^\mu (m^a D_\mu l_a) = n^\mu (o^A D_\mu o_A), \quad (\text{B.20})$$

$$\pi = -l^\mu (\bar{m}^a D_\mu n_a) = l^\mu (\iota^A D_\mu \iota_A), \quad (\text{B.21})$$

$$\mu = -m^\mu (\bar{m}^a D_\mu n_a) = m^\mu (\iota^A D_\mu \iota_A), \quad (\text{B.22})$$

$$\lambda = -\bar{m}^\mu (\bar{m}^a D_\mu n_a) = \bar{m}^\mu (\iota^A D_\mu \iota_A), \quad (\text{B.23})$$

$$\nu = -n^\mu (\bar{m}^a D_\mu n_a) = n^\mu (\iota^A D_\mu \iota_A), \quad (\text{B.24})$$

$$\epsilon = \frac{l^\mu}{2} (n^a D_\mu l_a - \bar{m}^a D_\mu m_a) = l^\mu (o^A D_\mu \iota_A), \quad (\text{B.25})$$

$$\beta = \frac{m^\mu}{2} (n^a D_\mu l_a - \bar{m}^a D_\mu m_a) = m^\mu (o^A D_\mu \iota_A), \quad (\text{B.26})$$

$$\alpha = \frac{\bar{m}^\mu}{2} (n^a D_\mu l_a - \bar{m}^a D_\mu m_a) = \bar{m}^\mu (o^A D_\mu \iota_A), \quad (\text{B.27})$$

$$\gamma = \frac{n^\mu}{2} (n^a D_\mu l_a - \bar{m}^a D_\mu m_a) = n^\mu (o^A D_\mu \iota_A), \quad (\text{B.28})$$

And we construct five complex Weyl scalars from Weyl tensors as

$$\Psi_0 \equiv -C_{abcd} l^a m^b \bar{c} m^d = -C_{ABCD} o^A o^B \bar{o}^C o^D, \quad (\text{B.29})$$

$$\Psi_1 \equiv -C_{abcd} l^a n^b \bar{c} m^d = -C_{ABCD} o^A o^B \bar{o}^C \iota^D, \quad (\text{B.30})$$

$$\Psi_2 \equiv -C_{abcd} l^a m^b \bar{m}^c n^d = -C_{ABCD} o^A o^B \bar{\iota}^C \iota^D, \quad (\text{B.31})$$

$$\Psi_3 \equiv -C_{abcd} l^a n^b \bar{m}^c n^d = -C_{ABCD} o^A \bar{\iota}^B \bar{\iota}^C \iota^D, \quad (\text{B.32})$$

$$\Psi_4 \equiv -C_{abcd} n^a \bar{m}^b n^c \bar{m}^d = -C_{ABCD} \bar{\iota}^A \bar{\iota}^B \bar{\iota}^C \bar{\iota}^D, \quad (\text{B.33})$$

where C_{ABCD} is the spinor representation of the Weyl curvature,

$$C_{ABCD} \equiv C_{abcd} \left(-\frac{i}{2} (\sigma^{ab} \epsilon)_{AB} \right) \left(-\frac{i}{2} (\sigma^{cd} \epsilon)_{CD} \right). \quad (\text{B.34})$$

We also define three complex Maxwell scalars from the electromagnetic field strength as

$$\phi_0 \equiv F_{ab} l^a m^b = F_{AB} o^A o^B, \quad (\text{B.35})$$

$$\phi_1 \equiv \frac{1}{2} F_{ab} (l^a n^b - m^a \bar{m}^b) = F_{AB} o^A \iota^B, \quad (\text{B.36})$$

$$\phi_2 \equiv F_{ab} (-n^a \bar{m}^b) = F_{AB} \iota^A \iota^B, \quad (\text{B.37})$$

where F_{AB} is the spinor representation of the electromagnetic field strength,

$$F_{AB} \equiv F_{ab} \left(-\frac{i}{2} (\sigma^{ab} \epsilon)_{AB} \right). \quad (\text{B.38})$$

It is convenient to classify the spin coefficients, Weyl and Maxwell scalars by their transformation laws under the transformations,

$$l^a \rightarrow r l^a, \quad n^a \rightarrow r^{-1} n^a, \quad (\text{B.39})$$

and

$$m^a \rightarrow e^{i\theta} m^a. \quad (\text{B.40})$$

Eqs.(B.39) and (B.40) correspond to a boost in (l, n) -plane and rotation in $(m, \bar{m})$ -plane, respectively. r and θ are functions of spacetime position. In terms of o^A and ι^A , they are given by

$$o^A \rightarrow \lambda o^A, \quad \iota^A \rightarrow \lambda \iota^A, \quad (\text{B.41})$$

where $\lambda = r^{1/2} e^{i\theta/2}$. A scalar F is called a scalar of type (p, q) for the above transformation, when it transforms as

$$F \rightarrow \lambda^p \bar{\lambda}^q F = r^{(p+q)/2} e^{i\theta(p-q)/2}. \quad (\text{B.42})$$

And $(p+q)/2$ and $(p-q)/2$ are called the boost and spin weight respectively.

The type of spin coefficients are as follows:

$$\begin{aligned} & \kappa(3, 1), \quad \sigma(3, -1), \quad \rho(1, 1), \quad \tau(1, -1), \\ & \pi(-1, 1), \quad \mu(-1, -1), \quad \lambda(-3, 1), \quad \nu(-3, -1), \end{aligned} \quad (\text{B.43})$$

where α, β, γ and ϵ are not listed because they do not transform as Eq.(B.42).

For the Weyl and Maxwell scalars, their type are given by

$$\Psi_0(4, 0), \quad \Psi_1(2, 0), \quad \Psi_2(0, 0), \quad \Psi_3(-2, 0), \quad \Psi_4(-4, 0), \quad (\text{B.44})$$

and

$$\phi_0(2, 0), \quad \phi_1(0, 0), \quad \phi_2(-2, 0). \quad (\text{B.45})$$

Therefore the quantities with the spin weight ± 2 are σ, Ψ_0 and λ, Ψ_4 . Hence we may expect that the physical modes of gravitational waves are expressed by them.

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Worldline quantization of gravity

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Abstract

The classical field equations of general relativity can be expressed as a single geodesic equation, describing the free fall of a point particle in superspace. We propose a “worldline” quantization of gravity, analogous to the Feynman-Schwinger treatment of particle propagation. The effective action induced at one loop for the supermetric, in certain minisuperspace models, is found to be stationary for vanishing cosmological constant. Finally, we suggest that the off-shell propagation of an interacting “particle” in superspace might be detectable as a new source of decoherence.

1 Introduction

In one of the classic papers of quantum electrodynamics, Feynman [1] (see also ref. [2]) suggested that relativistic electron propagation could be understood in terms of a sum over electron worldlines running both forwards and backwards in time. The evolution parameter was a path parameter, associated with the proper time of the electron worldlines, rather than the “clock time” of the laboratory. Here we extend Feynman’s worldline quantization of electron paths in spacetime to the quantization of a closed Universe propagating in superspace.

In the proper-time formulation for a relativistic, spinless particle of mass m , propagating freely on a background spacetime with metric $g_{\mu\nu}$, the classical motion (i.e. the geodesic equation) is derived from variation of the worldline action

$$S_p = -m \int ds = -m \int d\tau \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \quad (1)$$

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Removing the square-root by introduction of a Lagrange multiplier (lapse) N , and applying the usual Legendre transform, one obtains the first-order form

$$S_p'' = \int d\tau \left[p_\mu \frac{dx^\mu}{d\tau} - NH \right] ; \quad H = \frac{1}{2m}(g^{\mu\nu} p_\mu p_\nu + m^2) \quad (2)$$

Now go to the gauge $N = 1$. In this gauge, $\tau = s$ is the proper-time parameter of the classical equations of motion. Adopting s as the evolution parameter for the quantized theory leads to the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial s} = \mathcal{H}_s \psi \quad (3)$$

where, up to an operator-ordering, the Hamiltonian operator $\mathcal{H}_s$ is obtained from the classical Hamiltonian (with $N = 1$) via the usual replacement of c-number momenta by the corresponding operators.

The one-loop contribution to the gravitational effective action

$$S_{eff}[g_{\mu\nu}] = \frac{1}{2} i\hbar \text{Tr} \ln[\mathcal{H}_s] \quad (4)$$

is the trace logarithm of $\mathcal{H}_s$. We note that, although classically the mass-shell condition $H = 0$ is to be respected, the eigenvalues of the proper time Hamiltonian $\mathcal{H}_s$, such as those used in the evaluation of the effective action, can take on any value. In a quantum interacting theory the 4-momentum of a virtual particle is allowed to violate the mass-shell condition, and this condition may be relaxed.

2 The Worldline Action of a Closed Universe

Generalizing the proper-time approach to the case of gravity in combination with any number of interacting bosonic fields, we want to identify the gravitational action as

$$S_g = -\mathcal{M} \int ds \quad (5)$$

where s is an invariant length parameter in the space of all fields modulo spatial diffeomorphisms, i.e. superspace, and $\mathcal{M}$ is an arbitrary dimensionless parameter (see refs. [3] and [4]).

Let $\{q^a(x), p_a(x)\}$ represent the set of gravitational and other bosonic fields, and their conjugate momenta, with the fields scaled by an appropriate power of $\kappa^2 = 16\pi G_N$ so as to be dimensionless. In a condensed notation, the standard ADM action has the form

$$S_{ADM} = \int d^4x [p_a \partial_0 q^a - N \mathcal{H}_x - N_i \mathcal{H}_x^i] ; \quad \mathcal{H}_x = \kappa^2 G^{ab} p_a p_b + \sqrt{g} U(q) \quad (6)$$

where G^{ab} and U are, respectively, the metric and potential in superspace (and $\mathcal{H}_x^i$ is the supermomentum, linear in p_a). Solving Hamilton's equation and the Hamiltonian

constraint $\mathcal{H} = 0$ for the lapse function N in terms of velocities, one obtains the Baierlein-Sharp-Wheeler (BSW) form of the gravitational action [5]

$$S_{BSW} = - \int d^4x \sqrt{-\frac{1}{\kappa^2} \sqrt{g} U G_{ab} \partial_0 q^a \partial_0 q^b} \quad (7)$$

in the shift gauge $N_i = 0$.³ The BSW action is to serve as a proper-time parameter. We also introduce an arbitrary mass-scale σ in order to define an evolution parameter t with dimensions of time, so that

$$S_{BSW} = - \int ds = -\sigma \int dt \quad (8)$$

Choosing $x_0 = t$, comparing (8) with (7), and denoting $\tilde{N}$ the lapse function (derived by solving the Hamiltonian constraint) associated with the ('many fingered') time parameter t , it is possible to show [4] that $\tilde{N}$ must satisfy the global condition

$$\int d^3x \tilde{N} \sqrt{g} U = \frac{1}{2} \sigma \quad (9)$$

which is solved trivially by

$$\tilde{N} = \frac{\frac{1}{2} \sigma \mathcal{N}}{\int d^3x \mathcal{N} \sqrt{g} U} \quad (10)$$

where $\mathcal{N}$ is unrestricted.⁴ Inserting this form for $\tilde{N}$ back into (7), we find

$$ds^2 = - \int d^3x \left[\int d^3x' \mathcal{N} \sqrt{g} U \right] \frac{1}{\mathcal{N} \kappa^2} G_{ab} dq^a dq^b \quad (11)$$

Introducing a mixed discrete/continuous "coordinate index" (α, x) in superspace and extending the definition of superspace to include the non-dynamical field $\mathcal{N}(x)$, i.e.

$$q^{(\alpha x)} \equiv q^\alpha(x) = \begin{cases} \mathcal{N}(x) & \alpha = 0 \\ q^a(x) & \alpha = a \neq 0 \end{cases} \quad (12)$$

with associated (degenerate) supermetric

$$\begin{aligned} \mathcal{G}_{(\alpha x)(\beta y)} &= \left[\int d^3x' \mathcal{N} \sqrt{g} U \right] \frac{1}{\mathcal{N}(x) \kappa^2} G_{ab}(x) \delta^3(x - y) \\ \mathcal{G}_{(0x)(0y)} &= \mathcal{G}_{(\alpha x)(0y)} = \mathcal{G}_{(0x)(\beta y)} = 0 \end{aligned} \quad (13)$$

eq. (11) becomes

$$ds^2 = - \mathcal{G}_{(\alpha x)(\beta y)} dq^{(\alpha x)} dq^{(\beta y)} \quad (14)$$

The gravitational action then has the desired form

$$S_g = -\mathcal{M} \int ds = -\mathcal{M} \int d\tau \sqrt{-\mathcal{G}_{(\alpha x)(\beta y)} \frac{dq^{(\alpha x)}}{d\tau} \frac{dq^{(\beta y)}}{d\tau}} \quad (15)$$

³The supermomentum constraints $\mathcal{H}^i = 0$ are not lost by this choice, since they are still required for consistency of the Hamiltonian constraint with the equations of motion.

⁴This does not result in any restriction on the choice of foliation, but only on the time-label t associated with each hypersurface of a given foliation.

Variation of the action S_g w.r.t $q^{(\alpha x)}$ leads, in the usual way, to a geodesic equation

$$\mathcal{G}_{(\alpha x)(\beta y)} \frac{d^2 q^{(\beta y)}}{ds^2} + \frac{1}{2} \left(\frac{\delta \mathcal{G}_{(\alpha x)(\beta y)}}{\delta q^{(\gamma z)}} + \frac{\delta \mathcal{G}_{(\alpha x)(\gamma z)}}{\delta q^{(\beta y)}} - \frac{\delta \mathcal{G}_{(\beta y)(\gamma z)}}{\delta q^{(\alpha x)}} \right) \frac{dq^{(\beta y)}}{ds} \frac{dq^{(\gamma z)}}{ds} = 0 \quad (16)$$

Identifying $ds = \sigma dt$, it is straightforward to verify that the $\alpha \neq 0$ and $\alpha = 0$ components of (16) are identical, respectively, to the Hamilton's equations of motion and the Hamiltonian constraint obtained from the ADM action (6), with the gauge choice $N_i = 0$ and $N = \tilde{N}$. We have therefore interpreted the classical field equations of general relativity as describing the free fall of a point particle in superspace.⁵

We can now proceed as in the relativistic particle case to derive the proper-time Hamiltonian. Again introducing a Lagrange multiplier n to remove the square-root and going to the first-order form, we obtain

$$S_g'' = \int d\tau \left[p_{(ax)} \frac{dq^{(ax)}}{d\tau} - nH \right] ; \quad H = \frac{1}{2\mathcal{M}} [\mathcal{E} + \mathcal{M}^2] \quad (17)$$

where the expression

$$\mathcal{E} = \frac{\int d^3x \mathcal{N} \kappa^2 G^{ab} p_a p_b}{\int d^3x' \mathcal{N} \sqrt{g} U} \quad (18)$$

was introduced in ref. [6]. Variation of (17) with respect to $q^a(x, \tau)$, $p_a(x, \tau)$ and $\mathcal{N}(x, \tau)$, $n(\tau)$ give us, respectively, the set of Hamiltonian equations and constraints

$$\partial_\tau q^a(x) = \frac{n}{2\mathcal{M}} \frac{\delta \mathcal{E}}{\delta p_a(x)} , \quad \partial_\tau p_a(x) = -\frac{n}{2\mathcal{M}} \frac{\delta \mathcal{E}}{\delta q^a(x)} , \quad \frac{\delta \mathcal{E}}{\delta \mathcal{N}(x)} = 0 , \quad \mathcal{E} = -\mathcal{M}^2 \quad (19)$$

Setting $n = 1$, so that $\tau = s = \sigma t$, these equations are equivalent to the usual Hamiltonian equations of motion and Hamiltonian constraint

$$\begin{aligned} \partial_t q^a(x) &= \int d^3x' \tilde{N} \frac{\delta}{\delta p_a(x)} \mathcal{H} \\ \partial_t p_a(x) &= - \int d^3x' \tilde{N} \frac{\delta}{\delta q^a(x)} \mathcal{H} \\ \mathcal{H} &= \frac{\kappa^2}{\mathcal{M}} G^{ab} p_a p_b + \mathcal{M} \sqrt{g} U = 0 \end{aligned} \quad (20)$$

in the $N = \tilde{N}$, $N_i = 0$ gauge. These equations of motion and (Hamiltonian) constraint imply the remaining supermomentum constraint as a consistency condition.

The constant $\mathcal{M}$ is implicitly set to $\mathcal{M} = 1$ in the usual Hamiltonian formulation of general relativity, although there is no overwhelming reason to make this choice. The constant $\mathcal{M}$ appears as a constant multiplicative factor in the worldline action (15), as does the mass m in the worldline action (1). Both of these constants drop out of the corresponding geodesic equations, and there is no way of determining them from a given solution of the configuration-space field equations. In the context of the first-order formulation, the condition $\mathcal{E} = -\mathcal{M}^2$ is in every sense analogous to the particle mass-shell condition $g^{\mu\nu} p_\mu p_\nu = -m^2$, and we then dignify it with the title "mass-shell of the Universe".

⁵The degeneracy of the supermetric $\mathcal{G}_{(\alpha x)(\beta y)}$ in eq. (13) implies an infinite set of solutions for the geodesic between any two points in superspace. These solutions are related by D=4 time-reparametrizations, and have the same "proper time" interval in superspace between those two points.

3 Quantization

We now consider canonical quantization, in the “proper-time” gauge $n = 1$. The corresponding Schrödinger equation is

$$i\hbar \frac{\partial \Psi}{\partial s} = \mathcal{H}_s \Psi = \frac{1}{2\mathcal{M}}(\mathcal{E} + \mathcal{M}^2)\Psi \quad (21)$$

which has the general s -dependent solution

$$\Psi[q, s] = \sum_{\mathcal{E}\beta} a_{\mathcal{E}\beta} \Phi_{\mathcal{E}\beta}[q] e^{i(\mathcal{E} - \mathcal{M}^2)s/(2\mathcal{M}\hbar)} ; \quad \mathcal{E}\Phi_{\mathcal{E}\beta}[q] = -\mathcal{E}\Phi_{\mathcal{E}\beta}[q] \quad (22)$$

where the label β distinguishes among a linearly independent set of eigenstates of $\mathcal{E}$ with eigenvalue $-\mathcal{E}$. The classical constraint $\delta\mathcal{E}/\delta\mathcal{N} = 0$ becomes an operator constraint $\frac{\delta\mathcal{E}}{\delta\mathcal{N}}\Psi = 0$ which, upon insertion of the eigenstate expansion (22), leads to the infinite set of Wheeler-DeWitt equations

$$\left[-\frac{\hbar^2}{\mathcal{E}} \kappa^2 G^{ab} \frac{\delta^2}{\delta q^a \delta q^b} + \sqrt{g}U \right] \Phi_{\mathcal{E}}[q] = 0 \quad (23)$$

for each of the parameters $\mathcal{E}$ (quotation marks indicate the ordering ambiguity).

Finally, if we also impose the mass-shell constraint $\mathcal{E}\Psi = -\mathcal{M}^2\Psi$, then the only physical states are those with $\mathcal{E} = \mathcal{M}^2$, and the (classically indeterminate) constant $\mathcal{M}$ can be absorbed, via

$$\hbar_{eff} = \frac{\hbar}{\mathcal{M}} \quad (24)$$

into a rescaling of Planck’s constant. However, there are two ways in which the off mass-shell states, with $\mathcal{E} \neq \mathcal{M}^2$, may be physically relevant. First, instead of starting from the action (17), one might quantize the square-root action S_{BSW} directly, as was done in ref. [6], and this would lead to the dynamical equation (21) supplemented only by the constraints $(\delta\mathcal{E}/\delta\mathcal{N})\Psi = 0$. Second, it is possible that the Universe is *not* really in free fall, but there are actually interactions eventually associated with topology changing processes (for example, in the process of black hole evaporation, accompanied by the emission of a baby universe). Although at present a satisfactory description of topology-changing processes in a “third-quantized” theory of gravity does not exist yet, still it may be possible, by direct analogy to eq. (4), to evaluate the 1-loop contribution of virtual universes to the effective action, i.e.

$$S_{eff}[\mathcal{G}_{ab}] = \frac{i\hbar}{2} \text{Tr} \ln[\mathcal{E} + \mathcal{M}^2] \quad (25)$$

where the trace runs over a basis of states $\Phi_{\mathcal{E}}$ satisfying the one-parameter family of Wheeler-DeWitt equations (23). Of course, the supermetric $\mathcal{G}_{ab}$, unlike the ordinary spacetime metric $g_{\mu\nu}$, is not arbitrary; it is constrained to have the form (13). Therefore S_{eff} may be regarded as a functional of the potential term $U(q)$. But the form of $U(q)$ is also tightly constrained: it is the sum of all possible potential terms that could appear

in an ADM Hamiltonian. With this restriction, S_{eff} is just a function of the coupling constants of each possible interaction term, i.e.

$$S_{eff}[\mathcal{G}_{ab}] = S_{eff}[\lambda, e^2, g^2, \dots] \quad (26)$$

and the couplings are now viewed as dynamical variables. Variation of S_{eff} with respect to the couplings could, in principle, determine their phenomenological values, in a similar spirit to Coleman's "Big Fix" [7].

Let us illustrate this possibility with a minisuperspace toy model, in which the supermetric $\mathcal{G}_{ab}$ depends on one parameter only, namely, the cosmological constant λ . The starting point is the minisuperspace action representing a closed, homogenous and isotropic Friedman-Robertson-Walker (FRW) universe filled with a three-component, minimally coupled scalar field $\vec{\phi} \equiv (\phi_1, \phi_2, \phi_3)$, i.e.

$$S = \frac{1}{2} \int dt \left[-\frac{a\dot{a}^2}{N} + \frac{a^3 \vec{\dot{\phi}} \cdot \vec{\dot{\phi}}}{N} + N(a - \lambda a^3) \right] \quad (27)$$

where the 4-d invariant length is $ds^2 = \hat{\sigma}^2(-N^2 dt^2 + a^2 d\Omega_3^2)$ (with $\hat{\sigma}^2 \equiv 2G_N/3\pi$).

With the choice of coordinates $q^0 = a$, $q^i = \phi^i$, the supermetric for the corresponding worldline action reads $\mathcal{G}_{ii} = -a^2 \mathcal{G}_{00} = a^4(\lambda a^2 - 1)$, $i = 1, 2, 3$. Now, on general grounds of diffeomorphism invariance in minisuperspace, the effective action for a generic FRW universe will have a weak-curvature (adiabatic) expansion of the form

$$S_{eff}[\mathcal{G}_{ab}] = \int da \int d\vec{\phi} \sqrt{|\mathcal{G}|} [\Lambda_S + \kappa_S \mathcal{R} + O(\mathcal{R}^2)] \quad (28)$$

where Λ_S and κ_S are the (dimensionless) "supercosmological constant" and "super Newton's constant", respectively, and $\mathcal{R}$ is the scalar "supercurvature" of $\mathcal{G}_{ab}$.⁶

Regularizing the ranges of integration in (28) so that the scale factor runs as $a \in [0, \bar{a}]$ and the scalar fields run as $\phi_i \in [-\phi_{iM}, \phi_{iM}]$, and keeping only the leading term in the adiabatic expansion (28), the effective action (28) reads

$$\begin{aligned} S_{eff} &\simeq \Lambda_S \int_0^{\bar{a}} da \int_{-\phi_{1M}}^{\phi_{1M}} d\phi_1 \int_{-\phi_{2M}}^{\phi_{2M}} d\phi_2 \int_{-\phi_{3M}}^{\phi_{3M}} d\phi_3 a^7 (\lambda a^2 - 1)^2 \\ &= \left(\int d^3\phi \right) \Lambda_S \bar{a}^8 \left(\frac{\lambda^2 \bar{a}^4}{12} - \frac{\lambda \bar{a}^2}{5} + \frac{1}{8} \right) \end{aligned} \quad (29)$$

It is easy to check that S_{eff} is stationary at

$$\frac{dS_{eff}}{d\lambda} = 0 \implies \lambda \simeq \frac{6}{5\bar{a}^2} \quad (30)$$

with the result that $\lambda \rightarrow 0^+$ as $\bar{a} \rightarrow \infty$.

It is also possible to check [4] that the result of a vanishing cosmological constant still holds, for example, in the cases of a FRW universe minimally coupled to N_s massless

⁶In general, since Λ_S , κ_S are divergent at one loop, even in simple minisuperspace models, it must be assumed that either these constants are renormalized (and there exists a bare action $S_0[\mathcal{G}_{ab}]$), or else that there is a fundamental cutoff of some kind in the theory.

scalar fields (for $N_s = 1, 3$, also when the supercurvature term in the adiabatic expansion of S_{eff} is included), or to an odd ($N_s > 1$) number of massive scalar fields (with the stationary point being at zero mass for all the scalar fields).

It should be stressed that, although our proposal is reminiscent of Coleman's idea [7], in that we are deriving an effective action for the coupling constants, whose phenomenological values are then determined from the minimum of that action, in details the two programs are rather different. In particular, in our approach the metric of superspace is not the fixed DeWitt metric G_{ab} , but rather the modified metric $\mathcal{G}_{ab}$ and, as in ordinary 4-space, it is assumed that the superspace metric is a dynamical field in its own right. Coupling constants are dynamical because they parametrize a dynamical field in superspace (the supermetric), rather than becoming dynamical, as in Coleman's proposal, through the expectation values of baby-universe operators. The effective potential $S_{eff}[\mathcal{G}]$ for the supermetric is generated by loop effects, and in our example its adiabatic form is determined simply by diffeomorphism invariance in minisuperspace. There is no explicit appeal, in our approach, to topology-changing processes (although such processes are not ruled out).

4 A New Source of Decoherence?

It is also interesting to speculate how the possibility of an off-shell dynamics of the Universe might manifest itself, and if the effect would be large enough to be observable.

Consider a solution of the evolution equation (21) and constraints $(\delta\mathcal{E}/\delta\mathcal{N})\Psi = 0$, which is a superposition of two WKB states

$$\Psi(q, \tau) = \Psi_A(q, \tau) + \Psi_B(q, \tau) \quad (31)$$

of the form

$$\Psi_A(q, \tau) = \int d\mathcal{E} d\alpha F_A(\mathcal{E}, \alpha) \exp \left[\frac{i}{\hbar} \left\{ (\mathcal{E} - \mathcal{M}^2)\tau - \sqrt{\mathcal{E}} S[Q, \alpha] \right\} \right] \phi_A(q) \quad (32)$$

(and similarly for Ψ_B) where $\tau = s/2\mathcal{M}$ is the rescaled evolution (proper-time) parameter and $F_{A,B}$ are distributions concentrated at $\mathcal{E} = \mathcal{M}^2$ (with a rms uncertainty $\Delta\mathcal{E}$) and at parameter values $\{\alpha\} = \{\alpha_{A,B}\}$, the latter representing a set of integration constants for the the Hamilton-Jacobi equation of which the functional $S[Q, \alpha]$ is a solution.

Let us imagine that in some region of superspace where the amplitudes $\Psi_{A,B}$ are non-negligible, the phase difference

$$\delta S[Q'] = |S[Q, \alpha_A] - S[Q, \alpha_B]| \quad (33)$$

depends mainly on a small subset Q' of the semiclassical degrees of freedom Q .⁷

We now ask whether the two components Ψ_A and Ψ_B will interfere coherently in a measurement of $Q' \subset Q$. If $\Delta\mathcal{E} \neq 0$, then we must consider stationarity with respect

⁷For example, Q' might refer to the location of a particle recorded on a photographic plate, and δS refers to the difference in action, associated with two well separated particle paths in an interferometer, leading to the same final location.

to variation in $\mathcal{E}$. This leads to the components Ψ_A and Ψ_B being peaked at a given configuration Q at parameter times

$$\tau_A = \frac{S[Q, \alpha_A]}{2\sqrt{\mathcal{E}}} \quad , \quad \tau_B = \frac{S[Q, \alpha_B]}{2\sqrt{\mathcal{E}}} \quad (34)$$

respectively, with $\mathcal{E}$ evaluated at $\mathcal{E} = \mathcal{M}^2$. Interference of wavefunctions Ψ_A and Ψ_B is coherent, in the sense of physical optics, if the relative phase between the two wavefunctions is constant in the τ -interval $[\tau_A, \tau_B]$. In standard terminology the “linewidth” of the wavefunction is $\Delta\mathcal{E}/\hbar$, and the “coherence time” is $\Delta\tau = \hbar/\Delta\mathcal{E}$. The coherence criterion is then

$$\delta\tau < \Delta\tau = \frac{\hbar}{\Delta\mathcal{E}} \quad (35)$$

where $\delta\tau \equiv |\tau_A - \tau_B| \approx \frac{1}{2\sqrt{\mathcal{E}}} \delta S[Q]$, which means

$$\frac{1}{2\sqrt{\mathcal{E}}} \delta S < \frac{\hbar}{\Delta\mathcal{E}} \quad (36)$$

Observing now (cf. ref. [6] and eqs. (23) and (24)) that the $\mathcal{H}$ eigenvalues $\mathcal{E}$ can be absorbed into a rescaling of Planck’s constant, i.e. $\hbar_{eff}(\mathcal{E}) = \hbar/\sqrt{\mathcal{E}}$, with a dispersion

$$\delta\hbar = \left| \frac{d}{d\mathcal{E}} \hbar_{eff} \right|_{\mathcal{E}=\mathcal{M}^2} \Delta\mathcal{E} = \frac{1}{2} \hbar_{eff} \frac{\Delta\mathcal{E}}{\mathcal{E}} \quad (37)$$

using eqs. (36)-(37) the condition for coherent interference becomes

$$\frac{\delta S}{\hbar_{eff}} < \frac{\hbar_{eff}}{\delta\hbar} \quad (38)$$

The argument above is quite general, and applies to any WKB treatment of the evolution equation (21).⁸

A signature of finite dispersion $\delta\hbar$ in the effective value of Planck’s constant could be, e.g., an observed decoherence of particle beams in an ultra-sensitive particle interferometer, in a situation where standard time-energy considerations would imply that the beams should interfere coherently. In this case, the wavefunction Ψ_A (Ψ_B) represents the contribution to the full “wavefunction of the Universe” Ψ in which the particle travels through path A (B) of the interferometer, respectively, while $\delta S/\hbar_{eff}$ is a WKB phase difference associated with this difference in path. If the Universe propagates slightly off-shell, as has been suggested here, then the interference will be incoherent if the inequality (38) is violated.

Acknowledgements

J.G.’s research is supported in part by the U.S. Dept. of Energy, under Grant No. DE-FG03-92ER40711 and by the Director, Office of Energy Research, Office of Basic Energy Services, of the U.S. Department of Energy under Contract DE-AC03-76SF00098. A.C.’s research is supported by a JSPS postdoctoral fellowship, under contract No. P95196.

⁹In fact, accepting that there may be a stochastic uncertainty $\delta\hbar$ (of whatever origin) in the phenomenological value of Planck’s constant, then a condition of the form (38) can be easily deduced from the standard Feynman path integral in fixed background spacetime, cf. ref. [4].

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D-brane configuration and black hole thermodynamics

- *Bounds of entropy and temperature* -

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Abstract

Recently many authors investigated a configuration of strings and solitons (D-branes) in the type IIB superstring theory on $M^5 \times T^5$. Callan and Maldacena considered a configuration composed of single-wound D-fivebranes wrapped on T^5 and single-wound D-strings wrapped on S^1 of the torus. Maldacena and Susskind considered a configuration composed of a multiply-wound D-fivebrane and a multiply-wound D-string. In this report we consider a possible generalization of them: a general configuration composed of a set of arbitrarily-wound D-fivebranes and a set of arbitrarily-wound D-strings. For the configuration, it is shown that number of microscopic states is bounded from above by the exponential of the Hawking-Bekenstein entropy of the corresponding black hole and the temperature of closed string radiation from the D-branes is bounded from below by the Hawking temperature of the black hole. After discussing the necessary and sufficient condition to saturate these bounds, we give some speculations about black hole thermodynamics.

1 Introduction

Black hole thermodynamics [1] [2] is one of the most interesting topics in the theory of black holes. Recently Strominger and Vafa [3] gave a microscopic origin of the Hawking-Bekenstein entropy of a black hole in the framework of superstring theory by using the D-brane technology [4]. Hawking radiation was also explained in the D-brane picture [5].

In this report we consider a configuration of strings and solitons in the type IIB superstring theory on $M^5 \times T^5$. The corresponding classical black-brane solution was obtained in [6] and is characterized by four charges (Q_1, Q_5, n_L, n_R) . In the case of the extreme limit ($n_R = 0$), Callan and Maldacena have thought of it as a configuration composed of Q_5 D-fivebranes wrapped on the torus and Q_1 D-strings wrapped on S^1 of the torus, based on the fact that a D-string has charge $(Q_1, Q_5) = (1, 0)$ and a D-fivebrane has charge $(Q_1, Q_5) = (0, 1)$. (*Figure 1(a)* is a schematic picture of this situation for $Q_1 = 2$, $Q_5 = 3$.) On the contrary Maldacena and Susskind [7] regarded it as a configuration of a single long D-string and a single long D-fivebrane (*Figure 1(b)* is a schematic picture of this situation for $Q_1 = 2$, $Q_5 = 3$). In both interpretations $P_L \equiv n_L/R$ is the momentum

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along the circle which is a sum over left moving massless modes of open strings on the D-branes, where R is the radius of the circle. In this report we consider a more general configuration of D-branes and investigate open strings on it [8].

In Sec. 2 we give the number of microscopic states, which is related to the black hole entropy. In Sec. 3 temperature of a canonical ensemble of open strings on D-branes is obtained. It is related to the temperature of a decay of D-brane excitations to a closed-string mode, which seems to be the Hawking radiation. Finally in Sec. 4 we summarize this report and give some speculations.

2 Number of microscopic states

First let us consider a set of Q_1 D-strings wrapped on S^1 whose length is $2\pi R$. They may connect up [7] to form a set of multiply-wound D-strings, which is composed of $N_{q_1}^{(1)}$ D-strings of length $2\pi R q_1$ ($q_1 = 1, 2, \dots$). Next consider a set of Q_5 D-fivebranes wrapped on T^5 and a set of Q_1 D-strings wrapped on S^1 of the torus. The corresponding non-extremal supergravity solution is obtained by setting $N_1 = N_5 = 0$ in the solution found in [6] and is characterized by four parameters (Q_1, Q_5, n_L, n_R). Here Q_1 and Q_5 are charges of a U(1) gauge field; n_L (n_R) is left (right) moving momentum along S_1 multiplied by R . By suppressing reference to the other four directions we may think of the D-fivebranes as D-strings wrapped on the circle [7]. The D-fivebranes may also connect up to form a set of multiply-wound D-fivebranes, which is composed of $N_{q_5}^{(5)}$ D-fivebranes of length $2\pi R q_5$ ($q_5 = 1, 2, \dots$) along the circle. Note that $N_{q_1}^{(1)}$ and $N_{q_5}^{(5)}$ must satisfy the following constraints:

$$\begin{aligned} Q_1 &= \sum_{q_1} q_1 N_{q_1}^{(1)}, \\ Q_5 &= \sum_{q_5} q_5 N_{q_5}^{(5)} \end{aligned}$$

since a D-string which winds q_1 times has charge $(Q_1, Q_5) = (q_1, 0)$ and a D-fivebrane which winds q_5 times has charge $(Q_1, Q_5) = (0, q_5)$. For example, the situation in Figure 1(c) is an example for $Q_1 = 2, Q_5 = 3$, since $N_2^{(1)} = N_1^{(5)} = N_2^{(5)} = 1$ and others are zero.

There are several types of open strings on the D-brane configuration: both boundaries are on a common D-string; one boundary is on a D-string and another is on a different D-string; one is on a D-string and another is on a D-fivebrane; etc. Among them, as an approximation, we consider only strings which connect a q_1 -wound D-string and a q_5 -wound D-fivebrane and neglect contribution from other strings. The spectrum of the strings is

$$\begin{aligned} p_n(q_1, q_5) &= \frac{\pm n}{R(q_1, q_5)_{LCM}}, \quad (n = 1, 2, \dots) \\ ('+' \text{ for left moving} \quad ; \quad '-' \text{ for right moving}), \end{aligned} \quad (1)$$

where $(q_1, q_5)_{LCM}$ is the least common multiple of q_1 and q_5 , since the boundary condition of the strings is

$$X^5 \sim X^5 + 2\pi R(q_1, q_5)_{LCM}. \quad (2)$$

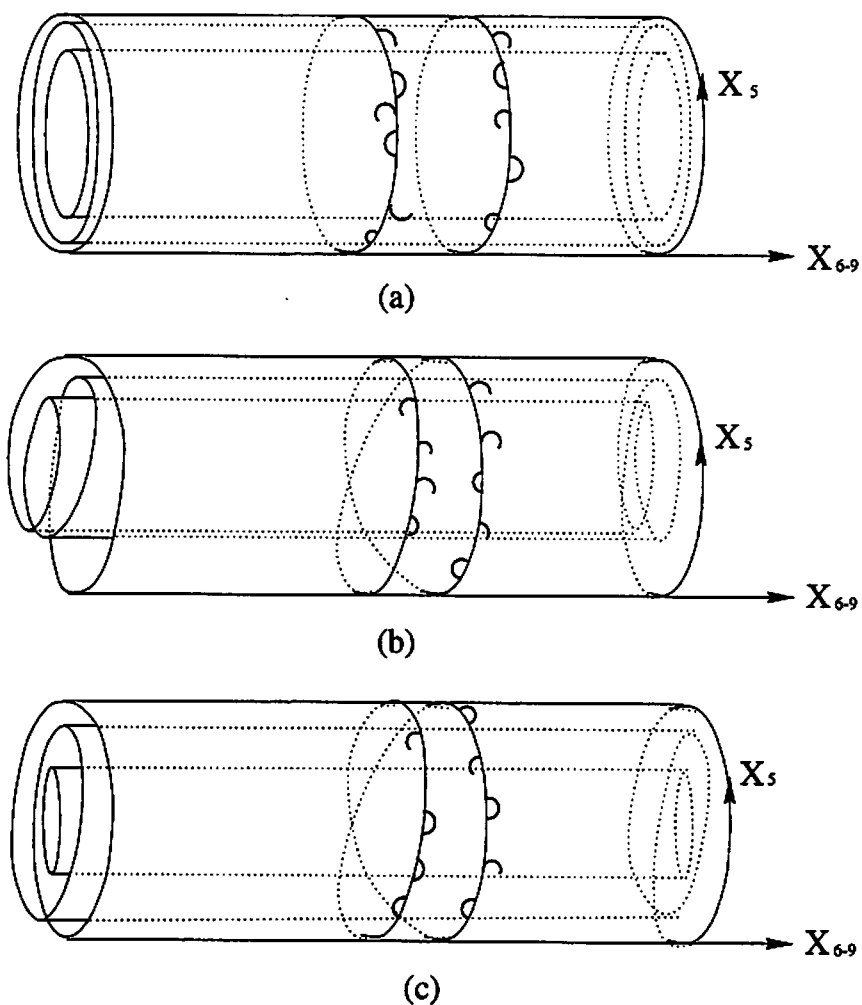


Figure 1: (a) The situation considered by Callan and Maldacena is shown for $Q_1 = 2$, $Q_5 = 3$. There are three D-fivebranes wrapped on T^5 (X_{5-9} are coordinates for the torus.) and two D-strings wrapped on S^1 of the torus. (X_5 is a coordinate for the circle.) (b) The situation considered by Maldacena and Susskind is shown for $Q_1 = 2$, $Q_5 = 3$. There are a triple-wound D-fivebrane and a double-wound D-string. (c) Another example of the generalized configurations of D-branes for $Q_1 = 2$, $Q_5 = 3$.

We also regard the momentum $P_L \equiv n_L/R$ ($P_R \equiv -n_R/R$) along the circle as a sum over the left (right) moving massless modes of the strings. Hence the number of microscopic states $d(n_L, n_R)$ is given by

$$d(n_L, n_R) = d(n_L)d(n_R),$$

$$\sum_N d(N)w^N = \prod_{q_1} \prod_{q_5} \prod_{n=1}^{\infty} \left[\frac{1 + w^{n/(q_1, q_5)_{LCM}}}{1 - w^{n/(q_1, q_5)_{LCM}}} \right]^{4N_{q_1}^{(1)} N_{q_5}^{(5)}}.$$

By using the asymptotic form of the θ -function and the saddle point method, we can obtain $d(n_{L,R})$ in the limit

$$\sqrt{\frac{Q_1 Q_5}{n_{L,R}}} \ll \min_{(q_1, q_5) \in A} (q_1, q_5)_{LCM}, \quad (3)$$

where $A \equiv \{(q_1, q_5) | N_{q_1}^{(1)} N_{q_5}^{(5)} \neq 0\}$. The result is

$$d(n_{L,R}) \approx \exp \left[2\pi \sqrt{n_{L,R} \sum_{q_1} \sum_{q_5} N_{q_1}^{(1)} N_{q_5}^{(5)} (q_1, q_5)_{LCM}} \right]$$

$$\leq \exp(2\pi \sqrt{n_{L,R} Q_1 Q_5}).$$

Therefore, for a configuration satisfying (3), $d(n_L, n_R)$ is bounded from above:

$$d(n_L, n_R) \leq \exp \left[2\pi (\sqrt{n_L} + \sqrt{n_R}) \sqrt{Q_1 Q_5} \right], \quad (4)$$

where the bound is saturated if and only if all $(q_1, q_5) (\in A)$ are relatively prime.

3 Canonical ensemble of open strings

In the D-brane approach to Hawking radiation [5] [9] [10], the thermal spectrum of the decay of non-BPS D-brane excitations to closed string modes was obtained by summing up decay rates over all consistent initial states of open strings on the D-branes. The summation was approximately performed by replacing the microcanonical ensemble by a canonical ensemble. The resulting temperature of the decay spectrum is as follows [10]:

$$T_{decay} = T, \quad (5)$$

where T is the temperature of the canonical ensemble.

Therefore, in this section we consider a canonical ensemble of open strings on the D-brane configurations introduced in Sec. 2. The partition function is

$$Z(\beta, \alpha) = Z_L(\beta, \alpha) Z_R(\beta, \alpha),$$

$$Z_{L,R}(\beta, \alpha) = \prod_{q_1} \prod_{q_5} \prod_{n=1}^{\infty} \left[\frac{1 + e^{-\beta e_n(q_1, q_5) - \alpha p_n(q_1, q_5)}}{1 - e^{-\beta e_n(q_1, q_5) - \alpha p_n(q_1, q_5)}} \right]^{4N_{q_1}^{(1)} N_{q_5}^{(5)}},$$

where p_n is given by (1) and $e_n = |p_n|$. By using the asymptotic form of the θ -function, we can obtain $Z_{L,R}(\beta, \alpha)$ in the limit (3). The result is

$$\ln Z_{L,R}(\beta, \alpha) = \frac{\pi^2 R}{\beta \pm \alpha} \sum_{q_1} \sum_{q_5} N_q^{(1)} N_q^{(5)}(q_1, q_5)_{LCM},$$

('+' for 'L' ; ' - ' for 'R').

β and α are fixed by

$$\begin{aligned} E_L &= P_L = n_L/R, \\ E_R &= -P_R = n_R/R, \end{aligned}$$

where E_L (E_R) is the left (right) movers' contribution to the expectation value of the energy and P_L (P_R) is the left (right) movers' contribution to the expectation value of the momentum:

$$\begin{aligned} E_L + E_R &= -\frac{\partial \ln Z(\beta, \alpha)}{\partial \beta}, \\ P_L + P_R &= -\frac{\partial \ln Z(\beta, \alpha)}{\partial \alpha}. \end{aligned}$$

Thus, for a configuration satisfying (3), the entropy S and the inverse-temperature $1/T = \beta$ are

$$\begin{aligned} S &= 2\pi(\sqrt{n_L} + \sqrt{n_R}) \sqrt{\sum_{q_1} \sum_{q_5} N_q^{(1)} N_q^{(5)}(q_1, q_5)_{LCM}}, \\ \frac{1}{T} &= \frac{\pi R}{2} \left(\frac{1}{\sqrt{n_L}} + \frac{1}{\sqrt{n_R}} \right) \sqrt{\sum_{q_1} \sum_{q_5} N_q^{(1)} N_q^{(5)}(q_1, q_5)_{LCM}}. \end{aligned}$$

They are bounded from above as follows:

$$S \leq 2\pi(\sqrt{n_L} + \sqrt{n_R}) \sqrt{Q_1 Q_5}, \quad (6)$$

$$\frac{1}{T} \leq \frac{\pi R}{2} \left(\frac{1}{\sqrt{n_L}} + \frac{1}{\sqrt{n_R}} \right) \sqrt{Q_1 Q_5}, \quad (7)$$

where the the bounds are saturated if and only if all $(q_1, q_5) \in A$ are relatively prime. Finally note that the following relation is trivially satisfied:

$$S \approx \ln d(n_L, n_R). \quad (8)$$

4 Summary and speculations

In summary we have investigated open strings on the general D-brane configurations. For configurations satisfying (3), the number of microscopic states $d(n_L, n_R)$ is bounded from above and the temperature of a decay of D-brane excitations to closed strings is bounded from below. Moreover, for any configurations not satisfying (3), the same bounds seem to

exist without saturation [7]. Therefore it is expected that the quantities calculated from the microscopic point of view are related as follows to the macroscopic quantities of the black hole.

$$\ln d(n_L, n_R) \leq S_{BH}, \quad (9)$$

$$T_{decay} \geq T_{BH}, \quad (10)$$

where S_{BH} and T_{BH} are the Hawking-Bekenstein entropy and the Hawking temperature of the black hole [6] [10]:

$$S_{BH} = 2\pi(\sqrt{n_L} + \sqrt{n_R})\sqrt{Q_1 Q_5},$$

$$\frac{1}{T_{BH}} = \frac{\pi R}{2} \left(\frac{1}{\sqrt{n_L}} + \frac{1}{\sqrt{n_R}} \right) \sqrt{Q_1 Q_5}.$$

Note that the Hawking-Bekenstein entropy and the Hawking temperature are defined by macroscopic quantities (area and surface gravity of the event horizon). The bounds (9) and (10) are saturated if and only if all $(q_1, q_5) \in A$ are relatively prime and (3) is satisfied.

Thus several speculations may be possible.

- Inside a black hole characterized by the four parameters (Q_1, Q_5, n_L, n_R) , some dynamical processes may occur. The processes may be described in the D-brane picture: D-branes repeat fission and fusion to settle down to one of the states for which all $(q_1, q_5) \in A$ are relatively prime and (3) is satisfied.
- During the process the microscopic entropy increases to reach the Hawking-Bekenstein entropy of the corresponding black hole. Moreover the temperature of closed string radiation from the D-branes decreases to reach the Hawking temperature of the black hole.

These speculations may be significant to investigate the microstates of dynamical black holes. For example let us consider a merger of two black holes B_1 and B_2 and suppose that B_1 corresponds to a D-brane configuration $\{N_{q_1}^{(11)}, N_{q_5}^{(15)}\}$ and B_2 corresponds to a configuration $\{N_{q_1}^{(21)}, N_{q_5}^{(25)}\}$. Just after merging, the large black hole B formed by the merger corresponds to a configuration $\{N_{q_1}^{(1)} = N_{q_1}^{(11)} + N_{q_1}^{(21)}, N_{q_5}^{(5)} = N_{q_5}^{(15)} + N_{q_5}^{(25)}\}$, provided that directions of the D-strings are the same for B_1 and B_2 . In general the last configuration does not saturate the bounds (9) and (10) even when the configurations for B_1 and B_2 saturate the bounds. Thus, in general just after the merger the microscopic entropy of B does not agree with the corresponding Hawking-Bekenstein entropy and the temperature of the closed string radiation does not agree with the corresponding Hawking temperature. However, after a sufficiently long time the D-branes' fission and fusion settle the entropy and the temperature to the Hawking-Bekenstein entropy and the Hawking temperature. To say something quantitative about the above process, we have to find a supergravity solution which represents the merger of branes and have to investigate the D-branes' fission and fusion in terms of the elementary interactions. However, without doing these, we could reach the above conclusion about the dynamical process inside the black hole by analyzing the microscopic entropy of open strings on D-branes.

Acknowledgements

I would like to thank Professor H. Kodama for continuous encouragement and discussions.

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The Spectral Representation of Spacetime Structures

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Abstract

We propose a quantitative representation of spacetime structures in terms of spectra of a suitable differential operator. To demonstrate the advantages of the representation, we construct a measure of ‘closeness’ between two spaces with, in general, different topologies, based on the representation. Under some assumptions, this measure of ‘closeness’ can be related with the reduced-density matrix element in quantum cosmology. In terms of the measure of closeness, therefore, we can discuss whether universes with different topologies interfere quantum mechanically. We show preliminary results on this line based on simple models.

1 Introduction

Ultimately we need to construct a physical theory of spacetime structures in a satisfactory manner. Here we immediately notice that the concept of spacetime structures is a complex mixture of two quite different ingredients:

- (a) Local structures, typically characterized by real numbers such as components of a metric g_{ab} .
- (b) Global structures, typically characterized by integers such as characteristic classes.

In order to describe a dynamics of spacetime structures as a whole, then, it is desirable to find out a unified representation of spacetime structures which can treat (a) and (b) on the same footing.

As one of promising representations suitable for such a purpose, we here consider a representation which may be called the ‘spectral representation’[1].

For simplicity, we shall only consider a compact Riemannian manifold without boundaries, $\mathcal{G} = (\Sigma, g)$, where g has a positive definite signature. (Therefore $\mathcal{G}$ can be regarded as a model either for a spatial section of the universe, or for an Euclideanized spacetime.)

Now we can set up an eigenvalue problem on $\mathcal{G}$ for an elliptic differential operator. As a typical example, let us choose the Laplace-Beltrami operator on $\mathcal{G}$, $\Delta = g^{ab}D_a D_b = \frac{1}{\sqrt{g}}\partial_a(\sqrt{g}g^{ab}\partial_b)$, where D_a stands for the covariant derivative operator. Thus the eigenvalue problem is

$$\Delta\phi_k + \lambda_k\phi_k = 0 \quad .$$

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Here, very roughly speaking, Δ reflects the local geometry ((a)) while the eigenfunction ϕ_k reflects the global geometry ((b)) of $\mathcal{G}$. Therefore the eigenvalue λ_k contains the information on both (a) and (b). Thus one obtains a quantitative representation of a geometry $\mathcal{G}$ in terms of a set of eigenvalues ('spectra'), $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots \uparrow \infty \equiv \{\lambda_k\}$. It can be said that this representation is identical to the conversion of the information of 'shape' into the information of 'sound'. Here the two words 'shape' and 'sound' may symbolize the spirit of the present framework: Interaction of mathematics and physics.

At this stage one may remember a famous problem in Riemannian geometry, which is usually stated as "Can one 'hear' the shape of a drum?" [2] Indeed our representation is closely related to this problem [1]. Following this phrase one can state the basic idea of the representation as "Let us 'hear' the shape of the Universe."

2 The spectral distance

2.1 Overview

In order to illustrate the utility of the spectral representation, let us concentrate on one tractable problem. Here we introduce some sort of 'distance' between two spaces $\mathcal{G}$ and $\mathcal{G}'$ based on the spectral representation. In the spirit of the spectral representation, such a 'distance' may be defined in terms of a difference in the 'sound' of $\mathcal{G}$ and $\mathcal{G}'$, so to say. Before formulating this idea quantitatively, let us note why it is significant to introduce such a 'distance' between spaces.

One of its strong motivations comes from the concept of scale-dependent topology (physical topology) [3, 4].

Mathematically, topology is a scale-independent concept, in the sense that objects of different scales, but continuously being deformed to each other, are identified. However, once topology is applied to spacetime physics it inevitably begins to carry more properties than its original concept [4]: Because of the limitation in the observational energy scale, thin objects smaller than E^{-1} cannot be observed. (E is the energy scale of observation.) Following Wheeler's spacetime foam picture [5], one can imagine that the universe is topologically complicated in the microscopic scale. For illustration suppose two kinds of small topological handles are attached to the universe, and one kind is of scale l and the other kind is of scale L , $l \ll L$. Firstly if our observational energy is between L^{-1} and l^{-1} , the smaller handles cannot be observed and the effective topology becomes simpler than the original one. Next if the energy scale is further decreased less than L^{-1} , even bigger handles cannot be observed, resulting in a much simpler effective topology. In this way we are led to the concept of scale-dependent topology (physical topology): Effective topology as a function of the observational energy scale. In other words, the scale-dependent topology is topology plus a distinction between big and small handles.

There is no known mathematical theory suitable for describing such a concept. To describe a sequential change in the effective topology of a space as a function of observational energy, we need to formulate 'closeness' between two spaces whose geometries are different globally as well as locally. One can say that quantum cosmology is indicating one direction in which some brave mathematicians should go sooner or later. Here, from the

viewpoint of spacetime physics, we shall suggest how to introduce the concept of closeness between two spaces.

2.2 The measure of closeness

Now we try to introduce a concept of closeness between spaces.

As a preliminary consideration, let us introduce a measure of closeness in a set $\mathcal{S} = \{a, b, \dots\}$. Now the basic procedure is as follows: Instead of comparing elements of $\mathcal{S}$ directly, we compare functions characterized by the elements of $\mathcal{S}$. Namely we first fix some function into which each element of $\mathcal{S}$ enters as a parameter, $\{p_a(\cdot), p_b(\cdot), \dots\}$. Here $a, b, c, \dots$ determine the detailed form of the function. For definiteness let us choose functions $p_i(\cdot)$ ($i = a, b, \dots$) such that $p_i(\cdot) \geq 0$ and $\int_{-\infty}^{\infty} p_i(x) dx = 1$. (If $\mathcal{S}$ is a set of positive numbers, for instance, we can choose $p_a(\cdot)$ as a Gaussian function with dispersion a .) Then an overlapping integral $\xi_{ij} = \int_{-\infty}^{\infty} (p_i(x)p_j(x))^{\frac{1}{2}} dx$ possesses nice properties as

- (i) $\xi_{ij} = \xi_{ji}$,
- (ii) $0 \leq \xi_{ij} \leq 1$,
- (iii) $\xi_{ij} = 0 \iff p_i(\cdot)p_j(\cdot) \equiv 0$, $\xi_{ij} = 1 \iff p_i(\cdot) \equiv p_j(\cdot)$.

Thus we can define a measure of closeness between i and j as

$$d(i, j) = -\ln \xi_{ij} = -\ln \int_{-\infty}^{\infty} (p_i(x)p_j(x))^{\frac{1}{2}} dx \quad .$$

Note that $d(i, j)$ takes some value between 0 (complete overlapping) and ∞ (complete non-overlapping).

The same procedure is followed for the case of geometry also:

- (I) Instead of comparing geometries directly, we introduce a suitable *functional* $P_{\mathcal{G}}[\cdot]$, where the information of a geometry $\mathcal{G}$ enters into this functional as a parameter and reflects on the shape of the functional.
- (II) Taking the overlapping functional-integral between $P_{\mathcal{G}}[\cdot]$ and $P_{\mathcal{G}'}[\cdot]$, we define the measure of closeness between two geometries $\mathcal{G}$ and $\mathcal{G}'$.

In order to probe the shape of a space in the form of 'sound', some medium should be utilized. As the simplest case, we choose a scalar field as a probe. This is the meaning of introducing a functional in the above procedure. It is clear that the above procedure can also be extended to other types of fields, though we do not discuss them here.

More definitely, we choose a functional $P_{\mathcal{G}}[\cdot]$ such that $P_{\mathcal{G}}[\cdot] \geq 0$ and $\int [df] P_{\mathcal{G}}[f] = 1$. Then we define a measure of closeness between $\mathcal{G}$ and $\mathcal{G}'$ as

$$d(\mathcal{G}, \mathcal{G}') = -\ln \int [df] (P_{\mathcal{G}}(f)P_{\mathcal{G}'}(f))^{\frac{1}{2}} \quad .$$

Thus $d(\mathcal{G}, \mathcal{G}')$ takes some value between 0 and ∞ .

We should note one subtlety included in this procedure. In taking the overlapping functional integral, some prescription for the identification of a function f defined on

$\mathcal{G}$ with *another* function f defined on $\mathcal{G}'$ should be fixed. This point is the essential difficulty which arises whenever we need to compare two spaces, e.g. the case of topology changes. Clearly detailed investigations on this point are needed from both physics and mathematics sides. Here we choose the following prescription: Let $\mathcal{G}$ and $\mathcal{G}'$ be compact Riemannian manifolds, and D be an elliptic differential operator. Let $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \cdots \uparrow \infty \equiv \{\lambda_k\}$ and $0 = \lambda'_0 < \lambda'_1 \leq \lambda'_2 \cdots \uparrow \infty \equiv \{\lambda'_k\}$ are spectra of D on $\mathcal{G}$ and $\mathcal{G}'$, respectively, numbered in the increasing order. Then, we obtain sets of eigenfunctions of D , $\{\phi_k(\cdot)\}$ and $\{\phi'_k(\cdot)\}$, corresponding to $\{\lambda_k\}$ and $\{\lambda'_k\}$, respectively. A function $f(\cdot)$ on $\mathcal{G}$ can be expanded in terms of $\{\phi_k(\cdot)\}$, $f(\cdot) = \sum_k \alpha_k \phi_k(\cdot)$. Then we identify $f(\cdot)$ on $\mathcal{G}$ with a function $\sum_k \alpha_k \phi'_k(\cdot)$ on $\mathcal{G}'$.

For each choice of $P_{\mathcal{G}}[\cdot]$ and the form of the overlapping functional-integral, one type of measure of closeness is defined. Therefore we need to pick up a measure amongst other possibilities which is suitable for physical applications.

From such a viewpoint, a distance $d(\mathcal{G}, \mathcal{G}')$ should satisfy several criteria: It should be diffeomorphism invariant with respect to $\mathcal{G}$ and $\mathcal{G}'$; It should be compatible with the spectral representation, so that it reflects the global as well as the local geometry of $\mathcal{G}$ and $\mathcal{G}'$; It should count larger scale behaviors of $\mathcal{G}$ and $\mathcal{G}'$ with heavier weight than their shorter scale behaviors. (This property is necessary for dealing with the scale-dependent topology[1].)

Taking these points into account, we are led to the choice for $P_{\mathcal{G}}[\cdot]$ as

$$P_{\mathcal{G}}[f] := \exp \left(-\frac{1}{2} \int_{\Sigma} g^{ab} \partial_a f \partial_b f \sqrt{g} \right) \quad ,$$

where $\mathcal{G} = (\Sigma, g)$.

Afterwards it is straightforward to get[1]

$$d(\mathcal{G}, \mathcal{G}') = \frac{1}{2} \sum_{k=1}^{\infty} \ln \frac{1}{2} \left(\sqrt{\frac{\lambda_k}{\lambda'_k}} + \sqrt{\frac{\lambda'_k}{\lambda_k}} \right) \quad .$$

Let us call this measure of closeness between $\mathcal{G}$ and $\mathcal{G}'$ the '*spectral distance*', for brevity.

Significantly enough, this spectral distance can be related with the reduced density-matrix element[6, 7, 8] $\rho_{\text{reduced}}(\mathcal{G}, \mathcal{G}')$ in quantum cosmology, under several assumptions[1]:

$$d(\mathcal{G}, \mathcal{G}') = -\ln \{ \rho_{\text{reduced}}(\mathcal{G}, \mathcal{G}') / \varphi(\mathcal{G}) \varphi^*(\mathcal{G}') \} \quad ,$$

where $\varphi(\mathcal{G})$ is a sort of wavefunction characterizing semiclassical behaviors of the universe.

This equality is quite suggestive. If it is read from left to right, it suggests that the measure of closeness between two spaces can be defined in terms of the degree of quantum decoherence between the two 'universes', if we find the satisfactory physical theory to analyze the latter. (This speculation can be regarded as a suggestion from physics to mathematics.) On the other hand, if the above equality is read from right to left, it suggests that the spectral distance introduced here, which can be always computed in principle, gives some insight on the quantum decoherence between two universes, even though at present we have no good theory to describe the latter. In the next section, we shall utilize the spectral distance in this way. (A suggestion from mathematics to physics.)

Here we note two properties of the spectral distance.

Firstly it is shown that

$$d(\mathcal{G}, \mathcal{G}') < \infty \implies \dim \mathcal{G} = \dim \mathcal{G}' \text{ and } \text{vol } \mathcal{G} = \text{vol } \mathcal{G}' ,$$

where $\dim \mathcal{G}$ and $\text{vol } \mathcal{G}$ are the dimension and the volume of $\mathcal{G}$, respectively. In other words, the spectral distance $d(\mathcal{G}, \mathcal{G}')$ becomes infinity when $\mathcal{G}$ and $\mathcal{G}'$ are of different dimensions or volumes. This result is very suggestive, bearing in mind the relation of $d(\mathcal{G}, \mathcal{G}')$ with quantum decoherence.

Secondly it turns out that the spectral distance does not satisfy the triangular inequality in general, i.e. there are many sets of three geometries $\{\mathcal{G}, \mathcal{G}', \mathcal{G}''\}$ for which a reversed inequality holds:

$$d(\mathcal{G}, \mathcal{G}') + d(\mathcal{G}', \mathcal{G}'') \leq d(\mathcal{G}, \mathcal{G}'') .$$

In this sense $d(\mathcal{G}, \mathcal{G}')$ is not a ‘distance’ if one respects a mathematical axiom of distance. However, not only $d(\mathcal{G}, \mathcal{G}')$ is still a nice measure of closeness between spaces, but also it has a tight link with physics, i.e. the reduced density-matrix element. Therefore it is worth while investigating various aspects of the spectral distance in detail.

3 Applications

Having obtained the spectral distance, we can at least speculate on a question: Whether do universes with different topologies decohere quantum mechanically?

We can imagine a possible answer to this question: They decohere because they ‘sound’ differently, resulting in a huge spectral distance.

Here, as the first step in this direction, we shall only investigate the simplest example. Let us calculate explicitly the spectral distances for various 2-tori (T^2) and Klein’s bottles ($\mathbf{RP}^2 \# \mathbf{RP}^2$). We take both T^2 and $\mathbf{RP}^2 \# \mathbf{RP}^2$ to be locally flat. Thus their local geometries are equivalent while their global structures (orientability in this case) are different. (As is well-known T^2 is orientable while $\mathbf{RP}^2 \# \mathbf{RP}^2$ is non-orientable.) Therefore the effects caused by the difference in orientability can be highlighted.

Usually the global structure of a 2-torus is represented by the Teichmüller parameters (τ^1, τ^2) ($\tau^2 > 0$). We here adopt the following convention. A pair of the parameters (τ^1, τ^2) define a parallelogram $OACB$ by $\vec{OA} = (\tau^1/\sqrt{\tau^2}, \sqrt{\tau^2})$ and $\vec{OB} = (1/\sqrt{\tau^2}, 0)$. By identifying two pairs of facing sides a torus is obtained, so that we understand that (τ^1, τ^2) represents this torus. The Laplacian on this torus becomes $\Delta = \frac{1}{\tau^2}(|\tau|^2 \partial_1^2 - 2\tau^1 \partial_1 \partial_2 + \partial_2^2)$, so that the spectra are $\lambda_{mn} = \frac{4\pi^2}{\tau^2}(|\tau|^2 m^2 - 2\tau^1 mn + n^2)$, where m and n are integers. (We also need to pay attention to multiplicities[1].)

When $\tau^1 = 0$, Klein’s bottle can also be constructed almost in the same manner, by a suitable way of identifying facing sides. (If $\tau^1 \neq 0$ a deficit angle appears by this construction of Klein’s bottle. We only consider the cases of $\tau^1 = 0$ below.)

Figure 1 shows a typical case of two tori with a unit 2-volume (area): $(\tau^1, \tau^2) = (0, 1)$ and $(\tau^1, \tau^2) = (0, 2)$.

Figure 2 shows the spectral distance between these two tori. In this figure, the scale-dependent spectral distance[1], d_Λ , is shown. Basically it means that the infinite summation in the definition of $d(\mathcal{G}, \mathcal{G}')$ is truncated at n such that $\lambda_n < \Lambda$ and $\lambda_{n+1} \geq \Lambda$. Roughly

speaking, $d_\Lambda(\mathcal{G}, \mathcal{G}')$ measures closeness between $\mathcal{G}$ and $\mathcal{G}'$, neglecting the difference in the scale smaller than $\Lambda^{-1/2}$. By taking $\Lambda \rightarrow \infty$, thus, $d_\Lambda(\mathcal{G}, \mathcal{G}')$ reduces to $d(\mathcal{G}, \mathcal{G}')$. *Figure 2* shows a nice convergence of $d_\Lambda((0, 1), (0, 2))$ as a function of Λ . We get the spectral distance as $d((0, 1), (0, 2)) \approx 0.219$.

Figure 3 indicates $d(T^2, T^2)$ for various cases. In each column, the first and the second lines indicate the Teichmüller parameters for two tori and the last line shows the spectral distance between the two tori.

Figure 4 is for Klein's bottles, $d(Klein, Klein)$.

As for the spectral distance between tori and Klein's bottles, the results are quite surprising. One may naturally expect that the spectral distances become very large in this case. However the results are not so. *Figure 5* shows the results for $d(T^2, Klein)$. In the figure, parameters (τ^1, τ^2) for a torus and Klein's bottle, and the spectral distance between them are indicated from top to bottom. If the parameters are fixed, $d(T^2, Klein)$ is of the same order as $d(T^2, T^2)$ and $d(Klein, Klein)$. One can infer the reason for these results. A torus T^2 corresponds to the covering space of Klein's bottle so that these two spaces are very similar in nature. Furthermore we used a scalar field as a 'probe' of shapes which does not 'feel' the difference in orientability so sensitively. If we use another field, e.g. the Dirac field, it will 'feel' the difference in orientability more clearly.

4 Concluding remarks

We have proposed the spectral representation of spacetime structures and applied it to introduce the spectral distance between spaces.

Though we have only discussed the case of using a scalar field as a probe, it is clear that the basic idea can be applied to any kind of fields. Each type of fields 'feels' some portion of the whole of geometrical structures of the space. From the viewpoint of this representation, a space is a total collection of what each field 'feels'.

Acknowledgments

This work was financially supported by the Yukawa Memorial Foundation, the Japan Association for Mathematical Sciences, and the Japan Society for the Promotion of Science.

Part of this work was also supported by the Indo-Japanese Cooperative Scientific Research Programme of the Japan Society for the Promotion of Science.

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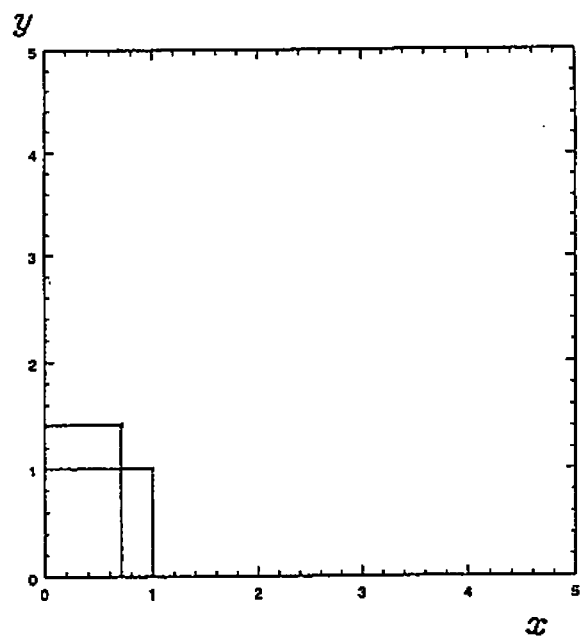


Figure 1

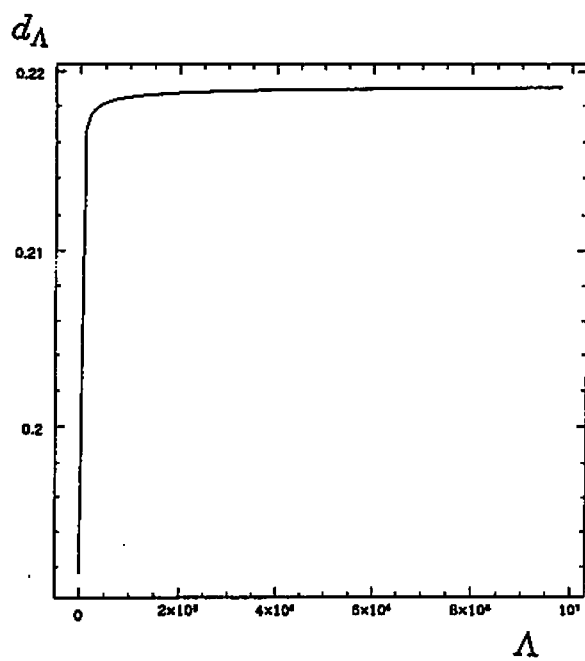


Figure 2

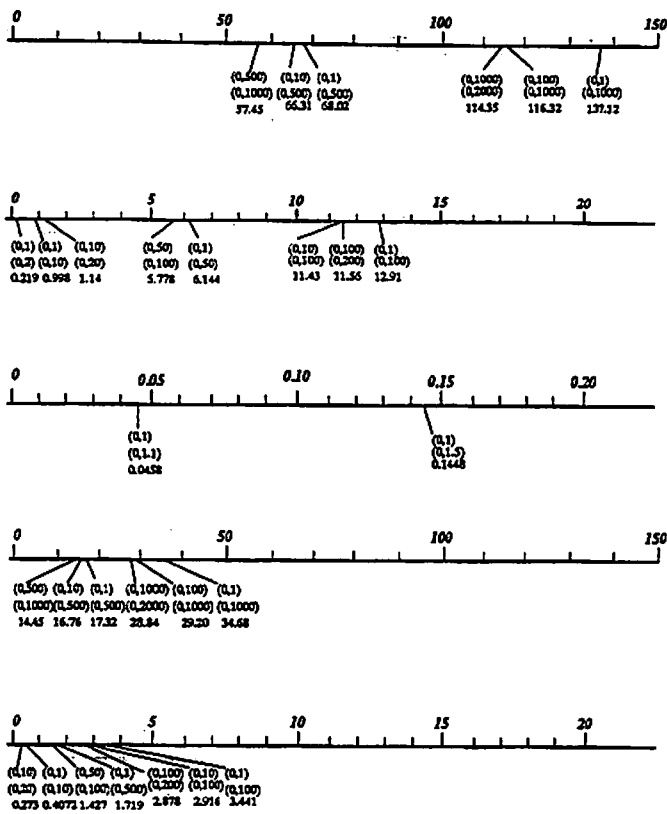


Figure 3

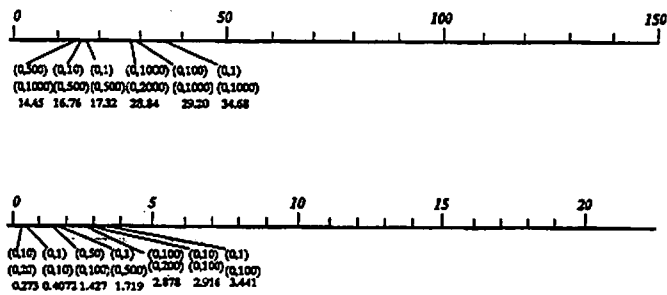


Figure 4

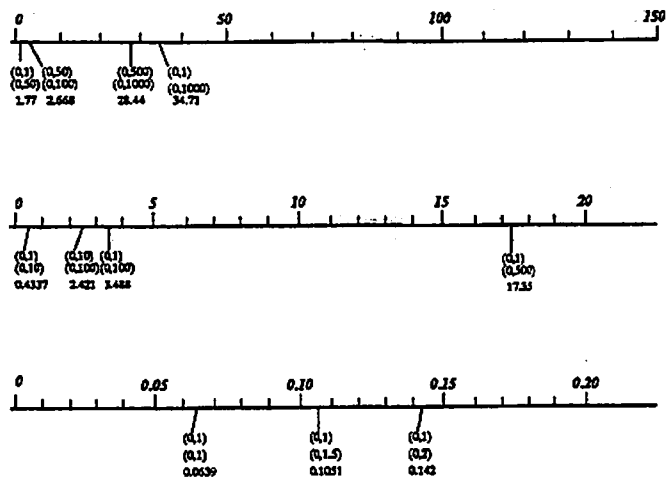


Figure 5

Hawking Radiation inside Apparent Horizon in Quantum Gravity

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Abstract

In this talk we would like to discuss the Hawking radiation inside the apparent horizon in quantum gravity. After constructing a canonical formalism for spherically symmetric geometry holding in the interior region of a black hole, we will apply for a specific model where the ingoing Vaidya metric is used as a simple model of an evaporating black hole. We will see that mass-loss rate of a black hole by thermal radiation is equal to the result obtained by Hawking in his semiclassical treatment. This talk is based on our recent work in the reference [1].

Until recent years, the region inside the event horizon of a black hole has been received little attention because it is physically of no relevance for external observers outside the horizon. However, this situation has seemed to be remarkably changed by the recent development of understanding the internal geometry near the Cauchy horizon inside the Reissner-Nordström black hole, what we call, the mass inflation [2,3,4], and the phenomenon of the smearing of black hole singularity in quantum gravity [5,6].

From this viewpoint, in this talk we will present the canonical formalism which describes the region inside the horizon of a spherically symmetric black hole. As one of applications of this canonical formalism, we shall consider the Hawking radiation [7]. The analysis follows the idea by Tomimatsu [8] that the Hamiltonian and supermomentum constraints have a simple and tractable form, and consequently it is possible to solve the Wheeler-DeWitt equation near the apparent horizon. The mass-loss rate by the black hole radiation will be shown to be equal to that evaluated by Hawking in the semiclassical approximation. Hawking's discovery of thermal radiation from quantum black holes was a pivotal event in the development of a quantum theory of gravity [9], thus the results obtained in this paper seems to encourage us to investigate various related problems further along the present canonical formalism.

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Following the conventional procedure, one needs to choose arbitrary spherically symmetric spacelike hypersurfaces to form a foliated structure of the spacetime. The important point is that the radial coordinate plays a role of time inside the horizon in the spherically symmetric coordinate system. As a simple choice, let us take a choice of $x^1 = \text{const}$ hypersurfaces to slice the spacetime. Later we will take the simplest choice $x^1 = r$. The analogous formalism which considers in the region outside the apparent horizon was constructed in the previous works [10,11].

The four dimensional action which we consider in this section is of the form

$$S = \int d^4x \sqrt{-^{(4)}g} \left[\frac{1}{16\pi} {}^{(4)}R - \frac{1}{4\pi} g^{\mu\nu} (D_\mu \Phi)^\dagger D_\nu \Phi - \frac{1}{16\pi e^2} F_{\mu\nu} F^{\mu\nu} \right], \quad (1)$$

where Φ is a complex scalar field, $F_{\mu\nu}$ is the corresponding field strength, and e is the electric charge of Φ . To clarify the four dimensional meaning we put the suffix (4) in front of the metric tensor and the curvature scalar. We follow the conventions adopted in the MTW textbook [12] and use the natural units $G = \hbar = c = 1$. The Greek letters $\mu, \nu, \dots$ take 0, 1, 2, and 3, on the other hand, the Latin ones $a, b, \dots$ do 0 and 1. Of course, the inclusion of other matter fields and the cosmological constants in this formalism is straightforward even if we limit ourselves to the action (1).

The most general spherically symmetric assumption for the metric is

$$\begin{aligned} ds^2 &= {}^{(4)}g_{\mu\nu} dx^\mu dx^\nu, \\ &= g_{ab} dx^a dx^b + \phi^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \end{aligned} \quad (2)$$

where the two dimensional metric g_{ab} and the radial function ϕ are the functions of only the two dimensional coordinates x^a . The substitution of (2) into (1) and then integration over the angular coordinates (θ, φ) leads to the following two dimensional effective action

$$\begin{aligned} S &= \frac{1}{2} \int d^2x \sqrt{-g} \left[1 + g^{ab} \partial_a \phi \partial_b \phi + \frac{1}{2} R \phi^2 \right] \\ &\quad - \int d^2x \sqrt{-g} \phi^2 g^{ab} (D_a \Phi)^\dagger D_b \Phi - \frac{1}{4} \int d^2x \sqrt{-g} \phi^2 F_{ab} F^{ab}, \end{aligned} \quad (3)$$

where

$$D_a \Phi = \partial_a \Phi + ie A_a \Phi. \quad (4)$$

Next let us rewrite the action (3) into the first-order ADM form. As mentioned before, we shall take the x^1 coordinate as time to cover the internal region of a black hole by spacelike hypersurfaces. The appropriate ADM splitting of (1+1)-dimensional spacetime is given by

$$g_{ab} = \begin{pmatrix} \gamma & \alpha \\ \alpha & \frac{\alpha^2}{\gamma} - \beta^2 \end{pmatrix}, \quad (5)$$

and the normal unit vector n^a which is orthogonal to the hypersurface $x^1 = \text{const}$ reads

$$n^a = \left(\frac{\alpha}{\beta\gamma}, -\frac{1}{\beta} \right). \quad (6)$$

The induced metric on the hypersurfaces, that is, the projection operator over $x^1 = \text{const}$ hypersurfaces, is given by

$$h^{ab} = g^{ab} + n^a n^b. \quad (7)$$

It is easy to check that h^{ab} is indeed the projection operator by substituting (5) and (6) into (7).

The extrinsic curvature K_{ab} , its trace K , and the scalar curvature R are given by [13]

$$K_{ab} = K_{ba} = h_a^c \nabla_c n_b, \quad (8)$$

$$K = g^{ab} K_{ab} = \nabla_a n^a = \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} n^a), \quad (9)$$

and

$$R = 2n^a \partial_a K + 2K^2 - 2\nabla_c (n^a \nabla_a n^c). \quad (10)$$

A straightforward calculation yields

$$K = -\frac{\gamma'}{2\beta\gamma} + \frac{\dot{\alpha}}{\beta\gamma} - \frac{\alpha}{2\beta\gamma^2} \dot{\gamma}, \quad (11)$$

and

$$R = 2n^a \partial_a K + 2K^2 - \frac{2}{\beta\sqrt{\gamma}} \partial_0 \left(\frac{\dot{\beta}}{\sqrt{\gamma}} \right), \quad (12)$$

where $\frac{\partial}{\partial x^0} = \partial_0$ and $\frac{\partial}{\partial x^1} = \partial_1$ are also denoted by an overdot and a prime, respectively. As a result, the action (3) can be written as

$$\begin{aligned} S = \int d^2x L = \int d^2x [& \frac{1}{2} \beta \sqrt{\gamma} \{ 1 - (n^a \partial_a \phi)^2 + \frac{1}{\gamma} \dot{\phi}^2 - K n^a \partial_a (\phi^2) \\ & + \frac{\dot{\beta}}{\beta\gamma} \partial_0 (\phi^2) \} + \beta \sqrt{\gamma} \phi^2 \{ |n^a D_a \Phi|^2 - \frac{1}{\gamma} |D_0 \Phi|^2 \} + \frac{1}{2} \beta \sqrt{\gamma} \phi^2 E^2] \\ & + \int d^2x [\frac{1}{2} \partial_a (\beta \sqrt{\gamma} K n^a \phi^2) - \frac{1}{2} \partial_0 \left(\frac{\dot{\beta}}{\sqrt{\gamma} \phi^2} \right)], \end{aligned} \quad (13)$$

where

$$\begin{aligned} E &= \frac{1}{\sqrt{-g}} F_{01}, \\ &= \frac{1}{\beta\sqrt{\gamma}} (\dot{A}_1 - A'_0). \end{aligned} \quad (14)$$

By differentiating the action (13) with respect to the canonical variables $\Phi(\Phi^\dagger)$, ϕ , γ_0 and A_0 we have the corresponding conjugate momenta $p_\Phi(p_\Phi^\dagger)$, p_ϕ , p_γ and p_A

$$p_\Phi = -\sqrt{\gamma} \phi^2 (n^a D_a \Phi)^\dagger, \quad (15)$$

$$p_\phi = \sqrt{\gamma} n^a \partial_a \phi + \sqrt{\gamma} K \phi, \quad (16)$$

$$p_\gamma = \frac{1}{4\sqrt{\gamma}} n^a \partial_a (\phi^2), \quad (17)$$

$$p_A = -\phi^2 E. \quad (18)$$

The Hamiltonian H is expressed by a linear combination of constraints as expected

$$H = \int dx^0 (\alpha H_0 + \beta H_1 + A_1 H_2), \quad (19)$$

where

$$H_0 = \frac{1}{\gamma} [p_\phi D_0 \Phi + p_{\phi^\dagger} (D_0 \Phi)^\dagger] + \frac{1}{\gamma} p_\phi \dot{\phi} - 2\dot{p}_\gamma - \frac{1}{\gamma} p_\gamma \dot{\gamma}, \quad (20)$$

$$\begin{aligned} H_1 = & \frac{1}{\sqrt{\gamma} \phi^2} p_\phi p_{\phi^\dagger} - \frac{\sqrt{\gamma}}{2} - \frac{\dot{\phi}^2}{2\sqrt{\gamma}} + \partial_0 \left(\frac{\partial_0 (\phi^2)}{2\sqrt{\gamma}} \right) \\ & + \frac{\phi^2}{\sqrt{\gamma}} |D_0 \Phi|^2 - \frac{2\sqrt{\gamma}}{\phi} p_\phi p_\gamma + \frac{2\gamma\sqrt{\gamma}}{\phi^2} p_\gamma^2 + \frac{\sqrt{\gamma}}{2\phi^2} p_A^2, \end{aligned} \quad (21)$$

$$H_2 = -ie(p_\phi \Phi - p_{\phi^\dagger} \Phi^\dagger) - \dot{p}_A. \quad (22)$$

Note that α , β and A_1 are non-dynamical Lagrange multiplier fields.

The action can be easily cast into the first-order ADM canonical form by the dual Legendre transformation. As Regge and Teitelboim pointed out [14], in order to have the correct Hamiltonian which produces the Einstein equations through the Hamilton equations, one has to supplement the surface terms to the Hamiltonian (19). In the present formalism, since we take the variation of all the fields to be zero at both the singularity and the apparent horizon we do not have to add any surface terms to the Hamiltonian.

We now turn our attention to an application of the canonical formalism constructed in the previous section for understanding the Hawking radiation [7] from the viewpoint in the internal region of a black hole in quantum gravity. A similar analysis was performed in the external region of a black hole by Tomimatsu [8].

To consider the simplest model of the Hawking radiation, let us turn off the electromagnetic field and treat the neutral scalar field by which the constraint H_2 generating the $U(1)$ gauge transformations identically vanishes. Moreover, we shall use the ingoing Vaidya metric to express the black hole radiation. The treatment of the case of the outgoing Vaidya metric can be made in a perfectly similar way. Here let us define the two dimensional coordinate x^a by

$$x^a = (x^0, x^1) = (v - r, r), \quad (23)$$

where v is the advanced time coordinate. Note that we have chosen $x^1 = r$ as mentioned above. And we fix the gauge freedoms corresponding to the two dimensional reparametrization invariances by the gauge conditions

$$\begin{aligned} g_{ab} &= \begin{pmatrix} \gamma & \alpha \\ \alpha & \frac{\alpha^2}{\gamma} - \beta^2 \end{pmatrix}, \\ &= \begin{pmatrix} -(1 - \frac{2M}{r}) & \frac{2M}{r} \\ \frac{2M}{r} & 1 + \frac{2M}{r} \end{pmatrix}. \end{aligned} \quad (24)$$

From these equations the two dimensional line element takes a form of the Vaidya metric

$$\begin{aligned} ds^2 &= g_{ab} dx^a dx^b, \\ &= -(1 - \frac{2M}{r}) dv^2 + 2dvdr. \end{aligned} \quad (25)$$

For a dynamical black hole, it is useful to consider the local definition of horizon, i.e., the apparent horizon, rather than the global one, the event horizon. The apparent horizon is then defined as

$$x^1 = r = 2M(x^0, x^1). \quad (26)$$

Following the idea in the reference [8], let us attempt to solve the Hamiltonian and supermomentum constraints only in the vicinity of the apparent horizon. Near the apparent horizon, by intuition we would be able to make an approximation

$$\Phi \approx \Phi(v), M \approx M(v), \phi \approx r. \quad (27)$$

From now on we shall use $\approx$ to indicate the equalities which hold approximately near the apparent horizon. Indeed one can prove the above assumptions (27) to be consistent with the field equations [1].

Near the apparent horizon (26), Eq.(24) yields

$$\alpha \approx +1, \gamma = \frac{1}{\beta^2} \approx 0, \quad (28)$$

and the canonical conjugate momenta (15)-(17) are given approximately as

$$\begin{aligned} p_\Phi &\approx -\phi^2 \partial_v \Phi, \\ p_\phi &\approx -\frac{1}{\gamma} \partial_v M, \\ p_\gamma &\approx -\frac{1}{2} \phi \approx -M. \end{aligned} \quad (29)$$

Moreover, the two constraints are proportional to each other

$$\begin{aligned} -\gamma H_0 &\approx \sqrt{\gamma} H_1, \\ &\approx \frac{1}{\phi^2} p_\Phi^2 + \gamma p_\phi. \end{aligned} \quad (30)$$

Here we have one important remark. Just at the apparent horizon γ becomes zero so that the various equalities approximately hold when we restrict our attention to the interior

region near but not at the apparent horizon. Under this sort of regularization γ takes a small but finite value.

An imposition of the constraint (30) as an operator equation on the state produces the Wheeler-DeWitt equation

$$i\frac{\partial\Psi}{\partial\phi} = -\frac{1}{\gamma\phi^2}\frac{\partial^2}{\partial\Phi^2}\Psi. \quad (31)$$

This Wheeler-DeWitt equation can be interpreted as the Schrödinger equation with the Hamiltonian $H_S = p_\Phi^2$ and the time $T_S = \frac{1}{\gamma\phi}$ in the superspace in hand. Here it is interesting to note that the superspace time T_S "stops" on the horizon because of gravitational time dilation.

Now it is easy to find a special solution of the Wheeler-DeWitt equation (31) by the method of separation of variables. The result is

$$\Psi = (Be^{\sqrt{A}\Phi} + Ce^{-\sqrt{A}\Phi})e^{-i\frac{A}{\gamma\phi}}, \quad (32)$$

where A , B , and C are integration constants. If one defines an expectation value $\langle \mathcal{O} \rangle$ of an operator $\mathcal{O}$ as

$$\langle \mathcal{O} \rangle = \frac{1}{\int d\Phi |\Psi|^2} \int d\Phi \Psi^* \mathcal{O} \Psi, \quad (33)$$

one can calculate $\langle \partial_\nu M \rangle$ by using either (29) or (30)

$$\langle \partial_\nu M \rangle = -\frac{A}{\phi^2}. \quad (34)$$

This equation shows the black hole radiation when one chooses the constant A to be a positive constant k^2 . Then the mass-loss rate becomes

$$\langle \partial_\nu M \rangle = -\frac{k^2}{4 \langle M \rangle^2}. \quad (35)$$

This result exactly corresponds to that calculated by Hawking in the semiclassical approach [7] and by Tomimatsu in the exterior region of the apparent horizon in quantum gravity [8]. Thus a black hole completely evaporates within a finite time. However, it is worth pointing out the difference between the Howking formulation and the present one. In the Hawking's semiclassical approach the gravitational field is fixed as a classical background and only the matter field is treated to be quantum-mechanical. By contrast, our present formulation is purely quantum-mechanical.

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On Lattice Regularization of 4D Quantum Gravity

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Abstract

A Langevin equation for 3-geometries is proposed in the Ashtekar's formalism to describe 4D Euclidean quantum gravity, in the sense that the corresponding Fokker-Planck hamiltonian recovers the hamiltonian in 4D quantum gravity exactly. The stochastic time corresponds to the Euclidean time in the gauge, $N = 1$ and $N^i = 0$. In this approach, the time evolution in 4D quantum gravity is understood as a stochastic process where the quantum fluctuation of "triad" is characterized by the curvature at the one unit time step before. The lattice regularization of 4D quantum gravity is presented in this context.

1 Introduction

In the course to study QCD_4 in terms of the Nicoli-Langevin maps [1][2], it is shown that stochastic quantization [3] of 3D Chern-Simons theory recovers the time evolution in 4D Euclidean Yang-Mills theory where the stochastic (or fictitious) time is interpreted as the Euclidean time [4]. There is also another example, found in stochastic quantization of matrix models with loop space hamiltonian at the double scaling limit, that the stochastic time corresponds to the Euclidean time coordinate in 2D quantum gravity [5][6]. These facts motivate us to interpret the stochastic time as the Euclidean time coordinate even in 4D quantum gravity. In this short note, I point out that the time evolution in 4D quantum gravity is described by a Langevin equation for 3-geometries in terms of the Ashtekar's canonical field variables [7][8] by showing that the corresponding Fokker-Planck hamiltonian operator recovers the hamiltonian of 4D Euclidean quantum gravity exactly [9]. The stochastic time is interpreted as the Euclidean time with the gauge $N = 1$ and $N^i = 0$, where N and N^i are the lapse function and the shift vector, respectively. The Hartle-Hawking type boundary condition is naturally imposed in this scheme by specifying the initial probability distribution functional. Then I propose the lattice regularization of this approach with 3D cubic lattice. The corresponding F-P hamiltonian defines the lattice regularization for the hamiltonian of 4D Euclidean quantum gravity in Ashtekar's formalism.

2 Basic Langevin Equation of Quantum Gravity

At first, I introduce the basic Langevin equation for 3-geometries in the Ashtekar's variables to recover the hamiltonian of 4D quantum gravity with the corresponding F-P

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hamiltonian defined latter. The simplest form of the Langevin equation is defined by,

$$\begin{aligned}\Delta A_i^a(x, \tau) &= \Delta \zeta_i^a(x, \tau), \\ < \Delta \zeta_i^a(x, \tau) \Delta \zeta_j^b(y, \tau) >_\zeta = \frac{\kappa}{2} \Delta \tau \epsilon^{abc} < F_{ij}^c(x, \tau) >_\zeta \delta^3(x - y),\end{aligned}\quad (1)$$

where $A_i^a(x, \tau)$ is a $SU(2)$ gauge field in Euclidean Ashtekar's canonical formalism². In this note, Latin indices "i,j,k,..." denote the spatial part of the spacetime coordinate indices. While the Latin letters "a,b,c,..." denote the spatial part of the internal indices. $x, y, \dots$ denote spatial spacetime coordinates. The one step time evolution is defined by $\Delta A_i^a(x, \tau) \equiv A_i^a(x, \tau + \Delta\tau) - A_i^a(x, \tau)$ in (1). $\Delta\tau$ is the unit of the discretized stochastic time and τ denotes the stochastic time after n steps, $\tau \equiv n\Delta\tau$. The discretization of the stochastic time is considered for convenience of stochastic calculus and for understanding the precise meaning of the noise correlation. The coupling constant κ is defined by $\kappa \equiv 16\pi G$ with G , the gravitational (Newton's) constant in the natural unit $\hbar = c = 1$. The noise variable in (1) is not a simple white noise. The expectation value of the R.H.S. of the noise correlation is understood to be taken with respect to the noises up to the one unit time step before, $\tau - \Delta\tau$, in the sense of Ito's stochastic calculus [10]. More precisely, if the expectation value is taken only at a specified stochastic time, τ , then the R.H.S. of the noise correlation is not the expectation value since it does not depend on the noise at τ . It is also equivalent to require $< \Delta \zeta_i^a(\tau) > = 0$ in Ito's calculus.

The basic Langevin equation is manifestly covariant under the $SU(2)$ local gauge transformation, $A_i^a(x) \rightarrow A_i^a + D_i^{ab}\omega^b(x)$. While it is not covariant under the spatial general coordinate transformation, $A_i^a(x) \rightarrow A_i^a + F_{ji}^a \xi^j(x)$, due to the divergent term, $\kappa \Delta\tau \delta^3(0) F_{ji}^a \xi^j$, which appears in the transformation of ΔA_i^a . This divergent term is formally (but not well-defined) cancelled by adding a term in the R.H.S. of the Langevin equation (1). The extra term comes from the invariant path-integral measure as we will see later.

3 Fokker-Planck Equation of Quantum Gravity

In terms of the solution of the Langevin equation, the following equality holds.

$$\begin{aligned}< \Pi_{z,i,a} \delta(A_i^a \zeta(x, \tau) - A_i^a(final)(x)) >_\zeta \\ &= < A_i(final) | e^{-\tau \tilde{H}_{FP}[\tilde{\pi}, \tilde{A}]} | A_i(initial) >, \\ &= < A_i(initial) | e^{-\tau \tilde{H}_{FP}[\tilde{A}, \tilde{\pi}]} | A_i(final) >.\end{aligned}\quad (2)$$

In the L.H.S., $A_i^a(x, \tau)$ denotes the solution of the Langevin equation with the initial condition, $A_i^a(x, 0) = A_i^a(initial)(x)$. In the R.H.S., $\tilde{H}_{FP}[\tilde{\pi}, \tilde{A}]$ is defined by,

$$\begin{aligned}\tilde{H}_{FP}[\tilde{\pi}, \tilde{A}] &= -\frac{\kappa}{4} \int d^3x \epsilon^{abc} \tilde{\pi}_a^i(x) \tilde{\pi}_b^j(x) \tilde{F}_{ij}^c(x), \\ \tilde{H}_{FP}[\tilde{A}, \tilde{\pi}] &= -\frac{\kappa}{4} \int d^3x \epsilon^{abc} \tilde{F}_{ij}^c(x) \tilde{\pi}_a^i(x) \tilde{\pi}_b^j(x).\end{aligned}\quad (3)$$

²I use the notation, $F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + \epsilon^{abc} A_i^b A_j^c$, and $D_i^{ac} = \delta^{ac} \partial_i + \epsilon^{abc} A_i^b$. The field variable $A_i^a(x)$ is real in the Euclidean Ashtekar's formalism.

To show the equality (2), the commutation relation,

$$[\hat{\pi}_a^i(x), \hat{A}_j^b(y)] = \delta_a^b \delta_j^i \delta^3(x-y), \quad (4)$$

and the vacuum, $|0\rangle$ with $\hat{\pi}_a^i|0\rangle = 0$, $\hat{A}_j^b|0\rangle = 0$, are assumed. The representation of the states is given by,

$$\begin{aligned} \langle A| &\equiv \langle 0| \exp\left[\int d^3x A_i^a(x) \hat{\pi}_a^i(x)\right], \\ |A\rangle &= \Pi_{x,a,i} \delta(\hat{A}_i^a(x) - A_i^a(x)) |0\rangle. \end{aligned} \quad (5)$$

The Fokker-Planck hamiltonians (3) are just the hamiltonian for 4D quantum gravity without cosmological term in Ashtekar's variables [7] with different operator orderings³. The annihilation operator $\hat{\pi}_a^i$ is interpreted as the "triad" in the Euclidean Ashtekar's formalism. Namely⁴, $\hat{\pi}_a^j(x) = -\frac{2i}{\kappa} \tilde{e}_a^j(x)$. The stochastic time evolution with the Langevin equation (1) corresponds to the Euclidean time evolution in 4D quantum gravity with the gauge fixing, $N = 1$ and $N^i = 0$ ⁵. In the stochastic process (1), the "triad" plays the role of the noise variable and the "quantum fluctuation" of the triad is characterized by the curvature. Though the two expressions in (2) are precisely equivalent, the second definition of the F-P hamiltonian in (3) has an advantage for further discussion. This is because the vacuum satisfies the (local) hamiltonian constraint with the operator ordering in H_{FP} , $\langle 0|\mathcal{H}(\hat{A}(x), \hat{\pi}(x)) = \mathcal{H}(\hat{A}(x), \hat{\pi}(x))|0\rangle = 0$, where $H_{FP}[\hat{A}, \hat{\pi}] \equiv \int d^3x \mathcal{H}(\hat{A}(x), \hat{\pi}(x))$. It should be noticed that the operator ordering of the F-P hamiltonian operator H_{FP} is deferent from that appeared in the F-P equation for the probability distribution.

Let us consider the initial distribution dependence of the probability distribution functional by averaging the expectation value (2) with respect to the initial probability distribution. It is defined by integrating out the initial configuration $A_i^{a(initial)}(x)$, on which the solution of the Langevin equation $A_{i\zeta}^a(x, \tau)$ depends, with the distribution, $P[A^{(initial)}, 0]$, in the L.H.S. of (2). The integration of $A_i^{a(initial)}$ gives a generalized form of the distribution functional $P[A, \tau]$, which is defined by $\langle O[A_\zeta](\tau) \rangle_\zeta \equiv \int \mathcal{D}A O[A] P[A, \tau]$, as follows

$$P[A, \tau] = \int \mathcal{D}A^{initial} \langle A^{initial} | P[\hat{A}, 0] e^{-\tau H_{FP}[\hat{A}, \hat{\pi}]} | A \rangle. \quad (6)$$

For an arbitrary observable $O[A]$, the average with respect to the initial value distribution also gives,

$$\langle O[A_\zeta(\tau)] \rangle_\zeta = \int \mathcal{D}A^{initial} \langle A^{initial} | P[\hat{A}, 0] e^{-\tau H_{FP}[\hat{A}, \hat{\pi}]} O[\hat{A}] | 0 \rangle. \quad (7)$$

In the definition of the expectation value in the L.H.S. of (7), the average is also taken with respect to the initial values with the distribution $P[A^{initial}, 0]$. For example, eq.(2)

³For a detailed derivation of the F-P hamiltonian operator from the Langevin equation, see also Ref. [6].

⁴ $\tilde{e}_a^i \equiv q^{1/2} e_a^i$, where $q = \det(q_{ij})$ with spatial metric $q^{ij} = e_a^i e_a^j$.

⁵In 2D quantum gravity, it is observed that this gauge fixing recovers the time evolution defined in non-critical string field theories[11].

is given for $O[A] = \Pi_{x,a,i} \delta(A_i^a(x) - A_i^{a(final)}(x))$, with the initial distribution, $P[A, 0] = \Pi_{x,a,i} \delta(A_i^a(x) - A_i^{a(initial)}(x))$.

Eq.(7) leads the time evolution equation for the expectation values of observables,

$$\frac{d}{d\tau} \langle O[A_\zeta(x, \tau)] \rangle_\zeta = - \langle H_{FP}[A_j^b(x), \frac{\delta}{\delta A_j^b(x)}] O[A_i^a(x)] |_{A_i^a(x)=A_{i\zeta}^a(x, \tau)} \rangle_\zeta, \quad (8)$$

as it is expected from the Langevin equation (1). I here notice that the initial condition dependence in (7) implies a constraint for initial distribution itself due to the equation,

$$\frac{d}{d\tau} \langle O[A_\zeta(x, \tau)] \rangle_\zeta = - \int \mathcal{D}A^{in} P[A^{in}, 0] H_{FP}[A^{in}, \frac{\delta}{\delta A^{in}}] \langle A^{in} | e^{-\tau H_{FP}[\hat{A}, \hat{\pi}]} O[\hat{A}] | A \rangle. \quad (9)$$

The existence of the equilibrium limit requires that the R.H.S. in (9) should be zero at the infinite stochastic time. A trivial solution of this constraint is the vanishing curvature at any spatial points, $P[A, 0] = \Pi_{x,a,ij} \delta(F_{ij}^a(x))$. This initial distribution, however, should be excluded because the R.H.S. of (8) is identically zero even at finite stochastic time and there is no time development. In general, if we choose a solution of the hamiltonian constraint as the initial value distribution, obviously there is no time evolution in the Fokker-Planck type equation (8). Especially, the constraint does not allow us to solve the Langevin equation with the initial condition $A_i^{a(initial)}(x) = 0$ for the existence of the time evolution.

To specify the physical boundary condition and to study a class of solutions for the hamiltonian constraint of 4D quantum gravity in this context, we may choose the initial condition which generates a nontrivial time evolution

$$P_{H-H}[A, 0] = \Pi_{x \neq z_0, a, i} \delta(F_{ij}^a(x)). \quad (10)$$

The spacetime point, $x = z_0$ at $\tau = 0$, is identified to the point where the 3D spatial manifold is absorbed into nothing in an analogous sense of the Hartle-Hawking type boundary condition [12].

As it is clear from (7), initial distributions and observables are the key quantities to specify 3-boundaries on quantized 4D spacetime manifolds in this approach. These quantities are the solution of the momentum constraint and the SU(2) Gauss law constraint. There are some candidates which may be useful to characterize 3-boundaries, such as the extrinsic curvature term, 3D Chern-Simons term, topological invariants [13] and loop variables [14] for which the present formalism can be applied. Under the appropriate choice of these observables, one can show the gauge invariance (gauge independence) of the expectation value of observables, following the standard method to fix the gauge in the Langevin equation [15][16]. For spatial general coordinate invariance and SU(2) gauge invariance, the gauge fixed Langevin equation is given by,

$$\Delta A_i^a(x, \tau) = -\Delta \tau \{ D_i^{ab} \Psi^b + F_{ij}^a \Phi^j \}(x, \tau) + \Delta \zeta_i^a(x, \tau), \quad (11)$$

with the same noise correlation as eq.(1). The gauge fixing corresponds to choose the multiplier fields, $\Psi^a(x)$ and $\Phi^i(x)$, as specified functionals of A_i^a 's. Then, the F-P hamiltonians are given by,

$$\tilde{H}_{FP}^{(g,f)}[\hat{\pi}, \hat{A}] = \tilde{H}_{FP} + \int d^3x \{ \Phi^i(x) \tilde{C}_i^M(x) + \Psi^a(x) \tilde{C}_G^a(x) \},$$

$$H_{FP}^{(g,f)}[\hat{A}, \hat{\pi}] = H_{FP} - \int d^3x \{ \Phi^i(x) C_i^M(x) + \Psi^a(x) C_G^a(x) \}, \quad (12)$$

instead of eq.(3), with

$$\begin{aligned} C_i^M(x) &\equiv \hat{F}_{ij}^a \hat{\pi}_a^j, \\ C_G^a(x) &\equiv \hat{D}_i^{ab} \hat{\pi}_b^i. \end{aligned} \quad (13)$$

C_j^M and C_G^a defines the momentum constraint and the Gauss law constraint, respectively. In the first expression in (12), $\tilde{C}_j^M$ and $\tilde{C}_G^a$, are defined by replacing the operator orderings in the original constraints so that the conjugate momentum, $\hat{\pi}_b^j$'s, are gathered to the left to A_i^a 's in the constraints. The probability distribution and the expectation value defined by the Langevin equation (11) are given by eqs.(6) (7) together with the F-P hamiltonian (12) instead of (3). As it is clear from (7) with the second expression of the F-P hamiltonian in (12), the insertion of constraints does not change the expectation value of the observables provided that the observables are the solution of these constraints, such as the extrinsic curvature term and 3D Chern-Simons term. We notice that the operator ordering in the second expression in (12) is essential for the local gauge invariance in our definition of the vacuum, namely the vacuum is gauge invariant, $\langle 0|C_i^M = C_i^M|0 \rangle = \langle 0|C_G^a = C_G^a|0 \rangle = 0$, with the operator ordering in (13). Therefore, (11) gives the same expectation value for arbitrary observables as one given by (1), provided that these observables are the solution of the constraints in (13). The gauge fixing in (1) corresponds to $N = 1$, $\Phi^i = \Psi^a = 0$, where Φ^i is essentially the shift vector. On the other hand, as for the partition function defined by the Langevin equation (11), which is given by (6) with (12), the gauge fixing procedure leads a gauge invariant (more precisely BRS invariant) partition function with a suitable choice of multiplier field, Φ^i and Ψ^a , as functionals of dynamical variables. To fix the gauge more generally for the reparametrization invariance of the time coordinate, one has to introduce the lapse function by multiplying the R.H.S. of the noise correlation (1) with it and choose it as a specified functional of A_i^a 's. Though the problem is not addressed in this note explicitly, it is possible following the general framework [16].

4 Lattice Regularization

Now let us consider the regularization procedure in this context. As it has been noticed in (1), an extra term is necessary for the covariance of the Langevin equation under the spatial general coordinate transformation. It is a direct consequence of the fact that the Ito's calculus picks up the Jacobian factor which comes from the change of variables in the path-integral measure [17][21]. One way to specify the path-integral measure in configuration space is to introduce a regularization for the noise correlation in (1). I here propose a lattice regularization of the Langevin equation and the noise correlation (1) in which the invariant property of the regularized Langevin equation in the sense of Ito's calculus naturally introduces the invariant path-integral measure in the configuration space of link variables.

The lattice regularized Langevin equation is defined by

$$\begin{aligned}\Delta U_l(\tau) &= \left\{ \Delta W_l(\tau) + \Delta\tau |\mathcal{G}|^{1/2} (D_L)_l^b (|\mathcal{G}|^{-1/2} \mathcal{G}_{ll'}^{ab} T^a) \right\} U_l(\tau) \\ |\mathcal{G}| &\equiv \det(\mathcal{G}_{ll'}^{ab}) .\end{aligned}\tag{14}$$

Here the dynamical variable and the noise variable, U_l and ΔW_l respectively, have been assigned on the link, $l \equiv (x, i)$, of 3-dimensional cubic lattice, which is specified by the site x and its nearest neighbor in the i -th direction $x + \hat{i}$. U_l is an element of $SU(2)$ group in the adjoint representation. While the noise, ΔW_l is algebra-valued in the adjoint representation, $\Delta W_l = \Delta W_l^a T^a$, with $[T^a, T^b] = \epsilon^{abc} T^c$. $(D_L)_l^a$ is the left Lie derivative on the group manifold defined by

$$[(D_L)_l^a, U_l] = \delta^{ll'} T^a U_l, \quad [(D_L)_l^a, (D_L)_l^b] = -\epsilon^{abc} (D_L)_l^c .\tag{15}$$

The regularized Langevin equation describes the one step time evolution of the link variables, $U_l(\tau + \Delta\tau) = U_l(\tau) + \Delta U_l(\tau)$. It should be noticed that $U_l(\tau + \Delta\tau)$ is also an element of $SU(2)$ group. The quantity $\mathcal{G}_{ll'}^{ab}$, is interpreted as the inverse of the "superspace" metric, here the superspace is spanned by the link variables $\{U_l\}$. The inverse of the superspace metric is given in the following regularized correlation of the noises defined on the links, $l = (x, i)$ and $l' = (y, j)$.

$$\langle (\Delta W_l)_{\alpha\beta}(\tau) (\Delta W_{l'})_{\gamma\delta}(\tau) \rangle = \Delta\tau \langle \mathcal{G}_{ll'}^{ab}(\tau) T_{\alpha\beta}^a T_{\gamma\delta}^b \rangle ,\tag{16}$$

where,

$$\begin{aligned}\mathcal{G}_{ll'}^{ab}(\tau) T_{\alpha\beta}^a T_{\gamma\delta}^b &= g_0 \delta_{x,y} \{ \delta_{\alpha\delta} (\Pi_{i \neq j} U)_{\beta\gamma} + \delta_{\beta\gamma} (\Pi_{i \neq j} U)_{\alpha\delta} - \delta_{\beta\delta} (\Pi_{i \neq j} U)_{\alpha\gamma} - \delta_{\alpha\gamma} (\Pi_{i \neq j} U)_{\beta\delta} \} , \\ (\Pi_{i \neq j} U)_{\alpha\gamma} &\equiv \frac{1}{2} \{ (U(x, i) U(x + \hat{i}, j) U(x + \hat{j}, i)^\dagger U(x, j)^\dagger)_{\alpha\gamma} \\ &\quad - (U(x, j) U(x + \hat{j}, i) U(x + \hat{i}, j)^\dagger U(x, i)^\dagger)_{\alpha\gamma} \} .\end{aligned}\tag{17}$$

g_0 is an arbitrary constant.

The lattice regularization has been determined from the following requirements. (i) The Langevin equation (14) and the noise correlation (16) should be covariant under the $SU(2)$ local gauge transformation,

$$\begin{aligned}U(x, i) &\rightarrow V^{-1}(x) U(x, i) V(x + \hat{i}) , \\ \Delta W(x, i) &\rightarrow V^{-1}(x) \Delta W(x, i) V(x) .\end{aligned}\tag{18}$$

(ii) The model recovers the Langevin equation and the noise correlation in (1) at the naive continuum limit defined by, $U(x, i) \approx 1 + A_i^b T^b a$ and $\Delta W(x, i) \approx \Delta \zeta_i^b T^b a$. At the limit, the coupling constant and the stochastic time are scaled, $g_0 a^2 \rightarrow \frac{\kappa}{2}$ and $\Delta\tau \rightarrow a^{-1} \Delta\tau$, as the lattice spacing a goes to zero, $a \rightarrow 0$. (iii) The regularized Langevin equation (14) is covariant under the "general coordinate transformation" in superspace, $\{U(x, i)\}$.

$$U(x, i)_{\alpha\beta} \rightarrow V(x, i)_{\alpha\beta} [U] ,\tag{19}$$

where $V(x, i)$ is also an element of $SU(2)$ group in the adjoint representation and an arbitrary functional of $U(y, j)$. The second term in the R.H.S. in (14) is necessary for the invariance in Ito's calculus [17]. The requirement (iii) is introduced to realize the spatial coordinate invariance in the lattice regularization. By using these requirements, it is straightforward to construct the model in the fundamental representation. It is given by simply replacing the inverse of the superspace metric (17) to the following one.

$$\mathcal{G}_{ll'}^{ab}(\tau) T_{\alpha\beta}^a T_{\gamma\delta}^b = g_0' \delta_{x,y} \left\{ \delta_{\beta\gamma} (\Pi_{i \neq j} U)_{\alpha\delta} - \delta_{\alpha\delta} (\Pi_{i \neq j} U)_{\gamma\beta} \right\}. \quad (20)$$

By applying the argument in Ref. [17], I have introduced the second term in the R.H.S. of (14) for the covariance of the lattice regularized Langevin equation under the transformation (19). The similar argument is formally possible even in the non-regularized version (1) if we identify the R.H.S. of the noise correlation in (1) to the inverse of the continuum superspace metric. Indeed, from the requirement (ii), the naive continuum limit of these equations, (14) (15) and (16), coincides with those in eq.(1) except a term which comes from the second term in the R.H.S. of the Langevin equation (14). The extra term has the same structure as the second term in the regularized Langevin equation (14) if we identify the noise correlation in (1) as the inverse of the superspace metric in Ashtekar's configuration variables. The extra contribution which comes from this term under the spatial general coordinate transformation formally cancels the divergent term, $\kappa \Delta \tau \delta^3(0) F_{ji}^a \xi^j$, which appears in the transformation in ΔA_i^a . The extra term appeared in the naive continuum limit of (14) actually represents the contribution from the path-integral measure and it is necessary for the invariant property of the Langevin equation (1) in a formal sense, however, it is not well-defined without regularization. A more detailed discussion, especially on the covariant property of the lattice regularization under the spatial general coordinate transformation in the present approach, will be published in elsewhere [18].

By using the same method developed in the continuum version, one can derive the regularized version of the expectation value, such as (6). I only comment on the three important consequences of the lattice regularized Langevin equation (14). The first is the corresponding lattice regularized F-P hamiltonian which is given by,

$$H_{reg} = -\frac{1}{2} \sum_{l,l'} |\mathcal{G}|^{1/2} (D_L)_l^a \{ |\mathcal{G}|^{-1/2} \mathcal{G}_{ll'}^{ab} (D_L)_{l'}^b \}. \quad (21)$$

The F-P hamiltonian defines the lattice regularization of the hamiltonian for 4D Euclidean quantum gravity in Ashtekar's formalism⁶. The second is the invariant measure which appears in the path-integral representation of the regularized expectation value of an observable (6) after integrating out the canonical momentum variables, $(D_L)_l^a$.

$$\mathcal{D}U \equiv \mathcal{G}^{-1/2} \Pi_l dU_l, \quad (22)$$

where dU denotes the left invariant Haar measure. The measure is invariant under the general coordinate transformation in superspace (19).

⁶In Refs. [19][20], similar lattice regularized hamiltonians have been discussed without the contribution of $|\mathcal{G}|$.

The third is the Schwinger-Dyson equation in this context. It is given by [6]

$$\langle |G|^{1/2} \{ (D_L)_\mu^b (|G|^{-1/2} G_{\mu\nu}^{ab}) \} T^a U_l \rangle = 0 , \quad (23)$$

at an equilibrium limit, $\lim_{\tau \rightarrow \infty} \langle \dot{U} \rangle = 0$.

5 Conclusion

In this note, I have pointed out that there exist a Langevin system for 3-geometries which describes the time evolution in 4D Euclidean quantum gravity. There have been some attempts to define 4D quantum gravity in the context of stochastic quantization [22][23][24][25] by introducing the stochastic time as the fifth time coordinate, however, the philosophy of the present approach is different from these previous works. The strategy I would like to adopt here is to characterize the 3-boundaries in the 4D space-time manifold by using the solution of both the momentum constraint and the Gauss law constraint to prepare the initial distribution and the observables. Then the remaining constraint is only the hamiltonian constraint which may be realized at the infinite stochastic time limit provided there exists an equilibrium limit. This is actually the idea which we have learned in stochastic quantization of matrix models in loop space for 2D quantum gravity, where the stochastic time is the proper Euclidean time and the hamiltonian constraint is realized as the Virasoro constraint in loop space [5][6]. In order to work in this program, one has to construct observables in the lattice regularization. It should be noticed that the Wilson loop is not a spatially general coordinate invariant observable in the present lattice regularization [18].

In the Langevin equation (1) and (14), the "triad" in Ashtekar's variables are realized as noise variables, which presumably represents the stochastic 3-geometries. The lattice regularized Langevin equation (14) may provide a possible basis for numerical simulation in 4D quantum gravity. The problem of the present scheme is that the noise correlation in the basic Langevin equations, (1) and (14), are not positive definite. It would force the Langevin equations to be complex. The field variables, though they are real in the Euclidean Ashtekar's formalism, would also become complex in the time evolution. The point would be main difficulty for the numerical analysis in this scheme. One way to deal with the problem may be to extend the Langevin equation (1) to a class of more general gauge fixing. It is always possible by multiplying the noise correlation in (1) with lapse function. Then one may choose the lapse function so that the noise correlation keeps the value to be positive definite. It is an open question if this gauge fixing procedure, a choice of non-trivial lapse function, make sense in the lattice regularization. It is also an open question how we introduce the cosmological term in (1) and (14). Apart from these questions, the description with the Langevin equation has a topological feature in the sense of Nicoli-Langevin map. Such a topological feature would relax another difficulty, renormalizability of quantum gravity. I hope that the approach is useful for deeper understanding of quantum gravity.

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A trick for passing degenerate points in Ashtekar formulation

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Abstract

We study a trick for passing a degenerate point, which is one of the advantages in the Ashtekar formulation of general relativity, within classical spacetime dynamics. Assuming that all dynamical variables are finite, we show that an essential mechanism for it is in complexifying variables rather than taking a special pair of dynamical variables. Using a degenerate solution derived by pull-back technique and taking some reality recovering conditions, we show some examples of such dynamical foliations by integrating dynamical equations numerically. We also discuss some features of these applications.

1 Introduction

One of the advantages of Ashtekar's formulation[1] of general relativity is the tractability of a degenerate point. Here, a 'degenerate point' is defined as the point in the spacetime where the density e of 3-space vanishes. This advantage comes from a fact that Ashtekar's formulation does not contain the inverses of the variables.

In this paper, we discuss the mechanism of passing a degenerate point in Ashtekar's formulation. We fix our spacetime to have the Lorenzian signature, and study what problems appear and how to solve them when we consider the dynamics through a degenerate point. Here, the dynamics means the evolution of the three-hypersurfaces expressed by Ashtekar's variables $(\tilde{E}_a^i, \mathcal{A}_i^a)$, fixing the gauge freedom of the lapse function N , shift vector N^i and triad lapse function $\mathcal{A}_0^a$ every time step. Assuming that both all dynamical variables and the calculation time are finite, we seek the condition which enables us to 'pass' degenerate points for the case of a vacuum.

We take mainly two approaches. The first one, named 'intersecting slice approach', means that a foliation passes a degenerate point directly. The second one, 'deformed slice approach', takes a complex path around a real degenerate point.

Notations are described in our previous paper[3].

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2 Definitions of ‘passing’

A ‘degenerate point’ is the point in the spacetime where the density e of 3-space vanishes (not 4-space). In the Ashtekar formulation, the density is defined as $e = \sqrt{\det \tilde{E}_a^i}$, this corresponds to the same condition in the ADM formulation as $e^2 = \det \gamma_{ij}$, where γ_{ij} is 3-metric. Such a ‘degenerate point’ does not mean always a physical singularity. At the degenerate points, one can not recover ADM variables from Ashtekar variables [2]. In order to say ‘pass’ degenerate points, we assume the following four conditions: (a) Ashtekar variables $\tilde{E}_a^i, \mathcal{A}_i^a, \tilde{N}, N^i, \mathcal{A}_0^a$ must remain finite throughout the calculation, (b) the spatial derivatives of them must also be finite throughout the calculation (because they appear in the equations of motion) (c) they satisfies all the constraints and the equations of motion.¹ and (d) the calculation must be finished in finite coordinate time.

3 Intersecting slice approaches

In this section, we describe the possibility of an ‘intersecting slice approach’, which intends to pass a degenerate point directly. To say our conclusion first, we face at least two problems, which we next explain in order. The detail of this section is already in [2].

3.1 Cusped lapse/density problem

The troublesome variable is the (inverse) densitized lapse $\tilde{N} := N/e_\Sigma$, where $e_\Sigma = \sqrt{\det \gamma_{ij}}$. Since $\tilde{N}$ is held finite [condition (a)], the ADM lapse $N = \tilde{N}e_\Sigma$ vanishes at the degenerate point $e_\Sigma = 0$. Notice that the ratio of the proper time τ to the coordinate time t is not $\tilde{N}$ but N . Thus we are afraid that the calculation exhausts an infinite amount of time, i.e., the condition (d) is violated. Let us take t and τ such that $\tau = t = 0$ at the degenerate point $e_\Sigma = 0$. Then the condition (d) is denoted by

$$\delta t = \int_{-t_0}^{t_1} dt = \int_{-\tau_0}^{\tau_1} \frac{d\tau}{N(\tau)} < \infty. \quad (1)$$

Here the range of the integral is arbitrary but includes the zero point $t = \tau = 0$. Note that $N \geq 0$ since it is a lapse function. This condition originally appeared in [4].

In terms of τ , the equation (1) is considerably restrictive because $N = 0$ at $\tau = 0$. For example, let us consider the form $N = |\tau|^s$ where s is constant. The condition for (1) is $0 < s < 1$, e.g., $N = \sqrt{|\tau|}$ satisfies (1). We see $dN/d\tau$ does not exist at $\tau = 0$. This is reason for calling ‘cusped lapse’.

But, in terms of t , the equation (1) is not restrictive: The only request is $N = 0$ at $t = 0$. So the lapse is not necessarily cusped in terms of t .

¹From the assumptions (b) and (c), we have that the time differentiation of dynamical variables $\dot{\tilde{E}}_a^i, \dot{\mathcal{A}}_i^a$ must also be finite.

Furthermore we will see that the density $|e|$ should also satisfy a condition which is similar to the cusped lapse condition. We call this problem the *cusped density problem*.

$$\infty > \int_{-\tau_0}^{\tau} \frac{d\tau}{N} = \int_{-\tau_0}^{\tau} \frac{d\tau}{e\tilde{N}} = \int_{-\tau_0}^{\tau} \frac{d\tau}{|e| |\tilde{N}|} > \frac{1}{M} \int_{-\tau_0}^{\tau} \frac{d\tau}{|e|}.$$

3.2 Connection divergence problem

Let us consider the quantity ω_i^{0a} which appears in the definition of $\mathcal{A}_i^a$. In order to examine this finiteness, we prepare

$$\omega_i^{0a} \tilde{E}_a^i = \frac{1}{\tilde{N}} \left[\frac{\partial_i(e^2)}{e^2} - \frac{\partial_j(e^2)}{e^2} N^j - 2\partial_i(N^i) \right]. \quad (2)$$

Now we assume the conditions (a)-(d).

Then we see (2) diverges at the degenerate point. If we assume the triad reality conditions, this fact tells us the passing is impossible.

Even if we do not impose the triad reality conditions, i.e. we impose the metric reality conditions or do not impose any reality conditions, this problem still exists. Therefore it is very difficult to avoid this serious problem.

4 Deformed slice approaches

In the previous section, we show that foliations through a degenerate point are difficult even within the Ashtekar framework. Therefore, in the second approach, we try to make spacetime foliation avoid a degenerate point by breaking reality conditions for basic variables. In this section, we examine what conditions are needed when we impose such locally deforming time evolutions. The coordinates $(t, \mathbf{x})$ of foliation are kept real.

In the following expression, we postulate that a degenerate point exists at $(t, \mathbf{x}) = (t_*, \mathbf{x}_*)$, but extending to many degenerate points is straightforward. In order to recover the Einstein spacetime, the foliations have to satisfy the following conditions, such as the ‘foliation recovering’ condition, which ensure that the foliation locally deforms into the imaginary region only in the vicinity of the degenerate point.

(A) Foliation recovering condition

We seek the foliation which starts from the real section, evolves in the complex region only in the vicinity of a degenerate point, and ends in the real section. We call this the ‘foliation recovering condition’. In the local deformation, the lapse function and the shift vector become complex valued to foliate in the complex region. Thus the ‘foliation recovering condition’ is expressed in terms of the gauge functions as:

$$\int_{t_-}^{t_+} \Im N(t, \mathbf{x}) dt = 0, \quad \int_{t_-}^{t_+} \Im N^i(t, \mathbf{x}) dt = 0, \quad (3)$$

where $t_- < t_*$ and $t_+ > t_*$ are defined in appropriate far time from a degenerate point at $t = t_*$.

(B) Asymptotic reality condition (spatial far limit)

Although we are considering deforming Einstein spacetime by using a complex metric in the time evolution, we require such deformations to be as local as possible. Therefore we impose two conditions ((B)conditions on spatial far limit, (C)conditions on time far limit) in order to ensure that the metric becomes complex valued only in the vicinity of the degenerate points. The former (B) is the condition imposed on the spatial far limit, expressed both for gauge function and metric as:

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0 \pm \Delta \mathbf{x}} \Im N(t, \mathbf{x}) = 0, \quad \lim_{\mathbf{x} \rightarrow \mathbf{x}_0 \pm \Delta \mathbf{x}} \Im N^i(t, \mathbf{x}) = 0, \quad \text{for } \forall t \quad (4)$$

and

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}_0 \pm \Delta \mathbf{x}} \Im [\tilde{E}_a^i \tilde{E}^{ja} / \det \tilde{E}(t, \mathbf{x})] = 0, \quad \text{for } \forall t \quad (5)$$

where $\Delta \mathbf{x}$ decides what we call the *far region* from a degenerate point, and this depends on the problem.

(C) Asymptotic reality condition (time far limit)

As well as the above conditions, we impose an asymptotic reality condition on the time far limit:

$$\lim_{t \rightarrow t_0 \pm \Delta t} \Im N(t, \mathbf{x}) = 0, \quad \lim_{t \rightarrow t_0 \pm \Delta t} \Im N^i(t, \mathbf{x}) = 0, \quad \text{for } \forall \mathbf{x} \quad (6)$$

and

$$\int_{t_-}^{t_+} \frac{d}{dt} \Im [\tilde{E}_a^i \tilde{E}^{ja} / \det \tilde{E}(t, \mathbf{x})] dt = 0. \quad (7)$$

The integrand is calculated using a dynamical evolution equation, so the explicit form of (7) is not given at an arbitrary time except in some simple analytic cases. The condition (7) becomes a boundary problem of the evolution both in initial and final time-slice, so that this method requires numerical iterations for finding gauge functions to satisfy all conditions (3)-(7).

In general, we cannot deny a possibility that two the different gauges satisfying (A)(B)(C) exist and each yields a different metric after deforming. However, when we know the natural evolution outside a degenerate point, then we expect some continuity conditions to fix the final time slices naturally.

5 Examples

5.1 Exact solution with degenerate point

One method to transform an exact solution is by pull-back technique. When the pull-back includes a singularity it is not a diffeomorphism, and the coordinate transformation from the non-degenerate metric to the degenerate metric is singular. This technique was also used by Horowitz[5]. Here, we derive a flat Minkowski spacetime but with one degenerate point at the origin, by using the pull-back method, for later numerical demonstration of deforming slices:

$$\begin{aligned} ds^2 = & -[1 - (f'(t)h(x))^2]dt^2 + [-2f'(t)h(x)(1 - f(t)h'(x))]dtdx \\ & + (1 - f(t)h'(x))^2 dx^2 + dy^2 + dz^2, \end{aligned} \quad (8)$$

where $f(t), h(x)$ are arbitrary functions which satisfy

$$\begin{aligned} f(\pm\infty) &= f'(\pm\infty) = 0, \quad -1 < f'(t) < 1, \\ -1 < f(t) &\leq 1, \quad \text{equality is satisfied only when } t = 0, \\ h(\pm\infty) &= h'(\pm\infty) = 0, \quad -1 < h(x) < 1, \\ -1 < h'(x) &\leq 1, \quad \text{equality is satisfied only when } x = 0. \end{aligned} \quad (9)$$

We will see that this metric has the following properties: (1°) flat, (2°) asymptotic Minkowski, (3°) degenerate at $t = x = 0$, and (4°) Lorentzian (t -time, x, y, z -space) elsewhere.

For example, $f(t) = e^{-t^2}$; $h(x) = xe^{-x^2}$ satisfy conditions (9). We start from the metric given by this example as:

$$\begin{aligned} ds^2 = & -[1 - (2txe^{-t^2-x^2})^2]dt^2 + 4txe^{-t^2-x^2}[1 - (1 - 2x^2)e^{-t^2-x^2}]dtdx \\ & + [1 - (1 - 2x^2)e^{-t^2-x^2}]^2dx^2 + dy^2 + dz^2. \end{aligned} \quad (10)$$

Hereafter, we call this the dM solution (a degenerate expression of Minkowski spacetime).

5.2 Numerical demonstration of deforming slices

Using the dM solution (10) as a background metric, we here show some numerical demonstration of our new proposal of *deforming slice* approach in time evolution.

Since the metric (10) has a degenerate point at $x = t = 0$, it is clear that we can not evolve the solution beyond a degenerate point because the inverse 3-metric diverges at the degenerate point within the ADM formulation. As we noted in §3, even if we take the Ashtekar formulation in time evolutions, we may say it is hard to ‘pass’ such a point. This feature is shown in Fig.1. The figure is a result of time evolutions of a solution (10) using Ashtekar’s dynamical equation. We set the gauge conditions (slicing conditions) the same as the exact solution (10), i.e., the lapse $N = 1$, the shift vector $N_i = 2txe^{-t^2-x^2}[1 - (1 - 2x^2)e^{-t^2-x^2}]$, and the triad lapse $\mathcal{A}_0^a = 0$. Initial data are constructed from the exact solution (10) taking triad e_i^a as the diagonal matrix $(\sqrt{\gamma_{xx}}, 1, 1)$. We see that the time evolutions do not work properly after intersecting the point, because of ‘connection divergence problem’.

Next, we show that a ‘deformed slicing’ approach works well. In order to make deformed slicing, we allow the shift vector to be complex. The imaginary parts of N_i are arbitrary but should satisfy conditions (3), (4) and (6). As an example, we choose the function:

$$\Im m N_i = atxe^{-b(t^2+x^2)}, \quad (11)$$

where a and b are arbitrary constants. We set $a = 0.00703$ and $b = 1$.² We take the lapse $N = 1$ and the triad lapse $\mathcal{A}_0^a = 0$, the simplest ones as before. And initial data is the same as before.

²Using these values, we can pass the degenerate point. A numerical parameter search and demonstrations by using other shift functions are now in progress.

As we mentioned in the previous section, a solution which recovered real section again appears discretely. In order to judge whether an evolution recovers what we expected, we check two values at the final slice of our evolution

$$f(t_{final}) := \max_x |\Re(e(t = t_{final}, x) - 1)| \quad (12a)$$

$$r(t_{final}) := \max_x |\Im e(t = t_{final}, x)| \quad (12b)$$

Those are related with asymptotic flatness and asymptotic reality, respectively. An evolution is successful in avoiding a degenerate point by deforming if $\sqrt{f(t_{final})^2 + r(t_{final})^2} < 0.05$. We show an example of evolution in Fig.2. In this figure, the density is plotted versus coordinate x and t . We see that the time evolution does work properly for all the evolution and that the 3-metric satisfies the asymptotic reality condition again.

Readers may think that if the essential trick of passing such a degenerate point exists in complexifying variables, then we might find success also passing a degenerate point by complexifying ADM variables. We find that this statement is true, as in Fig.3.

6 Conclusions and Discussions

We showed that a direct passing of degenerate point (named intersecting slice approach) is hard to work if we impose some natural assumptions on dynamical and gauge variables to express Lorentzian spacetime evolution. We showed that only a deformed slice approach, which violates reality conditions in the neighborhood of a degenerate point, enables as to 'pass' such a point. Since Ashtekar's variables are originally defined as complex variables, the deformed slice approach is a natural foliation within this formulation. But if we plan to apply this approach to a certain dynamical problem in real space (Lorentzian) spacetime, we need to impose some kind of reality recovering conditions after detouring degenerate points. We proposed such an idea and showed that this proposal works in an actual dynamical problem. Therefore we conclude that a trick for dynamical passing a degenerate point exists in using *complexified* spacetime.

This result suggests us that there is a new possibility for dynamical problems in classical spacetime such as focusing, shell-crossing coordinate points and singularities. However, since we are necessarily required to break reality conditions locally, the problem is turned to a difficult eigenvalue-like problem, we think there is little practical merit in applying this formulation to numerical relativity. This is somewhat disappointing.

We note that we can use this formulation in Euclidean spacetime. For example, we may apply it to Kodama's solution[7] to patch two Euclidean spacetimes through a degenerate point.

We showed only the case that a spacetime will recover its original metric. But in general, initial and final metrics are allowed to be different. Therefore an application of our proposal to a dynamical process of 'topology change' or 'signature change' is expected. If general relativity is extended to allow degenerate metric, the topology of spacetime has a possibility to change even within a classical picture [5]. If we allow locally broken reality conditions, then our classical path may connect the spaces with different topology dynamically.

We expect our approach is the first step towards understanding dynamics of signature changing processes, topology changing processes and causal structure in a complex manifold.

Acknowledgements

We thank Paul Haines for careful reading of our manuscript. This work was supported partially by the Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Science, Sports and Culture No. 07854014, by the Grant-in-Aid for JSPS Fellow (No. 3327), Waseda University Grant for Special Research Projects, and NSF 96-00507.

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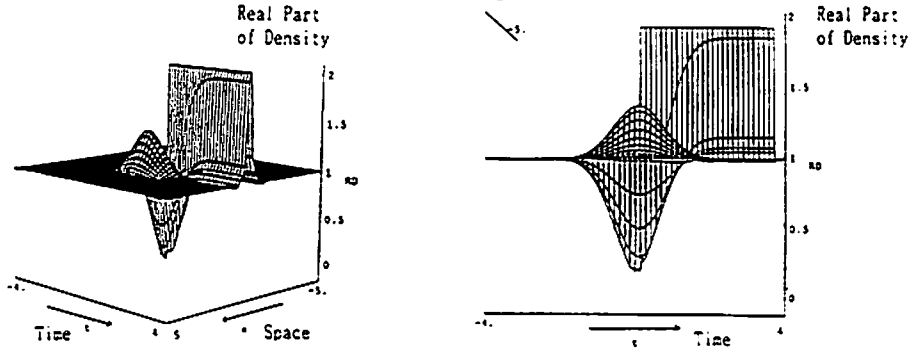


Figure 1: An example of *intersection approach* for the dM metric (10) using Ashtekar's dynamical equations in evolutions. The density is plotted versus spatial and time comoving coordinates x and t . A degenerate point exists at the origin ($t = x = 0$), and we see that the time evolutions do not work properly after intersecting the point.

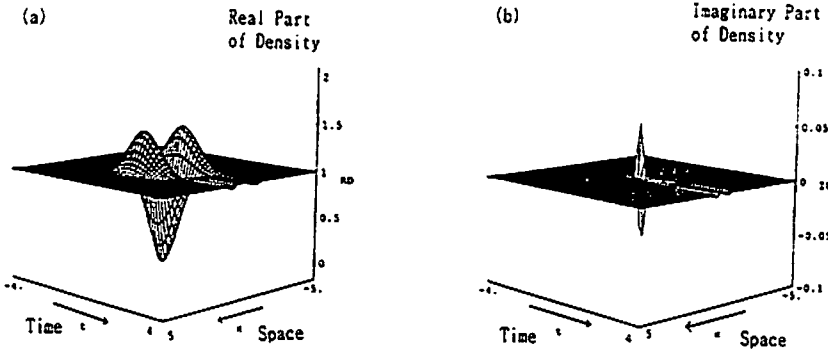


Figure 2: An example of *deformed slicing approach* for the dM metric (10) using Ashtekar's dynamical equation in evolution. The density [real part(a) and imaginary part(b)] is plotted versus coordinates x and t . A degenerate point exists at the origin ($t = x = 0$), and we see that the time evolution works properly for all the evolution and that the 3-metric satisfies the asymptotic reality condition again.

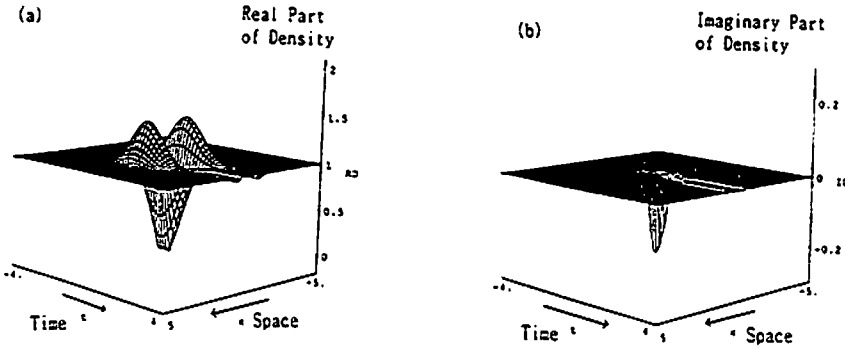


Figure 3: An example of *deformed slicing approach* for the dM metric (10) using ADM's dynamical equations in evolution. The density is plotted versus spatial and time comoving coordinates x and t . A degenerate point exists at the origin ($t = x = 0$), and we see that the time evolution does not work properly after intersecting the point.

Reality conditions for 2+1 dimensional gravity coupled with spinor field

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Abstract

We develop the canonical theory of three dimensional gravity coupled with spinor field. In the case of the gravity including the Dirac field, we find the new variables which change the Dirac brackets of the fundamental variables to the canonical ones, and also find that such variables do not exist in the case of the Majorana field.

1 Introduction

The aim of the present work is to describe the quantum theory of the 2+1 dimensional gravity coupled with spinor field. In the case of the 2+1 dimensional gravity including the Dirac field, Kim *et al* [1] showed that the constraint structure is similar to the 3+1 dimensional gravity [2], and found a new physical observable and its eigenstate. In [3], the solution of all quantum constraints based on the loop representation was found. However, since the non-Hermitian action for the Dirac field was used, suitable reality conditions have to be imposed on the canonical variables. But these conditions were not discussed. In previous paper [4], where we deal with the Hermitian action, it was shown that the Dirac brackets of the canonical variables are non-canonical. We also found the new variables which change the Dirac brackets of the canonical variables to the canonical ones. Because the new variables are complex, suitable reality conditions are needed. We showed how to treat the reality conditions. In this paper we review the canonical formalism of the gravity including the Dirac field, and study the canonical formalism for the Majorana field. In the case of the Majorana field the situation is different from the one of the case of the Dirac field. It is shown that there is no variable which changes the non-canonical Dirac brackets to the canonical ones. Finally, we also study quantum representation for the canonical gravity coupled with the Dirac field.

2 Canonical formalism of the gravity including the Dirac field

In this section we review the canonical formalism of the 2+1 dimensional gravity coupled with the Dirac field. For detail, see Ref. [4].

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2.1 Canonical formalism

The fundamental variables we use are the triad field e_μ^i and the dual spin connection

$$A_\mu^i = -\frac{1}{2}\epsilon^{ijk}\omega_{\mu jk}, \quad (1)$$

instead of the triad field and the usual spin connection $\omega_{\mu ij}$.

The action is written as

$$I = \int d^3x \left[\frac{1}{2}\epsilon^{\mu\nu\rho}e_\mu^i F_{\nu\rho i} - \frac{1}{2}e(\bar{\psi}\gamma^\mu D_\mu\psi - D_\mu\bar{\psi}\gamma^\mu\psi) - me\bar{\psi}\psi \right], \quad (2)$$

where $F_{\nu\rho i} = \partial_\nu A_{\rho i} - \partial_\rho A_{\nu i} - \epsilon_{ijk}A_\nu^j A_\rho^k$ is the curvature tensor of the spin connection, $\epsilon^{\mu\nu\rho}$ is the Levi-Civita antisymmetric tensor density and $D_\mu\psi = (\partial_\mu + \frac{1}{2}A_\mu^i\gamma_i)\psi$. The matrices γ_i generate the $SO(2,1)$ Lorentz group. The Dirac conjugation is defined by $\bar{\psi} := i\psi^\dagger\gamma_0$. We use Greek letters $\mu, \nu, \dots$ for curved indices in three dimensions and Latin letters $i, j, \dots$ for tangent space indices. The spacetime metric has a signature $(-, +, +)$ and we use the convention $\epsilon^{012} = 1 = -\epsilon_{012}$.

We decompose the spacetime metric following the ADM formalism. We assume that the spacetime manifold M has a topology $M = \Sigma \otimes R$, where Σ is a compact two dimensional manifold. We choose a time coordinate t on the manifold M so that M is foliated by two dimensional spacelike surfaces Σ_t each with the topology of Σ . One can define a timelike unit vector n^μ with $n^\mu n_\nu g_{\mu\nu} = -1$ which is normal to the Σ_t and a smooth time vector field t^μ which is chosen such that $t^\mu \nabla_\mu t = 1$. We then define the spatial metric $q_{\mu\nu}$ by $q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$; $n^\mu q_{\mu\nu} = 0$, the timelike part n^i of e_μ^i by $n^i \equiv n^\mu e_\mu^i$ and the projected part of e_μ^i into Σ_t by $E_\mu^i \equiv e_\mu^i (g_\mu^\nu + n_\mu n^\nu)$; $n^\mu E_\mu^i = 0$, $q_{\mu\nu} = E_\mu^i E_\nu^j \eta_{ij}$. The Levi-Civita density $\epsilon^{\mu\nu\rho}$ is related to a density on Σ_t , $\epsilon^{\mu\nu}$ by $\epsilon^{\mu\nu\rho} = 3N n^{[\mu} \epsilon^{\nu\rho]}$.

For canonical treatment, we compute the canonical momenta. Due to the fact that the Lagrangian density is a linear function of the velocities, the equations for momenta result in primary constraints. The consistency condition that the primary constraints are conserved requires secondary constraints. The consistency conditions of the secondary constraints become the combination of the secondary constraints, from which it follows that there are no tertiary constraints. In this case primary and secondary constraints become second and first class respectively.

After eliminating all the second class constraints, the non-vanishing Dirac brackets of the canonical variables are

$$\begin{aligned} \{\tilde{E}_i^\mu(x), A_\nu^j(y)\} &= q_\nu^\mu \eta_i^j \delta^2(x, y) \\ \{\psi_\alpha(x), \bar{\psi}_\beta(y)\} &= E^{-1}(n \cdot \gamma)_{\alpha\beta} \delta^2(x, y) \\ \{\psi_\alpha(x), A_\mu^i(y)\} &= \frac{1}{2}E^{-1}\epsilon_{\mu\nu}\epsilon^{ijk}\tilde{E}_j^\nu((n \cdot \gamma)\gamma_k\psi)_\alpha \delta^2(x, y) \\ \{\bar{\psi}_\alpha(x), A_\mu^i(y)\} &= \frac{1}{2}E^{-1}\epsilon_{\mu\nu}\epsilon^{ijk}\tilde{E}_j^\nu(\bar{\psi}\gamma_k(n \cdot \gamma))_\alpha \delta^2(x, y) \\ \{A_\mu^i(x), A_\nu^j(y)\} &= \frac{1}{2}\epsilon_{\mu\nu}n^i n^j \bar{\psi}\psi \delta^2(x, y) \end{aligned} \quad (3)$$

where $\tilde{E}_i^\mu$ and E are a vector density on Σ_t with $\tilde{E}_i^\mu \equiv \epsilon^{\mu\nu}E_{\nu i}$ and the determinant of E_μ^i . Unfortunately, A_μ^i does not commute with ψ , $\bar{\psi}$ and itself. In this respect the present

theory differs from the case that starts with non-Hermitian action [1], and the Dirac brackets are complicated at first sight. First class constraints are

$$\begin{aligned} H &= \frac{1}{2}\epsilon^{ijk}\tilde{E}_i^\mu\tilde{E}_j^\nu F_{\mu\nu k} + \frac{1}{2}E\epsilon^{ijk}\tilde{E}_i^\mu n_j\Psi_{\mu k} - mE^2\bar{\psi}\psi \approx 0 \\ H_\mu &= -\tilde{E}^{\nu i}F_{\mu\nu i} - \frac{1}{2}En_i\Psi_\mu^i \approx 0 \\ G_i &= D_\mu\tilde{E}_i^\mu + \frac{1}{2}En_i\bar{\psi}\psi \approx 0, \end{aligned} \quad (4)$$

where Ψ_μ^i is function defined as $\Psi_\mu^i := \bar{\psi}\gamma^i D_\mu\psi - (D_\mu\bar{\psi})\gamma^i\psi$. These constraints are called the Hamiltonian, the vector and the Gauss-law constraints respectively. The total Hamiltonian in the reduced phase space is described by these constraints as follows,

$$\mathcal{H} = -(NH + N^\mu H_\mu + A_0^i G_i). \quad (5)$$

where A_0^i and N are a time component of A_μ^i with $A_0^i \equiv t^\mu A_\mu^i$ and a lapse with density weight -1 respectively. Note that, since constraints and multipliers contained in (5) are real, this Hamiltonian is also real. So we do not have to consider the reality conditions.

Now we discuss the algebra of the constraints. Instead of the constraints (4), we use the following constraints smeared with suitable well-defined fields on Σ_t ,

$$\begin{aligned} H[N] &= \int d^2x N \left\{ \frac{1}{2}\epsilon^{ijk}\tilde{E}_i^\mu\tilde{E}_j^\nu F_{\mu\nu k} + \frac{1}{2}E\epsilon^{ijk}\tilde{E}_i^\mu n_j\Psi_{\mu k} - mE^2\bar{\psi}\psi \right\} \\ H_\mu[N^\mu] &= \int d^2x N^\mu \left\{ -\tilde{E}^{\nu i}F_{\mu\nu i} - \frac{1}{2}En_i\Psi_\mu^i \right\} \\ G_i[\Lambda^i] &= \int d^2x \Lambda^i \left\{ D_\mu\tilde{E}_i^\mu + \frac{1}{2}En_i\bar{\psi}\psi \right\}. \end{aligned} \quad (6)$$

where N , N^μ and Λ^i are smearing fields.

The constraint algebra are

$$\begin{aligned} \{G_i[\Lambda^i], G_j[\Gamma^j]\} &= G_i[\epsilon^{ijk}\Lambda_j\Gamma_k] \\ \{G_i[\Lambda^i], H[N]\} &= 0 \\ \{G_i[\Lambda^i], H_\mu[N^\mu]\} &= 0 \\ \{H_\mu[N^\mu], H_\nu[M^\nu]\} &= H_\mu[\mathcal{L}_{\vec{N}}M^\mu] + G_i[N^\mu M^\nu(F_{\mu\nu}^i + \frac{1}{2}\epsilon_{\mu\nu}n^i\bar{\psi}\psi(n \cdot G))] \\ \{H_\mu[N^\mu], H[N]\} &= H[\mathcal{L}_{\vec{N}}N] + G_i[NN^\mu\epsilon^{ijk}(\tilde{E}_j^\nu F_{\mu\nu k} + \frac{1}{2}En_j\Psi_{\mu k} \\ &\quad + mE\epsilon_{\mu\nu}\tilde{E}_j^\nu n_k\bar{\psi}\psi - \frac{1}{2}\epsilon_{\mu\nu}\tilde{E}_j^\nu n_k\bar{\psi}\psi(n \cdot G))] \\ \{H[N], H[M]\} &= H_\mu[\tilde{E}_i^\mu\tilde{E}^{\nu i}(N\partial_\nu M - M\partial_\nu N)] \end{aligned} \quad (7)$$

Note that in these algebra, some quadratic terms of constraint G_i appear in the right hand side, which do not appear in the case of the gravity including matter in 3+1 dimensions [2] and the non-Hermitian theory in 2+1 dimensions [1]. In classical level since these terms weakly vanish, the constraints are actually first class. So we think there is no problem. But we do not know whether these terms have an effect in quantum theory.

2.2 New variables

In the last section we found that the Dirac brackets of the canonical variables are non-canonical. This complication cause some difficulty. For example, we can no longer construct quantum theory in the connection representation.

In order to simplify the Dirac brackets of canonical variables, we introduce the following complex variables:

$$A'^i_\mu := A^i_\mu - \frac{1}{2}\epsilon_{\mu\nu}\epsilon^{ijk}\tilde{E}^\nu_j\bar{\psi}\gamma_k\psi, \quad (8)$$

which commutes with itself and ψ . However, it does not commute with $\bar{\psi}$, which then is not a suitable canonical variable. So we use $\pi := -E\bar{\psi}(n \cdot \gamma)$ as the canonical variable which replaces $\bar{\psi}$. The non-vanishing Dirac brackets of new variables are

$$\{\tilde{E}^\mu_i(x), A'^j_\nu(y)\} = q^\mu_\nu \eta^j_i \delta^2(x, y), \quad \{\pi_\alpha(x), \psi_\beta(y)\} = \delta_{\alpha\beta} \delta^2(x, y), \quad (9)$$

which are the same Dirac brackets in the non-Hermitian case. By means of this simplification, we can avoid the difficulty indicated above. So we can represent A'^i_μ by a multiplication operator in quantum theory.

The constraints in terms of the new variables are

$$\begin{aligned} H' &= \frac{1}{2}\epsilon^{ijk}\tilde{E}^\mu_i\tilde{E}^\nu_jF'_{\mu\nu k} - \tilde{E}^\mu_i\pi\gamma^iD'_\mu\psi - mE\pi(n \cdot \gamma)\psi + \frac{3}{4}(\pi\psi)^2 \\ H'_\mu &= -\tilde{E}^\nu_iF'^i_{\mu\nu} + \pi D'_\mu\psi \\ G'_i &= D'_\mu\tilde{E}^\mu_i - \frac{1}{2}\pi\gamma_i\psi, \end{aligned} \quad (10)$$

where $F'^i_{\mu\nu}$ is the curvature tensor of A'^i_μ and $D'_\mu\psi$ is the covariant derivative defined by $D'_\mu\psi := (\partial_\mu + \frac{1}{2}A'^i_\mu\gamma_i)\psi$. Note that the new constraints H'_μ and G'_i are almost the same as the ones of the non-Hermitian case in 2+1 dimensions [1] and Ashtekar formalism including the Dirac field in 3+1 dimensions [2]. The only difference is that $(\pi\psi)^2$ appears in the Hamiltonian constraint H' . The effect of this term will be discussed in section 4.

Using the new variables, the Hamiltonian is described as follows,

$$\mathcal{H} = -(NH' + N^\mu H'_\mu + \Lambda^i G'_i), \quad (11)$$

where

$$\Lambda^i \equiv A^i_0 + \frac{1}{2}N\epsilon^{ijk}n_j\pi(n \cdot \gamma)\gamma_k\psi - \frac{1}{2}N^\mu E^{-1}\epsilon_{\mu\nu}\epsilon^{ijk}\tilde{E}^\nu_j\pi(n \cdot \gamma)\gamma_k\psi. \quad (12)$$

The Hamiltonian takes a form of linear combination of the constraints with multipliers. Note that, as a result of getting the polynomial constraints, the multiplier of the Gauss-law constraint is different from the A^i_0 of (5).

Now we discuss the algebra of these constraints. Using the Dirac brackets (9), the constraint algebra are

$$\begin{aligned} \{G'_i[\Lambda^i], G'_j[\Gamma^j]\} &= G'_i[\epsilon^{ijk}\Lambda_j\Gamma_k] \\ \{G'_i[\Lambda^i], H'[N]\} &= 0 \\ \{G'_i[\Lambda^i], H'_\mu[N^\mu]\} &= 0 \end{aligned}$$

$$\begin{aligned}
\{H'_\mu[N^\mu], H'_\nu[M^\nu]\} &= H'_\mu[\mathcal{L}_{\tilde{N}}M^\mu] + G'_i\{N^\mu M^\nu F_{\mu\nu}^i\} \\
\{H'_\mu[N^\mu], H'[N]\} &= H'[\mathcal{L}_{\tilde{N}}N] \\
&\quad + G'_i[NN^\mu(\epsilon^{ijk}\tilde{E}_j^\nu F'_{\mu\nu k} - \pi\gamma^i D'_\mu\psi + m\epsilon_{\mu\nu}\epsilon^{ijk}\tilde{E}_j^\nu\pi\gamma_k\psi)] \\
\{H'[N], H'[M]\} &= H'_\mu[\tilde{E}_i^\mu\tilde{E}^{\nu i}(N\partial_\nu M - M\partial_\nu N)].
\end{aligned} \tag{13}$$

Note that, in contrast with the case based on the real variables, no quadratic terms of G'_i appear in the right hand side. This algebra (13) is similar to the non-Hermitian case except for mass term, so that the structure of the constraint algebra is not changed by the extra term $(\pi\psi)^2$ in (10).

As a result of introducing the new canonical variables, we constructed a theory which has simple Dirac brackets. However, since the complex variables are introduced, the reality of the theory is not manifest. So we must impose reality conditions.

2.3 Reality conditions

In order to ensure that a complex theory is equivalent to a real one, we must impose the reality conditions like in the 3+1 dimensional gravity. If we ignore the reality conditions, that is, if we consider A_μ^i and corresponding momentum $\tilde{E}_i^\mu$ as complex, we can no longer recover the real theory. So we must impose the reality conditions which restrict the phase space.

In our case, the canonical variables $\tilde{E}_i^\mu$ and A_μ^i are not independent of their complex conjugates but must satisfy the reality conditions

$$\tilde{E}_i^{\mu\dagger} = \tilde{E}_i^\mu, \quad A_\mu^{i\dagger} = A_\mu^i + E^{-1}\epsilon_{\mu\nu}\epsilon^{ijk}\tilde{E}_j^\nu\pi(n\cdot\gamma)\gamma_k\psi. \tag{14}$$

The latter condition of A_μ^i ensure that the original variable A_μ^i , which is related to A_μ^i by (8), is real. Furthermore, when spinor fields are included, additional conditions for the reality of observable currents are needed [2]. In our case the reality conditions for spinor fields are given by

$$(\pi\tilde{E}_i^\mu\gamma^i\psi)^\dagger = -\pi\tilde{E}_i^\mu\gamma^i\psi, \quad (\pi\tilde{E}_i^\mu\gamma^i\pi)^\dagger = E^2(\psi\tilde{E}_i^\mu\gamma^i\psi). \tag{15}$$

Using these conditions, the constraints H' and H'_μ are complex. Therefore the Hamiltonian takes a form of linear combination of the complex constraints. In order to satisfy the reality conditions at any time, the Hamiltonian must be real. So the Lagrange multipliers in the Hamiltonian are not totally arbitrary but must satisfy

$$\begin{aligned}
N^\dagger &= N, \quad N^{\mu\dagger} = N^\mu \\
\Lambda^{\dagger i} &= \Lambda^i + N\epsilon^{ijk}n_j\pi(n\cdot\gamma)\gamma_k\psi - N^\mu E^{-1}\epsilon_{\mu\nu}\epsilon^{ijk}\tilde{E}_j^\nu\pi(n\cdot\gamma)\gamma_k\psi.
\end{aligned} \tag{16}$$

If the reality conditions (14) and (15) are satisfied at initial time and the multipliers obey the relations (16), the equivalence between the real theory and the complex one is guaranteed. Note that the reality conditions (14) and the multiplier conditions (16) are similar to the ones in the case of pure gravity in 3+1 dimensions [5]. In both theories, complex connection is introduced to simplify the canonical structure. In our case the aim

is to simplify the Dirac brackets of canonical variables. In the case of pure gravity in 3+1 dimensions, on the other hand, the complex connection is used to get the constraints in polynomial form.

Now we consider reality conditions which are different from (19) in appearance. First we take Λ^i as an arbitrary complex function in the Hamiltonian. Therefore the Hamiltonian is in general complex. As new reality conditions, we impose that the triad $\tilde{E}_i^\mu$ is real,

$$\text{Im}\tilde{E}_i^\mu = 0 \quad (17)$$

and that the time derivative of $\tilde{E}_i^\mu$ is also real,

$$\text{Im}\dot{\tilde{E}}_i^\mu = 0. \quad (18)$$

As regards spinor fields, the spinor reality conditions (15) are required. Using the Gauss-law constraint, the spinor reality conditions (15) and the triad reality condition (17), it follows that (18) reduces to

$$\text{Im}A_\mu^i = -\frac{1}{2}E^{-1}\epsilon_{\mu\nu}\epsilon^{ijk}\tilde{E}_j^\nu\pi(n\cdot\gamma)\gamma_k\psi, \quad (19)$$

which is equivalent to the second condition in (14). On the other hand, it is possible to solve (18) with regard to Λ^i ,

$$\text{Im}\Lambda^i = \frac{1}{2}N\epsilon^{ijk}n_j\pi(n\cdot\gamma)\gamma_k\psi + N^\mu\text{Im}A_\mu^i. \quad (20)$$

This condition restricts a part of gauge freedom, and this situation is similar to the 3+1 dimensional gravity [6]. Using (19), we see that the condition (20) is the same as the last condition in (16). Thus the reality conditions (17) and (18) are equivalent to the those of (14) and (15).

3 Canonical formalism of the gravity including the Majorana field

In this section we consider the canonical gravity coupled with the Majorana field. The action is given by

$$I = \int d^3x \left[\frac{1}{2}\epsilon^{\mu\nu\rho}e_\mu^i F_{\nu\rho i} - \frac{1}{2}e(\bar{\psi}\gamma^\mu D_\mu\psi - D_\mu\bar{\psi}\gamma^\mu\psi) - m\bar{\psi}\psi \right], \quad (21)$$

where ψ is the Majorana field, i.e. $\bar{\psi} = -\psi^T C$ where C is the charge conjugation matrix.

Now we consider the canonical formalism in the same way as the case of the Dirac field. After calculating the Dirac brackets to eliminate second class constraints, the non-vanishing brackets of the canonical variables are

$$\{\tilde{E}_i^\mu(x), A_\nu^j(y)\} = q_\nu^\mu\eta_i^j\delta^2(x, y) \quad (22)$$

$$\{\psi_\alpha(x), \bar{\psi}_\beta(y)\} = \frac{1}{2}E^{-1}(n\cdot\gamma)_{\alpha\beta}\delta^2(x, y) \quad (23)$$

$$\{\psi_\alpha(x), A_\mu^i(y)\} = \frac{1}{2}E^{-1}\epsilon_{\mu\nu}\epsilon^{ijk}\tilde{E}_j^\nu((n\cdot\gamma)\gamma_k\psi)_\alpha\delta^2(x, y) \quad (24)$$

$$\{A_\mu^i(x), A_\nu^j(y)\} = \frac{1}{2}\epsilon_{\mu\nu}n^i n^j \bar{\psi}\psi\delta^2(x, y). \quad (25)$$

Note that A_μ^i does not commute with ψ and itself. The difference from the Dirac field is that a numerical factor which appears in (23) is a half of the case of the Dirac field. The first class constraints are

$$\begin{aligned} H &= \frac{1}{2}\epsilon^{ijk}\tilde{E}_i^\mu\tilde{E}_j^\nu F_{\mu\nu k} + \frac{1}{2}E\epsilon^{ijk}\tilde{E}_i^\mu n_j\Psi_{\mu k} - mE^2\bar{\psi}\psi \approx 0 \\ H_\mu &= -\tilde{E}^{\nu i}F_{\mu\nu i} - \frac{1}{2}En_i\Psi_\mu^i \approx 0 \\ G_i &= D_\mu\tilde{E}_i^\mu + \frac{1}{2}En_i\bar{\psi}\psi \approx 0, \end{aligned} \quad (26)$$

which are the same as the case of the Dirac field. The algebra of the constraints are equal to the one in (7).

Now we consider the transformation which changes the Dirac bracket of the canonical variables to the canonical ones. Unfortunately, we find that the variable which corresponds to A_μ^i in (8) does not exist. In analogy with (8), we proceed from the ansatz

$$A_\mu^i := A_\mu^i + B_{\mu\alpha\beta}^i\psi_\alpha\psi_\beta, \quad (27)$$

where $B_{\mu\alpha\beta}^i$ is an antisymmetric matrix with respect to spinor indices. In order to change the Dirac brackets to the canonical ones, we demand

$$\{A_\mu^i(x), \psi_\alpha(y)\} = 0. \quad (28)$$

However, from (28), we get $B_{\mu\alpha\beta}^i = B_{\mu\beta\alpha}^i$. Thus the matrix $B_{\mu\alpha\beta}^i$ which satisfies the condition (28) does not exist. So we can not change the Dirac brackets to the canonical ones.

4 Quantization

We look for a quantum representation for the canonical gravity coupled with spinor field. We must define the operators such that their commutator is i times the classical Dirac bracket.

For the case of the Dirac field, we can use two sets of the canonical variables. First, we use the canonical variables introduced in section 2.1. In this case, The Dirac brackets of the canonical variables are non-canonical. In particular, because of the fact that A_μ^i does not commute with ψ , $\bar{\psi}$ and itself, we can no longer represent A_μ^i by a multiplication operator. So we choose the $(\tilde{E}, \bar{\psi})$ -representation. The wavefunction becomes a functional $\Psi[\tilde{E}, \bar{\psi}]$. In order to satisfy the commutation relations, A_μ^i and ψ are given by the momentum operators

$$\hat{A}_\mu^i = -i\frac{\delta}{\delta\tilde{E}_i^\mu} - \frac{i}{2}E^{-1}\epsilon_{\mu\nu}\epsilon^{ijk}\tilde{E}_j^\nu(\bar{\psi}\gamma_k(n\cdot\gamma))_\alpha\frac{\delta}{\delta\bar{\psi}_\alpha}, \quad \hat{\psi}_\alpha = iE^{-1}(n\cdot\gamma)_{\alpha\beta}\frac{\delta}{\delta\bar{\psi}_\beta}. \quad (29)$$

Since these operators are Hermitian, no reality condition is needed. However, because of the complicated structure of the operators (29), it is difficult to solve the Wheeler-DeWitt equation. The complication of the operators is caused by the non-canonical structure of

the Dirac brackets of the canonical variables in the classical theory. If the new variables introduced in section 2.2 are used, we can obtain a simple quantum representation. In this case we can choose the (A, ψ) -representation, and the momentum operators are given by

$$\hat{E}_i^\mu = i \frac{\delta}{\delta A_\mu^i}, \quad \hat{\pi}_\alpha = i \frac{\delta}{\delta \psi_\alpha}. \quad (30)$$

The operators (30) are simpler than the ones of the $(\tilde{E}, \tilde{\psi})$ -representation. However, since A_μ^i is complex, we have to impose suitable reality condition in the quantum level. But we do not discuss this problem in this paper.

Now we discuss the action of the Hamiltonian constraint on the wavefunction. We take the ordering in which momentum operators are placed to the right. In the non-Hermitian case [1], as a solution including spinor fields for the Hamiltonian constraint, a trivial one $\psi^A \psi_A$ has been found. This solution corresponds to an eigenstate of the fermion number operator $\int d^2x \psi_A \pi^A$. In our Hermitian case, however, since the Hamiltonian constraint contains $(\psi\pi)^2$, $\psi^A \psi_A$ is no longer a solution.

Finally we consider the quantum gravity coupled with the Majorana field. In the last section we found that there is no variable which change the Dirac brackets to the canonical ones. So we can not employ the (A, ψ) -representation. Even if we take the representation in which $\tilde{E}_i^\mu$ is diagonal, it is also difficult to define the suitable momentum operators like the ones in (29). It seems that the quantum representation for the Majorana field does not exist.

5 Conclusion and Discussion

In this paper we studied the canonical formalism of the gravity coupled with spinor field. In the both case of the Dirac and the Majorana field, the Dirac brackets of the canonical variables are non-canonical. It was found that quadratic terms of constraint appear in the constraint algebra.

In the case of the Dirac field, we found the new variables which change the Dirac brackets of the canonical variables to the canonical ones. Because the new variables are complex, in order to ensure that the complex theory is equivalent to the real one, the reality conditions for the complex variables are needed. We presented the suitable reality conditions. In the case of the Majorana field, it was shown that there is no variable which changes the non-canonical Dirac brackets to the canonical ones.

We also discussed the quantum theory. In the case of the Dirac field, we found the $(\tilde{E}, \tilde{\psi})$ -representation which corresponds to the theory using the real variables. We also discussed the (A, π) -representation which corresponds to the theory using the complex variables. Since the Hamiltonian constraint contains $(\pi\psi)^2$, the solution of the non-Hermitian case [1] is no longer a solution in the Hermitian case.

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Structure of the Field Equations in $N = 1$ Chiral Supergravity

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Abstract

We study the structure of linearized field equations in $N = 1$ chiral supergravity (SUGRA) with a complex tetrad, as a preliminary to introducing additional auxiliary fields in order that the supersymmetry (SUSY) algebra close off shell. We follow the first-order formulation we have recently constructed using the method of the usual $N = 1$ SUGRA. In particular, we see how the real and imaginary parts of the complex tetrad are coupled to matter fields in the weak field approximation. Starting from the linearized (free) theory of $N = 1$ chiral SUGRA, we then construct a Lagrangian which is invariant under local SUSY transformations to zeroth order of the gravitational constant, and compare the results with the linearized field equations.

1 Introduction

Right- and left-handed supersymmetry (SUSY) transformations in $N = 1$ chiral supergravity (SUGRA) with a complex tetrad were introduced by Jacobson [1, 2], and their first-order formulation was then constructed using the two-form gravity [3, 4]. In a previous paper [5], following the method used in the usual $N = 1$ SUGRA [6, 7], we presented the explicit form of the first-order SUSY transformations in $N = 1$ chiral SUGRA for complex field variables; a complex tetrad e^i_μ , a self-dual connection $A^{(+)}_{ij\mu} = A^{(+)}_{[ij]\mu}$ which satisfies $(1/2)\epsilon_{ij}{}^{kl}A^{(+)}_{kl\mu} = iA^{(+)}_{ij\mu}$, and two independent (Majorana) Rarita-Schwinger fields $\psi_{R\mu}$ and $\bar{\psi}_{R\mu}$. * The SUSY transformation parameters are not constrained at all in contrast with the method of the two-form gravity. We showed that the SUSY algebra for $N = 1$ chiral SUGRA closes only on shell.

The full non-linear theory with auxiliary fields of the usual $N = 1$ SUGRA can be constructed from its linearized theory, making the rigid SUSY transformations local and

* We assume ψ_μ and $\bar{\psi}_\mu$ to be two independent (Majorana) Rarita-Schwinger fields, and define the right-handed spinor fields $\psi_{R\mu} := (1/2)(1 + \gamma_5)\psi_\mu$ and $\bar{\psi}_{R\mu} := (1/2)(1 + \gamma_5)\bar{\psi}_\mu$. The $\psi_{R\mu}$ and $\bar{\psi}_{R\mu}$ relate to the left-handed spinor fields $\psi_{L\mu}$ and $\bar{\psi}_{L\mu}$ respectively, because ψ_μ and $\bar{\psi}_\mu$ are Majorana spinors. The antisymmetrization of a tensor with respect to i and j is denoted by $A_{[i\dots j]} := (1/2)(A_{i\dots j} - A_{j\dots i})$. We shall follow the notation and convention of [5].

adding appropriate terms to the free Lagrangian order-by-order in the gravitational constant κ [8].[†] This suggests that if we can introduce additional auxiliary fields at linearized level, the full nonlinear theory with auxiliary fields will be constructed also for $N = 1$ chiral SUGRA. Motivated by this expectation, we consider the structure of the linearized field equations in $N = 1$ chiral SUGRA. In particular, we see how the real and imaginary parts of the complex tetrad are coupled to matter fields in the weak field approximation. We shall then modify the linearized (free) Lagrangian so that it be invariant under local SUSY transformations up to order κ^0 , and show that the modified Lagrangian correctly reproduces the field equations of $N = 1$ chiral SUGRA in the weak field approximation.

2 Real Lagrangian

In local field theory, the spinor field ψ usually appears in the kinetic Lagrangian forming a bilinear product with its own Dirac conjugate $\bar{\psi}$. However, in $N = 1$ chiral SUGRA, we must use the kinetic Lagrangian formed of the bilinear product of the two independent spinor fields $\psi_{R\mu}$ and $\bar{\psi}_{R\mu}$ in order to make the SUSY transformations compatible with the complex tetrad. Therefore, to recover the Hermiticity of the kinetic Lagrangian, we add the complex conjugate, $\overline{\mathcal{L}^{(+)}}$, to the chiral Lagrangian density $\mathcal{L}^{(+)}$: Namely, we define the real Lagrangian density,[‡]

$$\mathcal{L} := \mathcal{L}^{(+)} + \text{c.c.}, \quad (2.1)$$

where $\mathcal{L}^{(+)}$ is the sum of the chiral gravitational Lagrangian density and the chiral Lagrangian density of matter fields, and “c.c.” means “the complex conjugate of the preceding term”. The chiral gravitational Lagrangian density constructed from the complex tetrad and the self-dual connection is

$$\mathcal{L}_G^{(+)} = -\frac{i}{2} e^{\mu\nu\rho\sigma} e^i{}_\mu e^j{}_\nu R_{ij\rho\sigma}^{(+)}, \quad (2.2)$$

where e denotes $\det(e^i{}_\mu)$ and the curvature of self-dual connection $R^{(+)\,ij}{}_{\mu\nu}$ is

$$R^{(+)\,ij}{}_{\mu\nu} := 2(\partial_{[\mu} A^{(+)\,ij}{}_{\nu]} + A^{(+)\,i}{}_{k[\mu} A^{(+)\,kj}{}_{\nu]}). \quad (2.3)$$

In order to discuss the structure of linearized field equations as generally as possible, we suppose that the chiral Lagrangian density of matter fields take the form

$$\mathcal{L}_M^{(+)} := \mathcal{L}_M^{(+)}[e, A^{(+)}, \bar{\Psi}_R, D_\mu^{(+)}\Psi_R], \quad (2.4)$$

[†] The κ^2 is the Einstein constant: $\kappa^2 = 8\pi G/c^4$. Unless stated otherwise, we use units $c = 1 = \kappa^2$.

[‡] For pure gravity case, Peldán [9] used the real Lagrangian to derive the real Ashtekar-like theory based on the Lie-algebra $SO(3, 1)$, denoting the real Lagrangian density by $\mathcal{L}^{\text{real}}$.

which is analytic in the complex field variables, e^i_μ , $A_{ij\mu}^{(+)}$, Ψ_R and $\bar{\Psi}_R$. Here Ψ_R and $\bar{\Psi}_R$ are independent of each other, and denote collectively the matter fields; the $D_\mu^{(+)}$ means the covariant derivative with respect to $A_{ij\mu}^{(+)}$. The $\mathcal{L}_M^{(+)}$ of (2.4) is invariant under general coordinate and local Lorentz transformations, but its SUSY invariance is not necessarily satisfied. The matter fields under consideration include massless Majorana spin-1/2 and spin-3/2 fields. For example, the chiral Lagrangian density of (Majorana) Rarita-Schwinger fields is

$$\mathcal{L}_{RS}^{(+)} = -e \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_{R\mu} \gamma_\rho D_\sigma^{(+)} \psi_{R\nu}. \quad (2.5)$$

Let us express the real gravitational Lagrangian density, $\mathcal{L}_G := \mathcal{L}_G^{(+)} + \text{c.c.}$ by using only real variables. We define the real $SO(3, 1)$ connection by $\omega_{ij\mu} := A_{ij\mu}^{(+)} + \text{c.c.}$. Then the $R^{(+)\,ij}{}_{\mu\nu}$ of (2.3) becomes $R^{(+)\,ij}{}_{\mu\nu} = (1/2)(R^{ij}{}_{\mu\nu}[\omega] - (i/2)\epsilon^{ij}{}_{kl}R^{kl}{}_{\mu\nu}[\omega])$, where $R^{ij}{}_{\mu\nu}[\omega]$ is the curvature of the real connection $\omega_{ij\mu}$. Further we decompose the complex tetrad into the real and imaginary parts: $e^i_\mu = V^i_\mu + iW^i_\mu$. The real gravitational Lagrangian density $\mathcal{L}_G$ can then be written as

$$\mathcal{L}_G = -\frac{e}{4} \epsilon^{\mu\nu\rho\sigma} (V^i_\mu V^j_\nu - W^i_\mu W^j_\nu) \epsilon_{ij}{}^{kl} R_{kl\rho\sigma}[\omega] + e \epsilon^{\mu\nu\rho\sigma} V^i_\mu W^j_\nu R_{ij\rho\sigma}[\omega]. \quad (2.6)$$

The real Lagrangian density of matter fields, $\mathcal{L}_M := \mathcal{L}_M^{(+)} + \text{c.c.}$, can be written by using V^i_μ , W^i_μ and $\omega_{ij\mu}$ as $\mathcal{L}_M = \mathcal{L}_M[V, W, \omega, \bar{\Psi}, D_\mu[\omega]\Psi]$, where $D_\mu[\omega]$ denotes the covariant derivative with respect to $\omega_{ij\mu}$. Note that $\bar{\Psi}$ and Ψ appear in $\mathcal{L}_M$ instead of $\bar{\Psi}_R$ and Ψ_R . In fact, for massless (Majorana) Rarita-Schwinger fields, the real Lagrangian density can be expressed as

$$\begin{aligned} \mathcal{L}_{RS} = & -e \epsilon^{\mu\nu\rho\sigma} \{ V^i_\rho (\bar{\psi}_{R\mu} \gamma_i D_\sigma[\omega] \psi_{R\nu} - \bar{\psi}_{L\mu} \gamma_i D_\sigma[\omega] \psi_{L\nu}) \\ & + i W^i_\rho (\bar{\psi}_{R\mu} \gamma_i D_\sigma[\omega] \psi_{R\nu} + \bar{\psi}_{L\mu} \gamma_i D_\sigma[\omega] \psi_{L\nu}) \}. \end{aligned} \quad (2.7)$$

From (2.6) and $\mathcal{L}_M$, we can derive the field equations for V^i_μ , W^i_μ and $\omega_{ij\mu}$. Varying $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_M$ with respect to V^i_μ and W^i_μ yields

$$e \epsilon^{\mu\nu\rho\sigma} (W^j_\nu R_{ij\rho\sigma}[\omega] - \frac{1}{2} V^j_\nu \epsilon_{ij}{}^{kl} R_{kl\rho\sigma}[\omega]) + e T^{(1)\,i\,\mu} = 0, \quad (2.8)$$

$$e \epsilon^{\mu\nu\rho\sigma} (V^j_\nu R_{ij\rho\sigma}[\omega] + \frac{1}{2} W^j_\nu \epsilon_{ij}{}^{kl} R_{kl\rho\sigma}[\omega]) + e T^{(2)\,i\,\mu} = 0, \quad (2.9)$$

respectively. We note that there appear two energy-momentum tensors of matter fields, $T^{(1)\,i\,\mu}$ and $T^{(2)\,i\,\mu}$, defined by $T^{(1)\,i\,\mu} := e^{-1} (\delta \mathcal{L}_M / \delta V^i_\mu)$ and $T^{(2)\,i\,\mu} := e^{-1} (\delta \mathcal{L}_M / \delta W^i_\mu)$. For $\omega_{ij\mu}$, we have the field equation,

$$e \epsilon^{\mu\nu\rho\sigma} D_\rho[\omega] \text{Im} H^{(+)\,ij}{}_{\mu\nu} + \frac{e}{2} S^{ij\sigma} = 0, \quad (2.10)$$

where the spin tensor of matter fields is defined by $S^{ij\mu} := -2e^{-1} (\delta\mathcal{L}_M/\delta\omega_{ij\mu})$, and $\text{Im}H^{(+ij)}_{\mu\nu}$ is the imaginary part of $H^{(+ij)}_{\mu\nu} := e^i_{[\mu}e^j_{\nu]} - (i/2)\epsilon^{ij}_{kl}e^k_{[\mu}e^l_{\nu]}$.

If we impose the reality condition, $W^i_\mu = 0$ and $\bar{\Psi} = \bar{\Psi}$, then, in the case of spin-1/2 and spin-3/2 fields, substituting the solution of (2.10) into (2.8) and (2.9) yields the ordinary Einstein equation and the Bianchi identity respectively.

3 Weak field approximation

To see how the real and imaginary parts of the complex tetrad are coupled to matter fields, we apply the weak field approximation to the field equations (2.8), (2.9) and (2.10), assuming that V^i_μ and W^i_μ satisfy

$$V^i_\mu = \delta^i_\mu + a^i_\mu, \quad |a^i_\mu| \ll 1, \quad (3.11)$$

$$W^i_\mu = b^i_\mu, \quad |b^i_\mu| \ll 1. \quad (3.12)$$

It is convenient to decompose the self-dual connection $A^{(+)}_{ij\mu}$ as

$$A^{(+)}_{ij\mu} = A^{(+)}_{ij\mu}[e] + K^{(+)}_{ij\mu} \quad (3.13)$$

and to take $K^{(+)}_{ij\mu}$ as an independent variable instead of $A^{(+)}_{ij\mu}$. Here $A^{(+)}_{ij\mu}[e]$ is the self-dual part of the Ricci rotation coefficients $A_{ij\mu}[e]$. Since we need not distinguish Latin indices from Greek ones in the weak field approximation, we use Greek indices in the rest of this paper, which are raised and lowered with the Minkowski metric tensor $\eta_{\mu\nu}$. Now the real $SO(3,1)$ connection $\omega_{\mu\nu\lambda}$ can be expressed in terms of $a_{\mu\nu}$, $b_{\mu\nu}$ and $K^{(+)}_{\mu\nu\lambda}$ as follows:

$$\omega_{\mu\nu\lambda} = \omega_{\mu\nu\lambda}[a] + \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}\omega_{\rho\sigma\lambda}[b] + K_{\mu\nu\lambda}, \quad (3.14)$$

where

$$\omega_{\mu\nu\lambda}[a] := -\partial_\lambda a_{[\mu\nu]} - (\partial_\mu a_{(\nu\lambda)} - \partial_\nu a_{(\mu\lambda)}), \quad (3.15)$$

$$\omega_{\mu\nu\lambda}[b] := -\partial_\lambda b_{[\mu\nu]} - (\partial_\mu b_{(\nu\lambda)} - \partial_\nu b_{(\mu\lambda)}), \quad (3.16)$$

and $K_{\mu\nu\lambda}$ is defined by

$$K_{\mu\nu\lambda} := K^{(+)}_{\mu\nu\lambda} + \text{c.c.} \quad (3.17)$$

Therefore, the curvature $R_{\mu\nu\rho\sigma}[\omega]$ can be written as

$$R_{\mu\nu\rho\sigma}[\omega] = R_{\mu\nu\rho\sigma}[a] + \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta\rho\sigma}[b] + 2\partial_{[\rho}K_{\mu\nu|\sigma]}, \quad (3.18)$$

where

$$R_{\mu\nu\rho\sigma}[a] := 2\partial_{[\rho}\omega_{|\mu\nu|\sigma]}[a], \quad (3.19)$$

$$R_{\mu\nu\rho\sigma}[b] := 2\partial_{[\rho}\omega_{|\mu\nu|\sigma]}[b], \quad (3.20)$$

which are linear in $a_{(\mu\nu)}$ and $b_{(\mu\nu)}$ respectively.

Substituting (3.18) into (2.8) and (2.9), we can get the field equations in the weak field approximation. In (2.8), the term of $e \epsilon^{\mu\nu\rho\sigma} W^j{}_\nu R_{ij\rho\sigma}[\omega]$ can be neglected, and the term proportional to $R_{\mu\nu\rho\sigma}[b]$ vanishes due to the Bianchi identity. Therefore (2.8) becomes

$$G_{\mu\nu}[a] + \partial^\rho K_{\rho\nu\mu} + \partial_\mu v_\nu - \eta_{\mu\nu} \partial^\rho v_\rho = \frac{1}{2} T_{\mu\nu}^{(1)} \quad (3.21)$$

with $v_\mu := K_{\mu\nu}{}^\nu$. Here $G_{\mu\nu}[a]$ is the linearized Einstein tensor for $a_{\mu\nu}$,

$$G_{\mu\nu}[a] = -\{\Box \bar{a}_{(\mu\nu)} - \partial^\rho (\partial_\mu \bar{a}_{(\nu\rho)} + \partial_\nu \bar{a}_{(\mu\rho)}) + \eta_{\mu\nu} \partial^\sigma \partial^\rho \bar{a}_{(\rho\sigma)}\} \quad (3.22)$$

with

$$\bar{a}_{(\mu\nu)} := a_{(\mu\nu)} - \frac{1}{2} \eta_{\mu\nu} a, \quad a := \eta^{\mu\nu} a_{(\mu\nu)}, \quad (3.23)$$

and the d'Alembertian $\Box$ being defined by $\Box := \partial^\mu \partial_\mu$. Similarly, from (2.9) we have

$$G_{\mu\nu}[b] + \epsilon_{\lambda\rho\sigma\nu} \partial^\lambda K_\mu{}^{\rho\sigma} = -\frac{1}{2} T_{\mu\nu}^{(2)}, \quad (3.24)$$

where $G_{\mu\nu}[b]$ is the linearized Einstein tensor for $b_{\mu\nu}$. Further, the field equation (2.10) becomes

$$-K_{\rho[\mu\nu]} + \eta_{\rho[\mu} v_{\nu]} = \frac{1}{4} S_{\mu\nu\rho}. \quad (3.25)$$

We take the energy-momentum tensors, $T_{\mu\nu}^{(1)}$ and $T_{\mu\nu}^{(2)}$, and the spin tensor $S_{\mu\nu\rho}$ to lowest order in $a_{\mu\nu}$, $b_{\mu\nu}$ and $K_{\mu\nu\rho}$: Namely, they are independent of $a_{\mu\nu}$, $b_{\mu\nu}$ and $K_{\mu\nu\rho}$, and satisfy the conservation law,

$$\partial^\nu T_{\mu\nu}^{(1)} = 0, \quad \partial^\nu T_{\mu\nu}^{(2)} = 0, \quad (3.26)$$

and the Tetrad formula in special relativity,

$$T_{[\mu\nu]}^{(1)} - \frac{1}{2} \epsilon_{\mu\nu}{}^{\rho\sigma} T_{\rho\sigma}^{(2)} = \partial^\rho S_{\mu\nu\rho}. \quad (3.27)$$

Note that the antisymmetric parts of $T_{\mu\nu}^{(1)}$ and $T_{\mu\nu}^{(2)}$ are related with each other by

$$2T_{[\mu\nu]}^{(1)} = -\epsilon_{\mu\nu}{}^{\rho\sigma} T_{\rho\sigma}^{(2)}. \quad (3.28)$$

The linearized field equations (3.21), (3.24) and (3.25) are invariant under the gauge transformations,

$$a_{(\mu\nu)} \rightarrow a'_{(\mu\nu)} = a_{(\mu\nu)} + \partial_\mu \Lambda_\nu^1 + \partial_\nu \Lambda_\mu^1, \quad (3.29)$$

$$a_{[\mu\nu]} \rightarrow a'_{[\mu\nu]} = a_{[\mu\nu]} + \epsilon_{\mu\nu}^1, \quad (3.30)$$

and

$$b_{(\mu\nu)} \rightarrow b'_{(\mu\nu)} = b_{(\mu\nu)} + \partial_\mu \Lambda_\nu^2 + \partial_\nu \Lambda_\mu^2, \quad (3.31)$$

$$b_{[\mu\nu]} \rightarrow b'_{[\mu\nu]} = b_{[\mu\nu]} + \epsilon_{\mu\nu}^2 \quad (3.32)$$

where Λ_μ^a and $\epsilon_{\mu\nu}^a = \epsilon_{[\mu\nu]}^a$ ($a = 1, 2$) are arbitrary four and six functions, respectively. These transformations are the linearized version of complex general coordinate and complex local Lorentz transformations. Equations (3.30) and (3.32) mean that $a_{[\mu\nu]}$ and $b_{[\mu\nu]}$ can be eliminated. By means of the gauge freedom we can put the harmonic condition,

$$\partial^\nu \bar{a}_{(\mu\nu)} = 0, \quad \partial^\nu \bar{b}_{(\mu\nu)} = 0. \quad (3.33)$$

Then the remaining degrees of freedom are 6 for each $a_{\mu\nu}$ and $b_{\mu\nu}$, and the linearized Einstein tensors for $a_{\mu\nu}$ and $b_{\mu\nu}$ are written as

$$G_{\mu\nu}[a] = -\square \bar{a}_{(\mu\nu)}, \quad G_{\mu\nu}[b] = -\square \bar{b}_{(\mu\nu)}. \quad (3.34)$$

Equation (3.21) is decomposed into the symmetric and antisymmetric parts as

$$G_{\mu\nu}[a] + \partial^\rho K_{\rho(\mu\nu)} + \partial_{(\mu} v_{\nu)} - \eta_{\mu\nu} \partial^\rho v_\rho = \frac{1}{2} T_{(\mu\nu)}^{(1)}, \quad (3.35)$$

$$-\partial^\rho K_{\rho[\mu\nu]} + \partial_{[\mu} v_{\nu]} = \frac{1}{2} T_{[\mu\nu]}^{(1)}, \quad (3.36)$$

and similarly (3.24) into

$$G_{\mu\nu}[b] + \epsilon_{\lambda\rho\sigma(\mu} \partial^\lambda K_{\nu)}{}^{\rho\sigma} = -\frac{1}{2} T_{(\mu\nu)}^{(2)}, \quad (3.37)$$

$$-\epsilon_{\lambda\rho\sigma[\mu} \partial^\lambda K_{\nu]}{}^{\rho\sigma} = -\frac{1}{2} T_{[\mu\nu]}^{(2)}. \quad (3.38)$$

Both sides of (3.21) and (3.24) are divergenceless with respect to ν , and the divergence of (3.25) with respect to ρ yields (3.36) and (3.38) due to the Tetrad formula (3.27). Therefore, there are $(16 + 16 + 24) - (4 + 4 + 6 + 6) = 36$ independent equations for $6 + 6 + 24 = 36$ independent variables, $a_{(\mu\nu)}$, $b_{(\mu\nu)}$ and $K_{\mu\nu\rho}$.

We can rewrite (3.35) and (3.37) with the help of (3.25). Taking the divergence of (3.25) with respect to μ yields

$$\partial^\rho K_{\rho(\mu\nu)} + \partial_{(\mu} v_{\nu)} - \eta_{\mu\nu} \partial^\rho v_\rho = \frac{1}{2} \partial^\rho S_{\rho(\mu\nu)}, \quad (3.39)$$

or equivalently

$$\epsilon_{\lambda\rho\sigma(\mu}\partial^\lambda K_{\nu)}^{\rho\sigma} = -\frac{1}{2}\partial^\rho S_{\rho(\mu\nu)}^* \quad (3.40)$$

with the $*$ operation being the duality operation

$$S_{\mu\nu\rho}^* := \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}S_{\alpha\beta\rho}. \quad (3.41)$$

Using (3.39) and (3.40) in (3.35) and (3.37) respectively, we get

$$G_{\mu\nu}[a] = \frac{1}{2}T_{\mu\nu}^{(1)(\text{sym})}, \quad (3.42)$$

$$G_{\mu\nu}[b] = -\frac{1}{2}T_{\mu\nu}^{(2)(\text{sym})}, \quad (3.43)$$

where $T_{\mu\nu}^{(1)(\text{sym})}$ and $T_{\mu\nu}^{(2)(\text{sym})}$ are the symmetrized energy-momentum tensors:

$$T_{\mu\nu}^{(1)(\text{sym})} = T_{\mu\nu}^{(1)} - \frac{1}{2}\partial^\rho(S_{\mu\nu\rho} + S_{\rho\mu\nu} + S_{\rho\nu\mu}), \quad (3.44)$$

$$T_{\mu\nu}^{(2)(\text{sym})} = T_{\mu\nu}^{(2)} - \frac{1}{2}\partial^\rho(S_{\mu\nu\rho}^* + S_{\rho\mu\nu}^* + S_{\rho\nu\mu}^*). \quad (3.45)$$

Now we restrict ourselves to the case of (Majorana) Rarita-Schwinger fields. From the real Lagrangian density (2.7), $T_{\mu\nu}^{(1)}$ and $T_{\mu\nu}^{(2)}$ can be obtained as

$$\begin{aligned} T_{\mu\nu}^{(1)} &= -\epsilon_{\lambda\rho\sigma\nu}(\bar{\psi}_R\gamma_\mu\partial^\rho\psi_R^\sigma - \bar{\psi}_L\gamma_\mu\partial^\rho\psi_L^\sigma) \\ &= \epsilon_{\lambda\rho\sigma\nu}\bar{\psi}\gamma_5\gamma_\mu\partial^\rho\psi^\sigma, \end{aligned} \quad (3.46)$$

$$\begin{aligned} T_{\mu\nu}^{(2)} &= -i\epsilon_{\lambda\rho\sigma\nu}(\bar{\psi}_R\gamma_\mu\partial^\rho\psi_R^\sigma + \bar{\psi}_L\gamma_\mu\partial^\rho\psi_L^\sigma) \\ &= -i\epsilon_{\lambda\rho\sigma\nu}\bar{\psi}\gamma_\mu\partial^\rho\psi^\sigma. \end{aligned} \quad (3.47)$$

Since both of ψ_μ and $\tilde{\psi}_\mu$ are Majorana spinors, the $T_{\mu\nu}^{(1)}$ of (3.46) can be diagonalized as

$$T_{\mu\nu}^{(1)} = \epsilon_{\lambda\rho\sigma\nu}(\bar{\psi}^{1\lambda}\gamma_5\gamma_\mu\partial^\rho\psi^{1\sigma} - \bar{\psi}^{2\lambda}\gamma_5\gamma_\mu\partial^\rho\psi^{2\sigma}), \quad (3.48)$$

where $\psi_\mu^1 := (1/2)(\psi_\mu + \tilde{\psi}_\mu)$ and $\psi_\mu^2 := (1/2)(\psi_\mu - \tilde{\psi}_\mu)$. The minus sign in (3.48) means the appearance of negative energy: Namely, the positivity of $T_{\mu\nu}^{(1)}$ is not guaranteed. On the other hand, the $T_{\mu\nu}^{(2)}$ of Eq.(3.47) is not diagonalized by using ψ_μ^1 and ψ_μ^2 . However we have

$$T_{\mu\nu}^{(2)}[\psi, \tilde{\psi}] = T_{\mu\nu}^{(1)}[\psi', \tilde{\psi}], \quad (3.49)$$

$$T_{\mu\nu}^{(1)}[\psi, \tilde{\psi}] = -T_{\mu\nu}^{(2)}[\psi', \tilde{\psi}], \quad (3.50)$$

where we define $\psi'_\mu := \exp\{(i\pi\gamma_5)/2\}\psi_\mu$. Therefore $T_{\mu\nu}^{(2)}[\psi, \tilde{\psi}]$ is not positive definite, either.

If we impose the reality condition, $b_{(\mu\nu)} = 0$ and $\bar{\tilde{\psi}}_\mu = \bar{\psi}_\mu$, together with $\varphi = 0$, then the $S_{\mu\nu\rho}^*$ of (3.41) can be calculated as

$$S_{\mu\nu\rho}^* = -i\epsilon^{\alpha\beta}{}_{\rho[\mu}\bar{\psi}_{|\alpha|}\gamma_{\nu]}\psi_\beta, \quad (3.51)$$

by virtue of which the symmetrized energy-momentum tensor $T_{\mu\nu}^{(2)(\text{sym})}$ of (3.45) vanishes and the positivity of $T_{\mu\nu}^{(1)(\text{sym})}$ is recovered.

4 Local SUSY invariant Lagrangian to order κ^0

In the usual $N = 1$ SUGRA, the full nonlinear theory can be constructed from the linearized (free) theory, making the rigid SUSY transformations local and adding appropriate terms to the free Lagrangian order-by-order in the gravitational constant κ [8, 10]. For example, one can construct a local SUSY invariant Lagrangian to order κ^0 by adding an interaction term proportional to the energy-momentum tensor. Similarly, we can obtain a local SUSY invariant Lagrangian to order κ^0 also in $N = 1$ chiral SUGRA as we shall explain below. Hereafter we shall write the κ explicitly. In the linearized theory, the free Lagrangian of $N = 1$ chiral SUGRA is twice the usual $N = 1$ SUGRA: Namely, the free field limit of the real Lagrangian density, $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_{\text{RS}}$, is

$$L^0 = L_G^0 + L_{\text{RS}}^0, \quad (4.52)$$

where the linearized gravitational Lagrangian, L_G^0 , is

$$L_G^0 = -(a^{(\mu\nu)}G_{\mu\nu}[a] - b^{(\mu\nu)}G_{\mu\nu}[b]) \quad (4.53)$$

and the free Lagrangian of (Majorana) Rarita-Schwinger fields, L_{RS}^0 , is

$$L_{\text{RS}}^0 = \epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\rho\partial_\sigma\psi_\nu, \quad (4.54)$$

which can be diagonalized as

$$L_{\text{RS}}^0 = \epsilon^{\mu\nu\rho\sigma}(\bar{\psi}_\mu^1\gamma_5\gamma_\rho\partial_\sigma\psi_\nu^1 - \bar{\psi}_\mu^2\gamma_5\gamma_\rho\partial_\sigma\psi_\nu^2) \quad (4.55)$$

up to a total divergence term, where ψ_μ^1 and ψ_μ^2 are defined below (3.48).

The linearized theory of $N = 1$ chiral SUGRA possesses local gauge invariance and rigid SUSY invariance just like the usual $N = 1$ SUGRA: Indeed, the L^0 of (4.52) is invariant under local gauge transformations,

$$\begin{aligned}\delta a_{(\mu\nu)} &= \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x), \\ \delta b_{(\mu\nu)} &= \partial_\mu \eta_\nu(x) + \partial_\nu \eta_\mu(x), \\ \delta \psi_\mu &= \partial_\mu \epsilon(x), \\ \delta \tilde{\psi}_\mu &= \partial_\mu \tilde{\epsilon}(x).\end{aligned}\tag{4.56}$$

In contrast with the usual $N = 1$ SUGRA, on the other hand, L^0 is invariant under two kinds of rigid SUSY transformations. One is the following rigid SUSY transformations with supersymmetric partners being $(a_{(\mu\nu)}, \psi_\mu^1)$ and $(b_{(\mu\nu)}, \psi_\mu^2)$:

$$\begin{aligned}\delta a_{(\mu\nu)} &= -i\bar{\psi}_{(\mu}^1 \gamma_{\nu)} \alpha^1, \quad \delta \psi_\mu^1 = -2iS^{\rho\sigma} \partial_\rho a_{(\sigma\mu)} \alpha^1, \\ \delta b_{(\mu\nu)} &= -i\bar{\psi}_{(\mu}^2 \gamma_{\nu)} \alpha^2, \quad \delta \psi_\mu^2 = -2iS^{\rho\sigma} \partial_\rho b_{(\sigma\mu)} \alpha^2,\end{aligned}\tag{4.57}$$

where α^1 and α^2 are constant Majorana spinor parameters. In this case, if we make α^1 and α^2 local and add appropriate terms to the L^0 of (4.52) order-by-order in κ , then the resultant Lagrangian density is twice that of the usual $N = 1$ SUGRA: Namely, we have $\mathcal{L} = \mathcal{L}_{N=1\text{SUGRA}}^1 - \mathcal{L}_{N=1\text{SUGRA}}^2$.

The other is the rigid SUSY transformations,

$$\begin{aligned}\delta a_{(\mu\nu)} &= -\frac{i}{2}(\bar{\psi}_{(\mu} \gamma_{\nu)} \alpha + \bar{\psi}_{(\mu} \gamma_{\nu)} \tilde{\alpha}), \\ \delta b_{(\mu\nu)} &= \frac{1}{2}(\bar{\psi}_{(\mu} \gamma_5 \gamma_{\nu)} \alpha - \bar{\psi}_{(\mu} \gamma_5 \gamma_{\nu)} \tilde{\alpha}), \\ \delta \psi_\mu &= -2iS^{\rho\sigma}(\partial_\rho a_{(\sigma\mu)} + i\gamma_5 \partial_\rho b_{(\sigma\mu)})\alpha, \\ \delta \tilde{\psi}_\mu &= -2iS^{\rho\sigma}(\partial_\rho a_{(\sigma\mu)} - i\gamma_5 \partial_\rho b_{(\sigma\mu)})\tilde{\alpha}\end{aligned}\tag{4.58}$$

with α and $\tilde{\alpha}$ being constant Majorana spinor parameters. Although the gauge and SUSY transformations (4.56) and (4.58) form a closed algebra on shell, the rigid SUSY transformations (4.58) are not twice the usual $N = 1$ SUGRA.

Let us construct a local SUSY invariant Lagrangian to order κ^0 . If the spinor parameters α and $\tilde{\alpha}$ in (4.58) become spacetime dependent, i.e., $\alpha = \alpha(x)$ and $\tilde{\alpha} = \tilde{\alpha}(x)$, then the L^0 of (4.52) is no longer invariant: The variation of L^0 can be expressed as

$$\delta L^0 = (\partial_\mu \tilde{\alpha})(J^{1\mu}[a, \psi] + J^{2\mu}[b, \psi]) + (\partial_\mu \alpha)(\tilde{J}^{1\mu}[a, \tilde{\psi}] + \tilde{J}^{2\mu}[b, \tilde{\psi}]),\tag{4.59}$$

up to total divergence, because L^0 is invariant when α and $\tilde{\alpha}$ are constant. However, $J^{1\mu}[a, \psi]$, $J^{2\mu}[b, \psi]$, $\tilde{J}^{1\mu}[a, \tilde{\psi}]$ and $\tilde{J}^{2\mu}[b, \tilde{\psi}]$ are not uniquely fixed: As example, we shall

explain how the ambiguity arises in $J^{1\mu}[a, \psi]$. The variation of L^0 with respect to the transformations (4.58) is

$$\delta L^0 = (\partial_\mu a_{(\lambda\nu)})\epsilon^{\mu\nu\rho\sigma}\bar{\alpha}\gamma_5\gamma^\lambda\partial_\rho\psi_\sigma + \dots \quad (4.60)$$

If the spinor parameter $\bar{\alpha}$ is constant, the first term of (4.60) is just a total divergence. However, when $\bar{\alpha}$ is spacetime dependent, this term can be rewritten as

$$\begin{aligned} & (\partial_\mu a_{(\lambda\nu)})\epsilon^{\mu\nu\rho\sigma}\bar{\alpha}\gamma_5\gamma^\lambda\partial_\rho\psi_\sigma \\ &= (\partial_\mu \bar{\alpha})\{-pa_{(\lambda\nu)}\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma^\lambda\partial_\rho\psi_\sigma + (1-p)(\partial_\rho a_{(\lambda\nu)})\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma^\lambda\psi_\sigma\} \\ &+ [\text{a total divergence}] \end{aligned} \quad (4.61)$$

with p being an arbitrary real constant. Although the two terms proportional to p in (4.61) are combined to a total divergence, they do lead to non-trivial ambiguity in $J^{1\mu}[a, \psi]$. We can explicitly show that the remaining terms of (4.60) do not lead to any ambiguity in $J^{1\mu}[a, \psi]$. Thus we have

$$J^{1\mu}[a, \psi] = -pa_{(\lambda\nu)}\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma^\lambda\partial_\rho\psi_\sigma + (1-p)(\partial_\rho a_{(\lambda\nu)})\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma^\lambda\psi_\sigma + \dots \quad (4.62)$$

Note that similar ambiguity also appears in the usual $N = 1$ SUGRA. In the same manner, we get

$$J^{2\mu}[b, \psi] = iq b_{(\lambda\nu)}\epsilon^{\mu\nu\rho\sigma}\gamma^\lambda\partial_\rho\psi_\sigma - i(1-q)(\partial_\rho b_{(\lambda\nu)})\epsilon^{\mu\nu\rho\sigma}\gamma^\lambda\psi_\sigma + \dots, \quad (4.63)$$

$$\tilde{J}^{1\mu}[a, \tilde{\psi}] = -p'a_{(\lambda\nu)}\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma^\lambda\partial_\rho\tilde{\psi}_\sigma + (1-p')(\partial_\rho a_{(\lambda\nu)})\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma^\lambda\tilde{\psi}_\sigma + \dots, \quad (4.64)$$

$$\tilde{J}^{2\mu}[b, \tilde{\psi}] = -iq'b_{(\lambda\nu)}\epsilon^{\mu\nu\rho\sigma}\gamma^\lambda\partial_\rho\tilde{\psi}_\sigma + i(1-q')(\partial_\rho b_{(\lambda\nu)})\epsilon^{\mu\nu\rho\sigma}\gamma^\lambda\tilde{\psi}_\sigma + \dots \quad (4.65)$$

with p' , q and q' being arbitrary real constants.

Inspection of (4.59) shows that the invariance under transformations with local spinor parameter $\bar{\alpha}(x)$ is recovered to order κ^0 if the interaction term, $(-\kappa/2)\bar{\psi}_\mu(J^{1\mu}[a, \psi] + J^{2\mu}[b, \psi])$, is added to L^0 , and if we simultaneously make the gauge transformation of $\tilde{\psi}_\mu$ with $\tilde{\epsilon}(x) = (2/\kappa)\bar{\alpha}(x)$. On the other hand, the interaction term, $(-\kappa/2)\bar{\psi}_\mu(\tilde{J}^{1\mu}[a, \tilde{\psi}] + \tilde{J}^{2\mu}[b, \tilde{\psi}])$, is needed if the spinor parameter α becomes spacetime dependent. Since these two interaction terms must be the same, we require

$$\bar{\psi}_\mu(J^{1\mu}[a, \psi] + J^{2\mu}[b, \psi]) = \bar{\psi}_\mu(\tilde{J}^{1\mu}[a, \tilde{\psi}] + \tilde{J}^{2\mu}[b, \tilde{\psi}]). \quad (4.66)$$

Then the constants p , p' , q and q' are uniquely determined as $p = p' = 2$ and $q = q' = 2$. Since $J^{1\mu}[a, \psi]$, $J^{2\mu}[b, \psi]$, $\tilde{J}^{1\mu}[a, \tilde{\psi}]$ and $\tilde{J}^{2\mu}[b, \tilde{\psi}]$ are now fixed and satisfies (4.66), we modify the Lagrangian as

$$L^1 = L^0 - \frac{\kappa}{2}\bar{\psi}_\mu(J^{1\mu}[a, \psi] + J^{2\mu}[b, \psi]), \quad (4.67)$$

which is invariant to order κ^0 under the local SUSY transformations combined with the gauge transformations with the identification, $\epsilon(x) = (2/\kappa)\alpha(x)$ and $\tilde{\epsilon}(x) = (2/\kappa)\tilde{\alpha}(x)$.

The interaction term in the L^1 of (4.67) can be rewritten as $\kappa(a^{(\mu\nu)}T_{\mu\nu}^{(1)(sym)} + b^{(\mu\nu)}T_{\mu\nu}^{(2)(sym)})$. Thus (3.42) and (3.43) are derived from L^1 by taking variation with respect to $a^{(\mu\nu)}$ and $b^{(\mu\nu)}$ respectively. It should also be noted that the correction term of L^1 is not twice the usual $N = 1$ SUGRA, because ψ_μ and $\tilde{\psi}_\mu$ are contained both in $T_{\mu\nu}^{(1)}$ and $T_{\mu\nu}^{(2)}$ as shown in (3.46) and (3.47).

5 Conclusion

We have studied the structure of linearized field equations in the chiral formulation of gravity with the complex tetrad, and seen how the real and imaginary parts of the complex tetrad are coupled to matter fields in the weak field approximation. As an example, the explicit form of energy-momentum tensors has been calculated for (Majorana) Rarita-Schwinger fields. Starting from the linearized (free) theory of $N = 1$ chiral SUGRA, we have then obtained the Lagrangian which is invariant under local SUSY transformations to order κ^0 . The resultant Lagrangian just reproduces the field equations in the weak field approximation. We expect that the full nonlinear theory of $N = 1$ chiral SUGRA can be constructed by adding appropriate terms order-by-order in κ to the first-order Lagrangian we have obtained. We are also trying to introduce additional auxiliary fields into the linearized theory of $N = 1$ chiral SUGRA as the preliminary step to construct the full nonlinear theory with auxiliary fields of $N = 1$ chiral SUGRA.

Acknowledgements

We would like to thank the members of Physics Department at Saitama University for discussions and encouragement, Professor Y. Tanii and other members.

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Geometric Entropy and Conical Space

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Abstract

We revisit the renormalization coincidence problem between geometric entropy and gravitational constant.

1 Introduction

It has been recently claimed [2] that the ultraviolet divergence exactly coincides with the divergence of the gravitational constant in the perturbative quantum gravity. However, to justify the statement the geometric entropy needs to take a negative value in some case.

We discuss the definition of the entropy in the above argument from the view point of canonical formulation and show that the definition is not valid and that the renormalizations of the geometric entropy and the gravitational constant do not coincide in the non-minimally coupled scalar field theory.

Detailed analysis can be seen in our paper [1] and the similar conclusion can be also obtained in a rather different context [3].

2 Main Result

It might be imagined that geometric entropy can be naturally defined as follows.

$$S_\xi = \left(1 - n \frac{\partial}{\partial n}\right) \ln Z(n, \xi) \Big|_{n=1}, \quad (1)$$

where

$$Z(n, \xi) = \int_{\text{periodic}} D\phi \exp \left[-\frac{1}{2} \int_0^{2\pi n} d\tau \int d^{D-1}x \sqrt{g} \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \xi R \phi^2 \right) \right],$$

$$g_{\mu\nu} dx^\mu dx^\nu = N(\vec{x})^2 d\tau^2 + h_{ab}(\vec{x}) dx^a dx^b, \quad (0 \leq \tau \leq 2\pi n), \quad (2)$$

and the periodic condition, $\phi(\tau + 2\pi n) = \phi(\tau)$, is imposed.

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However we can show that the definition possesses an additional non-statistical contribution [1].

$$S_{\xi} = S_{geo} - 2\pi\xi A_{\perp}^{(D-2)} \frac{\langle 0_M | \phi(0)^2 | 0_M \rangle}{\langle 0_M | 0_M \rangle} \quad (3)$$

where S_{geo} is the geometric entropy and $|0_M\rangle$ is the Minkowskian vacuum state.

It is known [2] that the quantity S_{ξ} gets the same renormalization of the gravitational constant. Thus the renormalization of S_{geo} differs from that of the gravitational constant due to the correction term in eqn(3).

3 Conclusion

The renormalization of the geometric entropy does not in general coincide with that of the gravitational constant.

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Massless Limits of Massive Tensor Fields

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Abstract

Izawa's gauge-fixing procedure based on BRS symmetry is applied twice to the massive tensor field theory of Fierz-Pauli type. It is shown the second application can remove massless singularities which remain after the first application. Massless limit of the theory is discussed.

1 Introduction

In a previous paper [1] (referred to as I), we investigated how Izawa's gauge-fixing procedure [2] based on BRS symmetry works for infrared regularization of massive tensor fields. We studied two models for a linearized massive tensor field, the pure-tensor (PT) type model by Fierz-Pauli and the additional-scalar-ghost (ASG) type one. It turned out that Izawa's procedure is effective for the ASG model, but not for the PT model. In the case of the ASG model, Izawa's procedure can regularize the original massless singularities of second order. On the other hand, the original singularities contained in the PT model are of fourth order, and Izawa's procedure can only reduce them to second order.

What we have learned from the above exercise is that when Izawa's procedure is applied once, massless singularities are reduced by second order. Now comes the question what happens when we apply Izawa's procedure once more to the PT model. This is the issue to be discussed in the present paper. [3] We show that the second application of Izawa's procedure does regularize the remaining second-order massless singularities in the PT model.

In §2, the results obtained in I concerning the first application of Izawa's procedure are summarized. In §3, the second application of Izawa's procedure is performed to the PT model. It is shown that the second-order massless singularities which remain after the first application are in fact regularized. In §4, we discuss massless limit of the resulting theory to find a graviton field is contained in the theory as a combination of the basic fields. Summary and discussion are given in §5.

2 First application of Izawa's procedure

A massive tensor field is described by the Lagrangian²

$$L_h = \frac{1}{2} h^{\mu\nu} \Lambda_{\mu\nu,\rho\sigma} h^{\rho\sigma} - \frac{m^2}{2} (h^{\mu\nu} h_{\mu\nu} - a h^2), \quad (1)$$

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²Notations used in this paper are the same as in I.

where $\Lambda_{\mu\nu,\rho\sigma}$ is the operator defined by

$$\Lambda_{\mu\nu,\rho\sigma} = (\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\nu}\eta_{\rho\sigma})\square - (\eta_{\mu\rho}\partial_\nu\partial_\sigma + \eta_{\nu\sigma}\partial_\mu\partial_\rho) + (\eta_{\rho\sigma}\partial_\mu\partial_\nu + \eta_{\mu\nu}\partial_\rho\partial_\sigma), \quad (2)$$

and a is a real parameter taking the values

$$a = \begin{cases} \frac{1}{2} & \text{for the ASG model,} \\ 1 & \text{for the PT model.} \end{cases} \quad (3)$$

Two-point functions are calculated as

$$\begin{aligned} \langle h^{\mu\nu} h^{\rho\sigma} \rangle &= \frac{1}{\square - m^2} \left\{ \frac{1}{2} (\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}) \right. \\ &\quad \left. - \frac{1}{2m^2} (\eta^{\mu\rho}\partial^\nu\partial^\sigma + \eta^{\mu\sigma}\partial^\nu\partial^\rho + \eta^{\nu\rho}\partial^\mu\partial^\sigma + \eta^{\nu\sigma}\partial^\mu\partial^\rho) \right\} \delta \end{aligned} \quad (4)$$

for the ASG model, and

$$\begin{aligned} \langle h^{\mu\nu} h^{\rho\sigma} \rangle &= \frac{1}{\square - m^2} \left\{ \frac{1}{2} (\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}) \right. \\ &\quad - \frac{1}{2m^2} (\eta^{\mu\rho}\partial^\nu\partial^\sigma + \eta^{\mu\sigma}\partial^\nu\partial^\rho + \eta^{\nu\rho}\partial^\mu\partial^\sigma + \eta^{\nu\sigma}\partial^\mu\partial^\rho) \\ &\quad \left. + \frac{2}{3} \left(\frac{1}{2}\eta^{\mu\nu} + \frac{\partial^\mu\partial^\nu}{m^2} \right) \left(\frac{1}{2}\eta^{\rho\sigma} + \frac{\partial^\rho\partial^\sigma}{m^2} \right) \right\} \delta \end{aligned} \quad (5)$$

for the PT model. We see that the ASG model has second-order massless singularities, while the PT model fourth-order.

Applying Izawa's gauge-fixing procedure based on BRS symmetry, we obtain the following Lagrangian:

$$\begin{aligned} L_T &= \frac{1}{2} h^{\mu\nu} \Lambda_{\mu\nu,\rho\sigma} h^{\rho\sigma} \\ &\quad - \frac{m^2}{2} \left[\left(h_{\mu\nu} - \frac{1}{m} (\partial_\mu\theta_\nu + \partial_\nu\theta_\mu) \right)^2 - a \left(h - \frac{2}{m} \partial^\mu\theta_\mu \right)^2 \right] \\ &\quad + b^\mu \left(\partial^\nu h_{\mu\nu} - \frac{1}{2} \partial_\mu h + \frac{\alpha}{2} b_\mu \right) + i\bar{c}^\mu \square c_\mu, \end{aligned} \quad (6)$$

where an auxiliary vector field θ_μ , a Nakanishi-Lautrup (NL) field b_μ , a pair of Faddeev-Popov (FP) ghosts $(c_\mu, \bar{c}_\mu)$, and a gauge parameter α have been introduced. This Lagrangian is invariant under the following BRS transformation:

$$\delta h_{\mu\nu} = \partial_\mu c_\nu + \partial_\nu c_\mu, \quad \delta \theta_\mu = m c_\mu, \quad \delta \bar{c}_\mu = i b_\mu. \quad (7)$$

Putting $a = \frac{1}{2}$ in Eq. (6), we have for the ASG model

$$\begin{aligned} L_{T,a=\frac{1}{2}} &= \frac{1}{2} h^{\mu\nu} \Lambda_{\mu\nu,\rho\sigma} h^{\rho\sigma} \\ &\quad - \frac{m^2}{2} \left(h^{\mu\nu} h_{\mu\nu} - \frac{1}{2} h^2 \right) - 2m\theta^\mu \left(\partial^\nu h_{\mu\nu} - \frac{1}{2} \partial_\mu h \right) - \partial_\mu \theta_\nu \partial^\mu \theta^\nu \\ &\quad + b^\mu \left(\partial^\nu h_{\mu\nu} - \frac{1}{2} \partial_\mu h + \frac{\alpha}{2} b_\mu \right) + i\bar{c}^\mu \square c_\mu. \end{aligned} \quad (8)$$

This gives the following two-point functions:

$$\begin{aligned} \langle h^{\mu\nu} h^{\rho\sigma} \rangle &= \frac{1}{\square - m^2} \left\{ \frac{1}{2} (\eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho} - \eta^{\mu\nu} \eta^{\rho\sigma}) \right. \\ &\quad \left. - \frac{1}{2} \left[(1 - 2\alpha) \frac{1}{\square} + 2\alpha \frac{m^2}{\square^2} \right] (\eta^{\mu\rho} \partial^\nu \partial^\sigma + \eta^{\mu\sigma} \partial^\nu \partial^\rho + \eta^{\nu\rho} \partial^\mu \partial^\sigma + \eta^{\nu\sigma} \partial^\mu \partial^\rho) \right\} \delta, \end{aligned} \quad (9)$$

$$\langle h^{\mu\nu} b^\rho \rangle = \frac{1}{\square} (\eta^{\mu\rho} \partial^\nu + \eta^{\nu\rho} \partial^\mu) \delta, \quad (10)$$

$$\langle h^{\mu\nu} \theta^\rho \rangle = -\alpha m \frac{1}{\square^2} (\eta^{\mu\rho} \partial^\nu + \eta^{\nu\rho} \partial^\mu) \delta, \quad (11)$$

$$\langle b^\mu b^\rho \rangle = 0, \quad (12)$$

$$\langle b^\mu \theta^\rho \rangle = m \frac{1}{\square} \eta^{\mu\rho} \delta, \quad (13)$$

$$\langle \theta^\mu \theta^\rho \rangle = \frac{1}{2} \frac{1}{\square} \left(1 - 2\alpha \frac{m^2}{\square} \right) \eta^{\mu\rho} \delta, \quad (14)$$

except for the trivial one for $\langle c^\mu \bar{c}^\nu \rangle$.³ We see that the massless singularities in (4) have been regularized by this procedure.⁴

When $\alpha = 1$, the Lagrangian (6) gives

$$\begin{aligned} L_{T,\alpha=1} &= \frac{1}{2} h^{\mu\nu} \Lambda_{\mu\nu,\rho\sigma} h^{\rho\sigma} \\ &\quad - \frac{m^2}{2} (h^{\mu\nu} h_{\mu\nu} - h^2) - 2m\theta^\mu (\partial^\nu h_{\mu\nu} - \partial_\mu h) - \frac{1}{2} (\partial_\mu \theta_\nu - \partial_\nu \theta_\mu)^2 \\ &\quad + b^\mu \left(\partial^\nu h_{\mu\nu} - \frac{1}{2} \partial_\mu h + \frac{\alpha}{2} b_\mu \right) + i\bar{c}^\mu \square c_\mu. \end{aligned} \quad (15)$$

Thus two-point functions for the PT model are

$$\begin{aligned} \langle h^{\mu\nu} h^{\rho\sigma} \rangle &= \frac{1}{\square - m^2} \left\{ \frac{1}{2} (\eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho} - \eta^{\mu\nu} \eta^{\rho\sigma}) \right. \\ &\quad - \frac{1}{2} \left[(1 - 2\alpha) \frac{1}{\square} + 2\alpha \frac{m^2}{\square^2} \right] (\eta^{\mu\rho} \partial^\nu \partial^\sigma + \eta^{\mu\sigma} \partial^\nu \partial^\rho + \eta^{\nu\rho} \partial^\mu \partial^\sigma + \eta^{\nu\sigma} \partial^\mu \partial^\rho) \\ &\quad \left. + \frac{2}{3} \left(\frac{1}{2} \eta^{\mu\nu} + \frac{\partial^\mu \partial^\nu}{\square} \right) \left(\frac{1}{2} \eta^{\rho\sigma} + \frac{\partial^\rho \partial^\sigma}{\square} \right) \right\} \delta, \end{aligned} \quad (16)$$

$$\langle h^{\mu\nu} b^\rho \rangle = \frac{1}{\square} (\eta^{\mu\rho} \partial^\nu + \eta^{\nu\rho} \partial^\mu) \delta, \quad (17)$$

$$\langle h^{\mu\nu} \theta^\rho \rangle = \left\{ \frac{1}{6m} \frac{1}{\square} \eta^{\mu\nu} \partial^\rho - \alpha m \frac{1}{\square^2} (\eta^{\mu\rho} \partial^\nu + \eta^{\nu\rho} \partial^\mu) + \frac{1}{3m} \frac{1}{\square^2} \partial^\mu \partial^\nu \partial^\rho \right\} \delta, \quad (18)$$

$$\langle b^\mu b^\rho \rangle = 0, \quad (19)$$

³In Abelian cases which we are dealing with, FP ghosts decouple from all other fields to give trivial two-point functions. We omit to write down their explicit forms throughout the paper.

⁴Exactly speaking, the quantity $\square^{-2}$ is well-defined only when accompanied by derivatives. Therefore, the expression (14) is meaningful only in a formal sense. If the field θ_μ has some non-derivative couplings, α is to be set 0. On the other hand, if it has derivative couplings only, then any value of α is allowed.

$$\langle b^\mu \theta^\rho \rangle = m \frac{1}{\square} \eta^{\mu\rho} \delta, \quad (20)$$

$$\langle \theta^\mu \theta^\rho \rangle = \left\{ \frac{1}{2} \frac{1}{\square} \left(1 - 2\alpha \frac{m^2}{\square} \right) \eta^{\mu\rho} - \frac{1}{6m^2} \frac{1}{\square} \left(1 - \frac{m^2}{\square} \right) \partial^\mu \partial^\rho \right\} \delta. \quad (21)$$

We see the fourth-order massless singularities found in (5) for the original PT model have been driven away. However, new at-most-second-order singularities have appeared in the θ -sector (18) and (21). Can these singularities be driven away by applying Izawa's procedure once more? This is the issue to address in the next section.

3 Second application of Izawa's procedure

The starting point is the Lagrangian (15). The kinetic term of the vector field θ_μ shows this field is a kind of gauge field. It is expected from this fact that the second application of Izawa's procedure works. This is in fact the case as seen below.

We introduce a new set of variables (θ'_μ, φ) to perform a field transformation $\theta \rightarrow (\theta'_\mu, \varphi)$ such that

$$\theta_\mu = \theta'_\mu - \frac{1}{m} \partial_\mu \varphi, \quad (22)$$

$$\partial^\mu \theta'_\mu = 0. \quad (23)$$

The new variables (θ'_μ, φ) are first assumed to be independent of the old one θ_μ . Then the Lagrangian (15), which does not depend on the new variables, is invariant under the BRS transformation

$$\begin{cases} \delta' \theta'_\mu = c'_\mu, & \delta' \bar{c}'_\mu = i b'_\mu, \\ \delta' \varphi = m c, & \delta' \bar{c} = i b, \end{cases} \quad (24)$$

where the new FP ghosts (c'_μ, c) and $(\bar{c}'_\mu, \bar{c})$ as well as the new NL fields (b'_μ, b) have been introduced. To relate the old and new sets of variables, we supplement the Lagrangian (15) by adding the following BRS gauge-fixing term:

$$\begin{aligned} L'_B &= -i\delta' \left[\bar{c}^\mu \left(\theta_\mu - \theta'_\mu + \frac{1}{m} \partial_\mu \varphi \right) + \bar{c} \left(\partial^\mu \theta'_\mu - \frac{m}{2} h + \frac{\beta}{2} b \right) \right] \\ &= b^\mu \left(\theta_\mu - \theta'_\mu + \frac{1}{m} \partial_\mu \varphi \right) + b \left(\partial^\mu \theta'_\mu - \frac{m}{2} h + \frac{\beta}{2} b \right) \\ &\quad - i (\bar{c}^\mu + \partial^\mu \bar{c}) (c'_\mu - \partial_\mu c) + i \bar{c} \square c \end{aligned} \quad (25)$$

with the second gauge parameter β . The path integral is given as

$$\begin{aligned} Z &= \int \mathcal{D}h_{\mu\nu} \mathcal{D}\theta_\mu \mathcal{D}b_\mu \mathcal{D}c_\mu \mathcal{D}\bar{c}_\mu \mathcal{D}\theta'_\mu \mathcal{D}\varphi \mathcal{D}b'_\mu \mathcal{D}c'_\mu \mathcal{D}\bar{c}'_\mu \mathcal{D}b \mathcal{D}c \mathcal{D}\bar{c} \\ &\quad \times \exp i \int d^4x [L_{T,a=1} + L'_B]. \end{aligned} \quad (26)$$

Integrating over the variables $(b'_\mu, \theta_\mu, c'_\mu, \bar{c}'_\mu)$ and overwriting θ_μ on θ'_μ , we obtain

$$Z = \int \mathcal{D}h_{\mu\nu} \mathcal{D}\theta_\mu \mathcal{D}\varphi \mathcal{D}b_\mu \mathcal{D}b \mathcal{D}c_\mu \mathcal{D}\bar{c}_\mu \mathcal{D}c \mathcal{D}\bar{c} \exp i \int d^4x L'_{T,a=1}, \quad (27)$$

where

$$\begin{aligned}
L'_{T,a=1} &= \frac{1}{2} h^{\mu\nu} \Lambda_{\mu\nu,\rho\sigma} h^{\rho\sigma} \\
&\quad - \frac{m^2}{2} \left[\left(h_{\mu\nu} - \frac{1}{m} (\partial_\mu \theta_\nu + \partial_\nu \theta_\mu) + \frac{2}{m^2} \partial_\mu \partial_\nu \varphi \right)^2 \right. \\
&\quad \left. - \left(h - \frac{2}{m} \partial^\mu \theta_\mu + \frac{2}{m^2} \square \varphi \right)^2 \right] \\
&\quad + b^\mu \left(\partial^\nu h_{\mu\nu} - \frac{1}{2} \partial_\mu h + \frac{\alpha}{2} b_\mu \right) + i \bar{c}^\mu \square c_\mu \\
&\quad + b \left(\partial^\mu \theta_\mu - \frac{m}{2} h + \frac{\beta}{2} b \right) + i \bar{c} \square c \\
&= \frac{1}{2} h^{\mu\nu} \Lambda_{\mu\nu,\rho\sigma} h^{\rho\sigma} - \frac{m^2}{2} (h^{\mu\nu} h_{\mu\nu} - h^2) \\
&\quad - 2(m\theta^\mu - \partial^\mu \varphi)(\partial^\nu h_{\mu\nu} - \partial_\mu h) - \frac{1}{2} (\partial_\mu \theta_\nu - \partial_\nu \theta_\mu)^2 \\
&\quad + b^\mu \left(\partial^\nu h_{\mu\nu} - \frac{1}{2} \partial_\mu h + \frac{\alpha}{2} b_\mu \right) + i \bar{c}^\mu \square c_\mu \\
&\quad + b \left(\partial^\mu \theta_\mu - \frac{m}{2} h + \frac{\beta}{2} b \right) + i \bar{c} \square c. \tag{28}
\end{aligned}$$

This Lagrangian is invariant under the following BRS transformation:

$$\begin{cases} \delta h_{\mu\nu} = \partial_\mu c_\nu + \partial_\nu c_\mu, & \delta \bar{c}_\mu = i b_\mu, \\ \delta \theta_\mu = m c_\mu + \partial_\mu c, & \delta \bar{c} = i b, \\ \delta \varphi = m c. \end{cases} \tag{29}$$

We note the Lagrangian $L'_{T,a=1}$ has a smooth massless limit. This comes from the fact that the kinetic term of θ_μ contained in the Lagrangian $L_{T,a=1}$ (15) is the gauge-theoretic one $-\frac{1}{2} (\partial_\mu \theta_\nu - \partial_\nu \theta_\mu)^2$. On the contrary, the Lagrangian $L_{T,a=\frac{1}{2}}$ (8) for the ASG model has the non-gauge-theoretic kinetic term $-\partial_\mu \theta_\nu \partial^\mu \theta^\nu$. When performed the second application of Izawa's procedure, such a term yields at-most-second-order singular terms like $-\frac{1}{m^2} (\square \varphi)^2$.

Two-point functions obtained from $L'_{T,a=1}$ (28) are the following:

$$\begin{aligned}
\langle h^{\mu\nu} h^{\rho\sigma} \rangle &= \frac{1}{\square - m^2} \left\{ \frac{1}{2} (\eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho} - \eta^{\mu\nu} \eta^{\rho\sigma}) \right. \\
&\quad - \frac{1}{2} \left[(1 - 2\alpha) \frac{1}{\square} + 2\alpha \frac{m^2}{\square^2} \right] (\eta^{\mu\rho} \partial^\nu \partial^\sigma + \eta^{\mu\sigma} \partial^\nu \partial^\rho + \eta^{\nu\rho} \partial^\mu \partial^\sigma + \eta^{\nu\sigma} \partial^\mu \partial^\rho) \\
&\quad \left. + \frac{2}{3} \left(\frac{1}{2} \eta^{\mu\nu} + \frac{\partial^\mu \partial^\nu}{\square} \right) \left(\frac{1}{2} \eta^{\rho\sigma} + \frac{\partial^\rho \partial^\sigma}{\square} \right) \right\} \delta, \tag{30}
\end{aligned}$$

$$\langle h^{\mu\nu} b^\rho \rangle = \frac{1}{\square} (\eta^{\mu\rho} \partial^\nu + \eta^{\nu\rho} \partial^\mu) \delta, \tag{31}$$

$$\langle h^{\mu\nu} b \rangle = 0, \tag{32}$$

$$\langle h^{\mu\nu} \theta^\rho \rangle = -\alpha m \frac{1}{\square^2} (\eta^{\mu\rho} \partial^\nu + \eta^{\nu\rho} \partial^\mu) \delta, \tag{33}$$

$$\langle h^{\mu\nu}\varphi \rangle = \frac{1}{3} \frac{1}{\square} \left(\frac{1}{2} \eta^{\mu\nu} + \frac{\partial^\mu \partial^\nu}{\square} \right) \delta, \quad (34)$$

$$\langle b^\mu b^\rho \rangle = 0, \quad (35)$$

$$\langle b^\mu b \rangle = 0, \quad (36)$$

$$\langle b^\mu \theta^\rho \rangle = m \frac{1}{\square} \eta^{\mu\rho} \delta, \quad (37)$$

$$\langle b^\mu \varphi \rangle = 0, \quad (38)$$

$$\langle b b \rangle = 0, \quad (39)$$

$$\langle b \theta^\mu \rangle = -\frac{1}{\square} \partial^\mu \delta, \quad (40)$$

$$\langle b \varphi \rangle = m \frac{1}{\square} \delta, \quad (41)$$

$$\langle \theta^\mu \theta^\rho \rangle = \left[\frac{1}{2} \frac{1}{\square} \left(1 - 2\alpha \frac{m^2}{\square} \right) \eta^{\mu\rho} - \frac{1}{2} (1 - 2\beta) \frac{\partial^\mu \partial^\rho}{\square^2} \right] \delta, \quad (42)$$

$$\langle \theta^\mu \varphi \rangle = \frac{1}{2} (1 - 2\beta) \frac{m}{\square^2} \partial^\mu \delta, \quad (43)$$

$$\langle \varphi \varphi \rangle = \left[\frac{1}{6} \frac{1}{\square} \left(1 - \frac{m^2}{\square} \right) + \frac{1}{2} (1 - 2\beta) \frac{m^2}{\square^2} \right] \delta. \quad (44)$$

These expressions show that the massless singularities remaining in (18) and (21) have been regularized.⁵ Izawa's procedure does work in this case.

4 Massless limit

It has been found that the theory constructed in the previous section has a smooth massless limit. In this section, we investigate whether or not the limit is consistent with the ordinary massless tensor theory.

As m tends to 0, the Lagrangian $L'_{T,a=1}$ (28) reduces to

$$\begin{aligned} L = & \frac{1}{2} h^{\mu\nu} \Lambda_{\mu\nu,\rho\sigma} h^{\rho\sigma} + 2\partial^\mu \varphi (\partial^\nu h_{\mu\nu} - \partial_\mu h) \\ & + b^\mu \left(\partial^\nu h_{\mu\nu} - \frac{1}{2} \partial_\mu h + \frac{\alpha}{2} b_\mu \right) + i\bar{c}^\mu \square c_\mu \\ & - \frac{1}{2} (\partial_\mu \theta_\nu - \partial_\nu \theta_\mu)^2 + b \left(\partial^\mu \theta_\mu + \frac{\beta}{2} b \right) + i\bar{c} \square c. \end{aligned} \quad (45)$$

The $(h_{\mu\nu}, \varphi, b_\mu)$ -sector is completely decoupled from the (θ_μ, b) -sector. If it were not for the second term in the first line of the right hand side of (45), the $(h_{\mu\nu}, \varphi, b_\mu)$ -sector coincides with the ordinary massless tensor theory. Two-point functions are for the $(h_{\mu\nu}, \varphi, b_\mu)$ -

⁵The note stated in the second footnote on p. 465 holds here too. For $\langle \theta^\mu \theta^\rho \rangle$ in (42) to be well-defined, either the gauge parameter α should be set 0 or the field θ_μ should appear in company with derivatives in interaction Lagrangian. For $\langle \varphi \varphi \rangle$ in (44) to be well-defined, either the gauge parameter β should be chosen as $\frac{1}{3}$ or the field φ should be accompanied by derivatives in interaction Lagrangian.

sector

$$\begin{aligned}\langle h^{\mu\nu} h^{\rho\sigma} \rangle &= \frac{1}{\square} \left\{ \frac{1}{2} (\eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho} - \eta^{\mu\nu} \eta^{\rho\sigma}) \right. \\ &\quad - \frac{1}{2} (1 - 2\alpha) \frac{1}{\square} (\eta^{\mu\rho} \partial^\nu \partial^\sigma + \eta^{\mu\sigma} \partial^\nu \partial^\rho + \eta^{\nu\rho} \partial^\mu \partial^\sigma + \eta^{\nu\sigma} \partial^\mu \partial^\rho) \\ &\quad \left. + \frac{2}{3} \left(\frac{1}{2} \eta^{\mu\nu} + \frac{\partial^\mu \partial^\nu}{\square} \right) \left(\frac{1}{2} \eta^{\rho\sigma} + \frac{\partial^\rho \partial^\sigma}{\square} \right) \right\} \delta, \quad (46)\end{aligned}$$

$$\langle h^{\mu\nu} b^\rho \rangle = \frac{1}{\square} (\eta^{\mu\rho} \partial^\nu + \eta^{\nu\rho} \partial^\mu) \delta, \quad (47)$$

$$\langle h^{\mu\nu} \varphi \rangle = \frac{1}{3} \frac{1}{\square} \left(\frac{1}{2} \eta^{\mu\nu} + \frac{\partial^\mu \partial^\nu}{\square} \right) \delta, \quad (48)$$

$$\langle b^\mu b^\rho \rangle = 0, \quad (49)$$

$$\langle b^\mu \varphi \rangle = 0, \quad (50)$$

$$\langle \varphi \varphi \rangle = \frac{1}{6} \frac{1}{\square} \delta, \quad (51)$$

and for the (θ_μ, b) -sector

$$\langle \theta^\mu \theta^\rho \rangle = \frac{1}{2} \frac{1}{\square} \left\{ \eta^{\mu\rho} - (1 - 2\beta) \frac{\partial^\mu \partial^\rho}{\square} \right\} \delta, \quad (52)$$

$$\langle \theta^\mu b \rangle = \frac{1}{\square} \partial^\mu \delta, \quad (53)$$

$$\langle bb \rangle = 0. \quad (54)$$

The $(h_{\mu\nu}, \varphi, b_\mu)$ -sector does not reproduce the two-point functions for the ordinary massless tensor theory. If we didn't have the third line in the expression (46) for $\langle h^{\mu\nu} h^{\rho\sigma} \rangle$, and if $\langle h^{\mu\nu} \varphi \rangle$ were 0 in (48), then the whole set of two-point functions agrees with that of the ordinary massless tensor.

In order to see how the ordinary graviton field is contained in this model, we now introduce the following combination of the basic fields:

$$H_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu} \varphi. \quad (55)$$

For this new field the $(h_{\mu\nu}, \varphi, b_\mu)$ -part of the Lagrangian (45) becomes

$$\begin{aligned}L_{(H_{\mu\nu}, \varphi, b_\mu)} &= \frac{1}{2} H^{\mu\nu} \Lambda_{\mu\nu, \rho\sigma} H^{\rho\sigma} + b^\mu \left(\partial^\nu H_{\mu\nu} - \frac{1}{2} \partial_\mu H - \partial_\mu \varphi + \frac{\alpha}{2} b_\mu \right) + i \bar{c}^\mu \square c_\mu \\ &\quad - 3 \partial_\mu \varphi \partial^\mu \varphi, \quad (56)\end{aligned}$$

which is invariant under the following BRS transformation:

$$\delta H_{\mu\nu} = \partial_\mu c_\nu + \partial_\nu c_\mu, \quad \delta \varphi = 0, \quad \delta \bar{c}_\mu = i b_\mu. \quad (57)$$

The gauge-fixing and FP-ghost terms in (56)

$$\begin{aligned}L_{\text{GF+FP}} &= b^\mu \left(\partial^\nu H_{\mu\nu} - \frac{1}{2} \partial_\mu H - \partial_\mu \varphi + \frac{\alpha}{2} b_\mu \right) + i \bar{c}^\mu \square c_\mu \\ &= -i \delta \left[\bar{c}^\mu \left(\partial^\nu H_{\mu\nu} - \frac{1}{2} \partial_\mu H - \partial_\mu \varphi + \frac{\alpha}{2} b_\mu \right) \right] \quad (58)\end{aligned}$$

can be replaced with the conventional choice

$$\begin{aligned} L'_{\text{GF+FP}} &= -i\delta \left[\bar{c}^\mu \left(\partial^\nu H_{\mu\nu} - \frac{1}{2} \partial_\mu H + \frac{\alpha}{2} b_\mu \right) \right] \\ &= b^\mu \left(\partial^\nu H_{\mu\nu} - \frac{1}{2} \partial_\mu H + \frac{\alpha}{2} b_\mu \right) + i\bar{c}^\mu \square c_\mu \end{aligned} \quad (59)$$

without any physical effect. The choice of (59) gives

$$\begin{aligned} \langle H^{\mu\nu} H^{\rho\sigma} \rangle &= \frac{1}{\square} \left\{ \frac{1}{2} (\eta^{\mu\rho} \eta^{\nu\sigma} + \eta^{\mu\sigma} \eta^{\nu\rho} - \eta^{\mu\nu} \eta^{\rho\sigma}) \right. \\ &\quad \left. - \frac{1}{2} (1 - 2\alpha) \frac{1}{\square} (\eta^{\mu\rho} \partial^\nu \partial^\sigma + \eta^{\mu\sigma} \partial^\nu \partial^\rho + \eta^{\nu\rho} \partial^\mu \partial^\sigma + \eta^{\nu\sigma} \partial^\mu \partial^\rho) \right\} \delta, \end{aligned} \quad (60)$$

$$\langle H^{\mu\nu} b^\rho \rangle = \frac{1}{\square} (\eta^{\mu\rho} \partial^\nu + \eta^{\nu\rho} \partial^\mu) \delta, \quad (61)$$

$$\langle H^{\mu\nu} \varphi \rangle = 0 \quad (62)$$

to replace (46), (47) and (48). Thus the field $H_{\mu\nu}$ is identified with the ordinary massless *graviton* field.

5 Summary and discussion

It has turned out that:

- (1) The original ASG model for a massive tensor field has second-order massless singularities, which can be regularized by applying Izawa's procedure based on BRS symmetry once;
- (2) The original PT model develops fourth-order massless singularities, which can be regularized by applying Izawa's procedure twice.

The Batalin-Fradkin algorithm [4] is another powerful method for constructing gauge-invariant theories from non-gauge-invariant ones. The application of this procedure to a massive tensor field has been performed in Ref. [5], giving the same Lagrangians as obtained above, $L_{T,a=\frac{1}{2}}$ (8) for the ASG model and $L'_{T,a=1}$ (28) for the PT model.

We now have massive tensor theories equipped with BRS invariance as well as smooth massless limits. However, we are still in the linearized world. To construct complete nonlinear theories is a major problem to solve.

Acknowledgements

We would like to thank T. Kurimoto for discussion.

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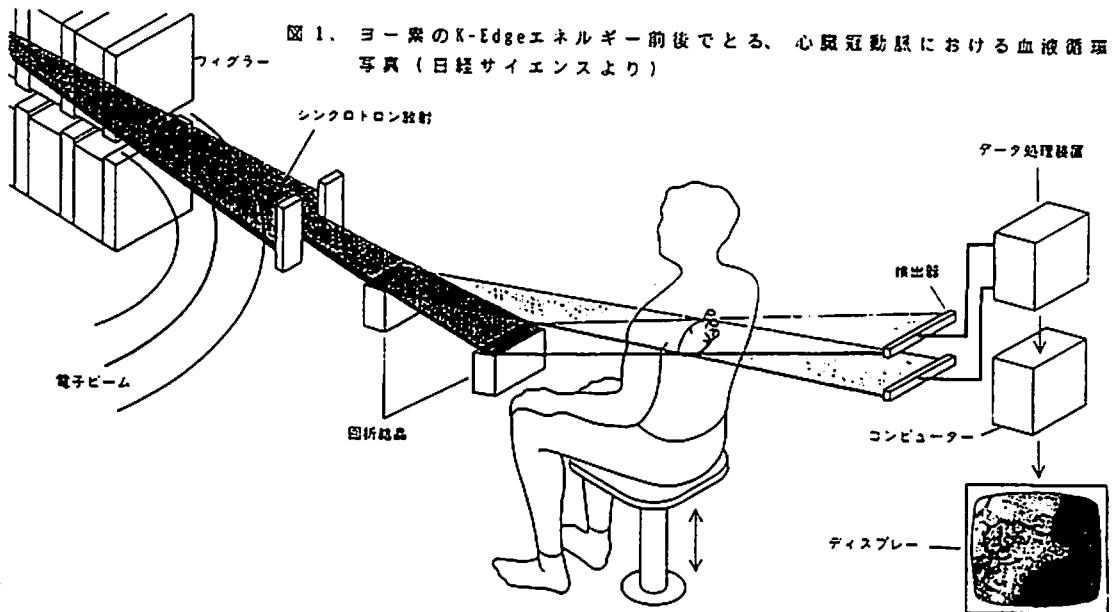
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ブラックホールのX線写真

京大（理）小山勝二

はじめに

約1世紀前、ドイツの科学者レントゲンは物質を透過する光線、X線を発見。その後、X線に関連した基礎応用科学は多方面で目ざましい発展と成果をうみだした。直接、間接的にX線が関与したノーベル賞級の科学成果は10指をこすであろう。なかでも我々にもっとも恩恵を与え馴染み深い分野は医学への応用であろう。発見後数日以内に撮った妻の左手のX線透過写真こそ医学への最初の応用を予見させるものであった。実際今日我々が経験する胃や胸部レントゲン写真は100年前の左手の写真となんら変わる技術ではない。しかし一方、X線自身、物質との相互作用の研究、関連諸技術の進展の結果、現在ではより高度なX線写真技術が医学に用いられるようになった。例えば軌道放射光で指向性のある強い連続X線を生成し、それを2つの近接した結晶ブラッグ分光し、特定元素のK-Edgeエネルギー前後での2枚の透過写真の差をとることにより、その元素のみの透過写真をとると言うような手法で血管中の血液のみのX線写真をとることが出来る（図1）。



冠動脈アンジオグラフィー 冠動脈の画像は、通常のX線透過の代わりにシンクロトロン放射を利用すれば、安全にはるかにシャープな像を得ることができる。まず患者の血液中にヨウ素を含む液体を注射する。そして特殊な回折結晶により、シンクロトロン放射からヨウ素の吸収端のエネルギー値の直上および直下の2種類の光子エネルギーのビームを取り出す。これらの2つのビームは患者の心臓に露光し、それから広がって別々の検出器

にあたる。軟らかい組織や骨による吸収は、高エネルギーのビームに対しても低エネルギービームの場合とほとんど同じである。しかし、ヨウ素による吸収は、高エネルギービームの方がはるかに大きい。したがって、2つの検出器で得られた像の差を取れば、患者の軟らかい組織や骨の像を差し引いて冠動脈およびその他の血管を強調した像が得られる。通常のX線透過を用いる方法では、感度が低いため、ヨウ素を直接動脈中に注入する必要がある。

X線の基礎科学への応用、あるいは基礎科学そのものとして、X線天文学は約30年前に誕生した。医学に例れば、つい最近まではマルタのX線写真の手法のX線天文学だったといってもよい。最近、宇宙X線の撮像と分光技術の進展にともない、X線天文学は血液写真のような”高度な”X線天文へと大きな変身と発展を遂げつつある。この発展にもっとも大きく寄与したのが日本のX線天文衛星“あすか”である。ここでは銀河の中心に存在する巨大ブラックホールの最新のX線写真を紹介する。医学の例とは少し異なり、X線天文ではK-edgeで吸収されたX線が再放出する特性X線を用いる。

銀河中心に存在するもの

銀河系には約1000億個の恒星が存在し、我々の太陽は銀河系の辺境に存在する1恒星にすぎない。宇宙全体ではこのような銀河がおよそ1000億も存在している。そのうち約1%の銀河はその中心から異常に強い電磁波—それは電波からγ線にいたるまで総ての波長を含む—を放射し、それ故に活動銀河と呼んでいる。その異常に強い放射強度や、特にX線で観測される急激な時間変動から、放射源としてもっともエネルギー変換効率の高いブラックホールが銀河中心に存在しているというのが通説である。その質量は太陽質量の100万から1億倍である。このブラックホールに周辺から物質が流入することにより、その静止質量の10%に近いエネルギーの解放が起こっているのである。

質量降着は、なんらかの外因により質量降着がトリガーされ、いずれ降着は休止する。降着の期間は約1億年くらいと推定され、宇宙年齢に比べればごく短時間である。質量降着がなければ、強い放射もないので活動銀河といえども大半は平穏な時期を過ごしているようである。活動銀河として分類されていない、残り99%の銀河—我々天の川銀河を含む—は実は活動銀河の休止期にすぎないと考える研究者も多い。実際、最近の観測で、多くの静かな銀河中心のごく狭い領域に太陽の100万から1億倍もの質量が集中している証拠—大質量ブラックホールの間接証拠—が見つかっている。さらに弱い活動性や、隠された活動性をもつ銀河が続々とX線観測にかかっている。

ブラックホールの質量 M 、光速度 c 、万有引力定数 G とし、 $R_s = 2MG/c^2$ で定義される半径をシュワルツシルド半径といい、これより内側の情報は外部には伝わらない。即ち R_s をブラックホールの半径だと思えばいい。後述のようにX線はほとんどは数倍の R_s 領域から放射される。このような狭い領域から放出されるX線の写真が本当に撮影できるのだろうか？

100万太陽質量のブラックホールの半径でも0.2天文単位（地球の公転軌道半径）ほどである。我々の隣人アンドロメダ星雲にこのようなブラックホールがあるとしてその視直径は 10^{-6} 秒角、我々の天の川中心にあっても 10^{-5} 秒角にすぎない。望遠鏡の角度分解能は原理的には回折限界できまり、我々が問題しようとしている鉄特性X線の波長 $2\text{\AA}$ では1 μ 口径のX線望遠鏡で 10^{-5} 秒角が回折限界である。しかしこれは理想的な鏡面形状、精度が達成されて初めて到達する限界であり、現状はとてことこまで及んでいない。すなわち、単なるブラックホールのX線写真では、それをいくら拡大現像しても点がひろがるだけでなんの構造もみることが出来ない。

たとえ直接解像できなくとも、ブラックホールの極近傍から放出されるX線が周辺に存在する物質からの相互作用をうける。その様子を観測することにより、ブラックホールがつくる時空構造、近傍の物質分布を推定、合成することができる。読者をだますような表題をかかげたが、ここでいうX線写真とは、最新のX線データからつくったそのような合成像である。

ブラックホール時空の写真

活動銀河中心の近傍のガスはケプラー運動しながら円盤状に降着する。(降着円盤)。古典的時空、すなわちニュートニアン空間では、与えられた質点の周辺、いかなる半径でも安定なケプラー軌道が存在できる。即ちその質点に降着するガスはその質点まで連続する降着円盤が存在できる。しかし、ブラックホールが周辺がつくる相対論的時空ではその空間のゆがみのため、必ずしも安定なケプラー軌道が存在するわけだけではない。安定なケプラー軌道がどこまで存在するかはブラックホールがつくる時空構造による。もっとも単純な時空は球対称、かつ角運動量をもたない質量 M のブラックホールがつくる時空すなわちシュワルツシルド時空である。この場合は、シュワルツシルド半径 $R_s (=2GM/c^2)$ の3倍 ($3R_s$)より内側では安定な軌道は存在せず、ガスははげしく衝突しX線を発生しながら、自由落下にちかいスピードで R_s の内側、すなわちブラックホールの中に吸収される。従って半径 $3R_s$ から R_s の範囲では濃いガス円盤は存在しない。ブラックホールの近傍で生成されたX線の一部は周辺のガス吸収されて特性X線を再放出する。この効率もガスの密度と、X線発生源をみこむ立体角に比例するので、降着円盤からの特性X線は有為な強度になる。

降着円盤は高速回転しているのでもともと単色の特性X線は赤方偏移と背方偏移をうけた2つの巾をもった線スペクトルに変化する。後者(背方偏移)の発生源は手前に高速で動いているので、ドブラーブースティングをうけ、前者(赤方偏移)より強度が強くみえる。即ち、X線スペクトルに高いエネルギー側により強い強度をもつ2連のピーク構造がつくられる。特性X線発生源での重力ポテンシャルが深ければ深いほど(つまり r がブラックホールに近ければ近いほど)全体が更に低いエネルギー側に重力赤方偏移する。

観測されたX線スペクトルは以上の効果を総て含んだ結果である。そこで逆に観測データの解析から、どの領域から特性X線がでていいるかがわかり、その結果どこまで降着円盤が存在しているかを知ることができる。図2に実際の観測例、活動銀河MCG-3-15-30の特性X線スペクトルとそれを再現する降着円盤の幾何構造での計算結果をもしめす。この結果は降着円盤は数倍の R_s 付近迄は存在指していることを物語る。即ちシュワルツシルド時空で説明できるわけである。しかし、もっと観測を進めていくうちに、より制限が加えられる可能性がでてきた。多くの活動銀河にみられるようにMCG-3-15-30からのX線もまた激しく時間変動している。図3はそのX線の強度変動を示したものである。これに付随して、特性X線のスペクトル構造も激しく変化していた(図3)。もっとも強度の弱いときのスペクトルは2連のピーク構造が無くなっている。これはどのように理解できるであろうか? もし特性X線がよりブラックホール近傍

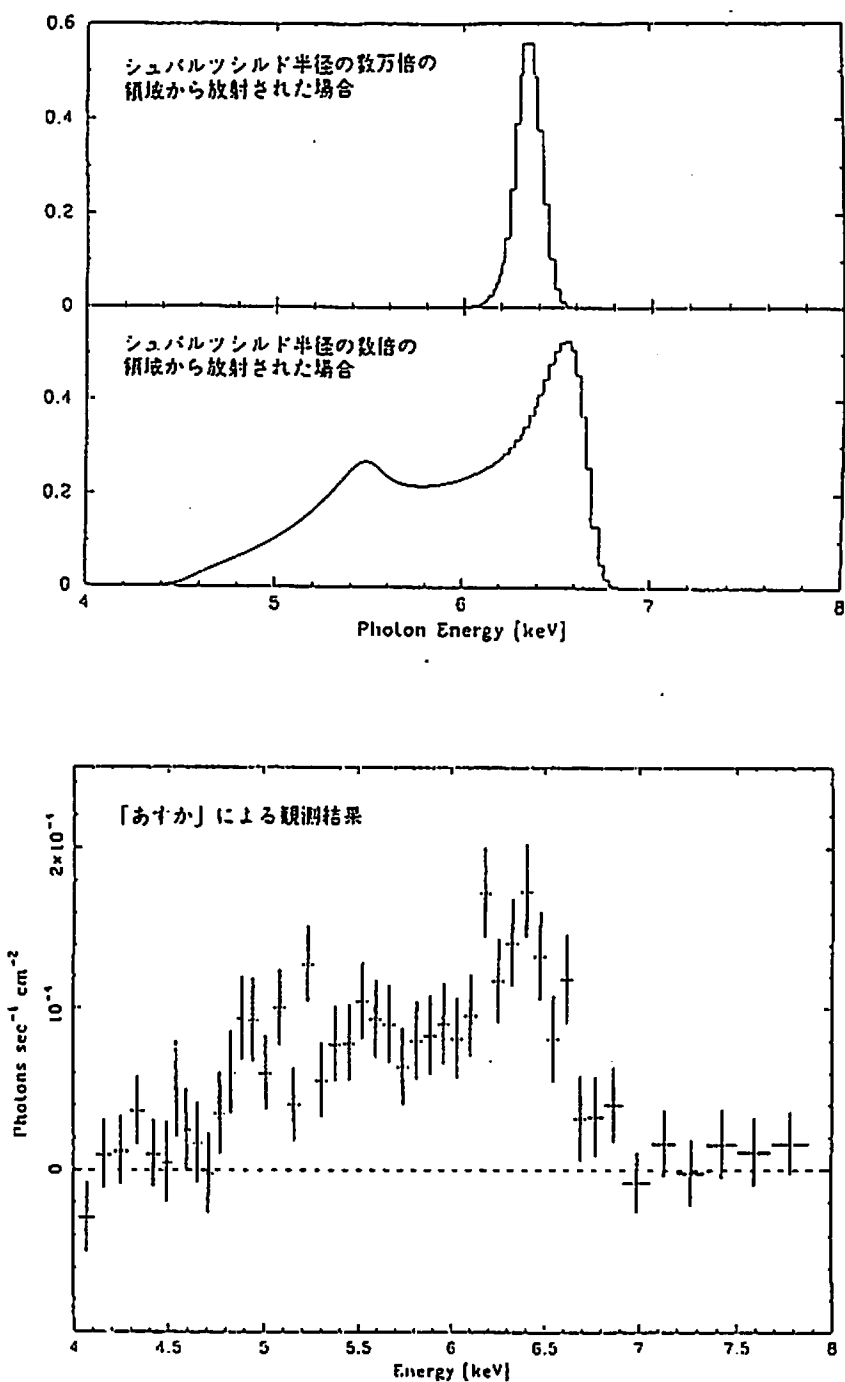


図2：活動銀河MCG-6-30-15から得られた、鉄の特性X線のエネルギー付近のX線スペクトル（下）と、回転している円盤から6.4keVの特性X線が発せられているときに予想されるX線スペクトル（上：特性X線の出ている場所が、ブラックホールの大きさ（シュバルツシルド半径）の数万倍の場所の場合と数倍の場所である場合の2つの予想スペクトルが示してある。）鉄の特性X線がシュバルツシルド半径の数倍の回転円盤から出ているときに観測されたスペクトルをよく再現できることがわかる。

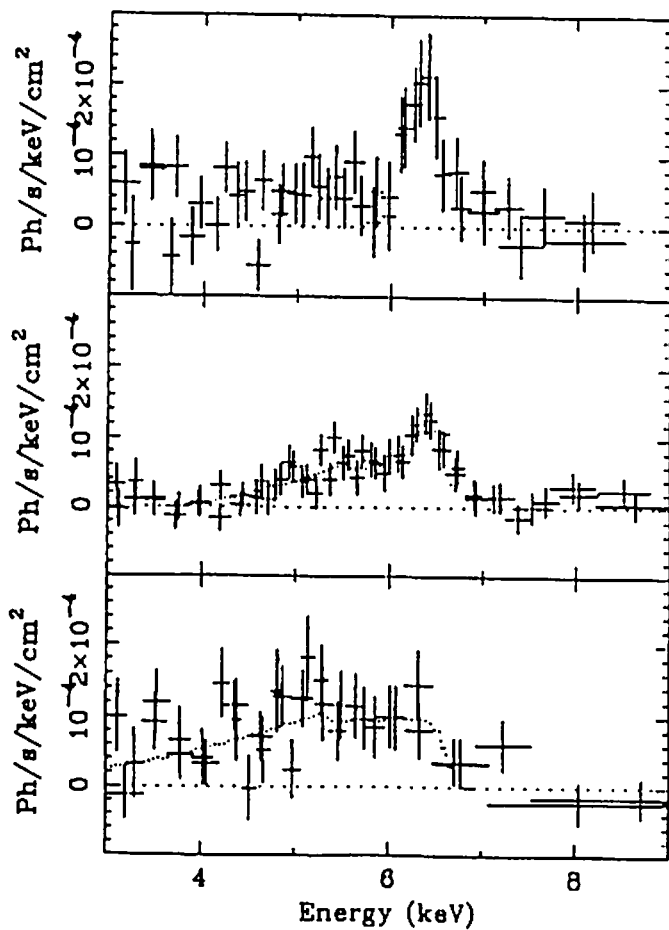
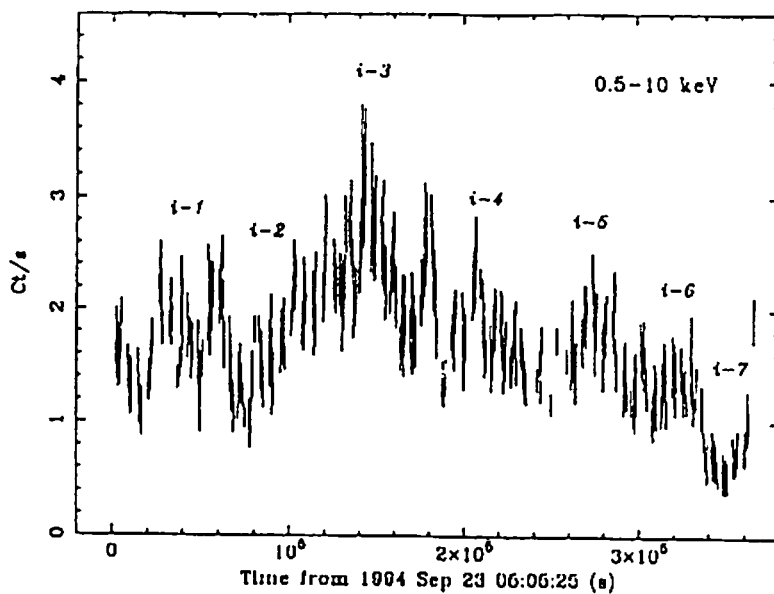


図 3 KCG-3-15-30のライトカーブ（上）とそれぞれの強度における特性X線スペクトル（下3）。もっとも強度の弱い時には鉄のピークが無くなっている（最下図）。これはカー・ブラックホール時空を示唆する。

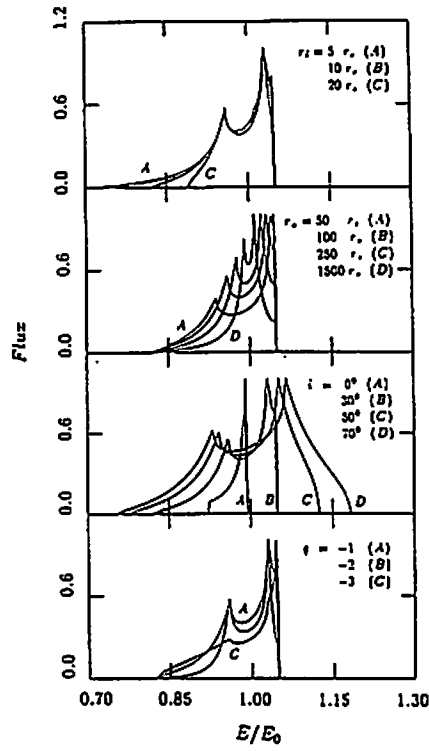


Figure 1. Line profiles computed using the method given in the Appendix. When not specified, the other parameters are fixed at $r_1 = 10 r_s$, $r_s = 100 r_s$, $i = 30^\circ$ and $q = -2$. Note that the blue horn is always brighter than the red one and a net redshift only occurs for low inclinations. The small spike on the high-energy side of the blue horn (see curves with $r_1 = 10 r_s$ and $i = 30^\circ$) is due to radiation emitted by the most blueshifted regions of the disk ($\sim 1.5 r_s$ for $i = 30^\circ$).

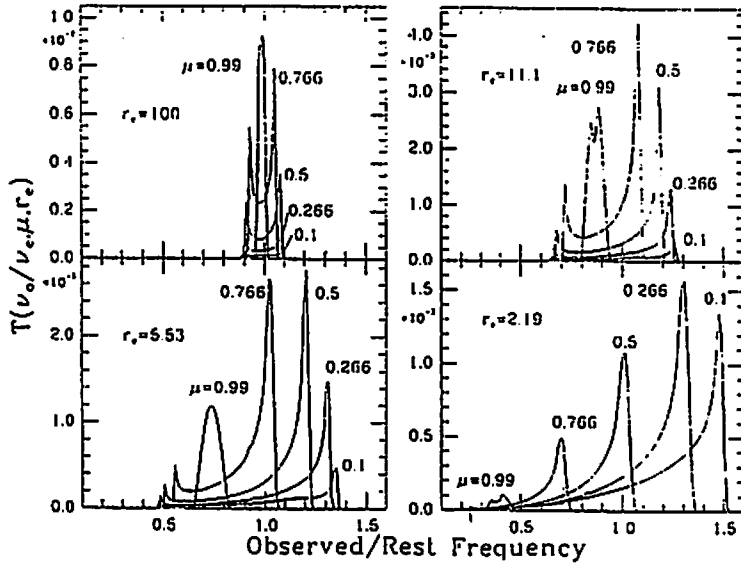


図4 相対論的降着円盤からの真銀構造のシミュレーション (Fabianらの計算)。

の円盤からでるなら、ドブラーブースティングの効果はより顕著になり、背方のみが顕著にある。また場所による視線速度の差が大きくなるので、線巾も広がる。一方全体はより大きな重力赤方偏移をおこす、その結果ダラーと広がった構造がでる。実際いくつかの例で計算された結果を図4にしめす。ブラックホールに近づくとつれて図3のような傾向がでてくるのがわかる。

ブラックホールはもともと物質が重力崩壊してできたものであり、そのとき程度の差こそあれその物質が角運動量を持ち込んだはずである。即ちより現実的な時空はシュワルツシルド時空でなく回転する（角運動量をもつ）ブラックホール（カーブラックホール）がつくる時空であろう（カー時空）。カー時空ではその角運動量の増加にともなって、安定なケプラー軌道は $3R_s$ から中にはいりこんでいき遂には R_s に到達する。つまり降着円盤もブラックホールの表面まで形成できる。図3と図4の比較から、MCG-3-15-30ではすくなくとも $1.5R_s$ まで降着円盤がのびる時があることを示唆する。このようなことはカー時空でなくてはできない。このようにして、特性X線観測結果の解析から、ブラックホール時空構造のX線写真というべきものが出来そうである。現在のところ残念ながらまだ観測例、統計がたりない。今後のデータの蓄積が期待される。

隠されたブラックホールのX線透過写真

可視光での振舞いから活動銀河は1型と2型に大別されている。前章でのべたMCG-3-15-30は1型の典型である。ここでは詳しくはのべないが、見かけ上（可視光でもX線でも）1型のほうがより活動的で2型は1型と通常の銀河との中間的な活動性をしめす。実は2型も1型とまったく同じであり、ただ2型銀河核は手前のガスで隠されているのみだという説がだされてきた。もしかくされているなら硬X線で見れば本体の1型が見えるかもしれない。なぜなら硬X線は圧倒的に物質透過力が強いからである。まさに医学でつかうX線写真そのものである。図5は2型銀河核NGC4507のX線スペクトルである。1型セイ銀河核のスペクトルとの大きな違いは、①低エネルギー側でX線強度が急激に減少すること、②鉄のK吸収端付近に顕著な吸収構造があること、そして③強くて巾の狭い鉄の特性X線の存在である。特徴①と②はブラックホールと我々の視線との間に濃い吸収ガスが存在することを意味する。まさに隠されたブラックホールだったのである。特徴③の狭い鉄輝線はそのガスがすくなくともあまりブラックホールの近傍ではないことを意味する（近傍だと前章でのべたよう複雑な形状になる）。また強い輝線はガスが大きな立体角でブラックホールを覆っていることをしめす。この結果と、従来からの光学観測など総合して描かれる像は図5（下）のようである。直径1pc程と推定されるドーナツ状のガスはいずれブラックホールに落ち込むガスの溜の役割をはたすのであろう。またドーナツの上下から覗くとブラックホールはそのまま見える。このような幾何構造の対象が1型活動銀河として統一的に理解できる。このような構造が活動銀河核に普遍的なものなら、ドーナツが覆う大きな立体角は隠されたブラックホールが裸のブラックホールよりたくさん存在することを示唆する。現在約1%の銀河が活動銀河と前に述べたが、隠されたブラックホールを考慮するとそれは数%以上に増えるだろう。

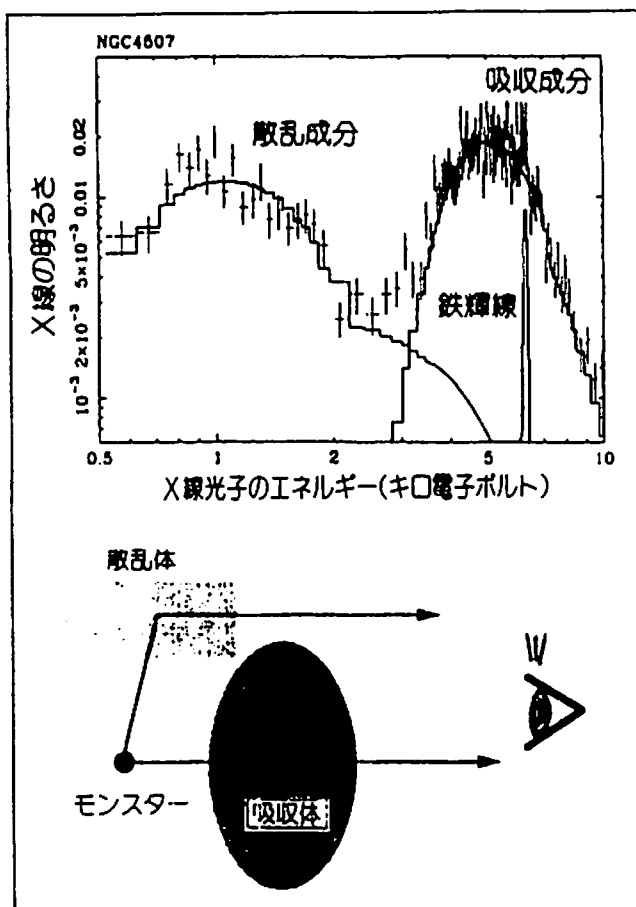


図 5、2 型銀河核 NGC4507 の X 線スペクトル。1 型セイ銀河核のスペクトルと異なり、低エネルギー側で X 線強度が急激に減少し、鉄の K 吸収端付近に顕著な吸収構造があり、また強くて巾の狭い鉄の特性 X 線が存在している。

天の川銀河中心のブラックホール

我々の銀河、天の川の中心は可視光では星間塵による減光でまったくみることができない。電波では明るい点源 (Sgr A*) がみつき、それが天の川中心天体と考えられている。しかし Sgr A* といっても活動銀核に比すれば随分電波強度は弱く、赤外線では天の川の他の領域よりはるかに暗い、従って天の川は到底活動銀河といえない。しかしこのことはブラックホールが存在しないことを意味するものではない。先にのべたように、もしガスの降着が現在なければ、外から見る限り、Sgr A* 付近は平穏な空間なのである。最近高空間分解能の赤外線観測が可能になり Sgr A* 周辺の星々の精度高い位置がきめられ、その固有運動が求まってきた。また赤外、電波分光から付近のガス雲のドブラー速度も観測されている。それらの速度はその天体より内側にある総質量をあらわすと考

えてよい。そこで横軸にSgrA*からの距離、縦軸にその距離より内部にふくまれる質量を太陽質量単位でしめたのが図6である。Sgr A*から0.1pcあたりよりなかで、質量が頭打ち ($2.5 \times 10^6 M_\odot$) になっているのは、まさに銀河中心にその質量が集中していることをしめす。巨大質量のブラックホールが存在することを示唆するもっとも重要な観測結果である。

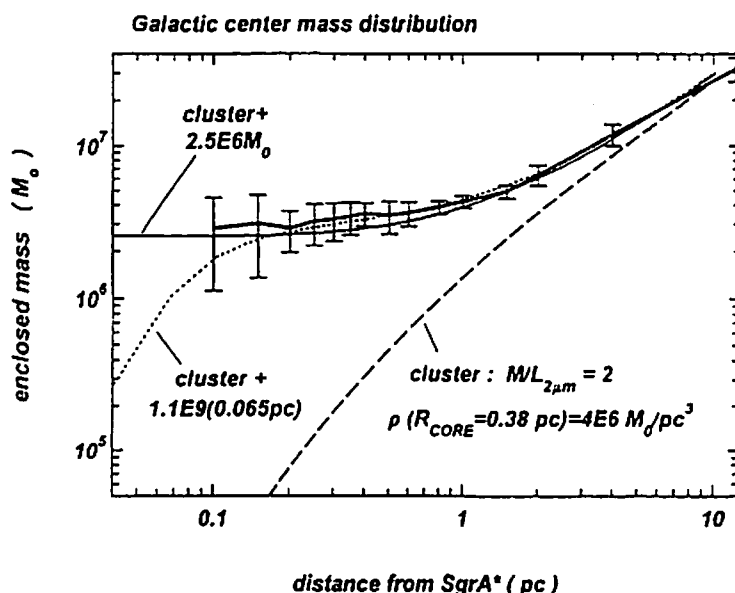


図6 銀河中心への質量集中をしめす (Mezger et al. より)。

“あすか”衛星は星間ガス透過して銀河中心のX線撮像を可能にした最初の衛星である。その“あすか”で撮った銀河中心付近約1平方度の広エネルギー帯域 (0.5-12 keV) のX線像を眺めると、Sgr A*付近はなにやら弱いボーとした像があるだけで、やはり活動銀河核ではないようだ。もちろんブラックホールの写真などとはおこがましくてとてもいえない。しかしその代わりに面白い事実を発見した。一つはSgr A*付近のボーとした像のスペクトルをみると高電離シリコン、硫黄、鉄の輝線がみつかったことである。これはこの領域が低密度 (ここは1-10/cc) の超高温のプラズマ雲 (銀河中心プラズマ) に覆われていることを示す。このようなプラズマは衝撃波加熱によって生成される。観測結果から、銀河中心プラズマは約1000前に 10^{51-52} erg程の爆発があったと推定される。一方約1平方度全体からのスペクトルも前記高電離輝線がみつかった (バルジプラズマ)。この場合は10万年程前の 10^{52-54} ergの爆発に相当する。不思議ことはこの超高温バルジプラズマ中に低電離鉄輝線がみつかったことである (図7の矢印)。図8にその低電離鉄輝線の空間分布を示した。低電離鉄輝

線は巨大分子雲(Sgr B2)から特に強くでていた。分子雲中の中性の鉄が強いX線放射場で蛍光X線を出したものと結論できる。しかし観測からはそのような強いX線放射場、それをつくる強いX線源は見あたらない。従って我々の視野方向からは遮蔽され見えないが、分子雲のほうには放射される強いX線天体が存在する可能性(2型活動銀河)がある。しかしこのような隠された活動銀河核があれば、そのX線を吸収して、強い赤外放射を起こすはずであるが、それも見あたらない。実はSgr A*から分子雲までの実距離は約300光年である。一方銀河中心プラズマを膨張速度(ほぼ音速)で逆に300年の過去に遡っていくとその時のX線強度は充分現在の低電離鉄輝線を生成するに足るほど明るくなる。即ちごく最近(約300年まえ)に銀河中心に活動性があったことを示唆する。バルジプラズマは10万年程昔、その他電波の膨張雲からは100万年程まえにやはり大爆発があったことを示唆している。これらはなんらかの外乱によって銀河中心の $2.5 \times 10^6 M$ ブラックホールにガスが間欠的に落ち込んだ結果の巨大なエネルギー放出したのであろう。

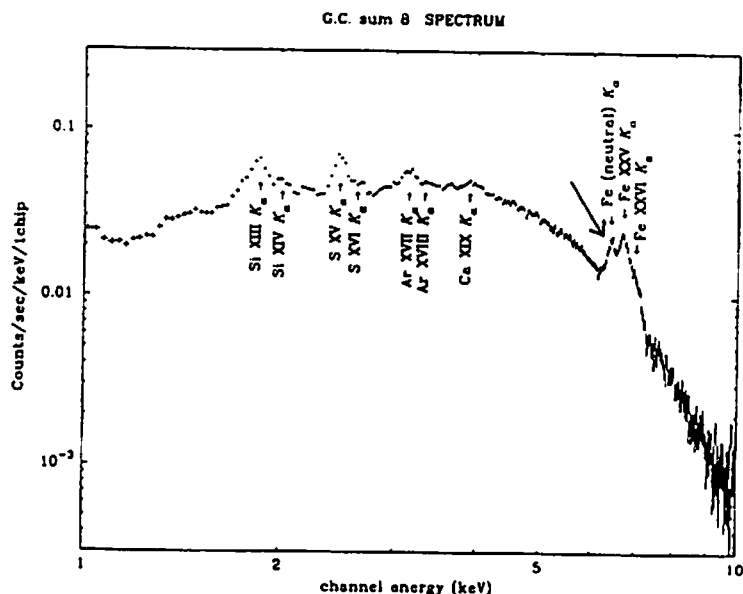


図7 超高温バルジプラズマのスペクトルと、低電離鉄輝線(矢印)。低電離鉄輝線放射は分子雲のような濃い低温ガスと強いX線放射場の両者の存在を要求する。

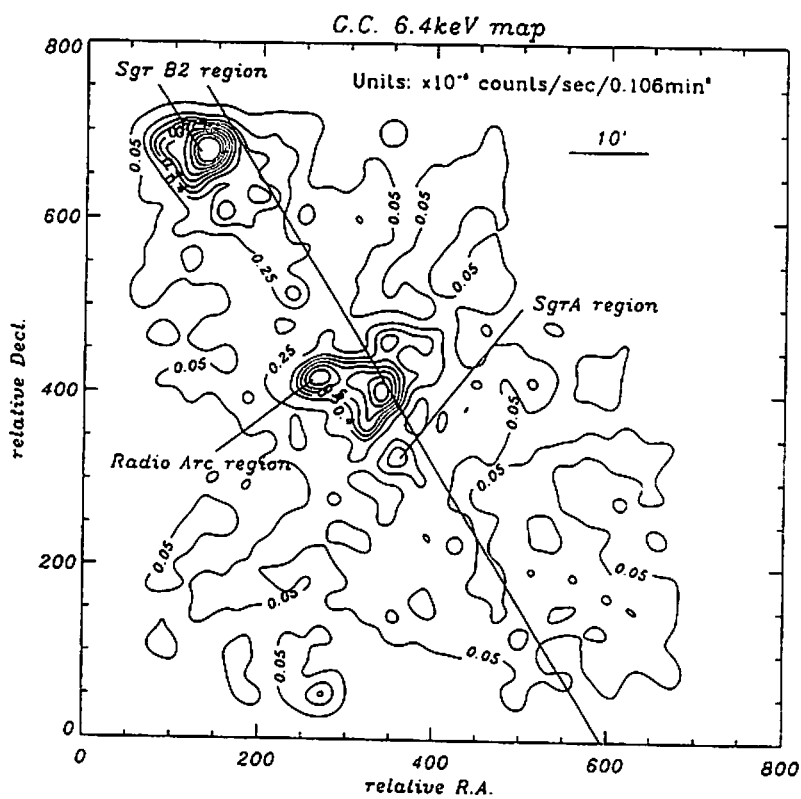


図8 低電離鉄輝線(6.4 keV)の空間分布。最も低電離鉄輝線が強い場所は巨大分子雲(Sgr B2)である。

