

**Proceedings of the Third Workshop
on
General Relativity and Gravitation**

Sanjo Conference Hall, The University of Tokyo

January 17-20, 1994

**Editors: K. Maeda
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T. Futamase
H. Ishihara
Y. Kojima
S. Yamada**

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on

General Relativity and Gravitation

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Volume 2: Gravitation
Volume 3: Gravitation
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PREFACE

The Third Workshop on General Relativity and Gravitation was held at Sanjo Conference Hall (University of Tokyo) from January 17 to 20, 1994. The main purpose of this series of workshops is to review the latest observational and theoretical work in gravitational physics and relativistic astrophysics. A further purpose is to promote lively and stimulating interactions between researchers working in fields related to relativity and gravity – including particle physics, astrophysics, and cosmology.

155 researchers participated from the above many fields, and 64 speakers gave 12 invited talks and 53 short contributions. The numbers of participants and talks are increasing every year since the first workshop in December, 1990. Hence, this year, the workshop was held for four days, but we are afraid that time for discussion might not be enough.

The workshop was supported by a Grant-in-Aid for Scientific Research on Priority Areas (No.05218101) from the Japanese Ministry of Education, Science and Culture. This proceedings volume was also partially supported by a Grant-in-Aid for Scientific Research on Priority Areas “Gravitational Wave Astronomy”(No.04234105). Although the Scientific Research on Priority Areas “Gravitational Wave Astronomy”, which was the financial base of this workshop, will finish this year, we may start next advanced project on “Gravitational Wave Astronomy”. We think that we should continue this series of workshops on general relativity and the related topics.

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THE X-RAY ASTRONOMY SATELLITE "ASCA" AND ITS EARLY RESULTS

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ASCA (ASTRO-D), the fourth X-ray astronomy satellite of ISAS, was successfully launched on February 20, 1993. ASCA is a high-throughput imaging and spectroscopic X-ray observatory with nested thin-foil X-ray mirrors and a set of CCD cameras and imaging gas scintillation proportional counters on the focal plane. The key elements are working well in orbit and many observations have already been done. Some of the early results are shown.

1. Introduction

The Institute of Space and Astronautical Science (ISAS) has so far launched four X-ray astronomy satellites. The first small X-ray astronomy satellite Hakucho (Kondo et al. 1981) was launched in 1979. Tenma (Tanaka et al. 1984), carrying for the first time the large-area gas scintillation proportional counters (Koyama et al. 1984), was launched four years later (1983). The third X-ray astronomy observatory Ginga (Makino et al. 1987) launched in early 1987 carried a large-area proportional counter array jointly prepared by the Japanese and the U.K. groups (Turner et al. 1989).

ASTRO-D, the fourth X-ray astronomy mission of ISAS, was launched on February 20, 1993, by the ISAS vehicle M-3SII into an approximately circular orbit of a 520 - 620 km altitude and was named ASCA. The Japanese characters for ASCA mean a "flying bird" and it is also an acronym of an "advanced satellite for cosmology and astrophysics". The schematic view of ASCA in orbit is shown in Fig.1. The spacecraft weighs 417 kg and is three-axis stabilized. The absolute pointing accuracy is approximately 10 arcseconds.

ASCA is designed to be a high-capability X-ray observatory. It is equipped with nested thin-foil mirrors which provide a large effective area over a wide energy range from below 1 keV up to 12 keV. Two different types of detectors, CCD cameras and imaging gas scintillation proportional counters (IGSPC), are employed for the focal plane instruments.

The key elements of ASCA were successfully turned on in orbit. The performance verifications and the test observations were carried out until October 1993 and the guest observations started after then.

In the guest observation phase, observing time on ASCA opens to competitive proposals. The observing time, after reserving its approximately 5% as an "Observatory Time" for recalibrations/readjustments of the instruments or spacecraft subsystems, is apportioned as follows:

60% for Japanese investigations,

15% for U.S. investigations

25% for Japan/U.S. collaborative investigations.

ASCA operations are managed mainly by Japanese astronomers at ISAS. They schedule observations, direct the satellite, collect the data and monitor the health of the spacecraft and its payload. Direct contact between the satellite and the ground stations is currently made for 10 of the 15 orbits per day on the average, 5 from ISAS Kagoshima Space Center and 5 from NASA Deep Space Network Stations (Madrid, Canberra and Goldstone). All the data from ASCA

are, together with satellite attitude and orbital information, stored at ISAS and are distributed to Japanese investigators. They are also sent to the U.S. ASCA Guest Observer Facility at the Goddard Space Flight Center and are distributed to U.S. guest observers.

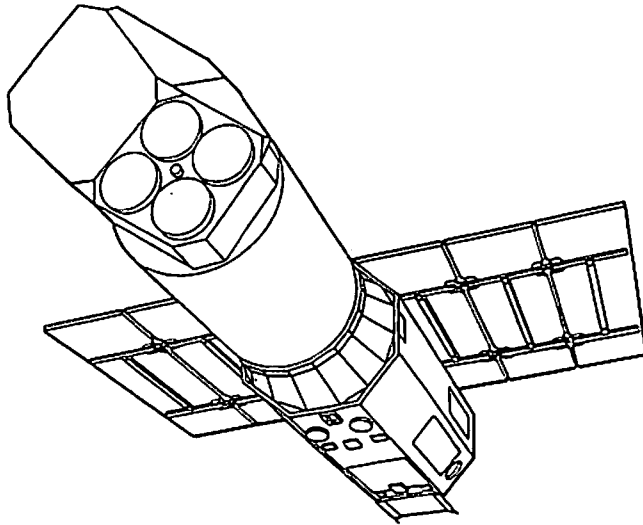


Fig.1 Schematic view of ASCA in orbit.

2. High Throughput X-ray Observatory

ASCA is a high-throughput X-ray observatory. The elements composing the observatory are schematically shown in Fig.2. The focal length is 3.5m. Four identical mirrors (XRT) are mounted on the top plate of the extendible optical bench (EOB), while the focal plane detectors, 2 units of CCD cameras (SIS) and 2 units of IGSPC's (GIS), are placed on the base plate. The detectors operate independently and simultaneously.

2.1 X-ray telescopes

The X-ray telescope consists of four sets of nested thin-foil conical mirrors. This technology is developed by Serlemitsos (1981). The mirror system of ASCA is prepared by a collaboration between P.Serlemitsos and his collaborators at NASA Goddard Space Flight Center and the Nagoya University/ISAS group. This technology can provide a large effective area over a wide energy range with a very light weight. The total on-axis effective area for four sets of mirrors of ASCA is 1500 cm^2 at 1 keV and 600 cm^2 at 7 keV.

2.2 Solid-state Imaging Spectrometer (SIS)

The CCD cameras have been prepared jointly by G.Ricker and his collaborators at MIT and the Osaka University/ISAS team. The Penn State University group has been also involved in the software development. A CCD camera head consists of the Lincoln Laboratory 840 x 840 hybrid CCD of front-side illumination (four 420×420 CCD chips abutted side by side, each read out by a separate preamplifier). They are operated at a modest temperature of about -60° C . The field of view coverage is 23×23 arcminutes². They are characterized by their good energy resolution. It is about 2% for 5.9 keV (Mn-K) X-rays.

2.3 Gas Imaging Spectrometer (GIS)

Each IGSPC consists of a gas chamber filled with Xe and sealed off with a thin beryllium window (10 micron), and a position-sensitive photo-multiplier. An energy resolution of 8% and a position resolution of 0.5 mm have been achieved for 5.9 keV photons. They cover the entire field of view of the telescope (larger than the coverage of the CCD camera), and have also a larger detection efficiency for higher energy photons above 5 keV than the CCD camera.

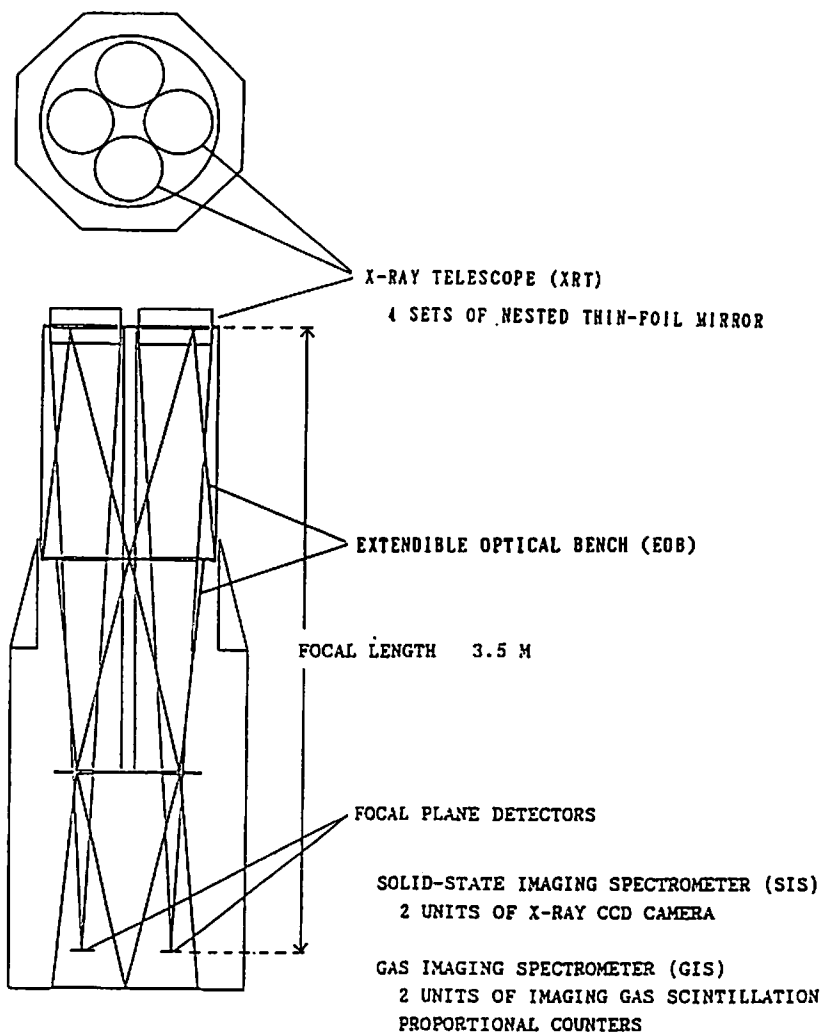


Fig.2 Elements composing a high throughput X-ray observatory on board ASCA.

3. Early results

3.1 X-rays from SN1993J

The supernova SN1993J appeared in the nearby galaxy M81 about 40 days after the launch of ASCA. This was similar to the case of Ginga in which the supernova SN1987A appeared in the Large Magellanic Cloud only 18 days after the launch. ASCA was pointed to the direction

of M81 about 1 week after the discovery and detected X-rays from this supernova (Tanaka et al. 1993) almost simultaneously with ROSAT (Zimmermann et al. 1993; 1994). This is the first detection of X-rays only about a week after the explosion.

The X-ray image obtained with ASCA is shown in Fig.3. The image size of the X-ray telescope on board ASCA is approximately 3 arcminutes half-power-diameter. However, the strongly peaked point spread function allow us to separate two X-ray sources with angular distance as close as 1 arcminute from each other, as shown in Fig.3. An X-ray source at the position of the supernova, SN1993J, is clearly distinguished from an X-ray source (see Fabbiano 1988) 1 arcminute separate from the supernova.

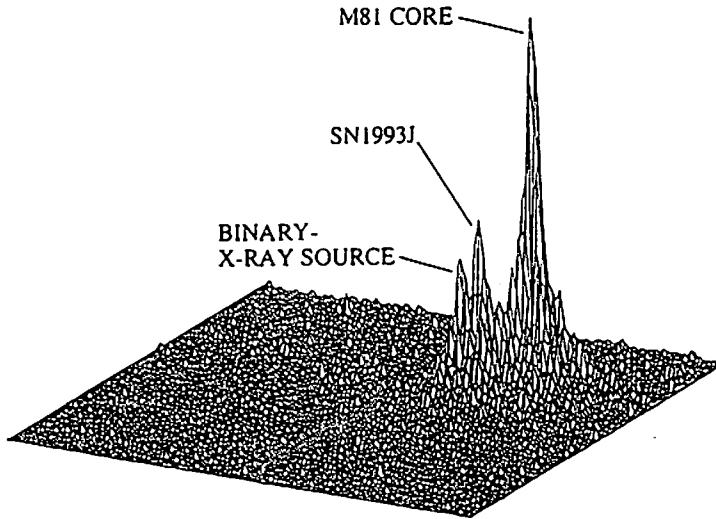


Fig.3 The X-ray intensity distribution taken with the SIS detector from the central part of M81 including SN1993J.

ASCA observed SN1993J seven times from April 5 to October 24 in 1993 (Kohmura 1994; Kohmura et al. 1994). The 1-10 keV luminosity was $1.2 \pm 0.3 \times 10^{40}$ erg s⁻¹ on 10 days after the explosion and decreased to $1.0 \pm 0.2 \times 10^{39}$ erg s⁻¹ on 210 days after the explosion. During this period, the spectral softening was clearly seen. The spectrum obtained from a sky region including SN1993J and the nearby X-ray source exhibits a prominent iron emission line and the correlation between the line flux with the SN1993J flux suggests that the iron emission line probably came from SN1993J. These observational results can be understood in the context of thermal X-ray emission from hot region behind the front shock due to the collision of the supernova ejecta with the pre-supernova stellar wind.

3.2 Prominent emission lines from young supernova remnants

Heavy elements in the universe are created in the cores of evolved massive stars and scattered into interstellar space through supernova explosions. X-ray spectral observations of supernova remnants (SNRs) reveal how the heavy elements are transported from the core of the star to the interstellar space via supernova explosions.

The CCD camera on board ASCA has an outstanding spectral resolution and has enabled us to make the finer spectroscopy of the SNRs than the previous experiments did. Figure 4 is an X-ray spectrum obtained with the CCD camera from the supernova remnant W49B

about a thousand years after the explosion (Fujimoto et al. 1994). The K_{α} line and K_{β} lines from helium-like ions and the K_{α} line from hydrogen-like ions of Si and S, the K_{α} line from helium-like ions and the K_{α} line from hydrogen-like ions of Ar, Ca, and the K_{α} line from hydrogen-like ions of Fe are clearly distinguished.

The X-ray images for the silicon-line photons and the iron-line photons are shown in Fig. 5. A remarkable finding is that the hot plasma emitting the iron lines fills the center of the remnant while that emitting the silicon lines seem to have a shell like structure around the central core. This may be due to the stratification of the heavy elements in the supernova ejecta as expected in the type II supernova.

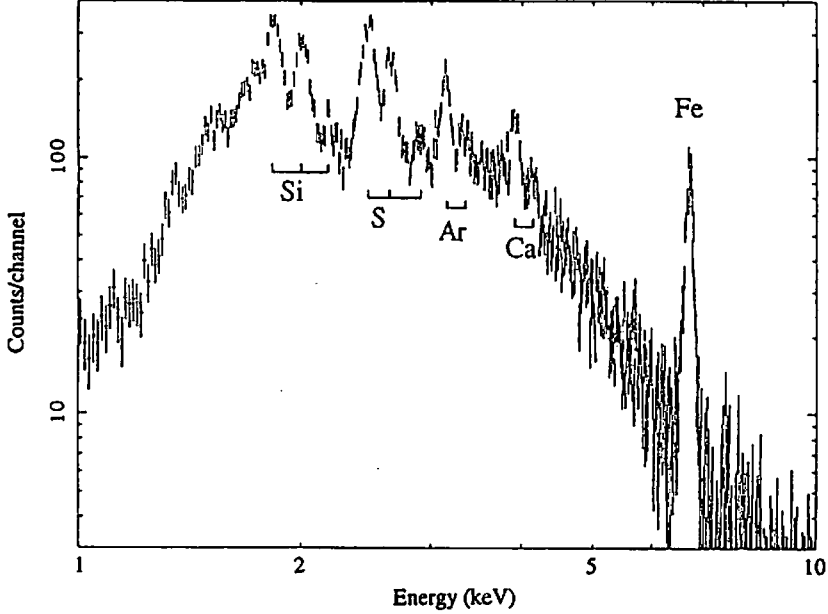


Fig.4. X-ray spectrum of the supernova remnant W49B obtained with the SIS detector on board ASCA.

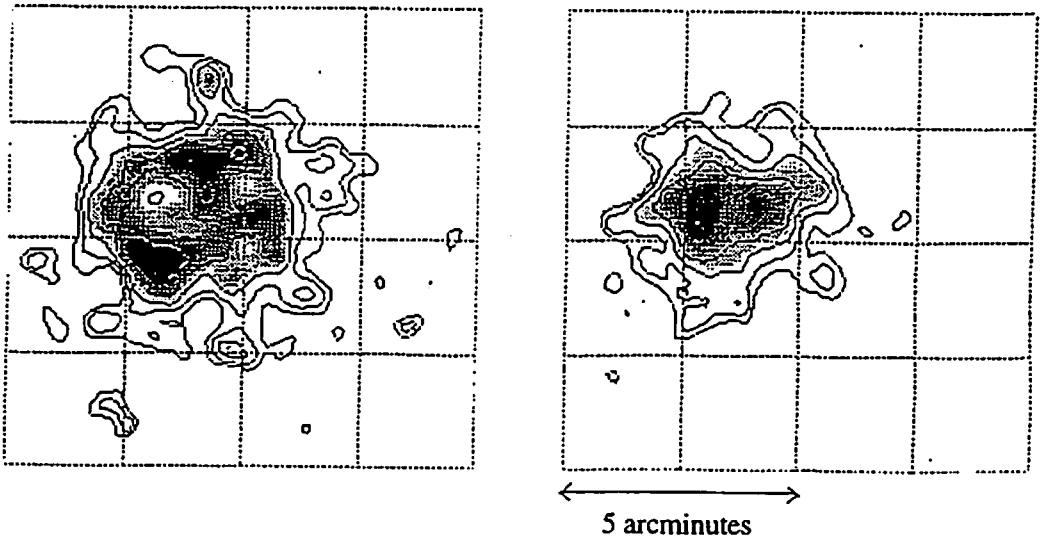


Fig.5 Images of silicon-line photons (left panel) and iron line photons (right panel) from W49B.

3.3 Deep survey of weak sources

With the instruments described above, ASCA is the first X-ray astronomy satellite capable of imaging the X-ray sky in the wide energy range over 0.5 to 12 keV. The sensitivity of ASCA is much better than that of non-imaging instruments such as the large area counters on board Ginga. In Fig.6, the expected detection limit of ASCA is plotted on the log N - log S relation in the 2-10 keV range. This high sensitivity of ASCA has roughly been confirmed in orbit and has enabled us to observe much fainter sources than the limit of previous missions.

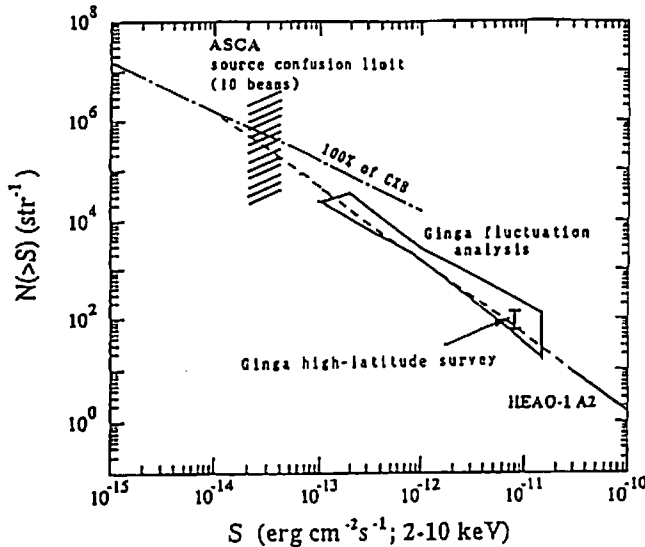


Fig.6 Expected sensitivity of ASCA plotted on the log N-log S plane for X-ray sources in the 2-10 keV range. The log N-log S relations obtained by the Ginga high latitude survey (Kondo 1991) and the Ginga fluctuation analysis (Hayashida 1990) are shown. The Ginga results are consistent with the $N \propto S^{-3/2}$ relation (the dashed line) extrapolated from the HEAO-1 A2 result (Piccinotti et al. 1982).

One of the most important scientific objectives of ASCA is to resolve the origin of the cosmic X-ray background (CXB). The origin of the extragalactic isotropic component of the X-ray background is still not fully solved almost 30 years after its discovery (Giacconi et al. 1962).

A simple solution is that the observed background flux is supplied by a collection of unresolved, weak point sources. The number-flux relation in the 2-10 keV range was estimated with Ginga for the 10^{-13} - 10^{-11} erg cm⁻² s⁻¹ flux level (Hayashida 1990; Kondo 1991) as shown in Fig.6. If the $N \propto S^{-3/2}$ relation is extrapolated down to 10^{-14} erg cm⁻² s⁻¹, the CXB flux can be explained as the collection of weak point sources. ASCA has the capability to resolve such sources down close to 10^{-14} erg cm⁻² s⁻¹.

AGNs are the largest contributor to the CXB among sources with flux levels 10^{-13} - 10^{-12} erg cm⁻² s⁻¹. The average spectral slope of AGNs in the flux range around 10^{-12} erg cm⁻² s⁻¹ (Hayashida 1990) is, however, 0.7 - 0.8, while the spectral slope of the CXB is 0.4 in the 2 - 10 keV range. Hence, sources which have the spectral slope as small as or smaller than 0.4 should more often be observed in the flux range around 10^{-13} erg cm⁻² s⁻¹ than in the higher flux range. In fact, one of two sources with flux of the order of 10^{-13} erg cm⁻² s⁻¹ found in the deep survey observation at the North Ecliptic Pole is brighter in the higher energy band as shown in Fig.7 (Makishima et al. 1994). This may be a typical example for

the common presence of the hard source in this flux range, although the systematic survey of serendipitous sources in this flux range is still under progress for many pointing observations performed so far.

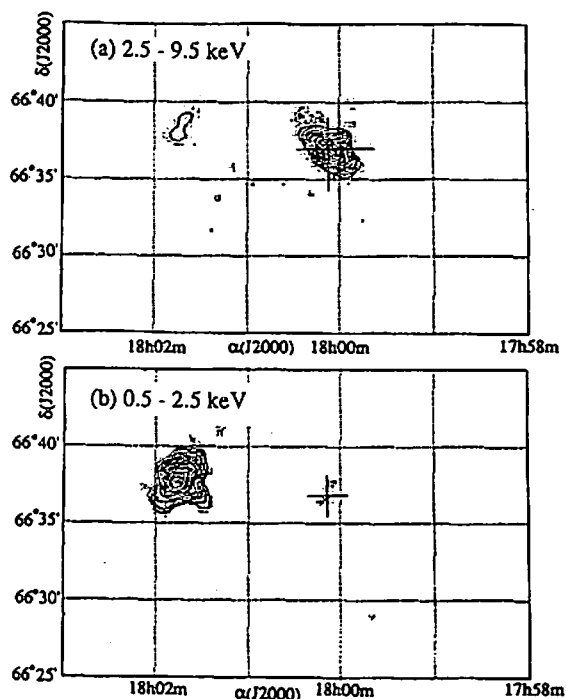


Fig.7 X-ray images of the North Ecliptic Pole region taken with the GIS on March 23-25, displayed on the J2000 sky coordinate. The harder source has been identified with the Seyfert 2 galaxy NGC6552 and its optical location is indicated with the cross. Panel (a) is in the 2.5 - 9.5 keV range, while panel (b) is in the 0.5 - 2.5 keV range. The source images have been smoothed but appear somewhat elongated due to the non axi-symmetric point spread function of the XRT.

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Cosmological Implications of X-Ray Emission from Clusters of Galaxies

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Abstract

Three topics related to hot gas in clusters of galaxies are discussed. First, I discuss the mass determination of various components of clusters and its cosmological implications. Second, I discuss the determination of the Hubble constant using the Sunyaev-Zeldovich effect, emphasizing uncertainties caused by the kinematic effect. Finally, I discuss on the origin of iron in the hot gas, where I point out that the uncertainty of the Hubble constant greatly influences the interpretation of X-ray observations.

1. Introduction

X-ray emission from clusters of galaxies was discovered more than 20 years ago, and has proved to be a strong cosmological tool. Soon after its discovery, it was established that the X-ray emission is due to thermal bremsstrahlung emission from a hot gas. The hot gas has been used to determine the mass of various components in clusters of galaxies. In mid 1970s, iron line emission was identified in the X-ray spectrum, which means that the gas includes at least a processed component ejected from stars in member galaxies. The hot gas has become an important probe of the chemical evolution of galaxies. Although much work has been done on these two topics, it is fairly surprising that most papers assumed the value of the Hubble constant to be 50km/s/Mpc in contrast to a continuing dispute over its value in other fields of observational cosmology. In this talk, I emphasize the importance of uncertainty of the Hubble constant in this field.

As soon as the discovery of X-ray emission from clusters, it was also recognized that the hot gas distorts the spectrum of the cosmic microwave background radiation by Compton scattering, which is called Sunyaev-Zeldovich effect. Most important point is that this distortion, combined with X-ray emission, provides a direct method of determining the Hubble constant. Although there are fairly large uncertainties, the Hubble constant estimated by this method has been relatively low, sometimes too low, compared with other standard determinations. In this talk, I emphasize the importance of a possible bias caused by the kinematic effect.

2. Mass Determination

The X-ray luminosity of hot gas in a cluster of galaxies is given by

$$L_X = \int \epsilon_0 n_e^2 T_e^{1/2} dV = 4\pi d^2 F_{obs}, \quad (1)$$

where N_e and T_e are the number density and temperature of electrons, and ϵ_0 is the coefficient of bremsstrahlung emission. The second equality represents the relation of the luminosity and the observed flux F_{obs} through the distance to the cluster d . Observations provide T_e and F_{obs} as well as the characteristic angular radius θ ; the characteristic radius of the gas distribution R is estimated by $R = d\theta$. Note that d and R are inversely proportional to h , where the Hubble constant is written as $H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$. Thus, we obtain $n_e \propto h^{1/2}$ and the dependence of the gas mass on h as

$$M_{gas} \propto n_e R^3 \propto h^{-5/2}. \quad (2)$$

On the other hand, stellar mass in member galaxies is estimated by multiplying the observed luminosity L_* with the average mass to light ratio $\langle M/L \rangle$. Since $L_* \propto h^{-2}$ and $\langle M/L \rangle \propto h$, we have the dependence of the luminous mass (total stellar mass in a cluster) on h as $M_* \propto h^{-1}$. The classical missing mass problem of clusters of galaxies is that the dynamical mass, which is required to gravitationally bind member galaxies with a random velocity of an order of 1000 km/s, greatly exceeds the luminous mass. The dynamical mass M_{dyn} is classically estimated by using the virial theorem as

$$M_{dyn} = 3\sigma_*^2 \langle R \rangle / G, \quad (3)$$

where σ_* is the one-dimensional velocity dispersion of galaxies and $\langle R \rangle$ is a suitably defined mean distance between galaxies. Note that $M_{dyn} \propto h^{-1}$. Since the estimate of

$\langle R \rangle$ is fairly uncertain, the value of M_{dyn} has been determined only roughly.

In this respect, X-ray imaging observations can provide a more reliable method of the determination of M_{dyn} . If we assume spherical symmetry, the equation of the hydrostatic equilibrium is given by

$$\frac{1}{\rho_{gas}} \frac{\partial P}{\partial r} = \frac{kT_e}{\mu m_p} \frac{\partial \ln \rho_{gas}}{\partial \ln r} \frac{1}{r} = -\frac{GM_r}{r^2}, \quad (4)$$

where r and M_r are the radial distance and the dynamical mass within r , respectively, and in the second equality we assumed that the gas is isothermal. Most observations to date have suggested that $\rho_{gas} \propto r^{-2}$, which leads to

$$M_r = \frac{2kT_e}{\mu m_p G} r. \quad (5)$$

This distribution seems to be a generic feature of the dark matter distribution.

Although much work has been done on this issue and several problems remain to be solved (e.g. Sarazin 1986; Bahcall 1993), I do not go into such details. Instead I just quote the recent result by White et al. (1993) using ROSAT observations of the Coma cluster. Their results are $M_* h = 1.0 \times 10^{13} M_\odot$, $M_{gas} h^{5/2} = 5.5 \times 10^{13} M_\odot$, and $M_{dyn} h = 10^{15} M_\odot$ within $rh = 1.5 \text{ Mpc}$. The ratio of the total baryon mass to the dynamical mass turns out to be around $0.01 \pm 0.055 h^{-3/2}$. Since the gas mass tends to increase with r , the second term may be somewhat larger. If we identify this ratio with the global ratio in the universe $\Omega_{baryon}/\Omega_{total}$ and if we adopt $\Omega_{baryon} h^2 = 0.015$ which is determined by the cosmological nucleosynthesis, we obtain $\Omega_{total} h^{1/2} \approx 0.2 \sim 0.3$. So the universe should be open unless the cosmological constant is invoked.

This argument assumes that no segregation has occurred between baryons and dark matter in the formation process of clusters of galaxies, which seems to be a good assumption for this scale. I hope that existing and future ASCA observations will provide more definitive data on this point.

3. Sunyaev-Zeldovich Effect

Hot gas in clusters Compton scatters cosmic microwave background photons, which results in a spectral distortion in the direction of the cluster. The degree of the distortion is given by

$$\frac{\delta T}{T} = [x \frac{e^x + 1}{e^x - 1} - 4] y, \quad (6)$$

where T is the radiation temperature, $x = h\nu/kT$ and y is the Compton parameter which is defined by

$$y = \int \sigma_T \frac{kT_e}{m_e c^2} n_e dl, \quad (7)$$

where σ_T is the Thomson cross section.

The above formula predicts that $\delta T/T = 0$ at 219GHz, that at lower frequencies

$$\frac{\delta T}{T} = -2y \quad (8)$$

which causes a diminution of the intensity of the cosmic microwave background and that at higher frequencies

$$\frac{\delta T}{T} = x(x - 4)y \quad (9)$$

which causes a submillimeter excess.

From observations we can estimate y , and if we combine this with X-ray observations, n_e and d (or R) can be determined independent of the Hubble constant. We have

$$n_e \propto \frac{F_{obs} T_e^{1/2}}{y \theta^2} \quad (10)$$

and

$$d \propto \frac{y^2 \theta}{F_{obs} T_e^{3/2}}. \quad (11)$$

Thus, we can determine the distance to the cluster directly, which makes it possible to determine the Hubble constant using the known recession velocity.

The value of the Hubble constant determined by this method is rather low (e.g., Birkinshaw and Hughes 1994) although many observational uncertainties have been pointed out. However, recent developments both in radio and X-ray observations make this method promising and worth of further theoretical attention. Here, I point out a possible importance of the kinematic effect due to the peculiar motion of the clusters. This effect is explained as follows; if the cluster has a positive peculiar velocity, scattered photons are redshifted, which results in a temperature decrease. In contrast, if the cluster has a negative peculiar velocity, scattered photons are blueshifted, which results in a temperature increase. The kinematic Sunyaev-Zeldovich effect gives rise to a temperature change of the cosmic microwave background, while keeping the shape of

a black body spectrum. When we discriminate the kinematic effect from the usual Sunyaev-Zeldovich effect, the latter is called the thermal Sunyaev-Zeldovich effect.

For the kinematic effect, we have

$$\frac{\delta T}{T} = -\frac{v_{pec}}{c} \sigma_T \int n_e dl, \quad (12)$$

where v_{pec} is the peculiar velocity of the cluster. In the Rayleigh-Jeans region where most detailed observations have been made, the observed decrement has contributions both from thermal and kinematic effects as

$$\frac{\delta T}{T} = -(2 + \frac{v_{pec}}{c} \frac{m_e c^2}{kT_e})y. \quad (13)$$

Thus the kinematic effect tends to decrease the value of y for a given observed value of $\delta T/T$. More quantitatively, the modification depends on the value of the peculiar velocity. If v_{pec} is as large as 1000km/s for $kT_e = 5\text{keV}$, y is by 17 percent smaller than and the Hubble constant is by 37 percent larger than that obtained without due consideration of the kinematic effect.

It should be also noted that if a large amount of cool ionized gas is associated with the cluster, it does not contribute to X-ray emission but does to the kinematic Sunyaev-Zeldovich effect. In this case, The true Hubble constant is further increased. For negative peculiar velocity, the true Hubble constant becomes smaller than that obtained without consideration of the kinematic effect.

My point of view is that the determination of the Hubble constant may suffer from kinematic effects if the number of observed clusters is too small. To discriminate the kinematic effect from the thermal one, accurate submillimeter spectral observations are needed, which seems to be difficult. It is more plausible to observe many clusters and take a statistical average to remove the kinematic effect. Once the true value of the Hubble constant is determined with sufficient accuracy, we can determine the peculiar velocity of clusters, which will exert a tremendous impact on the theory of structure formation.

4. Origin of Iron

As was mentioned in the Introduction, the origin of iron has been one of the most important problems in astrophysics of clusters of galaxies. The amount of iron in the hot gas is estimated by the strength of emission lines at 6.7keV. The abundance of iron

is determined independent of the Hubble constant and the average ratio of iron mass to gas mass turns out to be

$$\frac{M_{Fe}}{M_{gas}} = 5.7 \times 10^{-4} = 0.36 X_{Fe\odot}. \quad (14)$$

The average ratio of gas mass to the total stellar mass is

$$\frac{M_{gas}}{M_*} = 1.3 h^{-3/2}. \quad (15)$$

Although rich clusters have smaller iron abundance and higher gas fraction, a striking fact is that the ratio of iron mass to the total stellar mass is independent of the cluster richness:

$$\frac{M_{Fe}}{M_*} = 0.73 \times 10^{-3} h^{-3/2}. \quad (16)$$

Traditionally, community of X-ray astronomy has assumed $h = 0.5$, paying little attention to the uncertainty of the Hubble constant, and has taken that this ratio is 2×10^{-3} (Tsuru 1992; Arnaud et al. 1992); this value is much larger than that for the interstellar gas in our galaxy for which $M_{Fe}/M_* = 10^{-4}$. Even if we take account of the iron mass in stars, it seems to be a serious challenge for chemical evolution model of cluster galaxies to produce such a large amount of iron. But, this ratio actually depends on the Hubble constant and most problems are significantly alleviated if we adopt a higher value of the Hubble constant.

In our recent work (Mihara and Takahara 1994) we investigated this problem in detail and showed that a correct amount of iron can be produced by elliptical galaxies through type Ia supernovae if we adopt $h = 0.7 \sim 0.8$ even for conventional initial mass function. At the same time, we have predicted that stellar metallicity is lower than that previously believed and that the abundance of α -elements in the hot gas such as oxygen, magnesium and sulfur is underabundant compared with iron. Existing and future ASCA observations will provide an important clue to the chemical evolution of elliptical galaxies.

5. Conclusions

In this talk, I discussed cosmological implications of hot X-ray emitting gas in clusters of galaxies. It is used to determine the mass of various components of clusters and even the cosmological density parameter. At present, the observations favor

$\Omega_{total} \approx 0.2$. Combined with observations of the Sunyaev-Zeldovich effect, X-ray observations can provide a direct method to determine the Hubble constant. I here emphasized a possible importance of the kinematic effect, which may it difficult to measure the Hubble constant with sufficient accuracy. Finally, I discussed on the origin of iron in the hot gas and pointed out that it can be reproduced if we adopt $h = 0.7 \sim 0.8$ by a conventional chemical evolution model of elliptical galaxies. In all fields discussed here, ASCA satellite is expected to provide critical observational data in very near future.

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Trans-fast MHD Accretion onto Black Hole

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ABSTRACT

We study relativistic cold ideal magnetohydrodynamical (MHD) accretion in the stationary and axisymmetric magnetosphere of a black hole. The MHD accretion emanating from the inner region of a rapidly rotating disk or corona must pass through smoothly both the Alfvén and fast magnetosonic points. From the conditions at these critical points, the MHD accretion specifies the flow's energy and angular momentum, the accretion rate, the ratio of the electromagnetic contributions and fluid contributions, etc. In addition to these critical conditions, the properties of the accretion depend on the configuration of the magnetic field lines. To be clear the geometrical effects on the MHD accretion, we assume typical configurations of the field lines and demonstrate the trans-fast MHD accretion in the black-hole magnetosphere.

1. INTRODUCTION

To explain the activity of the active galactic nuclei or the compact X-ray sources, we consider a stationary and axisymmetric black-hole magnetosphere that is composed of a massive black hole with the surrounding fluids and magnetic fields. In this magnetosphere, we discuss trans-fast magnetohydrodynamical (MHD) accretion. This MHD accretion is ejected from plasma sources (e.g., accretion disk and its corona) with a low velocity, which is at least less than the Alfvén wave's speed, and accretes onto the black hole after passing through both the slow magnetosonic, the Alfvén point and the fast magnetosonic point smoothly. At these points, the poloidal velocity of plasma equals the speed of the slow magnetosonic, Alfvén and fast magnetosonic wave, respectively. The trans-fast MHD solutions are the only physical acceptable solution for accretion onto black hole because the flows ejected from the disk with a low velocity must be super-magnetosonic at the horizon. The structure of the trans-Alfvénic inflows in a black-hole magnetosphere was cleared by Takahashi et al. (1990). A highly accelerated trans-fast MHD wind which is driven away to a far distance by a centrifugal

force and carries the huge energy and angular momentum from the central active area has been discussed by Takahashi (1993).

For inflows accreting onto the black hole, the energy and angular momentum transport per unit mass is determined by these critical solutions. Because the MHD flow streams along the magnetic field line, the shape of magnetic flux tube is essential to discuss how to construct wind structures (e.g., velocity and density distributions). That is, geometry of the field lines in the meridional plane governs the positions of the critical points and in turn the energy and angular momentum loss rate for each stream line. Moreover, the field geometry has more significant influences on how much the flow is accelerated and on how the flow velocity and the density are distributed in the meridional plane. This is analogous to a hydrodynamical flow which passes through a de Laval nozzle. (This nozzle has a minimum in its area and allows a transition from subsonic to supersonic.)

Unfortunately, most workers have investigated the accretion onto a black hole with a radial configuration of magnetic field lines. However, when we research the global structure of magnetosphere between the plasma source and the black hole, we should note that the monopole geometry is inconsistent in the black-hole magnetosphere because this structure is not satisfied the relativistic trans-field equation (see Nitta et al. 1991). We must solve the poloidal equation with the trans-field equation consistently. (In a force-free magnetosphere, Blandford & Znajek (1977) discussed the configuration of magnetic field lines. In this strong magnetic field limit, the split-monopole geometry satisfies the relativistic trans-field equation.) In this paper, we will deal with geometric effects of magnetic fields on the magnetic fields on the trans-fast MHD wind.

2. CRITICAL CONDITIONS FOR TRANS-FAST MHD FLOWS

In this section we present the basic equations of a stationary and axisymmetric ideal MHD wind for a given magnetic field and derive the condition for a trans-fast MHD flow. To determine the magnetosphere's structure, we must solve the so-called Bernoulli equation along magnetic field lines (Camenzind 1986b, Takahashi et al. 1990) and the force-balance equation (e.g., Mobarri and Lovelace 1986, Camenzind 1987, Nitta et al. 1991) self-consistently. These equations are derived from the equation of motion for

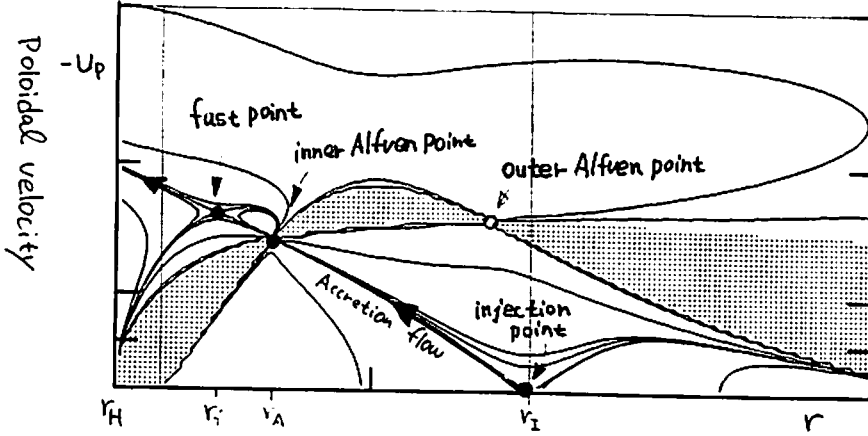


Figure 1: Trans-fast MHD Accretion Solution

plasma

$$T_{;\nu}^{\mu\nu} = \left[(\rho + P)u^\mu u^\nu - P g^{\mu\nu} + \frac{1}{4\pi} \left(F^{\mu\delta} F_\delta{}^\nu + \frac{1}{4} g^{\mu\nu} F^2 \right) \right]_{;\nu} = 0, \quad (1)$$

the conservation law of particle number $(nu^\mu)_{;\mu} = 0$, the ideal MHD condition $u^\mu F_{\mu\nu} = 0$ and Maxwell's equations. Here, ρ , P and n are the total energy density, the pressure of the plasma and the proper particle density. The electromagnetic field tensor $F^{\mu\nu}$ satisfies Maxwell's equations and u^μ is the four-velocity of the plasma. The background metric is written by the Boyer-Lindquist coordinates (with $c = G = 1$ and signature -2). The flow with the above assumptions stream along a magnetic field line $\Psi(r, \theta) = \text{constant}$, where the magnetic stream function Ψ measures the magnetic flux between the rotational axis and a given field line. Hereafter, the wind equation is only solved for cold flow (zero-temperature limit). This approximation would examine magnetic effects without thermal pressure.

An axisymmetric stationary MHD flow along a magnetic flux tube is specified by four constants of motion (e.g., Camenzind 1986a): the total energy $E(\Psi)$, total angular momentum $L(\Psi)$, angular velocity of the field line $\Omega_F(\Psi)$ and particle flux through a flux tube $\eta(\Psi)$. These conserved quantities are given by

$$E = \mu u_t - \frac{\Omega_F}{4\pi\eta} B_\phi, \quad (2)$$

$$L = -\mu u_\phi - \frac{1}{4\pi\eta} B_\phi, \quad (3)$$

$$\Omega_F = -\frac{F_{t\tau}}{F_{\phi r}} = -\frac{F_{t\theta}}{F_{\phi\theta}}, \quad (4)$$

$$\eta = \frac{nu_p}{B_p}. \quad (5)$$

where μ is the specific entalpy, B_ϕ is the toroidal magnetic field, u_p is the poloidal velocity defined by $u_p^2 \equiv -u_A u^A$ ($A = r$ or θ), B_p is the poloidal magnetic field seen by a lab-frame observer

$$B_p^2 \equiv -\frac{1}{\rho_w^2} [g^{rr}(\partial_r \Psi)^2 + g^{\theta\theta}(\partial_\theta \Psi)^2] \quad (6)$$

and $\rho_w^2 \equiv g_{t\phi}^2 - g_{tt}g_{\phi\phi}$.

The poloidal equation can be written as

$$\mu^2(1 + u_p^2) = (\gamma_g - 2M^2)f^2\rho_w^{-2} - K, \quad (7)$$

or its differential form is (Takahashi et al. 1990)

$$(\ln u_p)' = \frac{N}{D}, \quad (8)$$

where

$$D = (M^2 - \gamma_g)^2 [u_p^2(M^2 - \gamma_g) - (4\pi\mu\eta)^{-2}M^4B_\phi^2], \quad (9)$$

$$N = -\mu^{-2}\alpha^2M^4(\ln B_p)' - \frac{1}{2}\mu^{-2}M^4(M^2 - \gamma_g)K' - \frac{1}{2}\mu^{-2}(2KM^4 + 3M^2e^2 - \gamma_g e^2)\gamma_g', \quad (10)$$

with

$$\gamma_g \equiv g_{tt} + 2g_{t\phi}\Omega_F + g_{\phi\phi}\Omega_F^2, \quad (11)$$

$$-K \equiv g^{tt}E^2 - 2g^{t\phi}EL + g^{\phi\phi}L^2, \quad (12)$$

$$\alpha^2 \equiv e^2 + \gamma_g K, \quad (13)$$

$$e \equiv E - \Omega_F L \quad (14)$$

and the prime denotes the derivative $\partial_r - (\partial_r \Psi / \partial_\theta \Psi) \partial_\theta$ along a stream line. The Alfvén Mach-number M is defined by

$$M^2 \equiv \frac{4\pi\mu\eta u_p}{B_p}. \quad (15)$$

When $D = 0$, equation (8) becomes singular unless we require that N also vanish there in order that the flow smoothly pass through the critical point. This equation (8) has a critical point at the position $r = r_A$ defined by

$$M_A^2 = (\gamma_g)_A \quad (16)$$

and the critical condition for the trans-Alfvénic flow is given by $\alpha = 0$ or

$$\frac{L}{E} = - \left[\frac{g_{t\phi} + g_{\phi\phi}\Omega_F}{g_{tt} + g_{t\phi}\Omega_F} \right]_A. \quad (17)$$

If more critical points appear inside the Alfvén surface, these points coresspond to the fast magnetosonic points (r_f, θ_f) . In fact, for the accretion onto a black hole, the ingoing MHD flows must be super-magnetosonic. That is, the fast magnetosonic surface exists between the Alfvén surface and the event horizon.

For the super-Alfvénic flow which has passed through the Alfvén point, from the condition $D = 0$ and the equation (15), we have the critical condition for trans-fast MHD flows

$$\eta E = \left| \frac{B_p \sqrt{M^2 - \gamma_g}}{4\pi f} \right|_f, \quad (18)$$

where $M_f^2 \equiv M^2(r_f, \theta_f)$ is a solution of $N = 0$ and

$$f \equiv \frac{(g_{t\phi} + g_{\phi\phi}\Omega_F)E + (g_{tt} + g_{t\phi}\Omega_F)L}{\rho_w(M^2 - \gamma_g)} = \frac{\alpha}{(M^2 - \gamma_g)}. \quad (19)$$

Then, from poloidal equation (7), the critical energy is expressed in terms of (r_f, θ_f) and M_f^2

$$\left(\frac{E_f}{\mu} \right)^2 = \left[-K - f^2 \left(3M^2 + \frac{\gamma_g^2}{(M^2 - \gamma_g)} \right) \right]_f^{-1}. \quad (20)$$

From above relations, for a given magnetic field line geometry $\Psi(r, \theta) = \text{constant}$, we can say that the parameter x_f gives the total energy flux ηE , while parameter x_A gives the angular momentum of the flows. In the next section, we demonstrate the fast magnetosonic surface with some given configurations of the magnetic field.

3. Fast magnetosonic surface

Let us consider the accretion ejected from the accretion disk or its corona to the event horizon. Although it is very difficult to determine an electromagnetic structure by solving the transfield equation for $\Psi = \Psi(r, \theta)$, we can derive some important properties of the trans-fast MHD accretion. In the following, the poloidal magnetic field is given by

$$B_p^2 = B_I^2 \left(\frac{\rho_I^2}{\rho_w^2} \right) \left(\frac{r_I}{r} \right)^\beta \quad (21)$$

where the subscript I denotes the quantities at the injection point and β is the parameter of the magnetic flux tube. In figure 2, we have plotted ηE against x_f for a given x_A (or L/E) and some values of β . Then, we find the case that for certain values of the flow parameters, two X-type trans-fast regimes seem to exist with the same energy fluxes but different fast-magnetosonic radii.

The choice of the location of the fast-magnetosonic point depends on η (or E_f), and its location change discontinuously during the transition. The energy flux ηE is composed of the fluid and electromagnetic components. For the accretion passing through the inner X-type point (point A in figure 2) the electromagnetic contribution is dominated, while the fluid contribution is dominated for the outer X-type point (point B in figure 2), which is very close to the Alfvén point. We can conclude that there are obviously two phases — hydro-dominated and electromagnetic-dominated regimes — of accretion flow.

To demonstrate the fast magnetosonic surface, we consider the trans-fast MHD accretions that are originated from the disk surface or its corona and stream in the area between the equatorial plane and middle latitude region. We exclude the polar region because we expect that the accretion streaming in the polar region is originated from a divided plasma source from the accretion disk. Figure 3 shows schematic pictures of the fast surface, which are assumed a special case that the parameters of the accretion $\eta(\Psi)E(\Psi)$, r_A and $\Omega_F(\Psi)$ have same values for each of the stream lines, respectively. Figure 3a and 3c show the hydro-dominated and electromagnetic-dominated case, respectively. In figure 2b, we shows the case of the two phase trans-fast accretion coexisting. The accretion streaming in the lower-latitude region might include more hydrodynamical contribution (particle-rich) than that streaming in the higher-latitude region. So the fast surface would distribute

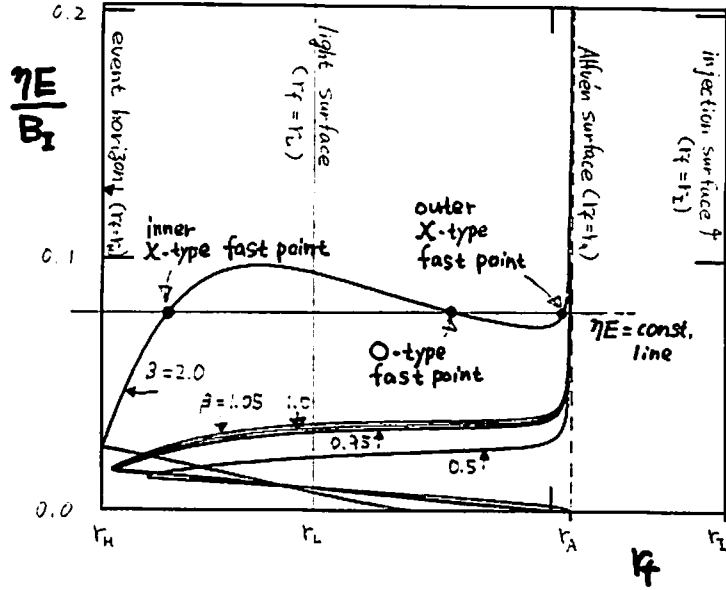


Figure 2: Energy flux vs location of the fast point : (a) Hydro-component dominate case and (b) Magnetically dominated case

very close to the Alfvén surface near the equator and locate near the horizon at the higher-latitude region.

4. Summary

We have discussed the condition for the trans-fast MHD accretion in the black-hole magnetosphere. For the accretion onto the black hole, two X-type transonic point can appear for certain values of the flow parameters. A wind passing through the outer fast point contains much plasma inertia effect, while a wind passing through the inner fast point is much electromagnetic.

Although our study of the trans-fast MHD winds does not satisfied the force-balance equation, we can understand quantitative deference between the various configurations of magnetic fields. Insight gained in the course of our analysis should be of use in further investigation.

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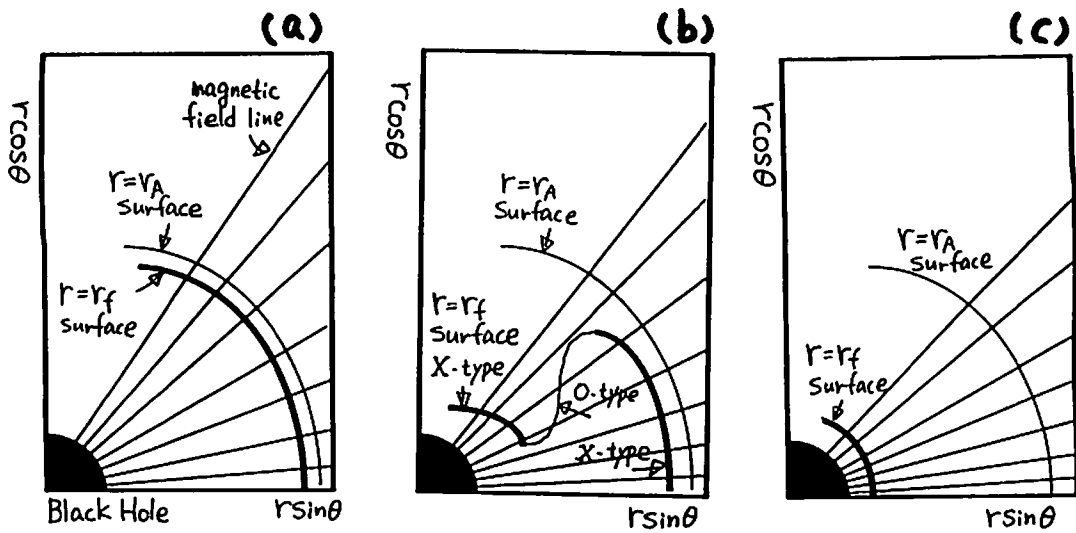


Figure 3: Schematic picture of the fast surface. Hydro-component dominate case [(a) continuous shell, (b) discontinuous shell] (c) Magnetically dominated case

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Collimated Propagation of Magnetohydrodynamic Waves near a Schwarzschild Black Hole

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Abstract

We study non-stationary and axisymmetric perturbations of a magnetohydrodynamic accretion onto a nonrotating (Schwarzschild) black hole. The unperturbed stationary inflow is assumed to be magnetically dominated, and our analysis is limited to the fast-magnetosonic perturbations produced near the black hole. The key point is that the plasma accretion can be highly disturbed as a result of a slight perturbation of the poloidal magnetic field. In fact we find an interesting perturbed structure of the plasma velocity with a large peak in some narrow region just beyond the fast-magnetosonic surface. If the critical surface has an oblate shape for radial poloidal field lines, this perturbation of plasma motion is induced by outgoing waves propagating from the equatorial super-fast-magnetosonic region to the polar sub-fast-magnetosonic one. Thus, the effective acceleration of particles due to the outgoing fast-magnetosonic waves will work mainly in the polar direction as a mechanism of trigger of jet production in active galactic nuclei.

1. Introduction

A few percent of the known galaxies in the universe contain hyperactive central region. Compared with the size of a galaxy the seat of this activity is extraordinarily compact, as indicated by rapid variability of non-thermal X-ray flux on time scales as short as minutes (e.g., Matilsky, Shrader, and Tannenbaum 1982; Kunieda et al. 1992). Emerging from the central compact source there is often a sharply focused jet, or two jets pointing in opposite directions. It is generally accepted that the jets are powered by the associated active galactic nuclei (AGNs), where accreting mass is converted into energy with several percent efficiency. We must therefore explain how the energy goes into a small fraction of the gas in the gravitational field of the prime mover, probably a supermassive black hole (e.g., Rees 1984).

Possible flows of this type were suggested by Shakura and Sunyaev (1973) in the context of a thick supercritical accretion disk which exhibited inflow along the equator and outflow near the poles. If the radiation-supported rotating gas adopts a hydrostatic toroidal configuration, then a pair of funnels will be defined which could be responsible for the production of jets along the rotational axis (Lynden-Bell 1978). Furthermore, the magnetohydrodynamic (MHD) disturbances produced at a galactic nucleus with a compact nuclear disk will be strongly collimated polewards resulting in jets, if the Alfvén velocity in the disk is much higher than its surroundings (Sofue 1980). Such a collimation will lead to an increase of wave amplitude resulting in shock waves which are conjectured to develop into a strongly compressed region of magnetic field. In this region, high energy particles are likely to be accelerated in the perpendicular direction to the equatorial nuclear disk.

The purpose here is to explore a little further into the problem whether MHD waves can convey energy from the equatorial region to the poles in the close vicinity of the central black hole of an AGN. Let us first describe a probable model of a rotating black hole magnetosphere and discuss the importance of considering the propagation of MHD waves in it. We expect that inflows start from an equatorial disk surface and that they stream along the magnetic field lines connecting the disk and the event horizon. Magnetic field lines are considerably bent because of the balance among inertial forces, electromagnetic forces, and gravitational forces (Wilson 1975; Nitta et al. 1991). Sufficient matter is expected to be supplied into the magnetosphere along these bent field lines from the dense disk so that we may adopt the MHD approximation. Causality requires that the MHD inflows should pass through the fast-magnetosonic point and become super-fast-magnetosonic at the horizon. The investigation of the so-called critical condition that the inflow should pass through this point smoothly is, in fact, the key to understand MHD interactions in a black hole magnetosphere.

The MHD interactions are expected to work most effectively in the magnetically dominated limit that the rest-mass energy density of particles is negligible compared with the magnetic energy density. In this limit, the fast-magnetosonic point is located very close to the horizon (e.g., Phinney 1983); as a result, general relativistic treatment is required. Analyzing the critical condition in a stationary and axisymmetric magnetically dominated black hole magnetosphere, Hirotani et al. (1992, hereafter paper I) showed that roughly 10% of the rest-mass energy and significant fraction of the initial angular momentum are transported from the fluid to the magnetic field during the infall. Furthermore, if a small-amplitude perturbation is introduced into the magnetosphere, a lot of perturbation energy is transported from the magnetic field to the fluid near the fast-surface in the short-wavelength limit; accordingly, the plasma accretion becomes highly disturbed there (Hirotani, Tomimatsu, and Takahashi 1993, hereafter paper II). If several portions of such a fluid disturbance, which is originally supplied from the equatorial disk, are transported polewards in the form of MHD waves, then the magnetic energy may be changed into bulk kinetic energy or fast-particle energy of outflows observed as jets in AGNs. This paper is therefore intended as an investigation of the structure and propagation of MHD waves in a magnetically dominated black hole magnetosphere.

The outline of this paper is as follows. We first briefly review an analysis of the stationary MHD accretion near the fast-magnetosonic point in the magnetically dominated limit in section 2.1. and next consider non-stationary axisymmetric perturbations of the MHD accretion in section 2.2. In the short-wavelength limit, it is known that the fluid becomes highly variable near the fast-magnetosonic point owing to the MHD interactions (paper II). In this paper, however, we do not adopt the short-wavelength limit; instead, we assume that characteristic scale of the radial variations of perturbed quantities is comparable with that of unperturbed quantities. Under this assumption, we can derive the wave equations of perturbed quantities on the poloidal plane. Solving the wave equations, we show in section 3 that the fluid becomes most variable at slight inside of the fast-surface; this result partly supports what was obtained in paper II in the short-wavelength limit. We further demonstrate in section 4 that the disturbance can escape outwards as fast-magnetosonic waves by propagating polewards.

2. Magnetically Dominated Accretion

We will begin by considering basic equations describing a magnetosphere around a non-rotating black hole. Since the self-gravity of the electromagnetic field and plasma around the black hole is very weak, the background geometry of the magnetosphere is described by the Schwarzschild metric,

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \quad (1)$$

where M is the mass of the hole. Throughout this paper we use geometrized units such that $c = G = 1$.

Under ideal MHD conditions that the electric field vanishes in the fluid rest frame, we have $F_{\mu\nu}U^\mu = 0$, where $F_{\mu\nu}$ is the electromagnetic field tensor satisfying Maxwell equations, $F_{[\mu\nu,\lambda]} = 0$ and U^μ is the fluid four velocity. The motion of the fluid in the cold limit is governed by the following equations of motion:

$$T^{\mu\nu}_{;\nu} \equiv \left[m_p n U^\mu U^\nu + \frac{1}{4\pi} (F^{\mu\rho} F_\rho{}^\nu + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}) \right]_{;\nu} = 0, \quad (2)$$

where the semicolon denotes a covariant derivative and m_p the rest-mass of a particle. The proper number density (n) obeys the continuity equation, $(nU^\mu)_{;\mu} = 0$. We adopt these basic equations for a description of a stationary and axisymmetric black hole magnetosphere in section 2.1 and for an analysis of perturbed states afterwards.

2.1 Unperturbed Magnetosphere

Before turning to a closer examination of a perturbed magnetosphere, a few remarks should be made concerning the unperturbed stationary and axisymmetric state. We first define the unperturbed electromagnetic field at a point (r, θ) as those measured by a distant observer at rest. They can be written as

$$\begin{aligned} E_r &= F_{r,t}^{(0)}, & E_\theta &= F_{\theta,t}^{(0)}, \\ B_r &= \frac{F_{\theta\phi}^{(0)}}{\sqrt{-g}}, & B_\theta &= \left(1 - \frac{2M}{r}\right) \frac{F_{\phi r}^{(0)}}{\sin \theta}, & B_\phi &= \sqrt{-g} F^{r\theta}_{(0)}, \end{aligned} \quad (3)$$

where $\sqrt{-g} \equiv r^2 \sin \theta$; the subscript and the superscript (0) indicate unperturbed quantities.

In order to look closely at the problem how the existence of the horizon affects the MHD interactions, we would like to focus attention on the analysis near the horizon ($1 - 2M/r \ll 1$). Since the poloidal magnetic field near the horizon tends to be bent into radial shape owing to fluid inertia even in the magnetically dominated limit (paper I), it may safely be assumed that the unperturbed field is radial ($F_{\phi r}^{(0)} = 0$). Because particles are frozen to the radial magnetic field lines, flow lines also become radial ($U^\theta_{(0)} = 0$). Moreover, from the Maxwell equations and the frozen-in conditions, we can see that the unperturbed electric field becomes meridional;

$$E_r = 0, \quad E_\theta = -\sqrt{-g} \Omega_F B_r = -\Omega_F \Psi_\theta, \quad (4)$$

where $\Psi_\theta (\equiv F_{\theta\phi}^{(0)})$ is the derivative of the ϕ component $\Psi(\theta)$ of the unperturbed electromagnetic vector potential with respect to θ , and the angular frequency Ω_F of a magnetic field line is conserved along a poloidal flow line $\Psi = \text{const.}$ (Bekenstein and Oron 1978; Camenzind 1986a,b). In addition to Ω_F , particle number flux per unit magnetic flux tube (η) is also conserved. Using η , we can express B_r in terms of U^r as

$$B_r = -\frac{\eta}{U^r}. \quad (5)$$

Furthermore, in a stationary and axisymmetric magnetosphere, total energy (E) and angular momentum (L) of MHD flows are also conserved along a poloidal flow line. They are composed of fluid and electromagnetic parts as follows:

$$E = m_p U_t^{(0)} - \frac{\sqrt{-g} \Omega_F}{4\pi \eta} F^{r\theta}_{(0)}, \quad (6)$$

and

$$L = -m_p U_\phi^{(0)} - \frac{\sqrt{-g}}{4\pi \eta} F^{r\theta}_{(0)}. \quad (7)$$

Note that in the magnetically dominated limit we have

$$\frac{m_p}{E} \ll 1, \quad \text{and} \quad \frac{m_p}{L \Omega_F} \ll 1, \quad (8)$$

so that the electromagnetic parts in equations (6) and (7) dominate the fluid parts. For a Schwarzschild black hole, we always have $E > 0$. These four conserved quantities (Ω_F , η , E , and L) are functions of Ψ only.

Before we leave the discussion over the unperturbed magnetosphere, we should notice that the fast-magnetosonic point ($r = r_F$) is located very close to the horizon ($r = r_H \equiv 2M$) in the magnetically dominated limit. This can be clearly seen if we write down its position as (paper I)

$$\frac{r_F - r_H}{r_H} = (-K_H) \sqrt{K_I - K_H} \frac{m_p}{E}, \quad (9)$$

where the nondimensional factor $\sqrt{K_I - K_H}$ is of order of unity; K_I and K_H are defined by $K_I \equiv (E - \Omega_F L)^2 / m_p^2$ and $K_H \equiv -4M^2 \Omega_F^2 \sin^2 \theta$, respectively.

2.2 Perturbed Magnetosphere

We next consider a non-stationary and axisymmetric perturbation superposed on the unperturbed state discussed in the last subsection. Let the actual poloidal component of the electric and magnetic fields, as a result of the disturbance, be $E_A + e_A$ and $B_A + b_A$ ($A = r, \theta$), respectively; the small letters (e_r , e_θ , b_r , and b_θ) are the Eulerian perturbations of the corresponding quantities. Let b_T denote a perturbation of $F^{r\theta}$. Since $F_{\phi t}$

vanishes for the unperturbed state, we use $F_{\phi t}$ as a perturbed toroidal electric field. Furthermore, we introduce u^r , u^θ such that the actual component of the poloidal fluid velocity field, as a result of the disturbance, becomes $U^A + u^A$. Let u_t and $-u_\theta$ be the Eulerian perturbations of fluid energy and angular momentum per unit mass, respectively. In what follows, we omit the subscript and the superscript (0), because no confusion arises.

Making use of (4), we can simplify t , ϕ , and θ components of the frozen-in conditions as follows:

$$-U^r e_r - U^\phi F_{\phi t} + \Omega_F \Psi_\theta u^\theta = 0, \quad (10)$$

$$-\left(1 - \frac{2M}{r}\right)^{-1} U^r F_{\phi t} + \sin \theta \left(1 - \frac{2M}{r}\right)^{-1} U^r b_\theta - \Psi_\theta u^\theta = 0, \quad (11)$$

$$\begin{aligned} & \left(1 - \frac{2M}{r}\right)^{-1} U^r (e_\theta - b_T) - \sqrt{-g} U^\phi b_r \\ & + \left(1 - \frac{2M}{r}\right)^{-1} (\Omega_F \Psi_\theta u_t + r^2 F^{r\theta} u^r) + \frac{\Psi_\theta}{r^2 \sin^2 \theta} u_\phi = 0. \end{aligned} \quad (12)$$

Our purpose here is to reduce these coupled perturbation equations to a single partial differential equation for one unknown function. If we assume that the characteristic scales of *meridional* variations in the perturbed state are much shorter than those in the unperturbed state, the perturbation equations reduce significantly. For axisymmetric perturbations, all perturbed quantities are therefore assumed to be proportional to $\exp(i\omega t - ik_\theta \theta)$, where $k_\theta \gg 1$. Under this approximation, the Maxwell equations reduce to

$$\sin \theta b_\theta = -\frac{i}{\omega} \frac{dF_{\phi t}}{dr_*}, \quad (13)$$

$$b_r = \frac{1}{\omega \sqrt{-g}} k_\theta F_{\phi t}, \quad (14)$$

$$e_r = \frac{i}{k_\theta} \left(1 - \frac{2M}{r}\right)^{-1} \left(-i\omega b_T + \frac{de_\theta}{dr_*}\right), \quad (15)$$

where r_* is the tortoise coordinate defined by

$$\frac{dr_*}{dr} \equiv \left(1 - \frac{2M}{r}\right)^{-1}. \quad (16)$$

It is convenient to introduce this coordinate when we describe waves near the horizon, because the interval (r_H, ∞) in the r -coordinate is stretched to $(-\infty, \infty)$ in r_* . We assume that the characteristic scale of the radial variations is comparable with that of the gravitational field, that is, $|df/dr_*| \approx |f/M|$, where f denotes some perturbed quantity. In this paper we do not take the short-wavelength limit ($|df/dr_*| \gg |f/M|$) which was adopted in paper II.

Let us finally consider the equations of motion. Eliminating perturbed fluid density by using the continuity equation and taking the leading orders in the expansions of $(1 - 2M/r) \ll 1$, we obtain the r , θ and ϕ components of the equations of motion after lengthy manipulations. They are written as

$$\begin{aligned} & 4\pi m_p n \left[\left(1 - \frac{2M}{r}\right)^{-1} U^r \left(-\frac{1}{2M} - i\omega + \frac{d}{dr_*}\right) \left(-i\omega + \frac{d}{dr_*}\right) u^r - ik_\theta \frac{\partial U_t}{\partial r} u^\theta \right] \\ & - \frac{\Omega_F \Psi_\theta \sin \theta}{\sqrt{-g}} \left(1 - \frac{2M}{r}\right)^{-1} \left(-\frac{1}{2M} - i\omega + \frac{d}{dr_*}\right) \left(-i\omega e_\theta + \frac{db_T}{dr_*}\right) = 0 \end{aligned} \quad (17)$$

$$\begin{aligned} & \left[K_H \left(1 - \frac{2M}{r}\right)^{-2} \left(-\frac{1}{M} - i\omega + \frac{d}{dr_*}\right) \left(-i\omega + \frac{d}{dr_*}\right) - \left(\frac{k_\theta}{2M}\right)^2 \right] \left(-i\omega + \frac{d}{dr_*}\right) F_{\phi t} \\ & - \omega k_\theta \Omega_F \sin^2 \theta \left(1 - \frac{2M}{r}\right)^{-1} \left(-\frac{1}{2M} - i\omega + \frac{d}{dr_*}\right) (e_\theta - b_T) = 0 \end{aligned} \quad (18)$$

$$4\pi m_p n \left[\left(1 - \frac{2M}{r}\right)^{-1} U' \left(-\frac{1}{2M} - i\omega + \frac{d}{dr_*}\right) \left(-i\omega + \frac{d}{dr_*}\right) u_\phi + ik_\theta \frac{\partial U_\phi}{\partial r} u^\theta \right] - \frac{\Psi_\theta \sin \theta}{\sqrt{-g}} \left(1 - \frac{2M}{r}\right)^{-1} \left(-\frac{1}{2M} - i\omega + \frac{d}{dr_*}\right) (-i\omega e_\theta + \frac{db_T}{dr_*}) = 0, \quad (19)$$

where U_t and $-U_\phi$ express unperturbed fluid energy and angular momentum respectively, n denotes the unperturbed fluid density. Furthermore, the definition of the proper time gives

$$u_t + u^r = \frac{1}{r^2 \sin^2 \theta} \left(1 - \frac{2M}{r}\right) \frac{1}{U'} \left[\frac{(U_\phi)^2 + r^2 \sin^2 \theta}{2U'} u^r - U_\phi u_\phi \right]. \quad (20)$$

Instead of solving the t component of the equations of motion, we use equation (20).

We analyze the system which is formed by 10 equations (3 frozen-in conditions, 3 Maxwell equations, and 4 equations of motion) in 10 unknown functions (b_r , b_θ , e_r , e_θ , b_T , $F_{\phi t}$, u_t , u^r , u^θ , and u_ϕ). Perturbed fluid density would be calculated if we solved the perturbed continuity equation. These 10 equations give some relations between the amplitude of arbitrary two perturbed quantities and are further combined into a single differential equation. As a relation between u^r and e_r , we present here

$$\frac{k_\theta}{2M} \frac{u^r}{U'} = iK_H \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{1}{2M} - i\omega + \frac{d}{dr_*}\right) \frac{e_r}{E_\theta/2M}. \quad (21)$$

If we combine (21) with an another independent relation between u^r and e_r , we obtain the single differential equation which u^r obeys. Since we are interested in the fast-magnetosonic mode, we obtain the following second-order single differential equation which is appropriate to the problem on hand;

$$\Lambda u^r = 0, \quad (22)$$

where the second-order differential operator Λ is defined by

$$\Lambda \equiv \left(\frac{1}{2M} - i\omega + \frac{d}{dr_*}\right) \left[-\frac{U_{FM}^2}{2M} - (U_p^2 + U_{FM}^2) i\omega + (U_p^2 - U_{FM}^2) \frac{d}{dr_*} \right] + U_{FM}^2 \left(1 - \frac{2M}{r}\right) \left(\frac{k_\theta}{2M}\right)^2. \quad (23)$$

The fast-magnetosonic speed U_{FM} near the horizon in the magnetically dominated limit can be written as (paper I)

$$U_{FM} \equiv \frac{\sqrt{K_I - K_H}}{(2M)^2 \Omega_F^2 \sin^2 \theta} \frac{E}{m_p} \gg 1. \quad (24)$$

To obtain the wave equation describing the oscillation of u^r , let us replace k_θ with $i\partial_\theta$ and ω with $-i\partial_t$. The resultant equation reads, introducing nondimensional variables $x \equiv (r - r_H)/r_H$ and $\tau \equiv t/(2M)$,

$$x^2(x - x_F) \frac{\partial^2 u^r}{\partial x^2} + x(3x - x_F) \frac{\partial u^r}{\partial x} + 2x_F x \frac{\partial}{\partial \tau} \frac{\partial u^r}{\partial x} - (x + x_F) \frac{\partial^2 u^r}{\partial \tau^2} + x u^r + x^2 \frac{\partial^2 u^r}{\partial \theta^2} = 0. \quad (25)$$

Considering only the spatial dependence, we can see that this equation is elliptic in the sub-fast region ($x > x_F$) while it becomes hyperbolic in the super-fast region ($x < x_F$). It must be noted that equation (25) is valid only for perturbations with $k_\theta \gg 1$, which ensures local analysis in the meridional direction. Since fluid's energy (u_t) and angular momentum ($-u_\phi$) obey the same differential equation as (25), it is entirely fair to say that this equation describes fluid's disturbance. In section 3 and 4, we shall therefore mainly discuss (25) in detail.

3. Structure of Fluid Disturbances

A study on non-stationary perturbations of magnetically dominated black hole magnetosphere was made in paper II, and it revealed that the plasma accretion should become highly variable near the fast-surface in the short-wavelength limit under the assumption of the neglect of meridional propagation of MHD waves ($k_\theta \ll \sqrt{E/m_p} k_{r,*}$). Considering meridional propagation offers, however, the key to an understanding of the energy transport from the equatorial region to the poles in the form of MHD waves. First of all, we therefore have to inquire into the question whether the same conclusions hold when the meridional propagation becomes important. It is shown in this section that though the relative amplitude among fluid quantities and electromagnetic field becomes slightly different from what was obtained in paper II, the conclusion that the fluid becomes highly variable near the fast-surface remains true. We next investigate the spatial structure of fluid's amplitude, which could not be treated in the framework of the short-wavelength limit, by solving equation (25).

Let us briefly examine (21) and (25) in the short-wavelength limit ($|x| |du^r/dx| \ll |u^r|$). In this limit, we cannot treat mode conversion, spatial confinement of perturbations, and so on. Nevertheless, this limit permits us to clarify how the MHD interaction affects relative amplitude among perturbed quantities at each point along a flow line. Replacing ∂_r , $x\partial_z$, and ∂_θ with $i2M\omega$, $-i2Mk_{r,*}$, and $-i2Mk_\perp$ respectively, and with the aid of the dispersion relation for the outgoing fast-magnetosonic mode obtained from (25), we find that (21) becomes

$$\frac{u^r}{U^r} = \frac{K_H}{x + x_F} \frac{k_{r,*} + \sqrt{k_{r,*}^2 + (x + x_F)k_\perp^2}}{k_\perp} \frac{e_r}{E_\theta/(2M)}. \quad (26)$$

Here, the outgoing waves propagate outwards in the fluid's co-moving frame. If we were to set $k_\perp \approx k_{r,*}$ in order of magnitude, which indicates that the θ -derivative term in (25) and hence the meridional propagation is negligible near the horizon, we would have the same relation $u^r \approx (E/m_p)[e_r/(E_\theta/2M)]$ which was presented in paper II.

We would like to consider, however, the case when the meridional propagation in the dispersion relation obtained from (25) is essential. Accordingly, let us postulate $k_\perp \approx \sqrt{E/m_p} k_{r,*}$. Then we have

$$u^r, u_t, -u_\phi \approx \sqrt{\frac{E}{m_p}} \frac{e_r}{E_\theta/(2M)} \quad (27)$$

in order of magnitude, in a somewhat extended region near the fast-surface ($x \approx x_F \approx m_p/E$). Moreover, setting $k_\perp \approx \sqrt{E/m_p} k_{r,*}$ in other relations among perturbed quantities, we can see that other well-behaved components of the electromagnetic field have the same or smaller amplitude compared with e_r in order of magnitude near the fast-surface as follows;

$$\begin{aligned} \frac{b_\theta/2M}{B_r} &\approx \frac{e_r}{E_\theta/2M} (\approx u^\theta), \\ \frac{e_\theta}{E_\theta}, \frac{b_r}{B_r}, \frac{b_T}{B_r} &\approx \sqrt{\frac{m_p}{E}} \frac{e_r}{E_\theta/2M}, \\ \frac{F_{\phi t}}{E_\theta} &\approx \frac{m_p}{E} \frac{e_r}{E_\theta/2M}. \end{aligned} \quad (28)$$

It follows from (27) and (28) that in the magnetically dominated limit ($E/m_p \gg 1$) the fluid suffer large amplitude disturbance compared with the electromagnetic field. This result partly supports what was obtained in paper II, in which the meridional propagation was neglected ($k_\perp \ll \sqrt{E/m_p}$ in order of magnitude). Although the perturbation energy is supplied mainly in the form of electromagnetic disturbances ($[e_r]^2 \approx [E/m_p] m_p n [u^r]^2 \gg m_p n [u^r]^2$) far from the horizon, the equipartition of energy ($[e_r]^2 \approx m_p n [u^r]^2$) is achieved near the fast-surface. In other words, a lot of perturbation energy is transported from the electromagnetic field to the fluid during the infall as a result of the effective MHD interactions. In a realistic magnetically dominated black hole magnetosphere, the fluid will become highly variable ($[u^r]^2 \approx [U^r]^2$) near the fast-surface for a very

small input of perturbation energy ($|e_r|^2 \approx (m_p/E)(E_\theta/2M)^2$) from the surroundings. This effect is due to the redshift effect near the horizon and disappears if the fast-magnetosonic point is located far from it.

Having got the special case of the short-wavelength limit out of the way, we may now turn to the real subject. To examine the spatial structure of $u'(x, \theta)$, let us return to (25) and write it as

$$\left\{ (1 - i\sigma + x \frac{\partial}{\partial x}) \left[x(x - x_F) \frac{\partial}{\partial x} + i\sigma(x + x_F) + x \right] + x^2 \frac{\partial^2}{\partial \theta^2} \right\} u' = 0, \quad (29)$$

where we have replaced ∂_r with $i\sigma$. Fluid's energy (u_r) and angular momentum ($-u_\phi$) also satisfy this equation. In general, solutions of this equation must be sought which satisfy appropriate boundary conditions. The fuller study of this subject by numerical method lies, however, outside the scope of this paper. Let us consider instead the subject from an analytic point of view and study the radial(x)-dependence of u' by neglecting the θ -derivative term as a first step. This approximation is valid only when meridional variations are not too large to violate the assumption $1/\sqrt{x} \gg k_\theta (\gg 1)$. Under this approximation, equation (29) gives the solution corresponding to the *outgoing* fast-magnetosonic mode, which is represented by the differential operator in the square bracket, as follows;

$$u' = C_1 \frac{x^{i\sigma}}{[x - x_F(\theta)]^{1+2i\sigma}}, \quad (30)$$

where C_1 is an integration constant. In order that the right-hand-side may not diverge at $x = x_F(\theta)$, the real part of $1 + 2i\sigma$ should be non-positive; this indicates that outgoing radial waves must decay, because no steady supply of perturbation energy across the fast-surface $x = x_F(\theta)$ is possible. In this paper, we postulate a steady excitation of perturbation induced by plasma injection from the equatorial disk. It requires that σ should be real. Then, to avoid the divergence at $x = x_F$, we must consider the θ -derivative term in (28) at least near the fast-surface.

In fact, unless the fast-surface is exactly spherical ($x_F = \text{const.}$), the derivative $\partial u'/\partial \theta$ for the solution (30) becomes important in the limit $x \rightarrow x_F$. In the narrow region $|(x - x_F)/x_F| \ll 1$, we must therefore modify the solution into to form

$$u' = C_1 \frac{x^{i\sigma}}{[x - x_F(\theta) + \epsilon(\theta)]^{1+2i\sigma}}, \quad (31)$$

where ϵ should be a complex function of θ , so that we may obtain a regular solution u' for real σ . Inserting (31) into (29) and evaluating the equation in the limit $|(x - x_F + \epsilon)/x_F| \ll 1$, we obtain a non-linear first-order differential equation for ϵ ,

$$\epsilon = \left(\frac{dx_F}{d\theta} - \frac{d\epsilon}{d\theta} \right)^2. \quad (32)$$

From equation (9) the variable $x_F \equiv (r_F - r_H)/r_H$ turns out to be of order of $m_p/E \ll 1$ in the magnetically dominated limit. Then we obtain the approximated solution

$$\epsilon = \left(\frac{dx_F}{d\theta} \right)^2 + i\epsilon_I, \quad (33)$$

where the imaginary part

$$\epsilon_I = C \exp \left(- \int_0^\theta \left[\frac{dx_F}{d\theta} \right]^{-1} d\theta \right) \quad (34)$$

is assumed to be much smaller than the real part. Note that for $dx_F/d\theta > 0$ the imaginary part ϵ_I rapidly decreases as θ increases, and the validity of equation (33) is assured for the high latitudes if the constant C is chosen to be of order of $(m_p/E)^2$. Interestingly the structure of the outgoing fast-magnetosonic perturbation crucially depends on the shape of the fast-surface (i.e., $dx_F/d\theta$): The sharp peak of the amplitude appears at the surface $x = x_F - (dx_F/d\theta)^2$ just beyond the fast-surface. Further, for the oblate shape ($dx_F/d\theta > 0$), the peak for the high latitudes is much smaller than that for the low latitudes. Various shapes of the critical surface

will be possible according to the stationary models of the MHD accretion. For example, if the ingoing winds are injected from the disk surface, the particle flux should be mainly supplied into the low latitudes. Then the Alfvén Mach number $\mathcal{M}_H(\theta) \equiv \sqrt{4\pi m_p \eta^2 / n}$ at the horizon becomes an increasing function of θ . Because x_F is equal to $\mathcal{M}_H^2(\theta)$ (paper I), the magnetosphere has an oblate critical surface. In fact, the ratio K_H/E in equation (9) may remain nearly constant, by virtue of the boundary condition at the horizon which gives $\eta E \propto \sin \theta \Psi_\theta \propto \sin^2 \theta$ for split monopole field (Blandford and Znajek 1977). Hence x_F depends on θ through the factor $\eta \sqrt{K_I - K_H}$, in which the particle flux η is an increasing function of θ , and the conserved quantity K_I equal to $(U_i^{(0)} + \Omega_F U_\phi^{(0)})^2$ is weakly dependent on θ if $U_i^{(0)} \approx 1$ and $\Omega_F U_\phi^{(0)} \ll 1$ at the injection points (the disk surface). Anyway, in this paper, we proceed with the study under the assumption $dx_F/d\theta > 0$ in relation to jet formation from the magnetosphere. We have found a large amplitude given by (31) at $x = x_F - (dx_F/d\theta)^2$ for the velocity perturbation. The important point is that this perturbation can extend to the sub-fast region ($x > x_F$) across the critical surface, where the solution (31) is regular. The infalling motion of plasma in the sub-fast region is mainly disturbed at slight outside this surface, because the outgoing waves disperse and the amplitude decays like $|u'| \sim x^{-1}$ for $x \gg x_F$. The steady supply of the outgoing waves is due to the meridional propagation from the super-fast equatorial region to the sub-fast polar one. A large part of the outgoing waves can escape from the narrow super-fast region in the range $x_F - (dx_F/d\theta)^2 < x < x_F$, by virtue of the oblate shape of the critical surface ($dx_F/d\theta > 0$), before reaching the polar region. Hence the peak amplitude of u' at $x = x_F - (dx_F/d\theta)^2$ exponentially decreases as θ decreases. To confirm the structure of the fast-magnetosonic perturbation found in the solution (31), in the next section, we will discuss the problem of the meridional propagation in detail.

4. Collimated Propagation of MHD Waves

The meridional propagation of outgoing fast-magnetosonic waves is essential to the perturbed structure near the fast-surface. Now we consider the problem from a different point of view, to explain the meridional propagation more clearly. Note that equation (29) becomes a hyperbolic type in the super-fast region. An useful approach is to calculate the characteristics of the wave equation (29), because in general they are identical with the ray paths along which disturbances propagate in the short-wavelength limit. If the short-wavelength limit breaks down, and as a result, if the wave can no longer be regarded as a plane wave, then the disturbance propagates within the "Mach cone" of which boundary is given by the characteristics.

The characteristics of equation (29) in the super-fast region are obtained by solving the non-linear differential equation,

$$\frac{dx}{d\theta} = \mp \sqrt{x_F(\theta) - x}. \quad (35)$$

Replacing $x_F - x$ with ϵ , we find that this equation is identical with (32). Now, to show the propagation to the critical surface $x = x_F$, we must obtain real solutions by exactly solving equation (35). For this purpose it is mathematically convenient to assume that $dx_F/d\theta (\equiv \Delta)$ is a positive constant. This assumption does not lose any essential feature of the characteristics. Then equation (35) yields two characteristics as follows;

$$\left[\Delta^{-1} \sqrt{x_F - x} + 1 \right] \exp \left[-\Delta^{-1} \sqrt{x_F - x} \right] = (\alpha + 1) e^{-\alpha} \exp \left[\frac{\theta_0 - \theta}{2\Delta} \right], \quad (36)$$

or

$$\left[\Delta^{-1} \sqrt{x_F - x} - 1 \right] \exp \left[\Delta^{-1} \sqrt{x_F - x} \right] = (\alpha - 1) e^{\alpha} \exp \left[\frac{\theta_0 - \theta}{2\Delta} \right], \quad (37)$$

where the integration constant is chosen so that the characteristic passes the point $x = x_0 \equiv x_F - \alpha^2 (dx_F/d\theta)^2 = x_F - \alpha^2 \Delta^2$ at $\theta = \theta_0$ ($\alpha > 0$). The further a disturbance is caused inside the fast-surface, the larger we must give the value to α .

In the super-fast region any waves must propagate in the inward direction $dx < 0$. Hence the propagation to the lower latitudes ($d\theta > 0$) and the higher ones ($d\theta < 0$) corresponds to (36) and (37), respectively. Further we have $d\theta < 0$ only for the characteristic (37) started from $x = x_0 > x_F - \Delta^2$ ($\alpha < 1$). On these ground we

have come to the conclusion: The solution (36) corresponds to the characteristic along which a disturbance caused at (x_0, θ_0) propagates into the equatorial region and at last is swallowed by the hole, while the solution (37) with $\alpha < 1$ corresponds to that along which the disturbance propagates into the polar region and at last escapes to a sub-fast region. Moreover, the difference of $dx/d\theta$ between (36) and (37) at (x_0, θ_0) is an increasing function of α . That is, the further a disturbance is caused inside the fast-surface, the smaller becomes the Mach angle. Figure 1 shows such a situation schematically. The angle $(\theta = \theta_X)$ at which the characteristic (37) meets the fast-surface ($x = x_F$) is given by putting $\sqrt{x_F - x} = 0$. The result is

$$\tan \theta_X = [(1 - \alpha)e^\alpha]^{1/\Delta} \tan \theta_0. \quad (38)$$

For the magnetically dominated limit the gradient Δ of order of m_p/E should be very small. Equation (38) means that the difference $\theta_0 - \theta_X$ also remains very small, unless α is nearly equal to unity. The meridional propagation to escape to the sub-fast region ($\theta > \theta_X$) is, therefore, very effective. If a disturbance is caused inside the maximum-amplitude surface (i.e., $\alpha > 1$), then the disturbance cannot propagate into the subsonic region; such a disturbance must be always swallowed by the hole. In other words, fast-magnetosonic waves can propagate from any point of the narrow region $x_F - (dx_F/d\theta)^2 < x < x_F$ into the sub-fast region by propagating polewards. The maximum-amplitude surface $x = x_F - (dx_F/d\theta)^2$ can be therefore referred to the fast-magnetosonic separatrix surface (Bogovalov 1992); one of the two characteristics which is oriented polewards as illustrated in Figure 1 leaves from this surface. This is the reason why $|u'|$ at this surface decreases with decreasing θ , as we noted in the last section. If MHD disturbances originate from the super-fast region, and thus the outgoing waves with $k_\theta < 0$ dominate in the sub-fast region, then the wave energy will collimate toward the polar axis.

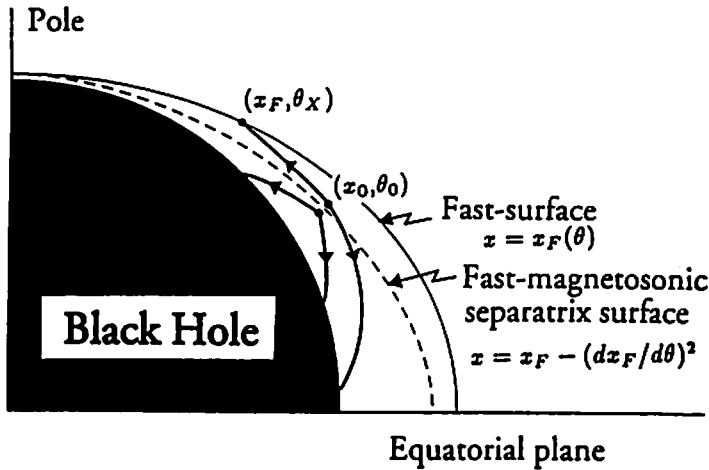


Fig. 1. Schematic figure (side view) of a black hole magnetosphere in the close vicinity of the event horizon. The thick solid curves denote characteristics of equation (28) which originate from the point (x_0, θ_0) . The maximum-amplitude surface ($x = x_F - [dx_F/d\theta]^2$), which is shown by the dashed line, coincides with the fast-magnetosonic separatrix surface. The poloidal magnetic field lines are not drawn for avoiding complication. The super-fast region is very thin in the magnetically dominated limit; however, the width of this region is exaggerated.

The question why only the waves propagating to the polar regions can escape (and the waves propagating to the equatorial region are swallowed by the hole) can be explained quite naturally in terms of the characteristics as a result of the oblate shape of the fast-surface. If the fast-surface were prolate, on the contrary, only the waves propagating into the equatorial region could escape. In this paper we have assumed a radial flow line for the unperturbed system. We emphasize here that the similar wave propagation can occur if the fast-surface is *not* perpendicular to the flow lines. Only if the fast-surface is perpendicular to the flow lines, any fast-magnetosonic waves can no longer escape.

Let us summarize the main points that have been made in this paper. We have studied non-stationary and axisymmetric perturbations of magnetically dominated accretion onto a non-rotating black hole. A slight fluctuation of the electromagnetic field is accompanied with the fluid motion which is highly disturbed in the super-fast region just beyond the fast-surface. Some portion of the energy of this fluid disturbance can escape to a sub-fast region by virtue of the meridional propagation and concentrate toward the polar sub-fast region in the form of outgoing fast-magnetosonic waves.

Viewed in this point, the following scenario can be established: Perturbation energy is supplied from the equatorial disk into the middle latitudes or low latitudes in the form of *ingoing* fast-magnetosonic waves. In a magnetically dominated magnetosphere, the perturbation energy is mostly supplied electromagnetically from the surroundings. During the propagation the ingoing waves will be scattered to produce the outgoing waves. Especially in the vicinity of the fast-surface, the active MHD interactions will establish some equipartition of energy between the fluid and the electromagnetic field ($m_p n [u^r]^2 \approx [e_r]^2$). In other words, a lot of perturbation energy is transported from the magnetic field to the fluid there. Therefore, some portion of the perturbation energy supplied originally from the equatorial disk can be finally transported to the polar regions in the form of *outgoing* fast-magnetosonic waves. It would be possible to argue (though the evidence is not compelling) that the poleward collimation of waves may bring about the acceleration of particles in jets. To understand how the wave energy is converted into particle's kinetic energy is an important problem for developing the MHD scenario of jet formation in the very central region of AGNs. For example, in addition to the heating of the surrounding plasma by the dissipation of wave energy into thermal energy, the wave stress exerted on the expanding atmosphere may result in a direct transfer of momentum and energy from the waves to the outgoing winds (Jacques 1977). This will be the subject of future investigations.

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THE AXISYMMETRIC NON-DEGENERATE BLACK HOLE MAGNETOSPHERE

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Abstract The nature of the axisymmetric steady magnetosphere surrounding a rotating black hole is studied, allowing for potential variations along magnetic field lines. Assuming split monopole geometry for the magnetic field, toroidal magnetic field and electric potential in the magnetosphere are solved from the Poisson-Maxwell equation in 3 + 1 formalism using Kerr's axisymmetric curved space. It is shown that the electric potential near the horizon would become zero because of the radiative boundary condition at the horizon and potential difference must exist between the plasma source and the horizon, then the positrons emitted inwardly from the shell plasma source surrounding the hole are accelerated to highly relativistic energies by electric force as the plasma goes in. It is suggested that these highly accelerated particles would be suffered with incoherent radiation drag yielding dissipation of plasma energy that might be observed from infinity.

1. Introduction

The electrodynamics of the rotating black holes was constructed by Blandford and Znajek (1977) for force-free magnetosphere in the contexts of the explanation of prime mover for AGNs. The basic idea of the Blandford-Znajek process goes back to pulsar electrodynamics. In their seminal work, Goldreich and Julian (1969) presented an electrodynamical model for rotation powered, aligned pulsars. Blandford and Znajek made an analogy of pulsar playing like a Faraday disc that the surface of neutron star was replaced with the event horizon and outside magnetic field was rotated by the general relativistic dragging of inertia frame. In this manner hole itself can emit its irreducible mass (i.e. rotation energy) asymptotically to the infinity through magnetic field line. The outgoing energy flow can be interpret as an ingoing negative energy flow, as in the Penrose process, an essentially general relativistic effect. On the other hands controversial arguments against the Blandford-Znajek mechanism were presented by Punsly and Coroniti (1990). They re-examined the analogies between pulsar physics and black hole physics and claimed that black holes behave just as passive receptor of information from outside. With some "gedankenexperiment" about mhd wind accreting onto the horizon, they insist that the boundary condition at the horizon used by Blandford and Znajek can not decide energy extraction rate because the argument implicitly assumes the causal relation between the horizon and the infinity, that is no longer valid in Punsly and Coroniti's opinion, and hence it can't be a unipolar inductor without closed current circuit. It seems to be required more careful studies in various aspects to make sure about this issue.

This work was motivated to clarify the nature of the black hole magnetosphere, especially the mechanism of electrodynamical plasma acceleration that should be crucial to establish the global closed circuit system. To see this effects one has to solve the Poisson-Maxwell equation and compute electric potential including the component that remains for observers who rotate with magnetic field line, *non-corotational* potential. This problem has been well examined in pulsar physics to show breakdown of Goldreich-Julian condition (i.e. $\mathbf{E} \cdot \mathbf{B} = 0$) near light-cylinder or to see plasma acceleration at polar cap (*cf.* Mestel *et al.* 1985, and references therein). We generalize this problem to general relativistic *non-GJ* black hole electrodynamics. The most significant difference in physics of the black holes from the neutron stars is that only inflow can exist near the horizon. This fact requires us to have a plasma

source region sitting between inner light-surface and outer light-surface. Inside the inner light-surface plasma can not flow outwardly because of inward centrifugal force. Likewise outside of the outer light-surface, plasma can flow only outwardly as a relativistic wind. In Section 2 of this paper we derive equations describing stationary axisymmetric electric field around a rotating Kerr-black hole with both the corotational term and a non-corotational term. These are applied in this work to a hole being surrounded with an equatorial plasma sheet and a plasma shell. Assuming split monopole geometry for magnetic field, the source term of the Poisson-Maxwell equation can be simplified with the current conservation law. We give toroidal component of magnetic field in term of total volume current Q . In Section 3, we discuss solutions for the electric potential and field. In Section 4 we look plasma acceleration near the horizon. Conclusion and prospect to the construction of global magnetosphere model are simply discussed in Section 5.

2. Basic equations

2.1 Steady axisymmetric equations of electrodynamics in absolute space

In the absolute-space/universal-time (3+1) formalism (Macdonald and Thorne 1982), the hole is characterized by an absolute three-dimensional space with curved geometry instead of a four-dimensional space-time geometry. For the special case of a rotating Kerr black holes of mass M and angular momentum parameter $a \equiv J_H/Mc$, a Boyer-Lindquist spatial coordinate system is

$$\begin{aligned} ds^2 &= (\rho^2/\Delta)dr^2 + \rho^2 d\theta^2 + \varpi^2 d\phi^2, \\ \rho^2 &\equiv r^2 + a^2 \cos^2 \theta, \quad \Delta \equiv r^2 - 2GM\tau/c^2 + a^2, \quad \Sigma^2 \equiv (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta, \quad m = GM/c^2, \\ \omega &= 2aGM\tau/c\Sigma^2 = 2acm/\tau^3, \quad \alpha = \rho\Delta^{1/2}/\Sigma, \quad \varpi = (\Sigma/\rho)\sin \theta, \quad g = -(c^2/2\alpha^2)\nabla\alpha^2, \end{aligned} \quad (1)$$

where ω is the angular velocity of the local zero angular momentum observers (ZAMOs) with respect to infinity that represents the dragging of the inertia frame, α is the lapse function, or the redshift factor between the ZAMOs and infinity, and ϖ is the length of Killing vector $\mathbf{m} = \varpi^2 \nabla\phi = \varpi \mathbf{e}_\phi = \varpi \hat{\mathbf{t}}$, where $\hat{\mathbf{t}}$ is the unit toroidal vector. The operator ∇ is the covariant derivative operator in absolute space, and $\mathbf{g}$ is the gravitational acceleration measured by a ZAMO. In this 3+1 formalism the observers who measure the electromagnetic field, the charge density and the current are the ZAMOs. The steady axisymmetric Maxwell equations are

$$\nabla \cdot \mathbf{E} = 4\pi \rho_e, \quad (2)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (3)$$

$$\nabla \times (\alpha \mathbf{B}) = 4\pi \alpha \mathbf{j}/c - (1/c)(\mathbf{E} \cdot \nabla \omega) \mathbf{m}, \quad (4)$$

$$\nabla \times (\alpha \mathbf{E}) = -(1/c)(\mathbf{B} \cdot \nabla \omega) \mathbf{m} \quad (5)$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields measured by the ZAMOs and the fields measured by the observer at infinity are $\alpha \mathbf{E}$ and $\alpha \mathbf{B}$. To know the electromagnetic field around a black hole, we need to solve these equations with a boundary condition on the field at the horizon, which requires that the field looks to ZAMOs near the horizon like an infinitely blueshifted, ingoing electromagnetic plane wave

$$E_r = O(1), \quad B_r = O(1),$$

$$E_{\parallel} = O(\alpha^{-1}), \quad B_{\parallel} = O(\alpha^{-1}),$$

$$E_{\parallel} - \mathbf{n} \times B_{\parallel} = O(\alpha). \quad (6)$$

where $\parallel$ denote the field components parallel to the horizon and $\mathbf{n}$ is an outwardly directed normal to the horizon. This radiation condition ensures that the electromagnetic field observed by an infalling observer is physically well-behaved (Znajek 1977).

By (3) the poloidal magnetic field B_P is described by the magnetic flux function $\Psi = \Psi(r, \theta)$, given by

$$B_P = \frac{\nabla \Psi \times \mathbf{m}}{2\pi\varpi^2}. \quad (7)$$

From (5) one can write the electric field surrounding the black hole in terms of the non-degenerate electric potential $\chi = \chi(r, \theta)$, the magnetic flux function $\Psi = \Psi(r, \theta)$, and the angular velocity $\Omega_F = \Omega_F(r, \theta)$ of a field-line seen from infinity,

$$\mathbf{E} = -\frac{1}{\alpha} \nabla \chi + \frac{(\omega - \Omega_F)}{2\pi\alpha c} \nabla \Psi. \quad (8)$$

The limit of no redshift ($\alpha \rightarrow 1$) and no frame-dragging ($\omega \rightarrow 0$) corresponds to the Newtonian expression, well known in pulsar physics (*cf.* Mestel *et al.* 1985). The continuity equation for the current is, from the Maxwell equation (4), $\nabla \cdot (\alpha \mathbf{j}_P) = 0$ so that one can write

$$\alpha \mathbf{j}_P = \nabla \times \left[\frac{I(S)}{2\pi\varpi} \hat{\mathbf{t}} \right] = \frac{dI}{dS} \left(\frac{\nabla S \times \hat{\mathbf{t}}}{2\pi\varpi} \right) \quad (9)$$

where $S = \text{constant}$ defines a stream line and I is the total current passing downward through a $\mathbf{m}$ -loop (circle of constant r and θ). In a charge-separated domain, we have $\mathbf{j} = \rho_e \mathbf{v}$, and then

$$\alpha \rho_e \mathbf{v}_P = \frac{dI}{dS} \left(\frac{\nabla S \times \hat{\mathbf{t}}}{2\pi\varpi} \right). \quad (10)$$

The non-dissipative equation of motion for a test particle of charge q and rest mass μ is

$$\alpha^{-1} [\partial/\partial t + (\alpha \mathbf{v} + \omega \mathbf{m}) \cdot \nabla] \mathbf{p} = \mu \Gamma \mathbf{g} + \alpha^{-1} [\omega (\mathbf{p} \cdot \nabla) \mathbf{m} - (\mathbf{p} \cdot \mathbf{m}) \nabla \omega] + q [\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B}] \quad (11)$$

where $\mathbf{v}$ is the particle's physical velocity relative to ZAMOs, $\mu \Gamma$ is its mass-energy and $\mathbf{p}$ is its momentum as measured by the ZAMOs, $\Gamma \equiv (1 - \mathbf{v}^2/c^2)^{-1/2}$, $\mathbf{p} \equiv \mu \Gamma \mathbf{v}$. In an axisymmetric steady state, (11) leads to

$$\nabla(\mu \Gamma c^2) - \mathbf{v} \times (\nabla \times \Gamma \mu \mathbf{v}) = -\Gamma \mu c^2 \frac{\nabla \alpha^2}{2\alpha^2} - \frac{\Gamma \mu \Omega \varpi^2}{\alpha} \nabla \omega + \left[\frac{(\omega - \Omega_F)}{2\pi\alpha c} \nabla \Psi - \frac{1}{\alpha} \nabla \chi + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right], \quad (12)$$

where $\Omega \varpi = v_\phi$. The scalar product of (13) with $\mathbf{v}$ yields the energy integral

$$\mathbf{v} \cdot \nabla \left[\alpha \mu \Gamma c^2 + (\Gamma \mu \Omega \varpi^2) \omega + \frac{q}{2\pi c} \int \Omega_F d\Psi + q\chi \right] = 0 \quad (13)$$

or

$$\alpha \mu \Gamma c^2 + (\Gamma \mu \Omega \varpi^2) \omega + \frac{q}{2\pi c} \int \Omega_F d\Psi + q\chi = q\chi^*(S) \quad (14)$$

which is constant on streamlines. Also from the toroidal component of (1)

$$\mathbf{v} \cdot \nabla \left(\mu \Gamma \Omega \varpi^2 + \frac{q}{2\pi c} \Psi \right) = 0 \quad (15)$$

or

$$\mu \Gamma \Omega \varpi^2 + \frac{q}{2\pi c} \Psi = \frac{q}{2\pi c} \Psi^*(S) \quad (16)$$

which is constant on streamlines. To consider the simplest case, if Ω_F is uniform for each field line then from (14) and (16), eliminate Ψ ,

$$\alpha \mu \Gamma c^2 + \Gamma \mu \Omega \varpi^2 (\omega - \Omega_F) + q\chi = q \left[\chi^*(S) - \frac{\Omega_F}{2\pi c} \Psi^*(S) \right]. \quad (17)$$

This is a generalization of the Bernoulli integral of the special relativistic pulsar problem (*cf.* Mestel *et al.* 1985 and reference therein). If inertial terms were zero, then (15) would become $\mathbf{v}_P \cdot \nabla \Psi = 0$, i.e. $\mathbf{v}_P$ parallel to B_P . The actual deviation from poloidal flow along B_P is measured by the ratio $\mu \Gamma \Omega \varpi^2 / (q \Psi / 2\pi c) \simeq \Gamma \Omega / (q B / mc)$ as in pulsar theory, showing that inertial drifts are important only for very high Γ values. Provided also dissipation terms remain small, then the quasi-mhd approximation $\mathbf{v}_P$ parallel to B_P remains a good approximation.

2.2 Special case: a radial poloidal magnetic field; slow rotation

A full treatment would relate the poloidal flux function Ψ to the toroidal currents, through the toroidal component of the Ampère-Maxwell equation. In this first study we shall bypass this, and instead adopt the simple split monopole form for B_P . Thus we take

$$\Psi = \begin{cases} \Psi_0(1 - \cos \theta) & \text{in } 0 < \theta < \pi/2, \\ -\Psi_0(1 - \cos \theta) & \text{in } \pi/2 < \theta < \pi. \end{cases} \quad (18)$$

Substitution of (18) into (7) yields a radial poloidal magnetic field

$$B_P = -\frac{\hat{\mathbf{t}} \times \nabla \Psi}{2\pi \varpi} = \pm \frac{\Psi_0}{2\pi r^2} \hat{\mathbf{r}}, \quad (19)$$

where the $+$ sign is for the northern hemisphere and the $-$ for the southern in the aligned case.

In this preliminary investigation we assume $\Omega_F = \text{constant}$ for simplicity. We make the slow rotation approximation, $\lambda = a/2m$ is small, and drop terms in a except the frame-dragging term in ω . The metric (1) now reduces to the Schwarzschild metric,

$$\begin{aligned} ds^2 &\simeq (1 - 2m/r)^{-1/2} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \\ \rho &\simeq r, \quad \Delta \simeq r^2 - 2GM r/c^2 = r(r - 2m), \quad \Sigma \simeq r^2, \quad \omega \simeq 2mca/r^3, \\ \alpha &\simeq \Delta^{1/2}/r = [(r - 2m)/r]^{1/2}, \quad \varpi \simeq r \sin \theta. \end{aligned} \quad (20)$$

Then from the Poisson-Maxwell equation (2) combined with expression (8) for E , with the metric approximated by (20), we have

$$4\pi \varrho_c = \nabla \cdot \mathbf{E} = -\frac{\Delta^{1/2}}{\rho^2 \varpi} \left[\frac{\partial}{\partial r} \left(\frac{\varpi \Delta^{1/2}}{\alpha} \frac{\partial \chi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\frac{\varpi}{\Delta^{1/2} \alpha} \frac{\partial \chi}{\partial \theta} \right) \right] + \frac{\Delta^{1/2}}{\rho^2 \varpi} \frac{\partial}{\partial \theta} \left[\frac{(\omega - \Omega_F) \Psi_0}{2\pi \alpha c} \frac{\varpi}{\Delta^{1/2}} \sin \theta \right]$$

$$\simeq \frac{\Delta^{1/2}}{r^3} \frac{1}{\sin \theta} \left[-\sin \theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial \chi}{\partial r} \right) + \frac{r^2}{\Delta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \chi}{\partial \theta} \right) - \frac{(\omega - \Omega_F) \Psi_0}{\pi} \frac{r}{\Delta^{1/2}} \sin \theta \cos \theta \right]. \quad (21)$$

In the “degenerate” case, with $\chi = \text{constant}$ - the analogue of the Goldreich-Julian approximation in pulsar theory - only the term in $(\omega - \Omega_F)$ survives on the right-hand side, and with the above approximations it is proportional to $\cos \theta$. Our main aim is to solve χ from (21) in the non-degenerate case.

We have assumed a split-monopolar field for B_P as shown in (19). This implies in the non-dissipative and quasi-mhd domain that the poloidal flow parallel to B_P be radial, and hence the stream function S in (9) is a function of θ only, to yield

$$2\pi \alpha j_r r^2 \sin \theta = -Q F(\theta) \sin \theta, \quad \frac{dI}{d\theta} = -Q F(\theta) \sin \theta \quad (22)$$

where the total volume current Q is a constant normalizing the current flux.

If the system is charge separated,

$$\mathbf{j}_r = j_r \hat{\mathbf{r}} = \varrho_e v_P \hat{\mathbf{r}} \quad (23)$$

and then

$$\varrho_e = -\frac{Q F(\theta)}{2\pi \alpha r^2 v_P}. \quad (24)$$

From Ampère's law (4), B_T is given by

$$\nabla \times (\alpha B_T) = \frac{4\pi}{c} \alpha j_r = \frac{4\pi}{c} \alpha \varrho_e v_P \hat{\mathbf{r}}. \quad (25)$$

With the above approximation, $\nabla \times (\alpha B_T)$ has only a radial component if $r \sin \theta \alpha B_T = G(\theta)$ so that

$$\frac{4\pi}{c} \alpha j_r = -\frac{1}{r^2 \sin \theta} G'(\theta) = -\frac{2Q F(\theta)}{c r^2} \quad (26)$$

and

$$G'(\theta) = -\frac{2Q}{c} F(\theta) \sin \theta. \quad (27)$$

This and $F(\theta)$ is known, $G(\theta)$ follows.

3. Solution of electric potential and field

3.1 Frozen-in inflow and velocity field

In general v_P and so also by (24) ϱ_e are complicate functions of both r and θ , and so the equation (21) for χ would not be separable. However, if $|v_T| \ll |v_P|$, the $V \equiv -v_P/c$, then will be approximately unity. We begin with the simplest case. The choice $F(\theta) = \cos \theta$ in (22), then (21) yields from (24) $\varrho_e \propto \cos \theta$, and there exists the solution of (21) of the form of $\chi = f(r) \cos \theta$. With the definition of non-dimensional quantities such as

$$f(r) \equiv \frac{\Psi_0 a}{4\pi m^2} L(R), \quad \omega \equiv \frac{ac}{4m^2} \frac{1}{R^3}, \quad \Omega_F \equiv \frac{ac}{4m^2} \bar{\Omega}_F, \quad Q \equiv \frac{\Psi_0 ac}{8\pi m^2} \bar{Q}, \quad \text{and} \quad \alpha = \sqrt{\frac{R-1}{R}},$$

we arrive at the differential equation

$$R(R-1)L'' + 2(R-1)L' - 2L = H, \quad H = H(R) \equiv \frac{1}{R^3} - \tilde{\Omega}_F - \frac{\tilde{Q}}{V}. \quad (28)$$

Also we have from (27), $G(\theta) = -(Q/c)\sin^2\theta$, and hence the associated toroidal field is

$$B_T = -\frac{Q}{c} \frac{1}{r\alpha} \sin\theta \hat{t} = -\frac{\lambda\Psi_0\tilde{Q}}{16\pi m^2 R\alpha} \sin\theta \hat{t} = -AB_P \sin\theta \hat{t}, \quad A \equiv \lambda\tilde{Q} \frac{R}{\alpha} \quad (29)$$

If inertia and dissipative effects are small, we can assume the poloidal and toroidal components of the plasma velocity follow

$$v_P = \kappa B_P, \quad v_T = \kappa B_T + v_F \quad (30)$$

(i.e. plasma slides along magnetic field-lines) where

$$v_F \equiv \frac{\varpi(\Omega_F - \omega)}{\alpha} \hat{t} = \lambda c \left(\tilde{\Omega}_F - \frac{1}{R^3} \right) \frac{R}{\alpha} \sin\theta \hat{t} = -cK \sin\theta \hat{t}, \quad K \equiv \lambda \left(\frac{1}{R^3} - \tilde{\Omega}_F \right) \frac{R}{\alpha}.$$

v_F is interpreted as the velocity of the magnetic field lines relative to the ZAMOs which can be far greater than the speed of light (Macdonald and Thorne 1982). The surface where $v_F = c$ corresponds to the outer light surface and $v_F = -c$ to the inner light surface. The plasma source must be located between these two surfaces. As long as $E < B$, the relation (30), which can be derived directly from the equation of motion (12), remains good approximation even in the domain where the dissipation is relevant but radiation drag balances Lorentz force (cf. Fitzpatrick, R. & Mestel, L. 1988). If there still remain the residual electric field components parallel to the magnetic field, the plasma (positron, in this case) can be accelerated to nearly the light speed and one can assume $v^2 = v_P^2 + v_T^2 \simeq c^2$ as a good approximation. From (30) one can write the formula for v_P as follows, adopting $v \simeq c$,

$$v_P = \frac{B_P}{B} \left(-\frac{B_T}{B} v_F - \sqrt{v^2 - \frac{B_P^2}{B^2} v_F^2} \right) \simeq -cV,$$

where we take $-$ sign before the root (cf. Okamoto 1991), and non-dimensional ingoing velocity V is

$$V \equiv \left[\frac{B_T}{B} \frac{B_P}{B} \frac{\varpi(\Omega_F - \omega)}{\alpha c} + \frac{B_P}{B} \sqrt{1 - \frac{B_P^2}{B^2} \frac{\varpi^2(\Omega_F - \omega)^2}{\alpha^2 c^2}} \right] = \frac{AK \sin^2\theta + \sqrt{1 + (A^2 - K^2) \sin^2\theta}}{1 + A^2 \sin^2\theta}.$$

In fact there still remain unseparable variable θ in V and this makes the problem too complicate to solve (28). To overcome this difficulty we take the average of $1/V$ in the meridian plane as an approximation for $1/V$, i.e. from (24)

$$\varrho_e = -\frac{Q \cos\theta}{2\pi\alpha r^2 v_P} \simeq \overline{\varrho_e(r, \theta)} = \int_0^{\pi/2} \frac{Q}{2\pi\alpha r^2 V} \sin\theta d\theta \cos\theta = \frac{Q}{2\pi\alpha r^2} \left\langle \frac{1}{V} \right\rangle \cos\theta, \quad (31)$$

and

$$\left\langle \frac{1}{V} \right\rangle = \int_0^{\pi/2} \frac{1}{V} \sin\theta d\theta = \frac{A}{K} - \frac{P-Z}{K^2}, \quad (32)$$

where P and Z are functions of R . Getting the asymptotic zero-order solution at the horizon, one can show that the "radiative boundary condition" at the horizon (6) is satisfied only when

$$\tilde{Q} = 1 - \tilde{\Omega}_F. \quad (33)$$

hence $V = 1$, $v_P = -c$, and $v_T = 0$ at the horizon. Equation (33) can be rewritten as $I = \frac{1}{2}(B_P \varpi^2)_H(\Omega_H - \Omega_F)$ and this is equivalent to the condition already provided for the force-free case (Blandford and Znajek 1977, Macdonald and Thorne 1982). Adopting this boundary condition, manipulation gives

$$Z = p \left[\frac{\pi}{2} - \sin^{-1} \left(\frac{1}{\sqrt{p^2 + 1}} \right) \right], \quad \text{and} \quad P = \begin{cases} \frac{A}{\sqrt{K^2 - 1}} \ln \left[\frac{1 + m\alpha(K + \sqrt{K^2 - 1})}{\sqrt{p^2 + 1}} \right] & \text{for } K > 1 \\ \frac{A}{\sqrt{1 - K^2}} \cos^{-1} \left(\frac{\alpha m K + 1}{\sqrt{p^2 + 1}} \right) & \text{for } 1 > K > 0 \\ -\frac{A}{\sqrt{1 - K^2}} \cos^{-1} \left(-\frac{\alpha m K + 1}{\sqrt{p^2 + 1}} \right) - \pi & \text{for } -1 < K < 0 \end{cases}$$

where $m \equiv \lambda(1 + R + R^2)/R$ and $p \equiv \lambda(1 - 2\tilde{\Omega}_F + R^{-3})^{1/2} (1 + R + R^2)^{1/2}$.

3.2 Method of variation of parameters and boundary condition

The homogeneous equation $R(R - 1)L'' + 2(R - 1)L' - 2L = 0$ has two solutions

$$L_1 = R - 1, \quad L_2 = -1 - \frac{R - 1}{R} - 2(R - 1) \ln \left(\frac{R - 1}{R} \right),$$

then one can apply the Lagrange's method (variation of parameters) and the general solution of (28) is

$$L = L_1(R)g_1(R) + L_2(R)g_2(R) \quad (34)$$

where, having the wroksian $W \equiv L_1 L_2' - L_1' L_2 = 1/R^2$,

$$g_1 = - \int \frac{L_2 H}{W R (R - 1)} dR = - \int \left[\frac{1 - 2R}{R - 1} - 2R \ln \left(\frac{R - 1}{R} \right) \right] H dR, \quad (35)$$

$$g_2 = \int \frac{L_1 H}{W R (R - 1)} dR = \int R H dR. \quad (36)$$

From L one can calculate E_θ ,

$$\begin{aligned} E_\theta &= -\frac{1}{\alpha}(\nabla \chi)_\theta + \frac{\omega - \Omega_F}{2\pi\alpha c}(\nabla \Psi)_\theta = \frac{1}{r\alpha} \left(-\frac{\partial \chi}{\partial \theta} + \frac{\omega - \Omega_F}{2\pi c} \frac{\partial \Psi}{\partial \theta} \right) = \frac{1}{\Delta^{1/2}} \left(f + \frac{\omega - \Omega_F}{2\pi c} \Psi_0 \right) \sin \theta \\ &= \frac{\Psi_0 a}{16\pi m^3} \left[L - \frac{1}{2} \left(\tilde{\Omega}_F - \frac{1}{R^3} \right) \right] \sin \theta. \end{aligned} \quad (37)$$

We now show that the boundary condition (6) and hence (33) require L to be zero at the horizon. The “surface” magnetic field at the horizon would be purely toroidal,

$$B_H = \alpha B_T|_{r \rightarrow 2m} = -\frac{Q}{cr} \sin \theta e_\phi|_{r \rightarrow 2m} = -\frac{\Psi_0 a}{16\pi m^3} (1 - \tilde{\Omega}_F) \sin \theta e_\phi.$$

The tangential component of the “surface” electric field, which is purely poloidal, measured by the ZAMOs and diverges at the horizon as $1/\alpha$, is

$$E_H = \alpha E_\theta|_{r \rightarrow 2m} = \frac{\Psi_0 a}{16\pi m^3} \left[\left(\frac{1}{R^3} - \tilde{\Omega}_F \right) - 2L \right] \sin \theta e_\lambda,$$

then “radiative boundary condition” at the horizon (6), i.e. $B_H = -n \times E_H$ require L to be zero at the horizon.

3.3 Behavior of simple solutions

As simple investigations, we consider the following two cases.

i) Series expansion of H

If R_0 is near the horizon, one can apply the series expansion of $H = R^{-3} - \tilde{\Omega}_F - \tilde{Q}\langle 1/V \rangle$ at the horizon and get $H \simeq a_0 + a_1(R-1) + a_2(R-1)^2 + O(R-1)^3$. The boundary condition $\tilde{Q} = 1 - \tilde{\Omega}_F$ guarantees $a_0 = 0$. Finally one can get

$$L = -C_1(R-1) + \left(\frac{a_1}{6} - \frac{a_2}{3}\right)R + \frac{a_2 R^2}{4} + \frac{2a_1 - a_1}{6} \left[\frac{1}{2R} - 1 + (R-1)\ln R\right], \quad (38)$$

$$E_r = -\frac{\Psi_0 a}{8\pi m^3} \frac{dL}{dR} \cos \theta = \frac{\Psi_0 a}{8\pi m^3} \left[C_1 - \frac{a_1 - a_2}{2} + \frac{a_2}{2}R + \frac{2a_1 - a_1}{12} \left(\frac{1}{R^2} + \frac{2}{R} - 2\ln R \right) \right] \cos \theta \quad (39)$$

where

$$C_1 = \frac{a_1 - a_2}{2} - \frac{a_2}{2}R_0 - \frac{2a_1 - a_1}{12} \left(\frac{1}{R_0^2} + \frac{2}{R_0} - 2\ln R_0 \right) \quad (40)$$

(See the Appendix in detail). Note this solution satisfies the condition that L is always zero at the horizon.

As a numerical example, for $\Omega_F = \Omega_H/2$, which makes the minimum sliding velocity of the plasma along the field lines to the speed of light in this magnetic field geometry (see p372 in Macdonald and Thorne 1982), and $\lambda = 0.1$, we get $a_1 = -3.01482$ and $a_2 = 6.01421$ in the case $\lambda = 0.1$, $R_0 = 1.2$, and $\tilde{\Omega}_F = 1/2$ is shown in Figure 1 (a) and (b).

ii) Applying the fitting function of $\langle 1/V \rangle$

When R_0 is not very close to the horizon, the above method seem to be inapplicable because the solution must be decided by the boundary condition at R_0 . In fact the curve for $\langle 1/V \rangle$ is monotonous. Replacing this with the function which has same behavior but simpler form, one may get an exact solution of the equation with corresponding non-homogeneous term by means of the procedure described in Section 3.2. One can easily show that, say, $\sqrt{(R_0-1)/(R_0-R)}$, satisfy this condition with only slight deviation from $\langle 1/V \rangle$. Using this function, then from the same manner in the previous case, one may arrive at

$$L = -C_1(R-1) - \frac{1}{2R^2} + \frac{R}{4} + \left(\frac{2R_0}{3R} - \frac{1+4R_0}{3}\right) \sqrt{(R_0-1)(R_0-R)} \\ - (R-1) \left(\frac{8}{3} R_0^{\frac{3}{2}} \sqrt{R_0-1} \tanh^{-1} \sqrt{\frac{R_0-R}{R_0}} + \frac{1+4R_0-8R_0^2}{3} \tanh^{-1} \sqrt{\frac{R_0-R}{R_0-1}} \right) \quad (41)$$

and then

$$E_r = -\frac{d\chi}{dr} = \frac{\Psi_0 a}{8\pi m^3} \left[-\frac{1}{4} + C_1 - \frac{1}{R^3} - \frac{R_0(4R_0-1)}{3R} \sqrt{\frac{R_0-1}{R_0-R}} + \frac{2R_0}{3R^2} \sqrt{(R_0-1)(R_0-R)} \right. \\ \left. + \frac{8}{3} R_0^{\frac{3}{2}} \sqrt{R_0-1} \tanh^{-1} \sqrt{\frac{R_0-R}{R_0}} + \frac{1}{3} (1+4R_0-8R_0^2) \tanh^{-1} \sqrt{\frac{R_0-R}{R_0-1}} \right] \cos \theta. \quad (42)$$

For the value $R_0 = 20$, $\Omega_F = \Omega_H/2$ and $\lambda = 0.1$, the potential and field profile is shown in Figure 2. (a) and (b).

4. Plasma acceleration by electric force

From (17) one can write

$$\alpha\Gamma = \frac{\alpha_0\Gamma_0(1 - D_0) + \nu(L_0 - L)\cos\theta}{1 - D} \quad (43)$$

where D represents the effect of the slingshot, which can be given from (29), (30)

$$D = \frac{\Omega(\Omega_F - \omega)}{c^2\alpha}\varpi^2 = \frac{v_T v_F}{c^2} = \left[\left(\frac{B_T}{B_P} \right) \left(\frac{v_P}{c} \right) + \frac{v_F}{c} \right] \frac{v_F}{c} = K(K - AV)\sin^2\theta \quad (44)$$

and, because of the boundary condition, even near the equator (but not exactly on it which is occupied by the sheet current disc in our model), D take just small finite value at the horizon. On the other hand the factor of electrostatic term $\nu = (q/\mu c^2)(\Psi\lambda/2m) = (q/\mu c^2)(4\pi)(G/c^2)M_B P\lambda$ is very large and the acceleration by the electric force is far more dominating than that by the slingshot effect.

5. Discussion and concluding speculations

In our model we dealt the total velocity as given to be nearly the light speed and solved the deferential equation for the electric potential with the poloidal velocity field along split-monopole magnetic field. For self-consistency we need to solve the deferential equation for the total velocity (or Lorentz factor Γ) which is non-linear and quite complicate. Even though we left it as a future work, we have seen, as preliminary step, that, yielding the potential variation along the magnetic field line, there could be the strong electromagnetic acceleration of ingoing plasma. This effect seems to be crucial in plasma dynamics near the horizon. This strong acceleration might cause the particle to have very high Γ value and suffer radiation drag near the horizon. This is interested in respect to the observed high energy radiation from the cores of AGNs. On the other hand, if this Γ , is large enough there exists a domain where dissipation dominate and equations (14) ~ (17) can not hold. To construct a global magnetosphere model and see how central black holes emit their angular momentum and rotation energy to supply the energy for highly collimated jet and strong radio activity, we are required to include the dissipation term in the general relativistic equation of motion.

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Appendix: method of variation of parameters for the series H function

Substituting $H = a_1(R - 1) + a_2(R - 1)^2$ into (35) and (36), one can get

$$g_1 = -C_1 - \frac{1}{6}(4a_1 - 5a_2)R + \frac{1}{12}(8a_1 - 13a_2)R^2 + \frac{a_2}{2}R^3 + \frac{1}{6}(2a_1 - a_2)\ln(R - 1)$$

$$-\left(a_1 - a_2 - \frac{2}{3}a_1 R + \frac{3}{2}a_2 R - \frac{a_2}{2}R^2\right) R^2 \ln\left(\frac{R-1}{R}\right),$$

$$g_2 = C_2 - \frac{1}{2}(a_1 - a_2)R^2 + \frac{1}{3}(a_1 - 2a_2)R^3 + \frac{a_2}{4}R^4$$

(C_1 and C_2 are the constants). Then from (34)

$$L = C_1 - 2C_2 + \frac{C_2}{R} + \left(\frac{a_1}{6} - \frac{a_2}{3} - C_1\right)R + \frac{a_2}{4}R^2 + (R-1)\left[\frac{2a_1 - a_2 - 12C_2}{6} \ln(R-1) + 2C_2 \ln R\right]$$

$$E_r = -\frac{d\chi}{dr} = -\frac{df}{dr} \cos \theta = -\frac{\Psi_0 a}{8\pi m^3} \frac{dL}{dR} \cos \theta$$

and

$$\frac{dL}{dR} = \frac{a_1}{2} - \frac{a_2}{2} - C_1 - \frac{C_2}{R^2} - \frac{2C_2}{R} + \frac{a_2 R}{2} + \frac{1}{6}(2a_1 - a_2 - 12C_2) \ln(R-1) + 2C_2 \ln R.$$

The constant C_1 may be decided by the condition at the source plasma sphere boundary $R = R_0$, $E_r(R_0) = 0$, hence $dL/dR = 0$. C_2 must be $a_1/6 - a_2/12$ to make E_r finite at the horizon.

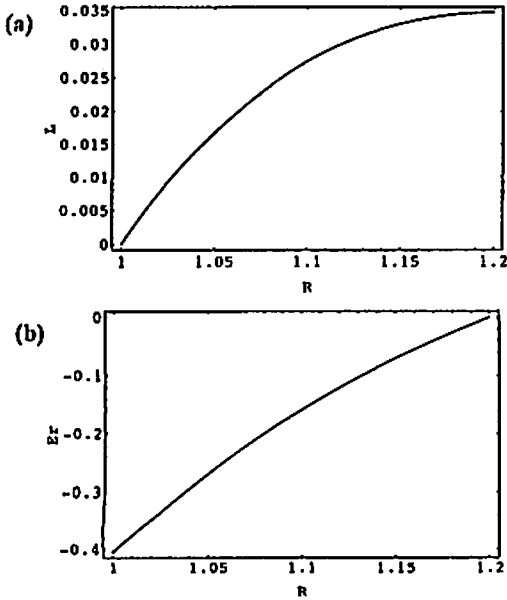


Figure 1. (a) Normalized Electric Potential L , and (b) Normalized Electric Potential E_r , for small R_0 (≈ 1.2)

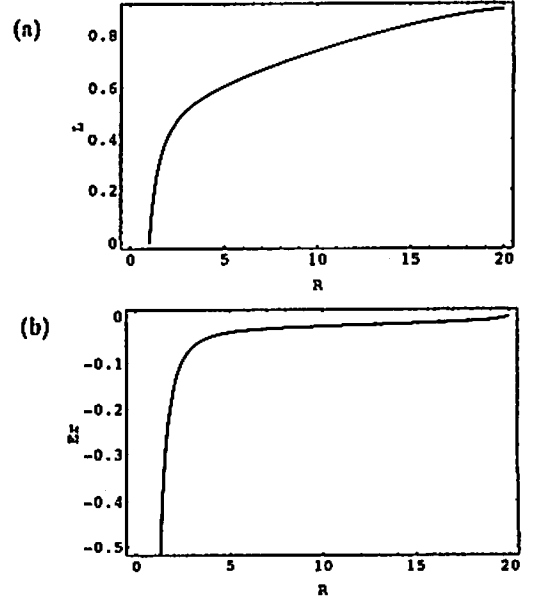


Figure 2. (a) Normalized Electric Potential L , and (b) Normalized Electric Potential E_r , for large R_0 (≈ 20)

A Role of Rotation and Neutrinos on Supernovae

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Abstract

Numerical simulations of convection in the supernova core were carried out in which axisymmetrically modified neutrino radiation from a rotating protoneutron star was assumed. Since neutrinos heat up matter around the rotational axis more intensively than around the equatorial plane, powerful and global convection of matter between the shock front and the protoneutron star is induced and moreover jet-like motion along the axis pushes the shock front into the prolate form. It is found that the hot bubble is distorted due to this quadrupolar convective motion and, consequently, that turbulent instability occurs behind the shock wave. This instability is considered to provide the initial fluctuation of Rayleigh-Taylor instability in the envelope, the seed of matter mixing in SN 1987A. We also found that three outstanding problems, namely, the seed of matter mixing, high velocity component of mixed elements, and observed asymmetry in the envelopes, can be explained simultaneously by those effects. The cooperative effect of rotation and neutrinos may be the key to the explosion mechanism problem of supernovae.

1. Introduction

There exist many unsettled problems in the type II supernova research principally because of lack of our knowledge on the explosion mechanism. Recently, however, we are getting many interesting observations on supernovae, which, we expect, guide us into the core of the explosion mechanism problem. Some of them are incompatible with theories based on the assumption of spherical symmetry. We here would like to pay attention to the importance of aspherical mechanisms and focus on the following four significant subjects: First of all, we do not have a satisfactory answer to the explosion energy problem yet; most simulations of core collapse either fail to reproduce an explosion or yield an anemic explosion with the explosion energy of by a few factor less than $1.5 \pm 0.5 \times 10^{51}$ erg as observed in SN 1987A. Some artificial assumptions, e.g., mixing length theory for convection (Wilson & Mayle 1988), are necessary to explain the explosion energy without exception so far. Second, in SN 1987A, several elements were observed to be globally mixed up during the explosion. Although the Rayleigh-Taylor instability is able to explain the gross properties

of this matter mixing, the problem is that large initial fluctuations of either 10% in the envelope or over 30% in the Si and Fe core are required (Den, Yoshida, & Yamada 1990; Yamada & Sato 1991). Third, the Rayleigh-Taylor instability in a spherical explosion alone cannot explain the large value of the mixing velocity; the resulting value from straightforward simulations is only half of the actually observed value (Herant & Benz 1991; Yamada & Sato 1991). Finally, there are convincing observations that indicate asymmetry in the envelope of SN 1987A (Cropper et al. 1988; Papaliolios et al. 1989). It is surprisingly interesting that the same kind of asymmetry was also observed in SN 1993J (Trammell, Hines, & Wheeler 1993). Hence it has become more probable that asymmetry is a general feature in type II supernovae.

Here we introduce our basic idea for the explanation of these subjects. As in the context of the delayed explosion mechanism (Wilson 1985), we consider that neutrino heating on the matter is crucial and regulates the dynamics of the explosion. Thereby it is greatly interesting to consider what would happen if the neutrino radiation is considerably aspherical, jet-like, not spherical as assumed in most of literatures, and thus the neutrinos heat matter mainly along the axis of symmetry. For, an increase in the neutrino luminosity by only a factor of 1.5 or so from an equilibrium value easily yields an energetic explosion in which the rate of energy gain is over $6 \times 10^{51} \text{ erg s}^{-1}$ (Shimizu, Yamada, & Sato 1993b; see also Wilson & Mayle 1988; Janka & Müller 1993; Burrows & Goshy 1993).

Then what is the cause of aspherical neutrino radiation? It is known that heavy main sequence stars, type II supernova progenitors, in general, rotate fast with the surface velocity of about 200 km s^{-1} by observations (e.g., Tassoul 1978). Although the angular momentum distribution in the star is unfortunately unknown, we can expect that the collapsed core may rotate as fast as millisecond pulsars unless some efficient mechanism of angular momentum transport works. If this is the case, since the spatial distributions of both density and temperature are deformed due to the centrifugal force under such fast rotation, the neutrino sphere is no longer a sphere but becomes a spheroid. Therefore neutrinos are emitted from the “neutrino spheroid” aspherically and preferentially to the vicinity along the rotation axis due to the difference in the apparent size of the spheroid seen from each position. In fact, in the context of the neutrino detection in SN 1987A, Janka & Mönchmeyer (1989a,b) discussed a possibility that the neutrino flux observed by an observer may differ by a factor of 3 with the inclination from the rotation axis. They used, in estimating the neutrino anisotropy, their rotating core collapse models in which the protoneutron star becomes oblate and its ratio between the semimajor and semiminor axes is between 1.5 and 2.0. In similar core collapse simulations by Yamada and Sato (1994), the produced neutron stars are also deformed with the axis ratio of about 1.5.

Although many numerical simulations were done on convection in the supernova core (Herant, Benz, & Colgate 1992; Burrows & Fryxell 1992; Janka & Müller 1993; Miller, Wilson, & Mayle 1993; Yamada, Shimizu, & Sato 1993; Shimizu, Yamada, & Sato 1993a,b), we further assumed here that the neutrino emission is enhanced along the rotational axis and suppressed along the equatorial plane. We will show below how such jet-like neutrino flow can solve the problems mentioned above.

2. Numerical Simulations of Convection: Asymmetric Neutrino Radiation

We have carried out three-dimensional simulations of convection in the shocked matter of the supernova core, assuming the neutrino radiation from the protoneutron star is radial but axisymmetric. Since there are many ambiguities in what to be assumed, we, as the first step, adopted a simple model as follows: The interaction of neutrinos with the matter were included in the simulations as heating and cooling terms, where ν_e , $\bar{\nu}_e$, e^- , e^+ captures on free nucleons are dominant terms. The neutrino flux at each point was estimated on the assumption of black body radiation from the surface of Maclaurin spheroid, which models the rotating hot neutron star. Its dimensions were determined so that the total luminosity and the black body temperature (L_ν and T_ν , respectively, for each individual neutrino species; T_ν is constant on the spheroid surface) would be fixed at the same values as in a spherically symmetric case. Then the local neutrino flux for each species at the point (r, θ) in spherical coordinates is given by a geometrical consideration as:

$$l_\nu = (7/16)\sigma T_\nu^4 a_x \sqrt{a_z^2 \cos^2 \theta + a_x^2 \sin^2 \theta} / r^2, \quad (1)$$

where σ , a_x , and a_z are the Stefan-Boltzmann constant, the semimajor and semiminor axes, respectively. Note that the neutrino fluxes along the rotational axis and the equatorial plane are proportional to the projected areas of the spheroid, πa_z^2 and $\pi a_x a_z$, respectively. One can easily recover the total luminosity as $L_\nu = (7/16)S(a_x, a_z)\sigma T_\nu^4$ by integrating equation (1) over the whole solid angle, where $S(a_x, a_z)$ is the area of the spheroidal surface.

We used the same numerical techniques and initial model as those developed to investigate convection in the core (Shimizu et al. 1993b); the numerical code and the physical inputs were unchanged except for the increased grid number of 100^3 and the aspherical neutrino flow. The initial model was produced consistently with the adopted physical inputs on the assumption that the shocked matter is in a steady state and spherically symmetric. These are sufficient ingredients in order to see what would happen as a result of such axisymmetric neutrino emission.

3. Results and Neutrino Jet

We have done four calculations, including a convergence test, for three models with different parameters. A calculation of the first model was carried out on the assumption that $T_\nu = 5.0$ MeV and the axis ratio, $a_x/a_z = 1.5$. Note that the ratio between rotational and gravitational energies, $T/|W| = 0.105$ for the Maclaurin spheroid. The values of a_x and a_z are determined in order that the total luminosity should take the same value as a spherical case in which $a_x = a_z = 50$ km. Figure 1a shows the density distribution and the velocity field on one of meridian planes for the first model at the time of 27 ms; the spherical stalled shock front is initially placed at a radius of 200 km. Since neutrinos heat up matter around the rotational axis more intensively than around the equatorial plane, powerful and global convection of matter between the shock front and the protoneutron star is induced and also jet-like motion along the axis pushes the shock front into the prolate form. Such a

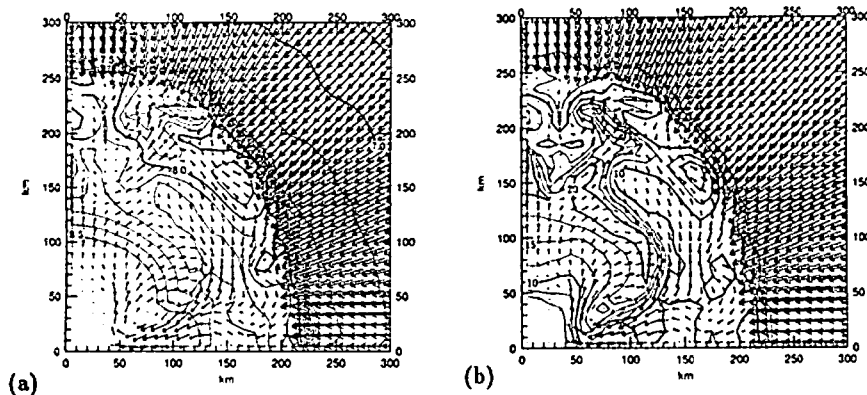


Figure 1: (a) Density contour map for the model of $a_x/a_z = 1.5$ at the time of 27 ms on a section that corresponds to one of meridian planes. The contour lines are spaced on a log scale. It shows that global convective motion is produced through the aspherical neutrino heating. The maximum convection velocity is about $2 \times 10^4 \text{ km s}^{-1}$. (b) The same as in Fig. 1a, except for the entropy distribution (linearly spaced). The high entropy hot-bubbles are highly deformed due to the global convective motion.

jet-like explosion is desirable to explain the asymmetry observed in SN 1987A as well as SN 1993J. Note that Ishikawa et al. (1992) found that the observed asymmetry (the axis ratio is around 1.5) requires either a jet-like explosion (the axis ratio of explosion energy should be more than 2) or an extremely distorted envelope and, furthermore, that the latter is unlikely.

Figure 1b is the same figure as Figure 1a except for the entropy distribution. It is clearly seen from this entropy contour map that the hot bubble (namely, high entropy region) is distorted due to the quadrupolar convective motion and, consequently, that turbulence-like instability of hot bubbles occurs behind the shock wave. Then such instability is considered to provide the initial fluctuation of Rayleigh-Taylor instability in the envelope, i.e., the seed of matter mixing in SN 1987A. Although the initial fluctuation as much as 30% was

simulation of Yamada & Sato 1994), aspherically convective and diffusive neutrino transfer inside the neutron star, temperature distribution on the neutrino spheroid, and neutrino-antineutrino pair annihilation just above the pole of the spheroid (note that its cross section is larger for head-on collisions).

Only slight energy gains due to the global convective motion were achieved for the first and second models, unlike the hot-bubble convection mechanism of Herant, Benz, Colgate (1992); this is because, we consider, the interior of neutron star should be included in our simulations. Nevertheless, our successful results with respect to other subjects should lead us to expect that a neutrino jet may be produced under some physical process inside the neutron star. One possible, important mechanism may be convective and diffusive neutrino transfer inside the neutrino star, which is also induced by the effect of rotation like above. Burrows and Fryxell (1993) recently demonstrated a remarkable effect of convective neutrino transfer on the enhancement of neutrino luminosity as well as explosion energy. Although the same effect is expected for a explosion with the neutrino jet, only radial neutrino transfer was included in their simulations, and thus the smoothing effect by the transverse neutrino diffusion was neglected. This effect does not seem to be negligible since it is shown that the time scale of diffusion is shorter than that of convective instability for the small-scale perturbation, using the random walk and Brünt-Väisälä analyses. On the other hand, it is considered that the jet created by the large-scale convective motion is free from such a difficulty, and that neutrino convective transfer and diffusion would cooperate with each other, producing a luminous neutrino jet.

The jet in a successful explosion is simulated in the third model in which only the black body temperature of neutrinos was increased by a factor of 4% compared with the second model. We confirmed that the global convective motion is enhanced rather than smoothed out, and that our above results are not altered even in the energetic explosion.

4. Conclusions

It has been suggested that three unsettled subjects in type II supernovae are solved and unified naturally on the only one assumption that the protoneutron star is rotating fast and thus the aspherical neutrino flow is induced by its rotation. (1) The seed of matter mixing is supplied by the turbulent instability induced in the global hot-bubble convection. (2) The jet-like explosion together with the large initial fluctuations in the core yields the high mixing velocity of elements in the envelope. (3) The jet-like explosion is the most reasonable interpretation for the asymmetry observed in both (!) SN 1987A and 1993J.

Considering these results, we can infer what the unified explosion model of type II supernovae should be as follows: The central core of the supernova progenitor rotates moderately and collapses sufficiently in order to release the gravitational energy in the form of neutrinos. Nevertheless, in the course of the collapse, the effect of rotation is turned on so that the high luminous neutrinos spout out of the protoneutron star along the rotational axis. This neutrinos heat up matter and drive part of the shocked core into a jet-like explosion (see also Sato, Shimizu, & Yamada 1993). We feel that we should not neglect the effect of either rotation or neutrinos in understanding the mechanism of supernova explosions.

In our scenario, the explosion energy problem is yet to be investigated. In order to check this, it is indispensable to carry out at least two-dimensional calculations that solve hydrodynamics consistently coupled with neutrino transports. Although solving equations of radiation hydrodynamics numerically in multi-dimensions is a very difficult task, we are now trying to develop a new scheme to solve this problem.

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Gamma-ray Bursts and Gravitational Collapse

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1 Introduction

Gamma-ray bursts have remained an unsolved problem in astrophysics for 15 years since the discovery. "Ginga" shows the sources have a strong magnetic field $10^{12}G$ which is consistent with the model of cyclotron absorption [10]. A neutron star is known as the only object which has such strong magnetic field. Then, before 1991, it was believed that the gamma-ray burst origin is near-by neutron stars. After the observation by GRO [4], however, coalescence neutron star binaries at cosmological distance becomes believed as the best candidate for the burst source. High energy relativistic plasma should be produced at the coalescence phase, called as fire ball model [3],[9],[11]. However, in this process, the energy of the emitting neutrino should be converted to the kinetic energy in the relativistic expanding fire ball with high efficiency and low baryon contamination. The low baryon contamination $< 10^{-6}M_{\odot}$ is strong constraint in the coalescence binary scenario. The strong magnetic field is one possibility for extracting the energy with low baryon contamination. In our consideration, the strong magnetic field can be generated by the strong convection motion induced from neutrino emission. In early phase of the binary coalescence, the released energy with gravitational collapse is partly stored into the strong magnetic field with low baryon contamination. In final phase, the system collapses into a black hole with the magnetic field. This field also falls into the hole. As the back reaction of this, the electromagnetic radiation can be emitted from the hole, well known as quasi normal mode (QNM). This process can supply strong electromagnetic wave with no baryon fraction. This is convenient to produce the baryon free fire ball. In this paper, I would like to consider physical model related to the relativistic high energy plasma process along the magnetic fire ball scenario.

2 Convection Dynamo and Strong Magnetic Formation

In the gravitational collapse of neutron stars, the outer part of a neutron star becomes hot due to shock heating in supernova explosion or the tidal heating in binary coalescence. In this process, the convection can be induced generally. We mainly summarize the convection process in supernova, which is well studied recently.

The heating with core bounce induces convection by an entropy gradient and the lepton density gradient Y_l . Convective instability occurs at

$$\frac{dS}{dr} + \left(\frac{\partial S}{\partial Y_l}\right)_{P,\rho} \frac{dY_l}{dr} < 0. \quad (1)$$

In previous works, two independent processes govern the convection. First, the outgoing shock weakens as it dissociates heavy nuclei. Second, the surface of the proto neutron star lose entropy and lepton number faster than the interior. Both process produce a negative radial gradient for the entropy.

In the gravitational collapse, the energy $\simeq 10^{53}$ ergs in the form of neutrinos. An atmosphere in pressure equilibrium on a neutron star is hotter at its base as it emits neutrinos. It drives the convection, which have a certain convective luminosity L_{con} at radius R and density ρ . According to the mixing-length theory [1], the convective velocity is estimated as

$$V_{con} = \left(\frac{\Gamma - 1}{2\Gamma} \frac{L_{con}}{4\pi R^2 \rho}\right)^{1/3} \simeq 10^8 \left(\frac{L_{con}}{10^{52} \text{ergs}^{-1}}\right)^{1/3} \left(\frac{\rho}{10^{14} \text{gcm}^{-3}}\right)^{-1/3} \left(\frac{R}{10 \text{km}}\right)^{-2/3} \text{cms}^{-1}. \quad (2)$$

We assumed the material is semidegenerate fermion fluid with $\Gamma = 5/3$. This strong convection induces the dynamo action for the amplification of magnetic field. This dynamo effect should be saturated when the magnetic energy becomes comparable to the kinetic energy of the convection. The saturated magnetic field B_{sat} is

$$B_{sat} \simeq (4\pi \rho V_{con}^2)^{1/2} \simeq 10^{15} \frac{V_{con}}{10^8 \text{cms}^{-1}} \left(\frac{\rho}{10^{14} \text{g}}\right)^{1/2} \text{G}. \quad (3)$$

In the upper convection zone, the ratio of the RMS magnetic field to the saturated magnetic field $f = (B_{RMS}/B_{sat})^2 \simeq 0.1$. Then, well developed convective motion can produce the strong magnetic field 10^{15-16} G.

3 Decay of Black Hole Magnetosphere and Gamma-Ray Bursts

As shown above, in early phase of the binary coalescence, the released energy with gravitational collapse is partly stored into the strong magnetic field with low baryon contamination. In final phase, the system collapses into a black hole with the magnetic field. This field also falls into the hole. As the back reaction of this, the electromagnetic radiation can be emitted from the hole, well known as quasi normal mode (QNM). This process can supply strong electromagnetic wave with no baryon fraction.

Electromagnetic Fields around Schwarzschild Black Hole We will formulate the dynamical approach of general relativistic plasma based on the “3+1” formalism. This approach has been extensively applied to numerical relativity. Thorne and Macdonald (1982) have formulated Maxwell equations with the familiar language for astrophysicists.

Around a Schwarzschild black hole (B.H), by using the "3 + 1" formalism, we can write the Maxwell's equations for the electromagnetic field,

$$\begin{aligned}\frac{\partial \mathbf{E}}{\partial t} &= \nabla \times (\alpha \mathbf{B}) - 4\pi \alpha \mathbf{j}, & \nabla \cdot \mathbf{E} &= 4\pi \rho, \\ \frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times (\alpha \mathbf{E}), & \nabla \cdot \mathbf{B} &= 0,\end{aligned}\quad (4)$$

where the lapse function $\alpha = (1 - 2M/r)^{1/2}$.

The electric and magnetic fields can be derived from a scalar potential ϕ and a vector potential $\mathbf{A}$, $\mathbf{B} = \nabla \times \mathbf{A}$, $\mathbf{E} = -\frac{1}{\alpha} \nabla \alpha \phi - \frac{1}{\alpha} \frac{\partial \mathbf{A}}{\partial t}$. The solenoidal gauge derives the final equation for the vector potential,

$$-\frac{1}{\alpha^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{1}{\alpha} \nabla \times (\alpha \nabla \times \mathbf{A}) - 4\pi \mathbf{j}. \quad (5)$$

Representation with Vector Spherical Harmonics

Any vector field can be expressed by the spherical vector harmonics. Then, with introducing the "tortoise coordinate" r^* as Regge and Wheeler (1957) defined $dr^* = \frac{dr}{1 - \frac{2M}{r}}$, $r^* = r + 2M \ln \left[\frac{r}{2M} - 1 \right]$. We obtain the equation for the solution with the radial dependent as

$$\left[\frac{\partial^2}{\partial r^{*2}} - \frac{\partial^2}{\partial t^2} - V(r) \right] \hat{g}_{lm}(r, t) = -4\pi \alpha^2 \hat{j}_{lm} = S(r, \omega), \quad (6)$$

$$V(r) = \left(1 - \frac{2M}{r} \right) \left[\frac{l(l+1)}{r^2} \right], \quad (7)$$

where $\hat{g}_{lm} = r\alpha_{lm}$ or $r\beta_{lm}$ and $\hat{j}_{lm} = j_{\alpha,lm} \frac{r^2}{\alpha}$ or $j_{\beta,lm} r$. The electric transverse mode β is important.

Boundary Condition on the Horizon We can cover only the exterior of the black hole and need the boundary condition on the horizon ($\alpha = 0$). This boundary condition was given by Znajek (1976,1978) and Domour (1978,1979,1980). We can also read excellent reviews (eq. Carter 1979, Thorne and Macdonald 1982).

The language of absolute space and universal time for the fiducial observer can express the boundary condition on the horizon where is the two-dimensional surface $\alpha = 0$. The charge density σ_H and the current density $\mathbf{J}_H$ per unit area on the horizon can be rewritten by the electromagnetic field.

$$\sigma_H \equiv \frac{\mathbf{E} \cdot \mathbf{n}}{4\pi} \quad (8)$$

$$\mathbf{J}_H \equiv \frac{1}{4\pi} \mathbf{n} \times (\alpha \mathbf{B}_{\parallel}) \quad (9)$$

where $\mathbf{n}$ is the normal unit vector on the horizon and $\mathbf{B}_{\parallel}$ is the parallel component of the magnetic field on the horizon.

On the horizon, any wave is ingoing, which means the poyinting vector is directed to the inside of the hall. Then, we can estimate parallel electric field,

$$\alpha \mathbf{E}_{\parallel} = \mathbf{n} \times \alpha \mathbf{B}_{\parallel} \quad (10)$$

Using this equation, we obtain an Ohm's law :

$$\mathbf{J}_H = \frac{1}{4\pi} \alpha \mathbf{E}_{\parallel}. \quad (11)$$

We introduce renomalized electromagnetic fields

$$\mathbf{E}_H = \alpha \mathbf{E}_{\parallel} \quad , \quad \mathbf{B}_H = \alpha \mathbf{B}_{\parallel}. \quad (12)$$

In free falling coordinate, the physical properties is the same as that in the flat space. The vector values with subscript H can be interpreted to the field observed by free falling observer, because Lorentz factor γ , which is related to the Lorentz compression of the parallel field to the horizon surface, is equal to $1/\alpha$. From Eqs.(11) and (12), the surface resistivity $R_H \equiv 4\pi = 377 \quad (\Omega)$.

We also obtain the boundary condition for the vector potential governed by Eq.(5). From Eqs. (6), (7), and (10) with solenoidal gauge condition,

$$-\frac{\partial \mathbf{A}}{\partial t} = \mathbf{n} \times (\alpha \nabla \times \mathbf{A}). \quad (13)$$

Quasi Normal Mode for Electromagnetic Field around B.H.

We can see the asymptotic behavior of the solutions α , β , and γ as $r \rightarrow \infty$,

$$\begin{aligned} \beta &= O\left(\frac{1}{r}\right), \\ \alpha \text{ and } \gamma &= O\left(\frac{1}{r^2}\right). \end{aligned}$$

So α and γ which is related to the electric parity solution is damped more rapidly than β related to the magnetic parity solution. Since we are interested in the physical value at the observer who is far from the source of the wave, the property of β is more important than that of α and γ . Then, we will mainly discuss the magnetic parity solution β .

Let consider the Fourier transform

$$\hat{g}(r, t) = \int_{-\infty}^{\infty} g(r, \omega) e^{i\omega t} d\omega$$

when $\hat{g}(r, t)$ is $\beta_{lm}(r, t)$.

Eqs.(6) and (7) have the following form,

$$\left[\frac{\partial^2}{\partial r^{*2}} + \omega^2 - V(r) \right] g(r, \omega) = 0. \quad (14)$$

Since the potentials $V(r)$ become 0 as $r_* \rightarrow \pm\infty$, we have two independent asymptotic solutions for this type equation,

$$g \propto \exp(\mp i\omega r^*) \quad \text{as} \quad r^* \rightarrow \pm\infty \quad (15)$$

which correspond to outgoing waves at infinity and ingoing waves at the horizon.

For a complex frequency $\omega = \omega^0 + i\Gamma$ with $\omega^0 \neq 0$, however, the solution, which is related to the sponenious emission from the region near horizon, can exist. The solution have the asymptotic solution,

$$g_{QNM}^{EM}(r, \omega) = \begin{cases} A_{out}(\omega)e^{+i\omega r^*} & : r^* \rightarrow +\infty \\ A'_{in}(\omega)e^{-i\omega r^*} & : r^* \rightarrow -\infty \end{cases} \quad (16)$$

It will be called the quasi-normal mode (QNM) in anology with the normal mode of oscillation of a star, since the situation is similar to that of gravitational wave emission which correspond to the free oscillation of B.H.

Flux conservation presents that QNM ω must be complex, $\omega^0 + i\Gamma$ with $\omega^0 \neq 0$ and $\Gamma > 0$ in this case that the average value of the potential is positive for a QNM,

$$\int_{-\infty}^{\infty} V(r^*) |g|^2 dr^* / \int_{-\infty}^{\infty} |g|^2 dr^* \geq 0.$$

The QNM should be conserved with the wave emission process with its frequency and damping rate in the perturbed phase as the last stage of the some event with the electromagnetic wave excitation by the collapse. We obtain the QNM for the approximated potential,

$$\begin{aligned} \omega_l &= \omega_l^0 + i\Gamma \\ &= (3\sqrt{3}M)^{-1} \left[\left(l(l+1) - \frac{1}{4} \right)^{1/2} + i \left(n + \frac{1}{2} \right) \right]. \end{aligned} \quad (17)$$

This shows the magnetic field decays in a light crossing time around B.H. Then, we consider the impulsive electro-magnetic radiation can be produced from the strong magnetic field which stretched into B.H. with the collapsed material. This also induce the plasma acceleration as the electromagnetic wave pulse works as a piston on the relativistic plasma. We can see this as the magnetic fire ball with low baryon contamination.

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Non-Abelian Black Holes I (Neutral Holes)

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Abstract

We re-analyze the globally neutral non-Abelian black holes and present a unified picture, classifying them into two types; Type I (black holes with massless non-Abelian field) and Type II (black holes with “massive” non-Abelian field). For the Type II, there are two branches: The black hole in the high-entropy branch is “stable” and almost neutral, while that in the low entropy branch, which is similar to the Type I, is unstable and locally charged. To analyze their stabilities, we adopt the catastrophe theoretic method, which reveals us a universal picture of stability of the black holes. It is shown that the isolated Type II black hole has a fold catastrophe structure. In a heat bath system, the Type I black hole shows a cusp catastrophe, while the Type II has both fold and cusp catastrophe.

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1 Introduction

The discovery by Bartnik and McKinnon of a non-trivial particle-like structure (BM particle) in the Einstein-Yang-Mills system is quite surprise[1], because it is known that there is no non-trivial solution in the Einstein theory and Yang-Mills theory respectively and because the work of Bekenstein, Hartle and Teitelboim shows that stationary black hole solutions are similarly hairless in a variety of theories coupling classical fields to Einstein gravity[2]. After this solution, a variety of self-gravitating structure with non-Abelian fields have been found. Besides the BM particle, the colored black hole[3], the Skymion[4, 5] or the Skyrme black hole[6, 5, 7], the dilatonic BM particle or dilatonic colored black hole[8, 7] the particle solution with massive Proca field (Procaon) or the Proca black hole[9], the monopole[10, 11, 12] or the black hole in monopole (monopole black hole)[11, 12, 13], and the sphaleron[14, 9] or sphaleron black hole[9] are discovered.

However, the colored black hole is unstable against the radial perturbations[15]. While, the Skyrme black hole solution really stirred our interest for the non-Abelian black hole, since it turns out to be a stable solution[16]. Hence, it can be the likeliest candidate of a counterexample for the black hole no-hair conjecture. Because of the non-Abelian gauge fields, each study revealed that in spite of a simple ansatz (spherical symmetry) those non-Abelian black holes have complicated structures and show a variety of properties. Are there any common properties in those non-Abelian structures? Can we find any universal understanding for them? In order to answer these questions we first classify non-Abelian black holes into two classes; one is globally neutral ones and the other is globally charged ones, because the charged monopole black hole is obviously different from others. We will discuss those two cases separately[17]. Here we just focus on neutral case. The neutral holes can be classified into two: one is a class of black holes with massless non-Abelian field (Type I), and the other is that with “massive” non-Abelian field (Type II).

One of the most important questions about the above self-gravitating non-Abelian structure or black holes is , are they stable? We answer this question via a catastrophe-theoretic analysis, which shows its power over various phenomena. The elementary catastrophe of Type II black hole is found

to be a fold type, which causes the existence of stable and unstable solutions. We also apply this to stability analysis of black holes in a heat bath. In §2, we provide a unified picture. The stability analysis of non-Abelian black holes via a catastrophe theory are given in §3. Section 4 includes discussions and some remarks.

2 Summary of Non-Abelian Black Holes

We have re-analyzed five non-Abelian theories, which are listed in Table 1. Each theory has non-trivial self-gravitating structure (particle solution) and non-trivial black holes. We summarize their properties in order to find out a universal picture.

We can classify non-Abelian black holes by their properties into two types. The colored black hole and the dilatonic colored black hole are classified as Type I. While the Proca black hole, the Skyrme black hole, and the sphaleron black hole are Type II. Those non-Abelian fields have a “mass” μ . $\mu = g_S f_S$ for the Einstein-Skyrme system, and $\mu = g_C \Phi_0$ for the Einstein-Yang-Mills-Higgs system. Each type of black hole has the following common properties.

1. Type I Black Hole

- (1) This type of black hole is in an equilibrium state by balance between a repulsive force by the massless gauge field and the gravitational force. It has no upper bound of their mass. When the mass gets large, however, the spacetime approaches to the Schwarzschild black hole. In the limit of $r_H \rightarrow 0$, we find a particle-like solution, which is the (dilatonic-) BM particle and cannot exist without gravity.
- (2) The field strength at the horizon $B_H \equiv (\text{Tr} F^2)^{1/2} \Big|_{\text{horizon}}$, may give us a naive idea about how much the black hole is locally charged. Here, the expression B_H has been used because only the radial component of magnetic part of non-Abelian field is finite at the horizon. Although they are globally neutral, their structure near the horizon is similar to that of the Reissner-Nordström black hole. When the black hole gets large, B_H decreases, that is, it becomes effectively less charged because it approaches the Schwarzschild solution.
- (3) These black holes have discrete mass spectrum, which is characterized by the node number of YM potential. All solutions are unstable against the radial linear perturbations. The number of the unstable modes increases as the node number increases[18].
- (4) The sign changes of the specific heat occur twice, but for the large the coupling constant ($\alpha \gtrsim 0.5$) in the dilatonic colored black holes, there is no sign change. This is because the strong coupling reduces its effective charge near the horizon.

2. Type II Black Hole

- (1) This type of black hole has an upper bound of the mass scale, beyond which no non-trivial solution exist. There are two branches(Fig.1): one is “stable” and the other is unstable. In the “low energy” limit ($\mu \rightarrow 0$), the stable branch approaches the Schwarzschild black hole, while the unstable one converges to the colored black hole. The typical mass scale of non-trivial particle in the unstable branch is about m_P/g_C .

We suspect that the mass of the non-Abelian field causes the existence of the upper bound of the black hole mass, though it has not been proved. The non-trivial structure balanced between a repulsive force by the massive gauge field and the gravitational force may have a characteristic size $1/\mu$, which is the Compton wave length. The horizon of non-Abelian black hole must exist inside of the structure, otherwise the black hole swallows whole non-trivial structure, resulting in the trivial

Schwarzschild black hole. The condition of $r_H \lesssim 1/\mu$ yields the upper bound of the black hole mass. Furthermore, when a mass of the non-Abelian field μ is large enough, the horizon scale ($\lesssim m_P/g_C$) becomes larger than $1/\mu$, and then the non-Abelian field is again swallowed into the black hole beyond the event horizon. For this reason, there is no non-trivial solution in the “high-energy” limit ($\mu \sim m_P$).

(2) Since the black hole entropy is given by $S = A/4 = \pi r_H^2$, where A is its area, the upper branch in Fig.1 has larger entropy than that in the lower branch. Hence, we shall call each of them, the high- and low-entropy branches respectively. The high-entropy branch is “stable”, while the low-entropy branch is unstable. This behavior of stability is well understood by a catastrophe theory (see §3.1).

(3) The specific heat in the high-entropy branch is always negative like the Schwarzschild black hole, while that in the low-entropy branch changes its sign a few times depending on the mass of the non-Abelian field μ .

(4) B_H is still small for the high-entropy branch. This black hole is approximately neutral. On the other hand, for the low-entropy branch, B_H is finite and rather large. This type of black hole is locally charged but globally neutral just as the colored black hole.

(5) As for the structure of Type II black holes, those in the low-entropy branch is quite similar to the Type I, including a distribution of the effective charge. However, the structure of black holes in the high-entropy branch may be different from that of Type I.

3 Stability of Black Holes via Catastrophe Theory

The catastrophe theory is a new mathematical tool to explain a variety of change of states in nature. In order to examine a stability with the catastrophe theory, we must first find control parameters, state variables and a potential function of the non-Abelian black hole system. They could be different depending on the type of the black hole (whether Type I or II) and on the environment around the black hole (whether the isolated system or that in a heat bath).

The next step to examine a stability of a certain system is to draw the equilibrium space M_V , which consists of extrema of the potential function, i.e., of a family of solutions of the system, in the space of control parameters and state variables. Then we project the equilibrium space onto a control parameter space (which may also be called the control plane in the 2-dimensional case as the present models) by a catastrophe map $\chi_V : M_V \rightarrow R^N$, where N is the number of the control parameters. There may exist singular points on the equilibrium space, where the Jacobian of the mapping χ_V vanishes. The image of the set of singular points is called the bifurcation set B_V . If such singular points exist, the mapping χ_V from M_V to the control parameter space is singular and then the number of solutions with the same control parameters changes beyond the bifurcation set B_V and then the stability also changes there. Hence, by looking at this bifurcation set, we can classify our models into several elementary catastrophe and show the properties about stability of the system as will be shown in below[19].

3.1 Isolated Black Holes

3.1.1 Type I non-Abelian black hole

We performed the above procedure for the dilatonic colored black hole. We find that its bifurcation set is an empty set, hence, the dilatonic colored black hole has no change of stability in any mass scale or for any coupling constant.

3.1.2 Type II non-Abelian black hole

Next we examine the Type II black hole. We show the equilibrium space and the bifurcation set of the Skyrme black holes in Fig. 2. We have here adopted M and the mass of non-Abelian field $\mu(=gsf_s)$ as control parameters, and B_H as a state variable, and S as a potential function, respectively. The equilibrium space looks like a piece of cloth folded and the catastrophe mapping from the equilibrium space to the control plane provides a line, which is the evidence of a fold catastrophe.

We keep our eyes on the change of black hole mass M . Fixing one control parameter μ , we find a smooth solution curve on the equilibrium space (which is called the equilibrium curve). There exist two solutions on this curve for each mass. The solution with the smaller value of B_H is stable against radial perturbations, i.e., in the high-entropy branch, while that with larger value of B_H is unstable, i.e., in the low-entropy branch.

How does a catastrophe come in this system? Assume that there is a stable black hole solution expressed by some point on the curve labeled e in Fig.2. When its mass gets large, the point shifts to the right along the curve e , and reaches eventually to the end point (a singular point). Then the solution point is forced to jump to other stable solution discontinuously as a solid arrow in Fig.2. This is a catastrophe. This phenomenon is explained by watching a potential function. The schematic forms of the potential function (the black hole entropy S) depend on the control parameters as shown in Fig.3. In the region of small μ and small M (the left hand side of B_V), the potential function has two extremal points, one of which is a maximal point corresponding to the stable high-entropy solution and the other is a minimal point corresponding to the unstable low-entropy type. When the mass of the black hole increases and the solution point reaches to the bifurcation set B_V , the maximal and the minimal points merge and turn to be a inflection point. Furthermore when the mass gets larger, i.e., the right hand side of B_V , there is no extremal point and no black hole solution. Catastrophe comes about with disappearance of a maximal and a minimal points.

3.2 Black Holes in a Heat Bath

In thermodynamics, we often consider two complimentary situations: one is an isolated adiabatic system and the other is an isothermal state in a heat bath. For the former case, the entropy S is the fundamental variable and the thermal equilibrium is realized at an entropy maximum, while for the latter case the Helmholtz free energy F is the fundamental variable and the equilibrium is obtained at a minimum point of the free energy.

Here we shall consider an ideal situation in which a reservoir keeps the temperature of a black hole in a heat bath unchanged. The thermal radiation around a black hole plays a role of heat reservoir. For example, the isolated Schwarzschild black hole is stable, however the Schwarzschild black hole in a box, which is filled by radiation, is always unstable when the box is infinitely large[20]. The reason why it is so may be understood by the fact that the specific heat is negative. As for the Reissner-Nordström black hole, there is a sign change of the specific heat at $Q = \sqrt{3}M/2$. This sign change may correspond to a kind of phase transition[21], i.e., we expect that the black hole in a heat bath becomes stable when $Q > \sqrt{3}M/2$. Although there is still a deep and unsolved problem about the interpretation of such a sign change of the specific heat and a stability in a “heat bath” [22], here we will deal with a black hole just as a conventional thermodynamical object.

3.2.1 Type I non-Abelian black hole

In the catastrophe theory of the dilatonic colored black holes in a heat bath, we adopt the temperature T , rather than mass M , and the coupling constant α as control parameters and the Helmholtz free energy F as a potential function instead of the entropy S (see Table 3). We plotted the equilibrium space in Fig.4. For the low coupling constant solutions, which includes the colored black hole as the case of $\alpha = 0$, two folds appear, while no fold is seen in the case of high coupling constant. This configuration gives a cusp on the control plane through a catastrophe map. The bifurcation set is also shown in Fig.4, which in fact has a cusp. Hence we can conclude that this type of stability belongs to a cusp catastrophe. As we can see from the behavior of the potential function shown in Fig.5, a stable solution exists only in the region ABC lying between two curves AC and BC, where two unstable solutions exist as well.

This type of catastrophe reveals some interesting properties. Suppose that there is a stable Type I black hole which is expressed by a point on the curve e in Fig.4. When the temperature of the black hole either increases or decreases, the solution point reaches to either C_1 or C_2 . Then the solution point may jump to the Schwarzschild black hole along one of the solid arrows. Though the final state are almost the same whichever the temperature increases or decreases, behaviors of the potential function are a little bit different as will be shown in below.

When the temperature of the black hole increases and the point reaches at C_1 , the maximal point m_1 and the minimal point m_2 on the bifurcation set (the curve BC) merge, leaving one maximal point m_3 . However, the black hole does not go to m_3 because it is unstable. Instead the solution will jump to more simple Schwarzschild black hole. When the temperature decreases, and the solution point crosses the curve AC, the minimal point m_2 and the maximal point m_3 merge, leaving the maximal point m_1 . The solution again jumps to the Schwarzschild black hole.

3.2.2 Type II non-Abelian black hole

The stability analysis of Type II black hole in a heat bath is more complicated. We plotted the equilibrium space of the Skyrme black hole and the bifurcation set on the control plane (Fig.6) as an example. The bifurcation set consists of two components, one of which is the curve corresponding to a fold catastrophe and the other is the cusp shape corresponding to a cusp catastrophe.

We showed the behaviors of the potential function in Fig.7. Fixing μ , it is interesting to examine how the potential function looks like as the temperature of the black hole changes. There are qualitatively four cases (a)~(d) as shown by dotted lines in Fig.7. The shape change of the potential function is shown in Fig.8. For the case (a), first one minimum m_1 and one maximum m_2 exist for the low temperature. When the temperature increases (the solution point moves to right), the point eventually reaches to the curve AB, where an inflection point appears. Beyond this point, since there are two minima and two maxima in the region ABCD, another maximum m_3 and minimum m_4 appears. Increasing the temperature further, the minimum m_2 and the maximum m_3 merge and turn to be a inflection point on the curve CD. The maximum m_1 and the minimum m_4 are left. Suppose that an initially stable solution is at the minimum m_2 . As the temperature increases, it will disappear and jumps to the minimum m_4 beyond the curve CD. A catastrophe occurs. Although the free energy at minimal point m_4 gets smaller than that of minimal point m_2 before the curve CD, the solution does not go to the point m_4 , because the point m_2 is locally stable. This is called the perfect delay convention. Note that quantum effects break this rule because of a quantum tunneling.

The case (b) after the curve EC is qualitatively not different from the case (a), except that new phase appears before the curve EC. For the case (c), the appearance of the maximum m_3 and the minimum m_4 is accomplished at BF earlier than that of m_1 and m_2 , which occurs on the line BC. The stable solution m_4 , which exists initially at the low temperature, continues to exist and there happens no catastrophe even if the temperature increases. Hence, a stable black hole at the low temperature can be different depending on the coupling constant α . When the temperature increases, some solutions continue to exist while a catastrophe occurs for other solutions and the spacetime structure changes discontinuously. This is interesting, because we do not know whether our solution is in the case (a) (or the case (b)) or in the case (c) only from the black hole structure at the low temperature. The last case (d) is easy to understand because the shape of a potential function changes only on the BF.

4 Concluding Remarks

We have re-analyzed five known globally neutral non-Abelian black holes and presented a unified picture by classifying into two types depending on whether the non-Abelian field is massless (Type I) or massive (Type II).

In order to analyze the stability of non-Abelian black hole, we adopt a catastrophe theoretic method. The Type II non-Abelian black hole shows a fold type of elementary catastrophe and the Type I black hole in a heat bath has a cusp catastrophe. In case of the Type II black hole in a heat bath, we find a fold type and a cusp type structures in the bifurcation set.

Although a catastrophe theory may be a good tool to examine a stability in nature, we should give some critical comments as well. When we discuss a stability change using a catastrophe theory, we usually focus on some specific modes though many modes have to be investigated. Hence even if we find a stable branch from the catastrophe theoretic analysis, it does not always mean that such a system is completely stable. For example, the equilibrium space of the sphaleron black hole is similar to that of the Skyrme black hole in Fig.2. From its structure we might conclude that the lower (small B_H) solutions in the equilibrium space are stable while the upper (large B_H) solutions are unstable. But this is wrong for the sphaleron black holes because both of solutions are unstable for the topological reason. What is true is that the lower solutions are stable for a certain mode while the upper solutions become unstable for the same mode. Hence although the analysis is very easy, we must be very careful to conclude about a stability of a system by using a catastrophe theory.

- Acknowledgments -

We would like to thank Gray W. Gibbons, Osamu Kaburaki, Fjodor V. Kusmartsev, and Ian Moss for useful discussions. This work was supported partially by the Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Science and Culture (No. 04640312 and No. 0521801), by the Grant-in-Aid for JSPS Fellows (053769), and by the Waseda University Grant for Special Research Projects.

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Black holes	non-Abelian fields	Higgs fields	particles	black hole modes	mass (entropy)	$C_\#$
1. colored BH	Yang-Mills field ($SU(2)$) $-\frac{1}{16\pi}\text{Tr}F^2$, $F = dA + gA \wedge A$	—	BM particle	1(unstable)	$m < \infty$	2
2. dilatonic colored BH	Yang-Mills field ($SU(2)$) $-\frac{1}{16\pi}e^{-\alpha\sigma}\text{Tr}F^2$, $F = dA + gA \wedge A$	dilaton field $-\frac{1}{16\pi}(\nabla\sigma)^2$	BM particle	1(unstable)	$m < \infty$	0 or 2
3. Skyrme BH	Skyrme field ($SU(2) \times SU(2)$) $-\frac{1}{32\pi^2}\text{Tr}F^2 - \frac{1}{4}f_\pi^2\text{Tr}A^2$, $F = -dA$	—	Skyrmion BM type	1(stable) 1(unstable)	finite (high) finite (low)	0 1 or 3
4. Proca BH	massive Yang-Mills (Proca) field $-\frac{1}{4\pi}\{\frac{1}{4}\text{Tr}F^2 - \frac{1}{2}\mu^2\text{Tr}A^2\}$	—	Procaon BM type	1(stable) 1(unstable)	finite (high) finite (low)	0 1 or 3
5. sphaleron BH	Yang-Mills field ($SU(2)$) $-\frac{1}{16\pi}\text{Tr}F^2$	(complex doublet) $-\lambda(\Phi^\dagger\Phi - \Phi_0^2)^2$	sphaleron BM type	1("stable") 1(unstable)	finite (high) finite (low)	0 1 or 3
6. monopole BH	Yang-Mills field ($SU(2)$) $-\frac{1}{16\pi}\text{Tr}F^2$	(real triplet) $-\frac{1}{4}(\Phi^2 - \Phi_0^2)^2$	monopole [BFM particle]	1(stable) [1(unstable)]	finite (high) [finite (low)]	0 [0]

TABLE 1: The properties of 6 models including non-Abelian fields. BFM particle means one of two non-trivial solutions found in [12], which is more massive than the usual monopole. $C_\#$ denotes how many times the sign of the specific heat changes in the branch.

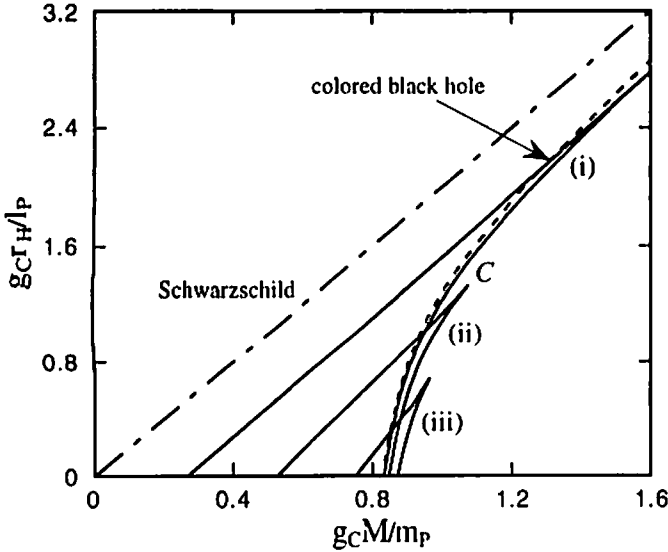


FIG 1: The mass-horizon radius diagrams for the Proca black hole with $\mu/g_C m_P =$ (i) 0.05, (ii) 0.10, and (iii) 0.15, C is a cusp, where the black hole has a maximal entropy. Beyond its entropy there is no non-Abelian black hole. The high-entropy (large horizon radius) branch is "stable", while the low-entropy (small horizon radius) branch is unstable.

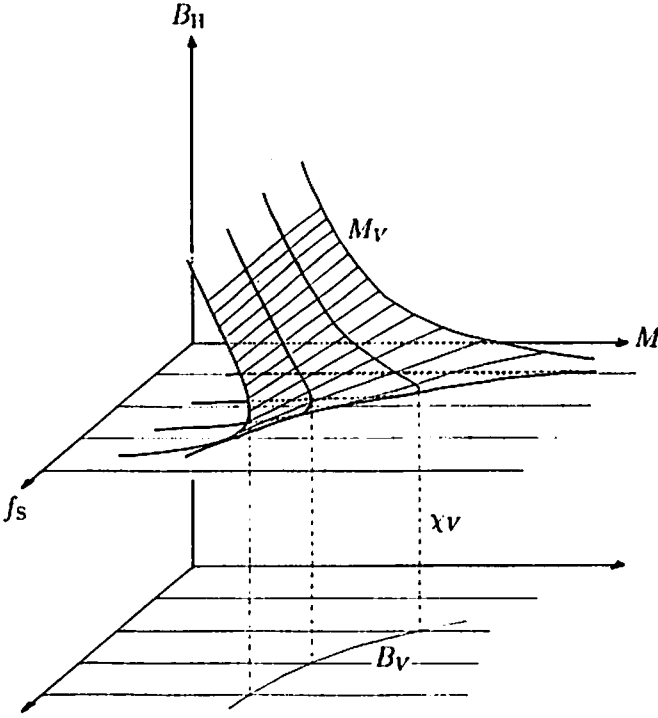


FIG 2: The equilibrium space of the Skyrme black hole. Upper side and lower side of the equilibrium space correspond to the unstable and stable solutions respectively. On the control plane, we draw the bifurcation set B_V , which shows a fold catastrophe.

FIG 3: The potential function S (entropy) of the Skyrme black hole. A maximal point and a minimal point exist in the left hand side of the bifurcation set B_V , while in the right hand side there is no extremal point.

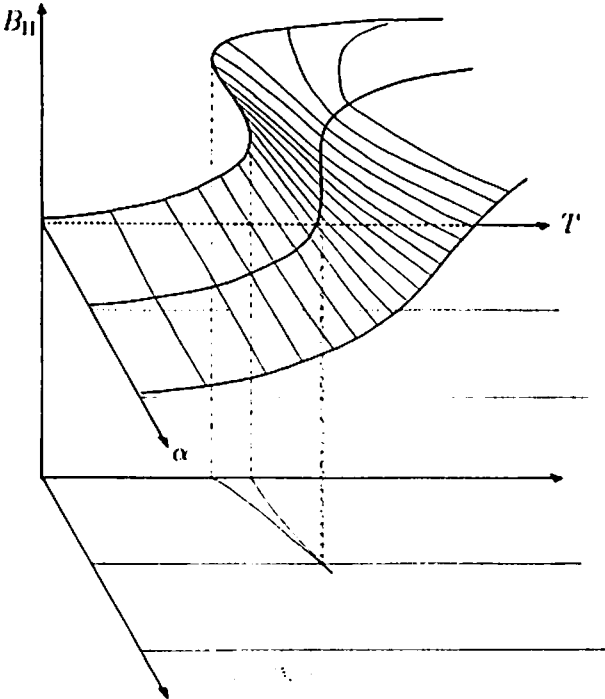
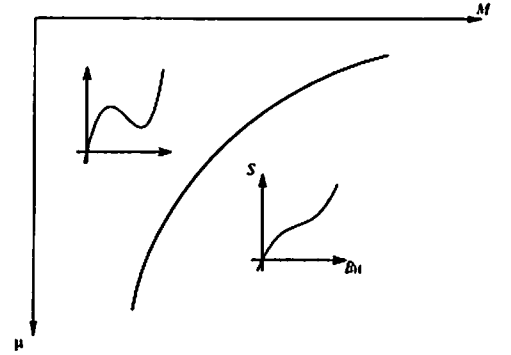
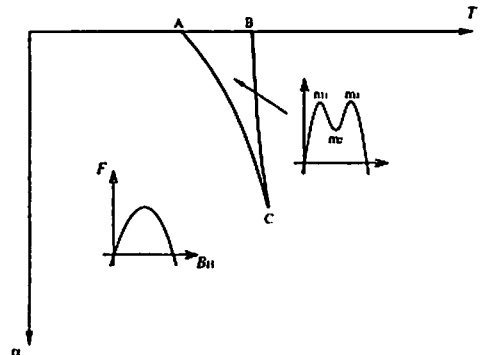


FIG 4: The equilibrium space and the bifurcation set of the dilatonic colored black hole in a heat bath. From the shape of the bifurcation set, we classify this system into a cusp catastrophe.

FIG 5: The potential function F (Helmholtz free energy) of the dilatonic colored black hole in a heat bath. In the interior of the region ABC, there are three extremal points m_1, m_2 , and m_3 , one of which (m_2) is stable and others (m_1, m_3) are unstable.



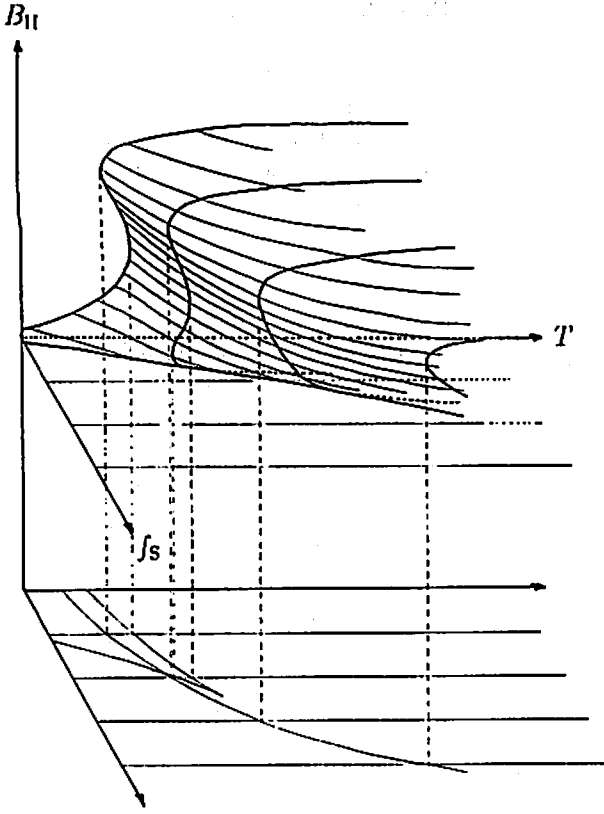


FIG 6: The equilibrium space and the bifurcation set of the Skyrme black hole in a heat bath. The bifurcation set consists of a cusp structure ECD and a smooth curve ABF .

FIG 7: The potential function F (Helmholtz free energy) of the Skyrme black hole in a heat bath.

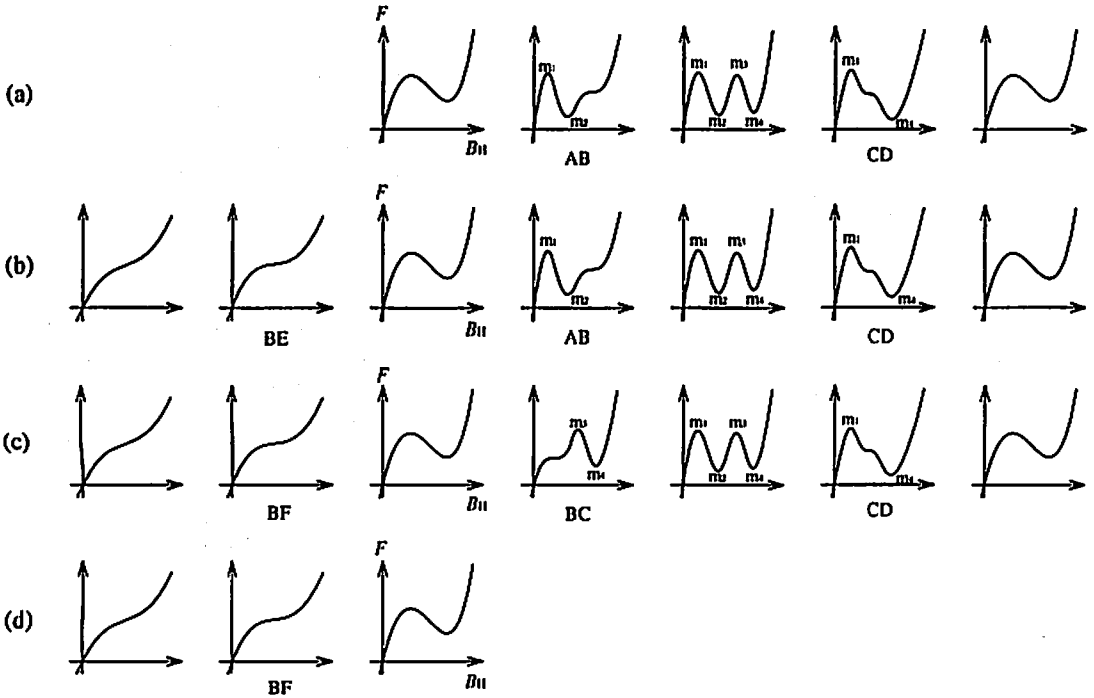
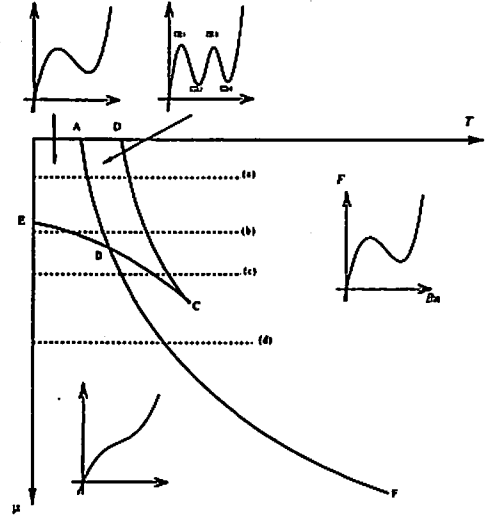


FIG 8: The behaviors of the potential function of the Skyrme black hole in a heat bath. There are four cases labeled (a)~(d) which correspond to the line labeled (a)~(d) in Fig.7.

Non-Abelian Black Holes II: Charged Case

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Abstract

We analyze a globally charged non-Abelian black hole, namely, the black hole in a magnetic monopole. The property of this black hole solution is quite different from that of neutral non-Abelian black hole. It is very complicated, but, we can still explain it by the swallow tail type in the catastrophe theory.

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We classify non-Abelian(NA) black holes into two types. Namely, globally neutral holes and globally charged ones. We discussed the neutral ones in detail in the previous paper (I) in this volume and in ref.[1]. Here, we consider only a magnetically charged one, which shows quite a different property from that of neutral one. A charged NA black hole means a non-trivial black hole which exists in a 't Hooft-Polyakov monopole (the so called monopole black hole)[2,3,4,5,6]. A 't Hooft-Polyakov monopole has non-trivial structure with a typical size. Thus, as a horizon radius of the black hole gets larger and eventually exceeds the monopole radius, the black hole swallows the NA structure, resulting in a trivial solution, i.e., the Reissner-Nordström (RN) black hole.

Now, the action is given by

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{16\pi} R + \mathcal{L}_{matter} \right], \quad (1)$$

with

$$\mathcal{L}_{matter} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - \frac{1}{2} D_\mu \Phi^a D^\mu \Phi^a - \frac{\lambda}{4} (\Phi^a \Phi^a - \eta^2)^2, \quad (2)$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - e\epsilon_{bc}^a A_\mu^b A_\nu^c, \quad (3)$$

$$D_\mu \Phi^a = \partial_\mu \Phi^a + e\epsilon_{bc}^a A_\mu^b \Phi^c, \quad (4)$$

where A_μ^a is a SU(2) Yang-Mills field, Φ^a is a triplet Higgs field transforming as isovector, Latin indices, a, b, c , referring to the internal SU(2) indices, D_μ is a gauge-covariant derivative, e is a gauge coupling constant, η is a Higgs vacuum expectation value at a broken symmetry state, and λ is a Higgs self coupling constant. We use the unit $c = G = \hbar = 1$. We consider a static and spherically symmetric solution. The metric is written as

$$ds^2 = - \left(1 - \frac{2m(r)}{r} \right) e^{-2\delta(r)} dt^2 + \left(1 - \frac{2m(r)}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (5)$$

For the matter fields, we adopt the ansatz given by

$$\Phi^a = \eta \hat{r}^a J(r), \quad (6)$$

$$A_i^a = \epsilon_{iak} \frac{1 - K(r)}{er}, \quad A_0 = 0. \quad (7)$$

The RN solution is, of course, a trivial solution in this system ($m(r) = m_0 - 8\pi/e^2 r^2$, $K(r) \equiv 0$, $J(r) \equiv 0$). As for a non-trivial solutions, we have to solve the basic equations, which are derived from above action, as an eigenvalue problem.

Fig.1 shows the relation between mass of non-trivial black holes and their horizon radius when $\lambda/e^2 = 1$. The mass of the RN black hole has a lower bound because the theory fixes its magnetic charge as $1/e$. Unlike the neutral case (see the paper I), we cannot find any cusp structure.

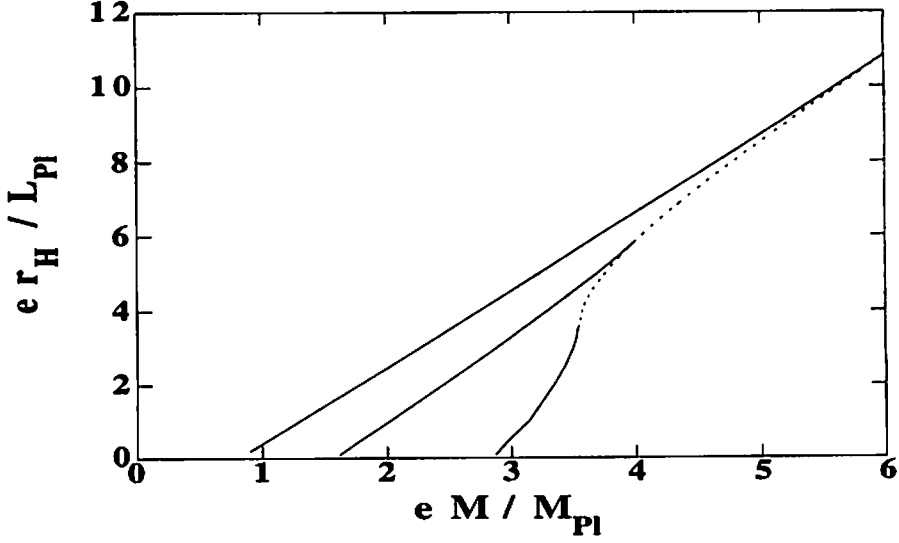


Figure 1: The mass-horizon radius diagrams in the case of $\lambda/e^2 = 1$ of Reissner-Nordström black hole (dashed line) and of monopole black hole (solid line) with $\eta/M_{Pl} =$ (A) 0.05, (B) 0.10, (C) 0.15.

Moreover, when $\lambda/e^2 = 0.1$, the relation becomes as shown in Fig.2(a). This looks quite similar to Fig.1 at a first glance. However, as shown in the enlarged figure 2(b), there appears cusp structure near the bifurcation point of the branch of the NA solutions and that of RN solutions.

We have two cases of the behavior of the solution depending on the value of λ . Then we can infer the stability of the solution as follows. In the first case as Fig.1, the NA solution is always stable. While, the RN solution is stable when the mass is larger than the critical value at the bifurcation point but becomes unstable if it is less than the critical one. In the second case as Fig.2, the NA solution is stable in the branch of larger horizon-radius, the stability changes at the cusp, while is unstable in the small horizon-radius branch. The RN solution is stable when the mass is larger than the critical value at the bifurcation point, but otherwise it is unstable.

Although we divide the solutions into two cases in the above, since our discussion is based on the numerical results, we cannot deny the possibility that a cusp structure exists in the case of Fig.1, but is so small that we cannot see it. Catastrophe theory, however, explains quite naturally the existence of two cases depending on λ and the change of the stability described above. Which elementary catastrophe explains these behaviors? We have found that it is a swallow tail catastrophe.

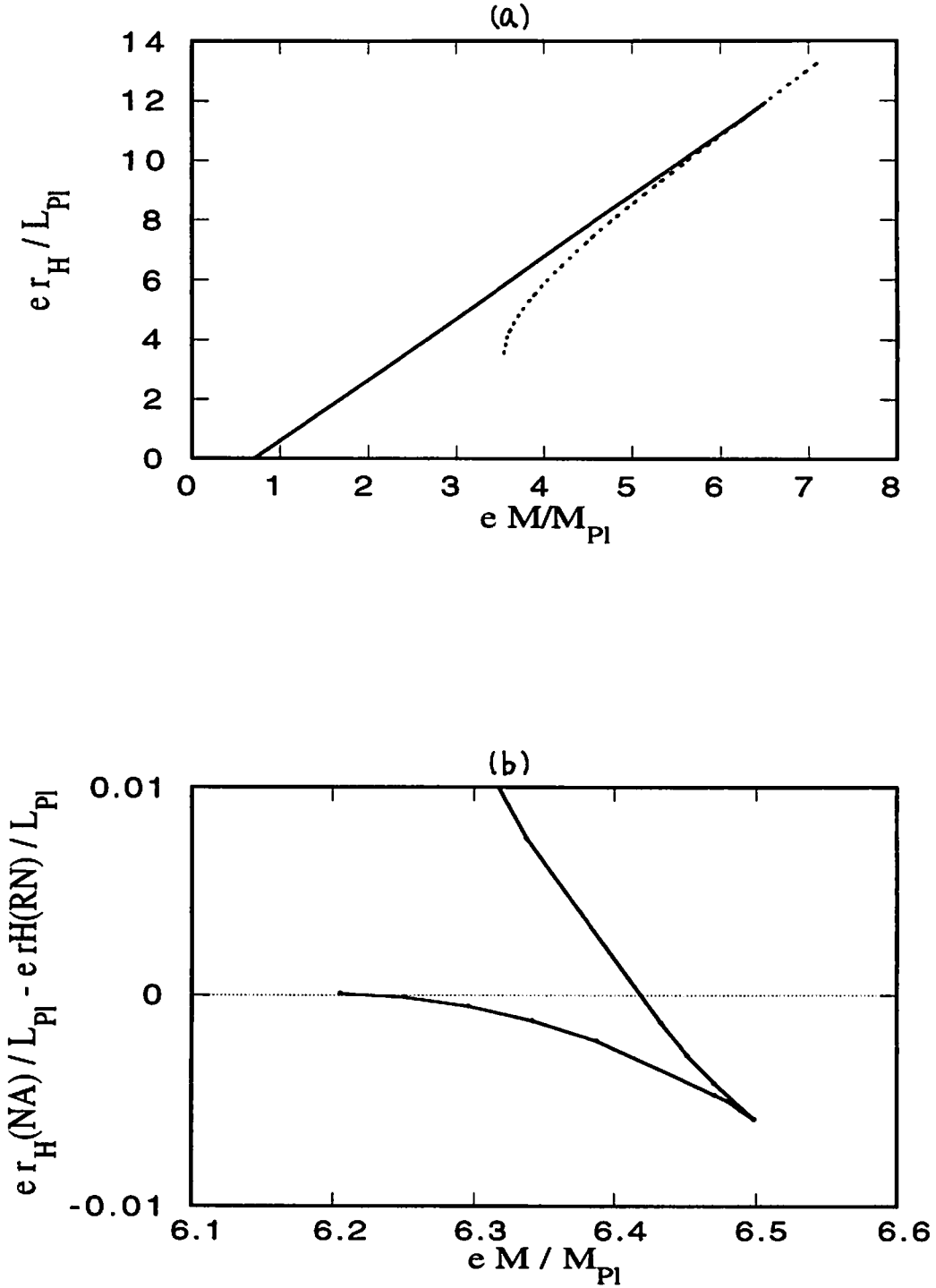


Figure 2: (a) The mass-horizon radius diagrams in the case of $\lambda/e^2 = 1$ of Reissner-Nordström black hole (dashed line) and of monopole black hole (solid line), with $\eta/M_{Pl} = 0.05$. (b) The difference between the horizon radius of non-Abelian black hole and that of Reissner-Nordström black hole is plotted. This shows that there exists a small cusp structure in Fig.(a).

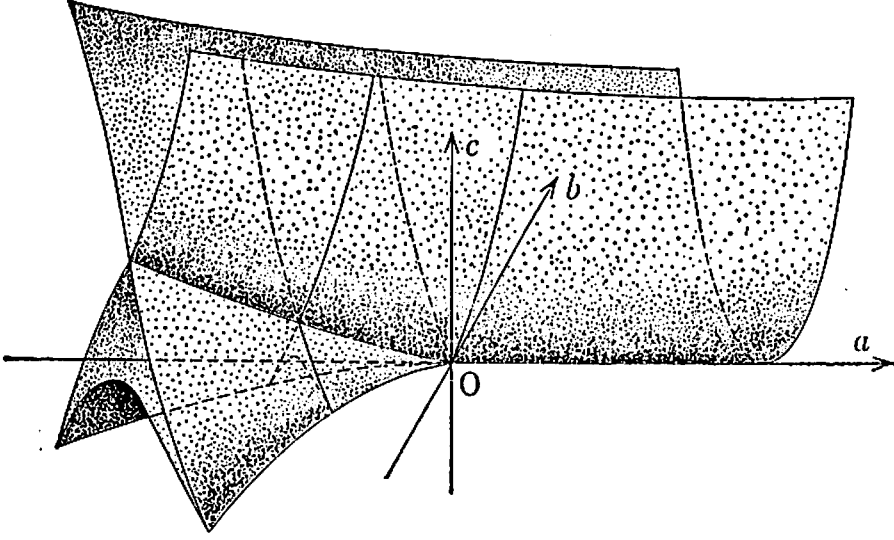


Figure 3: A catastrophe set of a swallow tail type.

Swallow tail catastrophe has three control parameters, a , b , c , and one state variable, x , and its potential function is described as

$$V = \frac{1}{5}x^5 + \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + cx. \quad (8)$$

A set of the points at which catastrophe occurs, that is, a catastrophe set is schematically shown in Fig.3, which looks like swallow's tail. We set $c = 0$ and reduce the three dimensional control parameter space to two dimensional space. We need this reduction in order that the RN solutions can always exist as trivial solutions. a and b correspond to the Higgs self coupling constant(λ/e^2), and the mass of the black hole(M), respectively. As we shall see below soon the condition of $a = 0$ divides the cases whether the cusp exists or not. That is to say, a cusp exists in the tail of the swallow, while does not exist in its body.

We denote the value of λ which corresponds to the case of $a = 0$ as λ_{crit} . Fig. 4 shows the schematic potential in the case of $a > 0$ (Namely, $\lambda > \lambda_{crit}$). The axis of abscissa represents a state variable. We adopt δ_H , which is the value of $\delta(r)$ at horizon, as the state variable. For the RN black hole, δ_H always vanishes, thus it seems to be a kind of the measure of the degree of non-Abelian. Furthermore, we cut down the negative part of δ_H as an unphysical region. We can easily show that δ_H is always positive from the basic equation for δ_H . The maximal points in Fig. 4 represent unstable solutions and the minimal points do stable ones. The origin is always a stationary point and it corresponds to the RN solution. Now, let us consider a physical process, for example, matter accretion, which increases b (M) from a

negative value. Initially, the RN solution is unstable and the NA solution is stable. (Fig. 4-1) As we increase the mass, the minimal value increases and the minimal point approaches to the origin (Fig. 4-2). When $b = 0$, the minimal and maximal points eventually merge. (Fig. 4-3) Then the NA solution disappears, and the RN solution turns to be stable. (Fig. 4-4)

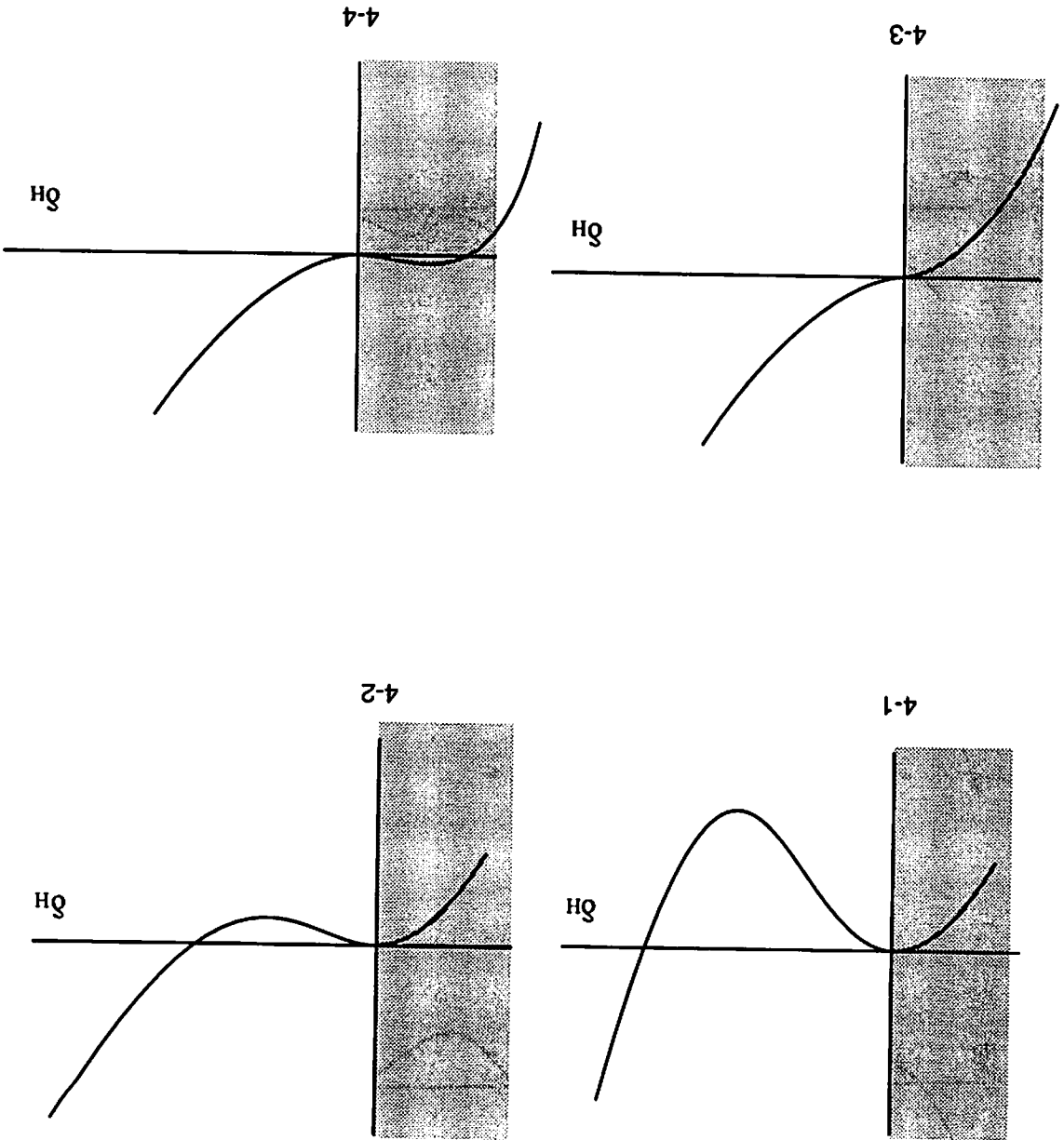
Next we consider the case of $a < 0$ ($\lambda < \lambda_{crit}$). In this case, a cusp structure exists. We again discuss a physical process which increases b from a negative value. The change of the potential form is shown in Fig. 5. Initially, one unstable RN solution and one stable NA solution exist. (Fig. 5-1) As we increase the mass, one more pair of maximal and minimal points come to appear in the unphysical region. (Fig. 5-2), (Fig. 5-3) When $b = 0$, the minimal point in the unphysical region and the maximal point at the origin merge, and the RN solution turns to be stable. (Fig. 5-4) But, since the NA solution is still stable, nothing happens to the NA black hole at this point. And one more unstable non-Abelian branch appears from the branch of the RN solutions. So, two stable solution and one unstable solution exist. This means the violation of the weak no-hair conjecture [6,7]. As b increases further, the branches of stable NA and unstable NA approach each other, (Fig. 5-5) and then they merge at an inflection point (Fig. 5-6). This causes catastrophe. The NA black hole jumps to a RN black hole discontinuously. Then, only stable RN solution remains. We can discuss the inverse process in which the b goes down, for example, by the Hawking evaporation mechanism. Suppose there exists initially only stable RN solution. As b decreases, the stable NA and the unstable NA solutions appear (Fig. 5-6). At this point, since the RN solution remains stable, nothing happens. If b decreases further and reaches 0, the stable RN and the unstable NA branches merge. The RN solution turns to be unstable (Fig. 4-4). The RN black hole jumps to a NA black hole discontinuously. The NA black hole eventually evaporate into a stable monopole.

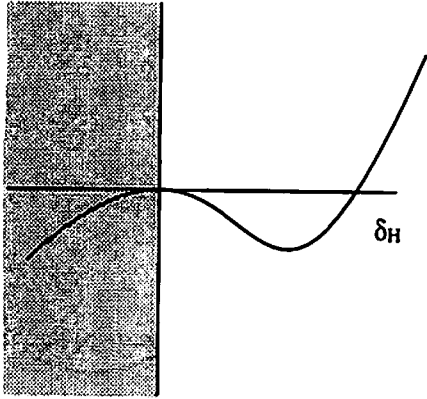
As you see, a swallow tail catastrophe explains the behavior of solution completely when we fix the vacuum expectation value of the Higgs field. Moreover, note that the RN branch and the NA branch always merge when $b = 0$ independently of the value of the λ . We can make sure of this fact from Fig. 6. It shows that the mass of the merging point of the RN branch and a NA branch is constant. It is consistent with the prediction from the catastrophe theory.

Next, we shall give a comment on the specific heat of the NA black hole briefly. Lee et al. [2,4] insisted that the specific heat of the NA black hole is always negative by use of their simplified model. But, our numerical result shows that the NA black hole with negative specific heat can exist when $\lambda < \lambda_{crit}$ (Fig. 7).

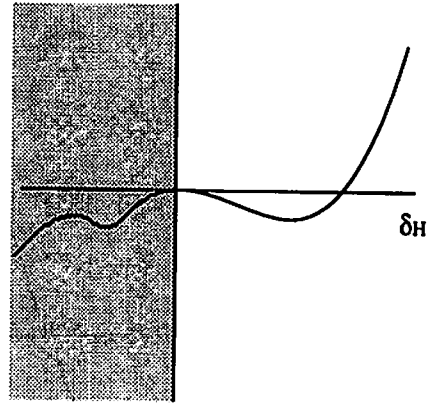
In summary, we found that swallow tail catastrophe can explain the behavior of the monopole black hole and this NA black hole can have positive specific heat in some parameter range.

Figure 4: The behavior of potential function for $\lambda/e^2 = 1.0$ and $\eta = 0.05$.

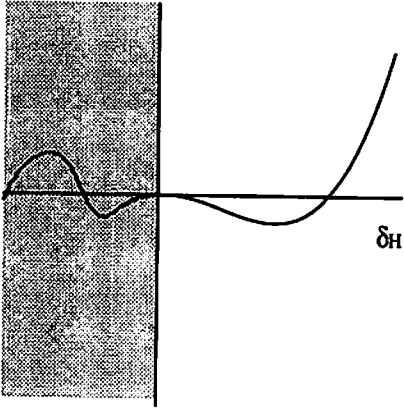




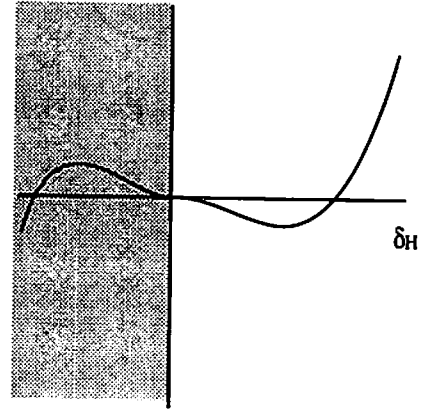
5-1



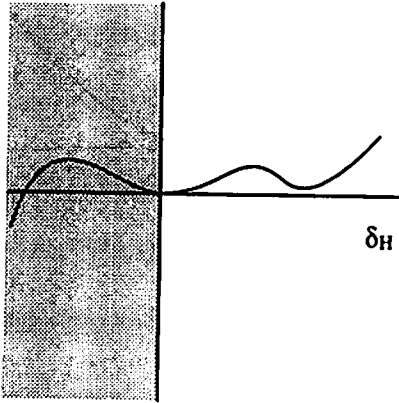
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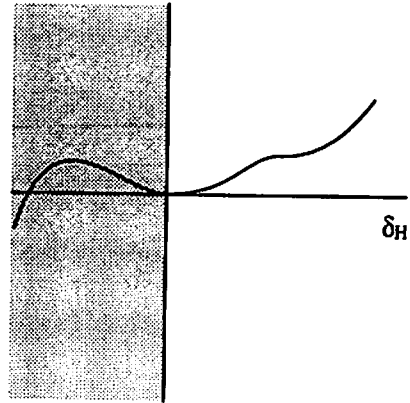
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Figure 5: The behavior of potential function for $\lambda/e^2 = 0.1$ and $\eta = 0.05$.

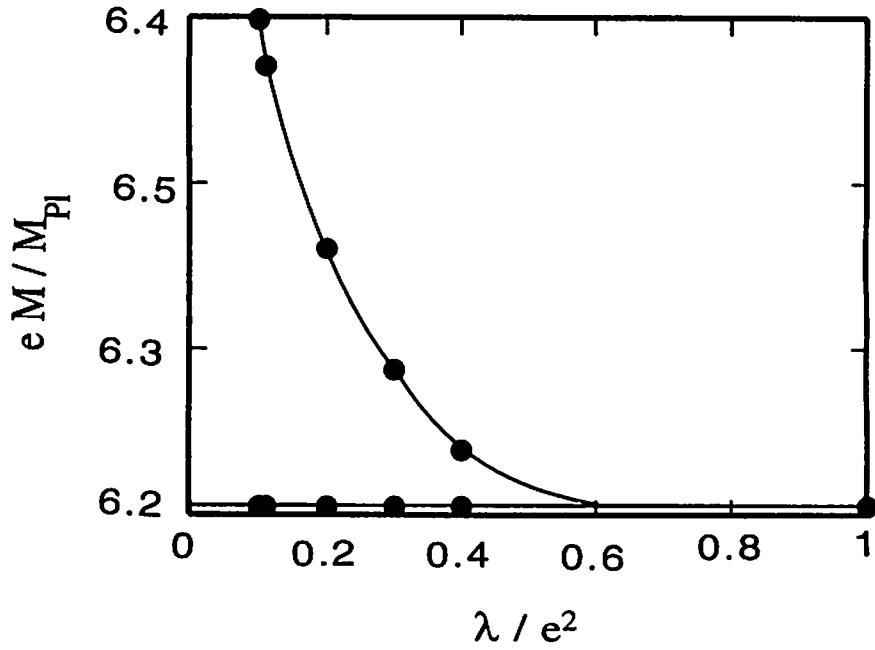


Figure 6: The value of mass M where catastrophe occurs. The mass where the NA branch and the RN branch merge is independent of λ .

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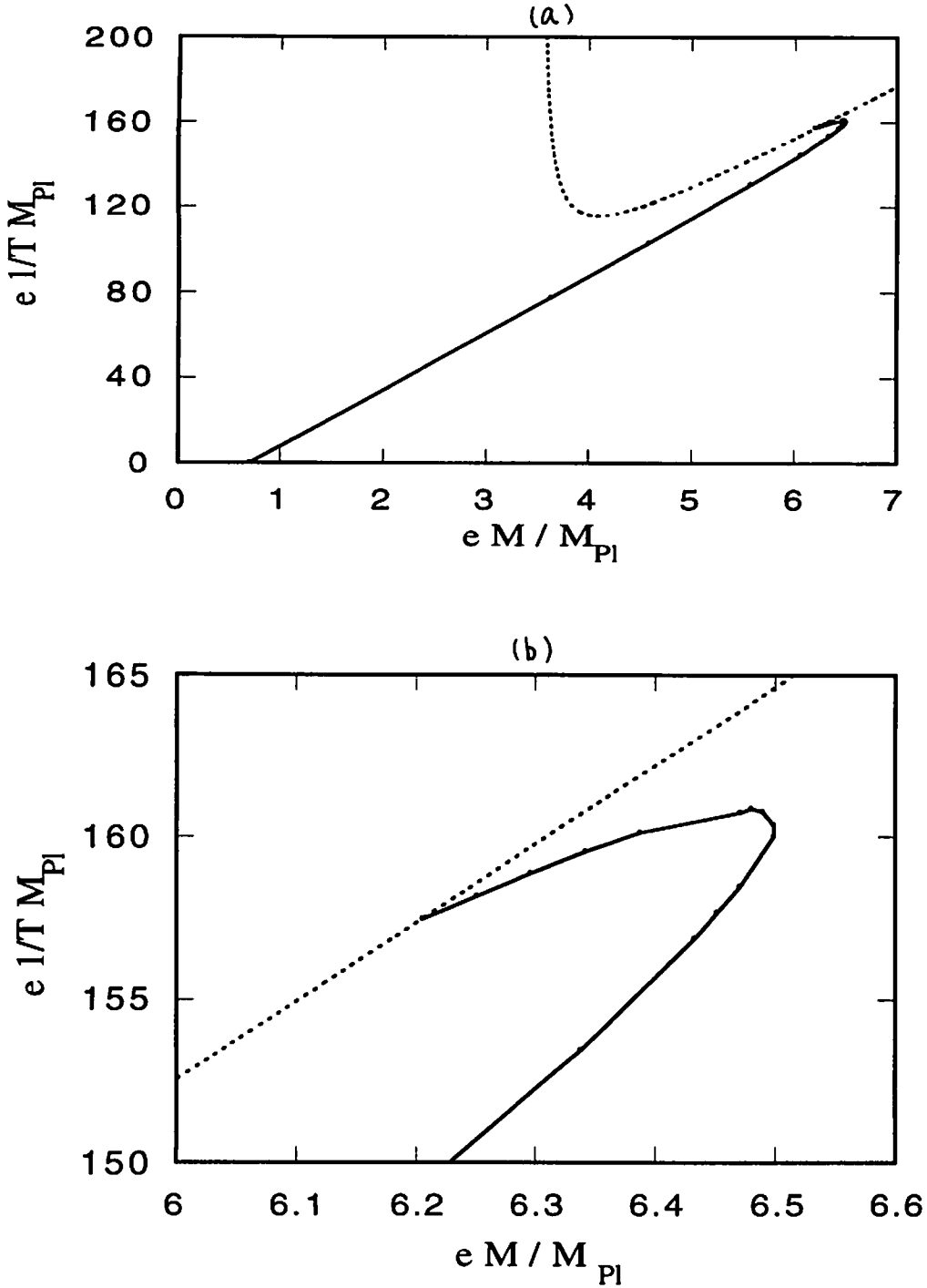


Figure 7: (a) The mass-inverse temperature diagrams of Reissner-Nordström black hole (dashed line) and of monopole black hole (solid line) for $\lambda/\epsilon^2 = 0.1$ and $\eta = 0.05$. (b) Enlarged figure near the bifurcation point of Fig.5(a).

Reissner-Nordstrom Black Hole with Non-Abelian Hair

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Abstract

We present a new class of black hole solutions in the theory with $SU(2) \otimes U(1)$ gauge field and calculate the Hawking temperature to discuss the thermodynamics.

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1 Introduction

Recently there has been great interest in non-trivial black hole solutions in theories with non-Abelian gauge fields. As the simplest example, a discrete family of spherically symmetric static black hole solutions was found by Bizon[1] in the Einstein- $SU(2)$ Yang-Mills(EYM) theory (These are called colored black holes). The solutions with non-trivial Yang-Mills field configurations can be seen as the singular solutions corresponding to a discrete family of particle-like ones found by Bartnik and MacKinnon[2]. The stability and the thermodynamics of such black holes have been discussed in connection with the no-hair conjecture. Though it turns out that the colored black holes are unstable [6], a variety of structures of the black holes with non-Abelian hair have been widely studied by many authors in the theories with non-Abelian fields coupled to other matter fields [3, 4]. It was suggested that these black holes have some universal properties due to the non-Abelian fields and the stabilities was discussed from a catastrophe-theoretic analysis of the black hole entropy[5]. It is necessary to study the physical properties in detail and cosmological implications.

In this paper we investigate the black hole solutions with $SU(2) \otimes U(1)$ gauge fields and their thermodynamics. The gauge fields coupled to gravity may arise more naturally from fundamental gauge theories. Such direct products of gauge fields appear in general in string theory[11]. We show that there are Reissner-Nordstrom solutions with Yang-Mills hair. Black hole with $U(1)$ charge is studied for monopole black holes [3, 9]. Since the temperature is of interest from aspects of black hole thermodynamics, we discuss this behavior for the black hole found in this paper.

2 Black Holes with Non-Abelian Hair

Before discussing the Einstein- $SU(2) \otimes U(1)$ gauge fields system, we will briefly review the colored black holes i.e. the static spherically symmetric, singular solutions in EYM

theory. The black hole solutions can be obtained by imposing the spherically symmetric static ansatz for the metric:

$$ds^2 = -f e^{-2\delta(r)} dt^2 + f^{-1} dr^2 + r^2 d\Omega^2, \quad (2.1)$$

where

$$f = \left(1 - \frac{2m(r)}{r}\right), \quad (2.2)$$

and 'tHooft ansatz for the Yang-Mills connection

$$dA = \frac{(1 - w(r))}{2g} U dU^{-1}, \quad (2.3)$$

where $U = \exp(i\frac{\tau}{2}\tau n)$, τ denotes the Pauli matrix and n is the radial unit vector. The geometrical units, $G = c = \hbar = 1$, is used throughout this paper. Note that the ansatz is assumed to be purely magnetic. The field equations for $m(r)$, $\delta(r)$ and $w(r)$ should be solved under the relevant boundary conditions: $m(r) \rightarrow M = \text{const.}$, $\delta = \text{const.}$ and $|w| = 1$ as $r \rightarrow \infty$ from suitable asymptotic behavior; The existence of a regular event horizon at $r = r_H$ requires that $m(r_H) = \frac{1}{2}r_H$ and $\delta(r_H) = \text{const.}$. We choose $\delta(r_H)$ to be zero as ref.[1, 2]. These equations admit the trivial solution of which the metric is the Reissner-Nordstrom type solution when $\delta(r)$ and $w(r)$ vanish identically.

It was shown that there are a discrete number of static solutions labeled by the node n of the Yang-Mills field. These solutions approach the Schwarzschild black holes as r is large and behave as the Reissner-Nordstrom black holes near horizons with magnetic charge of order unity. The black hole solutions do not have globally Yang-Mills charge but locally one which is exponentially damped. The black holes are unstable for a radial linear perturbations and may be interpreted as sphalerons in EYM theories.[7, 10] Such black holes with non-Abelian hair have been studied by many authors and a variety of solutions were founded in system consist of non-Abelian fields and other matter fields such as Higgs fields.

3 Colored Reissner-Nordstrom Black Holes

We consider the $SU(2) \otimes U(1)$ gauge fields coupled to gravity:

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} (-R + \frac{1}{4} G_{\mu\nu}^2 + \frac{1}{4} F_{\mu\nu}^2), \quad (3.1)$$

where $G_{\mu\nu}$ and $F_{\mu\nu}$ denotes the field strengths of $U(1)$ and $SU(2)$ gauge fields, B_μ and A_μ respectively. Since the gauge fields are only coupled through the metric, it is clear that there exists the solution that both gauge field configurations with $U(1)$ charge correspond to Reissner-Nordstrom black holes. We consider the static spherically symmetric solutions with $U(1)$ charge and Yang-Mills hair. Thus we impose the ansatz which $U(1)$ gauge field is the Coulomb type, $SU(2)$ Yang-Mills connection is given by Eq.(2.3) and the metric is the same form of Eq.(2.1) with

$$f = 1 - \frac{2m(r)}{r} + \frac{Q^2}{r^2}. \quad (3.2)$$

It is convenient to introduce the quantities scaled by the horizon radius: $r/r_H \rightarrow r$, $m/r_H \rightarrow m$, $q = Q/r_H^2$ and $l_H = gr_H$. Then the field equations are written in terms of $m(r)$, $\delta(r)$ and $w(r)$:

$$m' = \frac{1}{l_H^2} \left[\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) w'^2 + \frac{(1 - w^2)^2}{2r^2} \right], \quad (3.3)$$

$$\left(\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) e^{-2\delta} w' \right)' + e^{-2\delta} \frac{w(1 - w^2)}{r^2} = 0, \quad (3.4)$$

$$\delta' = -\frac{2w'^2}{l_H^2 r}. \quad (3.5)$$

The boundary conditions are the same as for the EYM system except for the relation from the regularity condition at the horizon:

$$m(r_H) = 2(1 + q^2). \quad (3.6)$$

We analyzed these equations for some fixed charge q and for the node $n = 1$. We found the solutions with $U(1)$ charge and $SU(2)$ Yang-Mills hair, which are not exhibited here (see ref.[8] in detail). The dependence of $w(r_H)$, $\delta(r_\infty)$ and M on q^2 are shown in Fig.1. For the maximal charged black hole ($q^2 = 1$), $w(r_h)$ approaches unity. In the extremal case, the derivative of $w(r)$ is diverge at the horizon. The solution presented here may be unique for fixed node n in Einstein- $SU(2) \otimes U(1)$ gauge field theory. The existence of the solution suggests that there exist at least the corresponding solutions in the theories with $SU(2) \otimes SU(2)$ gauge fields, i.e. one of those is a Reissner-Nordstrom type with $U(1)$ charge and another corresponds to one presented here.

4 The Black Hole Temperature

In this section, we examine the thermodynamical properties for the colored Reissner-Nordstrom black hole. From the Euclidean effective action, we can derive the relation:

$$S_E = \beta M - 4\pi m_H^2 - 8\pi\beta Q^2/r_H. \quad (4.1)$$

Note that the relation can be obtained for general non-rotating spherical symmetric black hole with charge Q (for EYM theory see ref[10]). Since the effective action can be interpreted as the thermodynamical potential F times inverse temperature β , i.e. βF , the black hole entropy is

$$S = 4\pi m_H = \pi r_H^2 (1 + q^2)^2, \quad (4.2)$$

and the electrical potential $\Phi = 8\pi Q/r_H$. The inverse temperature appeared as a period of the Euclidean action can be evaluated by the metric:

$$T = \frac{1}{4\pi r_H} e^{-\delta} (1 - q^2 - 2m'_H). \quad (4.3)$$

The dependence of the temperature on the charge q is shown in Fig.2. As the charge vanishes, this reduces to temperature of the colored black hole. From Eq.(3.3),

$$m'(r_H) = \frac{(1 - w^2(r_H))}{2l_H^2}, \quad (4.4)$$

For the extremal case(maximally charged case,i.e. $q = 1$), $w(r_H) = 1$ and r.h.s of Eq.(4.3) vanishes. Therefore the extremal Reissner-Nordstrom black hole with Yang-Mills hair has zero-temperature as the same for the Einstein-Maxwell theory. The non-Abelian black hole with zero-temperature are also expected for other theories, e.g. for monopole black holes [9].

5 Concluding Remarks

In this paper, we investigated the black hole solutions for Einstein- $SU(2) \otimes U(1)$ gauge field theory. We found the charged black hole solutions with Yang-Mills hair. We also calculated the black hole temperature. In the maximal charged case, $w(r_H) = 1$ and the black hole with Yang-Mills hair has zero-temperature.

The black hole solutions found in this paper are presented as a new class of solutions with non-Abelian hair. Charged black holes with non-Abelian hair may have interesting physical properties and therefore must be studied(e.g. ref.[9]). We also expect that the black holes solutions obtained in our paper is helpful to understand the non-Abelian black holes with $U(1)$ charge.

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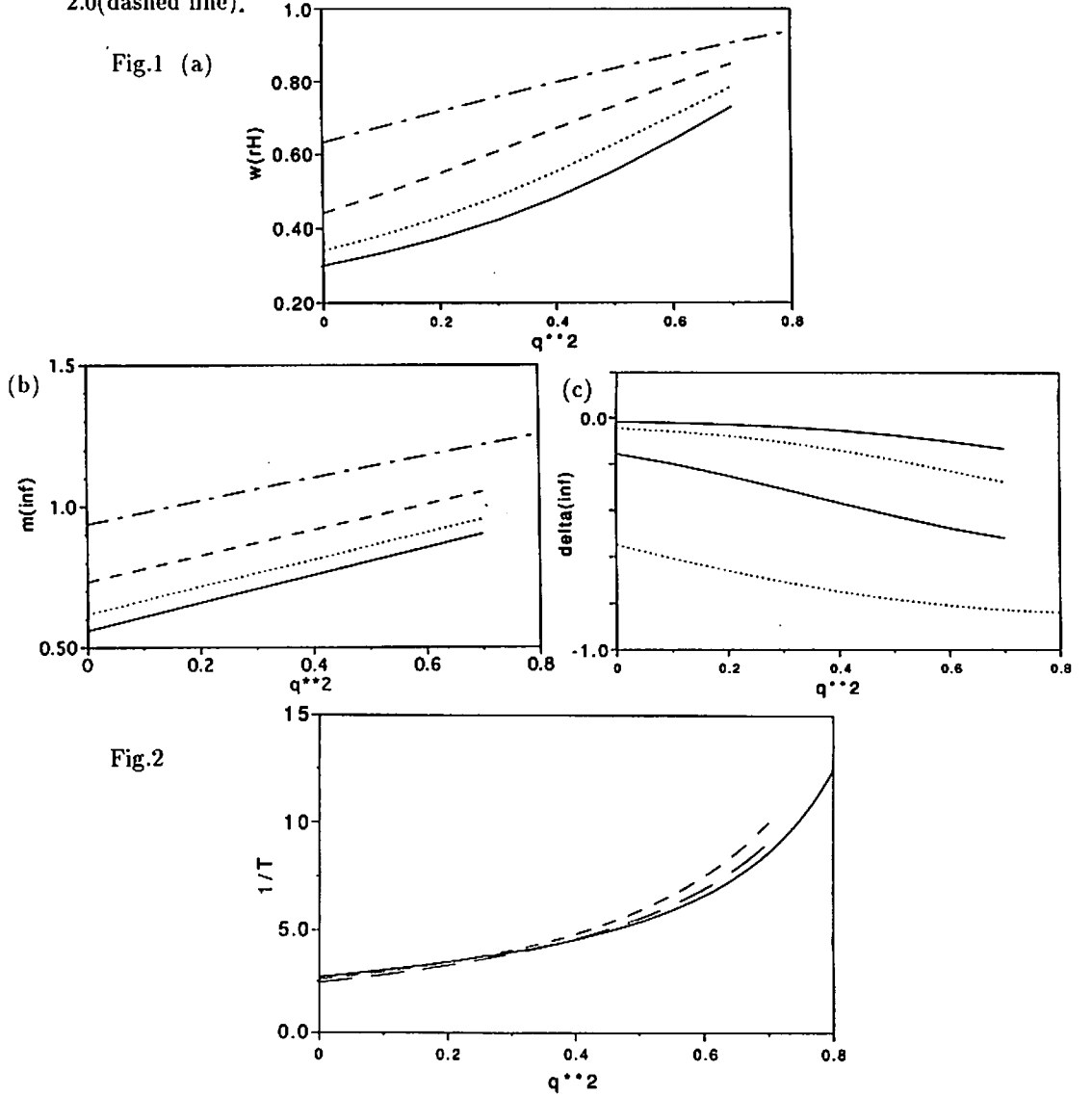
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Figure Captions

Fig.1 The q^2 -dependence of (a) $w(r_h)$, (b) $m(r = \infty)$, and (c) $\delta(r = \infty)$ for different values of l_H : $l_H = \sqrt{8}$ (solid line), 2.0(dotted line), $\sqrt{2}$ (dashed line), and 1.0(dashed-dotted line).

Fig.2 The inverse temperature of a colored Reissner-Nordstrom black hole is plotted as a function of the charge q^2 for l_H equal to 1.0 (solid line), $\sqrt{2}$ (dashed-dotted line), and 2.0(dashed line).



Equilibrium Condition in the 2-Charged-Weyl Solution [†]

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ABSTRACT

The inverse scattering method, which is one of the soliton techniques, is applied to solving the source-free Einstein-Maxwell equations. The 4-soliton solution that corresponds to the 2-charged- Weyl solution is investigated. The mass and charge of each charged Weyl are defined and the condition for equilibrium of two charged Weyls is studied. It is shown that the equilibrium is realized only in the extremal case.

Recently the exact solution to the source-free Einstein-Maxwell equations given by Majumdar[1] and Papapetrou[2] in 1947 has been often discussed in connection with dilaton black holes. This solution describes a system of point charged particles in static equilibrium in which the electrostatic repulsive forces exactly balance the gravitational attractive ones. This solution was studied also by Hartle-Hawking[3], and they revealed its space-time structure and that it can be interpreted as a system of extremal black holes each of which has an event horizon.

[†] Based on the work in collaboration with T. Koikawa.

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While the solution given by Majumdar and Papapetrou is obtained by assuming the conformastatic metric

$$ds^2 = f^{-1}(\vec{x})(dx^2 + dy^2 + dz^2) - f(\vec{x})dt^2, \quad (1)$$

the condition for equilibrium of two charged particles in a more general axisymmetric space-time was studied in the post-post-Newtonian approximation by Ohta-Kimura[4]. They obtained the following relation between the charge e_i and mass m_i of the i -th particle from their analysis of the static potential:

$$m_i = |e_i|, \quad (2)$$

where the units $c = G = 1$ are used. In the axisymmetric exact solution describing two Reissner-Nordström black holes, which was obtained by means of the electrovac generalization of Neugebauer's transformation, Tomimatsu[5] gave the same relation as in (2) as a sufficient condition for equilibrium. This result indicates that each black hole should be extremal and the space-time be Majumdar-Papapetrou type so that the equilibrium may be realized.

In this report we apply the inverse scattering method, which is one of the soliton techniques to obtain exact solutions to the Einstein equation, to solving the static and source-free Einstein-Maxwell equations. We find that the solution can be written in a compact form with "pole trajectories" in this method. We give the explicit expressions of all the metric functions and the electrostatic potential that correspond to the axisymmetric n -soliton solution. We show that the 4-soliton solution is interpreted as a 2-charged-Weyl solution that includes a 2-Reissner-Nordström solution in a special case. We define the mass and charge of each charged Weyl in terms of the surface integrals and consider the condition for static equilibrium in the 2-charged-Weyl solution. The word "static equilibrium" here means that there is no strut between two charged Weyls, and we have the relation (2) as a necessary and sufficient condition for equilibrium.

Let us start with the metric

$$ds^2 = f^{-1}[Q(d\rho^2 + dz^2) + \rho^2 d\phi^2] - f dt^2, \quad (3)$$

where f and Q are functions of ρ and z . The source-free Einstein-Maxwell equations then read

$$(\ln f)_{,\rho\rho} + (\ln f)_{,zz} + \rho^{-1}(\ln f)_{,\rho} = 2f^{-1}(\chi_{,\rho}^2 + \chi_{,z}^2), \quad (4)$$

$$(\ln Q)_{,\rho} = \frac{\rho}{2}[(\ln f)_{,\rho}^2 - (\ln f)_{,z}^2] - 2\rho f^{-1}(\chi_{,\rho}^2 - \chi_{,z}^2), \quad (5)$$

$$(\ln Q)_{,z} = \rho(\ln f)_{,\rho}(\ln f)_{,z} - 4\rho f^{-1}\chi_{,\rho}\chi_{,z}, \quad (6)$$

$$\chi_{,\rho\rho} + \chi_{,zz} + \rho^{-1}\chi_{,\rho} = \chi_{,\rho}(\ln f)_{,\rho} + \chi_{,z}(\ln f)_{,z}, \quad (7)$$

where χ denotes the electrostatic potential. We here assume the Reissner-Nordström-type solutions(i.e., the Weyl's electrovac or Papapetrou's class) for Eqs. (4)-(7), which implies the relation

$$f = 1 - 2c\chi + \chi^2, \quad (8)$$

where c is a constant. The differential equations we should solve are reduced to

$$\chi_{,\rho\rho} + \chi_{,zz} + \rho^{-1}\chi_{,\rho} = 2f^{-1}(\chi - c)(\chi_{,\rho}^2 + \chi_{,z}^2), \quad (9)$$

$$(\ln Q)_{,\rho} = 2\rho f^{-1}(c^2 - 1)(\chi_{,\rho}^2 - \chi_{,z}^2). \quad (10)$$

If we write χ by using the function $R(\rho, z)$ and the constants e and m ($m = ce$) as

$$\chi = \frac{e}{R + m}, \quad (11)$$

and introduce the transformation

$$R = d \frac{1 + \bar{f}}{1 - \bar{f}}, \quad (12)$$

then Eqs. (9) and (10) become

$$(\ln \bar{f})_{,\rho\rho} + (\ln \bar{f})_{,zz} + \rho^{-1}(\ln \bar{f})_{,\rho} = 0, \quad (13)$$

$$(\ln Q)_{,\rho} = \frac{\rho}{2}[(\ln \bar{f})_{,\rho}^2 - (\ln \bar{f})_{,z}^2]. \quad (14)$$

These expressions are exactly the same as those for the vacuum Einstein equation in the metric (3) with $\bar{f}$ instead of f , and therefore we can apply the inverse scattering method

to solving Eqs. (13) and (14). The n -soliton solution for $\bar{f}$ and the corresponding solution for Q are given by

$$\bar{f} = \prod_h^n (i\mu_h/\rho)^{q_h}, \quad (15)$$

$$Q = \frac{\rho^{D^2/2} \prod_{h>l} (\mu_h - \mu_l)^{2q_h q_l}}{\prod_h (\mu_h^2 + \rho^2)^{q_h^2} \prod_l \mu_l^{(D-2q_l)q_l} C^{(n)}}, \quad (16)$$

where μ_h 's are the pole trajectories

$$\mu_h = w_h - z \pm \sqrt{(w_h - z)^2 + \rho^2}, \quad (17)$$

and q_h 's, w_h 's, $C^{(n)}$ and $D (= \sum_h^n q_h)$ are constants. Through the relation (12) the electrostatic potential χ is given by Eq. (11) and the metric function f by

$$f = \frac{R^2 - d^2}{(R + m)^2}, \quad (18)$$

where $d^2 = m^2 - e^2$. We can find that the 2-soliton solution given by setting $n = 2$ in Eq. (15) is interpreted as a charged-Weyl solution with the distortion parameter $\delta (= q_1 = q_2)$ that is reduced to the Reissner-Nordström solution when $\delta = 1$.

We now consider the 4-soliton solution:

$$\bar{f} = (-\mu_1\mu_2/\rho^2)^{\delta_1} (-\mu_3\mu_4/\rho^2)^{\delta_2}, \quad (19)$$

$$Q = \frac{\rho^{2(\delta_1+\delta_2)^2} (\mu_2 - \mu_1)^{2\delta_1^2} (\mu_4 - \mu_3)^{2\delta_2^2}}{C^{(4)} (\mu_1\mu_2\mu_3\mu_4)^{2\delta_1\delta_2}} \times \frac{[(\mu_3 - \mu_1)(\mu_4 - \mu_1)(\mu_3 - \mu_2)(\mu_4 - \mu_2)]^{2\delta_1\delta_2}}{[(\mu_1^2 + \rho^2)(\mu_2^2 + \rho^2)]^{\delta_1^2} [(\mu_3^2 + \rho^2)(\mu_4^2 + \rho^2)]^{\delta_2^2}}, \quad (20)$$

where $\delta_1 = q_1 = q_2$ and $\delta_2 = q_3 = q_4$. The $\pm$ sign in the pole trajectories (17) and the constant $C^{(4)}$ should be determined so that the metric may satisfy the asymptotic flatness:

$$\begin{aligned} \mu_1 &= w_1 - z + \sqrt{(w_1 - z)^2 + \rho^2}, \\ \mu_2 &= w_2 - z - \sqrt{(w_2 - z)^2 + \rho^2}, \\ \mu_3 &= w_3 - z + \sqrt{(w_3 - z)^2 + \rho^2}, \\ \mu_4 &= w_4 - z - \sqrt{(w_4 - z)^2 + \rho^2}, \end{aligned} \quad (21)$$

$$C^{(4)} = [2^4(w_4 - w_2)^2(w_3 - w_1)^2]^{\delta_1\delta_2}. \quad (22)$$

We assume for the constants w_k 's that $w_1 < w_2 < w_3 < w_4$ and parameterize them as $w_1 = z_1 - \sigma_1$, $w_2 = z_1 + \sigma_1$, $w_3 = z_2 - \sigma_2$ and $w_4 = z_2 + \sigma_2$. The asymptotic behaviors of χ and $f(= -g_{00})$ at spatial infinity $\sqrt{\rho^2 + z^2} \rightarrow \infty$ are

$$\chi \sim \frac{e}{d} \frac{\delta_1 \sigma_1 + \delta_2 \sigma_2}{\sqrt{\rho^2 + z^2}}, \quad (23)$$

$$f \sim 1 - \frac{2m}{d} \frac{\delta_1 \sigma_1 + \delta_2 \sigma_2}{\sqrt{\rho^2 + z^2}}, \quad (24)$$

which imply that the total charge and mass in this solution are given by

$$e_{\text{total}} = \frac{e}{d} (\delta_1 \sigma_1 + \delta_2 \sigma_2), \quad (25)$$

$$m_{\text{total}} = \frac{m}{d} (\delta_1 \sigma_1 + \delta_2 \sigma_2). \quad (26)$$

The behaviors of the metric functions near the z -axis ($\rho \sim 0$) are as follows: when $\delta_1 = \delta_2 = 1$, $f \sim \rho^2$ and $Q \sim \rho^2$, and so $f^{-1}Q \sim \rho^0$ in the regions given by $w_1 < z < w_2$ and $w_3 < z < w_4$. These behaviors are the same as those of the metric functions on the horizon in the Reissner-Nordström solution. This fact implies that there are horizons in those two regions and the solution can be interpreted as a 2-Reissner-Nordström solution in this case. When $\delta_1 \neq 1$ and $\delta_2 \neq 1$, $f^{-1}Q \sim 0$ or ∞ in those regions, which means that there are naked singularities there and the solution should be called 2-charged-Weyl solution. While $Q = 1$ in the regions $z < w_1$ and $z > w_4$, $Q \neq 1$ in the region $w_2 < z < w_3$. This shows that the axis between w_2 and w_3 does not have locally Euclidean nature and there is a "strut" there.

We now have the interpretation that the solution constructed from the 4-soliton solution for $\bar{f}$ describes two charged Weyls. However, we do not have the charge and mass of each charged Weyl though we found the total values of them. Here we give the definitions of charge and mass. The charge e_i ($i = 1$ or 2) of each charged Weyl can be evaluated from the flux integral over a surface S_i around the region $w_1 < z < w_2$ or $w_3 < z < w_4$ on the z -axis:

$$\begin{aligned} e_i &= -\frac{1}{2} \oint_{S_i} [\rho f^{-1} \chi_{,\rho} dz + \rho f^{-1} \chi_{,z} d\rho] \\ &= \frac{1}{4} \frac{e}{d} \oint_{S_i} [\rho (\ln \bar{f})_{,\rho} dz + \rho (\ln \bar{f})_{,z} d\rho]. \end{aligned} \quad (27)$$

The surface integral appearing in the above second equation can be calculated over an arbitrary cylindrical surface S around the z -axis with the lower base at $z = z_a$, the upper base at $z = z_b$ and the side at $\rho = \rho_c$ if we use the properties of the pole trajectories (17):

$$\begin{aligned}
& \oint_S [\rho(\ln \bar{f})_{,\rho} dz + \rho(\ln \bar{f})_{,z} d\rho] \\
&= \int_{z_a}^{z_b} \rho(\ln \bar{f})_{,\rho} \Big|_{\rho=\rho_c} dz + \int_0^{\rho_c} \rho(\ln \bar{f})_{,z} \Big|_{z=z_a} d\rho - \int_0^{\rho_c} \rho(\ln \bar{f})_{,z} \Big|_{z=z_b} d\rho \\
&= \delta_1 (|w_1 - z_b| - |w_2 - z_b| - |w_1 - z_a| + |w_2 - z_a|) \\
&\quad + \delta_2 (|w_3 - z_b| - |w_4 - z_b| - |w_3 - z_a| + |w_4 - z_a|) \\
&\equiv I(z_a, z_b).
\end{aligned} \tag{28}$$

From this result we have

$$e_1 = \frac{e}{4d} I(z_a \leq w_1, w_2 \leq z_b < w_3) = \frac{e}{d} \delta_1 \sigma_1, \tag{29}$$

$$e_2 = \frac{e}{4d} I(w_2 < z_a \leq w_3, z_b \geq w_4) = \frac{e}{d} \delta_2 \sigma_2. \tag{30}$$

As for the mass m_i of each charged Weyl, we can also define it by the surface integral of the covariant derivative of the timelike Killing vector over S_i . Taking account of the contribution of electric field to the surface integral, we have

$$\begin{aligned}
m_1 &= \frac{1}{4} \oint_{S_1} [\rho f^{-1} (f - \chi^2)_{,\rho} dz + \rho f^{-1} (f - \chi^2)_{,z} d\rho] \\
&= \frac{1}{4} \frac{m}{d} \oint_{S_1} [\rho(\ln \bar{f})_{,\rho} dz + \rho(\ln \bar{f})_{,z} d\rho] \\
&= \frac{m}{4d} I(z_a \leq w_1, w_2 \leq z_b < w_3) = \frac{m}{d} \delta_1 \sigma_1, \tag{31}
\end{aligned}$$

$$m_2 = \frac{m}{4d} I(w_2 < z_a \leq w_3, z_b \geq w_4) = \frac{m}{d} \delta_2 \sigma_2. \tag{32}$$

Therefore we arrive at the interpretation that the solution describes two charged Weyls each of which has the charge e_i and mass m_i ($i = 1$ or 2) and the total charge and mass are given by the sums

$$e_{\text{total}} = e_1 + e_2, \tag{33}$$

$$m_{\text{total}} = m_1 + m_2. \tag{34}$$

When $\delta_1 = \delta_2 = 1$ the solution describes two Reissner- Nordström black holes with charges e_1 and e_2 and masses m_1 and m_2 .

Finally we consider the condition for static equilibrium of two charged Weyls on their charges and masses. As we mentioned before, in the solution there is a strut in general supporting two charged Weyls between w_2 and w_3 along the z -axis. This can be seen by evaluating the quantity

$$P_0^2 = \lim_{\rho \rightarrow 0} \left(\frac{g_{\phi\phi}}{\rho^2 g_{\rho\rho}} \right) = \lim_{\rho \rightarrow 0} Q^{-1}. \quad (35)$$

For the region $w_2 < z < w_3$ we have

$$P_0^2 = \left[\frac{(z_2 - z_1)^2 - (\sigma_2 - \sigma_1)^2}{(z_2 - z_1)^2 - (\sigma_2 + \sigma_1)^2} \right]^{2\delta_1\delta_2}. \quad (36)$$

From the condition $P_0 = 1$, which represents that the axis is locally Euclidean and therefore there is no strut along the axis, we obtain two conditions. One is $z_2 - z_1 = \sqrt{\sigma_1^2 + \sigma_2^2}$, which should be discarded because it contradicts the assumption $w_2 < w_3$ assuring that two charged Weyl are separated. The other is $\sigma_1\sigma_2 = 0$. This leads to $d = 0$ or $m = |e|$ because $m_1m_2 \neq 0$ and $e_1e_2 \neq 0$. Therefore we have the result that the condition for equilibrium should be $m_i = |e_i| (i = 1, 2)$.

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Multi Dilaton Black Holes [†]

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ABSTRACT

We study the axially symmetric static Maxwell-Einstein equations with a dilaton field. We show that the equations are decomposed into two groups, which enables us to obtain an infinite number of soliton solutions. The first non-trivial solution is the well known dilaton black hole. The second non-trivial solution is the two dilaton black holes with a strut.

[†] Based on the work in collaboration with T. Azuma

In this paper we show a series of soliton solutions to the Maxwell-Einstein equations with a dilaton field in the axially symmetric case. The first non-trivial solution of the series is the dilaton black hole[1]. The second non-trivial solution is the two dilaton black holes separated by a certain distance. The solution reminds us of the multi charged black holes known as MP solution[2]. It is known that the repulsive force between the same sign of electric charges is balanced by the gravitational attractive force. The electric charge has its maximal charge in order to sustain the static equilibrium. Here arises a question. What would happen if the charge does not satisfy the maximal charge condition? Is there any static solution? This is one of the motivations of our recent study on the multi black hole solutions[3]. Here we discuss the dilaton black holes. The reason why we study it is that the charged black hole solution can not be a classical solution to the low energy limit of the string theory. The constant dilaton field can not be the solution to the equations of motion as far as the dilaton field exists. Therefore we need to consider the non-trivial dilaton field in solving the equations. Furthermore we are interested in the static equilibrium of the multi dilaton black holes. It is known that the dilatonic force is attractive just like the gravitational force. It is interesting to know how the equilibrium condition would be in this case and what is more interesting is to know what would support the force if the equilibrium condition is broken.

In this paper we stress the structure of the equations or how we get the multi dilaton black holes. One should note that the solutions here are obtained without assuming the extremal(maximally charged) condition. The maximally charged condition brings about a great simplicity to the equations but it causes a loss of generality and destroys a physically interesting aspect.

We consider the Einstein-Maxwell action with a dilaton field:

$$I = \int d^4x \sqrt{-g} [-R + 2\partial_\mu \phi \partial^\mu \phi - e^{-2\phi} F_{\mu\nu} F^{\mu\nu}]. \quad (1)$$

Here ϕ is a dilaton field. The equations of motion are

$$\nabla_\mu (e^{-2\phi} F^{\mu\nu}) = 0, \quad (2)$$

$$\nabla^2 \phi - \frac{1}{2} e^{-2\phi} F^2 = 0, \quad (3)$$

$$R_{\mu\nu} + 2\nabla_\mu \phi \nabla_\nu \phi - e^{-2\phi} (2F_{\mu\rho} F_\nu{}^\rho - \frac{1}{2} g_{\mu\nu} F^2) = 0. \quad (4)$$

We solve the equations under the static, axially symmetric condition. We further assume that the U(1) gauge field has only electric charge and does not have magnetic charge. Under the condition the metric reads

$$-ds^2 = -f dt^2 + f^{-1} [Q(d\rho^2 + dz^2) + \rho^2 d\phi^2], \quad (5)$$

where $f = f(\rho, z)$ and $Q = Q(\rho, z)$. The equations of motion become

$$\nabla^2 (\ln f) = -e^{-2\phi} \frac{2}{Q} (\chi_{,\rho}^2 + \chi_{,z}^2), \quad (6)$$

$$\nabla^2 \phi = -e^{-2\phi} \frac{1}{Q} (\chi_{,\rho}^2 + \chi_{,z}^2) \quad (7)$$

$$\nabla^2 \chi = -\frac{f}{Q} [\chi_{,\rho} (\ln f)_{,z} + \chi_{,z} (\ln f)_{,\rho} + 2\chi_{,\rho} \phi_{,\rho} + 2\chi_{,z} \phi_{,z}], \quad (8)$$

$$(\ln Q)_{,\rho} = \frac{\rho}{2} [(\ln f)_{,\rho}^2 - (\ln f)_{,z}^2] - \frac{2\rho}{f} e^{-2\phi} (\chi_{,\rho}^2 - \chi_{,z}^2) + 2\rho (\phi_{,\rho}^2 - \phi_{,z}^2), \quad (9)$$

$$(\ln Q)_{,z} = \rho (\ln f)_{,\rho} (\ln f)_{,z} - \frac{4\rho}{f} e^{-2\phi} \chi_{,\rho} \chi_{,z} + 4\rho \phi_{,\rho} \phi_{,z}, \quad (10)$$

where $\nabla^2 = -\frac{f}{Q} (\partial_\rho^2 + \partial_z^2 + \rho^{-1} \partial_\rho)$ and χ is the zeroth component of the U(1) gauge field. When $\phi = 0$, the equations are simply the Maxwell-Einstein equation. When there is no gauge field, these are simply the Einstein equation coupled with a scalar field. When they all coexist it seems difficult to solve them. Despite the apparent difficulty that the fields are coupled in a complex way, there is a way to decompose them. We shall redefine the fields and then we can factorize the equations into two groups. Introducing F and $\bar{F}$ by

$$F = f e^{2\phi}, \quad \bar{F} = f e^{-2\phi}, \quad (11)$$

we obtain

$$\nabla^2 (\ln \bar{F}) = 0, \quad (12)$$

$$\nabla^2(\ln F) = -\frac{4}{Q}\sqrt{\frac{\bar{F}}{F}}(\chi_{,\rho}^2 + \chi_{,z}^2), \quad (13)$$

$$\nabla^2\chi = -\frac{1}{Q}\sqrt{F\bar{F}}[\chi_{,\rho}(\ln F)_{,z} + \chi_{,z}(\ln F)_{,\rho}], \quad (14)$$

$$(\ln Q)_{,\rho} = \frac{\rho}{4}[(\ln F)_{,\rho}^2 - (\ln F)_{,z}^2] - \frac{2\rho}{F}(\chi_{,\rho}^2 - \chi_{,z}^2) + \frac{\rho}{4}[(\ln \bar{F})_{,\rho}^2 - (\ln \bar{F})_{,z}^2], \quad (15)$$

$$(\ln Q)_{,z} = \frac{\rho}{2}(\ln F)_{,\rho}(\ln F)_{,z} - \frac{4\rho}{F}\chi_{,\rho}\chi_{,z} + \frac{\rho}{2}(\ln \bar{F})_{,\rho}(\ln \bar{F})_{,z}. \quad (16)$$

These equations are almost factorized except eqs.(15) and (16). We further introduce Q_{RN} , Q_s and A by

$$Q = \sqrt{Q_{RN}Q_s}, \quad (17)$$

$$\chi = A/\sqrt{2}. \quad (18)$$

Then eqs.(12)~(16) can be decomposed into two groups. We require that Q_s should satisfy the first group equations:

$$(\ln \bar{F})_{,\rho\rho} + (\ln \bar{F})_{,zz} + \frac{1}{\rho}(\ln \bar{F})_{,\rho} = 0, \quad (19)$$

$$(\ln Q_s)_{,\rho} = \frac{\rho}{2}[(\ln \bar{F})_{,\rho}^2 - (\ln \bar{F})_{,z}^2], \quad (20)$$

$$(\ln Q_s)_{,z} = \rho(\ln \bar{F})_{,\rho}(\ln \bar{F})_{,z}. \quad (21)$$

These are the vacuum Einstein equations for the metric

$$-ds^2 = -\bar{F}dt^2 + \bar{F}^{-1}[Q_s(d\rho^2 + dz^2) + \rho^2 d\phi^2]. \quad (22)$$

As far as the above equations are satisfied, Q_{RN} satisfy the second group equations. The equations satisfied by $\bar{F}$, A and Q_{RN} are

$$(\ln F)_{,\rho\rho} + (\ln F)_{,zz} + \frac{1}{\rho}(\ln F)_{,\rho} = \frac{2}{F}(A_{,\rho}^2 + A_{,z}^2), \quad (23)$$

$$A_{,\rho\rho} + A_{,zz} + \frac{1}{\rho}A_{,\rho} = A_{,\rho}(\ln F)_{,z} + A_{,z}(\ln F)_{,\rho}, \quad (24)$$

$$(\ln Q_{RN})_{,\rho} = \frac{\rho}{2}[(\ln F)_{,\rho}^2 - (\ln F)_{,z}^2] - \frac{2\rho}{F}(A_{,\rho}^2 - A_{,z}^2), \quad (25)$$

$$(\ln Q_{RN})_{,z} = \rho(\ln F)_{,\rho}(\ln F)_{,z} - \frac{4\rho}{F}A_{,\rho}A_{,z}. \quad (26)$$

Here A can be interpreted to denote the zeroth component of the $U(1)$ gauge field. We can regard the equations as the Maxwell-Einstein equation for the metric

$$-ds^2 = -Fdt^2 + F^{-1}[Q_{RN}(d\rho^2 + dz^2) + \rho^2 d\phi^2]. \quad (27)$$

Since both groups have the soliton solutions[4], we can construct an infinite number of solutions by combining them. The only point of which we need to take care is the boundary conditions. Since we impose the boundary conditions of the dilaton field ϕ and $f = -g_{00}$ as

$$f \rightarrow 1, \quad \phi \rightarrow \phi_0, \quad \text{as} \quad \sqrt{\rho^2 + z^2} \rightarrow \infty, \quad (28)$$

the boundary conditions of $F = fe^{2\phi}$ and $\bar{F} = fe^{-2\phi}$ become as

$$F \rightarrow e^{2\phi_0}, \quad \bar{F} \rightarrow e^{-2\phi_0}, \quad \text{as} \quad \sqrt{\rho^2 + z^2} \rightarrow \infty. \quad (29)$$

We can obtain the soliton solutions to both groups[3]. As for the vacuum Einstein equation we have already shown them in some papers, we here focus on the Maxwell-Einstein equation. The readers are recommended to refer to the paper by Azuma in this proceeding for the details[4]. In order to obtain the soliton solutions to the Maxwell-Einstein equation we assume that a part of the metric F is to be expressed in terms of the zeroth component of the gauge field A :

$$F = e^{2\phi_0} - 2cA + A^2, \quad (c = \frac{m}{e}e^{\phi_0}) \quad (30)$$

where m and e would be identified as mass and electric charge later. The equation for A is simplified by further introducing $\bar{f}$ by

$$A = \frac{ee^{\phi_0}}{R + m}, \quad (31)$$

with

$$R = d \frac{1 + \bar{f}}{1 - \bar{f}}, \quad (32)$$

where d is defined by

$$d^2 = m^2 - e^2. \quad (33)$$

Then the equation for $\bar{f}$ reads

$$(\ln \bar{f})_{,\rho\rho} + (\ln \bar{f})_{,zz} + \frac{1}{\rho}(\ln \bar{f})_{,\rho} = 0, \quad (34)$$

of which the N -soliton solutions is given by

$$\bar{f} = \prod_{k=1}^{N/2} \left(-\frac{\mu_{2k-1}\mu_{2k}}{\rho^2} \right)^{\delta_k}. \quad (35)$$

Here the deformation parameter δ_k are constants and μ_k called pole trajectory are given by

$$\mu_k = w_k - z + (-1)^{k+1} \sqrt{(w_k - z)^2 + \rho^2}, \quad (36)$$

where w_k are constants. We can now express both F and A in terms of $\bar{f}$ obtained above. F is expressed as

$$F = \frac{4d^2 e^{2\phi_0} \bar{f}}{[m + d - (m - d)\bar{f}]^2}. \quad (37)$$

Since we obtain both of the soliton solutions to the two groups, we can construct the solution to eqs.(12) $\sim$ (16). There is no mathematical reason why the parameters appearing in the pole trajectories of two groups are the same, but we equate them to extract physically interesting solutions. We also assume that the soliton numbers are the same in the combination of two soliton solutions and set the deformation parametes to be 1. We thus obtain the soliton solutions to the original action:

$$f = \frac{2d\bar{f}}{m + d - (m - d)\bar{f}}, \quad (38)$$

$$Q = \frac{\rho^{\frac{N^2}{2}} \prod_{k>l} (\mu_k - \mu_l)^2}{\prod_{k=1}^N (\mu_k^2 + \rho^2) \prod_{l=1}^N \mu_l^{N-2} C(N)}, \quad (39)$$

$$\phi = \phi_0 + \frac{1}{2} \ln \left[\frac{2d}{m + d - (m - d)\bar{f}} \right], \quad (40)$$

$$\chi = \frac{e e^{\phi_0} (1 - \bar{f})}{\sqrt{2} [m + d - (m - d)\bar{f}]}. \quad (41)$$

Here

$$\bar{f} = \prod_{k=1}^N \left(i \frac{\mu_k}{\rho} \right), \quad (42)$$

$$\mu_k = w_k - z + (-1)^{k+1} \sqrt{(w_k - z)^2 + \rho^2}, \quad (43)$$

where $w_i (i = 1, \dots, n)$ are constants.

In order to analyse the physical meaning of the solutions we rewrite the parameters w_i as

$$w_{i-1} = z_i - d_i, \quad w_i = z_i + d_i, \quad (44)$$

and introduce the coordinates (r_i, θ_i) by

$$\rho = \sqrt{(r_i - m_i)^2 - d_i^2} \sin \theta_i, \quad (45)$$

$$z - z_i = (r_i - m_i) \cos \theta_i. \quad (46)$$

By using the coordinates $\bar{f}$ is expressed as

$$\bar{f} = \prod_{i=1}^{N/2} \left(-\frac{\mu_{2i-1} \mu_{2i}}{\rho^2} \right) = \prod_{i=1}^N \left(\frac{r_i - m_i - d_i}{r_i - m_i + d_i} \right). \quad (47)$$

We are now ready to classify an infinite number of the solutions by their soliton numbers which correspond to the number of lumps in metric, dilaton field and electric field around the positions of their charges.

In order to find the soliton features of the solutions we first study the 2-soliton solution. It is given by

$$-ds^2 = -(1 - 2\frac{m+d}{2r})dt^2 + (1 - 2\frac{m+d}{2r})^{-1}dr^2 + r(r - m + d)d\Omega^2, \quad (48)$$

$$\phi = \phi_0 + \frac{1}{2} \ln(1 - \frac{m-d}{r}), \quad (49)$$

$$\chi = \frac{ee^{\phi_0}}{\sqrt{2r}}. \quad (50)$$

This shows that the mass is given by $M = (m + d)/2$, the dilatonic charge by $\Sigma = -(m - d)/2$ and the electric charge by $Q = ee^{\phi_0}/(\sqrt{2})$. Though there are

three charges, they are not independent. The condition that $d^2 = m^2 - e^2$ leads to

$$\Sigma = -\frac{(Qe^{-\phi_0})^2}{2M}. \quad (51)$$

The 2- soliton solution shown above is now to be expressed in terms of the three charges as

$$-ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r(r + 2\Sigma)d\Omega^2, \quad (52)$$

$$\phi = \phi_0 + \frac{1}{2}\ln(1 + \frac{2\Sigma}{r}), \quad (53)$$

$$\chi = \frac{Q}{r}. \quad (54)$$

This shows that the regular horizon exists at $r = 2M$ like the Schwarzschild solution and the area at constant r goes to zero when $r = -2\Sigma$ causing this surface to be singular[1]. This is a different point from the Schwarzschild solution. These two points in r shows that the solution has the event horizon as far as $M > -\Sigma$ holds, which turns out to give the maximal electric charge for the horizon to exist. The extremal value is given by $Q^2 = 2e^{2\phi_0}M^2$ and the difference from the value of the Schwarzschild solution can be interpreted as an effect due to the dilatonic attractive force extra to the gravitational attractive force.

We shall next consider the 4-soliton solution. Noting that

$$\bar{f} = \left(\frac{r_1 - m_1 - d_1}{r_1 - m_1 + d_1} \right) \left(\frac{r_2 - m_2 - d_2}{r_2 - m_2 + d_2} \right), \quad (55)$$

we obtain the asymptotic forms of the metric, dilaton field and the electric field around large r_1 and r_2 . The asymptotic expansions are

$$f \sim \begin{cases} 1 - \frac{2(\frac{m+d}{2}d_1)}{r_1}, & \text{for } 1 \ll r_1 \ll r_2 \\ 1 - \frac{2(\frac{m+d}{2}d_2)}{r_2}, & \text{for } 1 \ll r_2 \ll r_1 \end{cases} \quad (56)$$

$$\phi \sim \begin{cases} \phi_0 - \frac{(m-d)d_1}{2dr_1}, & \text{for } 1 \ll r_1 \ll r_2 \\ \phi_0 - \frac{(m-d)d_2}{2dr_2}, & \text{for } 1 \ll r_2 \ll r_1 \end{cases} \quad (57)$$

$$\chi \sim \begin{cases} \frac{ed_1}{\sqrt{2}dr_1}, & \text{for } 1 \ll r_1 \ll r_2 \\ \frac{ed_2}{\sqrt{2}dr_2}, & \text{for } 1 \ll r_2 \ll r_1 \end{cases} \quad (58)$$

The behavior around the zero of r_1 or r_2 is the same as that of 2-soliton solution. Furthermore the solution shows the solitonic feature that the lump around each zero disappears rapidly as the distance from the zero grows. These justify that masses, dilatonic charges and the electric charges are given by

$$M_i = \frac{m+d}{2d}d_i, \quad (i = 1, 2) \quad (59)$$

$$Q_i = \frac{e}{\sqrt{2}d}d_i, \quad (i = 1, 2) \quad (60)$$

$$\Sigma_i = -\frac{m-d}{2d}d_i. \quad (i = 1, 2) \quad (61)$$

Here these parameters are not independent. The condition that $d^2 = m^2 - e^2$ leads to

$$\Sigma_i = -\frac{(Q_i e^{-\phi_0})^2}{2M_i}. \quad (i = 1, 2) \quad (62)$$

Just as in the analysis in the vacuum Einstein equation, the Einstein equation with a scalar field or the Maxwell-Einstein equation we study the z-axis between two dilaton black holes and find the conical singularity which can be interpreted as a strut between the two black holes[4]. We can also show that the deficit angle disappears when the electric charge takes its maximal value. This may sound strange, but the equilibrium condition is realized when the electric charge takes its maximal value as in the MP solution[2]. This solution in this limit is the one already obtained by assuming the conforma-static metric[6].

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World-line SUSY in static axisymmetric spacetimes

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ABSTRACT

The conditions for the existence of Killing-Yano tensors, which are closely related to the appearance of non-generic world-line SUSY, are presented for static axisymmetric spacetimes. Imposing the vacuum Einstein equation, the set of solutions admitting Killing-Yano tensors is considered. In particular, it is shown that static, axisymmetric and asymptotically flat vacuum solutions admitting Killing-Yano tensors are only the Schwarzschild solution.

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One of the most remarkable properties of the Kerr black hole is that, in this background, particle motion is completely integrable. From the point of view of canonical analysis, this is a direct consequence of the existence of a non-trivial Killing tensor $K_{\mu\nu}$ [1-3], which gives rise to the associated constant of motion, $K^{\mu\nu}p_\mu p_\nu$, quadratic in the four-momentum, p_μ . This constant of motion completes the maximal number of constants of motion in conjunction with the other three well-known constants of motion: the energy coming from the time translation invariance of the Kerr geometry, the angular momentum coming from the axial symmetry of it, and the Hamiltonian. Moreover, various field equations, particularly the Dirac equation [4], separate in the Kerr geometry, and [5] this fact is related to the existence of the Killing-Yano tensor $f_{\mu\nu}$, which is defined as an antisymmetric second rank tensor satisfying the following Penrose-Floyd equation [6]:

$$\{\mu\nu\lambda\} \equiv 2D_{(\mu}f_{\nu)\lambda} = 0. \quad (1)$$

Here, D_μ represents the usual covariant derivative and, for later convenience, we defined braces. Contracting repeated Killing-Yano tensors, we obtain the Killing tensor associated with the Killing-Yano tensor: $K_{\mu\nu} = f_\mu{}^\lambda f_{\lambda\nu}$, $D_{(\mu}K_{\nu\lambda)} = 0$. In this sense, the Killing-Yano tensor is a square root of the Killing tensor.

Recently, it has been shown by considering supersymmetric particle mechanics that the Killing-Yano tensor can be understood as an object belonging to a larger class of possible structures which generate generalized supersymmetry algebras [7]. This novel aspect has renewed people's interest in the Killing-Yano tensor which has long been known to relativists.

It may be of some significance to ask conversely what kind of background gravitational field admits the Killing-Yano tensor. Are there any spacetime admitting a Killing-Yano tensor other than the Kerr spacetimes? While we already know that the existence of Killing-Yano tensor is directly related to the appearance of non-generic supersymmetry for the motion of spinning particle, such an investigation will give another facet of the

structural properties of spacetimes.

A formal answer to this question has been already given by Dietz and Rüdiger [8], where all metrics admitting Killing-Yano tensors were presented with no further assumptions restricting, say, the Ricci tensor. However, interpretation of those metrics seems far from trivial. In particular, the relation between the existence of Killing-Yano tensor and isometries is not known.

In this paper, we take another approach to the same problem by restricting from the beginning the metric to be static and axisymmetric. Using the prolate spheroidal coordinates (x, y, ϕ) , such metrics can be written in the form [9]

$$ds^2 = -e^{2\psi} dt^2 + \sigma^2 e^{-2\psi} \left[e^{2\gamma} (x^2 - y^2) \left(\frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right) + (x^2 - 1)(1 - y^2) d\phi^2 \right], \quad (2)$$

where

$$\psi = \psi(x, y), \quad \gamma = \gamma(x, y). \quad (3)$$

Note that the parameter σ can absorb possible constants added to ψ and γ , so that the additional constants can always be taken zero without loss of generality. We will first describe, in terms of $\psi(x, y)$ and $\gamma(x, y)$, the conditions for the existence of Killing-Yano tensor. Then we will see that the function $\psi(x, y)$ which satisfies the vacuum Einstein equation and admits a Killing-Yano tensor is determined by its derivatives, $\psi_x(p)$ and $\psi_y(p)$, at an arbitrary point $p = (x_0, y_0)$ (see (24)). Since at that time $\gamma(x, y)$ is determined according to $\psi(x, y)$ through the Einstein equation, the static axisymmetric vacuum solutions admitting Killing-Yano tensors form a three dimensional set parametrized by $\psi_x(p)$, $\psi_y(p)$ and σ . However, when contracting this set by all possible diffeomorphisms, we obtain, as the set of distinct solutions, a complicated one dimensional set. We then further restrict interest to asymptotically flat cases, and finally prove a uniqueness of the Schwarzschild solutions.

Before studying eq. (1) in terms of the metric (2), we comment upon three properties of the metric (2), which will be frequently used later.

First, we can immediately observe that the metric (2) is manifestly invariant under the interchange $x \leftrightarrow y$. This implies, for example, that if there is a correct expression containing ψ and/or γ , the expression obtained by interchanging $x \leftrightarrow y$ with $\psi_x \leftrightarrow \psi_y$, $\gamma_x \leftrightarrow \gamma_y$, $\psi_{xx} \leftrightarrow \psi_{yy}$, etc. will also be a correct expression. We call this interchange the "conjugation".

Second, the following formal transformation

$$(t, \phi) \rightarrow (i\sigma\phi, -i\sigma^{-1}t) \quad (4)$$

preserves the characteristic form of the metric (2), i.e., the metric obtained after the transformation is still designated by the (appropriately transformed) set of two functions and one parameter, $(\psi(x, y), \gamma(x, y), \sigma)$. In fact, this transformation is equivalent to the following transformation for ψ and γ :

$$\begin{aligned} (\psi, \gamma, \sigma) &\rightarrow (\psi, \gamma, \sigma)' = (\psi', \gamma', \sigma), \\ \psi' &\equiv -\psi + \frac{1}{2} \log(x^2 - 1)(1 - y^2), \\ \gamma' &\equiv \gamma - 2\psi + \frac{1}{2} \log(x^2 - 1)(1 - y^2). \end{aligned} \quad (5)$$

We call this transformation the t - ϕ transformation. It is obvious from (4) that if (ψ, γ, σ) is a solution of Einstein equation, then (ψ', γ', σ) is also a solution, which generally represents another spacetime. This also implies that Einstein equation is invariant under the t - ϕ transformation (5). Note that these transformations form a group isomorphic to $\mathbf{Z}_2$, i.e. $(\psi, \gamma, \sigma)'' = (\psi, \gamma, \sigma)$.

Finally, there exists a 2-parameter diffeomorphism which also preserves the characteristic form of (2);

$$\begin{aligned} (x, y) &\rightarrow (x', y'), \\ x' &\equiv \frac{1}{2} \left(\sqrt{s^2(x^2 + y^2) + 2s(a+1)xy + (a+1)^2 - s^2} \right. \\ &\quad \left. + \sqrt{s^2(x^2 + y^2) + 2s(a-1)xy + (a-1)^2 - s^2} \right), \end{aligned}$$

$$y' \equiv \frac{1}{2} \left(\sqrt{s^2(x^2 + y^2) + 2s(a+1)xy + (a+1)^2 - s^2} - \sqrt{s^2(x^2 + y^2) + 2s(a-1)xy + (a-1)^2 - s^2} \right), \quad (6)$$

where a and s are arbitrary parameters. Under this transformation, the metric (2) simply transforms as

$$(\psi(x, y), \gamma(x, y), \sigma) \rightarrow (\psi(x', y'), \gamma(x', y'), s\sigma). \quad (7)$$

We call these diffeomorphisms the characteristic-form-preserving diffeomorphisms (CPDs)².

We are now in a position to study static axisymmetric spacetimes which admit Killing-Yano tensors. Our strategy is quite straightforward— we simply write down all the components of the equation $\{\mu\nu\lambda\} \equiv D_{(\mu}f_{\nu\lambda)} = 0$ explicitly. So, it would be worth noticing how many independent components of eq. (1) exist. For this purpose, note first that, for the braces $\{\mu\nu\lambda\}$ defined in (1), we can read the following symmetries;

$$\{\mu\mu\mu\} = 0 \quad (\text{all indices coincide}), \quad (8)$$

$$\{\mu\mu\nu\} = -\{\nu\mu\mu\} = -\{\mu\nu\mu\} \quad (\text{repeated indices exist}), \quad (9)$$

$$\{\mu\nu\lambda\} = \{\nu\mu\lambda\} \quad (\text{no repeated indices exist}). \quad (10)$$

Now, it is easy to see that, for the braces of the types both (9) and (10), there are twelve independent components. Thus, the total number of independent ones of eq. (1) is 24, while that of $f_{\mu\nu}$ is six.

By explicit calculations of eq. (1), we can see that the equations for components $\{331\}$, $\{332\}$, $\{001\}$ and $\{002\}$ immediately implies that f_{12} and f_{21} must vanish, and therefore the equations for components $\{112\}$ and $\{221\}$ are identically satisfied. Similarly, the

²The geometrical meaning of the CPDs becomes clear if one views them in the cylindrical coordinates defined by $(\rho, z, \phi) = (\sigma(x^2 - 1)^{1/2}(1 - y^2)^{1/2}, \sigma xy, \phi)$. In these coordinates, the spatial part of the metric (2) takes the form $e^{-2\psi}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2]$, and the CPDs correspond to the diffeomorphisms which map $(\rho, z) \rightarrow (s\rho, sz + a)$. Therefore the parameter a is considered to represent the translations along the axis of symmetry, while s represents the similar transformations.

equations for components $\{013\}$, $\{310\}$, $\{023\}$, $\{320\}$, $\{301\}$, and $\{302\}$ demand that f_{30} and f_{03} vanish.

The remaining twelve components of eq. (1) are found to be divided into two sets, $\{113\}$, $\{312\}$, $\{223\}$, $\{123\}$, $\{231\}$ and $\{003\}$ which contain $f_{13}(= -f_{31})$ and $f_{23}(= -f_{32})$ only, and $\{110\}$, $\{012\}$, $\{220\}$, $\{120\}$, $\{201\}$ and $\{330\}$ which contain $f_{10}(= -f_{01})$ and $f_{20}(= -f_{02})$ only. For convenience we label these two sets "T" and "R"³, respectively. It is easy to check that T and R interchange by the t - ϕ transformation (5) with $f_{13} \leftrightarrow f_{10}$ and $f_{23} \leftrightarrow f_{20}$. This means that, if we solve T completely, the solution of R is automatically given. For this reason, we shall mainly consider T, since it is simpler than R.

Equation $\{003\} = 2D_0 f_{03} = 0$ obviously contains no derivatives of $f_{\mu\nu}$ due to the static nature of the spacetime. Its explicit form is

$$(x^2 - 1) \psi_x f_{13} + (1 - y^2) \psi_y f_{23} = 0. \quad (11)$$

The other equations in T contain derivatives with respect to x and/or y ;

$$\{113\} = 0 \Leftrightarrow$$

$$f_{13,x} = \left(\frac{x}{x^2 - y^2} - 2\psi_x + \gamma_x \right) f_{13} + \frac{1 - y^2}{x^2 - 1} \left(\frac{y}{x^2 - y^2} + \psi_y - \gamma_y \right) f_{23} \quad (12)$$

$$\{231\} = 0 \Leftrightarrow$$

$$f_{13,y} = - \left(\frac{2y}{1 - y^2} + \frac{y}{x^2 - y^2} + 3\psi_y - \gamma_y \right) f_{13} - \left(\frac{x}{x^2 - 1} - \frac{x}{x^2 - y^2} - \gamma_x \right) f_{23} \quad (13)$$

$$\{312\} = 0 \Leftrightarrow$$

$$f_{23,x} = \left(\frac{y}{1 - y^2} - \frac{y}{x^2 - y^2} + \gamma_y \right) f_{13} + \left(\frac{2x}{x^2 - 1} + \frac{x}{x^2 - y^2} - 3\psi_x + \gamma_x \right) f_{23} \quad (14)$$

$$\{223\} = 0 \Leftrightarrow$$

$$f_{23,y} = - \frac{x^2 - 1}{1 - y^2} \left(\frac{x}{x^2 - y^2} - \psi_x + \gamma_x \right) f_{13} - \left(\frac{y}{x^2 - y^2} + 2\psi_y - \gamma_y \right) f_{23}. \quad (15)$$

³As seen later, T and R are characterized by "constraints", which originate from, respectively, the "Time" translation invariance and the "Rotational" invariance of our metric.

The remaining equation $\{123\} = 0$ becomes trivial if eqs. (13) and (14) are satisfied. Note that T is a set of simultaneous first order partial differential equations with a “constraint” (11). For consistency, the derivative of (11) with respect to x must vanish. This yields

$$\Delta\gamma_y = (x^2 - 1) \left[\psi_x \psi_{xy} - \psi_y \psi_{xx} + \frac{y}{x^2 - y^2} (\psi_x^2 - \psi_y^2) \right] \quad (16)$$

with

$$\Delta \equiv [(x^2 - 1) \psi_x^2 + (1 - y^2) \psi_y^2]. \quad (17)$$

The derivative of (11) with respect to y yields the conjugation of (16), which reads

$$\Delta\gamma_x = (1 - y^2) \left[\psi_y \psi_{xy} - \psi_x \psi_{yy} + \frac{x}{x^2 - y^2} (\psi_x^2 - \psi_y^2) \right]. \quad (18)$$

We have also to impose the integrability condition for γ , which gives rise to a non-trivial condition:

$$\frac{\partial}{\partial x} \{(\text{the r.h.s. of (16)})/\Delta\} = \frac{\partial}{\partial y} \{(\text{the r.h.s. of (18)})/\Delta\}. \quad (19)$$

Moreover, the integrability of f_{13} must also be imposed. Using (16) and (18) to eliminate derivatives of γ , this implies

$$\begin{aligned} & 2\psi_x^3\psi_y + \psi_{xx}\psi_y [x(x^2 - 1)\psi_x^2 - x(1 - y^2)\psi_y^2 - 2y(x^2 - 1)\psi_x\psi_y] \\ & - \psi_{xy} [x(x^2 - 1)\psi_x^3 - 3y(x^2 - 1)\psi_x^2\psi_y] = \text{the conjugation of the l.h.s.} \end{aligned} \quad (20)$$

Here, we have supposed $\psi_x \neq 0, \psi_y \neq 0$. When imposing (20), we can see that the integrability of f_{23} is trivial.

Now, we can summarize the results of above calculations as follows. Once given a metric of the form (2), one can examine through (16)~(20) whether it admits non-trivial components $f_{13} = -f_{31}$ and $f_{23} = -f_{32}$ of a Killing-Yano tensor. Moreover, transforming the metric by (5) and reexamining (16)~(20) for the new metric ⁴, one can also know whether

⁴This operation is equivalent to transforming equations (16)~(20) by (5) and reexamining them for the original metric.

the original metric admits non-trivial components $f_{10} = -f_{01}$ and $f_{20} = -f_{02}$. As regards the other components of $f_{\mu\nu}$, they always vanish.

In order to get more definite consequences, we here impose the vacuum Einstein equation given by

$$\gamma_x = \frac{1-y^2}{x^2-y^2} \left[x(x^2-1) \psi_x^2 - x(1-y^2) \psi_y^2 - 2y(x^2-1) \psi_x \psi_y \right], \quad (21)$$

$$\gamma_y = \frac{x^2-1}{x^2-y^2} \left[y(x^2-1) \psi_x^2 - y(1-y^2) \psi_y^2 + 2x(1-y^2) \psi_x \psi_y \right], \quad (22)$$

$$0 = 2x\psi_x + (x^2-1) \psi_{xx} - 2y\psi_y + (1-y^2) \psi_{yy}. \quad (23)$$

The integrability of γ in the Einstein equation is guaranteed by eq. (23), so that we do not have to consider another integrability condition (19). Moreover, by a direct calculation eq. (20) is found to be trivial if eqs. (16), (18) and (21)~(23) are satisfied. As a result, it turns out that, for the present purpose, we need only eqs. (16) and (18) over the vacuum Einstein equation (21)~(23).

Eliminating γ_x and γ_y in (16) and (18) by substituting (21) and (22), we obtain three equations for ψ_x , ψ_y , ψ_{xx} , ψ_{yy} , and ψ_{xy} , and then we can algebraically solve these equations for ψ_{xx} , ψ_{yy} , and ψ_{xy} in terms of ψ_x and ψ_y (with x and y);

$$\begin{aligned} \psi_{xx} &= F(\psi_x, \psi_y, x, y), \\ \psi_{yy} &= F(\psi_y, \psi_x, y, x), \\ \psi_{xy} &= G(\psi_x, \psi_y, x, y), \end{aligned} \quad (24)$$

where F and G are functions with four arguments. Although we do not write down these lengthy expressions in explicit forms, it is evident that they do always exist (except for $\psi_x = \psi_y = 0$) and are unique because the three equations are linear in ψ_{xx} , ψ_{yy} , and ψ_{xy} .

Equation (24) implies that, given the first order derivatives ψ_x and ψ_y at an arbitrary point p , higher derivatives of ψ at p are all given. Thus, $\psi(x, y)$ and accordingly $\gamma(x, y)$ are determined at least in the neighbourhood of p in terms of two parameters $\psi_x(p)$ and $\psi_y(p)$.

This implies that the static axisymmetric vacuum solutions admitting Killing-Yano tensors form a three dimensional set parametrized by $\psi_x(p)$, $\psi_y(p)$, and σ , which we denote as Y_p . We must, however, recall the existence of two dimensional diffeomorphisms (CPDs), which contract Y_p and yield a connected one dimensional set, $\bar{Y}_p$, as the set of distinct solutions.

Our next step is to select out of $\bar{Y}_p$ all asymptotically flat solutions admitting Killing-Yano tensors. For this end, we expand ψ at spatial infinity ($x \rightarrow \infty$) as

$$\psi = -\frac{1}{x} + \frac{f_3(y)}{x^3} + \frac{f_4(y)}{x^4} + \dots, \quad (25)$$

where the first term of the r.h.s. has been taken to coincide with that of the Schwarzschild solution ψ_{Sch} :

$$\psi_{\text{Sch}} = \frac{1}{2} \log \frac{x-1}{x+1}, \quad (26)$$

which certainly admits a Killing-Yano tensor. This can be always done by choosing the parameter s in the CPDs appropriately. The second order term in the expansion, which corresponds to the dipole component of the system, has been taken zero by using the remaining freedom of the CPDs parametrized by a . Having restricted the form of the solutions to (25), ψ no longer has continuous freedom, because the remaining 1-parameter freedom of the metric is to be carried by the spatial conformal parameter σ . Uniqueness of the expression (24) suggests that the Schwarzschild solution (26) is the unique solution of the form (25). In fact, this is confirmed by directly solving the expanded equations of (24). This fact implies more precisely that the solutions of the form (25) admitting non-trivial components $f_{13} = -f_{31}$ and $f_{23} = -f_{32}$ of Killing-Yano tensors are only the Schwarzschild solutions parametrized by σ .

What we have to do next is to examine if there are asymptotically flat solutions which admit non-trivial components $f_{10} = -f_{01}$ and $f_{20} = -f_{02}$ of Killing-Yano tensors. However, it is found as follows that no such solutions exist. Namely, if there were such an asymptotically flat solution, the asymptotic form of the Schwarzschild solution $\psi \rightarrow -1/x$ should satisfy the asymptotic equations of the t - ϕ transformed version of equations (24).

However, it fails, and this proves the claim.

Gathering all results, we have proved that, *modulo diffeomorphisms, static, axisymmetric, and asymptotically flat vacuum solutions admitting Killing-Yano tensors are only the Schwarzschild solutions.*

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Thermodynamic Stability and Fluctuations of Black-Hole Equilibrium States

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The analysis of thermodynamic stability is reconsidered from a purely thermodynamic standpoint in order to provide a firm ground to Katz's turning point method. An interrelation between a change of stability and the cusp-type catastrophe is also pointed out. The results of stability theory are used to discuss the thermodynamic fluctuations, and as an application of the general theory, fluctuations of black-hole equilibrium states are calculated.

1. Introduction

We have systematically examined the thermodynamic stability of Kerr holes [1] and Kerr-Newman holes [2] which are placed in various thermodynamic environments. In judging their stability, we have used the turning point method developed by Katz [3]. Before describing his method, we explain the situation we are going to consider.

Suppose that an arbitrary thermodynamic system is placed in a given thermodynamic environment which fixes a set of variables, $\{x_i\}$, where $i = 1, 2, \dots, n$. We call them the control variables. For example, if the system is in contact with a heat bath then one of such control variables is its temperature, and if it is not, then the corresponding control variable is rather the system's thermal energy. As the thermodynamic theory shows, equilibrium states of the system are described by a Massieu function, Ψ , whose independent variables are the control variables, $\{x_i\}$. We call it the proper Massieu function to a given environment. The infinitesimal increment of Ψ is written generally as

$$d\Psi = \sum_{i=1}^n X_i dx_i, \quad (1)$$

and

$$X_i \equiv (\partial\Psi/\partial x_i)_{x_j} = X_i(\{x_k\}) \quad (2)$$

is called the conjugate of the variable x_i , where $\bar{x}_i$ denotes the set of the variables $\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}$. A series of equilibrium states thus forms a curve given by eq. (2) in the $2n$ -dimensional hyperspace $(\{x_i\}, \{X_i\})$ which may be called thermodynamic phase space.

In order to judge the stability of a series of equilibrium states, one plots, according to Katz, X_i versus x_i for each i with other control variables $\bar{x}_i$ being fixed. Katz's method is applicable to each portion of such curves as varies smoothly and has no bifurcations. He has proved [3] that a change of stability occurs only where the tangent of such a curve is vertical (see Fig. 1). In the neighbourhood of such a point of vertical tangent, the points on the curve having positive slopes are more stable than those having negative slopes. Such degrees of stability are kept unchanged in both directions along the curve until one meets another vertical tangent, where the same rule as the above again does apply.

There remain, however, a few questions about this method. They are:

- Is there any guarantee for treating each pair of (x_i, X_i) independently? (Katz's original paper [3] deals with only the cases in which $n = 1$.)
- Why the points of horizontal tangents have no relation to the change of stability? (The second-order variation of Ψ changes its sign not only at a vertical tangent but also at a horizontal one.)
- What relations do the slopes of tangent have with the eigenvalues of thermodynamic fluctuation modes, at an arbitrary point on an equilibrium curve? (Katz has given such relations only at the points in the close vicinity of a vertical tangent.)

2. Purely Thermodynamic Theory of Stability

Although Katz has discussed the problem of thermodynamic stability of equilibrium states from a standpoint of statistical mechanics, we present here a purely thermodynamic treatment of the same problem. The problem is to show what exactly one can say about the stability of a thermodynamic system in a given environment, based only on the knowledge about a series of its equilibrium states. Of course, a stable equilibrium is identified by a local maximum of the proper Massieu function.

The proper Massieu function is originally defined only along the equilibrium curve in the thermodynamic phase space. In order to perform a stability analysis, however, we have to know the *off-equilibrium* (i.e. off the equilibrium curve) behaviour of the Massieu function. Thus we are forced to introduce an extended Massieu function whose domain of definition is extended to include, at least, also near-equilibrium states, the states involved in a thin $2n$ -dimensional tube-like region around the equilibrium curve.

Our idea from a purely thermodynamic point of view is to introduce such a function, $\hat{\Psi}$, by the following procedure. First express the equilibrium Massieu function in terms of $\{x_i\}$ and $\{X_i\}$, and then release these variables from their equilibrium relations (2). Thus, $\hat{\Psi}$ has been defined as a function of both $\{x_i\}$ and $\{X_i\}$. This procedure is reminiscent of those used in introducing the Lagrangian and Hamiltonian in the classical mechanics. We assume that $\hat{\Psi}$ really describes the thermodynamic states of the system under consideration, not only on the equilibrium curve but also, at least, in its neighbourhood. The final results, however, turn out to include the equilibrium quantities only.

Let us remind here again that what is essential for a stability analysis is only the off-equilibrium behaviour of $\hat{\Psi}$ and that along the equilibrium curve is irrelevant. This means that we do not need to calculate the second-order variations of $\hat{\Psi}$ for small virtual displacements, in all of $2n$ independent directions in the phase space. Instead, we have only to do it in suitably selected n independent directions. As such a set of n independent displacements, we chose the one in which $\{x_i\}$ are fixed and only $\{X_i\}$ are displaced from their equilibrium values. There are two reasons for this. The first is, of course, because such a choice makes the calculations simple. The second is on the physical ground. As stated above, the control variables are fixed when a thermodynamic environment is specified. On the other hand, the conjugate variables can actually fluctuate around their equilibrium values. Thus, such a set of restricted displacements is not merely a set of virtual ones but may be considered as of actual ones performed by spontaneous fluctuations in the system.

Since the detailed calculations have been given in ref. [4], we summarize here only the results.

- The first-order variations evaluated on an equilibrium curve always vanish reflecting that the curve corresponds to a series of extrema:

$$(\delta\hat{\Psi})_{\{x_i\}} = 0. \quad (3)$$

- The second-order variations always have a form

$$(\delta^2\hat{\Psi})_{\{x_i\}} = - \sum_{k=1}^n \lambda_k (\delta X_k)^2, \quad (4)$$

where $\lambda_k = (\partial x_k / \partial X_k)_{\bar{x}_k} = (\partial^2 \Psi / \partial x_k^2)_{\bar{x}_k}^{-1}$ are the eigenvalues for fluctuation modes δX_k . The absence of the cross terms in the above second variation guarantees that each fluctuation mode for X_k may be treated independently from others. The eigenvalue of a fluctuation mode appears as the inverse of the slope of tangent, at any point, of the equilibrium curve projected on the eigenplane (x_k, X_k) of that mode.

- At a vertical tangent (i.e. a point of $\lambda_k = 0$) in each eigenplane of fluctuation mode, the stability for that mode changes in such a manner as described in section 1. This is because the surface of $\hat{\Psi}$ as a function of x_k and X_k shows a cusp-type catastrophe above the point of vertical tangent (Fig. 2). Corresponding to this fact, the equilibrium Massieu function Ψ as a function of x_k has a cusp-like folding there (Fig. 3). At a horizontal tangent, on the other hand, stability does not change there because the point corresponds only to an inflection of Ψ as a function of x_k (Fig. 3).

For the detailed results of stability analysis of black holes, the readers are referred to refs. [1] and [2].

3. Equilibrium Fluctuations of Black Holes

The results described in the previous section are directly applicable to the calculation of various correlations of equilibrium fluctuations, because the variations of $\hat{\Psi}$ given there represent actually possible displacements as spontaneous fluctuations of a thermodynamic system in a given environment. Before applying them to the case of black holes, we derive some general relations of use.

The macroscopic distribution function for equilibrium fluctuations is

$$P(\xi_1, \dots, \xi_n) d\xi_1 \dots d\xi_n \propto \exp [k_B^{-1} (\hat{\Psi} - \Psi)] d\xi_1 \dots d\xi_n, \quad (5)$$

where $\xi_i \equiv \delta X_i = X_i - X_i(\text{eq.})$ denotes a deviation of a conjugate variable from its equilibrium value and $\dot{\psi}$ represents an instantaneous value of the extended Massieu function evaluated on a non-equilibrium point in the phase space near an equilibrium curve. Recalling that the control variables $\{x_i\}$ are fixed at values which a given environment specifies, we have

$$P(\{x_i\}) \propto \exp [(2k_B)^{-1}(\delta^2 \dot{\psi})_{\{x_i\}}], \quad (6)$$

within the accuracy up to second order.

From eqs. (6) and (4) we can easily obtain

$$< \delta X_i > = 0, \quad (7)$$

$$< \delta X_i \delta X_j > = \frac{k_B}{\lambda_i} \delta_{ij} = k_B \frac{\partial^2 \dot{\psi}}{\partial x_i^2} \delta_{ij}. \quad (8)$$

The first equation is trivial but the second is different from other treatments appearing in the literature in the point that the cross-correlations ($i \neq j$) of fluctuations in the conjugate variables always vanish. The latter fact is reflecting that fluctuations in each conjugate variable can be regarded as an independent mode of fluctuations. The formula, however, gives the identical results with those of other treatments as for the auto-correlations ($i = j$) of fluctuating variables. It predicts a divergence of fluctuations at a point of vertical tangent since $\lambda_i = 0$ there. Of course, such a formula cannot be used where the eigenvalues are negative.

It is a straightforward task to calculate the correlations of fluctuations of various thermodynamic variables for black-hole equilibrium states according to eq. (8). As seen in ref. [2], there are 8 different thermodynamic environments which are suitably treated in our framework based on the Massieu functions, and fluctuations are different in every environment. We summarize below the results only for two typical cases, the completely canonical and microcanonical cases.

(1) completely canonical case

Black holes in this case are connected to all kinds of reservoir, i.e. thermal, angular momentum and electric charge reservoirs, so that their masses M , angular momenta J and electric charges Q can fluctuate while the control variables, $\beta \equiv T^{-1}$ (inverse temperature), $\alpha_J \equiv \Omega/T$ (Ω denotes angular velocity) and $\alpha_Q \equiv \Phi/T$ (Φ denotes electrostatic potential), are fixed by the reservoirs. The results are

$$\begin{aligned}
\langle (\delta M)^2 \rangle &= k_B T^2 C_{\alpha_I, \alpha_Q}, \quad \langle (\delta J)^2 \rangle = k_B T I_{\beta, \alpha_Q}, \quad \langle (\delta Q)^2 \rangle = k_B T K_{\beta, \alpha_J}, \\
\langle (\delta S)^2 \rangle &= k_B \left[C_{\alpha_I, \alpha_Q} + \frac{\alpha_J^2}{\beta} I_{\beta, \alpha_Q} + \frac{\alpha_Q^2}{\beta} K_{\beta, \alpha_J} \right], \\
\langle \delta M \delta S \rangle &= k_B T C_{\alpha_I, \alpha_Q}, \quad \langle \delta J \delta S \rangle = -k_B \Omega I_{\beta, \alpha_Q}, \quad \langle \delta Q \delta S \rangle = -k_B \Phi K_{\beta, \alpha_J},
\end{aligned}$$

where S is entropy and C , I and K are, respectively, heat capacity, moment of inertia and electrostatic capacity of a black hole as defined in ref. [2].

(2) microcanonical case

In this case black holes are considered as isolated systems, so that β , α_J and α_Q can fluctuate while M , J and Q are fixed once a system has been selected. The correlations of fluctuations are

$$\begin{aligned}
\langle (\delta \beta)^2 \rangle &= -k_B \frac{\beta^2}{C_{J,Q}}, \quad \langle (\delta \alpha_J)^2 \rangle = -k_B \frac{\beta}{I_{M,Q}}, \quad \langle (\delta \alpha_Q)^2 \rangle = -k_B \frac{\beta}{K_{M,J}}, \\
\langle (\delta \Omega)^2 \rangle &= -k_B \left[\frac{\Omega^2}{C_{J,Q}} + \frac{T}{I_{M,Q}} \right], \quad \langle (\delta \Phi)^2 \rangle = -k_B \left[\frac{\Phi^2}{C_{J,Q}} + \frac{T}{K_{M,J}} \right], \\
\langle \delta T \delta \Omega \rangle &= -k_B \frac{T \Omega}{C_{J,Q}}, \quad \langle \delta T \delta \Phi \rangle = -k_B \frac{T \Phi}{C_{J,Q}}, \quad \langle \delta \Omega \delta \Phi \rangle = -k_B \frac{\Omega \Phi}{C_{J,Q}}.
\end{aligned}$$

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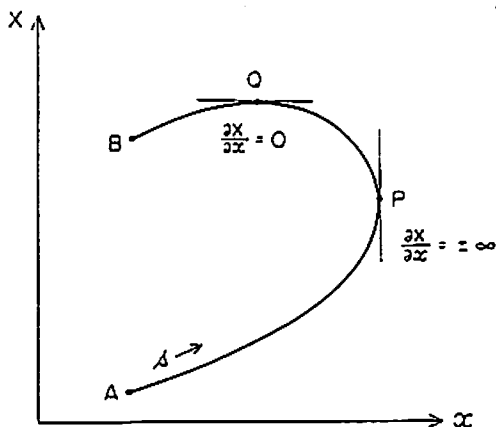


Fig. 1: An example of equilibrium series projected onto an eigenplane. It has a vertical and horizontal tangents at P (turning point) and Q , respectively.

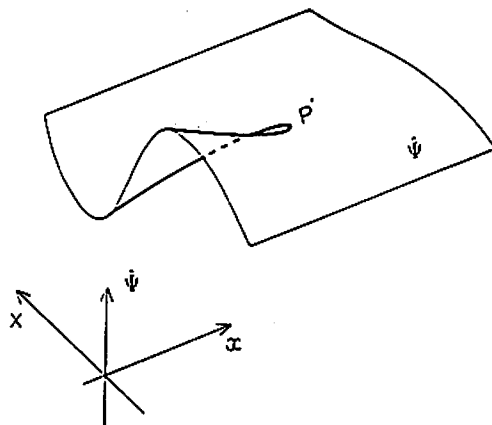
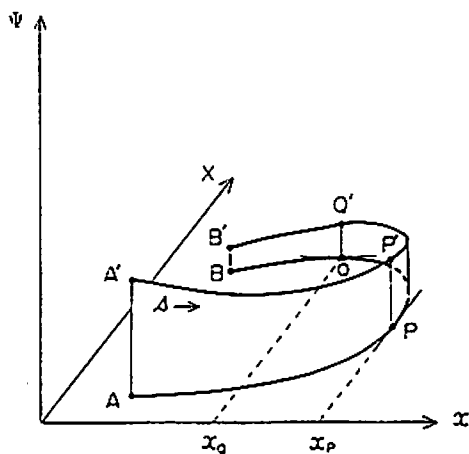
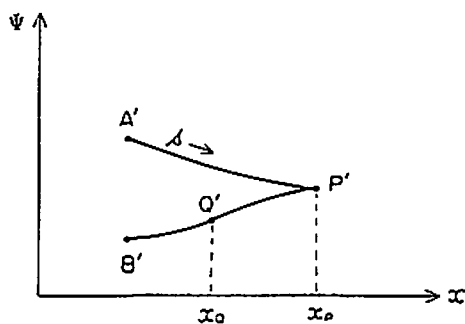


Fig. 2: An illustration of the extended Massieu function Ψ in a neighbourhood of P' (see Fig. 3). It forms a cusp-type catastrophe manifold whose fold curve (the loci of extrema) corresponds to the equilibrium Massieu function Ψ .



(a)



(b)

Fig. 3: (a) A proper Massieu function is plotted vertically to the eigenplane, along the equilibrium series. Although the eigenvalue change its sign both at P and Q , a change of stability occurs only at P . (b) The curve of the Massieu function against its intrinsic variable can be obtained by projecting the fold curve of Ψ onto a plane of constant X . In this process the curve is folded at P' to form a cusp. The point Q' is an inflection of the projected curve.

Thermodynamic stability, fluctuations and evaporation of a Schwarzschild black hole ¹

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Abstract. Thermodynamic stability, fluctuations and evaporation of a Schwarzschild hole in a perfectly reflecting container filled with radiation are discussed.

We consider thermodynamic stability and fluctuations of a hybrid system of a Schwarzschild black hole and black body radiation in a perfectly reflecting spherical container with fixed volume $V = \frac{4\pi}{3}l^3$. We use geometrical units, and then l is measured in the Planck length. The system is statistical-mechanically a microcanonical ensemble, because the total energy $E = \bar{E}_1 + \bar{E}_2 = \text{fixed}$, where $\bar{E}_1 = (1-x)E$ and $\bar{E}_2 = xE$ are energies of radiation and hole, respectively, and x is a fluctuating variable. The total entropy is given by

$$\tilde{S} = \tilde{S}(x; E, V) = S_C \left[\frac{2}{5} \left(\frac{5\bar{E}_1}{E_C} \right)^{0.75} + \frac{3}{5} \left(\frac{5\bar{E}_2}{4E_C} \right)^2 \right] = S_C \frac{15}{16} \left(\frac{y_C}{y} \right)^{1.6} [x^2 + y(1-x)^{0.75}] \quad (1)$$

(Hawking 1976; Gibbons and Perry 1978; Okamoto *et al.* 1994), where

$$E_C = \frac{(5aV)^{0.2}}{(32\pi)^{0.8}}, \quad S_C = \frac{10(aV)^{0.4}}{3(32\pi)^{0.6}}, \quad \frac{y}{y_C} = \left(\frac{E_C}{E} \right)^{1.25}, \quad y_C = \frac{32}{3 \cdot 5^{1.25}} = 1.4266 \quad (2)$$

and a is Stefan-Boltzmann constant [see *e. g.* Landsberg (1992) and Page (1992) for review articles]. Equilibrium is defined by $\partial_x \tilde{S} = 0$, from which we have

$$y(E) = \frac{8}{3}X(1-X)^{0.25}, \quad \text{or} \quad \frac{E}{E_C} = \left(\frac{5X}{4} \right)^{-0.8} [5(1-X)]^{-0.2} \quad (3)$$

where the solution $x = X(E)$ is no longer a fluctuating variable, but one of equilibrium variables. One can see that E has the minimum E_C at $X = 0.8$. The equilibrium entropy becomes

$$S = \tilde{S}(X; E(X), V) = S_C \frac{2 - \frac{5}{4}X}{\left(\frac{5X}{4} \right)^{0.6} [5(1-X)]^{0.4}}. \quad (4)$$

Figure 1 depicts $\tilde{S}(x; E, V)$ as a function of x for each value of E together with $S = S^\pm(X; V)$ as a function of X [see also Fig. 12.6 in Penrose (1979)]. It can be seen that there is no extremum of $\tilde{S}(x; E, V)$ and therefore no equilibrium state present for $E < E_C$. The curve $S = S^\pm(X)$ is the loci of the extrema of $\tilde{S}(x; E, V)$.

Stable equilibria must satisfy

¹The original Japanese title: "Thermodynamic fluctuation and stability: A simple example"

$$(\partial_x^2 \tilde{S})_e = -\frac{6S_C}{5} \left[\frac{X}{4(1-X)} \right]^{1.4} \left(\frac{5X-4}{X^3} \right) < 0. \quad (5)$$

This indicates that the branch of $S = S^+(E)$ for $X > 0.8$, passing through the maxima of $\tilde{S}$, gives stable equilibria and the branch of $S = S^-(E)$ with $X < 0.8$, passing through the minima, gives unstable equilibria.

The inverse temperature of the hybrid system is given by substituting (3) into the fluctuating one $\tilde{\beta} = \partial_E \tilde{S}$, i. e. ,

$$\beta = (\partial_E \tilde{S})_e = \beta_c \left[\frac{X}{4(1-X)} \right]^{0.2}, \quad \text{or} \quad X = \left[1 + \frac{1}{4} \left(\frac{\beta_c}{\beta} \right)^5 \right]^{-1} \quad (6)$$

where $\beta_c = \frac{32\pi}{5} E_C = \frac{3S_C}{2E_C} = (32\pi aV)^{0.2}$. Eliminating X between (6) and (3), we have

$$E = E_1(E) + E_2(E) = \frac{E_C}{5} \left[\left(\frac{\beta_c}{\beta} \right)^4 + 4 \left(\frac{\beta}{\beta_c} \right) \right] \quad (7)$$

where $E_1 = [1 - X(E)]E$ and $E_2 = X(E)E$ are the equilibrium energies of the radiation and the hole in the container. Eq (7) relates the control variable E to its conjugate variable β with respect to the equilibrium entropy S . The Poincaré turning point method provides a stability criterion for the linear series of $\beta = \beta(E)$ of the present system in terms of its slope [see *e. g.* Katz 1978, 1979]. The slope of the present linear series is given by

$$\left(\frac{\partial \beta}{\partial E} \right)_V = 8\pi \left[1 - \left(\frac{\beta_c}{\beta} \right)^5 \right]^{-1}. \quad (8)$$

This method tells us that a stability change occurs at the *critical* point with $\beta = \beta_c$ and $E = E_C$ where the linear series has vertical tangent, and the branch with positive slope gives us stable equilibria and that with negative slope does unstable equilibria (see Figure 2).

It is of particular interest to watch the energy-entropy relations; we can draw $S = S^\pm(E)$ for the equilibrium configurations as well as $S_1(E)$ for the radiation and $S_2(E)$ for the hole in the container, where

$$S_1 = \tilde{S}(0; E, V) = \frac{2S_C}{5} \left(\frac{5E}{E_C} \right)^{0.75}, \quad S_2 = \tilde{S}(1; E, V) = \frac{3S_C}{5} \left(\frac{E}{4E_C} \right)^2. \quad (9)$$

We see in Figure 3 that the stable and unstable branches of equilibrium configurations, $S = S^\pm(E)$, join together to form a cusp at the critical point (cf. Kaburaki 1994). In the region of $E_C < E < 1.3136E_C$ or $1.4266 > y > 1.0144$, we see that $S_1(E) > S^+(E)$ and hence this interval of the "stable" branch is in reality "metastable" (Gibbons and Perry 1977; Okamoto *et al.* 1994).

Let us evaluate the mean-square fluctuations of variables x , $\tilde{\beta}$, $\tilde{\beta}_1$ and $\tilde{\beta}_2$. Denoting $\Delta x = x - X$, we expand the entropy away from equilibrium

$$\tilde{S} = S + \frac{1}{2}(\partial_x^2 \tilde{S})_e(\Delta x)^2 + \dots \approx S - \frac{1}{2} \frac{(\Delta x)^2}{(\overline{\Delta x})^2} \quad (10)$$

where $(\partial_x \tilde{S})_e = 0$ and $(\overline{\Delta x})^2$ is the mean-square fluctuation of x [see Landau and Lifshitz 1980]. Then we have from (5) and (10)

$$\begin{aligned}
\frac{\overline{(\Delta x)^2}}{X^2} &= \frac{5}{6S_C} \left[\frac{X}{5X-4} \right] \left[\frac{4(1-X)}{X} \right]^{1.4} = \frac{1}{\beta_C^2} \left(\frac{\beta_C}{\beta} \right)^7 \left(\frac{\partial \beta}{\partial E} \right)_V, \\
\overline{(\Delta \beta)^2} &= (\partial_x \tilde{\beta})_c^2 \overline{(\Delta x)^2} = 8\pi \frac{\left(\frac{5X}{4} \right)^2 [4(1-X)]}{5X-4} = \left(\frac{\beta_C}{\beta} \right)^5 \left[\frac{1}{5} \left\{ 4 + \left(\frac{\beta_C}{\beta} \right)^5 \right\} \right]^{-2} \left(\frac{\partial \beta}{\partial E} \right)_V, \\
\overline{(\Delta \beta_1)^2} &= (\partial_x \tilde{\beta}_1)_c^2 \overline{(\Delta x)^2} = 8\pi \frac{X^2}{(5X-4)[4(1-X)]} = \left(\frac{\beta_C}{\beta} \right)^5 \left(\frac{\partial \beta}{\partial E} \right)_V, \\
\overline{(\Delta \beta_2)^2} &= (\partial_x \tilde{\beta}_2)_c^2 \overline{(\Delta x)^2} = 8\pi \frac{4(1-X)}{5X-4} = \left(\frac{\beta}{\beta_C} \right)^5 \left(\frac{\partial \beta}{\partial E} \right)_V,
\end{aligned} \tag{11}$$

It can be seen in (11) that the mean-square fluctuations are simply and equivocally related to onset of thermodynamic instabilities at the critical point (Okamoto *et al.* 1994). The root-mean-square fluctuations turn over from *real* in the stable branch to *imaginary* in the unstable branch (see Figure 2). Especially $\overline{(\Delta \beta)^2}$, *e. g.*, becomes infinitely large toward the critical point C at $\beta = \beta_C$.

When we quasi-statically remove energy out of the system situated at point A in Figures 1, 2 and 3, the system will evolve toward the point C beyond the point B. Note that $S(E_B) = S_1(E_B)$ at B, $S(E) > S_1(E)$ between B and C, and $\overline{(\Delta \beta)^2} \rightarrow \infty$ as the point C is approached. That is, thermodynamic fluctuations will make the hybrid system jump to the pure radiation state with higher entropy and higher temperature, somewhere between B and C where $\sqrt{\overline{(\Delta \beta)^2}} > \beta^+ - \beta_1$. It must be emphasized here that the radius l or volume V of the container so far appears only through the normalization constant of variables at the critical point C, *i. e.* E_C , S_C and β_C , whereas it appears explicitly in the normalized amplitudes of root-mean-square fluctuations, *e. g.*

$$\frac{\sqrt{\overline{(\Delta \beta)^2}}}{\beta_C} = \left(\frac{l_{\Delta \beta}}{l} \right)^{0.6} \frac{\left(\frac{\beta_C}{\beta} \right)^{2.5}}{\sqrt{1 - \left(\frac{\beta_C}{\beta} \right)^5 \left[\frac{1}{5} \left\{ 4 + \left(\frac{\beta_C}{\beta} \right)^5 \right\} \right]}}. \tag{12}$$

where $\sqrt{8\pi}/\beta_C \equiv (l_{\Delta \beta}/l)$ and $l_{\Delta \beta} = 2.2525$. This means that the normalized amplitudes depends upon the size of the container like $l^{-0.6}$. Figure 4 shows the normalized amplitudes of fluctuations of inverse-temperatures of the system, the radiation and the hole in the container, respectively, compared with the difference of the hybrid state from the pure radiation state. One can see that the hole contained in the cavity with sufficiently large size of $l \gg l_{\Delta \beta}$ can survive the “superheated” state almost up to the critical point (Okamoto *et al.* 1994).

In the container with $l \simeq l_{\Delta \beta}$, the normalized amplitude given by (12) becomes larger than the difference of $\beta^+ - \beta_1$ far before the critical point C. The size of the hole is now also of order of Planck length, and therefore backreaction of radiation and quantum gravity and fluctuation will be of significance (York 1985), so that the present semi-classical thermodynamic treatment may lose its validity. Also in the present analysis, we have neglected the effect of the hole’s gravity upon the radiation and temperature stratification. The latter effect must be taken into account in the analysis for the case of $l \gg l_{\Delta \beta}$.

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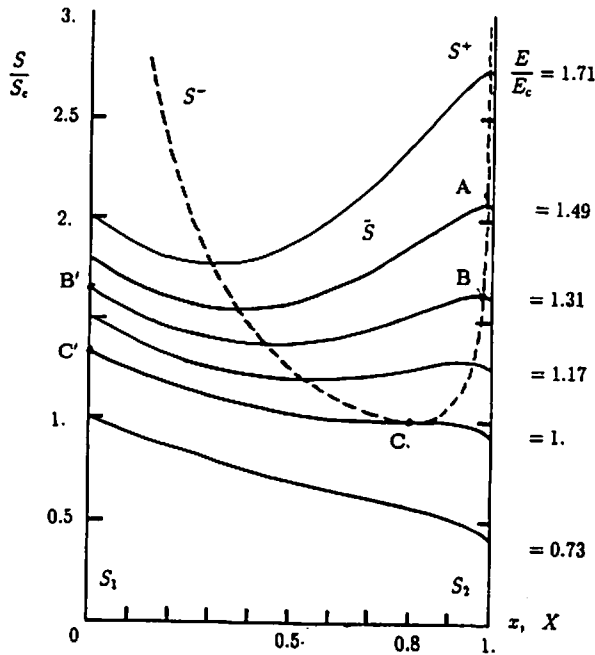


Figure 1. The entropy away from equilibrium $\tilde{S}(x)$ for each fixed energy E and the equilibrium entropy $S^\pm(X)$. For $E = E_C$, $\tilde{S}(x; E, V)$ has an inflection point at $x = 0.8$.

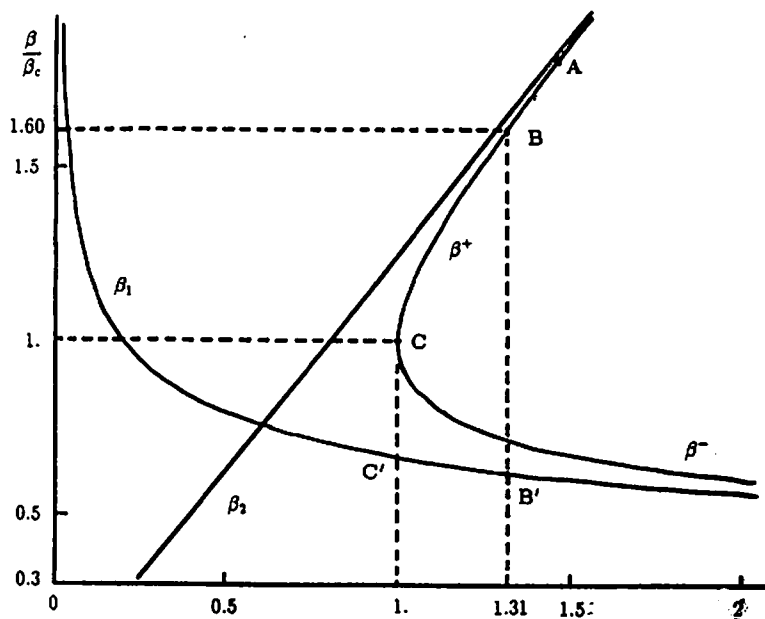


Figure 2. The linear series of equilibrium configurations. Stability change occurs at the critical point at C where the slope becomes vertical. The branch with positive slope gives stable configurations, while the branch gives unstable configurations.

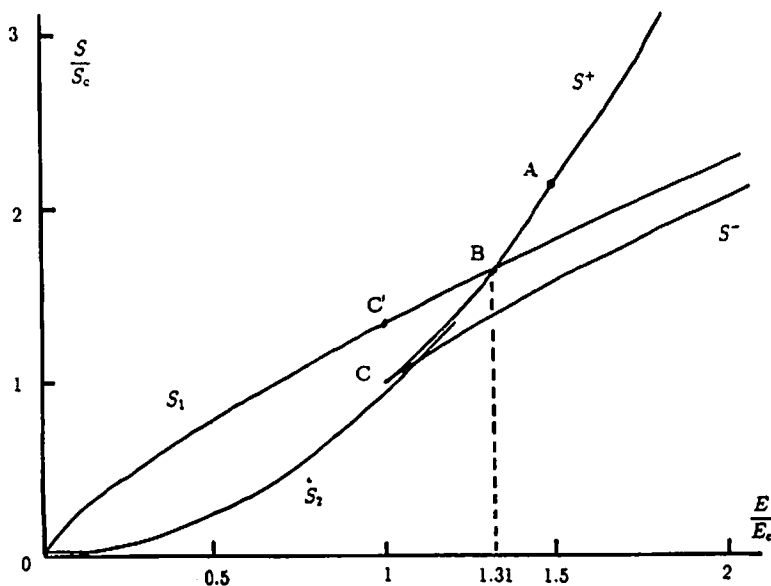


Figure 3. The energy-vs-entropy diagram. The two stable and unstable branches of equilibrium entropy $S^\pm$ join at the critical point C to form a cusp. The pure-hole entropy S_2 asymptotically approaches the stable branch S^+ .

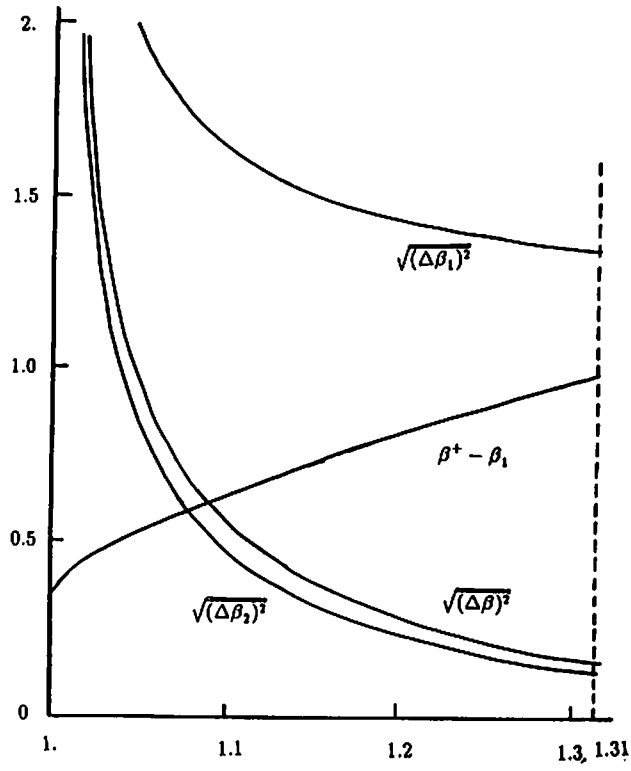


Figure 4. The normalized amplitudes of the root-mean-square fluctuations of inverse-temperatures in the container of $l = l_{\Delta\beta}$. For that with $l = 10^5 l_{\Delta\beta}$, the amplitudes become smaller by factor 10^{-3} . Thus $l \gg l_{\Delta\beta}$, the metastable state is almost stable.

3D Numerical Relativity

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ABSTRACT

Basic equations and numerical methods of a 3D numerical relativity code developed recently is shown. For a time slice and 3-space coordinate condition, conformal time slicing and pseudo-minimal shear condition are adopted. Preliminary numerical simulations for coalescing neutron stars using $(80)^3$ Cartesian grids are shown. Qualitatively the code works well without serious difficulties. Full scale simulations with at least $(200)^3$ grids are urgently required for quantitative arguments on the results including various test bed problems.

1. Basic Equations

Coalescing binary neutron stars (or black holes) is one of the most important sources of gravitational waves for LIGO/VIRGO projects.¹⁾ Comparing observed wave patterns in the last three minutes of coalescing binary neutron stars (or black holes) with theoretical templates, we may determine mass and spin of neutron stars.²⁾ For this purpose, accurate templates are needed in the last three minutes using the post-Newtonian expansion to high orders^{3),4)}. However in the final phase of coalescence numerical relativity is needed to analyse the wave forms. For this problem, Oohara and myself⁵⁾ have been doing Newtonian hydrodynamics simulation including the radiation reaction by gravitational waves. We have also developed full 3D code for the propagation of localized gravitational waves.⁶⁾ Combining these two codes, that is, the hydro code and the metric code, I tried to make a full 3D General Relativistic code for the final phase of coalescing binary neutron stars.

A) Initial Value Equations

In the (3+1)- formalism of the Einstein equations the four metric is expressed as

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \quad (1)$$

where α, β^i and γ_{ij} are the lapse function, the shift vector and the intrinsic metric of 3-space, respectively. The initial data should satisfy the constraint equations;

$$^{(3)}R + K^2 - K_{ij}K^{ij} = 16\pi\rho_H, \quad (2)$$

$$K_{i;j}^j - K_{;j} = 8\pi J_i \quad (3)$$

where $^{(3)}R, K_{ij}, K, \rho_H$ and J_i are scalar curvature, extrinsic curvature and its trace, energy density and momentum density of matter, respectively. In Eq.(3), $;$ means the covariant derivative with respect to metric γ_{ij} . Now we assume that γ_{ij} is conformally flat as

$$\gamma_{ij} = \phi^4 \tilde{\gamma}_{ij}, \quad (4)$$

where $\tilde{\gamma}_{ij}$ is the flat metric. We define the conformal transformation as

$$\begin{aligned} \tilde{K}_{ij} &\equiv \phi^2 K_{ij}, \tilde{K}_i^j \equiv \phi^6 K_i^j, \tilde{K}^{ij} \equiv \phi^{10} K^{ij} \\ \rho_B &\equiv \rho_H \phi^6 \text{ and } \tilde{J}_i \equiv J_i \phi^6. \end{aligned} \quad (5)$$

Now if we assume $K = 0$, Eq.(3) becomes

$$\tilde{K}_{ij}^j = 8\pi \tilde{J}_i, \quad (6)$$

where $|$ means the covariant derivative with respect to the flat metric $\tilde{\gamma}_{ij}$. We decompose $\tilde{K}_{ij}$ as

$$\tilde{K}_{ij} = \tilde{K}_{ij}^{TT} + (\tilde{W}_{ij} + \tilde{W}_{j|i} - \frac{2}{3} \tilde{\gamma}_{ij} \tilde{W}_{|l}^l), \quad (7)$$

where $\tilde{K}_{ij}^{TT}$ is the transverse traceless part of $\tilde{K}_{ij}$ and $\tilde{W}_i$ is the longitudinal part of $\tilde{K}_{ij}$. Assuming $\tilde{K}_{ij}^{TT} = 0$, we rewrite Eq.(6) as

$$\Delta_f \tilde{W}_i + \frac{1}{3} (\tilde{W}_{|l}^l)_{|i} = 8\pi \tilde{J}_i, \quad (8)$$

where Δ_f is the flat Laplacian.

Let define the divergence of W^i as

$$DIV \equiv W_{|i}^i. \quad (9)$$

Taking the divergence of Eq.(8), we have

$$\Delta_f (DIV) = 6\pi \tilde{J}_{|i}^i. \quad (10)$$

We first solve Eq.(10) for given $\tilde{J}_i$. Using DIV, we can rewrite Eq.(8) as

$$\Delta_f \tilde{W}_i = -\frac{1}{3} (DIV)_{|i} + 8\pi \tilde{J}_i. \quad (11)$$

Although Eq.(8) was the coupled elliptic type equation of $\tilde{W}_i$ originally, the above procedure enables us to decouple Eq.(8). The boundary conditions for Eq.(10) and Eq.(11) are

$$\tilde{W}_i = \frac{1}{r^3} \epsilon_{ijk} x^j M^k + O(\frac{1}{r^3}) \quad (12)$$

and

$$DIV = O(\frac{1}{r^4}), \quad (13)$$

where M^k is the constant related to angular momentum of the system. Now Eq.(2) is rewritten as

$$\Delta_f(\phi) = -\frac{2\pi\rho_B}{\phi} - \frac{1}{8}\phi^{-7}\tilde{K}_{ij}\tilde{K}^{ij}, \quad (14)$$

We solve Eq.(14) for given ρ_B under the boundary condition

$$\phi = 1 + \frac{M_G}{2r} + O(\frac{1}{r^3}), \quad (15)$$

where M_G is the gravitational mass of the system.

B) Relativistic Hydrodynamics

We assume the perfect fluid. Then the energy momentum tensor $T_{\mu\nu}$ becomes

$$T_{\mu\nu} = (\rho + \rho\epsilon + p)u_\mu u_\nu + P g_{\mu\nu}, \quad (16)$$

where ρ , ϵ , P and u_μ are the proper mass density, specific internal energy, pressure and the four velocity of the fluid, respectively. We define the projection tensor $h_{\mu\nu}$ as

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \quad (17)$$

where n_μ is the normal vector to the hypersurface. ρ_H , J_i and S_{ij} are defined as

$$\rho_H \equiv T_{\mu\nu} n^\mu n^\nu, \quad J_i \equiv -T_{\mu\nu} n^\mu h_i^\nu$$

and

$$S_{ij} \equiv T_{\mu\nu} h_i^\mu h_j^\nu. \quad (18)$$

Then the relativistic hydrodynamics equation $T_{\nu;\mu}^\mu = 0$ where ; is the covariant derivative with respect to $g_{\mu\nu}$ is written as

$$\begin{aligned} \frac{\partial}{\partial t}(\sqrt{\gamma} J_i) + \frac{\partial}{\partial x^l}(\sqrt{\gamma} J_i V^l) = & -\alpha\sqrt{\gamma} \frac{\partial P}{\partial x^i} - \sqrt{\gamma}(P + \rho_H) \frac{\partial \alpha}{\partial x^i} \\ & + \frac{1}{2}\alpha\sqrt{\gamma} \frac{\partial \gamma_{kl}}{\partial x^i} \frac{J^k J^l}{(P + \rho_H)} + \sqrt{\gamma} J_l \frac{\partial \beta^l}{\partial x^i}, \end{aligned} \quad (19)$$

$$\frac{\partial}{\partial t}(\sqrt{\gamma} \alpha u^0 \rho \epsilon) + \frac{\partial}{\partial x^l}(\sqrt{\gamma} \alpha u^0 \rho \epsilon V^l) = -P \frac{1}{\alpha\sqrt{\gamma}} \frac{\partial}{\partial x^\nu}(\alpha\sqrt{\gamma} u^\nu), \quad (20)$$

$$\frac{\partial}{\partial t}(\sqrt{\gamma} \alpha u^0 \rho) + \frac{\partial}{\partial x^l}(\sqrt{\gamma} \alpha u^0 \rho V^l) = 0, \quad (21)$$

$$\gamma = \det \gamma_{ij}, \quad (22)$$

and

$$V^l = \alpha J^l / (P + \rho_H) - \beta^l = u^l / u^0. \quad (23)$$

Left hand sides of Eq.(19),(20) and (21) are similar to the corresponding equations in

Newtonian hydrodynamics if we define the following quantities corresponding to variables in Newtonian hydrodynamics as

$$\rho_N \equiv \sqrt{\gamma} \alpha u^0 \rho$$

and

$$u_i^N \equiv J_i / (\alpha u^0 \rho). \quad (24)$$

Then left hand sides become as

$$\frac{\partial}{\partial t}(\rho_N u_i^N) + \frac{\partial}{\partial x^l}(\rho_N u_i^N V^l) = \dots\dots\dots, \quad (19)'$$

$$\frac{\partial}{\partial t}(\rho_N \epsilon) + \frac{\partial}{\partial x^l}(\rho_N \epsilon V^l) = \dots\dots\dots, \quad (20)'$$

and

$$\frac{\partial}{\partial t} \rho_N + \frac{\partial}{\partial x^l}(\rho_N V^l) = 0. \quad (21)'$$

Using the variables ρ_N , u_i^N and ϵ , we can apply Newtonian hydrodynamics code developed by Nakamura and Oohara⁵⁾ almost as it is. To complete relativistic hydrodynamics equations, we need the equation of state,

$$P = P(\epsilon, \rho)$$

C) Time evolution of the extrinsic curvatures

The evolution equations of the extrinsic curvatures are

$$\frac{\partial}{\partial t} K_{ij} - \frac{\partial}{\partial x^l} (K_{ij} \beta^l) = (SK)_{ij}^{main} + (SK)_{ij}^{non} + (SK)_{ij}^\beta, \quad (25)$$

where

$$(SK)_{ij}^{main} = \alpha \{ {}^{(3)}R_{ij} - 8\pi(S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - S_l^l)) \} - \alpha_{|ij}, \quad (26)$$

$$(SK)_{ij}^{non} = \alpha(KK_{ij} - 2K_{il}K_j^l), \quad (27)$$

and

$$(SK)_{ij}^\beta = K_{mj} \frac{\partial \beta^m}{\partial x^i} + K_{mi} \frac{\partial \beta^m}{\partial x^j} - K_{ij} \frac{\partial \beta^m}{\partial x^m}, \quad (28)$$

where $|$ means the covariant derivative with respect to γ_{ij} . In the left hand side of

equation (25), the transport term is treated numerically in the same way as that in the hydrodynamics equations.

D) Time Slice

Even for $t > 0$, we define the conformal factor ϕ as

$$\phi \equiv (\det \gamma_{ij})^{\frac{1}{12}}. \quad (29)$$

We define α as

$$\alpha = \exp(-2(\tilde{\phi} + \frac{\tilde{\phi}^3}{3} + \frac{\tilde{\phi}^5}{5})), \quad (30)$$

and

$$\tilde{\phi} \equiv \phi - 1.$$

Shibata and Nakamura⁷⁾ called this time slice as conformal time slicing because α is determined by the conformal factor ϕ . The detailed discussions on the nature of the conformal time slicing is given in Shibata and Nakamura.⁷⁾ Here we just note that in the conformal time slice the space outside the central matter quickly approaches the Schwarzschild metric. At the same time, it has the singularity avoidance nature similar to the maximal slicing. Therefore the conformal time slicing has the good points in both polar and maximal slicing in a sense. The conformal factor ϕ is determined by either solving the evolution equation for ϕ or Hamiltonian constraint equations as

$$\frac{\partial \phi}{\partial t} = \frac{\phi}{12}(-2\alpha K + \frac{1}{\phi^6} \frac{\partial}{\partial x^l}(\phi^6 \beta^l)), \quad (31)$$

or

$$\tilde{\Delta} \phi = -2\pi \rho_H \phi^5 - \frac{\phi^5}{8}(K_{ij} K^{ij} - K^2 - {}^{(3)}\tilde{R}), \quad (32)$$

where $\tilde{\Delta}$ is the Laplacian with respect to

$$\tilde{\gamma}_{ij} = \gamma_{ij} / \phi^4.$$

E) Spatial Coordinate

We define conformal transformation as

$$\begin{aligned}\tilde{K}_{ij} &\equiv \phi^{-4} K_{ij}, \quad \tilde{\beta}_i = \phi^{-4} \beta_i, \\ K &= \tilde{K}, \quad \tilde{K}^j_i = K^j_i \text{ and } \tilde{\beta}^i = \beta^i.\end{aligned}\tag{33}$$

Then the equation for metric becomes

$$\frac{\partial}{\partial t} \tilde{\gamma}_{ij} = (\tilde{\beta}_{i|j} + \tilde{\beta}_{j|i} - \frac{2}{3} \tilde{\gamma}_{ij} \tilde{\beta}^l_{|l}) - 2\alpha(\tilde{K}_{ij} - \frac{1}{3} \tilde{\gamma}_{ij} \tilde{K}) = A^T_{ij}\tag{34}$$

where $|$ means the covariant derivative with respect to $\tilde{\gamma}_{ij}$. We demand that the usual divergence of the right hand side of Eq.(34) to be zero, i. e.

$$\frac{\partial}{\partial x^j} A^T_{ij} = 0.\tag{35}$$

Then Eq.(35) becomes

$$\begin{aligned}\Delta_f \beta^i + \frac{1}{3} \frac{\partial}{\partial x^i} \left(\frac{\partial \beta^l}{\partial x^l} \right) &= \frac{\partial}{\partial x^j} \left(2\alpha \left(\tilde{K}_{ij} - \frac{1}{3} \tilde{K} \tilde{\gamma}_{ij} \right) \right) - \frac{\partial}{\partial x^j} \\ &[(\tilde{\gamma}_{jl} - \delta_{jl}) \frac{\partial \beta^l}{\partial x^i} + (\tilde{\gamma} - \delta_{il}) \frac{\partial \beta^l}{\partial x^j} - \frac{2}{3} (\tilde{\gamma}_{ij} - \delta_{ij}) \frac{\partial \beta^l}{\partial x^l} + \beta^l \frac{\partial}{\partial x^l} \tilde{\gamma}_{ij}]\end{aligned}\tag{36}$$

Eq.(36) is the similar to Eq.(8) in the initial value equations but in general Eq.(36) is the coupled equation for β^i . Instead of Eq.(35), if we demand

$$A^T_{i|j} = 0,\tag{37}$$

we have the minimal shear condition by York and Smarr.⁸⁾ In this sense we call Eq.(35) as “pseudo” minimal shear condition.

2. Numerical Methods

For simplicity we use a simple equation of state as

$$P = \rho\epsilon , \quad (38)$$

For hydrodynamics equations we use a certain kind of TVD(Total Variation Diminishing) method adopted in Oohara and Nakamura.⁵⁾ The hydrodynamic code has passed various checking problems details of which are shown in Oohara and Nakamura.⁵⁾ For the evolution equation of metric all the equations are written as

$$\frac{\partial Q}{\partial t} - \frac{\partial(Q\beta^l)}{\partial x^l} = S , \quad (39)$$

where S is the source term. We use a leap frog method for the time derivative with second order spatial derivatives in the source terms. As for the second term in the left hand side of Eq.(39), we use the same procedure as the hydrodynamics equations regarding $-\beta^l$ as V^l . We use Cartesian equal spacing grids of $(80)^3$. For the elliptic equations we use ICCG method shown in Oohara and Nakamura⁹⁾ in details. The size of source program is about 6000 lines in FORTRAN. The typical CPU time by VP2600 (5GFLOPS, 500Mbyte Machine) in the present test simulations is three hours. At present to make Eq.(36) simple, we use β^i in the previous step in r.h.s of Eq.(36). We rewrite Eq.(32) as

$$\Delta_f \phi = \Delta_f \phi - \tilde{\Delta} \phi - 2\pi\rho_H \phi^5 - \frac{\phi^5}{8}(K_{ij}K^{ij} - K^2 - {}^{(3)}\tilde{R}).$$

We use ϕ in the previous step in r.h.s of the above equation. By this treatment we do not need the iteration to solve β^i and ϕ . Moreover we can use the same subroutine in the initial value equations. In reality, we need iterations or different numerical techniques to determine true β^i and ϕ in future simulations. However the above treatment seems to be enough for the present test simulations to know the global behavior of the numerical schemes and the coordinate conditions.

3. Numerical Results

We use the units of

$$M = M_{\odot}, L = \frac{GM_{\odot}}{C^2} \text{ and } T = \frac{GM_{\odot}}{C^3}, \quad (40)$$

We put two spherical neutron stars of mass 1.5 and radius 5.7 with density distribution (ρ_N) of $n = 1$ polytrope at $x = 6$ and -6 . This means two neutron stars just touch with each other. We add half of the Keplerian rigid rotation to this system. As for Kepler velocity we use the angular velocity of two point Newtonian particles. The absolute value of outermost grid in each direction is 40. In Fig.1, we show the time evolution of density (left) and energy density of gravitational wave (right), that is, $(A_{ij}^{TT})^2 r^2 / (32\pi)$ (right). In each figure we show density contours in $x - y, y - z$ and $x - z$ plane. The maximum and minimum of density are shown as RHOMAX and RHOMIN. Each level of density contour is set as equal spacing in linear scale from RHOMAX to RHOMIN. IC is the number to distinguish various quantities, for example, IC=2 corresponds to ρ_N and IC=0 does $(A_{ij}^{TT})^2 r^2 / 32\pi$. In density contour (left), arrows mean the velocity vector V^i , while in gravitational wave energy contour (right), arrows mean the shift vector β^i . In the density contour, we see two neutron stars rotate about 90° as well as they coalesce. The final object will be a rotating black hole although I have not tried to determine apparent horizon yet. The evolution of gravitational wave energy (right) shows an interesting feature of generation and propagation of the wave. In the outermost grid point I put outgoing wave condition to $\tilde{\gamma}_{ij}$ and K_{ij} . However in the present simulation, $x_{\max}(= y_{\max} = z_{\max})$ is comparable to the wave length of the expected gravitational wave. I tried various boundary conditions (~ 10 different types) but none works well. So I stopped simulations when the gravitational wave reaches the outermost grid. A method to overcome this difficulty is to increase $x_{\max}$ so that the outgoing wave boundary condition becomes appropriate.

In Fig.2, I show evolution of α (left) and K_{xx} (right). It is clear that α therefore ϕ is also rotating in the same manner as density. This should be expected naively. Since ϕ is in a sense gravitational potential, it should be large where density is large. However if I use the maximal slicing, such a relation is not expected. In reality, I tried simulations using maximal slicing. What I found is that α becomes quickly small due to the increase of $K_{ij}K^{ij}$ so that the time does not proceed.

FIG 1-A

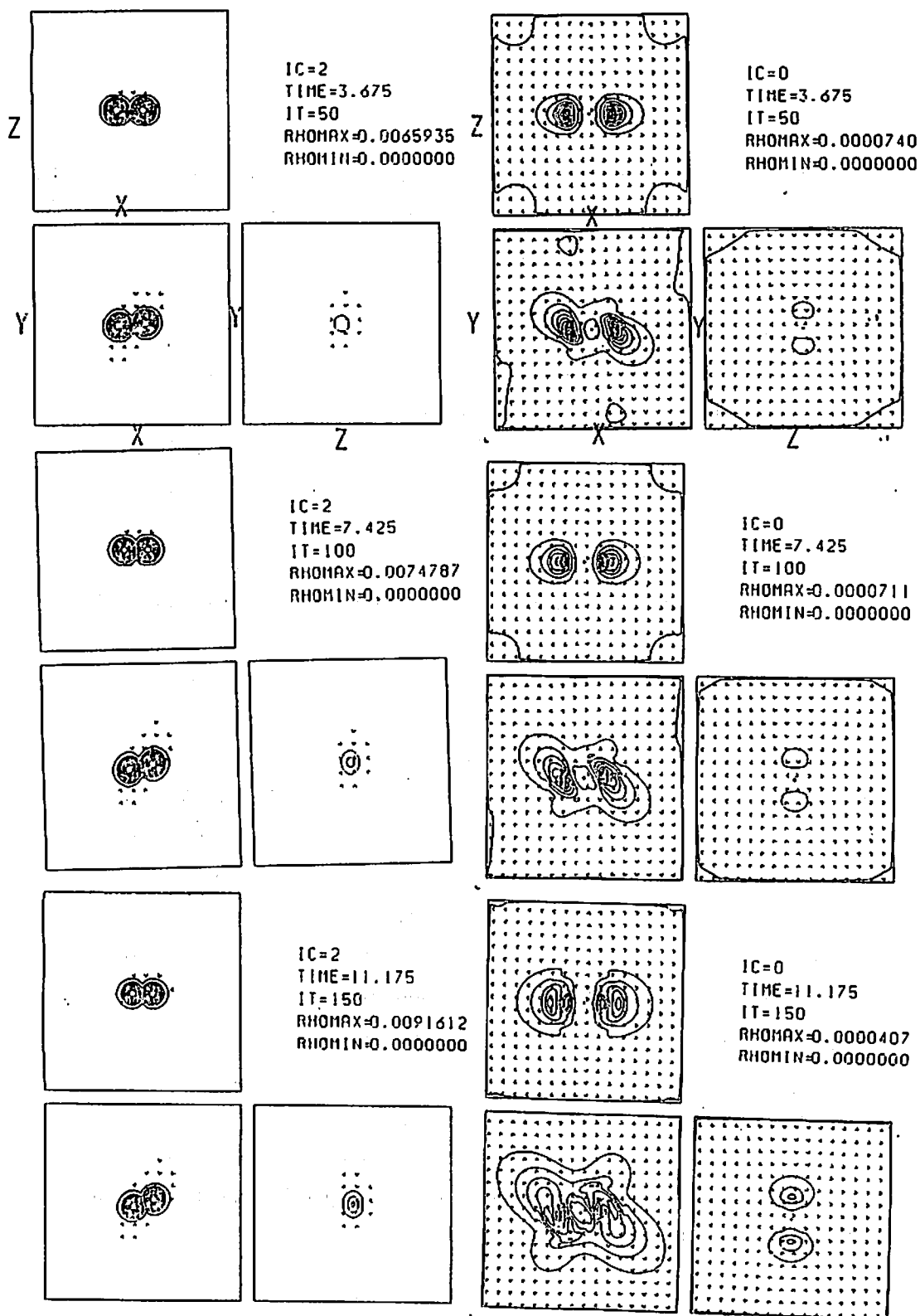


FIG 1-B

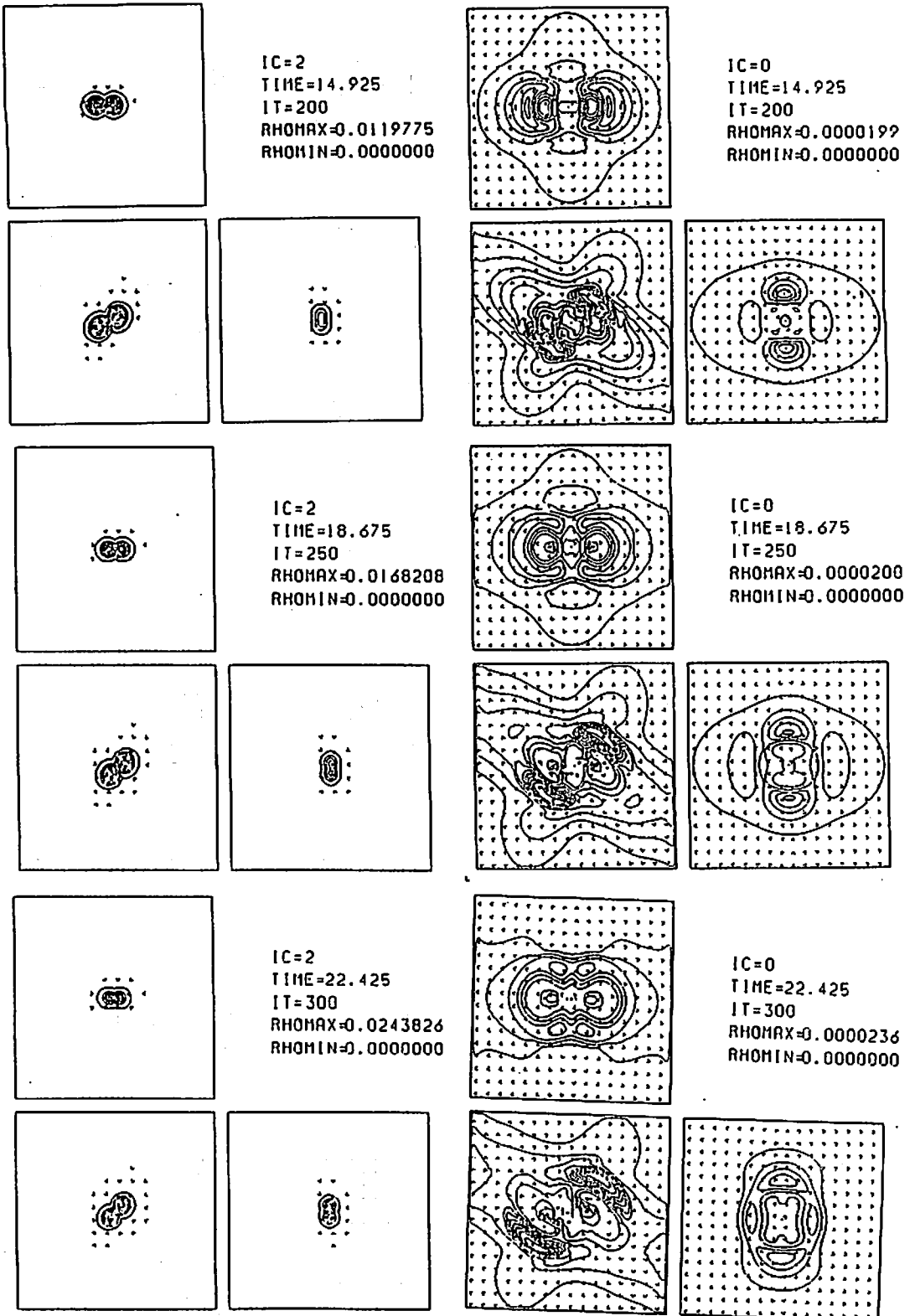


FIG 1-c

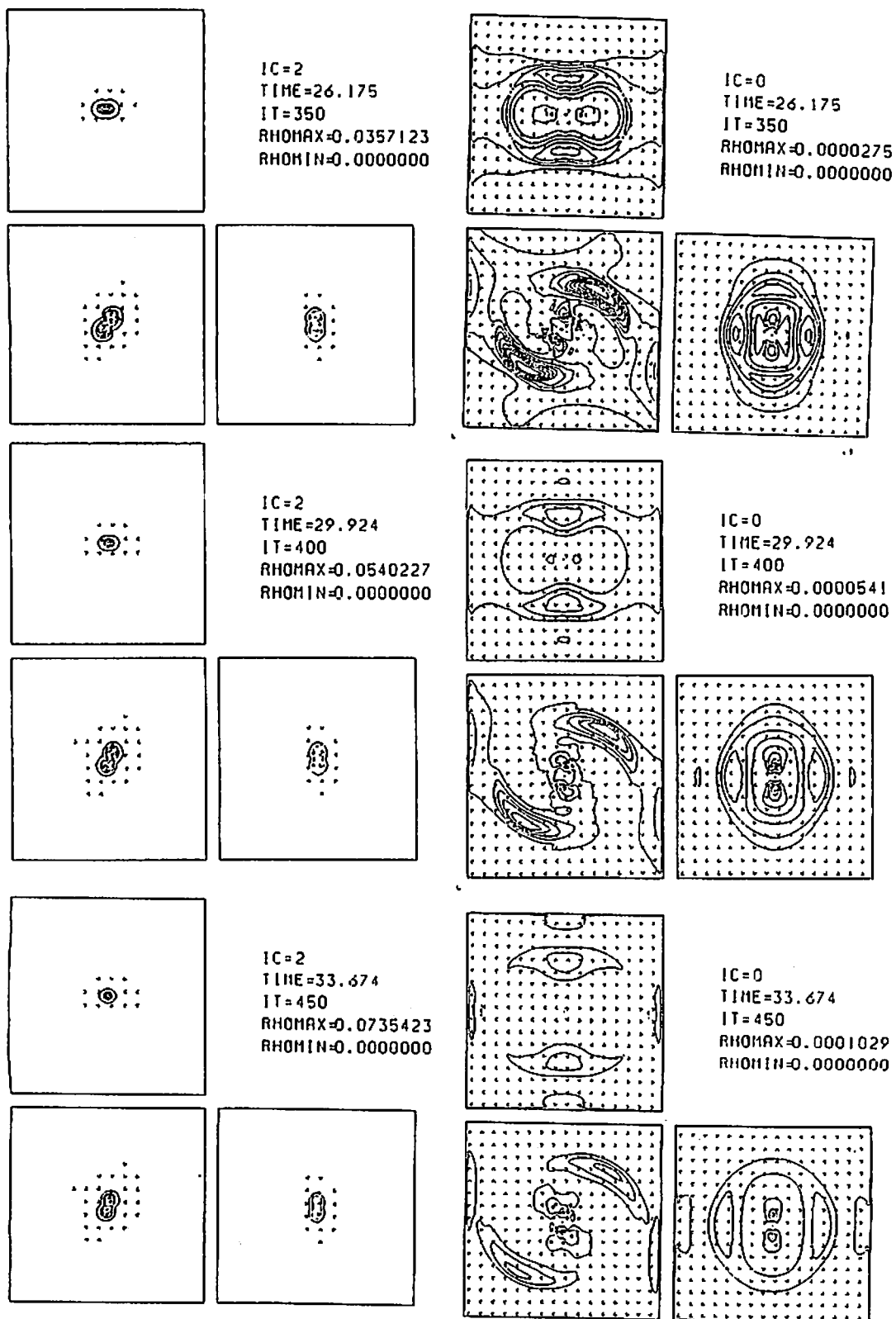


FIG 2-A

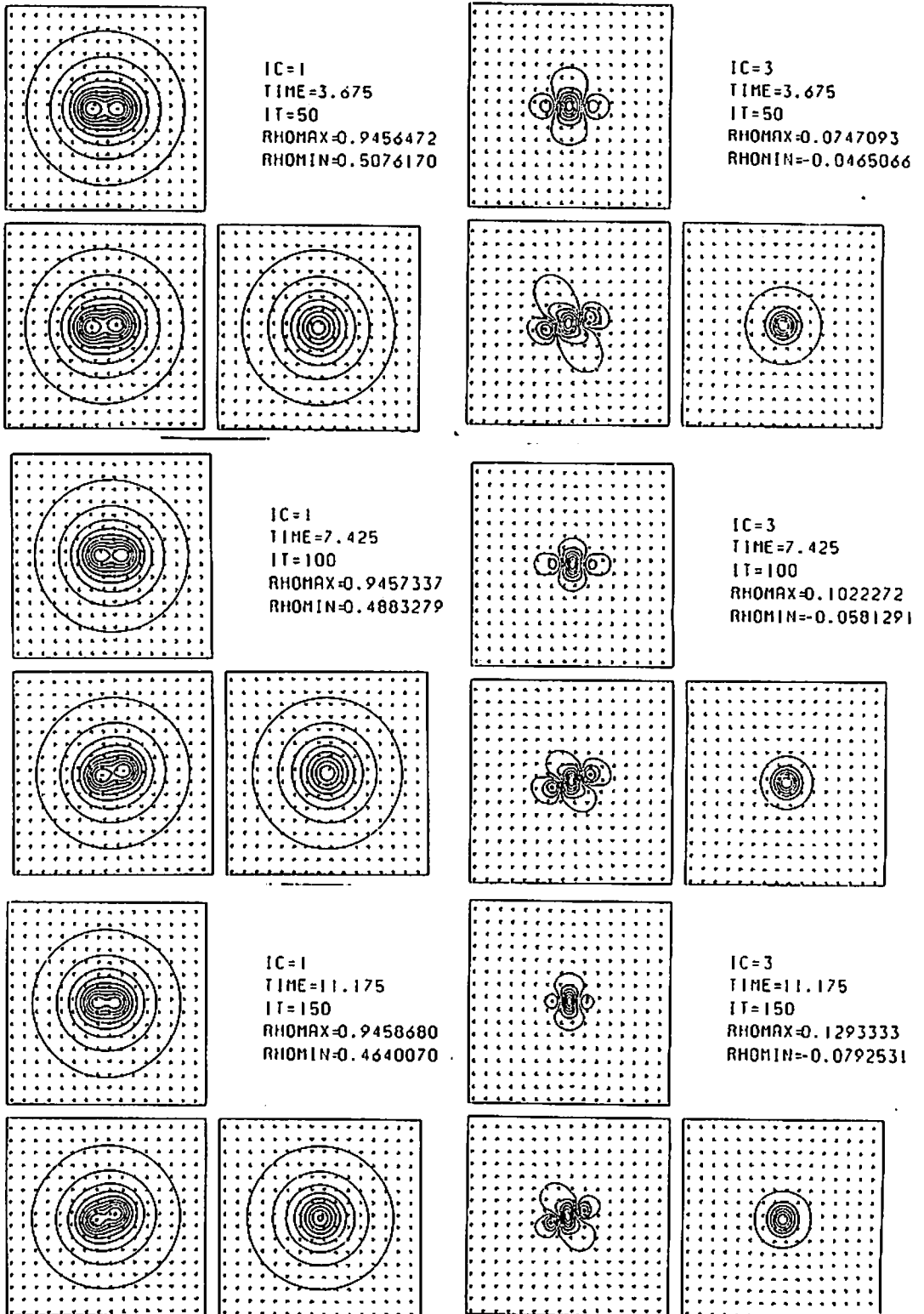
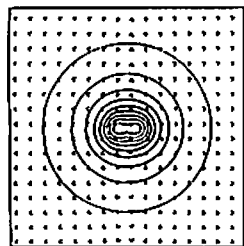
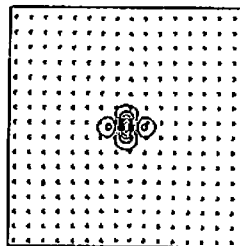


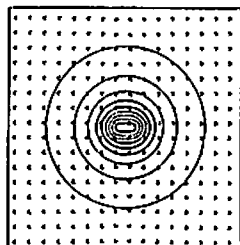
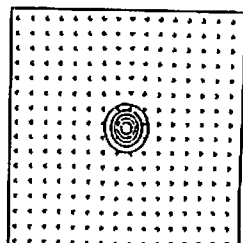
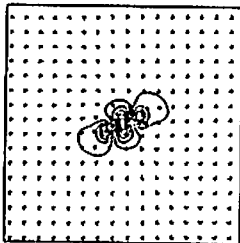
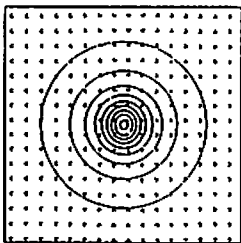
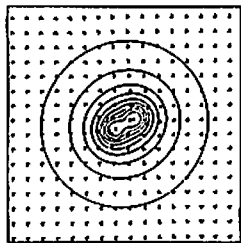
FIG 2-B



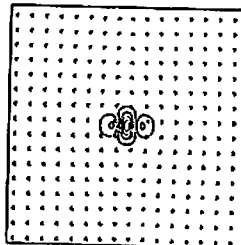
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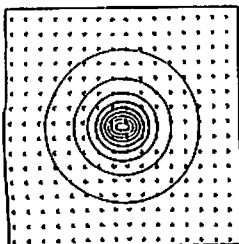
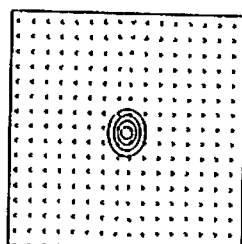
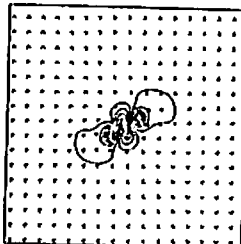
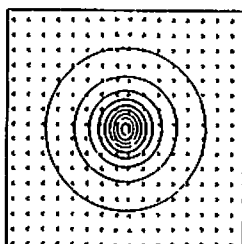
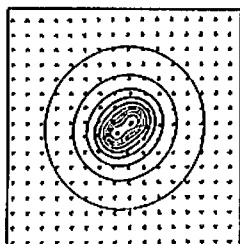
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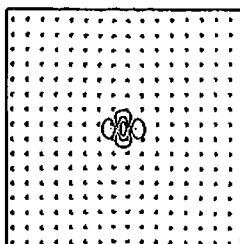
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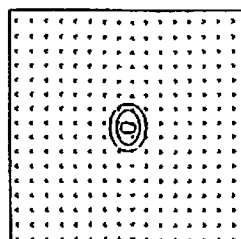
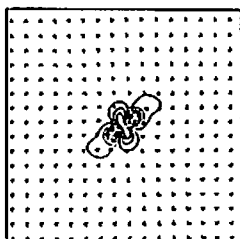
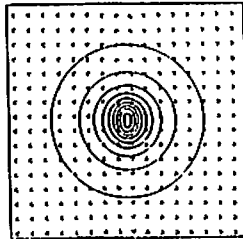
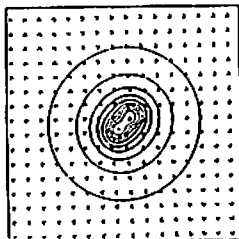
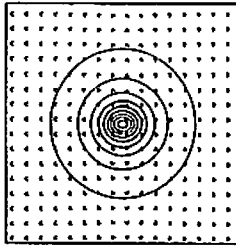
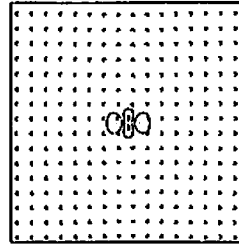


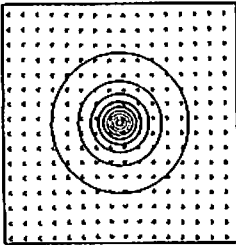
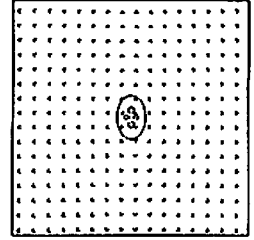
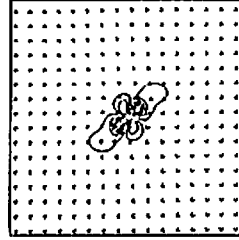
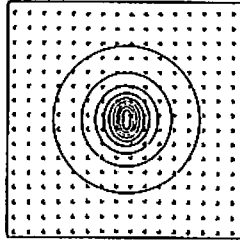
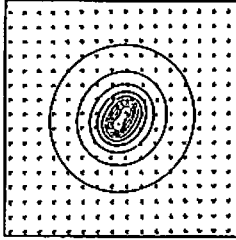
FIG 2-c



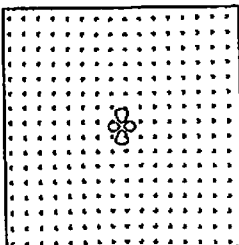
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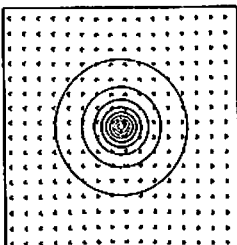
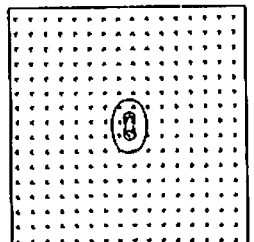
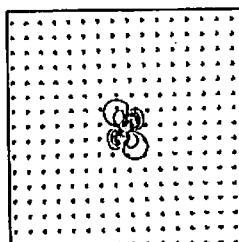
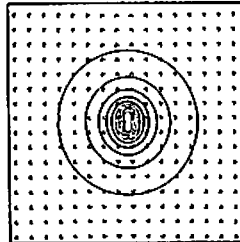
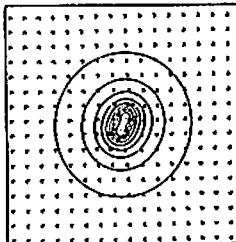
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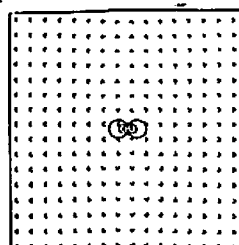
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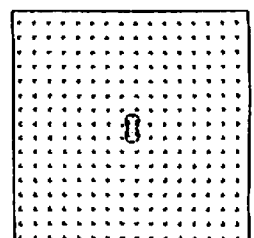
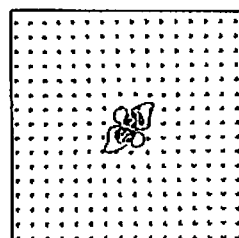
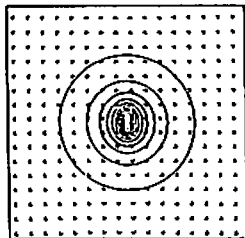
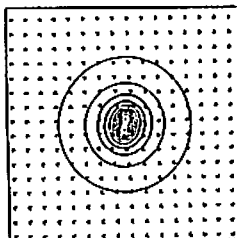


FIG 3

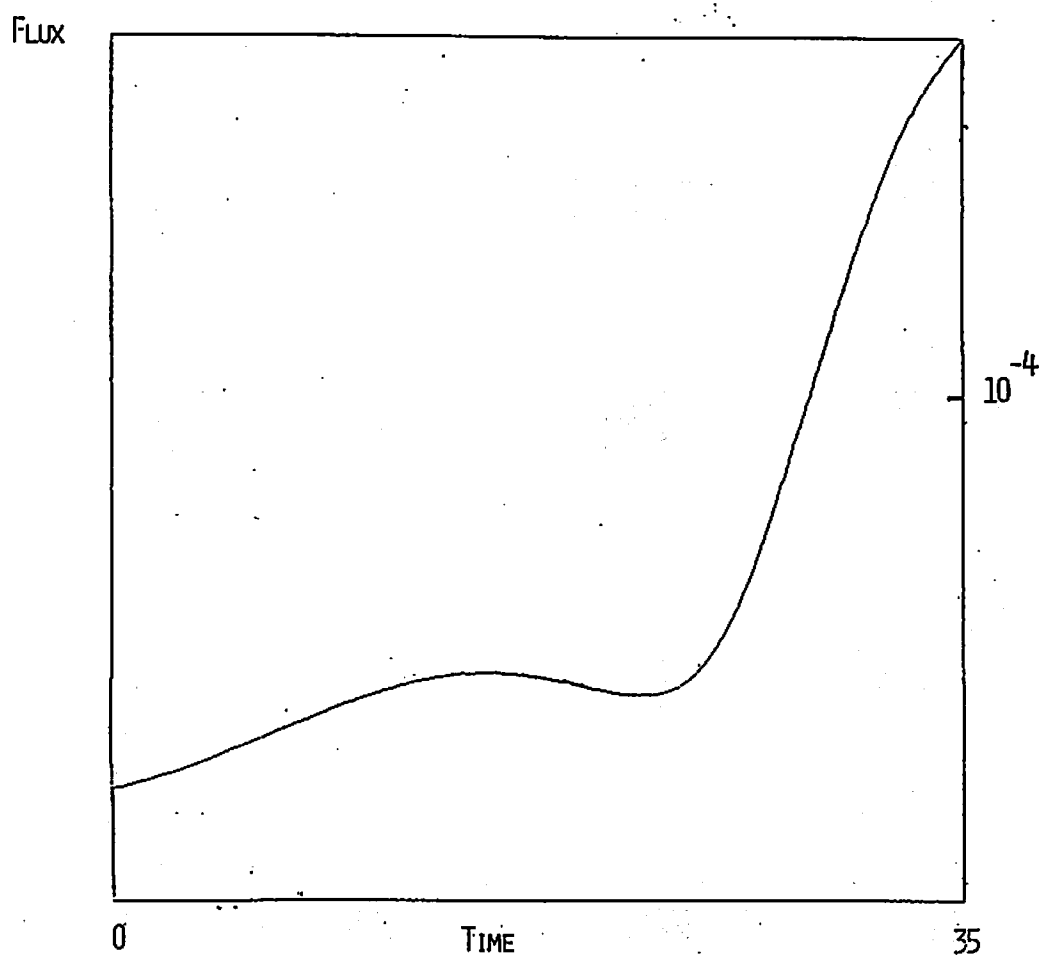


FIG 4-A

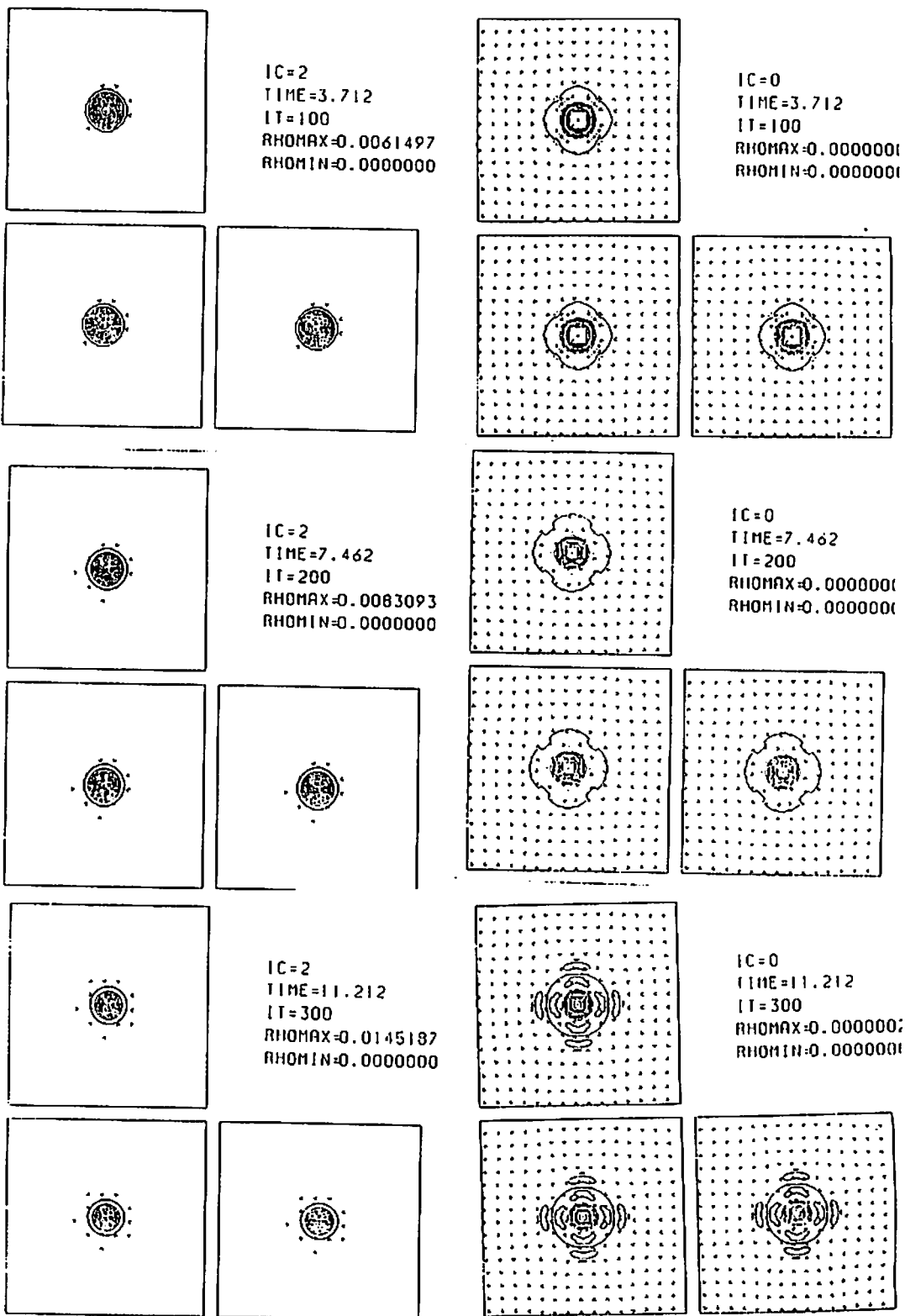


FIG 4-B

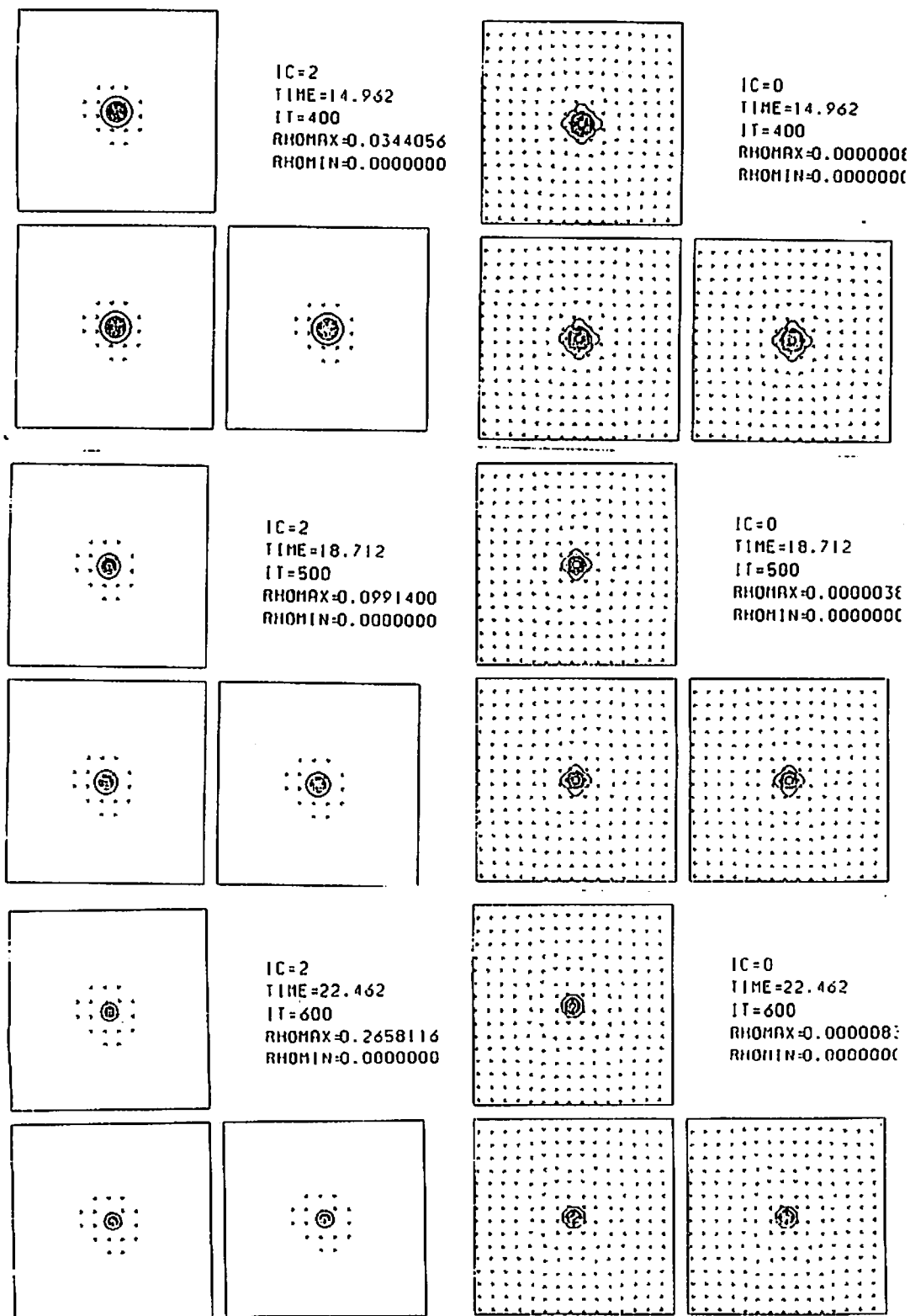


FIG 5-A

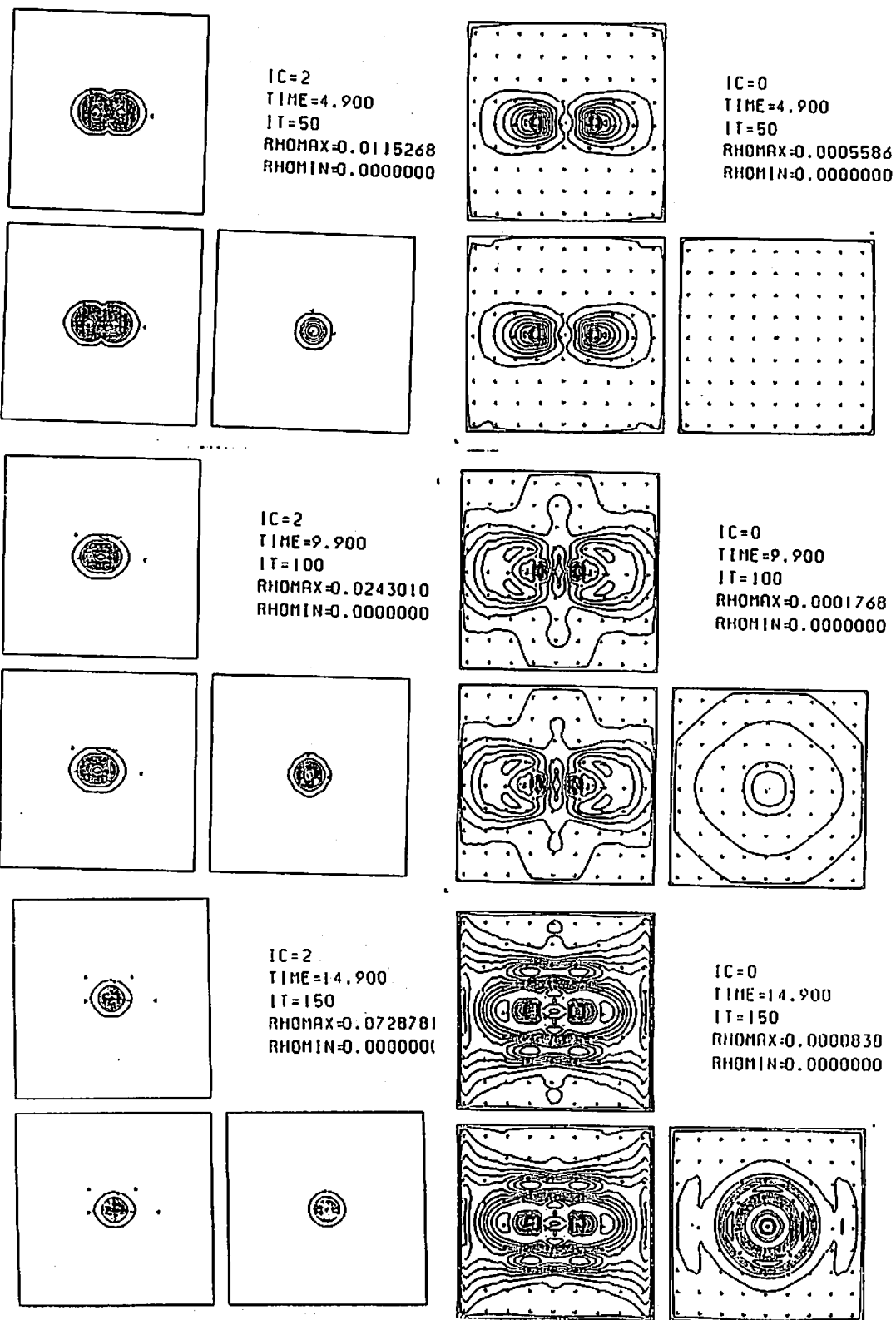


FIG 5-B

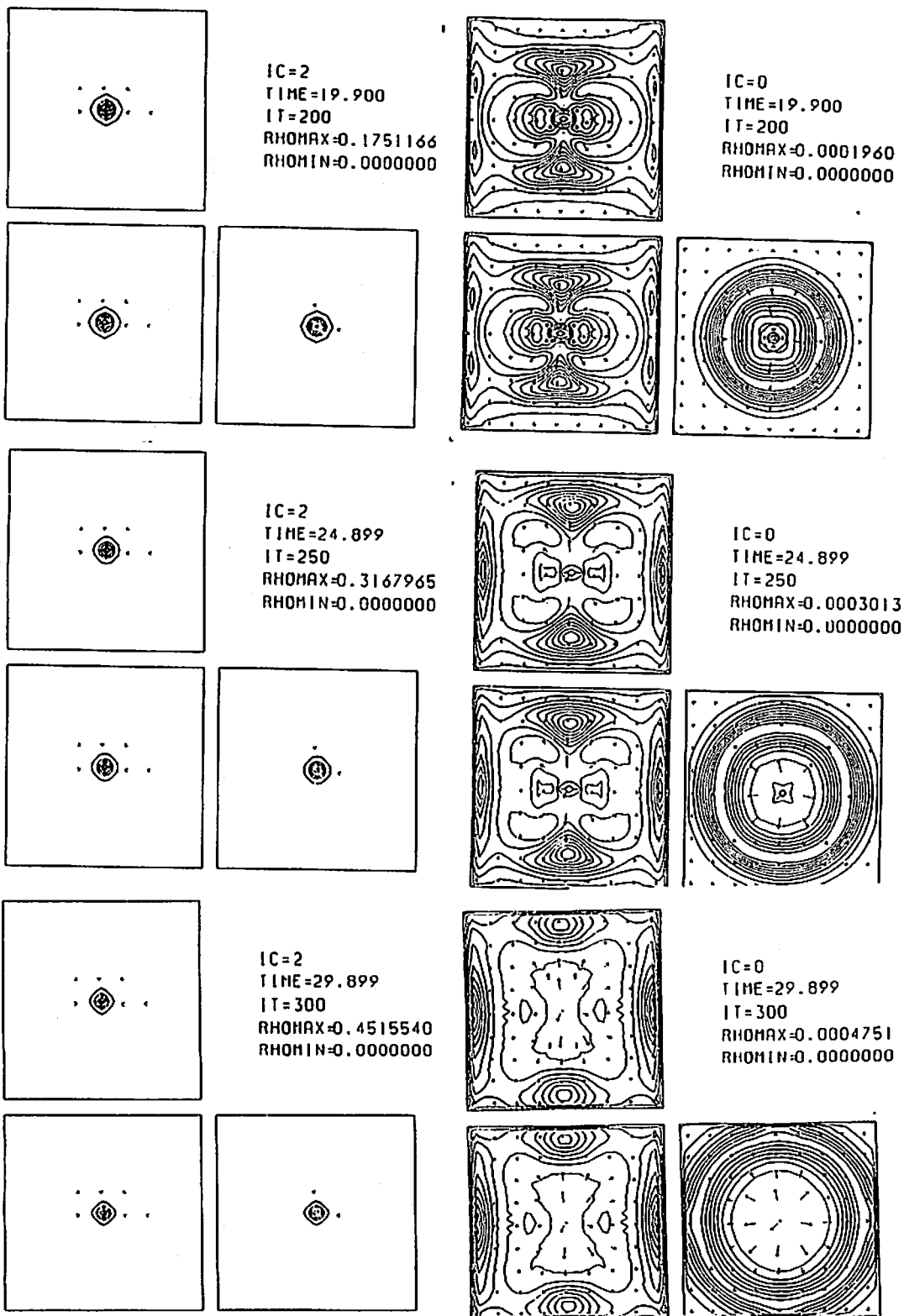


FIG 6-A

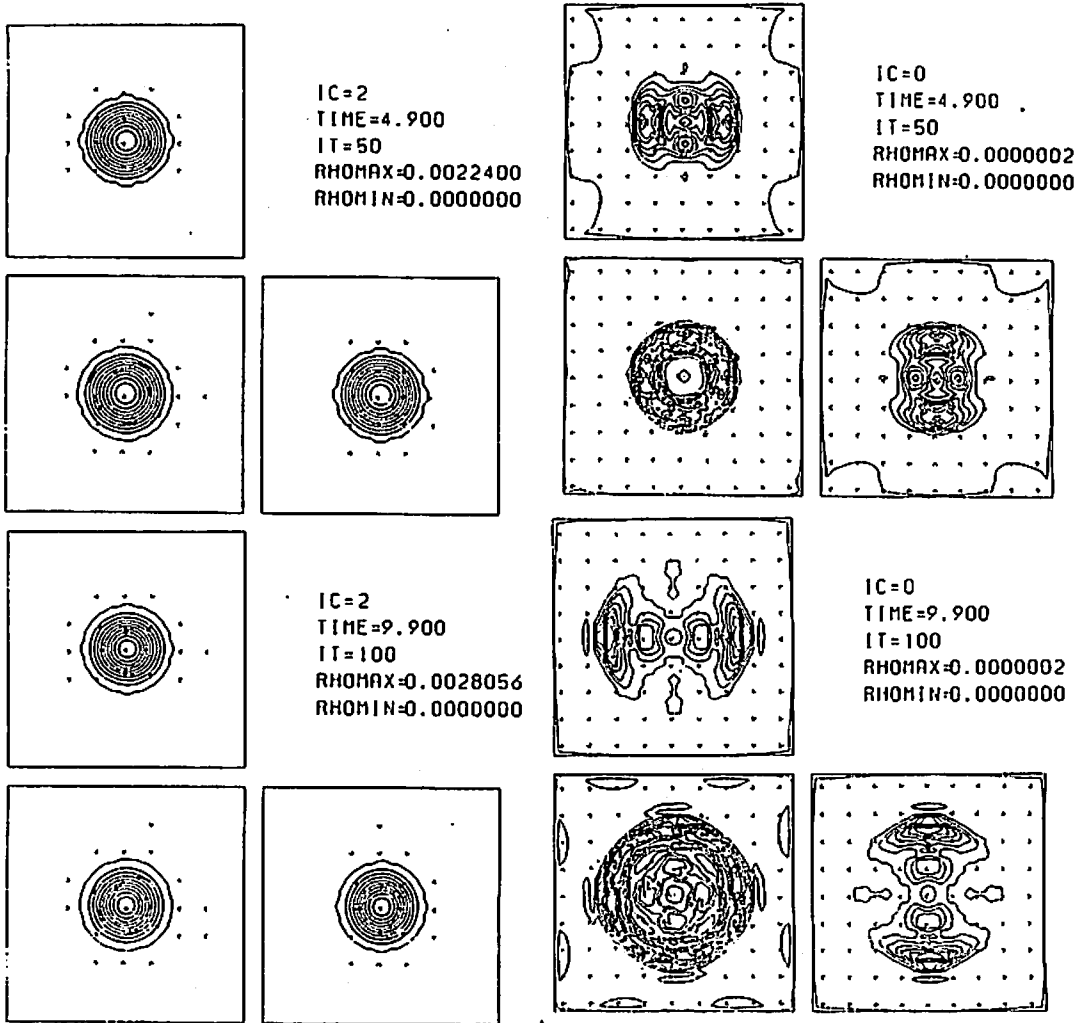
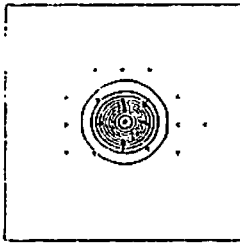
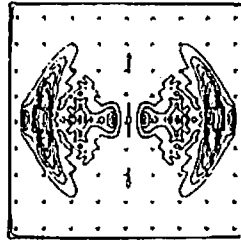


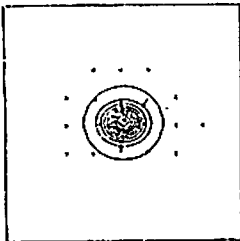
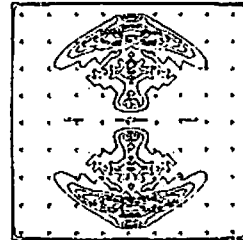
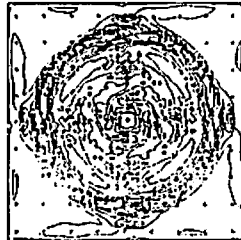
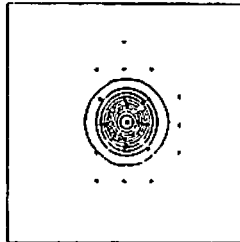
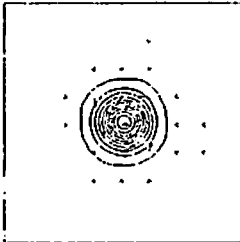
FIG 6-B



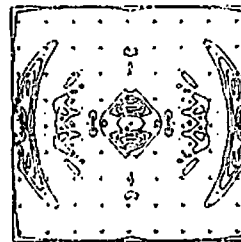
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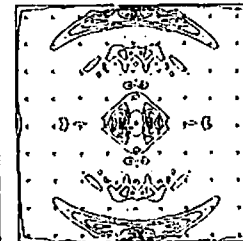
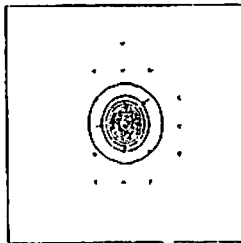
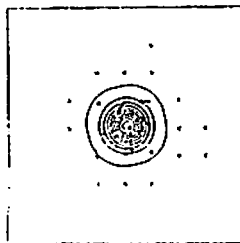


FIG 7-A

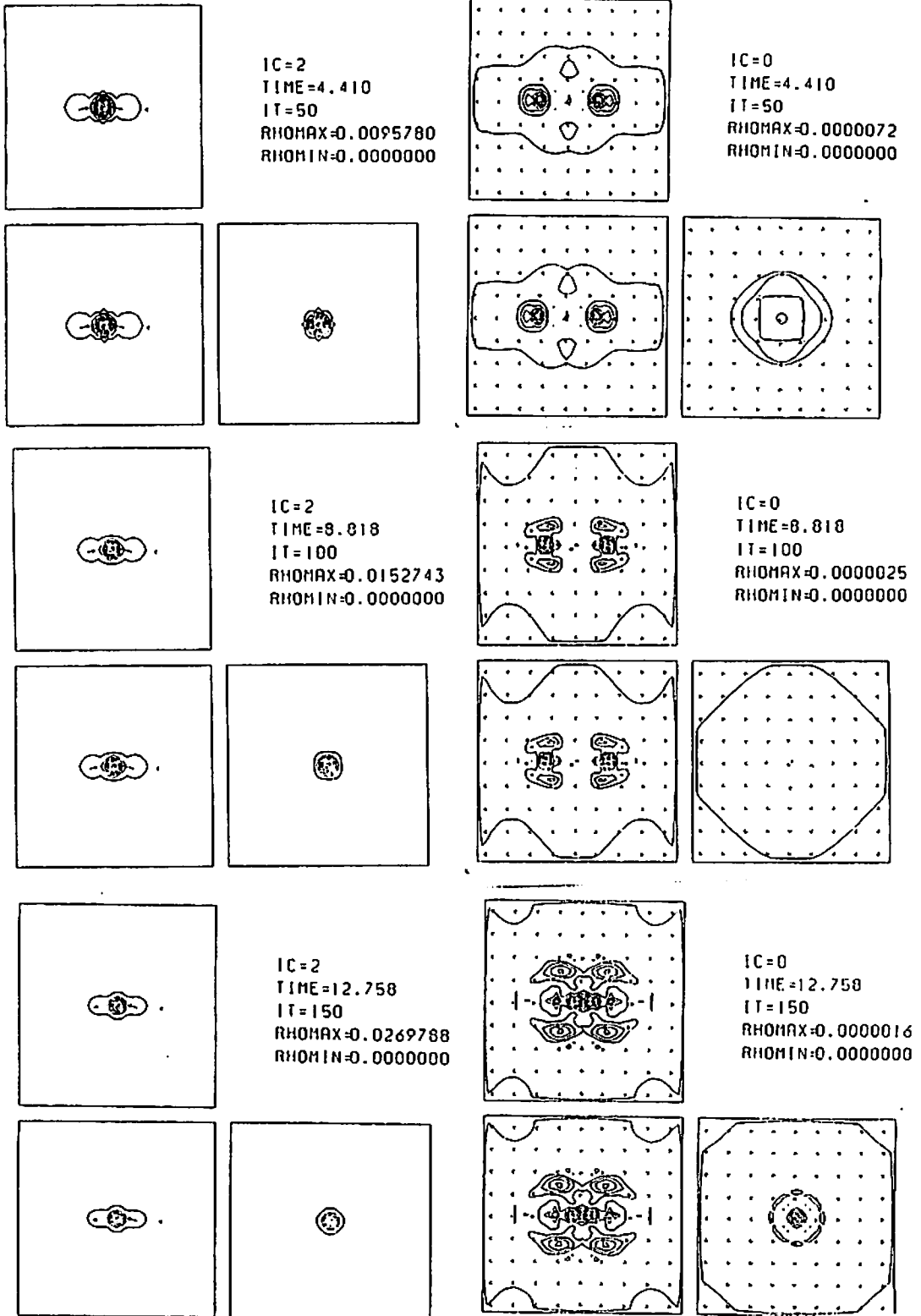


FIG 7-B

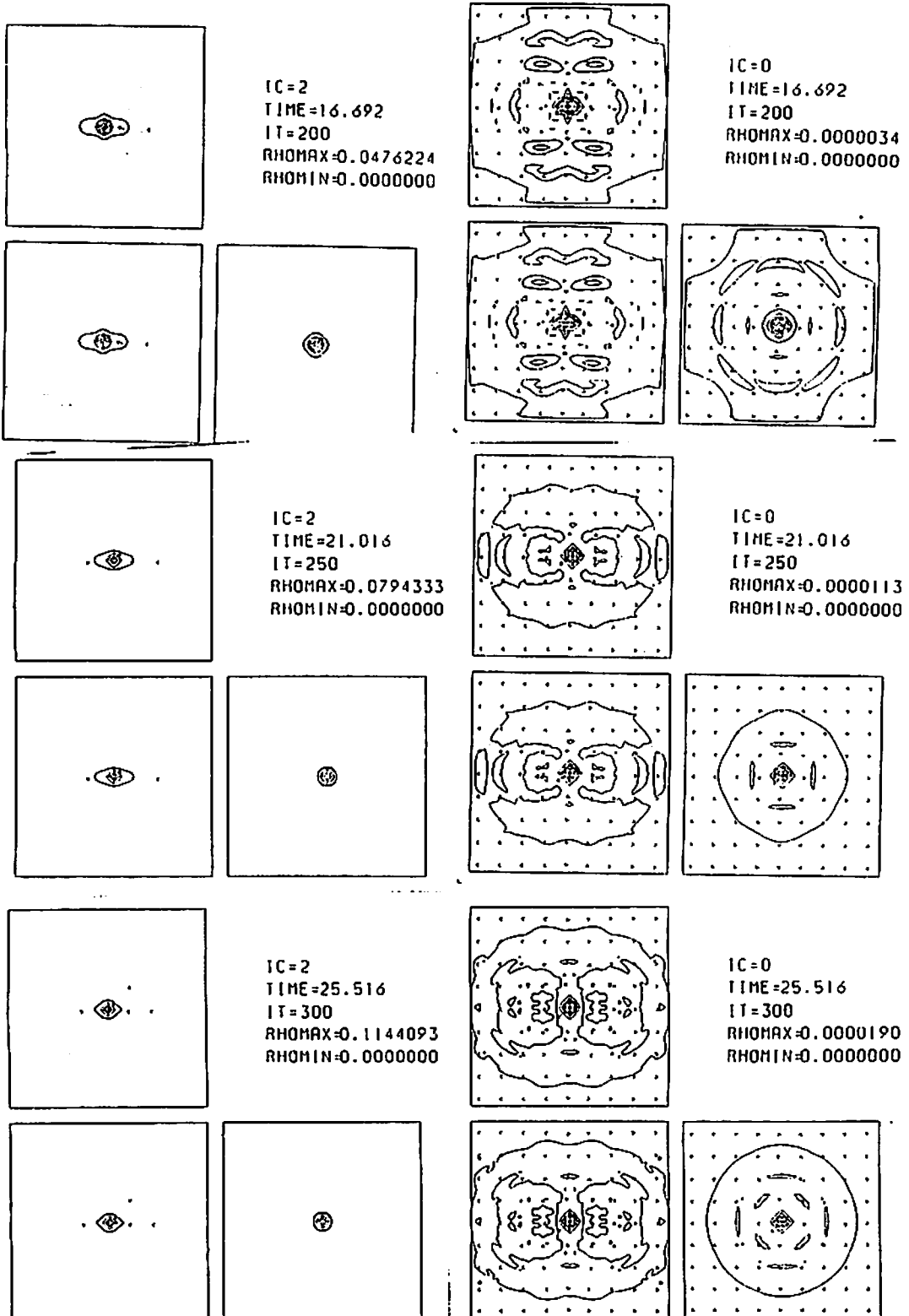
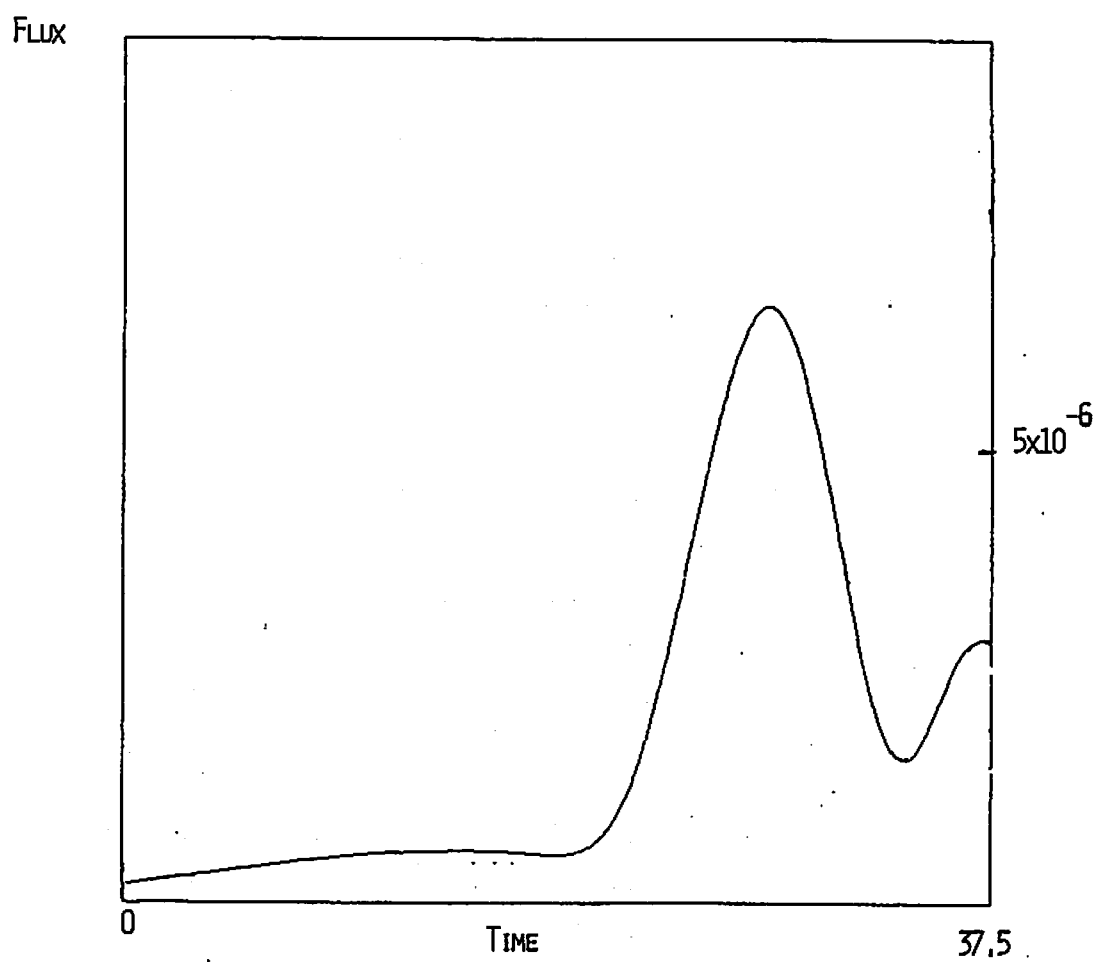


FIG 8



In the right I show the evolution of the extrinsic curvature K_{xx} . Contour of K_{xx} is completely different from that of gravitational wave energy in Fig.1 (right) in shape and value. This is because A_{ij}^{TT} is essentially the transverse traceless part of K_{ij} . Fig.3 shows the surface integration of gravitational energy flux at $r = 25$. Here the integration means

$$\frac{dE_{GW}}{dt} = \frac{1}{4} \int (A_{ij}^{TT})^2 d\Omega \Big|_{r=25}.$$

At $t = 35$, the total energy amounts to 2×10^{-3} .

To check whether A_{ij}^{TT} shows the gravitational waves degree of freedom. I performed the spherically symmetric dust collapse using the same 3D code with $(80)^3$ grids. The same figure as that of Fig.1 for spherically symmetric case is shown in Fig.4. Although A_{ij}^{TT} has the value at the center, it does not propagate outward, and its value is small compared with that in Fig.1. Since the meaning of transverse traceless is clear only in an asymptotically flat region, the result of Fig.4 is consistent with the idea that A_{ij}^{TT} represents gravitational waves.

I have also performed various test simulations using smaller $(46)^3$ grids. Fig.5 shows the same figure as Fig.1 for direct collision of two neutron stars, while Fig.6 shows the axi-symmetric collapse of rotating stars to black holes in 3D code. In Figs.5 and 6, we see qualitatively expected wave patterns. In Fig.7, I performed the simulations of spherically symmetric dust collapse with two smaller mass (10% of central mass each) falling on the central collapsed object. This may mimic two particles falling on to the black hole. In Fig.8, I show the time evolution of energy flux. If one compares this figure with that of Davis, Ruffini and Tiomno (Fig.1-d)¹⁰⁾, one finds that behavior of two curves are very similar, that is, precursor, sharp burst and ringing tail. In Fig.8, we see the beginning of the ringing tail.

4. Summary

One of the goals in numerical relativity is to perform simulations of general space times for any initial data. For this, construction of 3D code is essential. In this article I showed what I have been done for this purpose. At present my code is on a way of construction. I made some approximations to the solution of pseudo minimal shear condition (Eq.(36)) and conformal factor ϕ (Eq.(32)). The number of grids is not enough for the application of the outgoing wave boundary conditions at the numerical outermost boundary. Although all the results shown in §3 should be considered preliminary, qualitatively the code works well without serious difficulties. I think that the completion of the present 3D code will be done near future.

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Parallel Computation in Numerical Relativity

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Abstract

In numerical relativity, some of gauge equations as well as some of constraint equations are elliptic. Usually these elliptic equations must be solved on each time slice. We present a new scheme for solving them on a parallel computer. Results of bench mark tests on some parallel computers are shown.

1 Introduction

In numerical calculation for astrophysical problems with self-gravity using a finite difference method, we usually face a Poisson equation. Oohara and Nakamura¹⁾ used (1+2.5)-post-Newtonian hydrodynamics equations in numerical simulations of coalescing binary neutron stars. In these calculations, seven Poisson equations must be solved on each time step.²⁾ In numerical relativity, also, some of constraint equations are elliptic. In addition, some of gauge equations is likely to be elliptic. If the minimal shear condition is adopted to fix the shift vector, it leads a set of non-linear elliptic equations, which must be solved on each time step (see the Nakamura article in this volume). In these cases, solving Poisson or elliptic equations holds the greatest part of CPU consumption.

In 1980's, vector supercomputers are used for these calculations. We have fast and accurate, fully vectorizable, scheme of solving elliptic equations. Some of numerical relativists used the incomplete Cholesky decomposition conjugate gradient (ICCG) method.^{3),4),5)} Almost all of the numerical relativity codes have run on single processor computers. At the present time, vector supercomputers which have a peak performance of a few GFLOPS on a single processor and a maximum memory of several GBytes are available. They provide enough computing power and memory to perform 2-D fully relativistic calculations and 3-D post-Newtonian calculations. For 3-D post-Newtonian calculations, it allows a $150 \times 150 \times 150$ grid, which will give adequate results. On the other hand, for 3-D fully relativistic calculations, it only allows a $100 \times 100 \times 100$ or less grid, while twice more grid in each dimension would be needed to estimate the energy and wave form of gravitational waves radiated with a good accuracy. Thus a machine with more than a hundred GFLOPS of peak performance and more than several tens GBytes of memory will be needed to obtain 3-D results. A massively parallel supercomputer can provide performance and memory of this order of magnitude.

Unfortunately, however, the vectorized ICCG method we are using now is not suitable for massively parallel computation. The main reason is the hyperplane method for

vectorization.⁶⁾ A straightforward algorithm of the ICCG method cannot be fully vectorized since recursive calculations are included. The hyperplane method is technique for its vectorization. However, it divide a recursive loop into loops whose vector length varies from one to fractions of the original length. Some of processor nodes would not work for a greatest part of CPU time and the communication between nodes would be complicated and consume a lot of CPU time. Thus it is disadvantageous in massively parallel computing. The ICCG program can be vectorized with the red-black method or the multi-color method. Since it divides the recursive loop into two or more loops, the length of the loop on each processor node become short if it is parallelized. For vector-parallel supercomputer, the length of the loop should be sufficiently long to make the most of its power. What is worse, the number of the iteration till convergence will be usually greater than the original ICCG method for astrophysical problems. Note that the hyperplane method does not change the mathematical structure of the ICCG method and therefore the number of the iteration is the same as the original one.

In this article, we present a new scheme of the conjugate gradient method suitable for a (vector-) parallel computer. In the next section, we describe the scheme and a comparison of this with others is made in §3.

2 A Scaled, Taylor-Expanded Conjugate Gradient Method

Under the finite difference method, elliptic partial differential equations can reduce to large-scale, simultaneous linear equations,

$$A\mathbf{x} = \mathbf{b}, \quad (1)$$

where the matrix A is usually sparse and its dimension is very large. For a Poisson equation, A is positive definite and symmetric. The solution of Eq.(1) for such a matrix is the vector $\mathbf{x}$ that gives the minimum to a quadratic function

$$f(\mathbf{x}) = (\mathbf{x}, A\mathbf{x}) - 2(\mathbf{b}, \mathbf{x}). \quad (2)$$

The conjugate gradient (CG) method will search for the minimum of $f(\mathbf{x})$ in the direction conjugate to the matrix A to obtain the solution. In the CG method, in principle, the exact solution will be obtained with finite number of iteration. However, since it usually takes a large number of iteration to obtain the solution with reasonable accuracy, one should precondition Eq.(1) to accelerate the convergence. It is called the preconditioned conjugate gradient (PCG) method.

With a regular matrix $M = QQ^t$, Eq.(1) is modified to

$$(Q^{-1}A(Q^t)^{-1})(Q^t\mathbf{x}) = Q^{-1}\mathbf{b}, \quad (3)$$

or if MA is symmetric, it can be modified simply to

$$MA\mathbf{x} = M\mathbf{b} \quad (4)$$

If the matrix $A' \equiv Q^{-1}A(Q^t)^{-1}$ (or MA) is close to the identical matrix or the matrix A' has multiple or dense eigenvalues, the CG method converge fast. In general, an approximate matrix of A^{-1} is used as M .

The simplest one is the scaled CG method, where Q in Eq.(3) is a diagonal matrix $Q = Q^t = (\sqrt{a_{ii}})$. Defining $\mathbf{x}'$ and $\mathbf{b}'$ by $\mathbf{x}' = Q\mathbf{x}$ and $\mathbf{b}' = Q^{-1}\mathbf{b}$, we have

$$A\mathbf{x} = \mathbf{b} \Rightarrow Q^{-1}AQ^{-1}Q\mathbf{x} = Q^{-1}\mathbf{b} \Rightarrow A'\mathbf{x}' = \mathbf{b}'. \quad (5)$$

Note that each of diagonal elements of A' is unity. Since the matrix A that is obtained with the finite difference approximation of a elliptic equation is usually diagonally dominant, the absolute values of the off-diagonal elements are less than unity. Even SCG will, in most case, converge considerably faster than non-preconditioned CG.

Now define a matrix E by $A' = I - E$, where I is the identical matrix. Since the absolute value of each element of E is less than unity, it is expected that $A'^{-1} \approx (I + E)$ and the matrix A'' defined by

$$A'' = (I + E)(I - E) = I - E^2 \quad (6)$$

is closer to the identical matrix. Then we can apply the CG method to

$$(I + E)A'\mathbf{x}' = (I + E)\mathbf{b}' \Rightarrow A''\mathbf{x}' = \mathbf{b}'', \quad (7)$$

because A'' is symmetric if A' is symmetric. Hereafter we call this procedure a scaled, Taylor-expanded CG (STCG) method. Moreover it can be extended as

$$A'' = (I + E + \cdots + E^{m-1})(I - E) = I - E^m. \quad (8)$$

However, it increase the number of operation in each iteration while the number of iteration till convergence may be decreased. The best value of m is determined from the ratio between them. For the problem to be shown in next section, it is $m = 2$.

For an asymmetric matrix A , the same preconditioning can be applied to the conjugate residual method, the bi-conjugate gradient method and so on.

3 Numerical Experiments

Now we will compare the STCG method with the ICCG method on a vector computer and show that the STCG method gives a fairly good performance on a parallel computer. As a model problem, we solve the three-dimensional gravitational potential of a binary star;

$$\Delta\phi = 4\pi G\rho. \quad (9)$$

First we compare the STCG method with the ICCG method. Calculations are performed on HITAC S820/80. The ICCG method is vectorized with the hyperplane method (ICCG-h), the red-black (two-color) method (ICCG-2) and the three-color method (ICCG-3), while the STCG method can be vectorized straightforward. Table 1 is shown the number

ICCG-h		ICCG-2		ICCG-3		STCG	
ITR	CPU	ITR	CPU	ITR	CPU	ITR	CPU
55	0.866	194	2.108	209	3.040	163	1.462
1.0	1.0	3.5	2.4	3.8	3.5	3.0	1.7

Table 1: Comparison between the STCG method and the ICCG method. ICCG-h, ICCG-2 and ICCG-3 indicate results of the ICCG vectorized with the hyperplane method, the red-black (two-color) method and the three-color method, respectively. ITR is the number of iteration till convergence and CPU is its CPU seconds. The ratios to ICCG-h are shown in the lower row.

of iteration and the CPU seconds needed in solving Eq.(9) with grid size $91 \times 91 \times 46$. The number of iteration of STCG increases by three times compared with ICCG-h but the CPU seconds increase only by 1.7 times. Note again that ICCG-h is hard to be parallelized while STCG is not. Table 2 shows CPU seconds needed in solving Eq.(9) with grid size $32 \times 32 \times 16$ on some parallel computers. Calculations are performed with a single processor and 16 processors. AP1000 with 16 processors sustain 12.34 times performance of a single processor.

Although overhead in communication between processor elements become more serious for massively parallel computers with a few hundreds or more than several thousands processor elements, we can expected that STCG achieves a good performance on such computers.

	# of PE	CPU seconds	ratio
Intel iPSC860	1	28.17	1
	16	4.82	1/5.84
nCUBE2	1	119.11	1
	16	12.68	1/9.39
Fujitsu AP1000	1	108.63	1
	16	8.80	1/12.34

Table 2: Performance of STCG on parallel computers. CPU seconds with 1 and 16 processor elements are shown. The grid size here is $32 \times 32 \times 16$. Calculations are performed on Intel iPSC860, NCUBE nCUBE2 and Fujitsu AP1000.

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Gravitational waves from a black hole and logarithmic terms in the post-Newtonian expansion

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We found logarithmic terms in a post-Newtonian expansion of gravitational radiation induced by a particle traveling circular orbit of radius r_0 around a Schwarzschild black hole of mass M . We calculated gravitational wave luminosity using Teukolsky equation in high accuracy (~ 20 figures) and determined the coefficients of the post-Newtonian expansion by means of least square fitting. We found that there are terms proportional to $x^6 \ln x$ and $x^8 \ln x$ where $x = (M/r_0)^{1/2}$. We also examine the accumulated phase of coalescing compact star binaries by means of the post-Newtonian expansion as it sweeps through the bandwidth at which the future laser interferometric detectors have good sensitivity. For the details of this paper, see Tagoshi and Nakamura^[1]

One of promising sources of gravitational waves for the planned laser interferometer detectors^[2] is the last stage of inspiral of compact binaries. Accurate theoretical models of wave forms are needed to extract physical information from gravitational waves^[3]. Many theoretical efforts have already been done for this purpose. In the case when one of the stars in a binary is much more massive than the other, a linear perturbation theory is available. To investigate the effectiveness of a post-Newtonian method, Cutler et.al.^[5] numerically integrated the Teukolsky equation in high accuracy and fitted their numerical data for the gravitational wave luminosity to the post-Newtonian expansion form by means of least square fitting up to the sixth order.

In this paper, we try to extend the post-Newtonian expansion to the eighth orders by using fitting method. To do this it is essential to calculate the gravitational wave luminosity in higher accuracy than Cutler et.al.'s work in which numerical integrations were used to obtain a solution of the Teukolsky equation. In this paper we adopt a method which was originally proposed by Leaver^[6] to obtain higher accuracy. In this method, solutions of the

Teukolsky equations are represented in analytic manner.

We consider the case when a test particle of mass μ travels circular orbit around a Schwarzschild black hole of mass $M \gg \mu$. Then the specific energy $\tilde{E}$ and the angular momentum $\tilde{L}$ of the particle is given by

$$\tilde{E} = (\tau_0 - 2M) / \sqrt{\tau_0(\tau_0 - 3M)},$$

and

$$\tilde{L} = \tau_0 / \sqrt{\tau_0 - 3},$$

where τ_0 is orbital radius. The angular frequency is given by $\Omega = (M/\tau_0^3)^{1/2}$. To calculate the gravitational wave luminosity we use the Teukolsky equation^{[7][8]},

$$\left[\Delta^2 \frac{d}{dr} \left(\frac{1}{\Delta} \frac{d}{dr} \right) - U(r) \right] R_{\ell m \omega} = T_{\ell m \omega}(r), \quad (1)$$

where

$$U(r) = \frac{r}{r-2} [\omega^2 r^2 - 4i\omega(r-3)] - (\ell-1)(\ell+2), \quad (2)$$

and $\Delta = r(r-2M)$.

We solve Eq. (1) under the outgoing boundary condition at infinity and the ingoing boundary condition at the horizon. Then gravitational wave luminosity is given by

$$\frac{dE}{dt} = \sum_{\ell=2}^{\infty} \sum_{m=1}^{\ell} |Z_{\ell m}^{\text{out}}|^2 / 2\pi\omega^2, \quad (3)$$

$$Z_{\ell m}^{\text{out}} = \frac{1}{2i\omega B_{\text{in}}} \int_{2M}^{\infty} dr R_{\text{in}}(r) T_{\ell m \omega}(r) / \Delta^2, \quad (4)$$

where ℓ and m are the spherical harmonic degree and order. $R_{\text{in}}(r)$ is a solution of the

homogeneous Teukolsky equation and has a boundary condition

$$R_{\text{in}}(r) = \begin{cases} \Delta^2 e^{-i\omega r^*} & , r^* \rightarrow -\infty \\ r^3 B_{\text{out}} e^{i\omega r^*} + r^{-1} B_{\text{in}} e^{-i\omega r^*} & , r^* \rightarrow +\infty \end{cases} \quad (5)$$

where $r^* = r + 2M \ln(r/2M - 1)$. In the case of the circular orbit, $T_{lm\omega}$ contains only the terms which are proportional to a delta function and its derivatives. So Z_{lm}^{out} can be evaluated using $R_{\text{in}}(r_0)$ and its derivatives.

It is often convenient to use a Regge-Wheeler (RW) equation^[9] instead of the Teukolsky equation. If we transform R_{in} as^{[10] [11][12]}

$$R_{\text{in}}(r) = \Delta \left(\frac{d}{dr^*} + i\omega \right) \frac{r^2}{\Delta} \left(\frac{d}{dr^*} + i\omega \right) r X_{\text{in}}, \quad (6)$$

the homogeneous Teukolsky equation becomes

$$\left[\frac{d^2}{dr^{*2}} + \omega^2 - V(r) \right] X_{\text{in}}(r) = 0, \quad (7)$$

and

$$V(r) = \left(1 - \frac{2M}{r} \right) \left(\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3} \right).$$

The asymptotic form of X_{in} , after normalized appropriately, is given by

$$X_{\text{in}}(r) = \begin{cases} e^{-i\omega r^*}, & r^* \rightarrow -\infty \\ A_{\text{out}} e^{i\omega r^*} + A_{\text{in}} e^{-i\omega r^*}, & r^* \rightarrow +\infty. \end{cases} \quad (8)$$

To solve X_{in} we put Leaver's procedure^[6] into practice. The Teukolsky equation is classified into generalized spheroidal wave equations which can be computed by means of a power series expansion around the horizon or a series of Coulomb wave functions.

First, X_{in} can be represented as

$$X_{\text{in}} = \exp(-i\omega r^*) \sum_{n=0}^{\infty} a_n \left(1 - \frac{2M}{r}\right)^n. \quad (9)$$

This expansion is a power series expansion in variable $u = 1 - 2M/r$ around regular singular point $u = 0$. a_n 's are determined by a three term recurrence relation

$$(n+1-4iM\omega)a_{n+1} = \left(2n + \frac{\ell(\ell+1)-3}{n+1}\right)a_n - \frac{n^2-4}{n+1}a_{n-1}, \quad (n \geq 1) \quad (10)$$

$$a_1 = [\ell(\ell+1)-3]/(1-4iM\omega) \times a_0.$$

From the boundary condition (8), we must set $a_0 = 1$.

For convergence of (9), we need summation only up to $n \sim 10^5$ for $r_0 < 10^4 M$. As R_{in} is expressed by X_{in} and their derivatives, it also converges for $r_0 < 10^4 M$. Thus $Z_{\ell m}^{\text{out}}$ can be calculated almost analytically for $r_0 \lesssim 10^4 M$ except for the term B_{in} . To obtain B_{in} , we need to determine A_{in} . However since $r = \infty$ is an irregular singular point, a power series expansion with variable r^{-1} converges only asymptotically. So it is hard to obtain an accurate value of A_{in} using (9). To overcome this, we expand X_{in} using a series of Coulomb wave functions discussed by Leaver^[6]. That is

$$X_{\text{in}} = \left(A_{\text{out}} \sum_{L=-\infty}^{\infty} b_L u_{L+\nu}^{(+)}(z) + A_{\text{in}} \sum_{L=-\infty}^{\infty} b_L u_{L+\nu}^{(-)}(z) \right) \left(\frac{r}{r-2M} \right)^{2iM\omega} \quad (11)$$

where $u^{(\pm)}$ are a certain kind of Coulomb wave functions. In Eq. (11), ν is real and is determined so that a series of Coulomb wave functions converges.

All numerical calculations are done with precision of 32 figures. Checking the convergence, we confirmed that the accuracy of the summation in Eqs. (9) (11) and (4) has at least 20 figures. We compared our numerical data with those by Cutler et.al.^[5] and found that their data agree with ours up to $8 \sim 9$ figures which seems to be consistent with their error estimates.

Using these data, we perform the post-Newtonian expansion. First we assume that dE/dt can be expanded in powers of x ($= (M/r_0)^{1/2}$), that is, we assume that

$$\frac{dE}{dt} = \left(\frac{dE}{dt} \right)_N \sum_{k=0} a_k x^k, \quad (12)$$

where $(dE/dt)_N$ is derived from a quadrupole formula,

$$\left(\frac{dE}{dt} \right)_N = \frac{32}{5} \frac{\mu^2 M^3}{r_0^5}. \quad (13)$$

First four terms are known analytically^{[4] [14][15]}

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = -1247/336, \quad a_3 = 4\pi. \quad (14)$$

The next three terms are known approximately^[5],

$$a_4 \simeq 4.89(2\%), \quad a_5 \simeq -38(10\%), \quad a_6 \simeq 135(50\%), \quad (15)$$

where numbers in the parentheses denote the error. We calculated multipoles from $\ell = 2$ to $\ell = 6$. Then we can determine the coefficients up to the order x^9 because $\ell = 7$ multipole contributes from order x^{10} .

We determine the coefficients a_n by means of least square fitting implemented with the method of singular value decomposition^[13]. We used 120 data of dE/dt in the fitting. The results are shown in Table (a). Using these results we plot in Fig.1 the residual ζ_n of dE/dt which is defined by

$$\frac{dE}{dt} \times \zeta_n = \frac{dE}{dt} - \left(\frac{dE}{dt} \right)_N \sum_{k=0}^n a_k x^k. \quad (16)$$

We see that, at $n = 6$, $|\zeta_n|$ shows a strange behavior. In general, it is argued that the post-Newtonian expansion may contain terms proportional to $x^k (\ln x)^p$ (k, p : non-negative

integer)^{[4][5][16]}. To take into account this, we add $b_6 x^6 \ln x$ term to the model functions. That is

$$\frac{dE}{dt} = \left(\frac{dE}{dt} \right)_N \left(\sum_{k=0} a_k x^k + b_6 x^6 \ln x \right). \quad (17)$$

The result is shown in Table (b). It is clear that accuracy of the determination of first four values are improved about three orders. The residual of dE/dt including $x^6 \ln x$ term also becomes better. From the value in Table (b), we can understand the behavior of $|\zeta_6|$ in Fig.1. Since $b_6 x^6 \ln x + a_7 x^7$ becomes zero near $r \sim 20$, ζ_6 change sign near this point. All of these facts are consistent with the existence of the term proportional to $x^6 \ln x$ in the post-Newtonian expansion. From the analysis of the data dE/dt of each modes (ℓ, m) , we find that this term comes from $\ell = m = 2$ mode. We also tried model functions with $x^6 (\ln x)^n$ ($n \geq 2$) and found that the coefficients of these terms are less than 10^{-2} . This fact suggests that the terms proportional to $x^6 (\ln x)^n$ ($n \geq 2$) do not exist.

In a similar way, we examine the coefficients at the order x^7 and x^8 . We find that there seems to exist no $x^7 (\ln x)^n$ ($n \geq 1$) terms but there is $b_8 x^8 \ln x$ term. Fitting results are show in Table (c). We see that accuracy of the determination of first four values are improved about two orders compared with Table (b).

We also performed the fitting to the data in which first four terms are subtracted analytically and found that the post-Newtonian expansion has the form

$$\begin{aligned} \frac{dE}{dt} = \left(\frac{dE}{dt} \right)_N & \left(1 - \frac{1247}{336} x^2 + 4\pi x^3 - 4.9284611990 x^4 - 38.292835 x^5 \right. \\ & \left. + 115.730 x^6 - 16.3049 x^6 \ln x - 101.450 x^7 - 93 x^8 + 57 x^8 \ln x \right). \end{aligned} \quad (18)$$

To investigate the accuracy of these coefficients, we compare these results to the value in Table (c) and count the number of figures at which the value are not changed. We adopt these numbers as significant figures of the coefficients and the accuracy of these values are estimated as follows ; $a_4 : 10^{-5}\%$, $a_5 : 10^{-3}\%$, $a_6 : 0.1\%$, $b_6 : 0.1\%$, $a_7 : 1\%$, $a_8 : 30\%$, $b_8 : 20\%$.

Using these data and log terms, we plot in Fig. 2 the residual ζ'_n which is defined by

$$\frac{dE}{dt} \times \zeta'_n = \frac{dE}{dt} - \left(\frac{dE}{dt} \right)_N \sum_{k=0}^n (a_k x^k + b_k x^k \ln x), \quad (19)$$

where $b_k \neq 0$ for $k = 6$ and 8 . We see all ζ'_n except ζ'_7 show power law behavior. ζ'_7 changes sign around $r_0 \sim 15M$. This may suggest that the coefficients of x^9 may be very large (about 650). However for $n \geq 9$, more detailed analysis is needed.

Finally, we estimate the accuracy of the post-Newtonian expansion to construct the template wave form from an inspiraling compact binary. Consider that a binary starts to inspiral from orbital separation r_0 and that there are two template waveforms. One is constructed using correct $\dot{E}$ and the other is constructed using an approximate form for $\dot{E}$ (dot denote a time derivative). Then at the end of the inspiraling, accumulated phase difference $\Delta\Phi$ between two template waveforms is given by^[5]

$$\Delta\Phi/\Phi \sim \Delta\dot{E}(r_0)/\dot{E}(r_0), \quad (20)$$

where Φ is the total phase and right hand side refers to the relative error in $\dot{E}$.

The planned ground based laser interferometer detectors^[2] will observe inspiraling compact binaries in wave frequency $f \sim 10\text{Hz}$ to 1000Hz . In the case of a $1.4M_\odot$ neutron star binary, when the wave frequency is 10 Hz , r_0 is about 723km ($175M$) and the total accumulated phase from 10 Hz to 10^3 Hz is $\Phi \simeq 10^5$. We assume that the expansions for non-negligible mass μ/M will be subject to the same convergence problems as for negligible μ/M . Then from figure 2, we can estimate the accumulated phase error $(\Delta\Phi)_n$ when we include the post-Newtonian expansion of order x^n ; $(\Delta\Phi)_3 \sim 20$, $(\Delta\Phi)_4 \sim 7$, $(\Delta\Phi)_5 \sim 3$, $(\Delta\Phi)_6 \sim 0.2$, $(\Delta\Phi)_7 \sim 0.02$. This suggests that if we construct the template wave form which includes the post-Newtonian effect up to $n = 7$, we will obtain accuracy $\sim 3 \times 10^{-3}$ cycle after $\sim 1.5 \times 10^4$ cycles. Note that $b_6 x^6 \ln x$ term contributes to $\Delta\dot{E}(r_0)/\dot{E}(r_0)$ one third of $a_6 x^6$ term at $r_0 \simeq 175M$. This means the importance of $b_6 x^6 \ln x$ term in constructing the template wave form.

Recently Sasaki^[17] calculated analytically the post Newtonian expansion of the Regge Wheeler equation to v^8 order. Using his formula, we have checked above fitting result and confirmed our results. The details of which will be appear elsewhere^[18].

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Table The results of the least square fitting. The model functions of (a) are power in x . (b) contains $x^6 \ln x$ term and (c) contains both $x^6 \ln x$ and $x^8 \ln x$ terms.

Figure 1 The plot of the residual $|\zeta_n|$ of dE/dt using the coefficients in Table 2(a).

Figure 2 The plot of the residual $|\zeta'_n|$ of dE/dt using the coefficients in Eq.(18).

Table (a)

n	a_n
0	0.99999999999956737032
1	2.5540107268919450071E-10
2	-3.7113095967507652120
3	12.566384350926089206
4	-4.9304816403237679929
5	-37.999982610332799195
6	179.273234265923125902
7	-817.51080139077674299

Table (c)

n	a_n
0	1.000000000000000053030
1	-4.115960526368818532705E-15
2	-3.711309523807923530949
3	12.566370613929664016796
4	-4.928461103278450962933
5	-38.29285880105704321767
6	115.71947126712790878361
$x^6 \ln x$	-16.30735495755162544055
7	-101.1731705303648725845
8	-72.60876280963051767049
$x^8 \ln x$	64.21853039659554940097

Table (b)

n	a_n
0	0.99999999999999971856
1	1.92611218902394553128E-13
2	-3.7113095238743910921
3	12.5663706290971842005
4	-4.9284639028225238633
5	-38.292319775967105066
6	115.913907630116282263
$x^6 \ln x$	-16.262941934715851792
7	-104.97250734572709616
8	-259.25092983316886042

Fig. 1

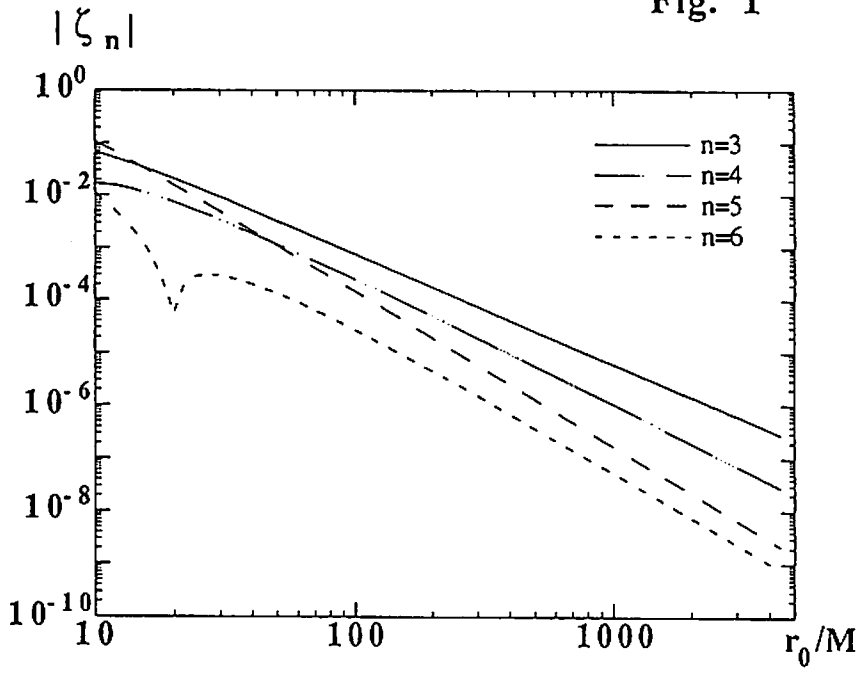
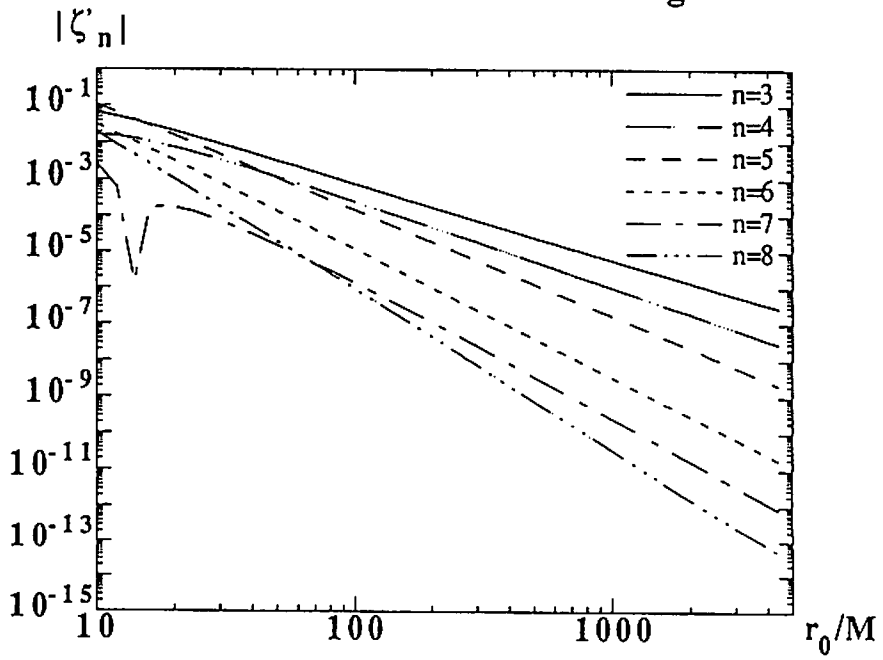


Fig. 2



Gravitational-Radiation Reaction in the Post Newtonian Approximation

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Abstract

The post Newtonian (PN) approximation is an only available approximation scheme at the moment to deal with the self-gravitating system, such as the compact binary system, in general relativity. It is previously suggested that the PN approximation contains the logarithmic divergences at some higher orders and the divergences might have an observable effect on the evolution of the system. We shall study the origin of these terms and how they affect the motion of the system. It is found that the logarithmic term appears because of the slow fall-off behavior of the gravitational radiation emitted by the source and affects the equation of motion at the 4PN order which is smaller by c^{-3} than the radiation reaction order and it is expressed as the integral on the past light cone. Thus if we take into account the effect of the radiation reaction up to a sufficiently higher order, then the equation of motion cannot be expressed in term of the matter variables at one instance.

1 Introduction

Recently, the LIGO project in U.S.A.[1], VIRGO project in France and Italy and a project also in Japan are under progress aiming at direct detection of the gravitational wave from the astrophysical objects. We will, in the near future, be able to study the universe by this new method. In this way, concerns about the gravitational wave are much increasing. The coalescing compact binary system is one of important sources of gravitational waves, expected to be detected directly. In order to understand the dynamics of such system including the reaction of gravitational radiation, we will be able to neglect other physics except gravity and must solve the Einstein's equations, which are coupled non-linear partial differential equations. Under the general condition it is impossible to solve the Einstein's equations not only exactly, but also numerically at the moment because of the non-linearity which plays an essential role in dynamics of the self-gravitating system such as the compact binary stars. Although it is difficult to describe the non-linear effects, the post Newtonian (PN) approximation is the most well known scheme to deal with such effects. In the approximation every quantities may be expanded in the power series of $1/c$ where c is the light velocity. However, many people pointed out the existence of the divergent integrals at higher orders in the PN approximation [2][3][4]. The appearance of the divergence may be easily seen as follows.

We usually adopt the harmonic coordinate condition, $\bar{h}^{\mu\nu}{}_{,\nu} = 0$ for $\bar{h}^{\mu\nu} = \eta^{\mu\nu} - \sqrt{-g}g^{\mu\nu}$, where $\eta^{\mu\nu}$ is the flat metric. Then Einstein's equation is written in the following form:

$$\begin{aligned}\square \bar{h}^{\mu\nu} &= -\frac{16\pi G}{c^4}(-g)(T^{\mu\nu} + t^{\mu\nu}) \\ &= -\frac{16\pi G}{c^4}\Lambda^{\mu\nu}\end{aligned}\tag{1}$$

where $\square = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the flat-space d'Alembertian and $t^{\mu\nu}$ is a non-linear quantity in $\bar{h}^{\mu\nu}$ which looks like $\bar{h}^{\mu\alpha}{}_{,\beta}\bar{h}^{\nu\beta}{}_{,\alpha} + \dots$. This equation is a sort of a wave equation and thereby it can be written in the form of a retarded integral.

$$\bar{h}^{\mu\nu}(t, x) = -\frac{4G}{c^4} \int \frac{\Lambda^{\mu\nu}(t - \frac{1}{c}|x - y|, y)}{|x - y|} d^3y\tag{2}$$

The range of the integration is whole inside the past-directed light cone whose vertex is

at (t, x) extending to the past null infinity. Here, we expand the retarded time by the Taylor's series and obtain the following equation

$$\bar{h}^{\mu\nu}(t, x) = -\frac{4G}{c^4} \sum_{n=0} \frac{1}{(-c)^n n!} \frac{\partial^n}{\partial t^n} \int \Lambda^{\mu\nu}(t, y) r^{n-1} d^3y, \quad (3)$$

where

$$r = |x - y| \quad (4)$$

and the integration is performed over all space. The point is that $\Lambda^{\mu\nu}$ does not always have a compact support because of the existence of the non-linear terms $t^{\mu\nu}$. For example, we consider the stationary space-time. Then $\bar{h}^{\mu\nu}$ is expanded as follows

$$\bar{h}^{\mu\nu} \sim \frac{M}{r} + O\left(\frac{1}{r^2}\right) \quad (5)$$

and then this leads to the asymptotic behavior of $t^{\mu\nu}$

$$t^{\mu\nu} \sim O\left(\frac{1}{r^4}\right). \quad (6)$$

Since $t^{\mu\nu}$ is independent of time in this case, we obtain

$$\frac{\partial^n}{\partial t^n} t^{\mu\nu} = 0 \quad (7)$$

Thus the above integrals in eq.(3) converge. However, when the gravitational radiation exists, $\bar{h}^{\mu\nu}$ is expanded as follows.

$$\bar{h}^{\mu\nu} \sim \frac{I(t)}{r} + O\left(\frac{1}{r^2}\right) \quad (8)$$

where $I(t)$ means the sum of the mass monopole and the mass quadrupole moment. Then this leads to the asymptotic behavior of $t^{\mu\nu}$

$$t^{\mu\nu} \sim O\left(\frac{1}{r^2}\right). \quad (9)$$

Since the time derivative of $t^{\mu\nu}$ does not vanish due to the time dependence in $I(t)$, we obtain

$$\frac{\partial^n}{\partial t^n} t^{\mu\nu} = O\left(\frac{1}{r^2}\right) \quad (10)$$

Therefore, at higher orders, the divergent integrals appear. For example, since $\Lambda^{\mu\nu} \sim c^{-4} r^{-2}$, we obtain

$$\bar{h}^{\mu\nu} \sim \left(\frac{1}{c}\right)^8 \left(\frac{1}{c}\right)^n \int \frac{1}{r^2} r^{n-1} r^2 dr \sim \left(\frac{1}{c}\right)^{8+n} \infty^n \quad (11)$$

Ehlers [5] discussed the relation between the divergences in the standard PN approximation and the asymptotic radiation condition, such as purely outgoing radiation condition at $\mathcal{I}^+$, future null infinity or no incoming radiation condition at $\mathcal{I}^-$, past null infinity. Schutz proposed the statistical approach to the radiation reaction problem based on the initial value problem with a spacelike Cauchy surface in the linearized theory [6].

Futamase and Schutz [7][8] applied this approach to the non-linear case and pointed out that the divergences are controllable and the so-called logarithmic terms appear at the 3PN level. The logarithmic terms arise since in the initial value approach the maximum value of the spatial distance, $|x - y|$, scales like τ/ϵ , where $\epsilon \sim v/c$, $d\tau = \epsilon dt$ and the initial time is taken as $\tau = 0$. Thus we have

$$\begin{aligned}\bar{h}^{\mu\nu} &\sim \sum_{n=0} \epsilon^8 \epsilon^n \int^{\tau/\epsilon} \frac{1}{r^2} r^{n-1} r^2 dr \\ &\sim \sum_{n=0} \epsilon^8 \epsilon^n \left(\frac{1}{\epsilon^n}\right) \sim \epsilon^8\end{aligned}\tag{12}$$

It is clear that these terms disappear if the system does not emit radiation and thus relate closely with the radiation-reaction problem.

Recently the interest in the logarithmic terms has been increasing in connection with the prediction of the evolution of the compact binary system and the correct wave forms from it. Therefore it seems urgent to investigate the effect of the logarithmic term on the equation of motion in detail.

2 Equation of Motion and the Logarithmic Term at the 3PN order

We wish to investigate these logarithmic terms in the equation of motion by the following two main reasons. One reason is that by this investigation we can know how these terms affect the evolution of the self-gravitating system, particularly the compact binary system and can predict the exact wave form from such a system which is needed in the near future to compare with the observed wave form by, say, LIGO. Another reason is that we can investigate whether our logarithmic terms are “physical” ones. In other words, it might be possible that fictitious terms appear at this order due to the harmonic coordinate condition which is a particular one among various gauge conditions. This has already

been remarked in the linearized case [9]. Therefore we investigate the effects of these logarithmic terms upon the equation of motion and the energy loss by the gravitational radiation.

We shall adopt the notation

$${}_nf(\tau, x^i) = \lim_{\epsilon \rightarrow 0} \frac{1}{n!} \frac{\partial^n}{\partial \epsilon^n} f(\tau, x^i, \epsilon) \quad (13)$$

We can deduce “the equation of motion” at the Newtonian level from $T^{\mu\nu}{}_{;\nu} = 0$.

$$\dot{\rho} + (\rho v^i)_{;i} = 0 \quad (14)$$

$$\rho \ddot{v}^i + P^{;i} = -\rho v^i v^j{}_{;j} - \rho \phi^{;i} \quad (15)$$

Here, $-1/\epsilon^2 + {}_0g_{\tau\tau}$ appears from $g_{\tau\tau}$, δ_{ij} from g_{ij} . g_{0i} does not appear yet.

At the 3PN order, ${}_6g_{ij}$ and ${}_6g_{\tau\tau}$ are needed. Among these, ${}_6g_{\tau\tau}$ is a type of an integral, $\int dr/r$, and thereby suggests a logarithmic divergence. This, however, has no dependences of the spatial coordinates and therefore vanishes due to the spatial derivative. At the 3PN order no logarithmic term appears in the equation of motion and thereby in the energy loss.

We have investigated the 3PN order under the different gauge condition [9]. We adopt the maximal slicing and the minimal distortion gauges there [10]. Although the logarithmic terms appear at the 3PN order of the lapse function in the case of the harmonic coordinate condition, such terms do not appear at the 3PN order in these conditions. This supports the above discussion in the case of the harmonic coordinate condition.

We consider the 3.5PN order. Here, g_{ij} never has the logarithmic term. $g_{\tau\tau}$ is expanded as follows

$${}_7g_{\tau\tau} \sim M \int^{\tau/\epsilon} I^{(5)} dr + M x_i \int^{\tau/\epsilon} I^{ij(5)} n_j \frac{dr}{r} + \text{finite} \quad (16)$$

where the first term is finite in the initial value approach and the second term seems to be a logarithmic term. It might be possible that the logarithmic term in ${}_7g_{\tau\tau}$ has the spatial dependence and thus remains in the equation of motion. However, this is killed by the angular integration due to the odd number of “n”. Thus no logarithmic term appears in the equation of motion at the 3.5PN order.

3 Non-Instantaneous Character of the Logarithmic Term

Then we must investigate the 4PN order. Here, we consider the meaning of the logarithmic term. The retarded integral is selected as the solution for the inhomogeneous wave equation satisfying the outgoing-radiation condition. It does not divergent by itself. The logarithmic terms arise at some higher orders from the Taylor's expansion of the retarded time, $t - \epsilon|x - y|$. The $\epsilon^k \log \epsilon$ term belongs to the C^k - class in the neighbourhood of $\epsilon \sim 0$. That is to say, the $(k+1)$ -th derivative of it does not exist. Then since we can not make an infinite series of Taylor's expansion, the so-called "remainder" is required. There remains the integral on the past-directed light cone in the four dimensional spacetime manifold. Thus "the equation of motion" is not written merely by the matter variable at one instance even if the system evolve slowly. The equation of motion is expected to take the following form,

$$\dot{\rho} + (\rho v^i)_{,i} = PN + \dots + \epsilon^8 \rho M x^k x^l \int \frac{I_{kl}^{(5)}(\tau - \epsilon r)}{r} dr \quad (17)$$

$$\rho v^i = N + \dots + \epsilon^8 \rho M x_k \int \frac{I^{ik(5)}(\tau - \epsilon r)}{r} dr \quad (18)$$

It is now urgent to seek the metrics up to 4PN and to obtain, in practice, the expressions of the energy loss in the near zone and the the energy carried out by the gravitational wave in the far zone. Calculation is now in progress and "the equation of motion" will be written down up to the 4PN order and thereby the energy loss associated with the logarithmic term will be acquired as the result in the near future.

The energy loss formula is expected to take the form,

$$\frac{dE}{dt} = (\epsilon^5) + (\epsilon^7) + \epsilon^8 M I_{ij}^{(2)} \int \frac{\mathcal{I}^{ij(5)}(\tau - \epsilon r)}{r} dr + O(\epsilon^9) \quad (19)$$

where

$$\mathcal{I}_{ij} = I_{ij} - \frac{1}{3} \delta_{ij} I,$$

and (ϵ^5) means the Quadrupole part and (ϵ^7) means the current Quadrupole and the Octapole parts. Here the tracelessness of the quadrupole moment should be expected since the gravitational radiation causes the "physical" logarithmic terms.

The following scenario is easily imagined for the origin of logarithmic terms at the 4PN order. The source has been radiating the gravitational wave in the past, producing the metric, $I_{ij}^{(2)}/r$. And the wave is scattered by the curvature caused by the gravitational potential, M/r . Then the effect propagates to the source over the past light cone. Since the above situation occurs at the every events on the past light cone, these are “integrated”.

4 Discussion

We have investigated the equation of motion to see the effect of the logarithmic term. The logarithmic terms appear at higher orders than the 3PN order in the harmonic coordinate condition. However, these terms turn out to disappear in the equation of motion at the 3PN and the 3.5PN order. The first non-trivial effect appears at the 4PN order. We can see that the logarithmic terms do not appear at the 3PN in the maximal slicing and the minimal distortion shift vectors. This supports the above discussion. Furthermore, it is found that the equation of motion is not written in terms of matter variables only at one instance.

At the just this order, Blanchet and Damour [11][12][13] have already pointed out the “hereditary” effects by the use of the combination of the Post Minkowskian approximation and the Post Newtonian approximation.

Recently, Wiseman [14] investigates the gravitational wave form from the circular-orbiting test particle by the use of their formalism. According to his calculation, the amplitude of the wave by the tail effect reaches about 40 percent of the quadrupole one. Therefore the non-linear effects including the tail effects should be investigated intensively for the precise prediction of the gravitational radiation from the coalescing compact binary system.

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Secular Instability of Rapidly Rotating Polytropes by Gravitational Radiation Reaction

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Abstract

Secular instability of rapidly rotating polytropic stellar models caused by Gravitational Radiation Reaction (GRR) has been investigated by taking account of realistic viscosities of neutron star matter. Whether the maximum angular velocity of neutron stars is limited by GRR effect has been discussed. The instability depends on the temperature of a star. In the relatively low temperature region the maximum angular velocity is limited by mass shedding condition. At higher temperatures, however, the secular instability limits the maximum rotation rate, which is the case for newly-born neutron stars. From our analysis the minimum rotation period is estimated to be about 1 msec. Although we have assumed the same physical situation as that of Ipser & Lindblom (1991), our treatment is different from theirs. Our results such as growing times of unstable modes differ from theirs. It is discussed that their treatment has some difficulties so that the present work gives the best results within the frame work of the Newtonian gravity.

1 Introduction

Since the discovery of millisecond pulsars, the maximum angular velocity of rotating neutron stars and the main cause of its existence have been investigated by several authors. In order to answer those questions, we need to consider not only the existence of rapidly rotating configurations but also their stability. There are three factors which will set limits to the rotational period: 1) mass shedding limit, 2) dynamical instability, and 3) secular instability. The first one is related to the existence of equilibrium, which should be the absolute upper limit of the maximum angular velocity. The dynamical instability is the overstable oscillation which depends on the dynamical structure of equilibrium models. Secular instability may work when there are dissipation mechanisms such as viscosities or radiations in the system considered. On the equilibrium sequence of rotating stellar models, dynamical instability occurs for models which are located on the sequence beyond the critical model for the secular instability (Tassoul[1]). Thus it is likely that the maximum rotation rate of a star is limited by the secular instability.

For highly relativistic cases such as neutron stars, the effect of Gravitational Radiation Reaction (GRR) becomes important (Chandrasekhar[2]). For uniformly rotating homogeneous spheroids, i.e. Maclaurin spheroids, this instability sets in at points on the sequence

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where stationary configurations sustained by the internal motion begin to bifurcate. The important feature of the appearance of these bifurcation points is that higher modes appear at slower rotation rate (Comins[3], Baumgart & Friedman[4]). Friedman & Schutz[5] and Friedman[6] discussed the generic nature of this instability and concluded that *all* rotating stars are secularly unstable against the GRR effect.

As Lindblom & Hiscock[7] pointed out, however, the existence of viscosity can stabilize this effect to some extent. Therefore, we need to investigate combined effect of the GRR and the viscosity to stability nature of rotating stars. Recently, Ipser & Lindblom[8] considered the effect of realistic viscosities as well as the GRR effect, and obtained detailed results for the stability limit. However there seems an improper treatment about the boundary condition on the surface in their paper so that we need to study the same problem by using different schemes.

In the present work we formulate a different scheme to this problem. Applying our new scheme to the same problem as that solved by Ipser & Lindblom, i.e. the same treatment of GRR and viscosities, we have obtained different results from theirs. We are convinced that their treatment has some defects because our results agree well with results of other calculation (Balbinski et al.[9]) but Ipser & Lindblom's results do not. Therefore, at the moment, our result in the present work is the best one available in the framework of 'Newtonian gravity + GRR correction'.

2 Formulation

2.1 Basic equations and Boundary conditions

Here we concentrate on the Newtonian polytropic stellar models. We assume the polytropic relation between the pressure and the density:

$$p = K\rho^{1+\frac{1}{N}},$$

where K is a constant to determine the strength of gravity of models.

Equilibrium configurations are obtained by the method developed by Eriguchi & M  ler[10]. Equilibrium models are assumed to be axially symmetric so that we can separate infinitesimal perturbations into modes that have harmonic dependence on the azimuthal coordinate. Thus the perturbations can be expressed as :

$$\delta f \sim \exp[-i\sigma t + im\varphi],$$

where the spherical coordinates (r, θ, φ) are used and m is an integer.

In order to include the effect of the gravitational radiation reaction, we only need to add the following gravitational radiation reaction potential to the equations of motion for the Newtonian dynamics. This has been shown by Thorne when gravitational fields are considered to be weak. For our problem, the GRR potential becomes as follows:

$$\begin{aligned} \Phi_{GRR} &= i \sum_{l=m}^{\infty} 8\pi^2 G \left(\frac{-\sigma}{c}\right)^{2l+1} \frac{(l+1)(l+2)}{l(l-1)((2l+1)!!)^2} \left(\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}\right)^{\frac{1}{2}} \\ &\quad \cdot \int \delta\rho(r', \theta') P_l^m(\cos\theta') r'^{l+2} dr' d(\cos\theta') r^l Y_{lm}(\theta, \varphi) \end{aligned}$$

As for viscosities, we adopt the same formulae as Ipser & Lindblom [8], which are obtained by Flowers & Itoh[12] and by Sawyer[13].

We use the following boundary conditions:

- Perturbed quantities vanish at the center of a star.
- Lagrangian derivative of the pressure vanishes on the surface of a star.
- Newtonian gravitational potential is regular at the center, continuous at the surface, and goes to zero at infinity.

2.2 Solving scheme for nonlinear eigenvalue problems

We are interested in the mode that becomes unstable for models with slowest rotation rate, i.e. the most nearest model to the sphere. The analysis of Maclaurin spheroids (Comins, Baumgart & Friedman) suggests that the mode that is the fastest to become unstable is so-called “ $l = m$ f-mode” (l and m denote the indices to classify the spheroidal harmonics). Therefore we concentrate on their polytropic version. For this purpose we formulate the eigenvalue problem in a little bit different way from the ordinary one.

In the linear eigenvalue problem, the number of the unknowns including the eigenvalue itself exceeds by one from that of the equations. This leads eigenvectors indefinite by a constant factor. Thus if the norm of the eigenvector is fixed, this problem becomes a proper nonlinear equation system. This nonlinear equation system can be solved by the method such as the Newton-Raphson schemes.

3 Results

3.1 Reliability of our calculation

The results obtained in this work are compared with those by others. In Table 1 we show our results together with results of two other groups. In this table eigenfrequencies of the $l = m = 2$ f-mode for spherical polytropes with $N = 1.0$ are shown.

	frequency(sec ⁻¹)	damping time(sec)
Balbinski et al. †	1.137×10^4	0.2275
Present results ‡	1.144×10^4	0.2287
Ipser & Lindblom *	1.115×10^4	0.030

Table[1] †,‡: The mass of the model is $0.802M_{\odot}$. The central density is 1.0×10^{15} g/cm³.

*: The values for the model with $1.5M_{\odot}$ are converted to those for the $0.802M_{\odot}$ model.

Balbinski et al. investigated the oscillation problem of spherical models by the standard scheme of decomposing the mode by using spherical harmonics. Thus they solved the 1D problem and adopted the quadrupole formula to estimate the damping time of oscillations. Therefore their results are very accurate. From this table it is clear that the damping time of Balbinski et al. and our value are in good agreement, but that Ipser & Lindblom’s value is different from that of Balbunski et al. by a factor 7.

This discrepancy seems to come from the improper treatment of the boundary condition in the formulation of Ipser & Lindblom. The boundary condition that they adopted on the stellar surface is unphysical because a function including “enthalpy” was extended to *vacuum region* to simplify the numerical treatment.

Good agreement of the present work with the simple 1D treatment suggests that our 2D scheme works as well as 1D code. Therefore results obtained here is the best one in the framework of the weak-field limit.

3.2 Neutron star models

We present here only the results of models with $N = 1$. The constant K is chosen so as to give a proper radius and a proper central density when the realistic mass for the model is taken. We choose its value to be $K = G \times 10^{12} (\text{g}^{-1} \text{cm}^2 \text{s}^{-1})$, where G is the gravitational constant. In this case, the mass of a spherical model is $1.4M_{\odot}$, its radius is 12.5km, and the central density is $1.11 \times 10^{15} \text{g/cm}^3$.

Figure 1 shows the temperature dependence of the imaginary part of the eigenfrequencies for the $m = 3$ mode. There are critical points where modes become unstable for temperature ranges roughly from 10^8K to 10^9K , while for lower and higher temperatures no secular instability occurs. For $m = 4$ & 5 modes, the same tendency appears but for different critical points and for different values for imaginary parts.

In Figure 2 values of the ratio of the rotational energy to the absolute value of the gravitational energy is plotted against the temperature for critical models with $m = 3, 4$ modes. As expected from Figure 1, modes cannot be stabilized at temperatures around 10^9K . It can be seen that the $m = 4$ mode is the most unstable modes compared with other modes such as $m = 3$ and $m = 5$. Therefore it is likely that the $m = 4$ mode instability sets limit to the maximum angular velocity.

4 Discussion

When we assume that models are in steady states with constant distribution of the temperature, the stability of neutron stars is determined from Figure 2. Models located above the curve showing critical angular velocity are unstable and cannot remain its permanent rotation. However, realistic neutron stars are *not* in steady states for the thermal structure. It must cool down due to a variety of microscopic processes. Therefore comparison of the cooling time with the growth time of instability is needed. We see from cooling calculations such as that by Nomoto & Tsuruta[14] that most part of the density region can be treated as isothermal at any moment of the cooling process. Thus we can simplify the argument with homogeneous temperature models.

Figure 3 displays contours of the growth time for modes with $m = 4$. Models that locate above each contour line have a lifetime shorter than the lifetime on the contour. As seen from Nomoto & Tsuruta, the cooling time at temperature $\sim 10^9 \text{K}$ is a few years which is compared with the timescale of the point A. Thus we can infer that the point A gives a good approximation for the critical angular velocity (or the critical period) of neutron stars which are thought to be born at a temperature $\sim 10^{10} \text{K}$. This critical period amounts to 1.2 msec for the present case.

5 Conclusion

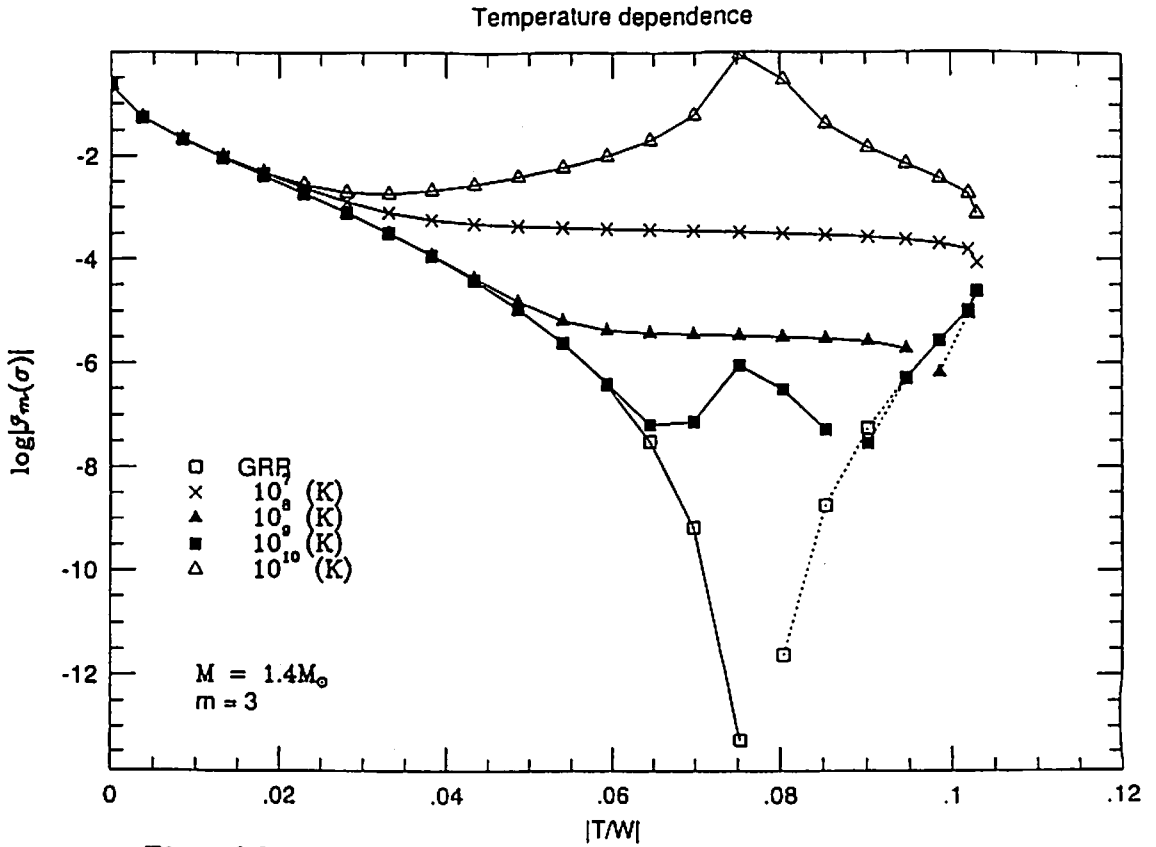
By considering both the GRR effect and the realistic viscosities, the stability of neutron star models has been investigated. It is shown that the minimum rotation period can be limited by GRR instability for the newly formed neutron stars. As for the so-called "recycled pulsars" which are thought to be produced by spin-up process from the accreted matter, GRR instability *may not* limit the rotation. In that case, the only condition that prevents neutron stars from rapid rotation must be mass shedding.

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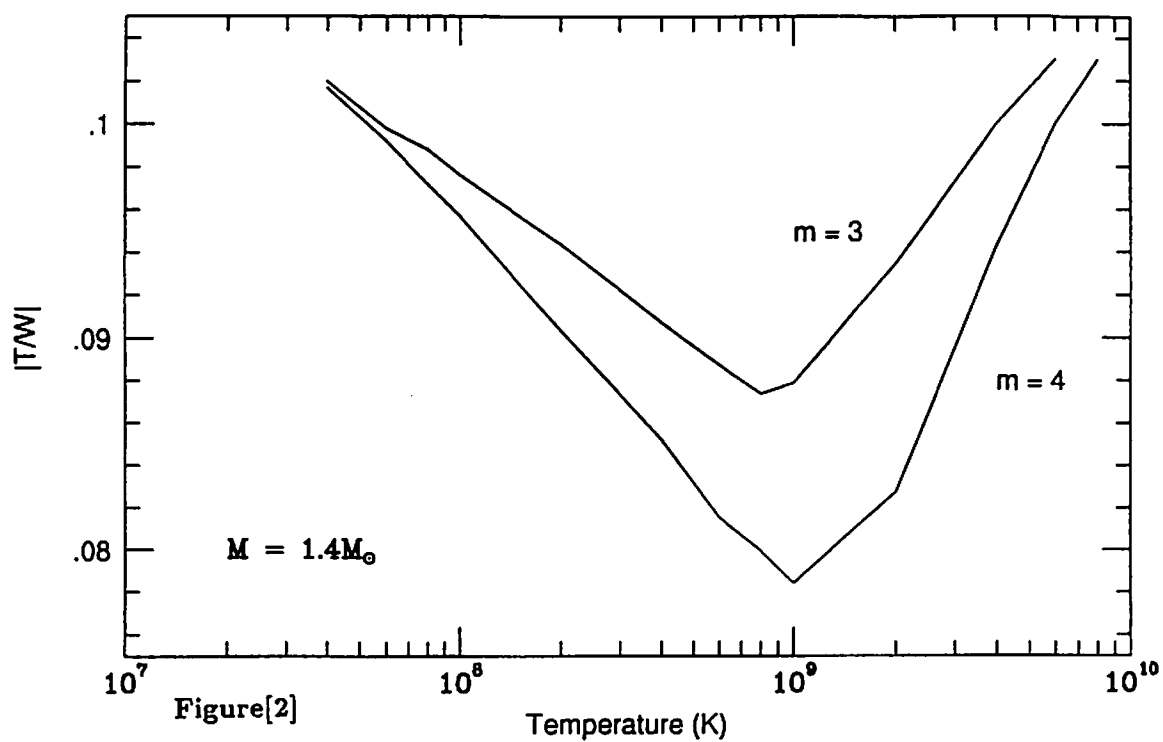
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Figure Captions

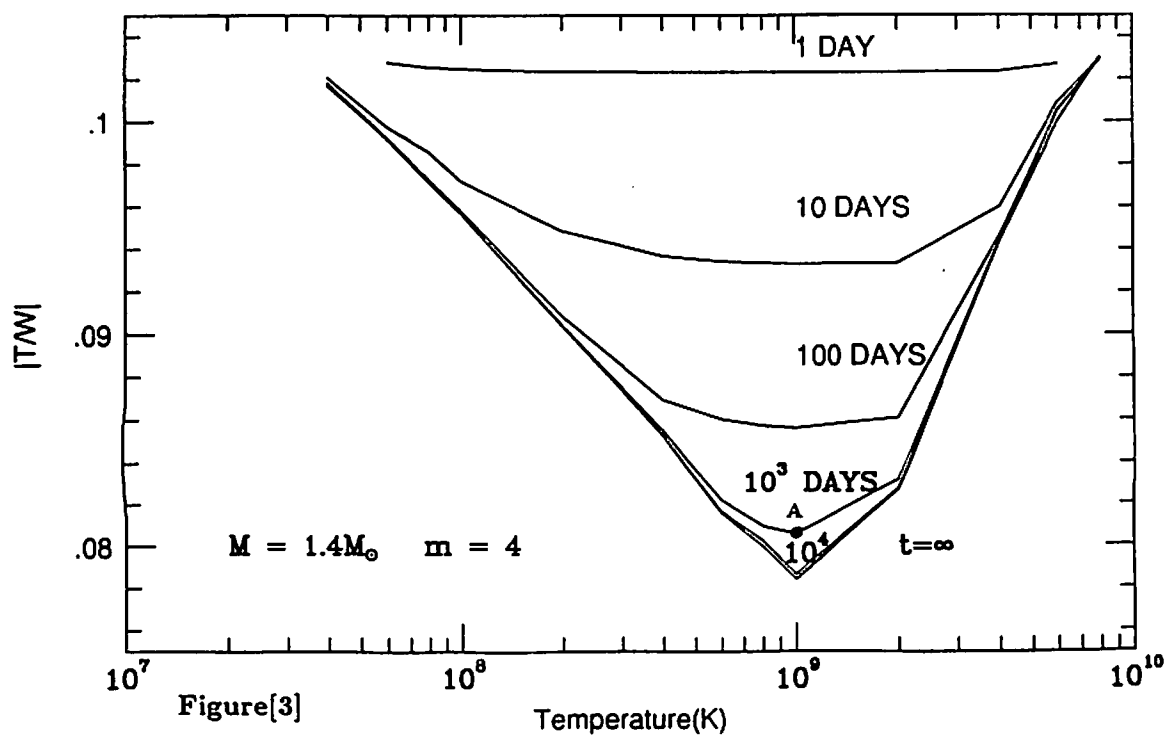
- **Figure 1:** Imaginary parts of the $m = 3$ mode eigenfrequencies with $N = 1, M = 1.4M_{\odot}$ model are plotted against the value of the ratio of the rotational energy T to the gravitational energy W . Solid lines correspond to stable branches, while dotted lines denote unstable branches. The curve with "GRR" represents models in which viscosities are not included.
- **Figure 2:** The values of $|T/W|$ at critical points where a mode becomes unstable are plotted against temperatures. The both sides of the two curves correspond to the mass shedding limit(i.e., no models are secularly unstable outside of them).
- **Figure 3:** Contours of the growth time of instability for $m = 4$ mode. The curve with $t = \infty$ curve corresponds to the $m = 4$ curve in Figure 2.



$|T/W|$ value at the critical point



growth time curve on Temperature - $|T/W|$ plane



Finding Principal Null Directions for Numerical Relativists

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abstract

We present a new method for finding principal null directions (PNDs). Because our method assumes as input the intrinsic metric and extrinsic curvature of a spacelike hypersurface, it should be particularly useful to numerical relativists. We illustrate our method by finding the PNDs of the Kastor-Traschen spacetimes, which contain arbitrarily many $Q = M$ black holes in a de Sitter back-ground.

1 Introduction

According to General Relativity, light propagating from a spherical source to a distant observer through spacetime curved by the mass-energy of intervening bodies conveys to that observer a distorted (i.e., non-circular) image of the source. The amount of observed distortion depends on both the direction and the distance to the source, and of course vanishes as that distance approaches zero. Interestingly, the distortion growth rate also vanishes in this limit, independently of the direction from which the observer approaches the source. We may sum all this up by writing $D = \frac{1}{2}C(\epsilon, \theta, \phi)\epsilon^2$, where D is a measure of the image distortion, ϵ is a measure of the distance to the source, and the coefficient $C(\epsilon, \theta, \phi)$ depends smoothly on ϵ (even at $\epsilon = 0$). Remarkably, for certain special directions (θ_i, ϕ_i) , it happens that $C(0, \theta, \phi) = 0$. In these directions, of which there can be at most four, the distortion vanishes unusually fast as the observer approaches the source. These approximately distortion-free directions are called *principal null directions* (PNDs) [1].

PNDs provide a great deal of gauge-invariant information about solutions to Einstein's equation. Indeed, in vacuum, they provide nearly as much information as the entire Riemann curvature. In the early 1960s, many researchers exploited this fact to obtain a

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large class of exact vacuum solutions with fewer than four distinct principal null directions at each point [2]. Some of these so-called algebraically special solutions were easy to interpret. Those with only one PND at each point, for example, clearly represented gravitational waves. But the significance of algebraically special solutions as a class remained somewhat obscure until Sachs proved his celebrated ‘peeling theorem’ [3]. This theorem asserts that the PNDs determined by the gravitational field of a bounded source gradually coalesce as distance from the source increases. The greater the distance, the more accurately we may approximate the field by ever more ‘special’ algebraically special solutions.

Sachs’s peeling theorem was only the first of many results linking the behavior of a spacetime’s PNDs to the physical situation that the spacetime models. More recently, for example, Arianrhod *et. al.* have shown how to infer some interesting physical characteristics of the static, cylindrically symmetric Curzon spacetime from the (admittedly somewhat complex) behavior of its PNDs [4].

Despite their proven usefulness in constructing and interpreting exact solutions to Einstein’s equation, PNDs have played almost no role in work on numerical solutions. A language barrier seems at least partly responsible for this unfortunate state of affairs. Spinors and null tetrads have traditionally been the tools of choice in work on PNDs, and neither of these tools fits naturally into the ‘3+1’ framework [6] that is the mainstay of numerical relativity.

In this paper, we present a new method for calculating PNDs tailored to the needs of numerical relativists. Our method assumes as input the induced metric and extrinsic curvature of a spacelike hypersurface, and produces as output the projections into that hypersurface of the PNDs.

In the next section, we present the details of our method, and in section 3 we apply it to obtain the PNDs of the Kastor-Traschen solutions. The result is a series of pictures analogous to those in Arianrhod *et. al.* showing how the PNDs of various Kastor-Traschen solutions vary from point to point within the spacelike hypersurfaces of a natural slicing.

2 A ‘3+1’ Method for computing PNDs

2.1 Our strategy

We consider vacuum spacetime with a cosmological constant Λ . We begin by fixing a triple (Σ, h_{ab}, p_{ab}) , where Σ is a smooth 3-manifold, h_{ab} is a Riemannian metric on Σ , and p_{ab} is an extrinsic curvature of Σ . The constraints on h_{ab} and p_{ab} are given by

$$R - p_{ab}p^{ab} + p^2 = 2\Lambda, \quad (1)$$

$$D_a(p^{ab} - ph^{ab}) = 0. \quad (2)$$

Here $p = p_{ab}h^{ab}$, D_a is the (unique) torsion-free derivative operator compatible with h_{ab} , and $R = R_{ab}h^{ab}$, where $R_{ab}v^b = -2D_{[a}D_{m]}v^m$ for all smooth v^a on Σ .

From (h_{ab}, p_{ab}) we construct two further tensor fields E_{ab}, B_{ab} as follows:

$$E_{ab} = R_{ab} - p_a{}^m p_{bm} + p p_{ab} - \frac{2}{3} \Lambda h_{ab}, \quad (3)$$

$$B_{ab} = \varepsilon_a{}^{mn} D_m p_{nb}, \quad (4)$$

where the tensor field $\varepsilon_{abc} = \varepsilon_{[abc]}$ satisfies $\varepsilon_{abc}\varepsilon^{abc} = 3!$. It follows from (1) and (2) that the fields E_{ab}, B_{ab} are both trace-free and symmetric.

The next step is to choose a unit vector field $\hat{z}^a$ on Σ , and to decompose E_{ab}, B_{ab} into components along and perpendicular to $\hat{z}^a$. We set

$$\begin{aligned} e &= E_{ab}\hat{z}^a\hat{z}^b, & b &= B_{ab}\hat{z}^a\hat{z}^b \\ e_a &= E_{bc}\hat{z}^b(\delta_a{}^c - \hat{z}_a\hat{z}^c), & b_a &= B_{bc}\hat{z}^b(\delta_a{}^c - \hat{z}_a\hat{z}^c) \\ e_{ab} &= E_{cd}(\delta_a{}^c - \hat{z}_a\hat{z}^c)(\delta_b{}^d - \hat{z}_b\hat{z}^d) + \frac{1}{2}e s_{ab}, & b_{ab} &= B_{cd}(\delta_a{}^c - \hat{z}_a\hat{z}^c)(\delta_b{}^d - \hat{z}_b\hat{z}^d) + \frac{1}{2}b s_{ab} \end{aligned} \quad (5)$$

where $s_{ab} = h_{ab} - \hat{z}_a\hat{z}_b$. Finally, we set

$$\Psi_0 = (-e_{ab} + J_a{}^c b_{bc})m^a m^b, \quad (6)$$

$$\Psi_1 = \frac{1}{\sqrt{2}}(e_a - J_a{}^c b_c)m^a, \quad (7)$$

$$\Psi_2 = \frac{1}{2}(e - ib), \quad (8)$$

$$\Psi_3 = \frac{1}{\sqrt{2}}(e_a + J_a{}^c b_c)\bar{m}^a, \quad (9)$$

$$\Psi_4 = (-e_{ab} - J_a{}^c b_{bc})\bar{m}^a \bar{m}^b. \quad (10)$$

where $J_a{}^b \equiv \varepsilon_a{}^{bc}\hat{z}_c$ is a rotation by 90 degree in the plane orthogonal to $\hat{z}^a$, and $m^a = \frac{1}{\sqrt{2}}(\hat{x}^a - i\hat{y}^a)$ for some pair of orthogonal unit vector fields that span that plane.

Given $\Psi_0, \dots, \Psi_4$, we need only solve the equation

$$\Psi_4 z^4 + 4\Psi_3 z^3 + 6\Psi_2 z^2 + 4\Psi_1 z + \Psi_0 = 0. \quad (11)$$

to get our final answer: for each root $z_k = \tan \frac{\theta_k}{2} e^{-i\phi_k}$ ($k = 1, \dots, 4$), the unit vector

$$P_{(k)}^a = \cos \theta_k \hat{z}^a + \sin \theta_k \cos \phi_k \hat{x}^a + \sin \theta_k \sin \phi_k \hat{y}^a \quad (12)$$

determines a principal null direction. By this we mean the following: at points of Σ in the 4-dimensional maximal evolution $\mathcal{M}$ of (Σ, h_{ab}, p_{ab}) , $t^a + P_{(k)}^a$ is a principal null vector for each k , where t^a is the unit normal to Σ in $\mathcal{M}$.

This completes the description of our method. In the following section, we recall the steps required to solve (11).

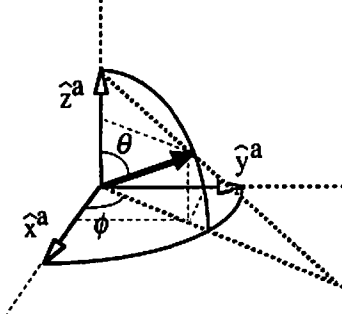


Figure 1. The solutions z_k (27)-(30) for each $k = 1, \dots, 4$ at each point determine the PND, respectively. If we write z_k as $z_k = r_k e^{-i\phi_k}$, where we fit $r_k = \tan \frac{\theta_k}{2}$, then each PND $P_{(k)}^a$ points to the direction (θ_k, ϕ_k) like the bold arrow in the figure, which will be expressed as (12).

2.2 Procedure for finding PNDs

We follow the d'Inverno-Russel-Clark method [5] to find the solutions z_i of (11). Upon setting $y = \Psi_4 z + \Psi_3$, (11) becomes

$$y^4 + 6Hy^2 + 4Gy + K = 0, \quad (13)$$

where H, G and K stand for the following combinations of the complex numbers $\Psi_0, \dots, \Psi_4$:

$$I \equiv \Psi_4 \Psi_0 - 4\Psi_1 \Psi_3 + 3\Psi_2^2, \quad (14)$$

$$J \equiv \det \begin{vmatrix} \Psi_4 & \Psi_3 & \Psi_2 \\ \Psi_3 & \Psi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \Psi_0 \end{vmatrix}, \quad (15)$$

$$K \equiv \Psi_4^2 I - 3H^2, \quad (16)$$

$$H \equiv \Psi_4 \Psi_2 - \Psi_3^2, \quad (17)$$

$$\text{and } G \equiv \Psi_4^2 \Psi_1 - 3\Psi_4 \Psi_3 \Psi_2 + 2\Psi_3^3. \quad (18)$$

The solutions of (13) are simply expressed in terms of the solutions of

$$\lambda^3 - I\lambda + 2J = 0, \quad (19)$$

which are

$$\lambda_1 = -(P + \frac{I}{3P}), \quad (20)$$

$$\lambda_2 = -(e^{i2\pi/3} P + e^{i4\pi/3} \frac{I}{3P}), \quad (21)$$

$$\lambda_3 = -(e^{i4\pi/3} P + e^{i2\pi/3} \frac{I}{3P}) \quad (22)$$

where $P = \{J + \sqrt{J^2 - (I/3)^3}\}^{1/3}$. From the λ_i ($i = 1, 2, 3$) we now determine three further complex numbers, α, β, γ , using the following equations:

$$\alpha^2 = 2\Psi_4\lambda_1 - 4H, \quad (23)$$

$$\beta^2 = 2\Psi_4\lambda_2 - 4H, \quad (24)$$

$$\gamma^2 = \alpha^2 + \beta^2 + 4H, \quad (25)$$

$$\text{and } \alpha\beta\gamma = 4G \quad (26)$$

(23) - (25) determine α, β, γ up to sign, and (26) determines the signs. Then the solutions of (13) are given as $\frac{1}{2}(\alpha + \beta + \gamma)$, $\frac{1}{2}(\alpha - \beta - \gamma)$, $\frac{1}{2}(-\alpha + \beta - \gamma)$ and $\frac{1}{2}(-\alpha - \beta + \gamma)$. Finally, from α, β, γ , we obtain the following four complex numbers:

$$z_1 = -\{\Psi_3 + \frac{1}{2}(\alpha + \beta + \gamma)\}/\Psi_4, \quad (27)$$

$$z_2 = -\{\Psi_3 + \frac{1}{2}(\alpha - \beta - \gamma)\}/\Psi_4, \quad (28)$$

$$z_3 = -\{\Psi_3 + \frac{1}{2}(-\alpha + \beta - \gamma)\}/\Psi_4, \quad (29)$$

$$z_4 = -\{\Psi_3 + \frac{1}{2}(-\alpha - \beta + \gamma)\}/\Psi_4. \quad (30)$$

These numbers are the solutions of (11).

3 PNDs of the Kastor-Traschen solutions

3.1 Kastor-Traschen solutions

The Kastor-Traschen (KT) solutions [7] to the Einstein's equation with cosmological constant represent arbitrary many $Q = M$ black holes that participate in an overall de Sitter expansion or contraction. In the $\Lambda \rightarrow 0$ limit, the KT solutions reduce to the Majumdar-Papapetrou solutions, in which the balance between gravitational attraction and electrostatic repulsion among the black holes causes each to maintain its position relative to the others eternally. To write the KT metric, we first choose $(x_i, y_i, z_i) \in \mathbf{R}^3$, $i = 1, 2, \dots, N$, and set $r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}$. Then

$$ds^2 = -\frac{1}{\Omega^2}dt^2 + a(t)^2\Omega^2(dx^2 + dy^2 + dz^2), \quad (31)$$

$$\text{where } \Omega = 1 + \sum_{i=1}^N \frac{M_i}{ar_i}, \quad a = e^{Ht} \quad \text{and} \quad H = \pm\sqrt{\frac{\Lambda}{3}}.$$

Naively, we interpret M_i as the mass of the i th black hole, although we have neither an asymptotically flat region nor event horizons available to convert this naive interpretation into a rigorous one.

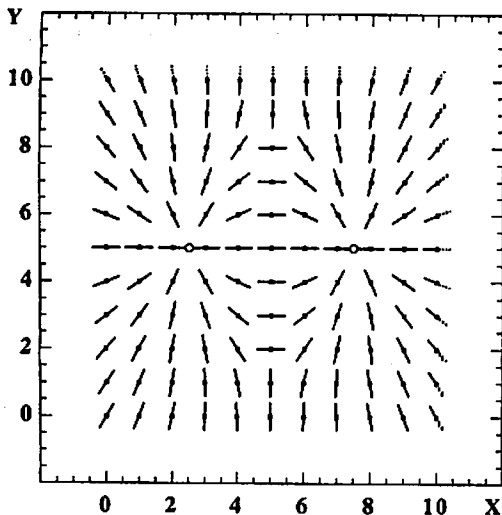


Figure 2. PNDs of the Kastor-Traschen solution for the case of 2 black holes of equal mass ($M_1 = M_2 = 1$) separated by a coordinate distance $d = 5$.

3.2 PNDs of Kastor-Traschen solutions

To apply the method of section 2 to the KT solutions, we must first choose spacelike hypersurfaces Σ and orthonormal triads $\hat{x}^a, \hat{y}^a, \hat{z}^a$ tangent to these Σ . The natural choice would seem to be $t = \text{const.}$ surfaces for Σ , and

$$\hat{x}^a = \frac{1}{a\Omega} \left(\frac{\partial}{\partial x} \right)^a, \quad \hat{y}^a = \frac{1}{a\Omega} \left(\frac{\partial}{\partial y} \right)^a, \quad \hat{z}^a = \frac{1}{a\Omega} \left(\frac{\partial}{\partial z} \right)^a \quad (32)$$

for triad. We keep this choice in force throughout in what follows.

We now select various values of the free parameters (x_i, y_i, z_i) and M_i appearing in the KT metric (31), and plot the PNDs our method generates. For simplicity, we take all $z_i = 0$ — that is, we confine all black holes to the $x - y$ coordinate plane — and plot the projections of the PNDs into that plane.

Fig.2 shows the PNDs of a KT solution containing 2 black holes of equal mass ($M_1 = M_2 = 1$) and separated by a coordinate distance $d = |x_1 - x_2| = 5$. Four PNDs are present at each point, but because these four PNDs coincide in pairs, with the coincident pairs pointing in opposite directions, we get a picture that looks like the electric field lines of equal charges. This indicates the spacetime is Petrov type D at all points. In Fig.3, we show the case of 2 black holes with different masses ($M_1 : M_2 = 1 : 10$) separated by (a) $d = 5$, (b) $d = 2$ and (c) $d = 0.5$. These cases have none of the symmetry of Fig.2, and as a result, the field of PNDs is algebraically general (Petrov type I). However, we also see that the spacetime near each black hole is almost Petrov type D, as is a single black hole, and that at great distances from the holes the PNDs exhibit the same almost type-D behavior. This is just as we would expect. Moreover, we see that the PNDs on the axis connecting the two black holes is Petrov type D.

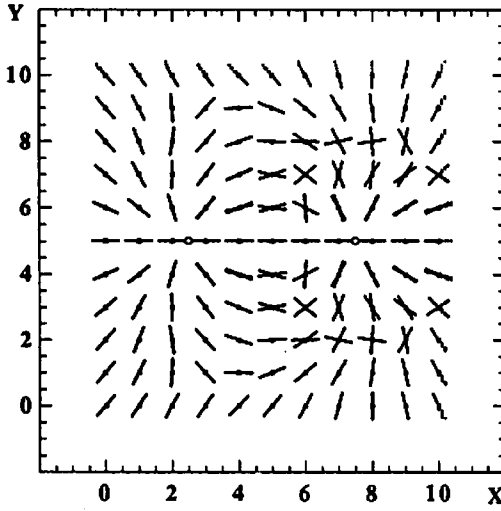


Fig. 3 (a)

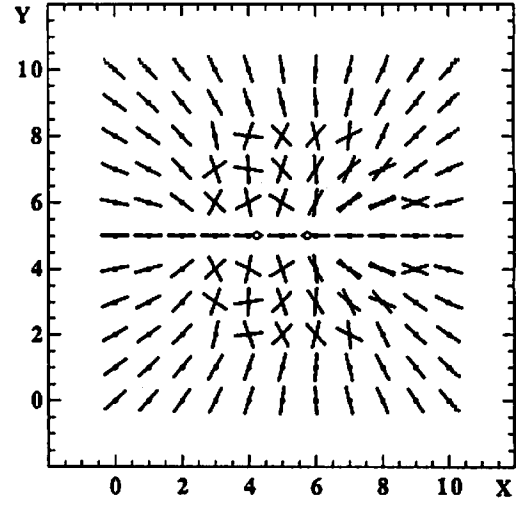


Fig. 3 (b)

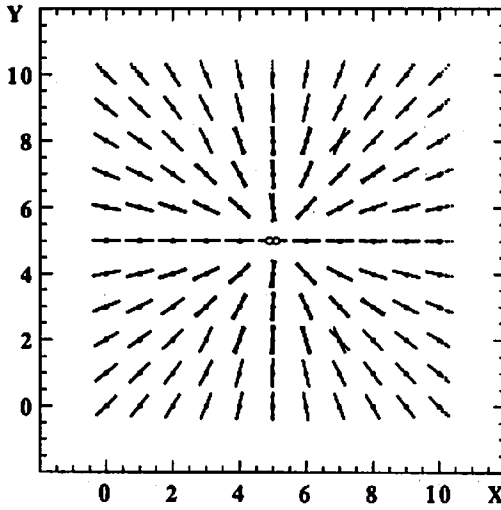


Fig. 3 (c)

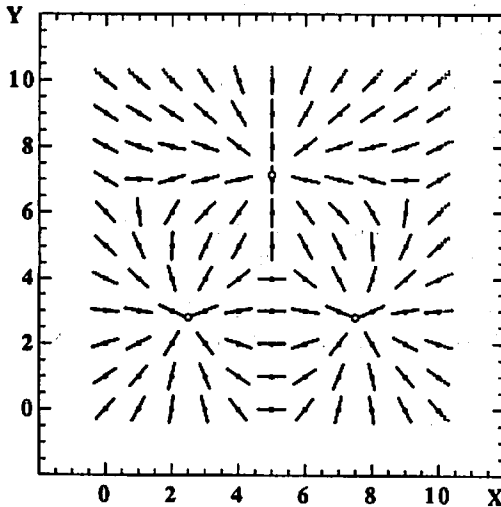


Fig. 4

Figure 3. PNDs of KT solution containing 2 black holes with different masses ($M_1 : M_2 = 1 : 10$) separated by a distance (a) $d = 5$, (b) $d = 2$ and (c) $d = 0.5$. In these cases, the symmetry of Fig.2 is absent, we see that the field is algebraically general (Petrov type I). We also see, however, that the space-time near each black hole and along the axis joining the two black holes is almost Petrov type D, and in regions far from the black holes, the spacetimes are also almost Petrov type D, like a single black hole.

Figure 4. PNDs of a KT solution with 3 black holes of equal mass at the vertices of an equilateral triangle.

We can also draw the pattern of PNDs even for KT solutions containing many black holes. As an example, we show in Fig.4, the PNDs of 3 black holes of equal masses located at the vertices of an equilateral triangle.

4 Conclusions and discussion

The results of the previous section illustrate the advantages of plotting PNDs over other methods of assessing how nearly algebraically special a spacetime is. Particularly when the spacetime is known only approximately, other methods are hard to apply, for they require us to determine whether certain scalar combinations of curvature components (e.g., $I^3 - 6J^2$) are exactly zero.

We hope the method presented in this paper will help numerical relativists to deal with at least two subtle issues. First, there is the problem of how to set initial data representing bounded sources free of incoming radiation. Because Sachs's peeling theorem assumes no incoming radiation, any failure of the PNDs computed from (Σ, h_{ab}, p_{ab}) to exhibit the proper peeling behavior shows that (Σ, h_{ab}, p_{ab}) is an unsatisfactory model of the instantaneous state of an isolated system. Our method thus allows us to recognize and reject initial data cluttered with spurious incoming radiation. This should be of some help. Second, there is the problem of detecting spurious gravitational waves in numerically generated spacetimes that result from physically inappropriate boundary conditions. We hope that such waves will manifest themselves in a pathological time dependence of the PNDs near numerical boundaries.

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Critical Behavior in Gravitational Collapse of a Massless Scalar Field

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We study a critical behavior of black hole formation using a self-similar model of spherically symmetric massless scalar field with a parameter p . The solution exhibits a critical behavior which found by Choptuik. The black hole formation is explicitly shown to occur only in the strong-field implosion of $p > 1$. The field evolution in the critical limit $p \rightarrow 1$ is discussed.

Gravitational collapse is one of important research areas in general relativity. In recent years, a large number of studies have been made on the dynamical behavior of a massless scalar field, in the context of spherically symmetric gravitational collapse. Fundamental question is that what initial conditions of gravitational collapse produces black hole. It has been shown that for a weak-field implosion scalar waves bounce and disperse to infinity, while a strong-field implosion produce black hole [1]. Furthermore, in numerical calculations, Choptuik [2] found an important behavior of a family of solutions which contain the parameter p characterizing the strength of the scalar field: For black hole formation there exists the critical limit $p \rightarrow p^*$ to give the gravitational mass satisfying a power law $M_{BH} \propto |p - p^*|^\gamma$ with a universal critical exponent $\gamma \simeq 0.37$. Such a critical behavior may be a general property of gravitational collapse of massless field, since imploding axisymmetric gravitational waves also show the similar behavior [3].

Our main purpose is to analyze the critical behavior of the spherically symmetric collapse of scalar waves. We present the self-similar solution of the spherically symmetric Einstein equations coupled to a massless scalar

field ψ which has a critical parameter p . This model explicitly show the black hole formation and the scalar field dispersion.

Let us impose a self-similarity on the massless scalar field such that $\psi(u, v) = \psi(u/v)$, where u and v are retarded and advanced times respectively. The line element in the spherical symmetric double-null coordinates is given by

$$ds^2 = -hdudv + r^2 d\Omega^2, \quad (1)$$

and we assume the metric to be $h(u, v) = h(u/v)$ and $r(u, v) = vf(u/v)$. For the energy-momentum tensor of the massless scalar field defined by

$$T_{\mu\nu} = \frac{1}{4\pi} (\partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \psi \partial_\alpha \psi), \quad (2)$$

the Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ give the unique solution as follows,

$$h = 1, \quad (3)$$

$$r^2 = \frac{1}{4} \{ (1 - p^2)v^2 - 2vu + u^2 \}, \quad (4)$$

$$\psi = \pm \frac{1}{2} \log \frac{(1 - p)v - u}{(1 + p)v - u}, \quad (5)$$

where the constant p can be chosen to be positive without loss of generality. We use units such that $G = c = 1$. This solution was previously discussed in the context of cosmic censorship [4]. The space-time is asymptotically flat at the past null infinity $u \rightarrow -\infty$. The scalar waves implode from the past null infinity with $\psi \simeq \mp pv/r$. The parameter p turns out to give the amplitude of the imploding scalar waves. This solution is useful to analyze the scalar wave collapse, if a pathological structure due to naked singularities is removed from the space-time. Because the Ricci scalar R has the form

$$R = \frac{p^2 uv}{2r^4}, \quad (6)$$

the curvature singularity is located at the origin $r = 0$. The space-time structure is crucially dependent on the parameter p . For $p > 1$ the singularity becomes time-like in the region $v < 0$ and spacelike in the region $v > 0$, while for $0 < p < 1$ it always becomes timelike. For $p = 1$ exactly, the curvature singularity at $r = 0$ in the region $v > 0$ becomes null. We do not consider

this case, and we will take the critical limit $p \rightarrow 1$ by keeping the condition $p > 1$.

The important point is that the apparent horizon exists only in the strong-field implosion of $p > 1$. In fact we obtain the expansion θ for outgoing null rays given by

$$\theta = \frac{\{(1 - p^2)v - u\}}{\{(1 - p)v - u\}\{(1 + p)v + u\}} \quad (7)$$

and the apparent horizon where $\theta = 0$ is found to be $u = (1 - p^2)v$. Thus the space-like singularity is hidden by the apparent horizon, while time-like one is not so. This result will be due to the gravitational mass M which can be locally defined in the spherically symmetric system as follows,

$$M \equiv \frac{r}{2}(1 - g^{\mu\nu}\partial_\mu r \partial_\nu r) = -\frac{p^2 uv}{8r}. \quad (8)$$

Near the singularity $u = (1 - p)v$ in the region $v > 0$, Eq. (8) is reduced to $M \simeq p^2(p - 1)v^2/r$. The mass turns out to be negative near the time-like singularity and positive near the space-like one. For a realistic evolution we must avoid such a singularity to set some regular initial data. Then we replace the region $v < 0$ by a flat space-time, which is smoothly matched to the region $v > 0$ (see. Fig 2). (There is no energy flux across the $v = 0$ line. and the mass is equal to zero on it.)

For the weak-field implosion ($0 < p < 1$) shown in Fig. 2a, the region corresponding to $u > 0$ and $v > 0$ is also replaced by a flat metric. This is the case that the dynamical scattering of the imploding waves can induce a complete bounce to the future null infinity. For the strong-field implosion ($p > 1$) shown in Fig. 2b. The self-similar model fails to give an asymptotically flat region at the future null infinity. Therefore, for gravitational collapse of wave packets, the self-similar solution is approximately valid in a strong-field region of $0 \leq v < v_0$.

Now let us consider the critical limit $p \rightarrow 1$ for black hole formation. To discuss a power law, we estimate the gravitational mass M_h on the apparent horizon. From Eq. (8) we obtain $M_h = p\sqrt{p^2 - 1}v/4$. Thus the black hole mass M_h estimated on the apparent horizon vanishes in the limit $p \rightarrow 1$, which gives the power law

$$M_h \simeq \left(\frac{p - 1}{8}\right)^{\frac{1}{2}} v. \quad (9)$$

This is slightly different from the power law with the critical exponent $\gamma^* \simeq 0.37$. We wish to emphasize that Eq. (9) exhibits a mass evolution in some dynamical stage of gravitational collapse instead of the asymptotic stage $v \rightarrow \infty$. If the self-similarity approximately holds in the stage $0 < v < v_0$ even for the wave-packet collapse shown in Fig. 3, the final mass will be given by Eq. (9) with v_0 in place of v . Then, to obtain the critical exponent $\gamma^* \simeq 0.37$, the advanced time v_0 should increase like $v_0 \propto |p - 1|^{-0.13}$ in the limit $p \rightarrow 1$. For the near-critical evolution the self-similar stage may continue for a long time.

We can point out another critical behavior in relation to the numerical result. Now let us introduce the time coordinate t defined by

$$\log(t/r) = - \int \left[\frac{\sqrt{p^2 X^2 + 4} - p^2 X}{2 - X\sqrt{p^2 X^2 + 4} + p^2 X^2} \right] dX, \quad (10)$$

where $X \equiv v/r$. The line element can be rewritten into the diagonal form

$$ds^2 = -\alpha(X)dt^2 + \beta(X)dr^2 + r^2 d\Omega^2, \quad (11)$$

where the metric coefficients $\alpha(X)$ and $\beta(X)$ are

$$\alpha(X) = \left(1 - \frac{p^2 X}{\sqrt{p^2 X^2 + 4}}\right) \left[\frac{2 - X\sqrt{p^2 X^2 + 4} + p^2 X^2}{\sqrt{p^2 X^2 + 4} - p^2 X} \right]^2 \times \exp\left(2 \int \left[\frac{\sqrt{p^2 X^2 + 4} - p^2 X}{2 - X\sqrt{p^2 X^2 + 4} + p^2 X^2} \right] dX\right), \quad (12)$$

$$\beta(X) = \frac{4}{\sqrt{p^2 X^2 + 4}(\sqrt{p^2 X^2 + 4} - p^2 X)}, \quad (13)$$

The scalar field ψ also depends on X only,

$$\psi = \pm \log \frac{1}{4} (\sqrt{p^2 X^2 + 4} + pX). \quad (14)$$

The variable X satisfies $X^2 < X_h^2 \equiv 4/p^2(p^2 - 1)$ outside of the apparent horizon. Then, in the critical limit $p \rightarrow 1$, there exists the strong-field region $1 \ll X^2 \ll X_h^2$ in a neighborhood of $r = 0$, where Eq. (10) gives the logarithmic relation

$$\frac{1}{(p-1)X^2} = \log t - \log r. \quad (15)$$

Note that the time "t" proportional to the proper time of a central observer which defined by $T = \int \sqrt{|g_{tt}(t, 0)|} dt \simeq \sqrt{|p-1|} t$. For the self-similar solution the scaling relation $\psi(\log r - \Delta, \log t - \Delta) = \psi(\log r, \log t)$ clearly hold with a continuous parameter Δ . For the wave-packet collapse, however, the scaling parameter Δ was found to be restricted to some discrete values $n\Delta^*$ ($\Delta^* \simeq 3.4$ and $n = 1, 2, 3, \dots$). This difference of the scaling will not be crucial, if we consider the region where both $\log r$ and $\log t$ are much larger than Δ^* . ($(\log r - \Delta^*)/\log r \simeq 1$.)

In summary, a self-similar model can give a simple picture of black hole formation. We obtain the critical behaviors which exhibit the power law and the logarithmic evolution. The scaling relation means that the self-similarity may appear even in wave-packet collapse with the parameter p sufficiently close to the critical value and can represent a dynamical growth of the apparent horizon. It is necessary to check the self-similar critical behaviors in numerical calculations of wave-packet collapse.

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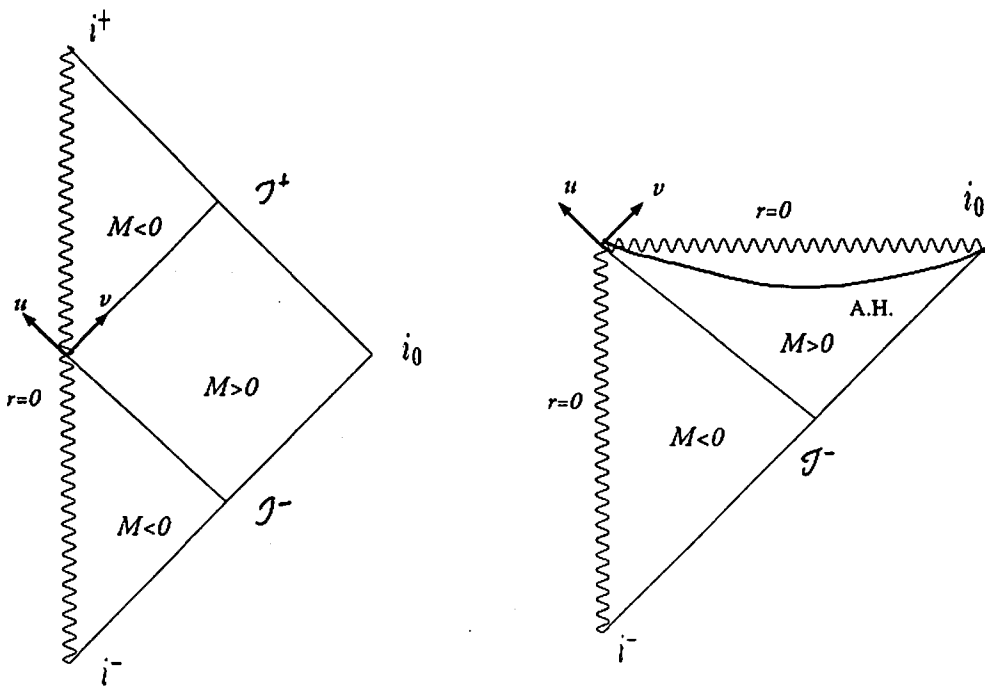


Fig. 1.

Penrose diagram for the self-similar solutions covering the whole space-time. The scalar waves implode from the past null infinity $\mathcal{J}^-$ with the amplitude denoted by the parameter p : (a) For the weak-field implosion $0 < p < 1$ the time-like singularity appears in the negative mass regions of $uv > 0$. (b) For the strong-field implosion $p > 1$ the apparent horizon (A.H.) is formed to surround the space-like singularity in the positive mass region. This space-time has no future null infinity $\mathcal{J}^+$ at $v \rightarrow \infty$.

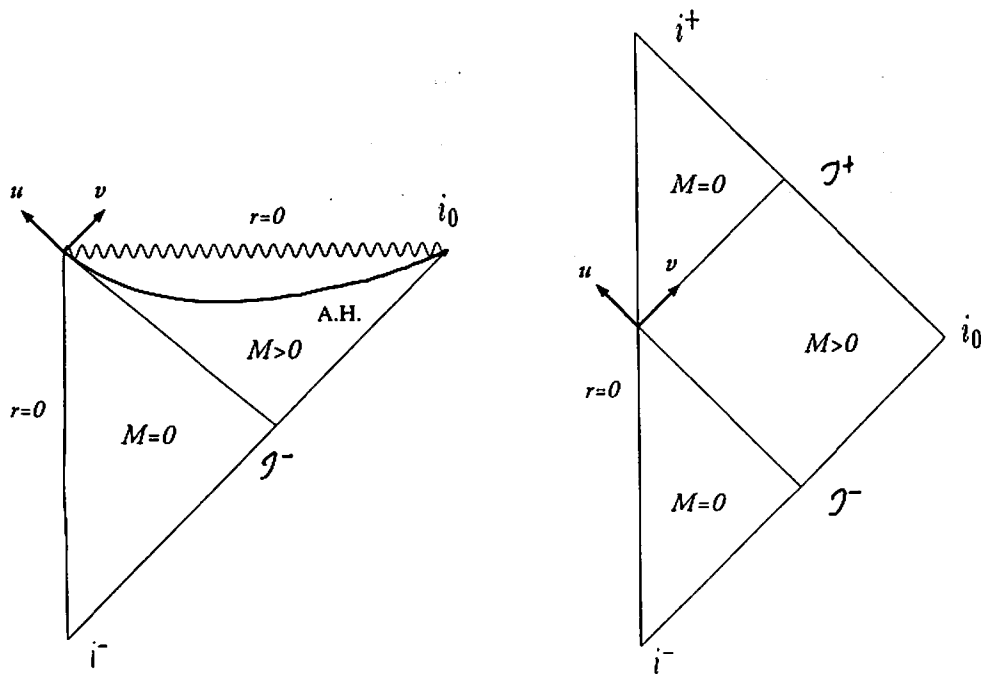


Fig. 2.

Penrose diagram for the self-similar solution, in which the negative mass region are replaced by the flat space-time with $M = 0$.

(a) For $0 < p < 1$ all the imploding waves bounce and disperse to the future null infinity $\mathcal{J}^+$. (b) For $p > 1$ we obtain a model of black hole formation due to the scalar wave collapse, though the mass infinitely increases as $v \rightarrow \infty$.

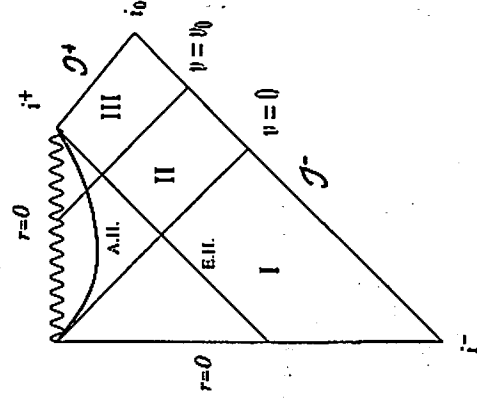


Fig. 3
 Penrose diagram for black hole formation due to wave-packet collapse. The region I ($v < 0$) is the flat space-time, and in the region III ($v > v_0$) outgoing waves dominate incoming ones. The self-similarity will remain valid in the region II ($0 < v < v_0$).

Instability of Naked Singularities

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18 January

1 Introduction

Recently many candidates of counterexample to cosmic censorship conjecture are found in spherically symmetric systems[1, 2, 3, 4], in which naked shell focussing singularities develop from regular initial data. One approach to save the cosmic censorship is to take account of an equation of state for matter. This approach has been partially successful. It is shown that for a hard equation of state the pressure effects prevent the formation of naked central singularities. For a soft equation of state, however, we observe a violation of cosmic censorship again. Alternative approach to save the cosmic censorship is therefore strongly needed. One possible approach is to consider the effects of the non-spherical symmetry of gravitational collapse. Non-spherical perturbations of a collapsing body which develops into a naked central singularity may generate a large amount of gravitational radiation. Such kind of unstable feature in a space-time may prevent the formation of naked singularity[5]. (For Weyl space-time Hara[6] found infinite flux of gravitational radiation from a freely falling particle, and Haugan and Liang[7] discussed the stability of nakedness of a singularity.)

In this paper we discuss this possible answer to cosmic censorship by applying the gauge invariant perturbation theory to the Lemaître-Tolman-Bondi (LTB) solution[2, 8, 9, 10] which has a naked central singularity.

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2 The basic equation

The perturbation equation in general spherically symmetric space-time is derived by Gerlach and Sengupta[11]. In this section we only present the perturbation equation for the LTB spacetime.

The background metric is given by

$$ds^2 = -dt^2 + b(t, R)^2 dR^2 + r(t, R)^2 d\Omega^2, \quad (1)$$

$$b(t, R) = \frac{r(t, R)'}{\sqrt{1 - 2m(R)/R}}, \quad (2)$$

$$r(t, R) = \frac{R}{2}(1 + \cos \eta), \quad (3)$$

$$t = \left(\frac{R^3}{8m(R)} \right)^{\frac{1}{2}} (\eta + \sin \eta), \quad (4)$$

where $m(R)$ is a positive non-decreasing function of R , which describes the mass distribution. In this paper we set $m(R)$ as $\rho(0, R) = m'(R)/4\pi R^2 = \rho_0 \exp [-(R/20M_\odot)^2]$, and the total mass of collapsing body is $1M_\odot$. Then, the basic equations for odd parity perturbations of in the LTB solution become

$$-\dot{h}_0 + \frac{1}{b^2}h_1' - \frac{\dot{b}}{b}h_0 - \frac{b'}{b^3}h_1 = 0, \quad (5)$$

$$\frac{1}{b^2 r^2} \left(\left(r^4 \left(\frac{h_0}{r^2} \right)' \right)' - \left(r^4 \left(\frac{h_1}{r^2} \right)' \right)' \right) + \frac{b'}{b} r^2 \left(\left(\frac{h_1}{r^2} \right)' - \left(\frac{h_0}{r^2} \right)' \right) = \frac{(l-1)(l+2)}{r^2} h_0 - 8\pi L_0, \quad (6)$$

$$\frac{1}{r^2} \left(\left(r^4 \left(\frac{h_1}{r^2} \right)' \right)' - \left(r^4 \left(\frac{h_0}{r^2} \right)' \right)' \right) + \frac{\dot{b}}{b} r^2 \left(\left(\frac{h_0}{r^2} \right)' - \left(\frac{h_1}{r^2} \right)' \right) = -\frac{(l-1)(l+2)}{r^2} h_1, \quad (7)$$

$$-(r^2 L_0)' + \frac{\dot{b}}{b}(r^2 L_0) = 0, \quad (8)$$

where $h_0(t, R)$ and $h_1(t, R)$ are the same as the gauge invariant perturbation variables of Regge-Wheeler[12], and $L_0(t, R)$ describes the velocity perturbation of a collapsing body. In the next section we numerically solve above equations for $l = 2$ and discuss the effects of non-spherical symmetry of a collapse.

3 Results

We first solve the equations (5)–(8) without matter perturbation ($L_0 = 0$). The initial wave patterns for h_0 ($= h_1$) at $t = 0$ are plotted in Fig.1. The initial conditions for $\dot{h}_0$ and

h_1 are determined by the equations (5) and (6). In Fig.2 we show the evolution of h_1 , where x-axis is the comoving radius R , y-axis is the time t , and z-axis is the amplitude of a metric perturbation, h_1 . The variable, h_1 , means the gravitational wave expanded with spherical harmonics. Fig.2 tells us that the initial gravitational wave located near the center soon radiates away and that the amplification of the gravitational wave at the last stage of collapse is very small. In Fig.3 we show the evolution of h_0 . The amplitude of h_0 diverges in the last stage of collapse, however, this divergence does not necessarily imply the instability of the space-time. The quantity, h_0 , is very similar to the Kerr parameter a and h_0 is related to the Kerr parameter a by the equation,

$$a \sim \frac{h_0 r}{m} \sim h_0 b. \quad (9)$$

In Fig.4 we show this effective Kerr parameter. (Here we exchange x-axis and y-axis in the Figure for convinience.) We can see that the Kerr parameter does have a very stable behavior.

We then solve the equations (5)-(8) with matter perturbation. Initial condition for L_0 is given by $L_0(0, R)/R^2 = h_0(0, R)$. Fig 5 and Fig.6 show the evolution of h_1 and h_0 , respectively. The qualitative behavior is similar to the case without matter perturbation though the amplitude of perturbations is larger than the previous one. Therefore the LTB space-time may be stable to the odd parity perturbations.

4 Discussion

We applied the gauge invariant perturbation theory to the LTB space-time and survey the instability of a naked singularity in the LTB space-time. For odd parity perturbations we could not find any unstable features of the space-time. However, the even parity perturbation seems to have unstable behaviors, because the energy density generated by matter perturbations will dominate the background energy density if we can ignore the gravitational wave production from the matter perturbations. Therefore the analysis of the even parity perturbations is important to examine the instability of a naked singularity in the LTB space-time.

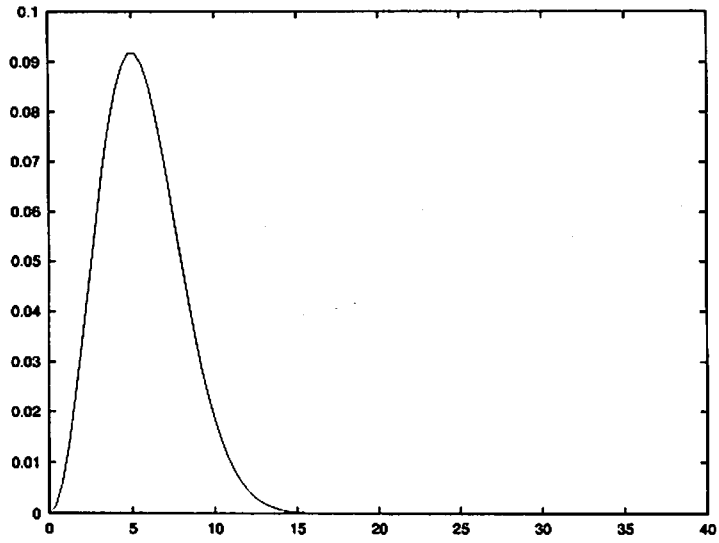


Figure 1: The initial metric perturbation amplitude.

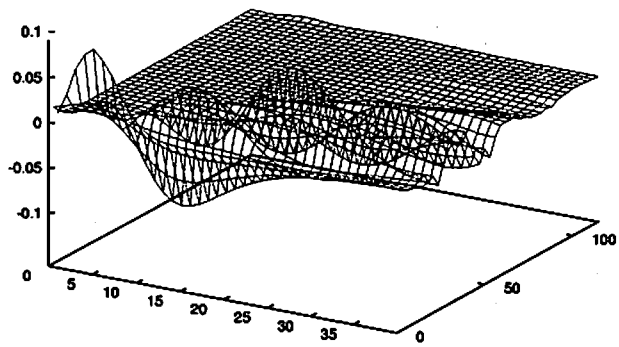


Figure 2: The evolution of h_1 .

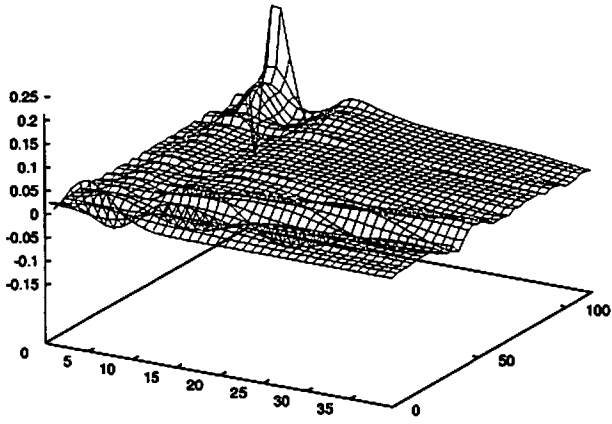


Figure 3: The evolution of h_0 .

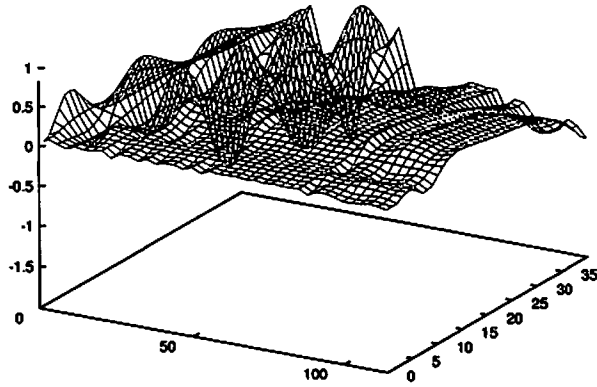


Figure 4: The evolution of the effective Kerr parameter.

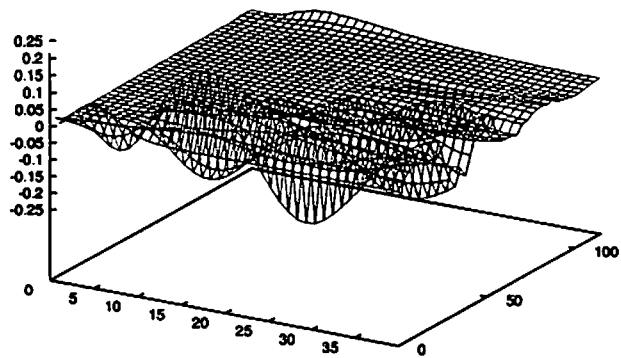


Figure 5: The evolution of h_1 .

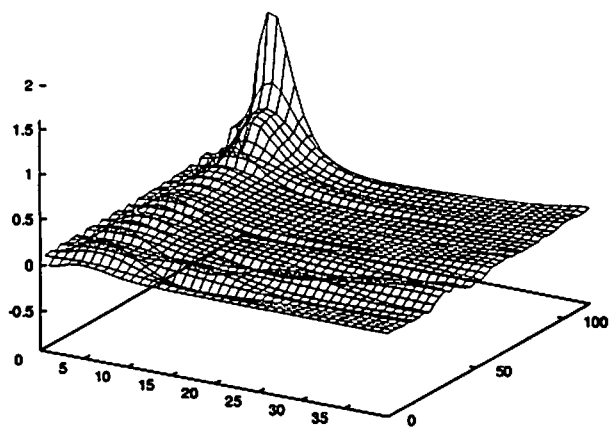


Figure 6: The evolution of h_0 .

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Current Status and Future of Gravitational Wave Detector in Japan

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Abstract

Several experiments are being continued aiming at developing gravitational wave detectors in Japan. I will report here the current status of the experiments and the future plan for the next step of the detectors.

1 Introduction

The direct detection of gravitational wave (GW) has been a dream of experimental physicists, so far. Now, two big projects to realize a “Gravitational Wave Observatory” have been approved; the projected sensitivity will be enough to catch a signal coming from drastic astrophysical events occurring within the Virgo cluster or at a farther distance. The main target is a coalescence of binary neutron stars which will be observed a few times in a year by the future detector. In Japan, the basic research project on this subject started in 1991, led by late Professor Satio Hayakawa; this project is funded by the Ministry of Education, Science and Culture, entitled “Gravitational Wave Astronomy”.

First, I will explain briefly what the interferometric GW detector is. After that, the current status of the experiments is reported. Finally, I will show the future plan for the next project in Japan

2 Interferometric GW detector

A schematic of an interferometric GW detector is shown in Fig. 1. All of optical elements such as mirrors and a beam splitter are suspended as a pendulum. These objects behave as a free mass for an incident GW of which the frequency is much higher than that of the pendulum resonance. The equivalent displacement induced by the GW (amplitude h with an appropriate polarization) is approximately given by

$$\delta L \sim \frac{1}{2} h L \quad (1)$$

for each arm (L is the length of the arm). Since the displacements in x and y arms are opposite in sign, the signal appearing in the change of the fringe $\delta\phi$ is presented as

$$\delta\phi = 2 \frac{4\pi\delta L}{\lambda} = \frac{4\pi L}{\lambda} h, \quad (2)$$

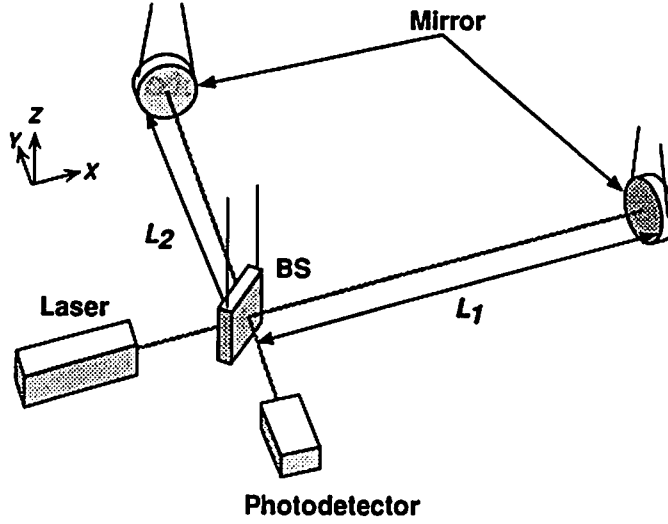


Figure 1: Schematic view of an interferometric gravitational wave detector; this is a Michelson interferometer, where all of the optical elements such as mirrors and a beam splitter are suspended as a pendulum. Both arms have nearly same length ($L_1 \sim L_2 \sim L$).

where λ is wavelength of the light. The optimum arm length L_{opt} is determined as

$$L_{opt} = \frac{c}{4f_{GW}}, \quad (3)$$

where c is the speed of light and f_{GW} is the frequency of the GW to be detected. For instance, if one wish to detect a GW of 1kHz, the arm length should be chosen as 75km. Since it is almost impossible to find a place for such an interferometer (even in U. S. A. or Australia), the light should be instead bounced back and forth, many times between two mirrors set in each arm. This is achieved by using a delay-line or a Fabry-Perot (FP) cavity (Fig. 2); adopting FP is, now, a trend in this business.

The fundamental noise sources of the detector are listed as follows:

- Seismic Noise ($\sim 10\text{Hz}$),
- Thermal Noise ($10\text{Hz} \sim 100\text{Hz}$),
- Shot Noise ($100\text{Hz} \sim$).

In order to improve the sensitivity of the detector, the following items are indispensable: an elaborated vibration isolation system against the seismic noise, a pendulum

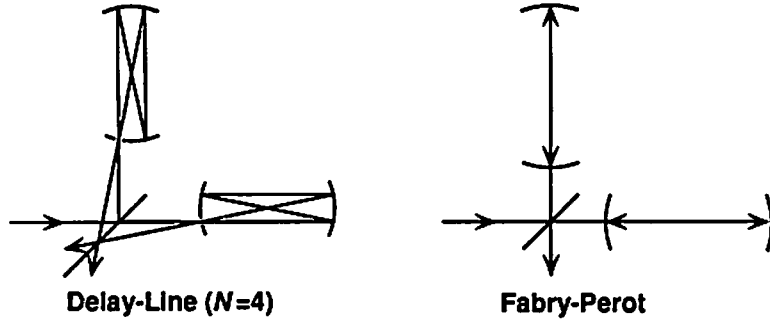


Figure 2: Delay-line and Fabry-Perot. The delay-line uses a different place on the mirror for each bounce, while the light is reflected at the same place on the mirror in the Fabry-Perot scheme.

with ultimately small loss to reduce the thermal noise, an incredibly high-power laser to eliminate the shot noise. While these techniques have been recently in rapid progress, further extensive efforts in technical development should be made to achieve the required performance.

For more detailed descriptions, see the review article [1] and references therein¹.

3 Big projects

Now, two big projects have been approved: LIGO and VIRGO. LIGO project is to construct two interferometers in the US by the collaboration of Caltech and MIT; it has been approved in 1991 and started in 1992 [2]. The total budget is about 250 million US\$. These detectors will have 4-km long vacuum pipes where several interferometers will be installed. The sites have been determined: Hanford, Washington and Livingston, Louisiana. The 40-m prototype detector in the campus of Caltech had the highest sensitivity among similar detectors in the world, which has been reconstructed and is now called Mark II. The reconstruction improved greatly its noise spectrum at around 100Hz.

VIRGO project is the collaborative project between France and Italy to construct an interferometer whose arm length is 3km; the site is in Pisa, Italy [3]. The total budget is about 90 million US\$. The outstanding feature of this detector is the design for detecting a GW of very low frequencies around 10Hz. In order to enhance the sensitivity at such a

¹The references cited in this article are quite limited because my purpose here is not to review the aspect of the detection of the GW from general points of view

frequency, an elaborated vibration isolation system is under development in Pisa. Orsay group in France is responsible for the optical system design, as well as for the development of low-loss optics and a stabilized laser of high-power.

These projects aim at starting their operation in 1998 or 1999. If we are lucky enough, we will see a GW within this century.

4 Current status of Japanese research

The first laser interferometer in Japan was built in Institute of Space and Astronautical Science (ISAS) in 1988 by Kawashima group. This one, named TENKO-10, was a delay-line type; the arm length was 10m and the light was bounced 102 times in each arm. Its strain sensitivity was $h \sim 1.6 \times 10^{-18}/\sqrt{\text{Hz}}$ at 2kHz; this was within twice of the shot noise level [4].

In 1991, a new project has started, involving ISAS, National Astronomical Observatory (NAO), National Laboratory for High Energy Physics (KEK), University of Electro-Communications (UEC), University of Tokyo (UTYO), Kyoto University and so on. The main purpose of the project is to investigate the technical feasibility of the interferometric GW detector. In this spirit, two prototype detectors with different schemes are being developed: one is a 100-m delay-line detector in ISAS and the other is a 20-m FP detector in NAO. The project also includes technical investigations required for the interferometer (done mainly in UTYO) and those concerning the laser and optics (in UEC).

4.1 100-m delay-line detector

The 100-m interferometer, named TENKO-100 has been built in Sagami-hara campus of ISAS. Its vacuum chambers are the same ones that were used in TENKO-10, while the pipes connecting the chambers are newly made for this detector. The delay-line is realized with big concave mirrors of 35-cm diameter and 8-cm thickness; their radius of curvature is close to 100m. The light source is an Ar ion laser, which was also used in TENKO-10. Its frequency and intensity noises are suppressed by stabilization circuits.

The installation of the system has been completed and its first operation was done in September, 1993; the measured bounce number was 110. The observed spectrum of the signal was dominated by the noise due to the scattered light in the delay-line. In order to eliminate the noise, the frequency modulation method was applied. As a result, the noise level was significantly reduced as shown in Fig. 3; intense efforts by ISAS group is gradually improving the performance. However, the noise level below 1kHz is still large. This might originate from the insufficient suppression of the scattered light effects because the amount of the scattered light is much larger than that expected. The slight tolerance

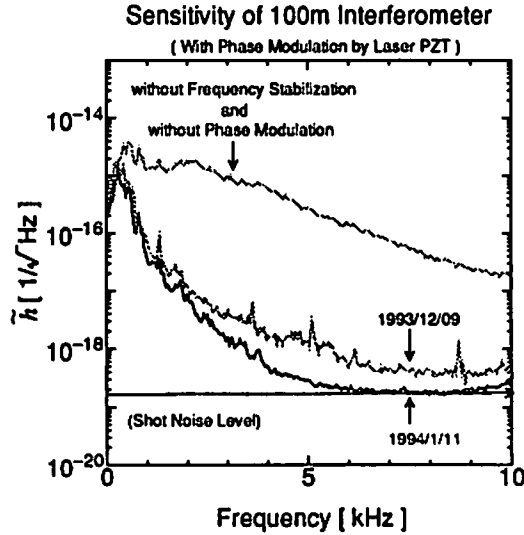


Figure 3: Noise spectrum of TENKO-100. The upper curve was observed without any counter method against the scattered light effect. The middle and the lower curves were measured with the frequency modulation method, which eliminate the coherence of the scattered light; the difference between two curves was due to the improvement of the laser performance.

of the mirrors might cause such a large amount of scattered light. Now, ISAS group is considering several plans to avoid the scattering noise more perfectly.

There are several other problems, for example, bad vibration environment at the site, pollution of the mirrors, and so on. I think, however, that the aimed sensitivity ($h \sim 2 \times 10^{-20}/\sqrt{\text{Hz}}$) will be attainable in the future, because all of the problems are, in principle, solvable.

4.2 20-m FP detector

The 20-m FP detector in NAO is the first attempt to adopt the recombined scheme in FP-type detectors, where the signal is extracted by means of the interference of the reflected lights from the FP cavities installed in the two arms. Since this scheme has quite significant advantages over the locked-FP scheme where the reflected lights from the cavities are separately detected, it is indispensable for the future km-scale interferometer. Figure 4 shows the schematic diagram of the detector; this design is based on the architecture required for the full-scale detector.

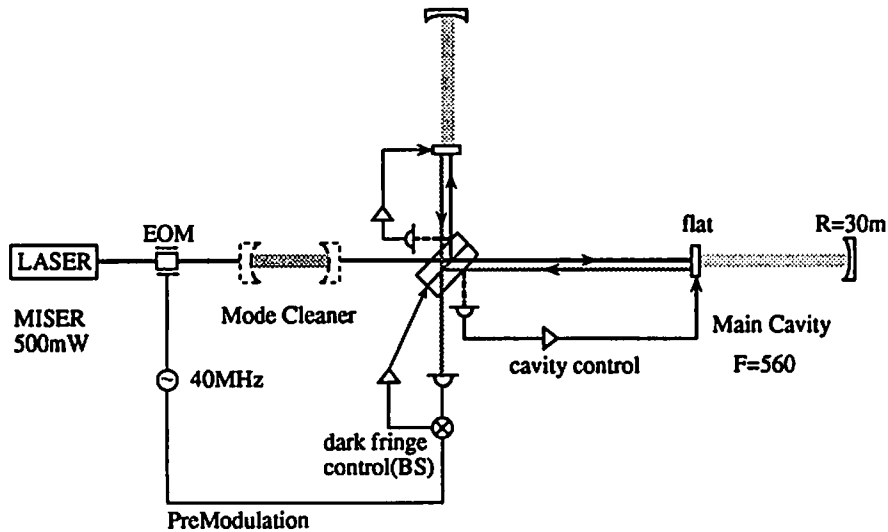


Figure 4: Optical system of NAO interferometer. The reflected lights from the main cavities are recombined and detected. In order to extract a signal due to the phase change, the pre-modulation method is adopted. Thus, the distance between the beam splitter and the near mirror of one arm is different from that of another arm so as to obtain an enough RF modulation sideband signal; the sideband is produced by a phase modulator (EOM) set at the input port of the beam splitter.

Now, the construction of the vacuum system has been completed. All of the chambers and the pipes are made of aluminum; these were fabricated by introducing the ultra-high vacuum technology using aluminum, which has been developed in KEK primarily for the application to a particle accelerator. The pressure attained in the empty chamber was on the order of 10^{-6} Pa; remarkably, this value could be achieved without any baking procedures. The optical system, including pendulum suspensions is being assembled. A laser-diode (LD) pumped Nd:YAG laser ($\lambda = 1.06\mu\text{m}$) is used as its light source. This is also the first attempt to use infrared (invisible) light for a prototype detector. NAO group has succeeded to lock one of the arm cavities to the laser light. I believe that we will be able to see interference between reflected lights from both the cavities soon.

4.3 Laser and optics

There is great progress in the laser and optics field. First of all, the technique to stabilize the LD pumped Nd:YAG laser has been well established. The Nd:YAG laser is quite

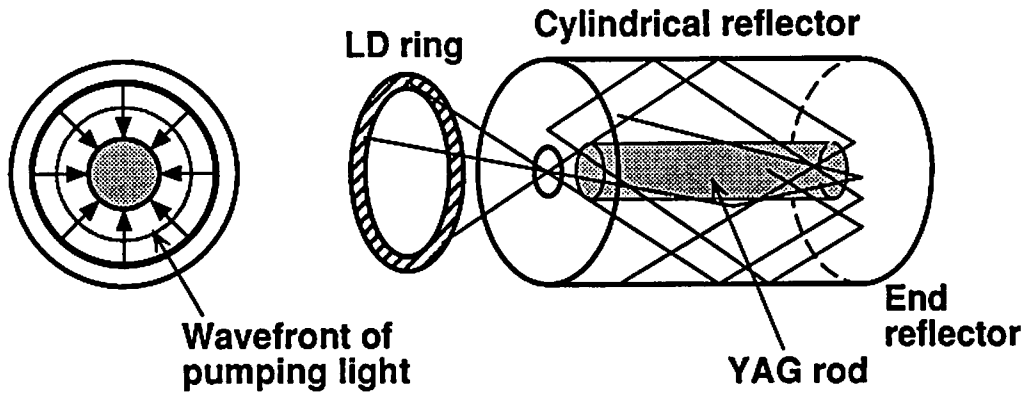


Figure 5: Virtual Point Source Clad Pumping (VSCP) system. Multi-LD beams forms a virtual point source on axis. The gain distribution can be controlled by converging beam pumping into the optically thin YAG rod.

promising because of its high efficiency and low noise. The frequency stability obtained in this project is one of the best values for solid state lasers; this was realized by combining a commercially available laser with an ultra-high quality FP cavity as a frequency reference [5, 6]. Through the experiment, in particular, in evaluating the performance of the FP, we strongly believe that we can utilize the technology offering ultra-low loss optics; this is one of the key technologies for the future detector.

As for a high power laser development, UEC group has demonstrated a new geometry, called virtual point source clad pumping. The significant feature of this geometry is to use an optically-thin YAG rod and multi-bouncing mode of pumping (Fig. 5). Since the Gaussian gain distribution allows a nice mode matching, the lasing of a single transverse mode can be achieved much more easily than the traditional pumping method. This is the remarkable advantage for the light source of the interferometer.

This method has been tested with 32-W pumping power and the quite promising result has been obtained. Now, UEC group is assembling the Nd:YAG laser involving 320-W pumping LD system. In the very near future, the lasing output over 100W will be possible.

4.4 Development of technical elements

There is another L-shaped chamber in UTYO, called "test bench". This is mainly for technical feasibility study on how to control suspended masses. Here, the multiple pendulum system has been tested; making use of strong permanent magnets to give enough

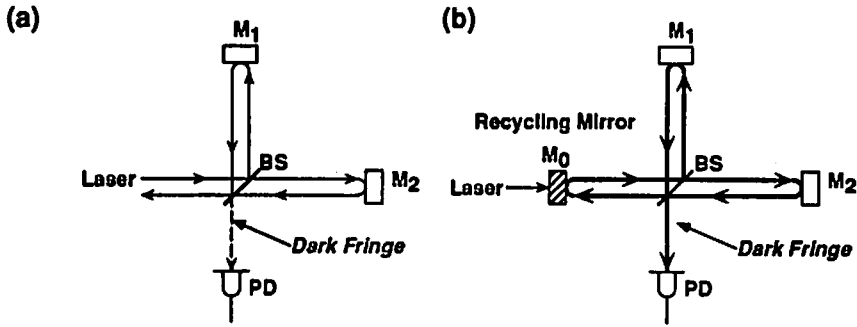


Figure 6: Schematic view of the recycling system. Since the fringe at the output port of the beam splitter (BS) is kept dark, the light is reflected back to the laser (a). If a recycling mirror is set between the laser and the interferometer, the system behaves like an optical cavity. As a consequence, the effective power in the interferometer is enhanced (b).

damping, the system is significantly simplified [7]. In this chamber, we have observed the interference fringe between the reflected lights from the two FP cavities which have been locked to the laser light. All of the system is quite stable; this may be partly due to the simple damping system mentioned above. This is the first demonstration of the recombined FP scheme using the suspended mirrors in the world.

By applying the technique, we have made the mode cleaner cavity consisting of the mirrors provided separately on the pendulums; the mode cleaner cavity is used to purify the spatial mode of the laser light and to suppress the beam jitter as well as to stabilize a frequency of the laser. This one is now moved to NAO for the 20-m detector.

We are also performing a recycling experiment. The recycling is the technique to enhance the effective power stored in the interferometer as show in Fig. 6. By setting the recycling mirror at the input port of the beam splitter, the entire system behaves like an optical cavity; the equivalent power inside the interferometer is much increased. The recycling gain is defined as the ratio of the enhanced power to the incident one. By the test system on the optical table, the recycling gain of 57 has been obtained; this value is one of the highest obtained so far. Now the system is changed to use pendulums in order to test the system in a more realistic condition.

I only report here a part of the work carried out in this project. Although the experiences that we have had are not enough to say "everything is OK for a full scale detector", the situation is quite promising for the future.

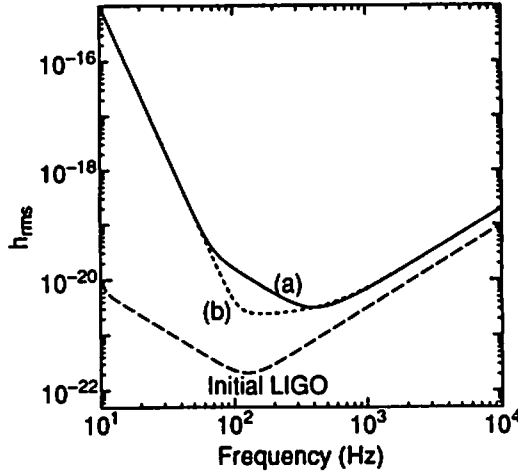


Figure 7: Expected noise spectrum of the 300-m detector. The curves (a) and (b) are calculated using different models for the thermal noise. The curve labeled as “initial LIGO” shows the projected sensitivity of the first LIGO detector. Here, as for the measurement bandwidth Δf , $\Delta f = f$ is assumed.

5 Future plan in Japan

The current project will end in 1995. As the next step, we have made a proposal to construct 300-m FP detector in Mitaka campus of NAO. The planned sensitivity is $h \sim 3 \times 10^{-21}$ at 500Hz; its expected noise spectrum is shown in Fig. 7. The required specifications are listed in Table 1; these values are much more realistic than those required for the full scale detector. The purpose of this project is to show the real feasibility for the full scale interferometer. The continuous operation is also planned to show the fact that the interferometer can really work for a long time. Since the detector will have the sensitivity enough to detect GWs from violent events in Andromeda Galaxy (M31), it might be possible to catch a blessing of nature.

This project can offer one of the most sensitive devises all over the world. I strongly expect that this facility should make fruitful contributions to various fields of physics as well as to the detection of the GW.

Table 1: Required specifications for the 300-m FP detector.

Base Line	300m
Laser	LD pumped Nd:YAG Laser
Effective Power	30W
Q of Pendulum	10^6
Q of Mirror Resonance	10^5
Vacuum Pressure	10^{-6} Pa

6 Conclusion

The interferometric detector is the most promising way to see a wave form of the GW; we can extract much information on astrophysical events, which can be obtained only by the GW. The gravitational wave astronomy, eventually, will extend our understanding of the Universe to its horizon. The detection of the GW is no longer a dream. This is just about to appear in front of us.

I would like to thank Masatake Ohashi and Shinji Miyoki for preparing the figures presented in this article. I would also appreciate to my colleagues for their useful comments.

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Scalar Type Gravitational Waves from Gravitational Collapse in Brans-Dicke Theory

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Abstract

To investigate the possibility of obtaining the evidence for scalar-tensor theories by laser interferometric gravitational wave observatory such as LIGO, we have performed numerical simulations of spherical symmetric gravitational collapse and investigated the wave form and amplitude of scalar type gravitational waves (SGW) in the Brans-Dicke theory. We have found that in the case of the dust collapse of mass $\sim 10M_{\odot}$ and the initial radius $\sim 100\text{km}$, the emitted SGW has the maximum amplitude $\sim 10^{-22}(500/\omega)(10\text{Mpc}/R)$ and the characteristic frequency $\sim 50 - 300\text{Hz}$. This means that if the gravity theory is the Brans-Dicke theory with $\omega \lesssim$ several thousands, the advanced LIGO may detect the signal of SGW.

1. Introduction

Can gravitational waves observatory such as LIGO^[1] be used to verify the Einstein gravity theory? This question will naturally arise from the physicist of gravity. Of course, there are not any experiments of counterevidence for the Einstein gravity^[2], but all tests except for the pulsar timing test^[3] are within the PPN framework, which has fundamental limitation because they are tests using objects in the weak gravity. For example, we can not rule out the other metric theory such as scalar-tensor theories and vector-tensor theory completely^[2], and such theories have the allowed region for their intrinsic parameter, such as ω in the Brans-Dicke theory^[4]. Thus, in order to verify the correctness of the Einstein theory more rigorously or to obtain an evidence for the other metric theory, we need other tests by means of a strong gravity source such as black holes and neutron stars.

The direct detection of gravitational waves, which may include various types of polarization^{[5] [6]}, is one method. For example, let us consider the Brans-Dicke theory (also scalar-tensor theory). In this case, there are three polarization modes; two of them are the same as those in the Einstein theory, that is, the so called + and

$\times$ modes, but there exists one additional mode, the scalar (spin zero) mode which is emitted even in the spherically symmetric spacetime. Thus, theoretically, if we could detect the scalar type gravitational waves (SGW) by the laser interferometric detector, we can verify that the scalar-tensor theory is correct. Also, if we detected only the $+$ and $\times$ modes and did not detect the scalar mode, it also has an important meaning because we can exclude the scalar-tensor theory within the detectability of the detector. In reality, it will be impossible to extract the scalar mode from the other modes unless we can not find the strong periodic source of gravitational waves^[2]. However as shown in the present paper, the characteristic amplitude and frequency of the scalar mode will be different from those of the other modes. Since the interferometric gravitational wave detector has the wide frequency band^[1], it has an ability to detect the different types of waves of the different characteristic frequencies and wave forms. Hence, we may be able to distinguish the scalar mode from the other modes.

From the above point of view, the supernova (SN) explosion, which is considered as a weak emitter of gravitational waves in the case of the Einstein theory^[7], becomes an important source of SGW because the scalar mode is emitted even in the case of spherical symmetric collapse. We can think as follows: If SGW from SN were detected, we can verify the scalar-tensor theory, and even if SGW were not detected, we may give the upper limit of the intrinsic parameter of the theory, *e.g.*, ω in the Brans-Dicke theory. Thus, *absence of signal leads to the restriction of the theory.*

From this interest, we perform the numerical simulation of the gravitational collapse, and investigate the amplitude and the frequency of SGW in the Brans-Dicke theory to see the detectability of gravitational waves other than the $+$ and $\times$ modes. Although there are many alternative scalar-tensor theories besides the Brans-Dicke theory which is consistent with the PPN experiment^{[2][8]}, we only pay attention to the Brans-Dicke theory because our intent here is to show it possible to detect SGW from the SN source. Hence, we do not also intend to calculate the detailed dynamics of the gravitational collapse and wave forms of SGW. The detailed calculations will be the future problems.

In section 2, we derive the equations for the spherical symmetric gravitational collapse in the Brans-Dicke theory. In section 3, we show the numerical results. Section 4 is devoted to the summary.

2. Basic Equations

Basic equation of the Brans-Dicke theory^[4] is originally expressed in the Brans-Dicke frame. However, as pointed out by Dicke^[9], the theory can also be expressed in the so called Einstein frame and for numerical calculation, the equations in this frame is easier to treat. Hence we adopt the equations in the Einstein frame as follows.

$$\begin{aligned} R_{ab} - \frac{1}{2}g_{ab}R &= 8\pi\phi^{-1}\hat{T}_{ab} + \omega'(\nabla_a\Phi\nabla_b\Phi - \frac{1}{2}g_{ab}g^{cd}\nabla_c\Phi\nabla_d\Phi), \\ \nabla_a\nabla^a\Phi &= \frac{4\pi}{\omega'}\hat{T}_a^a\phi^{-2}, \\ \nabla_a\hat{T}_b^a &= 2(\nabla_a\Phi)\hat{T}_b^a - \frac{1}{2}(\nabla_b\Phi)\hat{T}_a^a, \end{aligned} \quad (2.1)$$

where ω and ϕ are a parameter and scalar field in the Brans-Dicke theory and $\omega' = \omega + 3/2$, $\Phi = \log(\phi)$. g_{ab} is the metric in the Einstein frame, and R_{ab} , R and ∇ are Ricci tensor, Ricci scalar and covariant derivative with respect to g_{ab} , respectively. $\hat{T}_{ab}$ is the energy momentum tensor in the Brans-Dicke frame.

In numerical calculation, we use the (3+1)-formalism, and choose the line element as follows.

$$ds^2 = -\alpha^2 dt^2 + A^2 dr^2 + B^2 r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (2.2)$$

As for the time slice, we use the maximal slice, $K = 0$. In the above gauge, $K = -\partial_t(AB^2)/AB^2$, so that AB^2 at each coordinate radius is constant on any time slices.

The energy momentum tensor for the dust is written as

$$\hat{T}_{ab} = \hat{\rho}\hat{u}_a\hat{u}_b, \quad (2.3)$$

where $\hat{\rho}$ and $\hat{u}^a$ are the baryon mass density and the four velocity in the Brans-Dicke frame. This means that $-\hat{u}^a\hat{u}_a$ does not become unity, but ϕ in the Einstein frame. Furthermore, the equation for $\hat{\rho}$ does not become a flux conservative equation. Instead, it becomes

$$\nabla_a(\hat{\rho}\hat{u}^a) = 2\frac{\nabla_a\phi}{\phi}\hat{\rho}\hat{u}^a. \quad (2.4)$$

Thus, we define $\rho \equiv \hat{\rho}\phi^{-2}$. In this case the equation becomes the conservative form as

$$\nabla_a(\rho\hat{u}^a) = 0. \quad (2.5)$$

Taking into account these facts and also to guarantee the regularity at $r = 0$, we define

the fluid variables as follows:

$$\begin{aligned}\rho_h &= \rho AB^2(\alpha \hat{u}^0)^2, \quad S_1 = \alpha AB^2 \rho \hat{u}^0 \hat{u}_r / r, \quad D = \alpha AB^2 \rho \hat{u}^0, \quad S^0 = D \hat{u}^0, \\ V_1 &= \frac{\hat{u}_r \phi}{\hat{u}^0 A^2 r}.\end{aligned}\quad (2.6)$$

Then the equations of the dust become

$$\begin{aligned}\partial_t D + \frac{1}{r^2}(r^3 D V_1)_{,r} &= 0 \\ \partial_t S_1 + \frac{1}{r^3}(r^4 S_1 V_1)_{,r} &= -2S^0 \alpha \alpha_{,y} \phi^{-1} + 2 \frac{S_1^2 A_{,y}}{S^0 A^3} y \phi + \frac{D^2}{S^0} \Phi_{,y}.\end{aligned}\quad (2.7)$$

The equation of the scalar field becomes as follows:

$$\begin{aligned}\partial_t \Phi &= \frac{\alpha}{AB^2} \eta, \\ \partial_t \eta &= \frac{\alpha B^2}{A} \{4y \Phi_{,yy} + 6\Phi_{,y} + 4y \Phi_{,y} (\frac{\alpha_{,y}}{\alpha} + 2 \frac{B_{,y}}{B} - \frac{A_{,y}}{A})\} + \frac{4\pi D^2}{\omega' S^0}.\end{aligned}\quad (2.8)$$

As for the metric and extrinsic curvature, we solve the following equations.

$$\begin{aligned}\partial_t A &= -\alpha K_1 A, \\ B &= \psi^3 / \sqrt{A}, \\ 4y \alpha_{,yy} + 6\alpha_{,y} + 4y \alpha_{,y} (2 \frac{B_{,y}}{B} - \frac{A_{,y}}{A}) & \\ &= 4\pi \alpha (\frac{A \rho_h}{\phi B^2} + \frac{A S_1 V_1 y}{\alpha B^2}) + \frac{3}{2} \alpha K_1^2 A^2 + \omega' \alpha \frac{\eta^2}{B^4}, \\ K_1 &= r^{-3} B^{-3} \int_0^r dr r^4 B^3 [8\pi S_1 / AB^2 - 2\omega' \eta \Phi_{,y} / AB^2].\end{aligned}\quad (2.9)$$

The first equation is the evolution equation of A . The second and third equations are obtained from the maximal slice condition, $K = 0$. The fourth equation is obtained from the momentum constraint.

The initial condition is obtained from the Hamiltonian constraint. If we consider the conformal flat initial condition, $A = B = \psi^2$, it becomes

$$4y \psi_{,yy} + 6\psi_{,y} = -2\pi \frac{\rho_h \phi^{-1}}{\psi} - \frac{3}{16} \psi^5 K_1^2 - \frac{\omega'}{8} (\eta^2 \psi^{-7} + 4(\Phi_{,y})^2 y \psi). \quad (2.10)$$

Before proceeding further, we briefly review the behavior the metric in the wave zone^[6]. Since metric quantities, α , A and B , do not contain the wave like component

in the wave zone, in the Brans-Dicke frame(denoted by $d\hat{s}$), the line element in the wave zone becomes the following form,

$$d\hat{s}^2 = (1 + \Phi)ds^2, \quad (2.11)$$

where ds^2 is the component which does not contain the wave part. Therefore, in the wave zone, Φ is just the SGW, and to see SGW in the wave zone, we have only to see the Brans-Dicke scalar field.

3. Numerical Results

As for the density configuration(ρ_h), we adopt following type of initial conditions,

$$\rho_h = A[1 + \exp\{(y - y_1)/2y_2\}]^{-1}. \quad (3.1)$$

Since we treat the dust, we can freely scale the unit of mass. For convenience, we determine A to fix $M = 1$. Hence, y_1 and y_2 become free parameters. As for the initial condition of the scalar field, we set $\Phi = 0$.

Although we have performed numerical computations for a variety of parameters, y_1 , y_2 and ω' , in this article, we show only results of three models. We show the parameters as well as the density at the center ρ_c for selected models in table 1. Since the unit of the density is $6.17 \times 10^{17} \text{ g cm}^{-3} (M_\odot/M)^2$, the central densities in the initial conditions are $\sim 10^{11} - 10^{14} \text{ g cm}^{-3}$ for $M = 10M_\odot - M_\odot$.

In figure 1, we show the various wave forms for models 1-3. We note that the unit of the time t is $4.93 \times 10^{-5} (M/10M_\odot) \text{ second}$, where $M_\odot$ denotes the solar mass. Amplitudes of SGW are shown in units of $h = \Phi(R/M)$, where R is distance from the detector to the source. Hence the amplitude at the detector becomes

$$\Phi \sim 10^{-22} \left(\frac{h}{0.002} \right) \left(\frac{M}{10M_\odot} \right) \left(\frac{10 \text{ Mpc}}{R} \right). \quad (3.2)$$

It is found that the maximum h becomes ~ 0.002 for all the models. Therefore, we can expect that irrespective of the initial condition, the maximum amplitude of SGW becomes

$$\sim 10^{-22} \left(\frac{M}{10M_\odot} \right) \left(\frac{10 \text{ Mpc}}{R} \right) \left(\frac{500}{\omega} \right). \quad (3.3)$$

As for the wave forms shown in figure 1, we can see the following features: 1) In the case that the initial configuration of dust is more compact(y_1 is smaller), the wave

risers more sharply and the wavelength becomes shorter. 2) All the models have the same wave forms during their damping. Property 1) reflects the motion of the dust, while property 2) reflects the feature of the space-time, that is, we see the quasi normal mode(QNM) of the black hole.

In figure 2, we show the Fourier spectrum for models 1-3. Horizontal lines show that the $\log_{10} f$, and the unit of the frequency f becomes $2.03 \times 10^4 (10M_{\odot}/M)\text{Hz}$. The Fourier spectrum is defined as

$$h_f = \int_0^{t_{\max}} h(t) \exp(2\pi i f t) dt. \quad (3.4)$$

and in figures 2, we plot $f|h_f|/0.002 \equiv H_f$. Note that the unit of H_f is

$$10^{-22} \left(\frac{M}{10M_{\odot}} \right) \left(\frac{10\text{Mpc}}{R} \right) \left(\frac{500}{\omega} \right), \quad (3.5)$$

and it is the quantity to be compared with the root mean square(rms) noise(h_{rms}) at each frequency of the detector^[1]. From figures, it is found that H_f has the following features. 1) Below a characteristic frequency f_c , H_f damps linearly. 2) Irrespective of the model, $H_f \sim 10^{-0.5} = 0.3$ at $f = f_c$. 3) f_c is approximately proportional to $\sqrt{\rho_c}$. 4) For $f \geq f_c$, there are a few peaks in the spectrums, and value of some peaks as large as that at $f = f_c$, i.e., $0.2 \lesssim H_f \lesssim 0.3$.

As for feature 4), the peak frequencies are $200 - 400 (10M_{\odot}/M)\text{Hz}$ and they are not so different for each model. This means that they reflect the QNM of the black hole because the crude estimate of the QNM^[10] becomes

$$\lesssim 500 (M/10M_{\odot})\text{Hz}. \quad (3.6)$$

Hence, in the case of the neutron star formation, it may disappear. On the other hand, features 1), 2) and 3) mean that the characteristic frequency, f_c , of the Fourier spectrum is determined by the initial configuration although the maximum amplitude is not affected. Surprising feature is that H_f at $f = f_c$ is always greater than that at the frequency of the QNM. This is because the wave form is not dominated by the QNM contrary to the case of the + and $\times$ mode of gravitational waves^[11]. This, however, is useful property for the detection problem, since advanced LIGO^[1] has

the good sensitivity in the frequency band $\sim 50 - 300\text{Hz}$. In this case, f_c becomes $50 - 400(10M_\odot/M)\text{Hz}$, and the maximum amplitude becomes

$$3 \times 10^{-23} \left(\frac{M}{10M_\odot} \right) \left(\frac{10\text{Mpc}}{R} \right) \left(\frac{500}{\omega} \right). \quad (3.7)$$

Advanced LIGO^[1] has the ability to detect the signal of the amplitude 10^{-23} at $\sim 70\text{Hz}$, and 3×10^{-22} at $30 - 200\text{Hz}$, so that the characteristic frequency of the SGW by the gravitational collapse seems almost equal to the most sensitive frequency for LIGO.

Here, however, we should be careful to conclude that it is possible to detect SGW from SN by LIGO. This is because in the present calculation we consider the pressureless dust collapse and the formation of the black hole. In realistic SN, a neutron star will be formed almost in most cases, and we may overestimate the amplitude of SGW because in the case of the black hole formation, matter moves more rapidly. However, we think that the amplitude of SGW in the case of black hole formation and those in the case of the neutron star formation are not different so much due to the following reason: First, we estimate the amplitude of SGW by simple order estimate using the quadrupole-like formula(monopole formula?),

$$\Phi \sim \frac{500}{\omega} \frac{M}{R}. \quad (3.8)$$

Φ do not contain v , where v denotes the characteristic speed of the matter in the system, and is written only by the mass of the system. This is consistent with our numerical results.

On the other hand, the energy flux becomes^[6]

$$\frac{dE}{dt} = \frac{8\pi}{\omega'} \left| \frac{\partial \Phi}{\partial t} \right|^2 \sim \frac{1}{\omega} \Phi^2 \rho_c \quad \text{or} \quad \sim \frac{1}{\omega} \Phi^2 v^2 l^{-2}, \quad (3.9)$$

where l is a characteristic scale. Hence, if the characteristic density or scale and velocity are different for each model, the energy flux will be different. To see this, we show the time evolution of the energy flux for models 1-3 in figure 3 as well as the total energy flux(ΔE) for all models in table 2. We can see clearly that both ΔE and dE/dt are different for each model, and for models of more compact configurations of dust in the initial condition, ΔE becomes larger. This suggests that our crude estimate is qualitatively good, so that the estimate of the amplitude by (3.8) will be correct.

Thus, in the case of the neutron star formation, the total energy flux will be suppressed because of the factor v^2 , but the amplitude of SGW will be $\sim 10^{-22}(M/10M_\odot)(10\text{Mpc}/R)(500/\omega)$ irrespective of the model. In other word, the maximum amplitude is the same irrespective of the initial condition, but the characteristic frequency is different. From the present calculations, the characteristic frequency f is expected to become

$$f \sim 1.5\sqrt{\rho_c} \sim 40\text{Hz}(\frac{\rho_c}{10^{10}\text{g cm}^{-3}})^{1/2}, \quad (3.10)$$

at which LIGO has good sensitivity. Thus, we can conclude that if the Brans-Dicke theory is correct and ω is less than a few thousands, SGW from the SN in the Virgo cluster may be detected by advanced LIGO. And if nothing is detected by LIGO even though we find that the SN occurs in the Virgo cluster, we may say that ω is larger than a few thousands.

4. Summary

The results obtained above are as follows: Irrespective of initial conditions, the maximum amplitude of SGW seems to become

$$\Phi \sim 10^{-22}(\frac{500}{\omega})(\frac{M}{10M_\odot})(\frac{10\text{Mpc}}{R}), \quad (4.1)$$

while the characteristic frequency, f_c , depends on the initial condition, and is approximately proportional to $\sqrt{\rho_c}$. At this frequency, the characteristic amplitude become

$$\Phi_f f \sim 3 \times 10^{-22}(\frac{500}{\omega})(\frac{M}{10M_\odot})(\frac{10\text{Mpc}}{R}). \quad (4.2)$$

Using the results of the present calculation, we can estimate that in the realistic gravitational collapse f_c becomes

$$f_c \sim 1.5\sqrt{\rho_c} \sim 40\text{Hz}(\frac{\rho_c}{10^{10}\text{g cm}^{-3}})^{1/2}. \quad (4.3)$$

These mean that advanced LIGO may detect SGW from the Virgo cluster if ω is less than a few thousands.

This result is interesting in the light of the current resurgence of the scalar-tensor theories as either the source of the inflationary scenario or the dilaton gravity. These theories predict that the scalar-tensor theories are correct, so if we detect SGW, not only we will know the correct theory of gravitation, but also we will obtain a clue to know the very early universe.

Although we did not mention in this paper, the detection of gravitational waves from coalescing black hole-neutron star binaries or binary neutron stars of large mass ratio by laser interferometric detector can also be used as a test of the theory of gravities. This is because the back reaction to the binaries of gravitational waves are different in each metric theory, so the evolution of them are different in each theory^[2]. Compared the Einstein theory with the Brans-Dicke theory with $\omega = 500$, as an example, the difference of the accumulated cycles becomes about 3 in the case of $10M_{\odot} - 1.4M_{\odot}$ black hole-neutron star binary^[12], which we will be able to measure by the matched filter technique^[13].

Thus, gravitational waves have a possibility to investigate 'the first three minutes' of the universe as well as 'the last three minutes' of coalescing compact binary. Hence, detailed study of the gravitational collapse and coalescing compact binary in the various scalar-tensor theory will be needed when the gravitational astronomy starts.

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Table 1. Initial conditions for models 1-3.

Model	ω'	$\sqrt{y_1}$	$\sqrt{y_2}$	ρ_c
1	500	15	0.4	7.4×10^{-5}
2	500	30	0.4	9.0×10^{-6}
3	500	5	0.4	2.1×10^{-3}

Table 2. The total energy emitted by the scalar waves for each models.

Model	ΔE
1	7.2×10^{-5}
2	5.4×10^{-5}
3	1.3×10^{-4}

Figure Captions

Figure 1(a)-(c). Wave forms of the scalar wave observed in the wave zone are shown for models 1-3. The unit of the time is $4.93 \times 10^{-5}(M/10M_\odot)$ second.

Figure 2(a)-(c). Fourier spectrum of the scalar wave for models 1-3. The unit of the frequency is $2.03 \times 10^4(10M_\odot/M)$ Hz. The horizontal line and the vertical line show $\log_{10} f$ and $\log_{10}(f h_f/0.002)$, respectively.

Figure 3. Time evolution of the energy flux of the scalar waves dE/dt observed in the wave zone are shown for models 1-3. A vertical line shows $\log_{10} dE/dt$. The solid line, dashed line and the dotted dashed line correspond to models 3, 1 and 2, respectively.

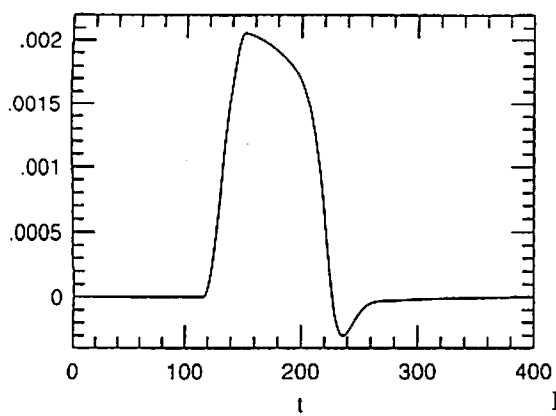


Figure 1(a)

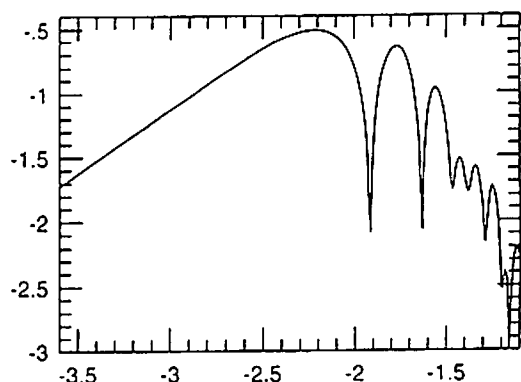


Figure 2(a)

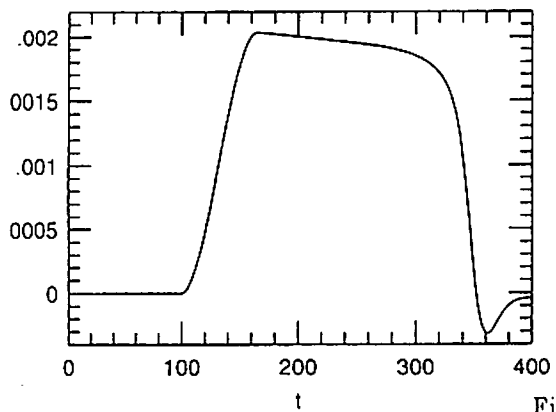


Figure 1(b)

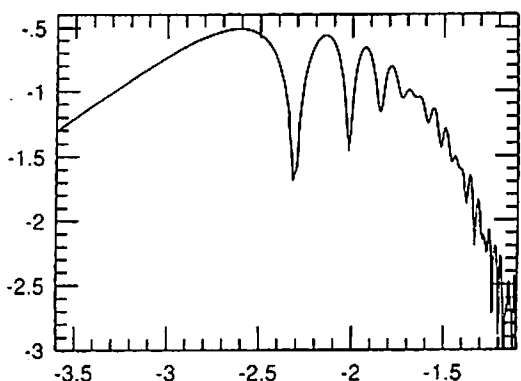


Figure 2(b)

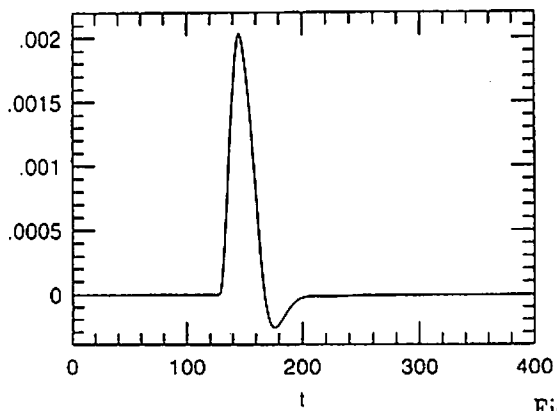


Figure 1(c)

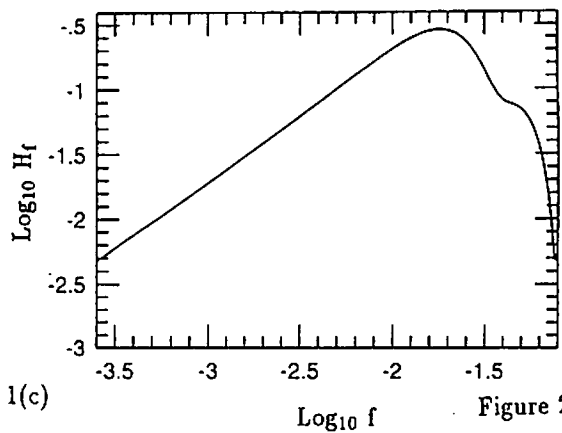


Figure 2(c)

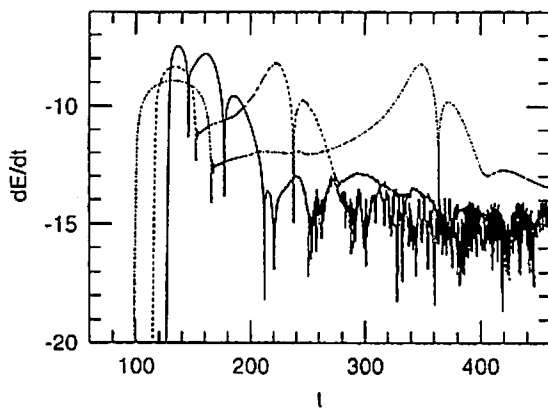


Figure 3

NORMALIZED POWER SPECTRAL DENSITIES OF TWO X-RAY COMPONENTS FROM GS1124-683

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ABSTRACT

Time variations of X-rays from the black hole candidate GS1124-683 are investigated.

In the high state, X-rays consist of two X-ray energy spectral components, i.e. a power law component and a disk blackbody component. Power spectral densities (PSD) of these two energy spectral components have their inherent shape functions. The PSD of the power law energy spectral component has a flat top shape function (FT noise function), which is flat at frequencies less than about 1 Hz, and the PSD of the disk blackbody component has a power law shape function (PL noise function). The FT noise functions normalized by the power law energy spectral component (the normalized FT noise function) are almost the same in cases when the photon count fraction of the power law energy spectral component (power law component fraction) is larger than about 20 %, although there are a few exceptions. The PL noise functions normalized by the disk blackbody component (the normalized PL noise function) are also almost the same when the power law component fraction is less than about 10 %. However, the normalized PL noise functions have a tendency to increase with the power law component fraction, when this fraction is larger than about 10 %. This shows that time variations of the disk blackbody component are influenced by the power law energy spectral component.

In the high-to-low transition state, the normalized PSD (NPSD) of the power law energy spectral component (the normalized FT noise function) is 4 times larger than the value in the high state, although the energy spectrum of this component is the same to that in the high state.

In the low state, the X-rays consist of mainly a power law energy spectral component and this component has a harder energy spectrum than the power law energy spectral component in the high state. Moreover, the NPSD of X-rays in the low state has also a flat top at the frequencies less than about 0.1 Hz and has larger absolute values than those of the normalized FT noise function of the power law energy spectral component in the high state. As the phase lags between time variations of different energy X-rays are also different in these two states, the power law energy spectral components in the high and the low state should have different production processes.

Models of producing these energy spectral components are also discussed.

Subject headings: accretion, accretion disk -- binaries: close -- black hole physics -- stars: individual (Nova Muscae 1991, GS1124-683) -- X-rays: stars

1. Introduction.

Mainly due to observations of Cyg X-1, it has been believed that black hole candidate X-ray stars have two states; a high X-ray intensity state (high state) and a low X-ray intensity state (low state), and in the low state X-rays from the black hole candidates have a hard power law energy spectrum and show rapid time variations. In the high state, the X-rays have a very soft energy spectrum and consist of a disk blackbody component (Mitsuda et al. 1984) and a power law energy spectral component (see for instance, Oda 1977; Liang and Nolan 1984; Tanaka 1989, and the references therein).

To investigate the rapid time variations of X-rays, power spectral density (PSD) has been used by Terrell (1972), Weisskopf, Kahn, & Sutherland (1975), Nolan et al. (1981), and others (see for instance, Oda 1977; Liang & Nolan 1984; Makishima 1988 and the references therein). Recently the normalized power spectral density (NPSD) has been used to compare the PSD of Cyg X-1 at different occasions (Belloni & Hasinger 1990), and of GX339-4 and other different sources (Miyamoto et al. 1991, 1992; Kitamoto et al. 1992 a, b). Phase lags have been used to investigate time differences between time variations of different energy X-rays (van der Klis et al. 1987; Miyamoto et al. 1988, 1991, 1992, 1993). The NPSD is the power spectral density normalized to its intensity, where the Poisson noise due to statistical fluctuation is subtracted. The NPSD is

defined as the square of the ratio of the root mean square amplitude of the two-sided power spectral density in unit frequency band (Hz^{-1}) to the total signal photon number, as shown in the followings;

$$\text{NPSD}_k = \frac{\{a_k^2 + b_k^2 - \frac{1}{n}\sigma^2\}T}{(\bar{x}-B)^2}, \quad a_k = \frac{1}{n} \sum_{j=0}^{n-1} x_j \cos\left(\frac{2\pi k j}{n}\right), \quad b_k = \frac{1}{n} \sum_{j=0}^{n-1} x_j \sin\left(\frac{2\pi k j}{n}\right), \quad k=0, \dots, n/2,$$

where, x_j is a j -th datum of n bins time series data, $\bar{x}$ (c/bin) is their mean value and their background is B (c/bin), σ^2 is the estimated power due to Poisson statistics of the data ($= \bar{x}$), T is $n \Delta T$ (s) (ΔT is the bin length), and the frequency f (Hz) is given by $f=k/T$.

We found that the previous definition of NPSD had an ambiguity of a factor of two if the power spectrum was one sided or two. So we re-define NPSD as mentioned above. All values we published previously (Miyamoto et al. 1991, 1992, 1993; Kitamoto et al. 1992 a, b) are based on this definition.

Very recently, it was found that the high state in GX339-4 can be divided into two sub-states, i.e. a very high state and a high quiet state (Miyamoto et al. 1991). In addition to the above two sub-states, a high-to-low transition state was recognized in GS1124-683 (Miyamoto et al. 1993).

In the low states, energy spectra and time variations of X-rays from the black hole candidates Cyg X-1, GX339-4, GS2023+338, and GS1124-683 are the same; the NPSD and the phase lags between different energy X-rays are the same (Canonical time variations; Miyamoto et al. 1992). In the very high state, the X-rays have two energy spectral components; a power law energy spectral component and a disk blackbody component, and show rapid time variations including QPO (Miyamoto et al. 1991). At some periods in this state, GX339-4 and GS1124-683 showed similar time variations, i.e. the same peculiar phase lags and similar NPSD. In the high quiet state, the main X-ray component is the disk blackbody component and time variations are smallest, i.e. the NPSDs have the lowest values (Miyamoto et al. 1993).

In the very high state, the NPSDs of GS1124-683 change with time and their shapes depend on the photon count fraction of the power law energy spectral component to the total X-ray counts (power law component fraction). For instance, when the power law component fraction is about 30%, the NPSD has a flat top shape, and when the power law component fraction is about 15%, the NPSD has a power law shape. In the high quiet state, where the power law component fraction is less than 3%, the NPSD has a power law shape (Miyamoto et al. 1993).

The results described above suggest that the two X-ray energy spectral components of the black hole candidate in the high state have their own inherent NPSD. To confirm this we investigated followings.

2. The NPSD for the two X-ray energy spectral components.

For this investigation we used LAC (Large Area Counter) data of GS1124-683 observed with Ginga in 1991 from January 11 to July 23 (11th and 204th days in 1991). The X-ray energy spectra were fitted with a two component model consisting of a disk blackbody component (Mitsuda et al. 1984) and a power law energy spectral component with a broad iron K-edge absorption structure (Ebisawa et al. 1994). We also found that a model with a Compton scattering component instead of a power law energy spectral component could be fitted well to the X-ray energy spectra. For details of long term variations of these two components, see Miyamoto et al. (1993).

We calculated the normalized power spectral densities (NPSD) of the observed X-ray data of GS1124-683 and some of the results are shown in Fig.1. As one can see, in the high and the high-to-low transition state (from January 11 to May 17 (137th days)), there is a tendency that the NPSD has a flat top shape if the power law component fraction is large, and these results suggest that there are inherent shape functions of the NPSDs of the two components; the disk blackbody component and the power law component.

To investigate this, we fitted the NPSDs with a two PSD shape functions model; a power law shape function (PL noise function) and a flat top shape function (FT noise function). (The term "shape function" is used to mean the shape only, i.e. it does not mean to include both the shape and the absolute value.). In the high state, the PL noise function has a power law index (γ) of -0.7 and the FT noise function is a Lorentzian function $\text{FTN}(f)$ of zero central frequency, i.e.

$$\text{FTN}(f) = A (\Gamma/2)^2 / (f^2 + (\Gamma/2)^2),$$

where f is the frequency and $\Gamma/2$ is the half width. With γ and $\Gamma/2$ fixed at $\gamma = -0.7$ and $\Gamma/2 = 4$ Hz, we could fit the NPSDs in the high state and the high-to-low transition state. In the low state (after June 13, 164th day), shapes of the NPSD changed clearly from the high state and the NPSD could

not be fitted with the parameters mentioned above. Some of the fitting results are shown in Fig.1 and Table 1. In this table, fitting results of the energy spectra with the two energy spectral components model (the disk blackbody component and the power law energy spectral component) are also included. There are several cases which show QPO. In these cases, we added a Lorentzian function to express QPO (For QPO of GS1124-683, see Takizawa et al. 1993). Some of the fitting results of both the energy spectra and the power spectra are not allowed to be accepted formally. This means that more detailed structure might exist in these spectra. However, our aim of these fittings is to decompose the spectra to the two component and to find that the simple model we used can simulate the data except for small deviations. Thus we could find that in the high and the high-to-low transition state, both the energy spectra and the power spectra could be described approximately by the sum of the disk black body component and the power law energy spectral component, and by the sum of the PL noise function and the FT noise function, respectively. In the low state, the PSD shape function is not equal to those in the high and the high-to-low transition state.

Next, we examined if there are inherent NPSD functions for the two components. If we assume that the NPSDs are composed of two different NPSD functions, one inherent to the power law energy spectral component and the other to the disk blackbody component, each normalized by its respective component, and that these are independent of each other, the total NPSD function of these two should be expressed by

$$\begin{aligned} \text{NPSD}(f) &= (\text{NPSD}_{\text{PL}}(f) x^2 + \text{NPSD}_{\text{DB}}(f) y^2) / (x+y)^2 \\ &= \text{NPSD}_{\text{PL}}(f) X^2 + \text{NPSD}_{\text{DB}}(f) (1-X)^2 \end{aligned} \quad (1)$$

where $\text{NPSD}_{\text{PL}}(f)$ and $\text{NPSD}_{\text{DB}}(f)$ are the NPSDs of the power law energy spectral component and the disk blackbody component at frequency f , x and y are the X-ray counts of the power law and the disk blackbody energy spectral component respectively, and X is the power law component fraction ($X = x / (x+y)$).

For the values of the $\text{NPSD}(f)$ in the high and the high-to-low transition state, the summed values at 0.3 Hz of the two PSD shape functions fitted above are used. For the values in the low state, a power law function is fitted to the NPSD between 0.1 to 1.0 Hz, and the values at 0.3 Hz are used. The values are plotted against the power law component fractions, and the results are shown in Fig.2. The frequency 0.3 Hz is chosen as a reference frequency only to represent the absolute values of the NPSDs. The NPSD values at 0.3 Hz increase rapidly with the power law component fraction at the fractions of above 0.1. In the high-to-low transition state (May 17 (137th day)) and in the low state (after June 13 (164 th day)), the NPSDs at 0.3 Hz are much larger than those in the high state.

The best fit of the formula (1) to the observed values at 0.3 Hz for the data up to March 30 (89th day) is shown by a dotted curve in Fig. 2. In this case, the values of $\text{NPSD}_{\text{PL}}(0.3)$ and $\text{NPSD}_{\text{DB}}(0.3)$ of the dotted curve are $(9.4 \pm 0.9) \times 10^{-4}$ and $(1.36 \pm 0.53) \times 10^{-5}$ respectively, and the reduced χ^2 is 25 (the degree of freedom is 18). The curve qualitatively fits the data, but observed values of the NPSD differ from the best fit by factors of up to 2. The discrepancy between the best fit and the data suggests that NPSD_{PL} and NPSD_{DB} are not constant from observation to observation. Thus we could find that the simple relation expressed by Eq.(1) is not valid, i.e. although the shape functions of the NPSDs of the two energy spectral component are approximately expressed by the two shape functions obtained above, the absolute values of the two shape functions are not constant.

It is natural to assume that the PL noise function corresponds to the disk blackbody component and the FT noise function to the power law component, because the NPSD becomes flat top noise (FT noise) when the power law component fraction is large, and the NPSD becomes power law noise (PL noise) when the fraction of the disk blackbody component is large. With this assumption, the values of the fitted PL noise function and the fitted FT noise function were normalized by the squared intensity (counts) of the disk blackbody component and that of the power law energy spectral components, respectively, and the values of the normalized functions (the normalized PL noise function and the normalized FT noise function) at 0.3 Hz are shown in Fig.3.

When the power law component fraction is less than about 10 % (i.e. when the disk blackbody component is larger than about 90 %), the values of the PL noise function at 0.3 Hz normalized by the disk blackbody component are consistent with a constant value of $(1.49 \pm 0.13) \times 10^{-5}$ with a reduced χ^2 values of 0.65 (d.o.f. = 10), as shown in Fig.3(a). However, when the power law component fraction is larger than 10 %, the normalized PL noise values have a tendency to increase with the power law component fraction.

In the high state and when the power law component fraction is larger than 20 %, the values of the FT noise function at 0.3 Hz normalized by the power law energy spectral component are about $(9.3 \pm 0.2) \times 10^{-4}$ except for Jan. 16 and Feb. 6 (37th days), although the assumption of a constant value is not accepted formally. In the high-to-low transition state (May 17, 137th day), the normalized FT noise function is about 4 times larger than this value, and on Feb. 6, the normalized FT noise function is about 0.2 of this value as shown in Fig.3 (b). When the power law component fraction was less than about 20 %, we could not determine the normalized FT noise function because of their large errors due to swamping of FT noise with PL noise.

Although there are several exceptions, probable values of the normalized PL noise function and the normalized FT noise function at 0.3 Hz are 1.49×10^{-5} and 9.3×10^{-4} respectively as mentioned above. Choosing $\text{NPSD}_{\text{DB}}(0.3)$ and $\text{NPSD}_{\text{PL}}(0.3)$ as 1.49×10^{-5} and 9.3×10^{-4} respectively, the values of Eq. (1) are plotted in Fig. 2 by a solid curved line. This curved line also shows observed values of the NPSD approximately.

As the shape functions and their typical normalized values at 0.3 Hz are known, one can get typical noise functions i.e. the normalized PL noise function and the normalized FT noise function in the high state, although there are some exceptions. The results are shown in Fig. 4. From this figure one can see that time variations (NPSD) of the power law energy spectral component are larger by a factor of more than 100 than those of the disk blackbody component in the frequency range of about 0.5-10 Hz.

In the low state (after June 13, 164th day), the NPSD could not be fitted with the two shape functions with the same parameters in the high state mentioned above: the shape is different from those in the high and the high-to-low transition state. The values are also much larger than those in the high state as shown in Fig.2 and 4. In the low state, the NPSDs are the same at different occasions in the frequency range above about 1 Hz, and they are the same to those of other black hole candidate X-ray stars as reported by Miyamoto et al. (1992).

3. Discussions.

1) On the shape function of the NPSD and X-ray emission mechanism.

We recognized two types of PSD shape functions in the high state and the high-to-low transition state. One is the FT noise function corresponding to the power law energy spectral component, and the other is the PL noise function which corresponds to the disk blackbody component. In the low state, X-rays consist mainly of a power law energy spectral component. This component has a photon number index of about -1.6, which is different from that of -(2-3) of the power law energy spectral component in the high state (Miyamoto et al. 1993).

These two power law energy spectral components, in the high and the low state, have different PSD shape functions, and the NPSD in the low state is much larger than that in the high state and has two turning points as shown in Fig.4. However, there are some similarities between them. At frequencies higher than about 5 Hz, both the NPSDs have a power law shape of index about -2, and both have turning points of about 2-5 Hz. At lower frequencies, they have a flat top shape, although the turning points to the flat top are different. In the high state, the turning point to the flat top is around 2-5 Hz as mentioned above, and in the low state there is the second turning point around 0.05 - 0.5 Hz to the flat top. The disk blackbody component in the high state may have its turning point at less than 0.01 Hz, because it cannot be observed in the power spectra in Fig.1 with minimum frequencies of around 0.01 Hz. These frequencies and their corresponding time constants are shown in Table 2. Because the PL noise function associated with the disk blackbody component has a slope of -0.7, it must be cut off at higher frequency to prevent divergence.

If an X-ray light curve is the sum of a sequence of exponentially decaying shot s, its PSD will be a power law of index -2, with a low frequency cut off determined by the time constant of the shots (see for instance, Miyamoto & Kitamoto 1989; Lochner, Swank & Szymkowiak 1991). Thus the NPSD shape difference of the two power law energy spectral components in the high and the low state can be explained if X-rays in the high state consist of many exponentially decaying shot s, whose time constants are shorter than those in the low state by a factor of about ten. In the low state, there are many shots which have the maximum time constant $(1/(2\pi f))$ of about 1 sec. In the high state the maximum time constant is about 0.03 - 0.08 sec. The difference of these time constants may correspond to extension of the emission region of the power law energy spectral X-ray components which is given by $c \times (\text{time constant})$, i.e. in the low state the region may extended out to about 3×10^{10} cm, and in the high state to about 10^9 cm.

It is interesting that the extent of the inverse-Compton cloud in the very high state, estimated from the time lags between time variations of different energy X-rays (Miyamoto & Kitamoto 1991; Miyamoto et al. 1991) is of the same order as the extent in the high state mentioned above.

It is of interest also to note that in the low state the main component is the power law energy spectral component and the disk blackbody component contributes a small fraction of the X-rays. The emission region seems to extend to about 3×10^{10} cm as mentioned above. These suggest that in the low state a stable accretion disk does not exist in the X-ray emitting region around the black hole. Matter is falling into the black hole in the forms of geometrically thin optically thick rings and geometrically thick optically thin very hot rings or tori chaotically (Miyamoto & Kitamoto 1989). Low energy X-rays from the optically thick rings are Compton scattered in the hot tori and become the hard power law energy spectral component. Increase of the electron energy in the hot tori as the tori fall into the black hole is the cause of the hard X-ray time lag observed by Miyamoto et al. (1988). The above process causes the large time variations (large NPSD) in the low state.

For the disk blackbody component, the shape of its PSD is a power law, and its index is about -0.7, which can not be explained by exponentially decaying shots of a single time constant (Lochner, Swank & Szymkowiak 1991). To explain the shape of the PSD by exponentially decaying shots, we must assume many different time constants. This suggests that the accretion disk is unstably moving on various time scales at various radii, although the amplitude of the motion is small as one can observe from Fig.4.

A sketch of the above physical processes is shown in Fig. 5. In the high state, large high energy electron clouds are formed due to the rotating magnetic field trapped in the accretion disk, which is similar to the case of the star formation process (Uchida & Shibata 1985). Inverse Compton scattering of low energy X-rays from the inner edge of the disk produces the power law energy spectral component (Miyamoto & Kitamoto 1991). In the low state, the disk is unstable in the X-ray emitting region, and accretion matter forms rings or tori, causing the X-ray shots in the low state (Miyamoto & Kitamoto 1989).

2) The correlation between the power law energy spectral component and the disk blackbody component.

As shown in Fig.3(a), when the power law component fraction is smaller than 10% (i.e. the disk blackbody component is larger than 90%), there seems to exist an inherent NPSD function for the disk blackbody component (the normalized PL noise function); the shape and the amplitude are constant. In this case, the value of the normalized PL noise function is about 1.49×10^{-5} at 0.3 Hz.

However, when the power law energy spectral component is larger than about 10%, the normalized PL noise function has a tendency to increase with the power law component fraction as shown in Fig.3(a). Thus the power law energy spectral component seems to induce short term variations of the disk blackbody component, if the power law energy spectral component is larger than about 10%. On the other hand, these two components show independent long term variations as shown by Miyamoto et al. (1993).

The correlation of short term variations and the independence of long term variations between these two components can be explained by the model proposed by Miyamoto & Kitamoto (1991) that the power law energy spectral component in the very high state is due to the inverse Compton scattering of low energy X-rays (seed photons) in large hot electron clouds at the upper and the lower side of the accretion disk. This model is proposed to explain the peculiar phase lags observed in GX339-4 in its very high state (Miyamoto et al. 1991). In this model the long term variations are explained to be due to variation of the large hot electron clouds. These hot electron clouds are connected to the disk through magnetic field, because the magnetic field lines thread through the disk as shown in Fig.5. Thus large disturbance in the hot electron clouds can be transferred to the accretion disk through the magnetic field, activating the short time scale variation in the accretion disk, which is reflected in the increase of time variations of the disk blackbody component. The disturbance in the hot electron clouds is induced by the incident photons.

The peculiar phase lags in GX339-4 in the very high state were also observed in GS1124-683 in the very high state (Miyamoto et al. 1993). It is of interest to note that these peculiar large phase lags were observed also in Cir X-1, an X-ray binary neutron star, just after it flared up (Makino 1993). Its NPSD was also similar to those of GX339-4 and GS1124-683, although the X-ray energy spectrum of Cir X-1 was composed of two blackbody radiation spectra with kTs of about 1 keV and 2 keV. The 1 keV and the 2 keV blackbody radiation component correspond to the disk blackbody component and the power law component respectively of GX339-4 and GS1124-683. These suggest that the production process of the seed photons of the power law energy spectral

component in the black hole candidate X-ray stars is similar to that of the 2 keV blackbody radiation component in X-ray binary neutron stars, and the difference between the energy spectra of the X-ray binary neutron stars and black hole X-ray stars in the high state may be due to difference of the temperature of the Compton scattering electron clouds.

3) Thermal instability and viscous instability.

Time variation of X-rays should be related to some instability in the accretion disk. Thermal instability and viscous instability of the accretion disk were predicted by Pringle, Rees & Pacholczyk (1973), Lightman & Eardley (1974), Shibazaki & Hoshi (1975), Shakura & Sunyaev (1976), and the transition from the high state to the low state was discussed by Ichimaru (1977). According to Shakura & Sunyaev (1976), there should be two types of time variations due to the thermal instability and the viscous instability when the ratio of the radiation pressure to the total pressure ($\beta_0 = p_r/(p_r + p_g)$) is larger than 3/5, and the growth-rate (Ω_T) of the thermal instability is given by

$$\Omega_T/\alpha\omega = 6(5\beta_0 - 3)/A(\beta_0),$$

where $A(\beta_0) = 8 + 51\beta_0 - 3\beta_0^2$, and the growth rate (Ω_V) of the viscous instability is given by

$$\Omega_V/\alpha\omega = 2(5 - 3\beta_0)(H_0/\Lambda)^2/(3(5\beta_0 - 3)),$$

where α is the parameter characterizing the level of turbulence, ω is the angular velocity of the Kepler motion of the accreting matter, H_0 is the half thickness of the disk, and Λ is the wave length. The values of $\Omega_T/\alpha\omega$ and $\Omega_V/\alpha\omega$ are about 0.1-0.2 and $(1.7-0.67)(H_0/\Lambda)^2$ respectively for the case of $\beta_0 \sim 0.8-1$. Thus assuming that H_0/Λ is about 0.1, the growth rate of the thermal instability is larger than that of the viscous instability by a factor of about 10 at the same radius. As ω is about 1000 rad/s and 40 rad/s at $r = 10^7$ and 10^8 cm ($3R_s$ and $30R_s$ (Schwarzschild radius) for the black hole of 10 Mo) respectively, one get $\Omega_T = 120s^{-1}$ and $3.7s^{-1}$, and $\Omega_V = 1.2s^{-1}$ and $0.04s^{-1}$ for these radii respectively assuming α is about 1.

Thus, comparing the time constants in Table 2 with these values, one can see that in the high state, the time variation of the disk blackbody component, i.e. PL noise, is related to the viscous instability of the accretion disk at around 10^8 cm ($1/\Omega_V = 25$ s), and the time variation of the power law energy spectral component (FT noise) is related to the thermal instability ($1/\Omega_T = 0.01$ s) at the inner edge (10^7 cm) of the disk. These are also shown in Table 2. It must be remarked, however, that these values are based on the assumption that H_0/Λ is about 0.1 and α is about 1.

In the high state, the disk component should have many different time constant as discussed in 1) in this section, and the possible maximum time constant seems to be larger than 16 sec. We could not find large change of the inner radius of the accretion disk from the observed data of the disk blackbody component during large decrease of the power law energy spectral component by a factor of about 200 (Ebisawa et al. 1994). And the NPSD of the disk blackbody component (the normalized PL noise function) is small. Thus the time variations of the disk blackbody component should be due to the small amplitude viscous instability in various part of the accretion disk. In this state, time variations of the power law energy spectral component may be explained as follows. Low energy seed photons of the power law energy spectral component are emitted from the inner edge of the accretion disk (10^7 cm), whose rapid time variations are due to the thermal instability of the disk at the inner edge. These photons become higher energy X-rays by the inverse-Compton scattering in the high energy electron clouds as mentioned in 2) in this section.

In the low state, the power law component seems to have two time constants; 0.03 s - 0.08 s and 0.3 s - 3 s as mentioned in 1) in this section. Shorter one is due to the thermal instability around 10^7 cm ($1/\Omega_T = 0.01$ s), which is similar to that in the high state. Another larger time constant is due to the thermal instability at larger radii of the disk ($1/\Omega_T = 0.3$ s), which makes clumps of the accreting matter, and causes the large X-ray shots in the low state. Thus in the low state, X-ray shots of larger time constants are superposed on the short time variations similar to those in the high state. The radial drift velocity of the accretion clumps is related to the hard X-ray lags of the X-ray time variations as discussed by Miyamoto & Kitamoto (1989).

In the high-to-low transition state (May 17), the NPSD at 0.3 Hz is about 4 times larger than the best fit curve in Fig. 2, although the index of the power law energy spectral component in the high-to-low transition state is the same to that in the high state. This suggests that in the high-to-low transition state the instability around the inner edge of the disk, which will extend to larger radii in the low state, started already. Extension of this instability to larger radii is the transition to the low state.

Recently Nowak (1993) proposed that in some case the low state is possible to be the more luminous state than the high state, and the very high state is a transition phase between the high and the low states, and the PL noise is due to the viscous instability of the disk and the FT noise is due to the thermal instability. We can not agree that the low state is the more luminous state than the high state, and the very high state is a transition phase between the high and the low states, because clearly the low state is less X-ray luminous than the very high state and the high quiet state exists between the very high state and the high-to-low transition state. However, his view that in the high state the PL noise is due to the viscous instability and the FT noise has a relation to the thermal instability of the accretion disk seems to be probable as mentioned above, although there are some discrepancies in the region of the instability from our model mentioned above.

4. Conclusions.

In the high state, the NPSD consists of the PL noise function and the FT noise function. The PL noise function is due to the disk blackbody component and the FT noise function is due to the power law energy spectral component. Typical NPSDs for these two energy spectral components, i.e. the normalized PL noise function and the normalized FT noise function, are shown in Fig.4. Time variations of the power law energy spectral component can be larger than those of the disk blackbody component by a factor of more than about 100 as shown in Fig.4. However, time variations of the disk blackbody component increase with the fraction of the power law energy spectral component if this fraction is larger than 10 % as shown in Fig.3 (a). This may be due to the connection between the hot electron clouds and the disk by magnetic field lines.

In the high-to-low transition state, the NPSD is about 4 times larger than those in the high state as shown in Fig.2, although the index of the power law energy spectral component in the high-to-low transition state is the same to that in the high state. This suggests that the high-to-low transition state is a different state from the high state; some activity at the inner edge of the accretion disk is increased.

In the low state, X-rays have a power law energy spectrum and its NPSD is larger than the normalized FT noise function (the NPSD of the power law energy component) in the high state by a factor of more than 10 at the frequencies less than about 1 Hz. Though both the power law energy spectral components in the low and the high state have the flat top shape noise function, the flat top in the low state is at the frequencies less than about 0.1 Hz and in the high state the FT noise function is flat at the frequencies less than about 2 Hz. The energy spectrum in the low state is also harder than the power law energy spectral component in the high state. Besides, the phase lags between time variations of different energy X-rays are different between the low state and the high state. Thus the production processes of the power law energy spectral components in the low state and in the high state should be different.

In the high state, time variations of the disk blackbody component (PL noise) are due to the viscous instability of the accretion disk. In this state, the power law energy spectral component is supposed to be produced as follows. Low energy X-rays (seed photons) of the power law energy spectral component are emitted from the inner edge of the accretion disk with rapid time variations due to the thermal instability of the disk at this edge. These photons become higher energy X-rays by the inverse-Compton scattering in the large high energy electron clouds around the disk as shown in Fig.5. The long term large flux change of the power law energy spectral component in this state is due to the change of the large high energy electron clouds.

In the low state, the accretion disk is unstable in the X-ray emitting region around the black hole, and matter is falling into the black hole in the forms of geometrically thin optically thick rings and geometrically thick optically thin very hot rings or tori as shown in Fig.5. Low energy X-rays from the optically thick rings become the hard power law energy spectral component by the Compton scattering in the hot tori. This is the cause of the rapid time variations in the low state and the phase lags observed in the low state.

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Table 2. Time constants of the NPSD and their possible candidates.

X-ray intensity state	Energy spectrum component	NPSD	Turning point frequency (f) (Hz)	Time constant $1/2\pi f$ (s)	Time constants of possible candidate of disk instability at	
					$R_{in.}$ (10^7 cm)	R_{disk} (10^8 cm)
High state	disk b.b.	PLN	< 0.01	> 16		$1/\Omega_v = 25s$
High state	power law	FTN	2-5	0.03-0.08	$1/\Omega_T = 0.01s$	
Low state	power law	FTN	2-5	0.03-0.08	$1/\Omega_T = 0.01s$	
Low state	power law	FTN	0.05-0.5	0.3 - 3	$1/\Omega_T = 0.3s$	

Table 1. X-ray Spectral Components and Power Spectral Components

Data #	Start Day	End Time	Total day in 1991	Mode	Power Law Energy Spectral Comp.			Disk Black Body Component			Power Law Component Fraction	χ^2 /D.O.F. Fitting	NPSD of PL noise at 0.3 Hz	NPSD of FT noise at 0.3 Hz	NPSD at 0.3 Hz	Normalized PL noise at 0.3 Hz	Normalized FT noise at 0.3 Hz	χ^2 /D.O.F. of PSD Fitting
					Index	Photons/sec/keV	Photons/sec/keV	Rise (sec) ⁻¹ keV	keV	Photons/sec/keV								
11	1/11/91	05:58:16	20:41:48	11.5	MPC2	2.27±0.02	1.03E-07	18.5	4.3±0.4	0.909±0.027	1.72E-08	4.8	0.82	42/39				
	1/11/91	20:49:00	24:00:32	11.9	MPC3													
2	1/14/91	19:48:28	22:50:44	14.9	MPC2	2.44±0.01	1.05E-07	21.3	6.4±0.1	1.070±0.004	8.14E-08	20.8	0.48	64/38	(1.40±0.28)E-4	(6.85±0.29)E-4	(8.25±0.40)E-4	(4.10±0.81)E-3
3	1/16/91	18:46:44	23:26:28	16.9	MPC2	2.45±0.02	6.58E-08	13.5	10.0±0.1	0.985±0.006	1.36E-07	36.2	0.27	56/38	(2.13±0.31)E-5	(2.64±0.17)E-4	(2.84±0.29)E-4	(8.04±0.75)E-5
4	1/17/91	16:22:40	21:56:32	17.8	MPC2	2.40±0.03	2.52E-08	4.97	10.7±0.1	0.973±0.005	1.49E-07	39.8	0.11	53/39	(5.12±0.62)E-5	(3.07±0.38)E-5	(8.19±0.73)E-5	(9.51±1.15)E-5
5	1/18/91	17:29:40	21:28:44	18.8	MPC2	2.52±0.03	5.13E-08	10.9	9.3±0.2	0.996±0.006	1.23E-07	32.5	0.24	48/39	(5.13±0.55)E-5	(0.78±0.60)E-6	(5.20±0.61)E-5	(6.53±0.70)E-5
6	1/20/91	14:59:51	17:14:45	20.7	MPC2	2.53±0.03	5.79E-08	12.4	8.9±0.2	0.993±0.007	1.12E-07	29.8	0.28	37/39	(1.43±0.26)E-5	(5.06±0.18)E-5	(6.49±0.31)E-5	(2.47±0.44)E-5
7	1/22/91	19:03:31	19:35:39	22.8	MPC3													
8	1/25/91	12:48:43	13:46:35	25.6	MPC3													
9	2/5/91	10:39:39	11:35:19	36.5	MPC3													
10	2/6/91	7:10:15	8:55:56	37.3	MPC2	2.48±0.05	4.06E-08	8.44	8.4±0.2	0.904±0.008	6.43E-08	17.9	0.27	40/37	(2.91±0.64)E-5	(6.05±0.35)E-5	(8.96±0.73)E-5	(5.61±1.23)E-5
	2/6/91	10:08:39	12:13:23	37.5	MPC3													
11	2/13/91	5:07:31	8:35:11	44.3	MPC2	2.38±0.04	2.18E-08	4.26	9.9±0.2	0.826±0.005	5.84E-08	16.8	0.23	72/39	(4.21±1.14)E-5	(2.99±0.10)E-4	(3.41±0.15)E-5	(6.29±0.58)E-5
12	2/14/91	4:37:26	5:31:51	45.2	MPC2	2.22±0.02	1.02E-08	1.79	11.6±0.1	0.785±0.003	6.35E-08	18.7	0.11	107/39	(1.46±0.41)E-5	(5.43±0.30)E-5	(6.89±0.51)E-5	(1.04±0.68)E-5
13	2/14/91	6:11:47	8:05:59	45.3	MPC3													
	2/20/91	23:32:43	24:27:35	52.0	MPC2	2.29±0.04	7.42E-09	1.37	10.9±0.2	0.772±0.004	5.25E-08	15.6	0.10	45/39	(2.99±0.40)E-5	(1.64±0.33)E-5	(4.64±0.52)E-5	(3.79±0.51)E-5
14	2/21/91	1:06:39	3:35:35	52.1	MPC3													
	2/22/91	0:36:07	1:31:11	53.0	MPC2	2.22±0.04	4.54E-09	0.792	10.6±0.1	0.772±0.004	4.95E-08	14.7	0.067	68/39	(1.44±0.28)E-5	(0.01±1.62)E-6	(1.44±0.32)E-5	(1.79±0.35)E-5
15	2/25/91	20:59:07	21:51:35	56.9	MPC2	2.19±0.07	2.46E-09	0.421	10.9±0.1	0.749±0.003	4.47E-08	13.5	0.042	62/39	(1.12±0.27)E-5	(2.44±2.27)E-6	(1.36±0.35)E-5	(1.28±0.31)E-5
16	2/26/91	20:28:03	21:19:55	57.9	MPC2	2.19±0.05	3.73E-09	0.636	10.6±0.1	0.746±0.003	4.16E-08	12.5	0.066	52/39	(1.20±0.68)E-5	(1.25±6.10)E-6	(1.32±0.92)E-5	(1.31±0.74)E-5
	2/26/91	0:07:11	22:58:56	57.5	MPC3													
17	3/8/91	16:03:43	19:16:35	67.7	MPC2	2.91±0.16	1.30E-09	0.338	11.1±0.1	0.701±0.003	3.41E-08	10.6	0.033	36/39	(1.38±0.24)E-5	(0.01±1.02)E-5	(1.38±1.05)E-5	(1.58±0.28)E-5
	3/8/91	19:42:35	19:54:39	67.8	MPC3													
18	3/10/91	14:56:23	17:16:09	69.7	MPC2	2.97±0.03	9.52E-10	0.253	11.3±0.1	0.683±0.001	3.08E-08	9.6	0.027	51/39	(1.62±0.78)E-5	(0.01±4.36)E-6	(1.62±0.80)E-5	(1.74±0.72)E-5
19	3/12/91	13:32:15	16:08:31	71.6	MPC3													
20	3/18/91	12:11:28	13:22:10	77.5	MPC2	2.83±0.22	7.26E-10	0.182	10.8±0.1	0.685±0.003	2.85E-08	8.9	0.022	67/39	(1.35±0.62)E-5	(0.09±5.00)E-6	(1.36±0.80)E-5	(1.43±0.66)E-5
	3/18/91	13:45:30	15:23:16	77.6	MPC3													
21	3/20/91	12:38:34	13:16:42	79.5	MPC2	2.44±0.03	5.37E-10	0.109	10.7±0.1	0.695±0.001	2.99E-08	9.3	0.015	80/39	(2.03±0.01)E-5	(0.38±3.26)E-8	(2.03±0.01)E-5	
	3/20/91	9:40:26	10:35:38	79.4	MPC3													
22	3/28/91	6:29:18	6:54:55	87.3	MPC2	2.41±0.10	5.09E-10	0.101	10.8±0.1	0.721±0.002	3.66E-08	11.2	0.012	95/39	(1.63±0.58)E-5	(1.59±2.46)E-8	(1.63±0.63)E-5	(1.68±0.60)E-5
	3/28/91	8:20:26	8:33:34	87.4	MPC3													
23	3/29/91	4:25:06	6:04:56	88.2	MPC2	2.60±0.07	2.74E-10	0.061	10.4±0.1	0.725±0.001	3.49E-08	10.7	0.0068	84/39	(1.32±0.83)E-5	(2.05±6.69)E-6	(1.52±1.06)E-5	(1.35±0.85)E-5
	3/29/91	7:28:02	7:57:30	88.3	MPC3													
24	3/30/91	3:48:38	5:28:47	89.2	MPC2	2.42±0.05	2.96E-10	0.059	10.1±0.1	0.726±0.001	3.35E-08	10.2	0.0075	80/39	(1.20±0.38)E-5	(7.77±0.28)E-8	(1.20±0.38)E-5	(1.21±0.38)E-5
	3/30/91	6:52:22	7:21:42	89.3	MPC3													
25	4/2/91	2:50:38	5:28:58	92.2	MPC1	2.40±0.12	3.11E-10	0.062	10.3±0.1	0.715±0.002	3.25E-08	10.0	0.0081	75/39	(1.34±0.45)E-5	(0.01±2.77)E-6	(1.34±0.53)E-5	(1.35±0.46)E-5
26	4/19/91	19:34:33	20:29:45	109.8	MPC1	2.61±0.01	4.88E-10	0.109	12.6±0.1	0.593±0.001	1.90E-08	6.3	0.025	103/39				
27	5/17/91	3:13:04	8:11:48	137.2	MPC1	2.32±0.01	1.86E-09	0.347	12.2±0.1	0.502±0.001	7.46E-09	2.6	0.21	128/39	(1.34±0.45)E-5	(0.01±2.77)E-6	(1.34±0.53)E-5	(1.35±0.46)E-5
	5/17/91	6:09:40	6:39:38	137.3	MPC3													
28	6/13/91	13:19:42	16:45:02	164.6	MPC1	1.66±0.01	1.75E-09	0.190	4.4±0.2	0.344±0.005	1.07E-10	0.042	0.95	40/39	(3.12±2.08)E-5	(1.79±0.19)E-4	(2.11±0.28)E-4	(5.06±3.36)E-5
	6/13/91	17:22:56	18:17:56	164.7	MPC3													
29	6/21/91	11:07:46	13:41:42	172.5	MPC1	1.60±0.02	1.50E-09	0.155	3.1±0.1	0.337±0.047	4.91E-11	0.019	0.97	56/39				
30	7/21/91	17:17:04	22:12:32	202.8	MPC1	1.57±0.02	2.31E-09	0.230	1.7±1.2	0.413±0.076	5.16E-11	0.019	0.98	29/39				
31	7/22/91	18:00:52	18:11:20	203.8	MPC1	1.57±0.01	2.16E-09	0.217	3.4±0.4	0.341±0.013	6.05E-11	0.024	0.99	30/39				
	7/22/91	19:33:20	19:52:00	203.8	MPC3													
32	7/23/91	17:10:12	18:03:16	204.7	MPC1	1.59±0.01	2.13E-09	0.219	2.0±0.1	0.384±0.005	4.27E-11	0.016	0.98	31/39				
	7/23/91	18:42:48	21:08:28	204.8	MPC3													

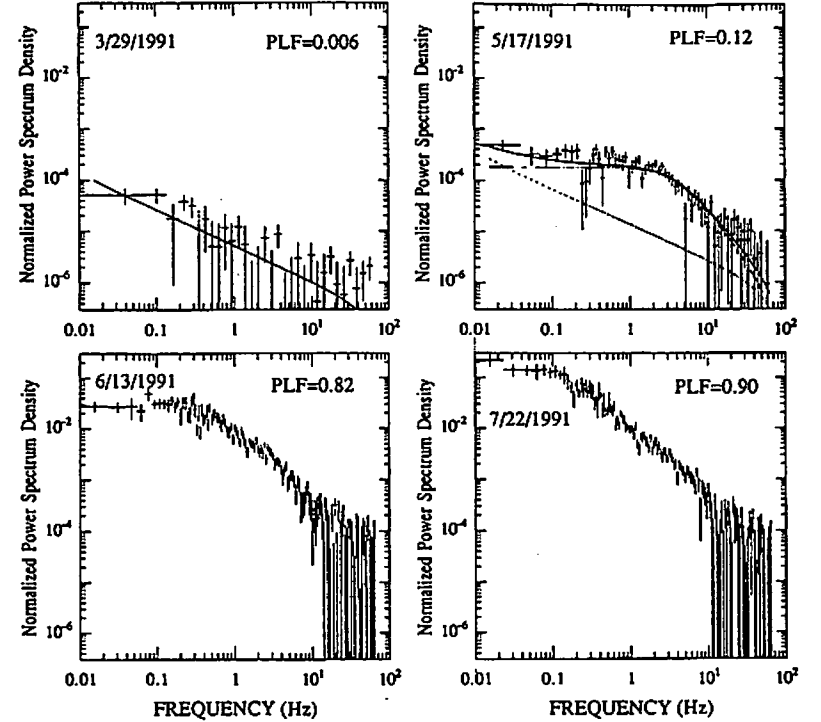
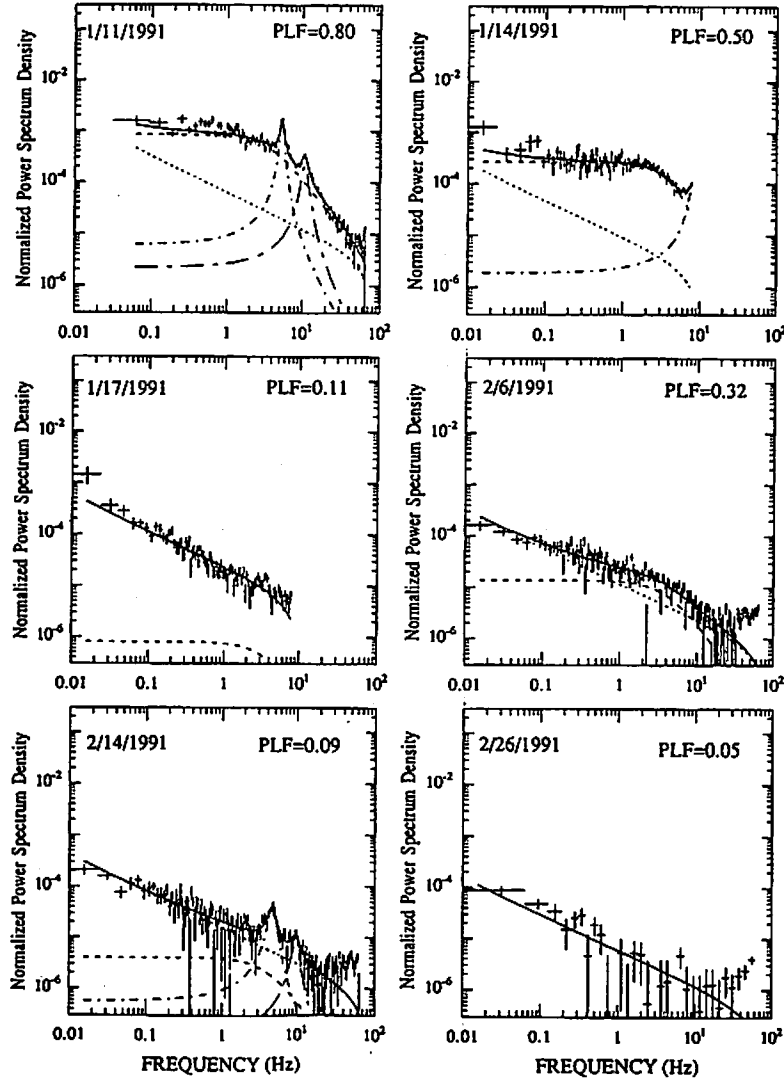


Fig. 1. Some examples of the NPSDs of GS1124-683. The values are fitted with a two component model; a power law noise function (PL noise function, the dotted curve) and a flat top Lorentzian noise function (the FT noise function, the dashed curve), and the results are shown by solid curves. QPOs are also shown by the dash-dotted curves. (PLF is the power law component fraction)

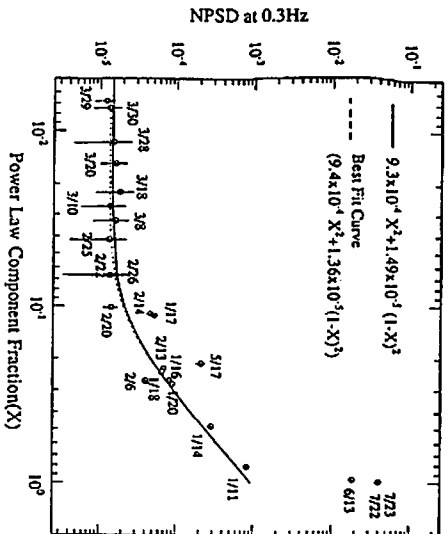


Figure 2

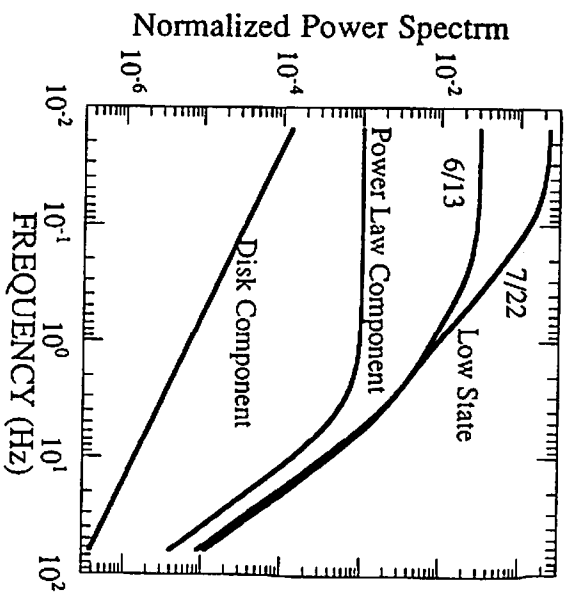


Figure 3

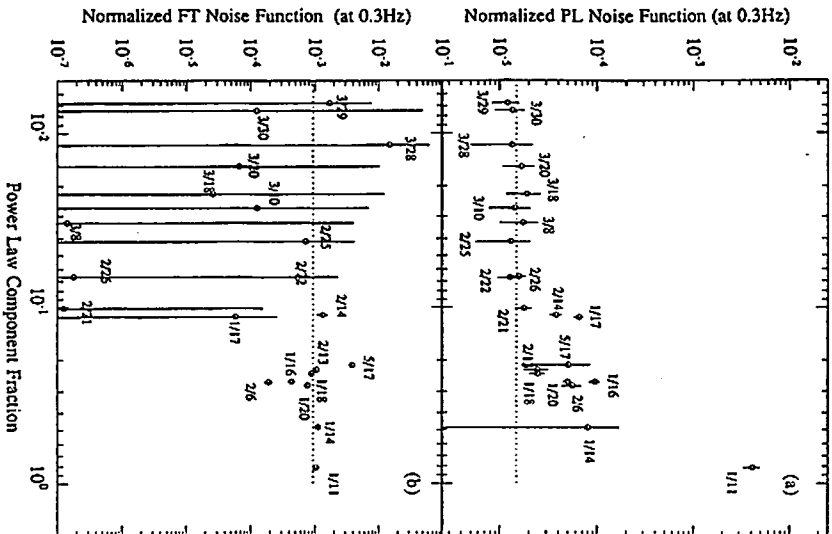


Figure 4

Fig. 2. The values of the NPSD of GS1124-683 at 0.3 Hz are shown against the power law component fraction (count fraction of the hard power law energy spectral component to that of the total X-rays). The errors are quoted by the one parameter 90% error.

Fig. 3. The values at 0.3 Hz of the PL noise function normalized by the disk blackbody component and the FT noise function normalized by the power law energy spectral component in the high state. The errors are quoted by the one parameter 90% error. The dotted lines are 1.49×10^{-5} and 9.3×10^{-4} for the normalized PL noise function (a) and for the normalized FT functions (b), respectively.

Fig. 4. Typical PL noise function and FT noise function normalized by the disk blackbody component and the power law energy spectral component respectively (the normalized PL noise function and the normalized FT noise function). The NPSDs in the low state are also shown for comparison.

Fig. 5. A sketch of the proposed X-ray production process in the accretion disk in the high and the low state in the black hole candidates.

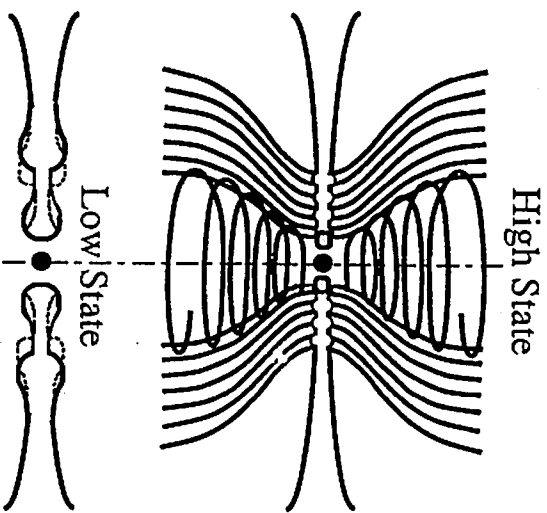


Figure 5

Chaos in Static Axially Symmetric Vacuum Spacetime

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Abstract

We examine a relationship between chaos of test particle in static, axially symmetric vacuum spacetimes and a local instability of the orbit estimated by the Riemann curvature. Although the local instability may become the necessary condition for the geodesic to be chaotic if there exists some symmetry, there is in general no one-to-one correspondence. Instead, we find that the existence of unstable periodic orbit, which does not appear in Newtonian system, may be better criterion for strong chaos in GR.

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1 Introduction

Chaos is now one of the most important idea to explain various non-linear phenomena in nature, e.g., the Kirkwood gap in the asteroid belt is explained by chaotic behaviors. We know many features of chaos in the Newtonian dynamics. However, we do not know, so far, so much about those in general relativity (GR). If gravity is strong, we have to use the Einstein's theory of gravitation. We may find a new type of chaotic behaviors in strong gravitational fields, which does not appear in Newtonian mechanics. Since the Einstein equations are non-linear, we also expect that not only the particle motion [1] but also the gravity (the metric or the curvature) itself [2] show chaotic behaviors. Although there many works on Bianchi IX spacetime, we still do not know whether chaos really occur in such a spacetime because a choice of time coordinate is arbitrary in GR. Before understanding this new type of metric chaos, we should study more about the chaotic motion of test particle in a fixed curved background, in which we may not worry about the coordinate choice ambiguity. Here we will analyze the motion of test particle in static axisymmetric vacuum spacetime and search for a criterion for chaos to occur.

One may expect that if there is a local instability, which means a deviation of initial position diverges exponentially in some local region, it could be either a

sufficient or necessary condition for chaos. In Newtonian mechanics, there are two approaches to examine a relation between the local instability and chaos. The first one is to examine the eigenvalues of tidal-force matrix appeared in the deviation equation, which is obtained by linearizing the equation of motion. The second one is to examine a sign-distribution of the curvature of a fictitious space obtained by a conformal transformation, which converts the orbit into a geodesic motion .

In some systems, the local instability by the above two methods can be used as a criterion for strong chaos. For example, in Henon-Heiles system it was shown that only the orbits which pass through a negative eigenvalue region or a negative-curvature region become chaotic [3]. The local instability is the necessary condition for chaos. In Bianchi IX model, Szydowski applied the similar method and concluded that Bianchi IX spacetime is chaotic[4].

In Newtonian mechanics, however, it turns out that both of these two approaches cannot always be a criterion for the chaotic behavior of test particles [5]. Therefore, it is important to study whether the above criterions can be applied in GR to confirm the Szydowski's conclusion. Here we examined whether or not a chaotic behavior of test particles in GR can be determined by a local instability of the orbit. The orbit of test particle is described by a geodesic in a fixed curved background. Then we can determine the local instability from the sign of the curvature. We do not need a conformal transformation to find a geodesic motion as in Newtonian mechanics. Although we use the Riemann curvature, this method corresponds to the first one in Newtonian system because the Riemann curvature becomes the tidal-force matrix in Newtonian limit.

2 Chaos around the Locally Unstable Region

The equation of a geodesic deviation n^a is given by the Riemann curvature as

$$\frac{D^2 n^a}{D\tau^2} = -R_{bcd}^a u^b n^c u^d \quad (2.1)$$

where u^a is the 4-velocity of test particle and τ is its proper time. From (2.1), we see that the norm of the deviation vector n^a satisfies

$$\frac{d^2}{d\tau^2} \|n\|^2 = -2K(u, n) \|n\|^2 + 2g_{ab} \frac{dn^a}{d\tau} \frac{dn^b}{d\tau} \quad (2.2)$$

where $K(u, n)$ is the sectional curvature of 2-surface spanned by two vector u^a, n^a [4]. From (2.2), a local instability of geodesic is determined by the sign of the

sectional curvature $K(u, n)$. If $K(u, n)$ is negative, the geodesic deviation n^a is exponentially unstable. To determine the signature of $K(u, n)$, we define a bivector V_{ab} as follows

$$V_{ab} \equiv u_{[a} n_{b]}.$$

$K(V)$ takes a critical value when

$$\frac{\partial K(V)}{\partial V^A} = 0 \quad \text{for } A = 1 \sim 6 \quad (2.3)$$

$$\leftrightarrow R_B^A V^B = \kappa V^A \quad (2.4)$$

where 6-vector V^A and 6×6 matrix R_B^A are those described by a bivector formalism, and κ is the eigenvalue of the Riemann tensor R_B^A . The critical value of $K(V)$ is equal to $-\kappa$. Hence the eigenvalue of the Riemann curvature determines the local instability of geodesic. In the case of vacuum spacetime, the Riemann tensor R_B^A is equal to the Weyl tensor C_B^A , and it is decomposed into two 3×3 symmetric matrices ; an 'electric part' E and a 'magnetic part' H as follows.

$$(C_B^A) = \begin{pmatrix} E & H \\ -H & E \end{pmatrix} \quad (2.5)$$

From this form, it can be easily shown that six eigenvalues of Weyl tensor are composed of three independent eigenvalues and their complex conjugates. In a static spacetime, six eigenvalues degenerate into real three independent eigenvalues because H vanishes. Since the trace of the electric part E vanishes and then the sum of these eigenvalues becomes zero, we can divide the spacetime into two regions by the sign distribution of the eigenvalues. The one is a region where two eigenvalues are negative and one is positive and the other is a region where two eigenvalues are positive and one is negative. The positive eigenvalue of Riemann tensor means negative critical value of $K(V)$, which leads to the exponential deviation. Hence we call these two regions a locally stable (LS) and a locally unstable (LU) regions, respectively.

In a static axially symmetric spacetime, the Einstein equations become simple, and some technical methods to get exact solutions have been developed [6], [7] and then we can easily construct exact solutions with any mass distribution. Here we examine a system with n -point singularities on symmetric axis [6], which we call n -Curzon spacetime because $n = 1$ case corresponds to the Curzon spacetime.

We have mapped the LS and LU regions in some of these exact solutions and examined the chaotic behaviors of test particles passing around there by the Poincaré map and the Lyapunov exponent. We have to define a measure of the deviation vector n in the phase space in order to calculate the Lyapunov exponent. As the measure, we have adopted

$$D(\tau)^2 = g_{ab}n^a n^b + g_{ab} \frac{Dn^a}{D\tau} \frac{Dn^b}{D\tau} \quad (2.6)$$

with τ being the proper time of test particle, which seems to be plausible because it is covariant and positive definite. Then the Lyapunov exponent λ defined by this measure is as follows,

$$\lambda = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \ln \frac{D(\tau)}{D(0)} \quad (2.7)$$

For a spacetime with a single point-mass singularity, such as Curzon [8], the LU region appears only around singularity. The orbit of test particles is trapped only in the LS region (Fig1a). In that case, we did not find any chaotic behavior (Fig1b). On the other hand, for the spacetime with several-equal-mass singularities [6], [7], the LU region appears not only around the singularities but also in the region distant from the singularities. The orbit of test particles can be trapped there (Fig2a, 2b, 3a). We then find a strong chaotic behavior only for trapped orbits passing through the LU region, and any orbits moving only in the LS region do not show any stochastic behavior (Fig2c, 2d, 2e, 2f, 3c). Therefore, in these cases, we can conclude that a passage through the LU region is a necessary condition for a orbit to be chaotic such as the Henon-Heiles system.

In the case with two different masses, e.g. the mass ratio is 1:2, however, chaos occurs even for the orbit which is completely included in the LS region (Fig4). Consequently, our criterion for chaos is not generic and could be used to a limited dynamical system with some symmetry.

3 Concluding Remarks

We have seen that an occurrence of chaos is closely related to the distribution of eigenvalues of Riemann curvature if the background spacetime has some symmetry such as our equal-mass system. The chaotic behavior of test particles always appears only around the LU region where two eigenvalues are negative. It seems that the local instability is closely related to a global instability.

As we also showed, however, this result is no longer valid for the system with different masses. We find that there is no one-to-one correspondence between an occurrence of chaos and a local instability. Instead of local instability, we find that the existence of unstable periodic orbit, which does not appear in Newtonian system, may be better criterion for strong chaos in GR[10].

Although our criterion by the local instability is limited to the case with equal masses, we can predict where strong chaos appears exactly. If our result is still valid for a rotating system with two equal masses, we can predict strong chaotic behavior of test particle around the equal-mass binary systems. Such an application may be important and useful to understand some physical process in relativistic astrophysics.

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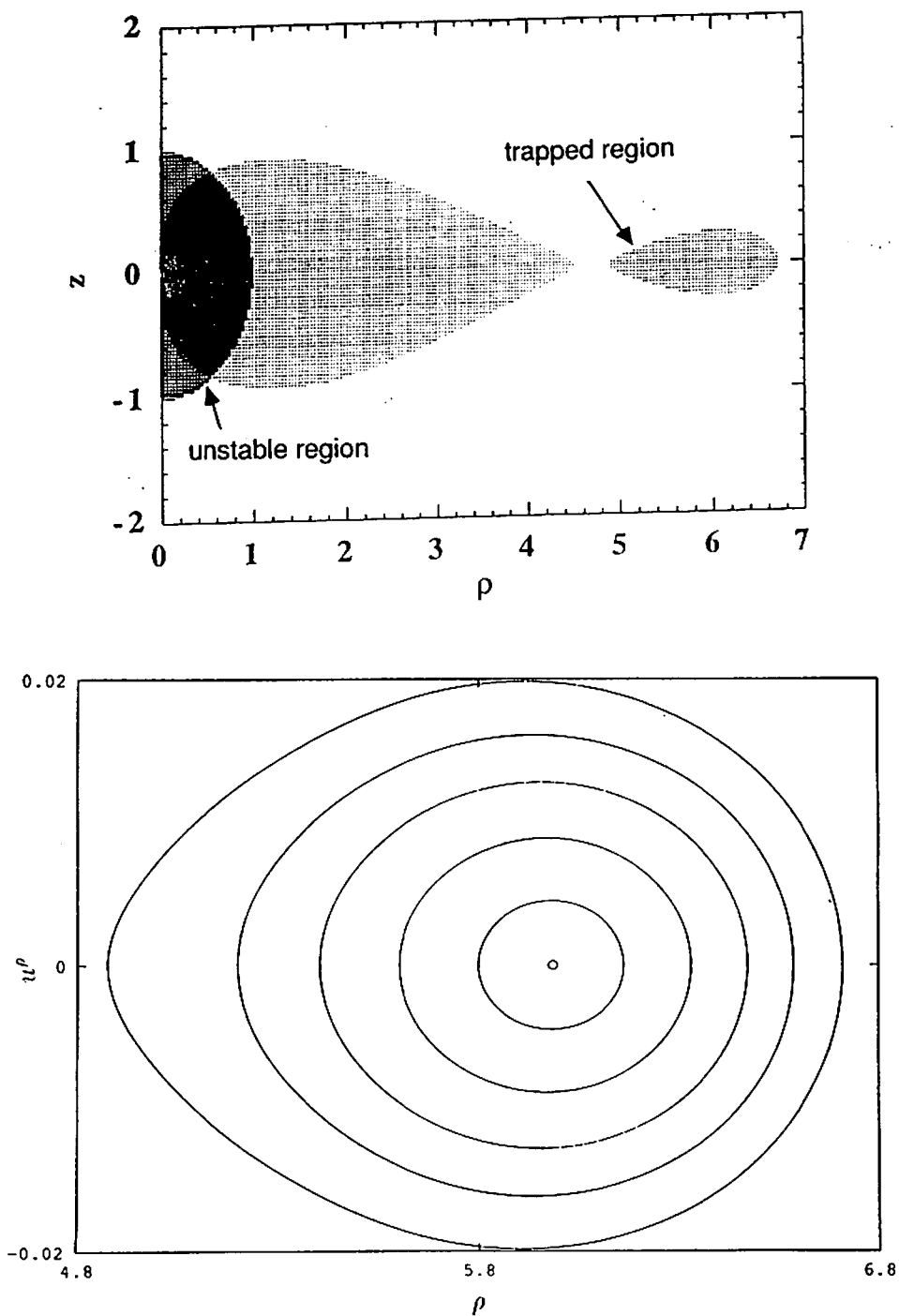


Figure 1: (a) is the unstable region and the trapped region of Curzon spacetime. In this picture, unstable region can be seen only around singularity, and trapped orbit can go through only stable region. (b) is Poincaré Map of the orbits which is completely included stable region. All of the tori are not broken.

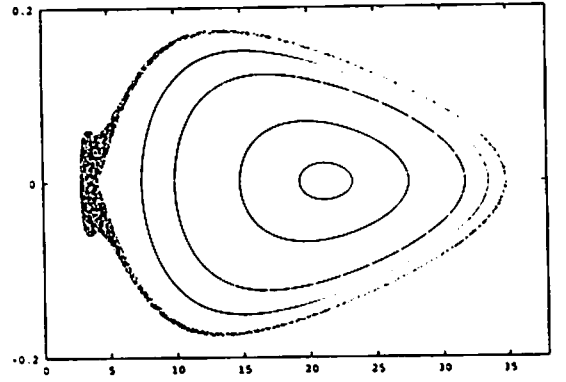
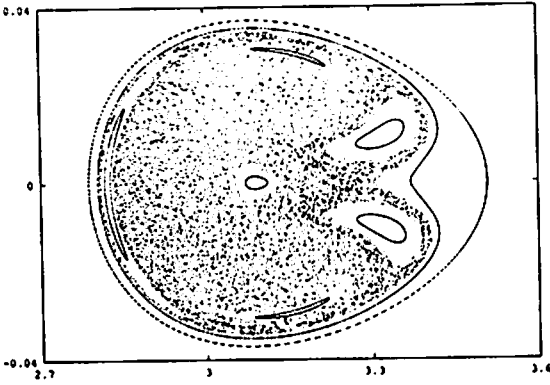
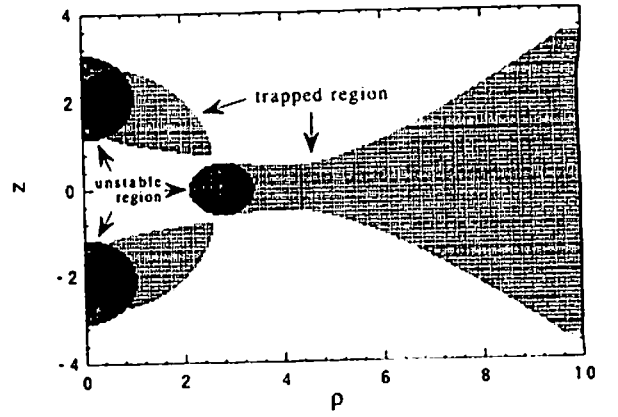
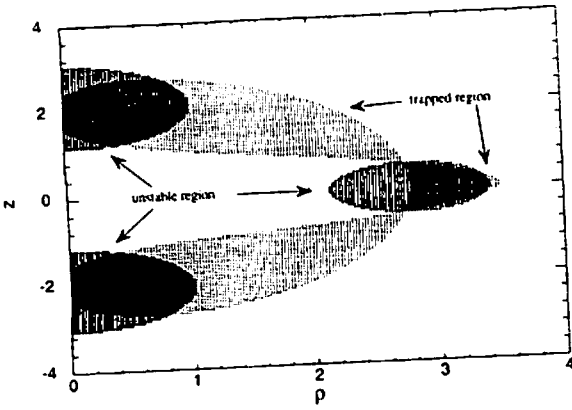


Figure 2: Chaos in the 2-Curzon spacetime with two equal masses. (a) and (b) are the unstable region and the trapped region for $E^2 = 6.8$ and $L = 0.88957$, and for $E^2 = 7.0$ and $L = 0.924$, respectively. Some bounded orbits pass through the unstable region. (c) is the Poincaré map of the orbits of (a) which are completely included in the unstable region. The tori are completely broken. (d) is the Poincaré map of the orbits of (b). Only the torus whose orbit passes through the unstable region is broken.

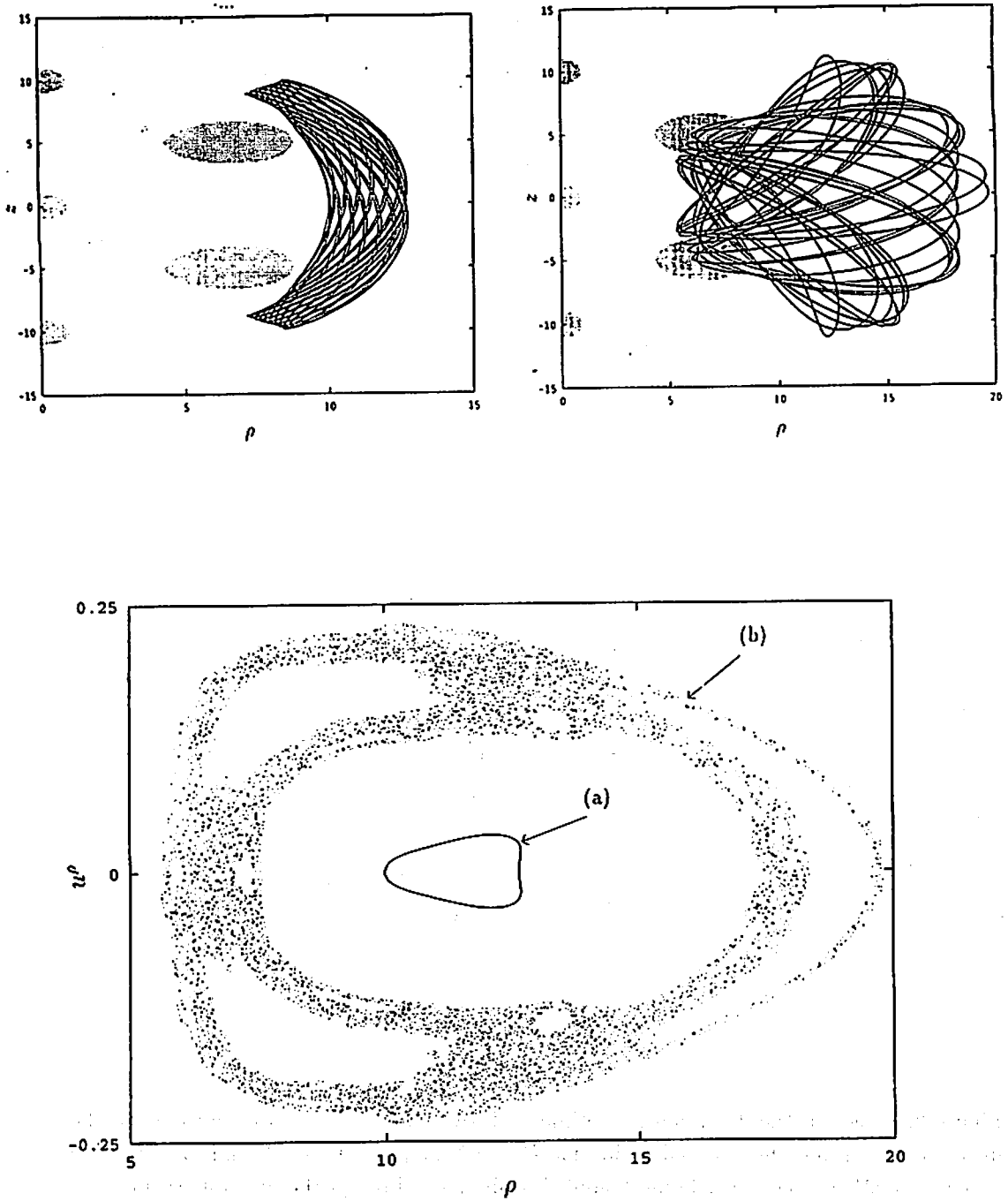


Figure 3: Chaos in the 3-Curzon spacetime with three equal masses. (a) and (b) show the orbits of test particle with $E^2 = 6.0$ and $L = 0.81$ in the configuration space. The shaded area means unstable region. The orbit which passes through the unstable region (case(b)) becomes stochastic. (c) is the Poincaré map of (a) and (b). Only the orbit of (b) is chaotic.

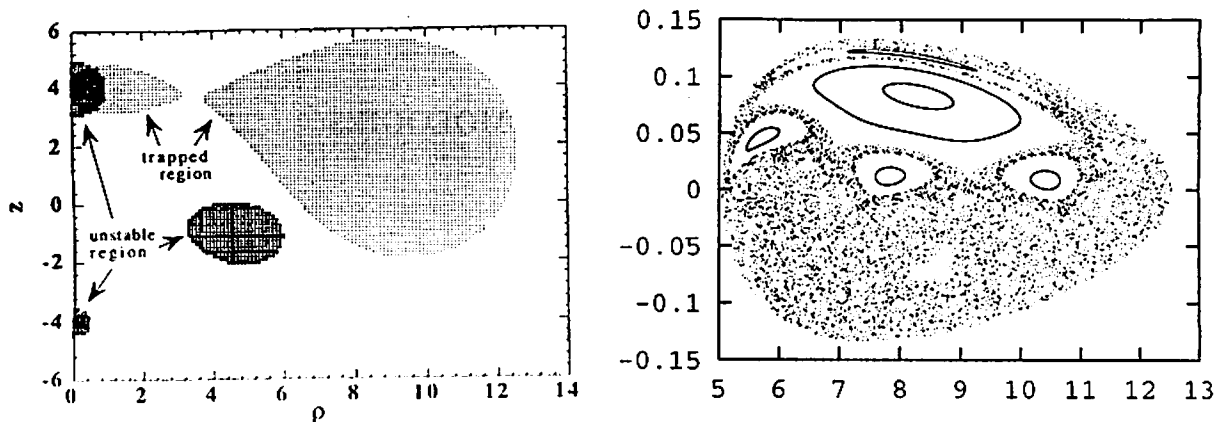


Figure 4: Chaos in the 2-Curzon spacetime with two different masses. The mass ratio is 2:1. (a) shows the unstable region and the trapped region. (b) is the Poincaré map of the orbits of (a). Although the trapped region do not overlap with the unstable region, the orbit of test particle is chaotic.

EXTENDED ROBERTSON-WALKER COSMOLOGIES

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Abstract. Some physical effects of a non-trivial topological structure of space-time are pointed out. We discuss particle creation, backscattering, anisotropy in the microwave background, parity violation and how all this relates to the multiple connectivity of the open spacelike slices.

I. INTRODUCTION

Cosmology - so far as the global structure of the universe is concerned - has always been somewhat at odds with the basic criterion of verifiability, that we rightly impose, since Galilei's time, onto a physical discipline.

This has lead to a very cautious attitude of many physicists toward cosmological modeling. So, for example, the idea that the universe is infinite has been rejected on the grounds that we will not be able to look at infinity, and to verify what is happening there. Other quite attractive arguments, like the atomicity of matter, or Mach's principle, were put forward to plead for the closure of space. A noticeable exception is Ref. 1. I think it is also fair to say, that closed universes are handier for heuristic reasoning.

However, the predictive records of such universes, tied to the principles of perfect homogeneity and isotropy, are extremely poor. It may also be appropriate to mention here that over the centuries most natural philosophers attached to the concept 'Universe' the attribute 'infinite', and some kind of non-trivial evolution toward a non-trivial end.

Although we are not able to look at infinity, there could nevertheless be the possibility to test infinite cosmological models, provided that local, microscopic phenomena are influenced by the global topological and metrical structure of the universe. Consequently, if we start with this assumption, we must abandon the reasoning 'we do not know the global structure, and therefore we assume the simplest possible'. Instead we have to try different

topological and metrical scenarios, and to find traces of them in the quantum fields, and then we have to see what makes the difference.

It is largely agreed today that the rough overall evolution of the universe should be described by Riemannian geometry, but virtually nothing is known about the topological structure of space-time. Though allusions to this question emerge in the literature for a long time, cf. Refs. 2-4, neither philosophers nor scientists have been much inspired by it.

In Refs. 5-7 we suggested a practical way to gain more insight into the global topological structure of the universe. We assumed a given topological scenario, and tried to find physical effects of the topology. Do particles, rays, fields, and galaxies behave qualitatively differently in infinite universes of different connectivity? In the long run they do.

Our basic assumption is that the universe is open, infinite, without boundaries, and locally endowed with a Robertson-Walker line element of negative spatial curvature. The topology of the spacelike slices that we investigated was either $I \times D_N$ or $I \times S$, where I is a finite open interval, D_N is a disk with N smaller disks cut out, $N \geq 1$, and S is a Riemann surface of genus $g \geq 2$. The product $I \times S$ is topologically a thickened surface, $I \times D_N$ is a solid handlebody. These structures are the two major classes of topological three-manifolds that can be used as spacelike slices of universes with the above mentioned properties. These spaces have a very compact topological representation, but they are actually infinite, without physical boundaries, because the metric on them gets singular at the topological boundaries.

Though contemporary cosmology originated from general relativity, the input that comes from this theory is rather small and qualitative: the existence of a space-time line-element, the geodesic principle, the principle of general covariance. Additional assumptions are needed to fix the actual form of the line-element: the principles of homogeneity and isotropy assure that the 3-space is of constant curvature at any given instant of time. That determines the line-element apart from a time-dependent expansion factor $a(\tau)$ that sets the length scale in the 3-space, and apart of the sign of the curvature of the 3-space.

What have Einstein's equations then to contribute to cosmology? Clearly they do not determine the metric, even not locally, because for that we would have to know the energy-momentum tensor of the universe, which is not the case. If we insert the RW-metric into the Einstein equations we can however express the energy and the pressure density of the universe as a function of $a(\tau)$ and its derivatives. The requirement of positivity of energy and pressure gives then restrictions on the asymptotic behavior of $a(\tau)$, for example in the case of negative curvature positivity would require $a(\tau) \sim \Lambda\tau$ for $\tau \rightarrow \infty$. (See Ref. 1 for striking criticism on these predictions). In the limit $\tau \rightarrow 0$ (or $\tau \rightarrow -\infty$, if one believes in an infinite past) positivity arguments of this kind are not reliable, because in this limit either pressure or energy density diverge to infinity, and the classical Einstein equations are not supposed to give even qualitatively correct results in this regime. Anyhow, the expansion factor, the pressure and the energy density remain unknown functions, and even if one postulates ad hoc and in addition a universal thermodynamic equation of state this situation is not remedied, particularly not in an infinite space. I do not want to deny here the considerable heuristic and phenomenological value of such supplements, however we are here more interested in the underlying structure of space-time, and on its bearing on the microscopic dynamics, and about that these equations have little to say.

Up to now we have only addressed the local, metric structure, the line element. In cosmology a much more important question is the global structure of space-time, the global topology, which is clearly left totally undecided by general relativity, being a purely metrical theory. Usually this question is settled in textbooks by postulating maximal symmetry, the existence of a continuous six-parameter symmetry group transitively acting on the space-like slices. This postulate restricts then the possible topologies to be either Euclidean space, or the 3-sphere, or projective 3-space, or a shell of the Minkowski hyperboloid. Moreover there is only one metric of constant curvature that one can impose on these topologies. Maximal symmetry implies perfect homogeneity and isotropy.

This principle of maximal symmetry of the 3-space is of course very convenient, because it fixes once and for all both the topology and the geometry. However we regard it as too rigid, too static, unnecessary restrictive, and we suggest to relax it by requiring instead of maximal symmetry constant negative curvature. Moreover we require that the 3-space is infinite. As pointed out in the preceding papers this gives rise to a variety of possible topologies. Moreover the topology and the requirement of constant curvature does not any more determine the metric uniquely, which therefore may itself vary in time, apart from the scaling with the expansion factor. Locally, on sufficiently small coordinate patches we have still the six-parameter group of isometries, and the space is locally homogeneous and isotropic. Finally I mention that as soon as the topology is non-trivial there does not any more exist a continuous group of global isometries, in a sense it is replaced by the discrete covering group of the 3-space manifold.

Because the actual impact of Einstein's equations is rather small, one has to pose the question in how far Riemannian geometry enters at all in cosmologies with constantly curved spaces. Though it provides the underlying structure of the principle of general relativity, the actual use of Riemannian geometry is trivialized by the simplicity of the RW-line element. At first, the time-coordinate in the line-element is extremely distinguished, which highly encourages the use of 3-dimensional formalism, using time as a parameter. The second reason is that the constant sectional Gaussian curvature of the 3-space makes the use of the Riemann curvature tensor superfluous, and renders the 3-space more to an object of elementary geometry.

In the following Sections we report on some phenomena that arise in infinite and multiply connected cosmologies. For the technical details the reader is referred to Refs. 5-9.

II. TOPOLOGICAL BACKSCATTERING AND PARTICLE PRODUCTION IN ELECTROMAGNETIC AND NEUTRINO FIELDS.

In Ref. 7 we discussed how an open hyperbolic 3-manifold can undergo global metrical deformations, without changing its constant curvature. In simply connected RW-cosmologies it is known for a long time, cf. Ref. 10, that variations of the expansion factor can lead to particle creation in quantum fields, and to backscattering in classical fields. I mention here, that one does not need a rapidly varying expansion factor to obtain this effect, linear expansion causes in a simply connected open RW-cosmology particle creation in a massive Dirac field, cf. Ref. 9.

These effects cannot happen in conformally coupled fields, like neutrinos and electromagnetic fields, because in the solutions of the corresponding wave equations the expansion factor scales out with a simple power law, but global metrical deformations of the 3-space manifold do create particles even in conformally coupled fields, Fig. 1 illustrates how this comes about. We divide the time axis τ for simplicity into three intervals. In I and III the metric ds^2 on the spacelike slices is time independent, apart perhaps from a time dependence via the expansion factor, that is irrelevant here. The wave equation is time separable, and because the field is also conformally coupled, positive and negative frequencies can be unambiguously defined.

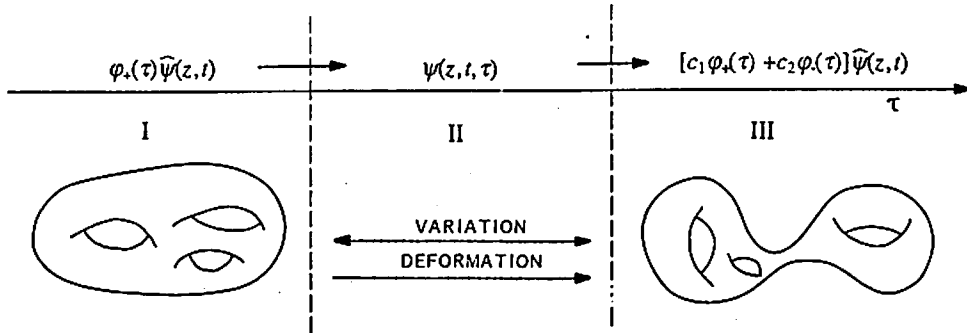


Figure 1. Global metrical deformations of the spacelike slices- without a change in the local curvature- lead to particle creation and backscattering. A wave packet, initially composed of positive frequencies, receives an admixture of negative frequencies during the deformation.

In the second interval a global metrical deformation takes place; in this period the wave equation is not separable, and the particle/antiparticle concept loses its meaning here. Finally, in interval III, the general solution of the wave equation is a linear combination of positive and negative frequency modes. If the variation in II is generic, none of the coefficients c_1, c_2 will be zero. Therefore antiparticles are created in the quantum case, and a backscattered wave appears in the case of an electromagnetic field.

III. ANGULAR FLUCTUATIONS IN THE TEMPERATURE OF THE MICROWAVE BACKGROUND

Angular anisotropy in the Planck distribution is a possible consequence of global metrical deformations of the spacelike slices, cf. Ref. 7. Assume for the moment that the spacelike slices are just copies of hyperbolic space H^3 , the upper shell of the Minkowski hyperboloid. Because H^3 is homogeneous, it happens that the eikonal of geometric optics appears in the phase of the spectral elementary waves of the Maxwell equations. This relation is in other systems only true approximately, semiclassically, here and in homogeneous

spaces in general it is exact. We use the Poincaré half space realization of H^3 , i. e. the upper half space with rectangular coordinates (z, t) , z complex, $t > 0$.

Geometric optics does not know the concept of momentum. However, because of this relation we attach to the rays a momentum via the Einstein relation $p_\mu = \hbar k_\mu$. So we obtain a vector field on H^3 , $p_\mu(z, t, \tau; \eta, s)$, describing the momentum of a horospherical bundle of classical flow lines. τ denotes time, s is a spectral parameter, labeling the elementary waves. The directions are labeled by η , cf. Fig. 2, a degeneration index in spectral theory. This flow coming onto an observer at (z, t) , is obviously isotropic.

Spacelike slices that are multiply connected hyperbolic manifolds we represent as a polyhedron F in H^3 , the universal covering space. The covering group Γ is generated by the face-pairing transformations of F . We project the horospherical bundles together with the vector fields attached to them into the 3-manifold (F, Γ) , cf. Ref. 8. The spectral parameter η is now restricted to the boundary at infinity of the 3-manifold, cf. Ref. 7. It is easy to see that for any given $p_\mu(z, t)$ in F there is exactly one ray coming from the boundary, that has this momentum at (z, t) . Thus the flow stays isotropic, and we have again the same Planck distribution $\rho(h\nu/kT)$ as in Minkowski space.

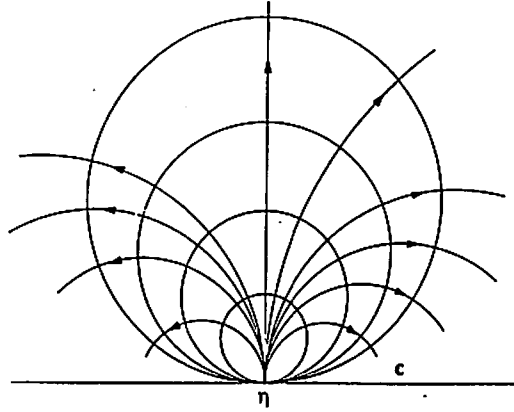


Figure 2. Wave fronts of horospherical elementary waves generated at a point η at infinity of H^3 (the horizon). They appear as the surfaces of constant phase in the eigenfunctions of the Maxwell equations. The bundle of rays issuing from η comprises their orthogonals. Accordingly the surfaces of constant action and phase coincide for rays and elementary waves emanating from a point at infinity.

Let us finally switch on adiabatically a global deformation $\tilde{g}_{ij}$ of the 3-space metric. The perturbed horospherical eikonal for rays issuing at some point η on the horizon, cf. Fig. 2, is $\tilde{\psi}^r = \psi^r + \chi(z, t, \tau; \eta)$, χ is a periodic scalar field with respect to the covering group. The perturbed frequencies are $\tilde{\nu} = \nu \left(1 + \frac{1}{\omega} \frac{\partial \chi}{\partial \tau} \right)$, which means that we have to replace in

$\rho(h\nu/kT)d\nu$ the temperature by $\tilde{T} = T \left(1 - \frac{1}{\omega} \frac{\partial \chi(z, t, \tau; \eta)}{\partial \tau} \right)$, which amounts to an angular variation η of the temperature in the distribution, that remains otherwise unchanged. The adiabatic time dependence of the temperature reminds us that we have here only a first approximation to a non-equilibrium process. A further nice feature of these infinite cosmological models is that we can carry out the thermodynamic limit in them, so that the usual convexity properties of the thermodynamic quantities hold true.

IV. CLASSICAL AND QUANTUM DISPERSION

One of the most remarkable features of simply connected RW cosmologies of negative spatial curvature is the instability of the classical geodesic trajectories, the probabilistic character of world lines. The most efficient and quantitative way to describe such systems that are highly sensitive with respect to the choice of the initial conditions is that of statistical mechanics.

In Ref. 6 we developed a probabilistic description of this instability, similar to the Liouville equation, but in a manifestly covariant, non-Hamiltonian form. To achieve this we introduced the concept of a horospherical geodesic flow of expanding bundles of parallel world lines. We constructed an invariant measure and a covariant evolution equation for the probability density on which this flow acts.

The orthogonal surfaces to these bundles of trajectories are horospheres, closed surfaces in 3-space, touching the boundary at infinity of hyperbolic space, where the flow lines emerge, cf. Fig.2. These horospheres are just the wave fronts of spherical waves which constitute a complete set of eigenfunctions of the Klein-Gordon equation. This fact suggests to compare the evolution of the quantum mechanical density with the classical one on which expanding bundles of geodesic flow lines act. We found asymptotic identity in the asymptotically flat region and in periods of adiabatic expansion, when no particle production processes occur. This lead us furthermore to study the time behavior of the dispersion of the energy and the coordinates and of the energy-time uncertainty relation, and we found again identity in the late stage of the cosmic evolution. This identity can persist in the early phase of the expansion with a rapidly varying scale factor, provided the fields are conformally coupled to the curvature, and provided the variation permits to disentangle positive and negative frequencies, so that particles can emerge.

V. PARITY VIOLATION DUE TO SELF-INTERFERENCE OF SPACE-REFLECTED WAVES

The universal covering space construction provides here again an easy and explicit way to define a space reflection on the multiply connected 3-space, see Ref. 9. At first we define the covering projection $\pi^\Gamma: H^3 \rightarrow F$, $\pi^\Gamma(z, t) = \gamma^{-1}(z, t)$, if $(z, t) \in \gamma(F)$. We refer here to the tiling $\Gamma(F)$ of H^3 by images $\gamma(F)$, $\gamma \in \Gamma$, of the fundamental polyhedron F , cf. Sec. III. Fig. 4 shows a typical tiling induced by $\Gamma(F)$ on the boundary at infinity of H^3 . The space reflection in (F, Γ) we define finally by $P^\Gamma(z, t) = \pi^\Gamma(P(z, t))$. P is a space reflection in H^3 .

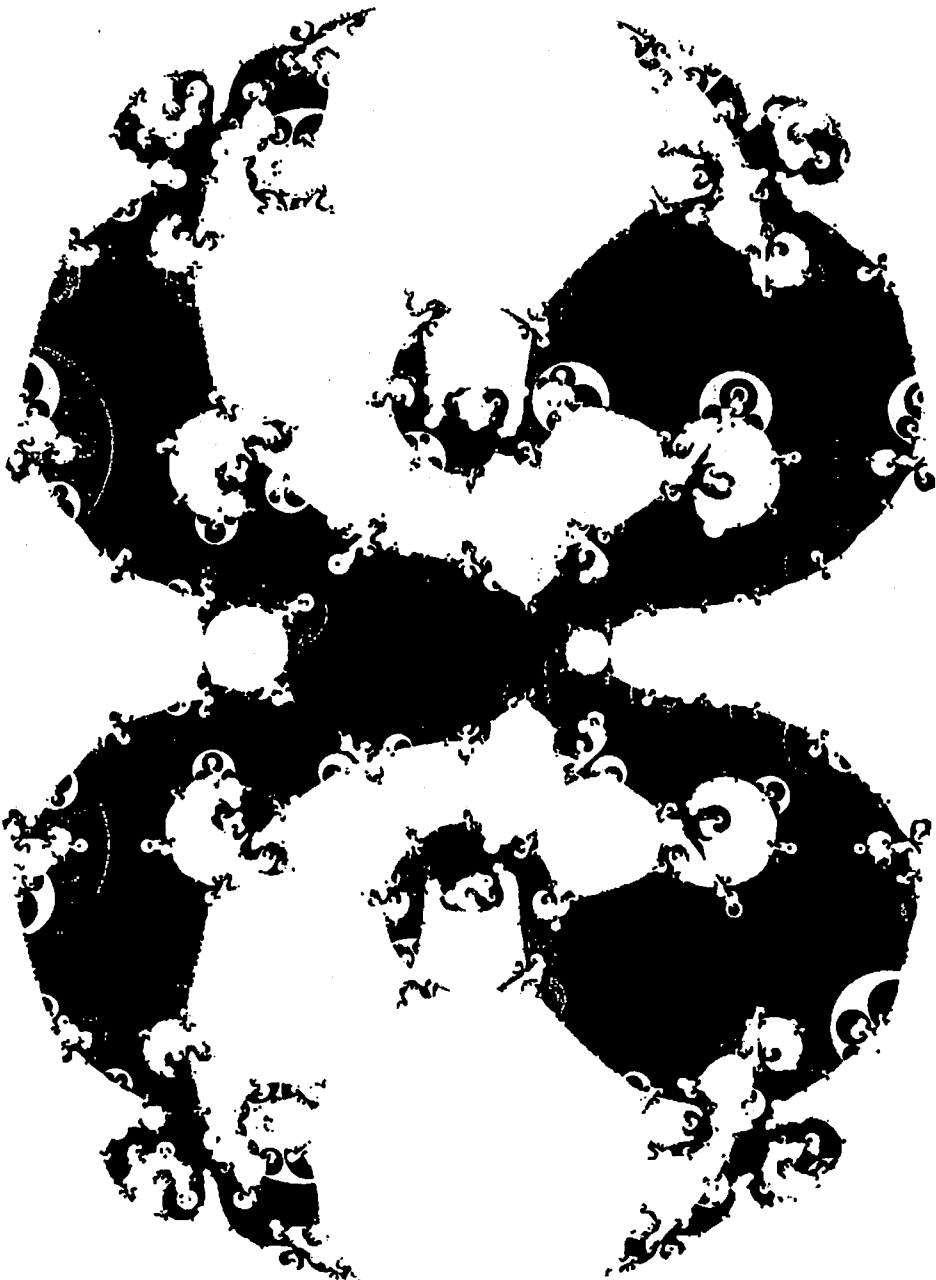


Figure 3. Tiling on the plane at infinity of the Poincaré half-space H^3 . A spacelike slice (F, T) is realized in H^3 as a polyhedron F with a face-identification. The identifying transformations generate a discrete group T . This group T applied to the polyhedron gives a tessellation $T(F)$ of H^3 with polyhedral images. This tessellation induces a tiling on the boundary of H^3 . There is also a tiling exterior to the Jordan curve. Figures 3 and 4 stem from manifolds which are topologically equivalent but non-isometric ($l \times S$, $g(S) = 19$, $g(A \text{ of Fig.3}) \approx 1.3$, $g(A \text{ of Fig.4}) \approx 1.4$).

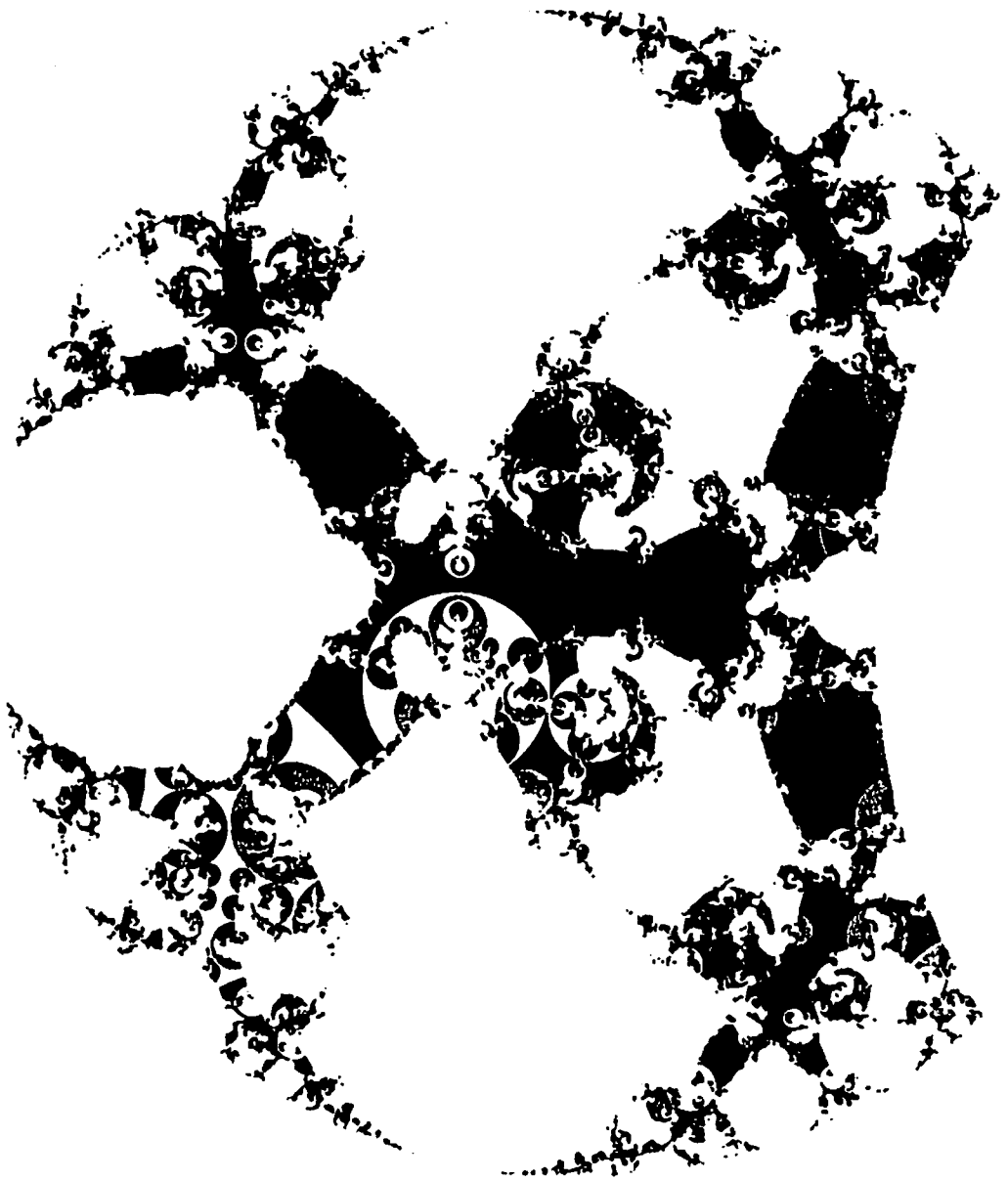


Figure 4. The singular curve $\Lambda(\Gamma)$ is the set of accumulation points of the polyhedral tiling $\Gamma(F)$ of H^3 . The half-space H^3 is the universal covering space of a multiply connected three-manifold (F, Γ) that represents a spacelike slice of the universe at a given instant of time. The centre $C(\Lambda) \setminus \Gamma$ of the three-manifold is the intersection of the hyperbolic convex hull $C(\Lambda)$ of $\Lambda(\Gamma)$ with the polyhedron F . Its boundaries on the polyhedral faces are identified by elements of the covering group Γ . Trajectories that are mixing in the centre of the spacelike slices are unstable and have covering trajectories with an end point in $\Lambda(\Gamma)$.

The classical geodesic equations are reflection invariant, but the situation is quite different concerning wave mechanics. Imagine a wave packet concentrated on a finite domain in the manifold (F, Γ) . If we apply P^Γ , it can happen that the reflected wave wraps around a handle of the manifold, cf. Fig. 5. The wave packet starts to interfere with itself, and its norm is not preserved.

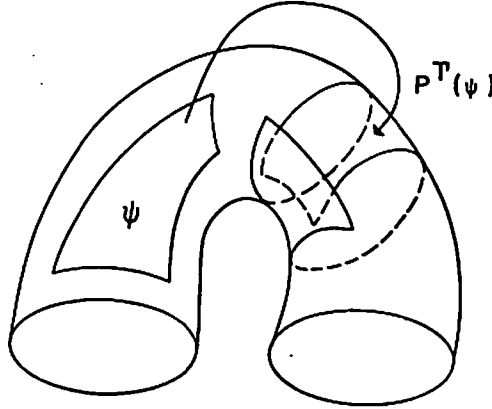


Figure 5. In two dimensions: a wave packet concentrated on a strip. The space reflection wraps the strip around the handle, so that it overlaps with itself. That results in self-interference.

Because of this self-interference the space reflection symmetry is already broken on the level of the free Dirac equation. The T -symmetry is likewise broken because of the time dependence of the metric. C is still a good symmetry, but all combinations of C , P , and T are broken. That is remarkable, because usually one has to attach on purpose symmetry breaking interactions in order to achieve this. One can speculate if the baryon asymmetry is topologically generated. Clearly CP violation is a pure quantum effect, unthinkable in the classical context. It is interference, not so much dispersion, that distinguishes quantum mechanics from classical dynamics, cf. Sec. III. Accordingly it seems perfectly natural to associate CP violation with an interference phenomenon.

VI. CONCLUSION

The phenomena reviewed here indicate that from the investigation of the topological structure of spacetime further surprising consequences can be expected.

ACKNOWLEDGMENTS

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A Few Remarks on the Effects of Nonlinear Inhomogeneity in the Universe

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Abstract

Our understanding of the universe is based on the working hypothesis that the homogeneous and isotropic models give a successful description on a large scale, in spite of the nonlinear inhomogeneity of the matter distribution. What is the theoretical basis upon which our conviction relies? In order to examine this compatibility problem, we present a scheme to construct inhomogeneous universe which are nevertheless homogeneous and isotropic on average. Nonlinear evolution of the density contrast in such a universe is studied in comparison with that in linear theory. We also investigate the possible observational effects in a globally inhomogeneous, but locally Friedmann model. In particular, it is pointed out that the Hubble parameter H_0 may depend on the scale of observation. This implies that a locally ($< 100\text{Mpc}$) large ($\sim 100\text{km/s/Mpc}$) value of H_0 does not necessarily represent the global expansion rate of the universe.

1 Introduction

The remarkable isotropy of the cosmic microwave background now strongly supports the idea that the universe is homogeneous and isotropic in some averaged sense and well described by the Friedmann-Lemaître-Robertson-Walker (FLRW) models. The universe in reality is, however, neither homogeneous nor isotropic, and the density contrast is larger than unity up to the scale of superclusters. Although the FLRW models provide a successful description of the averaged overall behaviour, the averaging procedure is still not understood in detail mainly because of the non-linearity of the Einstein equations.

As a first step in order to describe a realistic inhomogeneous universe, we construct an inhomogeneous model universe which is homogeneous and isotropic on average in Section 2. The model can be regarded as an exact perturbation of the FLRW models. In Section 3, we investigate the nonlinear evolution of the density contrast in the inhomogeneous universe in comparison with that in linear perturbation theory. In particular, we explicitly show the asymptotic behaviour of the density contrast δ in “open” models and “flat” models with a positive cosmological constant is different from the result in linear theory in the sense that δ can go infinite.

Section 4 gives an alternative viewpoint on the effect of inhomogeneity in the universe on a different scale. If we accept the idea that the universe is described by the FLRW models, the evolution is completely determined by a few parameters, i.e. the Hubble “constant” H_0 , the density parameter Ω_0 (and possibly the normalized value of the cosmological constant $\lambda_0 \equiv \Lambda/3H_0^2$). Recent observations indicate relatively larger value $H_0 \sim 75 - 100 \text{ km/s/Mpc}$ on scales, say, $< 100 \text{ Mpc}$. This “locally” determined value of H_0 , if it is universal, can be reconciled with the cosmic age indicated by globular cluster evolution only with a finite positive cosmological constant. Another observations, however, sometimes indicate smaller value of $H_0 \sim 50 \text{ km/s/Mpc}$ or even less. One of the examples is the time delay measurement of the gravitational lens system QSO0957 + 561. The red-shifts of the lens and the source are $z_L = 0.36$ and $z_S = 1.4$, respectively, therefore the distance scale is essentially thousands Mpc. Although the true value of H_0 is still the subject of lively debate, the different results of H_0 on different scales leads us to the following question: is H_0 really a universal constant? If the universe is inhomogeneous, it is likely that the Hubble parameter may depend on scale. We investigate the above possibility by using a simple inhomogeneous model. Finally in Section 5, we summarize possible implications of nonlinear inhomogeneity in observational cosmology.

2 Inhomogeneous, but on-average Friedmann cosmology

We consider models which contain irrotational dust with density ρ and 4-velocity u^μ and a cosmological constant Λ . In comoving coordinates with $u^\mu = (1, 0, 0, 0)$ and $g_{0i} = 0$, the Einstein equations read

$$\frac{1}{2} \{ {}^{(3)}R + (K^i_i)^2 - K^i_j K^j_i \} = 8\pi G\rho + \Lambda, \quad (1)$$

$$K^i_{j|i} - K^i_{i|j} = 0, \quad (2)$$

$$\dot{K}^i_j + K^k_k K^i_j + {}^{(3)}R^i_j - (4\pi G\rho + \Lambda)\delta^i_j = 0, \quad (3)$$

and the energy equation $u_\mu T^{\mu\nu}_{; \nu} = 0$ gives

$$\frac{\partial}{\partial t} \rho + \rho K^i_i = 0, \quad (4)$$

where K^i_j is the extrinsic curvature, a dot ($\dot{}$) denotes $\partial/\partial t$, $|$ denotes the covariant derivative with respect to g_{ij} , ${}^{(3)}R^i_j$ is the Ricci tensor of the 3-metric, and ${}^{(3)}R = {}^{(3)}R^i_i$. From the Bianchi identity, we have the propagation equation for ${}^{(3)}R^i_j$;

$$\frac{\partial}{\partial t} {}^{(3)}R^i_j + 2K^i_\ell {}^{(3)}R^\ell_j = K^i_{\ell j}{}^{|\ell} + K^\ell_j{}^{|\ell}{}_i - K^i_j{}^{|\ell}{}_\ell - K^\ell_\ell{}^{|\ell}{}_j. \quad (5)$$

If the expansion is locally isotropic, the temporal change of the separation of neighbouring galaxies is proportional to the separation length and does not depend on the direction. Therefore

$$\frac{\partial}{\partial t} (g_{ij} dx^i dx^j) \propto (g_{ij} dx^i dx^j). \quad (6)$$

It implies that the metric is conformally static,

$$g_{ij}(t, x^k) = a^2(t, x^k) \gamma_{ij}(x^k), \quad (7)$$

which leads to $K^i_j = \dot{a}(t, x^k)/a(t, x^k) \delta^i_j$. However, from Eq. (2) it turns out that the a does not depend on x^k : a is the function of only t . Using this result and Eq. (5), we obtain

$${}^{(3)}R^i_j \propto \frac{1}{a^2(t)} \delta^i_j \equiv \frac{2k}{a^2} \delta^i_j, \quad (8)$$

which means the 3-space is of constant curvature and the universe is described by the FLRW models.

A conventional approach to treat small inhomogeneities is the linear perturbation theory on the FLRW background. In the case of a nonlinearly inhomogeneous universe, it should also be helpful if we could define the “background” spacetime. Let us define some quantities to describe the average behaviour of an inhomogeneous universe. The mean (or “background”) density ρ_b is obtained by averaging ρ over any sufficiently large region V :

$$\rho_b = \langle \rho \rangle \equiv \frac{1}{\int_V \sqrt{g} d^3x} \int_V \rho \sqrt{g} d^3x. \quad (9)$$

The scale factor $a(t)$ is defined from the “average” volume expansion related to ρ_b :

$$\dot{\rho}_b + 3\frac{\dot{a}}{a}\rho_b = 0, \rightarrow \frac{8\pi G}{3}\rho_b = \frac{1}{a^3}. \quad (10)$$

If $a(t)$ satisfies Friedmann’s equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{1}{a^3} + \frac{\Lambda}{3}, \quad (11)$$

then and only then an inhomogeneous model is “FLRW on average.”

The deviations from the FLRW models are therefore expressed by the peculiar part of the extrinsic curvature

$$V^i_j \equiv K^i_j - \frac{\dot{a}}{a}\delta^i_j, \quad (12)$$

which represents the deviation from the uniform Hubble expansion, and the deviation of the spatial curvature tensor

$$\mathcal{R}^i_j \equiv {}^{(3)}R^i_j - 2\frac{k}{a^2}\delta^i_j, \quad (13)$$

and the density contrast

$$\Delta \equiv \frac{\rho - \rho_b}{\rho}, \quad (14)$$

which is here more convenient than the conventional definition

$$\delta \equiv \frac{\rho - \rho_b}{\rho_b} = \frac{\Delta}{1 - \Delta} \quad (15)$$

in the sense that it obeys the following second order equation very similar to that for δ in linear theory:

$$\ddot{\Delta} + 2\frac{\dot{a}}{a}\dot{\Delta} - 4\pi G\rho_b\Delta = -(1-\Delta)\{(V^i_i)^2 - V^i_j V^j_i\}. \quad (16)$$

Let us investigate the case

$$(V^i_i)^2 - V^i_j V^j_i = 0. \quad (17)$$

It means that the collapse is locally one-dimensional. In this case, Δ obeys the exactly same equation as δ in linear theory, and the solutions can be extrapolated from the result of the linear perturbation theory. Using the growing mode $\Delta^+(t)$ and the decaying mode $\Delta^-(t)$ solutions, we obtain

$$\Delta = \frac{\beta(x)}{A(x, y, z)}\Delta^+(t) + \frac{\mu(x)}{A(x, y, z)}\Delta^-(t) \quad (18)$$

for the density contrast, the metric has the following form

$$ds^2 = -dt^2 + a^2(t)\{A^2(x, y, z)(1-\Delta)^2 dx^2 + \phi(y, z)dy^2 + \varphi(y, z)dz^2\}, \quad (19)$$

which was first found by Szekeres from a different context. The metric contains five arbitrary functions of x . Among them, $\beta(x)$ and $\mu(x)$ represent the amplitude of the growing mode and the decaying mode respectively. If $\beta(x) = \mu(x) = 0$, then the spatial metric becomes conformally static and the spacetime is FLRW. The only property we shall assume here is that the mean values of $\beta(x)$ and $\mu(x)$ vanish, i.e.,

$$\int_{V_*} \beta(x) dx = \int_{V_*} \mu(x) dx = 0. \quad (20)$$

This leads to the interesting consequence that the spacetime is FLRW on average:

$$\langle \rho \rangle = \rho_b, \quad \langle K^i_j \rangle = \frac{\dot{a}}{a}\delta^i_j, \quad \langle {}^{(3)}R^i_j \rangle = 2\frac{k}{a^2}\delta^i_j. \quad (21)$$

3 Non-linear evolution of the inhomogeneities

The solution given in the previous section can be regarded as exact perturbations of the FLRW models. In this section, we shall investigate the physical properties of the solution,

paying particular attention to the nonlinear evolution of the inhomogeneity in a FLRW “background”.

A characteristic behaviour of the density contrast in open models without Λ is that the growth is suppressed at later stages. For sufficiently large values of t , the universe on average expands almost “with the speed of light,” $a(t) \simeq t$, and the density contrast “formally” approaches a constant value

$$\lim_{t \rightarrow \infty} \Delta = \frac{\beta}{A} \quad (22)$$

with respect to time. However, this does not in general mean that the structure formation at later stages is prevented in open models. It is apparent that the density contrast $\delta = \Delta/(1 - \Delta)$, hence ρ , can go to infinity in a finite time if $\beta/A > 1$. (Note that $\Delta < 1$ by definition. Hence the limit Eq. (22) should also be less than unity, if it exists. $-\beta/A > 1$ means the time t never go to infinity.)

A similar situation occurs for $k = 0, \Lambda > 0$ models. Δ again approaches a constant

$$\lim_{t \rightarrow \infty} \Delta = \Delta_\infty \equiv -\frac{2 \Gamma(\frac{5}{6}) \Gamma(\frac{2}{3})}{3\sqrt{\pi}} \left(\frac{3}{\Lambda}\right)^{\frac{1}{3}} \frac{\beta}{A} - \sqrt{\frac{\Lambda}{3}} \frac{\mu}{A}. \quad (23)$$

Therefore, if the five arbitrary functions are chosen such that formally $\Delta_\infty > 1$, such regions can develop a curvature singularity in a finite time even if the scale factor asymptotically shows the exponential expansion $a(t) \propto \exp\left(\sqrt{\frac{\Lambda}{3}}t\right)$. For regions with $\Delta_\infty < 1$, the density fluctuations are eventually frozen, but do not fade away, although the matter density itself becomes negligible compared to Λ .

4 Observational effects in globally inhomogeneous, but locally Friedmann cosmology

In this section, we examine the effect of nonlinear inhomogeneity from an alternative point of view. The typical scale of inhomogeneity under consideration here is much larger than that in the previous section. The motivation is as follows. We usually accept that the universe is described by the FLRW models whose evolution is completely specified by a

few parameters. Among them, the Hubble “constant” H_0 determines the size and the age of the observable universe, and the precise measurement of such a fundamental quantity has been a major undertaking over the past 60 years. Recent observations indicate larger value $H_0 \sim 75 - 100 \text{ km/s/Mpc}$ on relatively small scales. This “locally” determined value of H_0 might be in conflict with the cosmic age indicated by globular cluster evolution, without introducing a cosmological constant. However, another observations on a large scale, such as the time delay measurement of the gravitational lens system, indicate smaller value of $H_0 \sim 50 \text{ km/s/Mpc}$ or even less. Although the true value of H_0 is still the subject of lively debate among observers, it might be interesting to consider the H_0 problem from another point of view. If the universe “locally” look like FLRW but is globally inhomogeneous, the effect of the inhomogeneity might be translated to the value of H_0 depending on the scale of observation. We shall investigate this possibility by using a simple inhomogeneous model in general relativity.

The model we consider is the spherically symmetric dust model, known as the Lemaitre-Tolman-Bondi model. The metric is expressed as follows:

$$ds^2 = -dt^2 + \frac{[\partial_r(a r)]^2}{1 - k(r^2/R^2(r))} dr^2 + a^2 r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (24)$$

It resembles to that of the FLRW models, except that the “curvature radius” R is a function of the radial coordinate and the “scale factor” a now depends on both t and r . The model has been discussed in a number of textbooks and papers, but they often have focused on the positive curvature ($k = +1$) case of a collapsing dust ball. In this paper, we consider the models with negative $k = -1$ or flat $k = 0$ curvature. The Einstein equations for $k = -1$ are

$$\dot{a}^2 a - \frac{a}{R^2} = A, \quad (25)$$

$$\frac{8\pi G}{3} \rho a^2 \partial_r(ar) = A, \quad (26)$$

where A is a constant. The solutions are obtained for $k = -1$

$$a(\eta, r) = \frac{1}{2} R^2(r) A (\cosh \eta - 1), \quad (27)$$

$$t(\eta, r) = \frac{1}{2} R^3(r) A (\sinh \eta - \eta) + t_B(r), \quad (28)$$

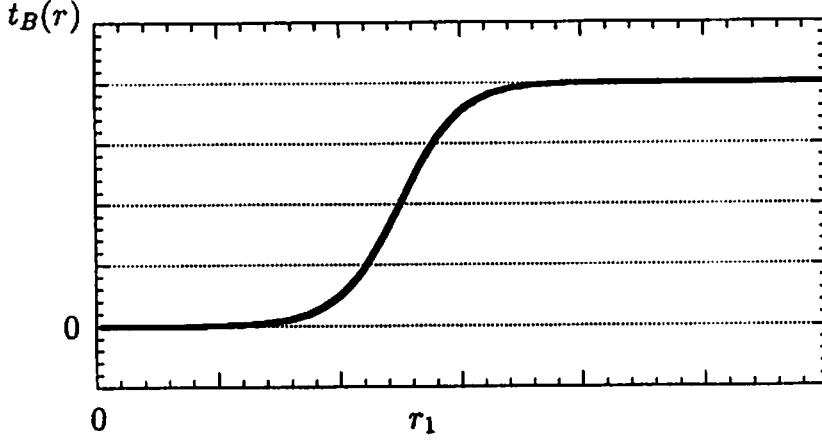


Figure 1: A schematic plot of $t_B(r)$.

and for flat models

$$a(t, r) = \left(\frac{3}{2}(t + t_B(r)) \right)^{2/3} A^{1/3}. \quad (29)$$

The model is specified by the two functions of r , $R(r)$ and $t_B(r)$. The former represents the “curvature radius” and the latter is the “bang time”.

As a simple example, hereafter we consider a flat model. If the bang time function $t_B(r)$ behaves like in Fig. ??, then a local observer at the origin would regard the universe homogeneous and isotropic. However, the Hubble “constant” determined by the observer using only local data is different from that determined by use of distant ($\gg r_1$) data;

$$H_0 \equiv \left. \frac{\dot{a}}{a} \right|_{t_0, r=0} > \left. \frac{\dot{a}}{a} \right|_{t_0, r \gg r_1}. \quad (30)$$

We have calculated the luminosity distance in such an inhomogeneous universe, and obtained the result that the Hubble parameter determined from the distance-redshift relation may depend on scale of observation, and can vary from, say, 100 km/s/Mpc near the origin to 50 km/s/Mpc on sufficiently large scales.

5 Possible implications

The result obtained in the previous section suggests some interesting observational implications. First, the locally determined value of H_0 may not necessarily represent the global rate of the cosmological expansion. If so, locally large H_0 might not always in conflict with the cosmic age suggested by the globular cluster evolution, and the compelling reason to introduce a positive cosmological constant might disappear. Second, there is also a possibility that the average density, or the density parameter Ω_0 is not globally constant. Therefore, the locally observed value of Ω_0 (~ 0.2) for baryonic matter does not necessarily require the non-baryonic dark matter component in order to reconcile the inflationary paradigm of $\Omega_0 = 1$.

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Non linear Inhomogeneity and Gradient Expansion

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Non linearity of gravity plays an important role in many situation. For example, black hole formation, structure formation, realization of inflationary phase, etc. To investigate these problems, one method is to use numerical calculation (numerical relativity). The other method is to use some approximation schemes. In this article, we use the gradient expansion approximation. This scheme treats large-scale (super horizon scale) inhomogeneity as background and includes short wave length effects through a non-linear perturbation. We apply it to the problem of cosmic no-hair conjecture: how the non-linear inhomogeneity affects the onset of inflationary phase.

We consider the system of Einstein gravity with minimally coupled scalar field. The action is

$$\begin{aligned} I &= \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa} R^{(4)} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) \\ &= \int d^4x \left(\pi^\phi \dot{\phi} + [\pi^{ij}] \dot{\gamma}_{ij} - N\mathcal{H} - N^i \mathcal{H}_i \right), \end{aligned} \quad (1)$$

where $\kappa = 8\pi/m_{pl}^2$ and $\mathcal{H}, \mathcal{H}_i$ are

$$\begin{aligned} \mathcal{H} &= 2\kappa\gamma^{-1/2} [\pi^\gamma]^{ij} [\pi^\gamma]^{kl} (\gamma_{jk}\gamma_{li} - \frac{1}{2}\gamma_{ij}\gamma_{kl}) - \frac{1}{2\kappa}\gamma^{1/2}R, \\ \mathcal{H}_i &= -2(\gamma_{il}[\pi^\gamma]^{lk})_{,k} + [\pi^\gamma]^{lk}\gamma_{lk,i} + \pi^\phi\phi_{,i}. \end{aligned} \quad (2)$$

The time evolution of the system is obtained by solving the dynamical equations with the Hamiltonian and the momentum constraints:

$$\begin{aligned} \dot{\phi} &= \frac{\delta H}{\delta \pi^\phi}, \quad \dot{\gamma}_{ij} = \frac{\delta H}{\delta \pi^{\gamma}}, \quad \dot{\pi}^\phi = -\frac{\delta H}{\delta \phi}, \quad \dot{\pi}^\gamma = -\frac{\delta H}{\delta \gamma}, \\ \mathcal{H} &= 0, \mathcal{H}_i = 0. \end{aligned} \quad (3)$$

To solve these equations, we use the Hamilton-Jacobi method. The generator S that transforms the Hamiltonian to identically zero satisfies the following Hamilton-Jacobi equation:

$$\begin{aligned} 2\kappa\gamma^{-1/2} \frac{\delta S}{\delta \gamma_{ij}} \frac{\delta S}{\delta \gamma_{kl}} (\gamma_{jk}\gamma_{li} - \frac{1}{2}\gamma_{ij}\gamma_{kl}) - \frac{1}{2\kappa}\gamma^{1/2}R + \frac{1}{2}\gamma^{-1/2} \left(\frac{\delta S}{\delta \phi} \right)^2 + \gamma^{1/2}V(\phi) &= 0, \\ -2(\gamma_{ik} \frac{\delta S}{\delta \gamma_{kj}})_{,j} + \frac{\delta S}{\delta \gamma_{lk}} \gamma_{lk,i} + \frac{\delta S}{\delta \phi} \phi_{,i} &= 0, \end{aligned} \quad (4)$$

and conjugate momentums are given by $[\pi^\gamma]^{ij} = \frac{\delta S}{\delta \gamma_{ij}}$ and $\pi^\phi = \frac{\delta S}{\delta \phi}$. The evolution equations become

$$\begin{aligned}\frac{1}{N}(\dot{\phi} - N^i \phi_{,i}) &= \gamma^{-1/2} \frac{\delta S}{\delta \phi}, \\ \frac{1}{N}(\dot{\gamma}_{ij} - N_{i|j} - N_{j|i}) &= \frac{1}{2\kappa} \gamma^{-1/2} (2 \frac{\delta S}{\delta \gamma^{ij}} - \gamma_{ij} \frac{\delta S}{\delta \gamma}).\end{aligned}\quad (5)$$

We solve these equations by using the gradient expansion. For the simplicity of the calculation, we use $\phi(t, x)_{,i} = 0$ gauge with $N^i = 0$. The action S is expanded by the order of the spatial gradient:

$$S = S^{(0)} + S^{(2)} + S^{(4)} + \dots, \quad (6)$$

where

$$S^{(0)} = -\frac{2}{\kappa} \int d^3x \gamma^{1/2} H(\phi), S^{(2)} = \int d^3x J(\phi) R. \quad (7)$$

Using the relation

$$\begin{aligned}\frac{\delta S^{(0)}}{\delta \gamma_{ij}} &= -\frac{1}{\kappa} \gamma^{1/2} \gamma^{ij} H(\phi), \quad \frac{\delta S^{(2)}}{\delta \gamma_{ij}} = -\gamma^{1/2} (R^{ij} - \frac{1}{2} \gamma^{ij} R) J(\phi), \\ \frac{\delta S^{(0)}}{\delta \phi} &= -\frac{2}{\kappa} \gamma^{1/2} \frac{dH}{d\phi}, \quad \frac{\delta S^{(2)}}{\delta \phi} = \gamma^{1/2} R \frac{dJ}{d\phi},\end{aligned}\quad (8)$$

the 0th and the 2nd order of Hamilton-Jacobi equations become

$$\begin{aligned}H(\phi)^2 &= \frac{2}{3\kappa} \left(\frac{dH}{d\phi} \right)^2 + \frac{\kappa}{3} V(\phi) \quad (\text{the 0th order}), \\ H(\phi) J(\phi) - \frac{1}{2\kappa} - \frac{2}{\kappa} \frac{dH}{d\phi} \frac{dJ}{d\phi} &= 0 \quad (\text{the 2nd order}).\end{aligned}\quad (9)$$

The momentum constraint is trivially satisfied because we use $\phi_{,i} = 0$ time slice:

$$\mathcal{H}_i = \frac{\delta S}{\delta \phi} \phi_{,i} = 0. \quad (10)$$

Once the function $H(\phi)$ and $J(\phi)$ is obtained, the evolution of the metric and the matter field can be calculated by integrating the following evolution equations:

$$\begin{aligned}\frac{1}{N} \dot{\phi} &= -\frac{2}{\kappa} \frac{dH}{d\phi} + R(\gamma) \frac{dJ}{d\phi}, \\ \frac{1}{N} \dot{\gamma}_{ij} &= 2H\gamma_{ij} + 4\kappa J \left(\frac{1}{4} \gamma_{ij} R(\gamma) - R_{ij}(\gamma) \right).\end{aligned}\quad (11)$$

As an example, we consider the chaotic inflationary model with $V(\phi) = \frac{1}{2} m^2 \phi^2$. The 0th order Hamilton-Jacobi equation has the approximate solution in the slow rolling regime $\dot{\phi}^2 \ll m^2 \phi^2, m_{pl} \ll \phi$:

$$H^2 \approx \frac{\kappa}{6} m^2 \phi^2. \quad (12)$$

By eliminating the lapse function N , the evolution equation becomes

$$\frac{d\gamma_{ij}}{d\phi} = -\kappa\phi\gamma_{ij}, \quad (13)$$

and the solution is

$$\gamma_{ij}(\phi) = \exp\left(-\frac{\kappa}{2}\phi^2\right)k_{ij}(\vec{x}), \quad (14)$$

where $k_{ij}(\vec{x})$ is the arbitrary function of the spatial coordinate. The approximate solution of the 2nd order Hamilton-Jacobi equation in the slow rolling regime is given by

$$J(\phi) \approx \frac{1}{2\kappa}\left(\frac{6}{\kappa m^2}\right)^{1/2}\frac{1}{\phi}. \quad (15)$$

Using these solution, we can discuss the effect of inhomogeneity on the inflationally phase. By comparing the first and the second term in the right hand side of the evolution equation,

$$\frac{\frac{2}{\kappa}\left(\frac{dH}{d\phi}\right)}{R(\gamma)\left(\frac{dJ}{d\phi}\right)} \sim \frac{\kappa m^2}{R(\gamma)/\phi^2} \sim \frac{H^2}{R(k)}. \quad (16)$$

Therefore, for the spatial inhomogeneity that satisfies $H^2 \leq R(k)$, the slow rolling phase is broken. In other word, the inhomogeneity of the spatial curvature whose scale is smaller than the horizon scale will prevent the system from the inflationary phase. To obtain more definite result, we must clarify when this approximation breaks down and this is now under studying.

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Covariant Approach to Cosmological Perturbations

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Abstract

In a series of recent papers by Ellis et al., an alternative approach called covariant approach to cosmological perturbations are developed. In this report, we give a brief review of the covariant approach, and show its attractive features. Then, as an example, we present gauge-invariant perturbation equations in generalized Einstein theories (GETs) and discuss the possibility of an enhancement in the growth rate of density perturbations in GETs.

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1 Introduction

When we study cosmological perturbations, an appropriate choice of gauge becomes important. In order not to pick up unphysical modes by a bad gauge choice, Bardeen proposed a gauge-invariant (GI) formalism for density perturbations[1]. Although Bardeen's GI formalism is one of the most elegant and attractive approaches to cosmological density perturbations, it requires some tedious calculations. Therefore, it is difficult to apply the formalism to complex systems like that in generalized Einstein theories (GETs).

On the other hand, Ellis and his collaborators have recently developed an alternative to Bardeen's formalism[2]-[6]. Their covariant approach is rather simple and straightforward, and it seems to have advantages in particular to apply it to a complex system.

In what follows, we briefly review the definition of the GI variables and the derivation of the evolution equations. Then, as an example, we derive GI perturbation equations in GETs, which include the Brans-Dicke (BD) theory[7], theories with a non-minimally coupled scalar field and certain curvature-squared theories. Furthermore, in order to show the possibility of an enhancement in the growth rate of density perturbations in GETs, we analyze perturbations particularly in the BD theory.

2 Covariant Approach to Cosmological Perturbations

The covariant approach to cosmological density perturbations is originally introduced by Hawking[8] to analyze cosmological perturbations. At that time, the theories which perturb the metric was plagued by difficulties in interpretation,

particularly concerning a choice of gauge. Furthermore, what we observe is not the metric but its second derivatives. Then, He thought that it is desirable to deal with perturbations of curvature tensor rather than those of the metric.

In general relativity, the curvature of spacetime is represented by the Riemann curvature tensor R_{abcd} [9]. This can be split into its trace part and traceless part i.e., the Ricci tensor R_{ab} and the Weyl tensor C_{abcd} . Furthermore, the Weyl tensor is separated into its electric part E_{ab} and magnetic part H_{ab} , which are defined as

$$E_{ab} = -C_{apbq}u^p u^q, \quad H_{ab} = -\frac{1}{2}C_a{}^{pqr}\eta_{qrs}u_p u^s, \quad (2.1)$$

where u^a is any arbitrary timelike unit vector, and η^{abcd} is totally-skew pseudotensor defined as

$$\eta^{abcd} = \eta^{[abcd]}, \quad \eta^{0123} = (-\det g_{ab})^{-\frac{1}{2}}. \quad (2.2)$$

The electric part of the Weyl tensor corresponds to tidal force field in Newtonian theory, and magnetic part has no Newtonian counterpart.

The Ricci tensor is determined algebraically by the Einstein equation:

$$R_{ab} = (T_{ab} - \frac{1}{2}Tg_{ab}). \quad (2.3)$$

The 4-velocity of the fluid can be different from the timelike vector u^a in (2.1). We have a freedom of choice of the observer's frame.

The evolution of E_{ab} and H_{ab} is determined by the Bianchi identity;

$$R_{ab[cd;e]} = 0. \quad (2.4)$$

The evolution of the first derivative of u^a involved in (2.4) is determined by the Ricci identity;

$$2u_{a[d}u_{c]} = R_{abcd}u^b \quad (2.5)$$

In order to understand the geometrical meaning, it is convenient to decompose $\nabla_b u_a$ as

$$\nabla_b u_a = \omega_{ab} + \sigma_{ab} + \frac{1}{3}\theta h_{ab} - a_a u_b, \quad (2.6)$$

where $a_a \equiv \dot{u}_a$, $\theta \equiv \nabla_a u^a$, $\omega_{ab} \equiv {}^{(3)}\nabla_{[b}u_{a]}$, and $\sigma_{ab} \equiv {}^{(3)}\nabla_{(b}u_{a)} - \frac{1}{3}h_{ab}\theta$ are the acceleration, the expansion, the vorticity tensor and the shear tensor, respectively¹.

¹ $h_{ab} = g_{ab} + u_a u_b$ is the projection tensor into the 3-space Σ orthogonal to u^a . The time derivative along u^a (denoted by a dot) of any rank of tensor $T^{ab\cdots}_{cd\cdots}$ and its spatial covariant derivative in the 3-space Σ (denoted by ${}^{(3)}\nabla_a$) are defined by

$$\dot{T}^{ab\cdots}_{cd\cdots} \equiv u^e \nabla_e T^{ab\cdots}_{cd\cdots},$$

and

$${}^{(3)}\nabla_a T^{bc\cdots}_{de\cdots} \equiv h_a{}^m h_i{}^b h_j{}^c \cdots h_d{}^k h_e{}^l \cdots \nabla_m T^{ij\cdots}_{kl\cdots},$$

respectively, where ∇_a denotes the covariant derivative with respect to g_{ab} .

Finally, we can determine the evolution of the matter by the energy-momentum conservation:

$$T^{ab}{}_{;b} = 0. \quad (2.7)$$

From the equations (2.3)~(2.7), we can derive the evolution equations and the constraint equations for E_{ab} , H_{ab} , σ_{ab} , ω_{ab} , a^a , θ and μ . Among the variables, E_{ab} , H_{ab} , σ_{ab} , ω_{ab} and a^a totally vanish in the homogeneous and isotropic universe. Thus, they are covariantly defined perturbations themselves. Furthermore, in perturbed spacetimes, they are gauge-invariant because they are invariant under a general coordinate transformation, i.e., for any choice of correspondence between a homogeneous and isotropic background spacetime and the physical inhomogeneous universe.

Hawking based his analysis of density perturbations on the gauge-dependent density contrast $\delta\mu/\mu$. Ellis and Bruni[2] extended his formalism and introduced GI variables corresponding to density contrast as

$$\Delta = \frac{a^2}{\mu} {}^{(3)}\nabla^2 \mu, \quad (2.8)$$

where a is the typical length scale defined from the expansion θ :

$$\frac{\dot{a}}{a} \equiv H \equiv \frac{1}{3}\theta, \quad (2.9)$$

and ${}^{(3)}\nabla^2 \equiv {}^{(3)}\nabla^a {}^{(3)}\nabla_a$. This is GI because in the background μ is homogeneous and ${}^{(3)}\nabla^a \mu$ vanishes².

Obtaining the evolution equations for Δ is rather simple. First, take spatial derivative (by operating $a^{(3)}\nabla_a$) of the energy conservation. Then, take the (co-moving) divergence (by operating $a^{(3)}\nabla^a$ to that equation) constructing GI variables such as Δ defined above. The evolution of Δ , to first order of perturbations, is written as

$$\dot{\Delta} - 3H \frac{p}{\mu} \Delta + \left(1 + \frac{p}{\mu}\right) \mathcal{Z} = 0, \quad (2.10)$$

where $\mathcal{Z}$ is GI perturbation variable for θ . The evolution equation for $\mathcal{Z}$ is derived in the same way as Δ .

The procedure to define GI perturbations and to derive the evolution equations as described above is called the covariant approach. The approach has some attractive features; (1) Perturbations are defined in a covariant and gauge-invariant manner. (2) Many of fundamental equations are already perturbation equations

² Δ is related to Bardeen's GI variable $\epsilon_m (= \delta\mu/\mu$ in velocity-orthogonal slicing) in [1] as follows[5]:

$$\Delta = - \sum_n n^2 \epsilon_m^{(n)} Q^{(n)}, \quad \epsilon_m = \sum_n \epsilon_m^{(n)} Q^{(n)},$$

to first order of perturbations, where n is a wave number of the harmonics $Q^{(n)}$.

themselves. (3) We can start with a complete set of exact equations. Then, linearizing or imposing symmetry conditions, we can apply them to wide range of universe models. (4) The point of view adopted in this formalism is basically that of comoving observers. That is, we may say it is a Lagrangian approach. For these reasons, the covariant approach is not only adequate to be an alternative to standard approach, but also more powerful for some kind of relativistic perturbation analysis.

3 Applications

After the work by Ellis and Bruni, various applications have been performed. Hwang and Vishniac[10] and Dunsby[4] applied the approach to the case with imperfect fluid. Bruni, Ellis and Dunsby[11] extended it to perturbations in universes dominated by a scalar field coupled minimally to gravity. Dunsby, Bruni and Ellis[12] considered the systems of interacting multi-component fluids. Moreover, application to the anisotropic universe was recently performed by Dunsby[6].

Here, we present GI perturbation equations in GETs, as one of the applications. By GETs, we can expect an important difference in the growth rate of cosmological density perturbations, which might resolve the problem of structure formation. There have been attempts at this kind of formulation with standard approach before, but never completed owing to the complexity of the problem[13].

3.1 Formulation of Perturbation Equations in GETs

The action of GETs we consider here is

$$S = \int d^4x \sqrt{-g} \left[F(\phi, R) - \frac{\epsilon(\phi)}{2} (\nabla\phi)^2 + L_m \right], \quad (3.1)$$

where $F(\phi, R)$ is an arbitrary function of a scalar field ϕ and a scalar curvature R , $\epsilon(\phi)$ is an arbitrary function of ϕ , and L_m is the Lagrangian of ordinary matter, for which we assume a perfect fluid i.e.,

$$T_{ab}^M = \mu u_a^M u_b^M + p h_{ab}^M, \quad (3.2)$$

where μ , p and u_a^M represent the energy density, the pressure and the 4-velocity of the matter fluid, respectively, and h_{ab}^M is the projection tensor into the 3-space orthogonal to u_a^M .

Although we may formulate the equations leaving the present variables as they are, we prefer to circumvent the complexity by changing variables using the conformal transformation[14]

$$\hat{g}_{ab} = e^{2\omega(x)} g_{ab}. \quad (3.3)$$

Hereafter, variables with $\hat{\cdot}$ (a caret) denote those with respect to $\hat{g}_{ab}$. The equation (3.3) gives the relations between the length scales and Hubble parameters as

$$\hat{a} = e^{\omega} a, \quad (3.4)$$

$$\hat{H} = e^{-\omega} (H + \dot{\omega}). \quad (3.5)$$

Using these relations, we can recover all the original variables in the physical frame from new ones in the conformal frame. In particular, the density perturbation Δ^M observed along fluid flow u_a^M is the same as that in terms of the conformal variables.

$$\Delta^M = \hat{\Delta}^M, \quad (3.6)$$

When we discuss the results, of course, we transform all the conformal variables back to the physical ones (see §3.2).

For simplicity, we will present here the basic equations explicitly only for the following two interesting cases:

$$\text{Case (A) : } F(\phi, R) = f(\phi)R - V(\phi), \quad (3.7)$$

$$\text{Case (B) : } F(\phi, R) = L(R) \quad \text{and} \quad \epsilon(\phi) = 0. \quad (3.8)$$

and other theories with a non-minimally coupled scalar field, while Case (B) includes R^2 -gravity[16]. Formulation for further general cases is straightforward (see [17]).

The field equations become

$$\begin{aligned} \hat{G}_{ab} &= \kappa^2 \left\{ \hat{\nabla}_a \varphi \hat{\nabla}_b \varphi - \frac{1}{2} \hat{g}_{ab} (\hat{\nabla} \varphi)^2 - U(\varphi) \hat{g}_{ab} \right\} + \kappa^2 N(\varphi) \{ \mu \hat{u}_a^M \hat{u}_b^M + p \hat{h}_{ab}^M \} \\ &\equiv \kappa^2 (\hat{T}_{ab}^\varphi + \hat{T}_{ab}^M), \end{aligned} \quad (3.9)$$

$$\square \varphi - U'(\varphi) + \kappa^2 M(\varphi) \{ \mu - 3p \} = 0, \quad (3.10)$$

where $\kappa^2 = 8\pi G$, ' (a prime) $= d/d\varphi$, and new variables and functions in these equations are defined as follows:

Case (A) :

$$\kappa \varphi = \int d\phi \left[\frac{\epsilon(\phi)f(\phi) + 3(df(\phi)/d\phi)^2}{2f^2(\phi)} \right]^{\frac{1}{2}}, \quad (3.11)$$

$$U(\varphi) = \frac{(\text{sign})}{(2\kappa^2 f(\phi))^2} V(\phi), \quad (\text{sign}) = \frac{f(\phi)}{|f(\phi)|}, \quad (3.12)$$

$$M(\varphi) = \frac{df(\phi)/d\phi}{(2\kappa^2 f(\phi))^2 [2\kappa^2 \{ \epsilon(\phi)f(\phi) + 3(df(\phi)/d\phi)^2 \}]^{\frac{1}{2}}}, \quad (3.13)$$

$$N(\varphi) = \frac{(\text{sign})}{(2\kappa^2 f(\phi))^2}. \quad (3.14)$$

Case (B) :

$$\kappa\varphi = \frac{\sqrt{6}}{2} \ln \left(2\kappa^2 \left| \frac{\partial L}{\partial R} \right| \right), \quad (3.15)$$

$$N(\varphi) = (\text{sign}) \exp \left[-\frac{2}{3} \sqrt{6} \kappa \varphi \right], \quad (\text{sign}) = \frac{\partial L}{\partial R} / \left| \frac{\partial L}{\partial R} \right|, \quad (3.16)$$

$$U(\varphi) = N(\varphi) \left[\frac{\partial L}{\partial R} R - L \right], \quad (3.17)$$

$$M(\varphi) = \frac{\sqrt{6}}{6\kappa} N(\varphi), \quad (3.18)$$

where $R = R(\varphi)$ and $L = L(\varphi) \equiv L(R(\varphi))$ through the relation

$$\begin{aligned} \frac{\partial L}{\partial R}(R) &= \text{a given function of } R \\ &= (\text{sign}) \frac{1}{2\kappa^2} \exp \left[\frac{2}{\sqrt{6}} \kappa \varphi \right]. \end{aligned} \quad (3.19)$$

The 4-velocity and projection tensor with carets are defined as

$$\hat{u}_a^M = e^\omega u_a^M, \quad \hat{u}_M^a = e^{-\omega} u_M^a, \quad (3.20)$$

$$\hat{h}_a^{Mb} = h_a^{Mb}, \quad \hat{h}_{ab}^M = e^{2\omega} h_{ab}^M, \quad \hat{h}_M^{ab} = e^{-2\omega} h_M^{ab}. \quad (3.21)$$

In what follows, we will formulate equations in terms of the new conformal variables. We omit $\hat{}$ (a caret) for brevity.

Applying the covariant approach to the above equations, we can find the first order perturbation equations. Here, we define the scalar GI perturbation variables for the matter and scalar field in the frame where the energy flux of the total energy-momentum tensor, $T_{ab}^* \equiv T_{ab}^\varphi + T_{ab}^M$ in equation (3.10), vanishes, that is, in the center of mass frame³:

$$\Delta^E = \frac{a^2}{\mu} {}^{(3)}\nabla_E^2 \mu, \quad \mathcal{Y}^E = a^{2(3)} \nabla_E^2 p, \quad (3.22)$$

$$\mathcal{Z}^E = a^{2(3)} \nabla_E^2 \theta \quad (3.23)$$

$$\Phi^E = a^{2(3)} \nabla_E^2 \varphi. \quad (3.24)$$

On describing the resultant equations, we expand the dynamical variables as

$$\Delta = \sum_n \Delta_{(n)}(t) Q^{(n)}, \quad (3.25)$$

³The index E denote the variables defined in the center of mass frame. We can relate the variables in the matter frame and the center of mass frame. As for the density perturbations

$$\Delta^M = \Delta^E + \frac{\dot{\mu}}{\mu} \frac{\dot{\varphi}}{N(\mu + p)} \Phi^E.$$

where $Q^{(n)}$ is an eigenfunction of the Laplace-Beltrami operator ${}^{(3)}\nabla^2[5]$:

$${}^{(3)}\nabla^2 Q^{(n)} = -\frac{n^2}{a^2} Q^{(n)}. \quad (3.26)$$

Finally, we obtain the basic equations for the harmonic components of perturbed quantities as follows (We drop the suffix (n) for brevity):

$$\begin{aligned} \dot{\Delta}^E = & 3w \left(H - \kappa^2 \frac{M}{N} \dot{\varphi} \right) \Delta^E + 3\kappa^2 \frac{M}{N} \frac{\dot{\varphi}}{\mu} \mathcal{Y}^E - (1+w) \mathcal{Z}^E \\ & + \left[\frac{3H\dot{\varphi}}{N\mu} - \kappa^2 \frac{M}{N} (1-3w) - \frac{N'}{N} \right] \dot{\Phi}^E \\ & + \left[-\frac{3H}{N\mu} (U' - N'\mu w) + \dot{\varphi} \left\{ \frac{n^2}{a^2} \frac{1}{N\mu} - \left(\frac{N'}{N} \right)' - \kappa^2 \left(\frac{M}{N} \right)' (1-3w) \right\} \right] \Phi^E, \end{aligned} \quad (3.27)$$

$$\begin{aligned} \dot{\mathcal{Z}}^E = & -\frac{1}{2} \kappa^2 N\mu \Delta^E - \frac{1}{2\mu(1+w)} \left(\kappa^2 \dot{\varphi}^2 - \frac{2n^2}{a^2} + \frac{6k}{a^2} \right) \mathcal{Y}^E - 2H \mathcal{Z}^E \\ & - \frac{\dot{\varphi}}{2} \left[\kappa^2 + \frac{1}{N\mu(1+w)} \left(\kappa^2 \dot{\varphi}^2 - \frac{2n^2}{a^2} + \frac{6k}{a^2} \right) \right] \dot{\Phi}^E \\ & - \frac{1}{2} \left[\kappa^2 (U' + N'\mu) - \frac{(U' - N'\mu w)}{N\mu(1+w)} \left(\kappa^2 \dot{\varphi}^2 - \frac{2n^2}{a^2} + \frac{6k}{a^2} \right) \right] \Phi^E, \end{aligned} \quad (3.28)$$

$$\begin{aligned} \ddot{\Phi}^E = & -\frac{1}{\dot{\varphi}^2 + N\mu(1+w)} \left[3 \left(H + \kappa^2 \frac{M}{N} \dot{\varphi} \right) N\mu(1+w) + \frac{1+c_s^2}{1+w} \dot{\varphi}^3 \left\{ \frac{N'}{N} + \kappa^2 \frac{M}{N} (1-3w) \right\} \right. \\ & \left. - \frac{N'}{N} \dot{\varphi}^3 + \dot{\varphi} \left\{ 3(c_s^2 - 1)H\dot{\varphi} - 4U' - 12\kappa^2 M\mu w + N'\mu w \right\} \right] \dot{\Phi}^E \\ & - \frac{1}{\dot{\varphi}^2 + N\mu(1+w)} \left[\left\{ U'' - \kappa^2 M'\mu(1-3w) + \frac{n^2}{a^2} + \frac{3c_s^2 H N' \dot{\varphi}}{N} \right\} N\mu(1+w) \right. \\ & \left. - (U''' - N''\mu w) \dot{\varphi}^2 + c_s^2 \frac{N'}{N} \mu \dot{\varphi}^2 \left\{ N' + \kappa^2 M(1-3w) \right\} \right. \\ & \left. + (U' - N'\mu w) \left\{ \frac{N'}{N} \frac{w - c_s^2}{1+w} \dot{\varphi}^2 - \kappa^2 \frac{M}{N} (1-3w) \left(2N\mu + \frac{1+c_s^2}{1+w} \dot{\varphi}^2 \right) \right. \right. \\ & \left. \left. - (3c_s^2 H \dot{\varphi} - 2U'') \right\} \right] \Phi^E \\ & + \frac{\kappa^2 M N \mu^2 (1+w)}{\dot{\varphi}^2 + N\mu(1+w)} \Delta^E - \frac{N\dot{\varphi}}{\dot{\varphi}^2 + N\mu(1+w)} \mathcal{Y}^E \\ & - \frac{1}{\dot{\varphi}^2 + N\mu(1+w)} \left[\kappa^2 M N \mu (5-3w) + N(3c_s^2 H \dot{\varphi} - 2U') \right. \\ & \left. + \frac{1+c_s^2}{1+w} \dot{\varphi}^2 \left\{ N' + \kappa^2 M(1-3w) \right\} \right] \mathcal{Y}^E \\ & - \frac{N\mu(1+w)\dot{\varphi}}{\dot{\varphi}^2 + N\mu(1+w)} \mathcal{Z}^E, \end{aligned} \quad (3.29)$$

where we have used notations $w \equiv p/\mu$, $c_s^2 \equiv \dot{p}/\dot{\mu}[1]$.

3.2 Density Perturbations in the Brans-Dicke Theory

In order to see how the coupling term to the curvature scalar in GETs changes the growth rate of density perturbations, we consider the BD theory[7] as an example. In the BD theory, the reciprocal of the scalar field, $1/\phi$, corresponds to an “effective” gravitational constant $G(\phi)$. If a flat dust universe ($k = 0$, $p = 0$) satisfies a specific initial condition

$$\frac{\mu_0 t_0^2}{\phi_0} = \frac{(3 + 2\omega)}{4\pi(4 + 3\omega)}, \quad (3.30)$$

where ω is a constant (the BD parameter) and the suffix 0 represents the present time, then the BD scalar field ϕ , the energy density of matter μ and the scale a evolve in time as

$$\phi = \phi_0 \left(\frac{t}{t_0} \right)^{2/(4+3\omega)}, \quad (3.31)$$

$$a = \left(\frac{t}{t_0} \right)^{(2+2\omega)/(4+3\omega)}, \quad (3.32)$$

$$\mu = \mu_0 a^{-3} = \mu_0 \left(\frac{t}{t_0} \right)^{-3(2+2\omega)/(4+3\omega)}. \quad (3.33)$$

de-Sitter universe ($\phi = \text{constant}$, $a \propto t^{2/3}$).

In our formalism, the BD theory corresponds to Case (A) with the following functions

$$f(\phi) = \frac{\phi}{16\pi}, \quad (3.34)$$

$$\epsilon(\phi) = \frac{\omega}{8\pi\phi}, \quad (3.35)$$

$$V(\phi) = 0. \quad (3.36)$$

initial condition (3.30), we can solve the perturbation equations analytically. As for the density perturbation of dust, we obtain

$$\begin{aligned} \Delta^E = & c_1 \left\{ \left(\frac{\hat{t}}{t_0} \right)^{\frac{2(2+\omega)}{5+3\omega}} + \frac{4(8+5\omega)}{(5+3\omega)^2} \left(\frac{5+3\omega}{4+3\omega} \right)^{\frac{1+\omega}{5+3\omega}} \frac{1}{t_0^2 n^2} \right\} \\ & + c_2 \left(\frac{5+3\omega}{4+3\omega} \frac{\hat{t}}{t_0} \right)^{-\frac{4+3\omega}{5+3\omega}} \\ & + c_3 \left(\frac{\hat{t}}{t_0} \right)^{-\frac{4+3\omega}{2(5+3\omega)}} J_\nu(nT) + c_4 \left(\frac{\hat{t}}{t_0} \right)^{-\frac{4+3\omega}{2(5+3\omega)}} J_{-\nu}(nT), \end{aligned} \quad (3.37)$$

J_ν is the Bessel function of the ν -th order, and

$$\nu \equiv \frac{4+3\omega}{2(2+\omega)}, \quad T \equiv \frac{4+3\omega}{2+\omega} \left(\frac{5+3\omega}{4+3\omega} \frac{\hat{t}}{t_0} \right)^{\frac{2+\omega}{5+3\omega}}. \quad (3.38)$$

The density perturbation Δ^E has four independent modes. The first two terms have their counterparts in the Einstein de-Sitter model in the limit of $\omega \rightarrow \infty$, while the last two originate from the BD scalar field. The growing mode (the first term in equation (3.37)) evolves asymptotically

$$\left(\frac{\hat{t}}{t_0} \right)^{\frac{2(2+\omega)}{5+3\omega}} \propto \left(\frac{t}{t_0} \right)^{\frac{2(2+\omega)}{4+3\omega}} \propto a^{\frac{2+\omega}{1+\omega}}, \quad (3.39)$$

which shows that the growth rate of Δ^E in this specific BD model is somewhat higher (by $a^{\frac{1}{1+\omega}}$ times) than that in the Einstein de-Sitter model. We have also analyzed perturbations in more generic background models numerically[17]. So far, our calculations have implied that it is not the variation of effective gravitational constant itself but its change rate that is the key to the acceleration of the perturbation evolution.

4 Concluding Remarks

In the first part of this article, we have reviewed the covariant approach to cosmological perturbations. This approach makes the formulation straightforward, and the intuitive meanings of the naturally introduced gauge-invariant variables are easy to obtain. Now that its correspondence to the conventional Bardeen's approach is well understood[5], this approach is becoming established and will be applied in various situations to formulate relativistic perturbation theories.

As one of such applications, we have presented gauge-invariant cosmological perturbation equations in GETs which are characterized as equations (3.7) and (3.8) with a perfect fluid. Particularly, applying the derived equations to perturbations in BD cosmology, we have obtained an analytic solution for a specific background model and also performed numerical analysis for more generic initial conditions. Both analyses have shown an enhancement in the growth rate of perturbations.

We are now engaged in further studies of theories with non-minimally coupled scalar fields and perfect fluid matter, applying the results presented here. Through those analysis, we aim to clarify the correlation between the growth rate of perturbations and generalization of Einstein's theory.

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Generalized Scalar-Tensor Theory and Cosmological Term

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Abstract. It is shown that a cosmological term depends on both a scalar field and the energy-momentum tensor of matter. It means that the term can be different for successive cosmological phases though it varies continuously with time in each phase.

1. Introduction

Now it is a big problem how to reconcile a nearly or completely vanishing cosmological "constant" in the present epoch with that being inferred to have been a large value of some 100 orders of magnitude of its observed upper bound before the electroweak phase transition in order to allow inflation to occur. There have been many approaches proposed so far to solve the cosmological term problem. They are reviewed and then surveyed by Weinberg(1989). Since all of them are likely to have a much wider impact on other areas of physics or astronomy, they are not their final versions and remain interesting.

Recently Fujii and Nishioka(1990) proposed a theoretical model of a decaying cosmological constant, with their crucial ingredients of an extended theory of a new scalar field theory and a careful analysis of conformal transformations. They pointed out that the smallness of the present value of the cosmological constant is due simply to the fact that our Universe is old. But they also commented whether their mechanism applies effectively to a series of cosmological phase transitions, each of which creates a new cosmological constant.

One of us (T.F.) proposed a modified Brans-Dicke model of the Universe by a variable cosmological term which varies as t^{-2} in terms of the cosmic time(Endo and Fukui 1977). Then it results in the cosmological term at the GUT phase 10^{-39} s. having been 10^{112} times larger than that at the present epoch 10^{17} s. There it is also shown that the cosmological term can change drastically between

where T_{ik} is the energy-momentum tensor of matter.

$$R_{ik} - \frac{1}{2} R g_{ik} + \Lambda g_{ik} = \frac{8\pi}{c^4} T_{ik} + \frac{\phi}{\omega} (\phi_{,i} \phi_{,k} - \frac{1}{2} g_{ik} \phi_{,l} \phi_{,l}) + \frac{1}{\phi} (\phi_{,i} \phi_{,k} - g_{ik} \phi_{,l} \phi_{,l}), \quad (1)$$

Then the field equation for metric field is obtained as

$$L = \phi [R - 2\Lambda(\phi)] + \frac{c^4}{16\pi} L_\phi - \frac{\phi}{\omega(\phi)} \phi_{,i} \phi_{,i}.$$

a variable Λ is given here as

The Lagrangian L for a generalized scalar-tensor theory modified by 2. Field Equations

possibly on the energy-momentum tensor.

deriving a functional form of Λ which depends on a scalar field and and Burd and Coley(1992), because the present main concern is about only a particular functional form of ω taken by Van den Bergh(1982)

previous work to the generalized scalar-tensor theory by assuming

was studied by Banerjee et al. (1985), here we just extend our

simple functional form of Λ in a generalized scalar-tensor theory of a scalar field ϕ . Although a cosmological model with a specified cosmological and the Brans-Dicke coupling "constants" are functions term in the generalized scalar-tensor theory where both the

Therefore we study here the behaviour of the variable cosmological

Wagoner(1970), and Nordtvedt(1970) to employ.

of the generalized scalar-tensor theory proposed by Bergmann(1968).

us to infer that ω is not constant but is variable. This reminds us model remains to be seen and the discrepancy in the value of ω leads model of the Universe. Only the coupling constant problem in our

adjusted to obtain the abundances of nuclear elements in the standard the model except for the Brans-Dicke coupling constant can be

Furthermore Arai et al. (1987) could show that the parameters in

Nishitoka's comments and the cosmological term problem as well.

procedure in a modified Brans-Dicke model may explain the Fujii and

different physical states of the Universe. This implies that our

While the wave equation for ϕ is obtained as

$$R - 2\Lambda - 2\phi\Lambda,$$

$$(2) \quad = -\frac{\omega}{2}\phi_{,1}\phi_{,1} + \frac{\omega}{2}\phi_{,2}\phi_{,2} - \frac{\phi}{2\omega}\square\phi,$$

where (') is differential with ϕ .

3. Application to Cosmological Problems

In the homogeneous and isotropic Universe, we employ the Robertson-Walker metric

$$ds^2 = -c^2 dt^2 + a^2(t) \left\{ \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right\},$$

and the energy-momentum tensor of a perfect fluid

$$T_{ik} = (p + \epsilon)u_i u_k + pg_{ik}.$$

Then the time-time components of Eq. (1) gives

$$(3) \quad \left(\frac{x}{\omega}\right)^2 + \frac{a^2 x^2}{kc^2} - \frac{3}{Vc^2} = \frac{3\phi c^2}{8\pi} \epsilon - \frac{x}{\omega} \frac{\phi}{\phi} + \frac{6}{\omega} \left(\frac{\phi}{\phi}\right)^2,$$

where x is a scale factor as to that of the present time a_0 . The space-space components of Eq. (1) give the single equation,

$$(4) \quad 2\frac{x}{\omega} + \left(\frac{x}{\omega}\right)^2 + \frac{a^2 x^2}{kc^2} - Vc^2 = -\frac{8\pi}{c^2} p - 2\frac{x}{\omega} \frac{\phi}{\phi} - \frac{\phi}{\phi} - \frac{2}{\omega} \left(\frac{\phi}{\phi}\right)^2.$$

From Eq. (2) we obtain

$$\frac{x}{\omega} + \left(\frac{x}{\omega}\right)^2 + \frac{a^2 x^2}{kc^2} - \frac{3}{Vc^2} - \frac{3}{Vc^2} \frac{\phi}{\phi} = \frac{x}{\omega} \frac{\phi}{\phi} + \frac{3}{\omega} \frac{\phi}{\phi} - \frac{6}{\omega} \left(\frac{\phi}{\phi}\right)^2 + \frac{6}{\omega} \frac{\phi}{\phi} \left(\frac{\phi}{\phi}\right)^2.$$

$$(5)$$

4. Brans-Dicke Coupling "Constant"

Since there is no physical principle to suggest a particular form

The most important results we'd like to emphasize is that Λ depends explicitly on T as well as ϕ . The term Λ varies continuously with t through ϕ but has possibility of changing drastically for the different physical states of the Universe through T . Then it explains

5. Discussion and Comments

Eq. (9) shows that Λ varies with ϕ and T . It means that Λ is continuously with time during each phase. different for successive cosmological phases though it varies

where $\Phi = (2 + n/m)\omega_0 \phi^n + 3$.

$$\Lambda = \frac{4\pi}{1-n} \frac{\phi^{n-1}}{c^4} (1 - \frac{\phi}{n}) T, \quad (9)$$

Eqs. (1) and (4) can be solved with Eqs. (6)-(8) to be model without Λ . Eq. (8) also tells how ϕ varies with t for each T . where n measures how much the present model with Λ deviates from the

$$[(2 + \frac{m}{n})\omega_0 + 3] \square \phi = \frac{8\pi}{c^4} n T. \quad (8)$$

Here we extend our previous procedure in a modified Brans-Dicke model to that in the generalized scalar-tensor model. We set

$$\frac{\ddot{\phi}}{\phi} + 3 \frac{\dot{\phi}}{\phi} \frac{\dot{\omega}}{\omega} = m (\frac{\phi}{\phi_0})^2. \quad (7)$$

As is well known, the relation of the present theory with a higher dimensional theory having flat extra dimensions in vacuum (Ishihara 1984) makes it possible by introducing a constant m to assume (Van den Bergh 1982).

constant to the standard Bergmann-Wagoner-Nordvedt form more general theories with varying masses or varying gravitational where ω_0 and n are constants. The form was assumed to reduce the

$$\omega = \omega_0 \phi^n. \quad (6)$$

for $\omega(\phi)$ (Nordvedt 1970), there have been several assumptions made so far on the form (Schwinger 1970, Bishop 1976, Barker 1978). Here we follow the assumption made by Van den Bergh (1982) and Burd and Coley (1992)

the case why a new cosmological "constant" is created by successive cosmological phase transitions. Eq.(9) tells that the radiation state is a critical one for Λ changing its sign from negative to positive or vice versa because T itself turns its sign from positive for a stiff matter state ($p=\epsilon$) through 0 for a radiation state ($p=\epsilon/3$) to negative for a matter state ($p=0$).

In order for the present procedure to survive observations, it needs to see the effects of a scalar field ϕ and so of ω , Λ , and G , on observable quantities in cosmology through numerical calculation. Then it becomes possible to study a lower limit of ω which is now set at 500 from the results of radar time-delay experiments and at 600 from the analyses of lunar laser-ranging data(Will 1993).

Once density and temperature are obtained explicitly as a function of time, then we can trace the evolution of chemical abundances through the nuclear reaction network which is described in detail by Hashimoto and Arai(1985). Those numerical consideration will be made soon elsewhere.

Here we analyze just the behaviour of ϕ for the preparation of the numerical calculation of nucleosynthesis. In the period which nucleosynthesis is concerned, we can solve Eq.(8) with the help of Eq.(7) as

$$\frac{n+2m}{n+m} \omega_0 K \phi^{n+m} + 3K \phi^n = (2\omega_0 + 3)(A\phi + B), \quad (10)$$

where $A = (8\pi\mu\epsilon_{00}) / [(2\omega_0+3)c^2]$. ϵ_{00} is the present energy density of matter ϵ_0 , and K and B are constants.

Following Brans and Dicke(1961) and Weinberg(1972), the gravitational "constant" is given by the weak field approximation as

$$G = \frac{1}{3-n} \left(4 - \mu - n + \frac{3\mu-n}{\Phi} \right) \frac{1}{\phi}. \quad (11)$$

It should be noted that G still includes the factor being inversely proportional to ϕ , but Φ , the effect of a variable ω , complicates the dependence of G on ϕ . In the case of $\mu=1$ and $n=0$, Eq.(11) reduces to the locally measurable, Newtonian "constant" G (Eardley 1975). The present value of G enables us through Eq.(11) to judge whether the values of the parameters adopted will be suitable or not. The following are typical cases for three different values of n .

I. $n = 0$, i.e. $\Phi_1 = 2\omega_0 + 3$;

This is the case of a constant Brans-Dicke coupling parameter and so all equations correspond to those shown by Arai et al. (1987). This case with $\mu = 1$ reduces to the Brans-Dicke model of the Universe.

$$\dot{\phi} x^3 = At + B,$$

$$\Lambda = \frac{4\pi}{\phi c^4} \frac{1-\mu}{3} T,$$

$$G = \frac{1}{3} \left(4 - \frac{2\omega_0}{2\omega_0+3} \mu \right) \frac{1}{\phi}.$$

II. $n = -2m$, i.e. $\Phi_1 = 3$;

This is the case where the quantities don't depend on ω . It's quite interesting because it helps to bridge the gap in the values of ω between observation and theory.

$$\dot{\phi} x^3 = A't + B, \quad (A' = \frac{8\pi}{3c^2} \mu \varepsilon_{00}),$$

$$\Lambda = \frac{4\pi}{\phi c^4} \frac{1-\mu}{3} T,$$

$$G = \frac{4}{3} \frac{1}{\phi}.$$

III. $n = -m$, i.e. $\Phi_1 = \omega_0 \phi^{-m} + 3$;

This is the case where Eq. (10) can not be employed. By going back to Eq. (8), we get the following relations

$$\phi^{K/m} \exp\left(\frac{3K}{\omega_0} \phi^m\right) = B \exp(A'' t), \quad (A'' = \frac{8\pi\mu}{\omega_0 c^2} \varepsilon_{00}),$$

$$\Lambda = \frac{4\pi}{\phi c^4} \frac{1-\mu}{(3+m)} \left(1 + \frac{m}{\Phi_1}\right) T,$$

$$G = \frac{1}{3+m} \left(4 - \mu + m + \frac{3\mu+m}{\Phi_1} \right) \frac{1}{\phi}.$$

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Gravitational Lens Effects

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After briefly summarizing some basic formulae in gravitational lens theories, two subjects are reviewed. First, Snapshot Survey of gravitational lensed QSOs with HST is summarized, and its implications to the cosmological constant limit is briefly discussed. Second, the theoretical outline of H_0 determination with Q0957+561 is described, and observational data of this lens system are summarized in order to estimate the model-dependent value of H_0 with the lens model by Falco et al.

Basic formulae

A basic tool used in theories of gravitational lenses is a lens equation which is derived by the Newtonian approximation to general relativity. When one can regard inhomogeneities which cause gravitational lens effects, as a thin mass sheet, the deflection of light paths occurs instantaneously when they pass the sheet. In this case, one immediately finds the following geometrical relation:

$$\vec{\psi} = \vec{\theta} - \frac{D_{LS}}{D_S} \vec{\alpha}, \quad (1)$$

where the meaning of each notation is found in figure 1. The validity of equation (1) and meaning of distance factors was discussed by Sasaki(1993). In the usual astronomical situations, say, gravitational lensing of QSOs by foreground galaxies, the thin lens approximation may be good unless there are inhomogeneities on the large scale compared to the cosmological horizon $\sim 3000\text{Mpc}$.

When one assumes a point lens with the mass, M , the deflection angle formula is

$$\vec{\alpha} = \frac{4GM}{D_L} \frac{\vec{\theta}}{\theta^2}. \quad (2)$$

Without loss of generality, one can assume $\vec{\psi} = (\psi, 0)$, and the lens equation is easily solved to obtain $\theta = \theta_+, \theta_-$, given by

$$\theta_{\pm} = \frac{1}{2} \left\{ \psi \pm \left(\psi^2 + \frac{16GM D_{LS}}{D_L D_S} \right)^{\frac{1}{2}} \right\}. \quad (3)$$

When a source is approximately regarded as a point-like source, one has generically two point image: one is an even parity image, θ_+ , and another is an odd parity image, θ_- . The

case, $\psi = 0$, is peculiar, and one has a ring image (Einstein ring) with an angular radius, θ_E , where

$$\theta_E = \left(\frac{4GM D_{LS}}{D_L D_S} \right)^{\frac{1}{2}} \approx 2 \left(\frac{Mh}{10^{12} M_\odot} \right)^{\frac{1}{2}} \left(\frac{H_0^{-1} D_{LS}}{D_L D_S} \right)^{\frac{1}{2}} \text{ arcsec.}, \quad (4)$$

where $H_0 = 100h \text{ km/s/Mpc}$ is the Hubble parameter.

When one has multiple images, there is difference of arrival time (time delay) between images. The time delay consists of two parts: the geometrical time delay due to the path length difference, and the gravitational time delay. We examine a simple order estimation of the gravitational time delay below:

$$\begin{aligned} \Delta t_{grav} &\sim \Delta \int \Psi dt \sim \int \frac{\partial \Psi}{\partial \theta} \theta dt \\ &\sim \Psi(H_0 \theta) \sim 1 \text{ yr.} h^{-1} (\Psi/10^{-5}) (\theta/2 \text{ arcsec.}). \end{aligned} \quad (5)$$

When we examine the similar order estimation for the geometrical time delay part, we find that it is the same order of magnitude as the gravitational time delay. The exact time delay formula derived by Borgeest & Refsdal(1984) is

$$\Delta t_{BA} = D_L \int_A^B \left(\vec{\alpha}(\vec{\theta}) - \frac{1}{2}(\vec{\alpha}_A + \vec{\alpha}_B) \right) \cdot d\vec{\theta}, \quad (6)$$

where the integration is done along an arbitrary path connecting images A and B.

Due to recent progress in radio astronomy, image deformation, more precisely a mapping matrix from image A to image B, is one of important ingredient in determining model parameters of lens systems (Gorenstein et al. 1988a). A deformation matrix is defined as

$$M = \left(\frac{\partial \vec{\psi}}{\partial \vec{\theta}} \right)^{-1}, \quad (7)$$

and a determinant of M is an amplification factor, A . When A is positive (negative), one has an even (odd) parity image.

HST Snapshot Survey

The Snapshot Survey of gravitational lens candidates with HST (Bahcall et al. 1992, Maoz et al. 1992a, b, 1993a, b) was partially motivated by a pioneering paper on the

gravitational lens statistics by Turner et al.(1984). One of the important theoretical predictions made by Turner et al. was that about half of gravitational lensed QSOs have a small image splitting angle less than 1 arc second. On the contrary, the image resolution in optical regions on the ground is constrained by seeing and is typically 2 arc seconds or much worse. Therefore, observations with high resolution is essential in efficiently searching gravitational lensed QSO candidates.

Target sources (502 QSOs) are selected from Veron catalog (Veron-Cetty & Veron 1987) under the following criteria: (1) $z > 1$, (2) $V < -25.5$, (3) $|b| > 10$ deg, where absolute V magnitude is evaluated using $h = 1$, $\Omega_0=1$, $\lambda_0 = 0$, and k-corrections assuming an optical power-law spectral index of -0.5, and Galactic absorption corrections. Since pointing errors were very large, say 25 arc seconds at most, efforts for the corrections were made to finally achieve the pointing accuracy of (14 ± 8) arc seconds. The detection limit of Snapshot survey depends on both angular separation of images, $\Delta\theta$, and their magnitude difference, Δm . The observational detection function can be well described by the following form (Maoz & Rix 1993):

$$D(\Delta\theta, \Delta m) = 1 \quad \text{for } \Delta\theta > 0.1 \text{arc sec. and } 10^{0.4|\Delta m|} < 19(\Delta\theta/1 \text{arc sec.})^{0.85}, \\ = 0 \quad \text{otherwise.} \quad (8)$$

For example, $\Delta\theta > 0.2 \text{arc sec.}$ for $|\Delta m| \lesssim 1.0$, and $\Delta\theta > 1.0 \text{ arc sec.}$ for $|\Delta m| \gtrsim 3.0$. Gravitational lensed QSO candidates were then searched within $\bar{r}$ arc sec. from a target QSO (Bahcall et al. 1992). The limiting magnitude of Snapshot Survey is considerably brighter than $m_V = 23$.

In the target QSOs, there are 5 previous known gravitational lensed QSOs (including 2 candidates) and one newly found candidates, Q1208+101 (Maoz et al. 1992a). Redshift and number of images of the new candidate are, respectively, 3.8 and 3. The apparent V magnitude of images are, respectively $m_V = 19.8$ for image A and image C and $m_V = 20.0$ for image B. The image configuration is sketched in figure 2. The lensing frequency can be compared with the theoretical predictions by the standard method of Fukugita & Turner (1991) in order to obtain the cosmological constant limit. One may consider the more sophisticated and complicated model for lensing galaxies (e.g., Maoz & Rix 1993), but one cannot significantly change the λ_0 limit. In cases of $\Omega_0 + \lambda_0 = 1$ models, the predicted numbers of gravitational lenses in 502 QSOs are 24 for $\lambda_0 = 0.9$, 15 for $\lambda_0 = 0.8$, and 11 for $\lambda_0 = 0.7$. By taking into account the statistical fluctuations of $\sqrt{N}$, the cosmological models with $\lambda_0 > 0.8$ may be excluded with the confidence level greater than 90%. Maoz & Rix examined the $\Delta\theta$ distribution, however it is safe for us to conclude that we can do nothing further with these few lenses less than 10. (Maoz & Rix examined the gravitational lens statistics of $\Delta\theta$ with only 4 lenses.)

Hubble parameter determination with Q0957 + 561

As seen in equation (6), the time delay is proportional to the distance factor, D_L , and therefore inversely proportional to H_0 . Since the deflection angle, $\tilde{\alpha}$, is a local quantity,

the observationally accurate determination of the time delay may enable us to evaluate the value of H_0 if we can uniquely fix the lens model. However, the time delay is typically a quantity of order of 1 year, and one should monitor the flux variability during 10 years or more in order to accurately measure the time delay.

The first discovery of a gravitational lensed QSO, Q0957+561 (images A and B), was in 1979 (Walsh et al.), and it is an only lens to use the H_0 determination at present. Detailed observational investigations of this system in optical region were done by e.g., Young et al.(1980, 1981), and the results are summarized in table 1, where G denotes a elliptical galaxy found in the field and may be a dominant gravitational source. In the CCD frame, there were 146 galaxies for $m_R \lesssim 24.5$, and the most of their redshifts are highly unknown due to their faint apparent magnitude, typically $m_R \gtrsim 21$.

These faint galaxies may be member of a cluster of galaxies containing G. If this is the case, the model of this system may be very complicated. However, there are only one galaxy within 7 arc sec. from G, and they can be therefore represented in a simply parametrized form. The lens model proposed by Falco et al.(1991) assumes the following deflection angle form:

$$\vec{\alpha} = \vec{\alpha}_G + \vec{\alpha}_B + \vec{\alpha}_C, \quad (9)$$

where the terms with subscripts, G, B, and C are, respectively contribution from G, a possible massive black hole at the center of G, and the cluster of galaxies around G. From the optical observations, the first term is well approximated by a King-type mass profile parametrized with a velocity dispersion, σ , and a finite core size, θ_c , assuming that the light profile well traces the mass profile. Its explicit form is found in Young et al.(1981). The rest terms are, respectively,

$$\vec{\alpha}_B = \frac{4GM_c\vec{\theta}}{D_L\theta^2}, \quad \vec{\alpha}_C = \vec{\alpha}_0 + Q_S\vec{\theta} + Q_A\omega[\gamma]\vec{\theta}, \quad (10)$$

where $\vec{\alpha}_0$ is a constant vector, Q_S , Q_A , and γ are constants, and $\omega[\gamma]$ is a 2×2 matrix with components, $\omega_{22} = -\omega_{11} = \cos 2\gamma$, $\omega_{12} = \omega_{21} = \sin 2\gamma$.

The model parameters are chosen such that all of observational quantities are well reproduced by the lens model. However, not all of these parameters can be determined uniquely due to the degeneracy of the parameters, as was pointed out by Falco et al.(1985) and Gorenstein et al.(1988a). Falco et al.(1985) found that the lens equation (1) and time delay formula (6) are unaffected by the following invariant transformation:

$$\begin{aligned} \vec{\alpha} &\rightarrow \epsilon\vec{\alpha} + (1 - \epsilon)\frac{D_S}{D_{LS}}\vec{\theta}, \\ \vec{\psi} &\rightarrow \epsilon\vec{\psi}, \quad H_0 \rightarrow \epsilon^{-1}H_0. \end{aligned} \quad (11)$$

In the model by Falco et al.(1991), the transformation of $\vec{\alpha}$ can be interpreted as, say, $\sigma \rightarrow \epsilon^{1/2}\sigma$.

Now, since Q_S will be small, we express it as

$$Q_S = (1 - \epsilon) \frac{D_S}{D_{LS}}, \quad (12)$$

where ϵ is a parameter. The condition, $\epsilon \leq 1$, should be imposed if we assume the positivity of mass in the cluster. By comparing equations (11) and (12), we find that the parameter Q_S cannot be uniquely determined unless we can directly measure the value of, say, σ . This fact implies that we cannot uniquely determine the value of H_0 with Q0957+561 even when we can accurately determine the time delay. The direct measurement of one (or all, if possible) of the model parameters is therefore essential. When one cannot determine the value of ϵ , one will have the model-dependent upper limit on H_0 by assuming $\epsilon = 1$.

The procedure to determine the value of H_0 proceeds as follows: First, the best fit parameters of Q0957+561 are obtained by assuming the arbitrary value of ϵ , say, $\epsilon = 1$, and let the estimated value of σ be σ_1 . Next, the velocity dispersion, σ_0 , should be observationally determined. Then, by comparing σ_1 and σ_0 , we can uniquely determine the value of ϵ as

$$\epsilon = \left(\frac{\sigma_1}{\sigma_0} \right)^2. \quad (12)$$

Now, all of the model parameters are fixed, and the observational data of the time delay, Δt enables us to determine the model-dependent value of H_0 .

In determining the model parameters in equation (10), Falco et al.(1991) used not only the total magnitude ratio but also relative image shape difference of the radio map by Gorenstein et al.(1988b). The best fit parameters obtained by assuming $\epsilon = 1$ is

$$\begin{aligned} \sigma &= 390 \pm 4 \text{ km/s} (= \sigma_1), \quad \theta_c = (2.9 \pm 0.1) \text{ arc sec.}, \\ M_c &= (11.1 \pm 0.1) 10^{11} M_\odot, \quad Q_A = 0.30 \pm 0.01, \quad \gamma = (63.3 \pm 0.6) \text{ deg.}, \end{aligned} \quad (13)$$

and the estimated value of H_0 is

$$H_0 = (90 \pm 10) \left(\frac{\sigma_0}{390 \text{ km/s}} \right)^2 \left(\frac{1 \text{ year}}{\Delta t} \right) \text{ km/s/Mpc.} \quad (14)$$

It should be noted that the origin of σ_0^2 factor is not the deflection angle due to G, which is proportional to σ^2 , but is the degeneracy of the model parameters.

The direct measurement of Δt and σ_0 is very difficult. Especially, as for Δt , two distinct values have been obtained by two groups, even though they analyzed the same

observational data (e.g., Vanderriest et al. 1989, Lehar et al. 1992). The obtained values by Press et al.(1992a, b) and Pelt et al.(1994) are, respectively,

$$\begin{aligned}\Delta t(\text{optical}) &= 536 \pm 5 \text{days}, \\ \Delta t(\text{radio}) &= 536 \pm 5 \text{days},\end{aligned}\tag{15}$$

and

$$\begin{aligned}\Delta t(\text{optical}) &= 415 \pm 32 \text{days}, \\ \Delta t(\text{radio}) &= 409 \pm 23 \text{days}.\end{aligned}\tag{16}$$

Since the data are very noisy and sampled inhomogeneously in time, Δt of Q0957+561 will not converge soon. The observation for σ_0 was done by Rhee(1991). He said “the lines of the Mg1b triplet are clearly visible at 5174Å”, and the line width might be used in evaluating σ_0 . The obtained value is $\sigma_0 = 303 \pm 50 \text{km/s}$, and he is an only person who has shown us the value of σ_0 till today.

Taking into account the estimated value of Δt , σ_0 , and their errors, we have

$$\begin{aligned}H_0 &= 50.0 \left(\frac{\sigma_0}{303 \text{km/s}} \right) \left(\frac{415 \text{days}}{\Delta t} \right)^2 \text{ km/s/Mpc} \\ &= 24.3 \left(\frac{\sigma_0}{253 \text{km/s}} \right) \left(\frac{556 \text{days}}{\Delta t} \right)^2 \text{ km/s/Mpc}, \\ &= 98.2 \left(\frac{\sigma_0}{353 \text{km/s}} \right) \left(\frac{386 \text{days}}{\Delta t} \right)^2 \text{ km/s/Mpc}.\end{aligned}\tag{17}$$

As mentioned before, we have large errors in Δt and σ_0 . Therefore, we can obtain either large $h \sim 1$ or small $h \sim 0.2$ by taking the error bars at most, as is shown in equation (17).

Discussion

The usual method of gravitational lens survey is the detail imaging of sources listed in existing QSO catalogs. However, at present, the crucial difficulty of observationally examining the gravitational lens statistics is the lack of QSO catalogs with high quality. That is, a good catalog should be well defined and contain a lot of QSOs such that it contains a lot of gravitational lensed QSOs. The present available catalogs are either highly incomplete (e.g., Veron catalog) or containing small number (~ 500) of QSOs (e.g., Boyle et al. 1987).

When Digital Sky Survey program will succeed, the situation will be significantly improved. The survey covers a half of the northern sky with seeing ~ 0.8 arc sec. and limiting magnitude ~ 25 . Then about 10^5 QSOs will be found. Furthermore, with aids of SUBARU telescope, the photometry of these QSOs with $m_V < 24 \sim 25$ may be possible with

high angular resolution ~ 0.2 arc sec. Then significant number $\sim 10^2$ of new gravitational lensed QSOs may be found with Japanese observational programs, and the gravitational lens statistics will be a powerful tool not only to limit the cosmological constant but to accurately evaluate the value of H_0 .

However, it should be noted that the estimated value of H_0 with gravitational lenses may be highly model-dependent. One of important tasks in gravitational lens theories will be Lens Clean, that is, the inversion problem from observed quantities to reconstruct a reliable lens model as possible (e.g., Kochanek & Narayan 1992). Future observational studies of gravitational lenses with new observational tools will also play an important role as much as theoretical investigations.

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figure 1

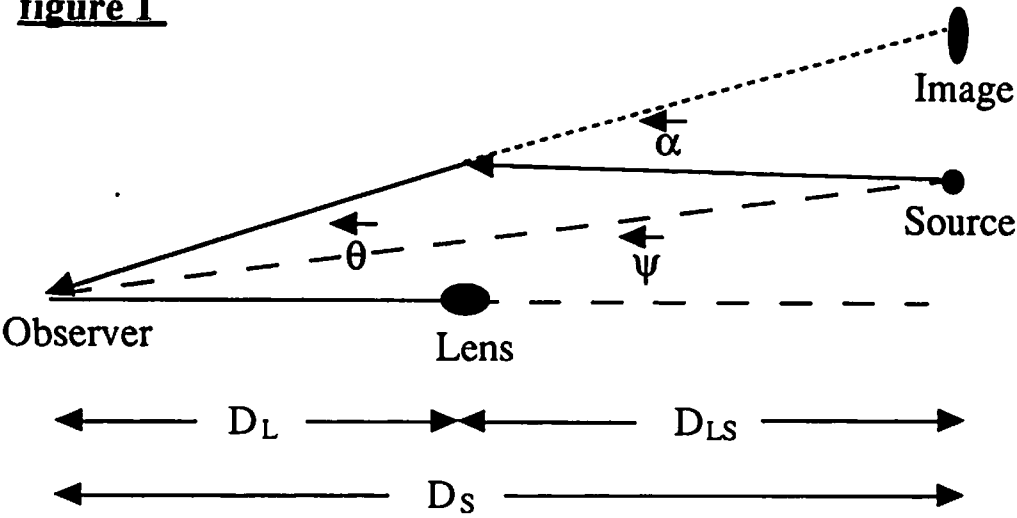


figure 2

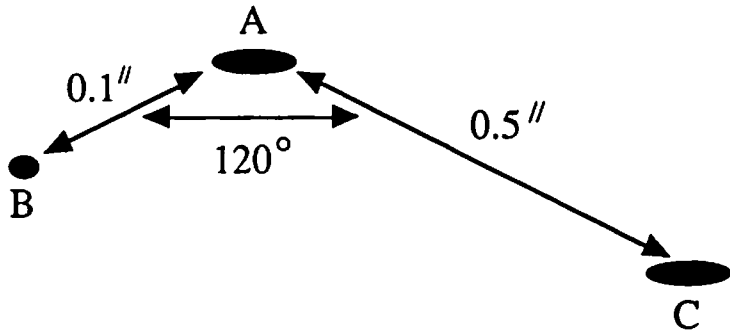


table 1

	Q0957+561		
	image A	image B	G
sky position relative to B	(1.2,6.0)	(0,0)	(0.0,0.8)
apparent magnitude	17.3	17.6	18.5
redshift	1.41	1.41	0.39

MACHO !

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Abstract

The observations and theories of MACHO/EROS/OGLE experiments are reviewed. It has been deduced from the HI rotation curve measurements that there are very massive dark objects in the halo of our Galaxy. Optical observations showed that the stars of the mass of about $0.1 M_{\odot}$ are dominant constituents in the halo. Proposals have been made to detect the compact halo objects utilizing the gravitational microlensing in the directions of 10^6 stars in the Magellanic Clouds and the Galactic bulge. MACHO/EROS/OGLE experiments detected 5 microlens events and provided the evidence of existence of the massive compact halo objects. The mass and number density of the MACHOs are estimated from these observations. A new proposal to detect the microlensing at radio frequency utilizing water maser sources is also presented.

I. Observational facts on the rotation curve and small mass stars

Masses of spiral galaxies have been estimated by measuring the rotation curve with the HI 21 cm line (Bosma 1981). It is well known that the rotational velocities of spiral galaxies do not decrease with radius with the $r^{-1/2}$ law which is expected from the Kepler motion (a mass concentrated at the center of the system). From this fact, it is deduced that spiral galaxies have very massive halo. Though the rotation curve of our Galaxy can be measured only inside of the solar circle with the radius of about 8 kpc (the distance from the galactic center to the Sun ; see Verschuur and Kellermann 1988), the rotation curve outside the solar circle can be obtained from the measurement of the radial velocity of HII regions with known distance (Rohlfs & Kreitschmann 1987). By this method the mass of our galaxy inside inner 20 kpc is estimated to be about $2 \cdot 10^{11} M_{\odot}$ and the mass of the Galaxy increases with the radius approximately linearly.

There is an alternative evidence on the massive halo of galaxies. The brightness distribution of spiral galaxies consists of two components : the spheroidal component and the disk component. The spheroidal components further divided into the bulge and halo. The brightness of the disk component falls exponentially with radius. However, it is known that the brightness of the spheroidal component

falls very slowly (called as the $r^{1/4}$ law), expressed approximately as $\exp[-(r/r_0)^{1/4}]$ (For example, Kodaira et al. 1990). From this fact, it has been deduced that this extended component of spiral galaxies are low mass stars.

The direct observation of the very low mass stars with mass of about $0.1 M_\odot$ in the Galactic halo has been made by Richer et al. (1992). They made a deep CCD exposures in three bands (V, R, and I to the 24.5 mag.) in the direction of the high galactic latitude ($b=73^\circ$) and distinguished the star images from galaxies. In their investigations, 39 objects among 5000 images were picked up as stars. The color analyses of these objects showed that mass density of these low mass halo stars between the mass range dM is proportional to $M^{-3.5 \pm 1.2}$. This fact have shown that mass of the halo is composed of the very low mass stars of approximately equal to or less than $0.1 M_\odot$.

All these observational facts lead that the large part (more than 90%) of the mass of our Galaxy lies in the halo and the visible disk component is only a minor constituent of the Galaxy in term of the mass. However, we did not have any good observational tools to probe these massive halo components. Paczynski (1986) pointed out a possibility of observing these massive compact halo objects of low mass with gravitational microlensing.

II. Theory of Microlensing

Reviews of theories of gravitational lensing have been made in number of articles (Blandford & Narayan 1992; Kayser et al. 1992; Mellier et al. 1990; Moran et al., 1989; Bliokh and Minakov 1989). The probability of the lensing can be written in terms of the optical depth as

$$\begin{aligned} \tau &= \pi a_0^2 n L = (4 \pi D G / c^2) \rho L \\ &= 1.5 \cdot 10^{-6} (\rho / 1.6 \cdot 10^{-25} \text{ g}) (L / 10 \text{ kpc}) \end{aligned} \quad (1)$$

where a is the Einstein radius of the gravitational lens

$$\begin{aligned} a_0^2 &= 4 G M D / c^2 \\ &= (4.3 \cdot 10^{13} \text{ cm})^2 (M / 2 \cdot 10^{31} \text{ g}) (D / 10 \text{ kpc}) \end{aligned} \quad (2)$$

and n is the number density of the lens with mass M , ρ the mass density of the lens, L the distance to the light source, D the average distance ($= D_0 (L - D_0) / L$), and the D_0 distance from observer to the lens (see for example, Deguchi and Watson 1987 & 1988). The splitting of the two images created by the microlens is approximately

$$\theta \approx a_0 / D = 2.9 \cdot 10^{-4} \text{ arcsec} (M / 2 \cdot 10^{31} \text{ g})^{0.5} (D / 10 \text{ kpc})^{-1} \quad (3)$$

This angle is too small and cannot be observed as two separated images except by the VLBI techniques at radio frequencies which will be described in section IV.

The optical depth is proportional to the mass density of the lens and does not depend on the distribution of the mass spectrum of lensing objects. The probability of observing the single lens event with looking a distant ($L > 10$ kpc) star as a light source is about 10^{-6} for lensing objects in our Galaxy. This number is quite small and no chance for us to observe lensing events when we observe one particular star as a light source. However, Paczynski (1986) pointed out that when we observe the 10^6 stars in Magellanic clouds at once, we can find one lens event. Stars in the Magellanic clouds are resolvable by medium-size optical telescopes and the brightness variations of 10^6 stars can be monitored relatively easily.

Griest (1991) calculated the probability of finding lens events per one year. The probability of finding lens events per unit time is proportional to

$$P \approx 2 a_0 v T n L \quad (4)$$

where v is the relative velocity of the lens to the light source and T is the time length of observations. Note that the difference of this expression to equation (1) is $(2 a_0 v)$ instead of (πa_0^2) . The relative velocity is approximately 200 km s^{-1} for stars in our Galaxy. Then, in one year, the vT is approximately $6.3 \cdot 10^{14} \text{ cm}$. This number is comparable with the size of the Einstein length of the mass of $0.2 M_\odot$. So that the probability of finding lens events for the lens mass less than $0.1 M_\odot$ is slightly larger than the value deduced from the optical depth. The probability is proportional to $M^{-0.5}$, and increases with decreasing the lens mass.

Note that the light bending due to gravitational lens does not depends on the frequency of the light (when the diffraction is negligible, see Deguchi and Watson, 1986). So no aberration is expected in the gravitational microlens event (except a special case which is described in section 4). So the color of stars does not change by the gravitational lens amplification.

III. MACHO/EROS/OGLE

Stimulated by the Paczynski's proposal, three groups of astronomers in the world started the projects to find the gravitational microlens events. One (Livermore-Mt. Strolmo) is called MACHO group, which stands for Massive Astrophysical Compact Halo Object (about this terminology, see Griest 1992). They looked the Large Magellanic Cloud with the 2×2 mosaic of 2048×2048 CCD camera at Mt. Strolmo observatory, Australia (Bennett et al. 1993). The second group (Saclay-IAP) is called EROS (Expérience de Recherche d'Objects Sombres). They also look the LMC with normal plates and the 16 mosaic of 578×400 CCD camera at European Southern Observatory, Chile (Magneville et al. 1993). The third group (Warsaw-Carnegie-Princeton) is called OGLE (Optical Gravitational Lensing Experiment), using CCD camera at Las Campanas Observatory, Chile (Udalski

et al. 1993a). The summary of their instrumentations are given in table 1.

They found several gravitational microlens events almost at the same time. The amplification factor for the event found by the MACHO group is 8 (Alcock et al. 1993) and those for the other groups are less than 3 (Aubourg et al. 1993; Udalski et al. 1993b). The typical brightening time scales were about 3 weeks. No color change of the brightened stars were observed. One of these observations are shown in Figures 1. As shown in equation (1), the mass of the lens and the average distance to the lens always appears to be multiplied in the same equation. It is difficult to know the distance of the lens (though it is deduced approximately about 10 kpc). So the masses of the lens which causes these lens events are approximately estimated to be between $3 \cdot 10^{-2}$ and $1 M_{\odot}$, most probably $0.1 M_{\odot}$.

There are a great number of variable stars which exhibit time variation in their brightness. These are Cepheid variables (several days), Mira-type variables (a year), irregular variables (months), cataclysmic variables (10 years), novae and recurrent novae (1000 years). Normally they are periodic, and show some change of colors. To distinguish the gravitational lens events from the brightness variations of variable stars, MACHO groups set three criteria:

- (1) no color change
- (2) symmetric increase and decrease of the brightness with respect to the peak.
- (3) no repeat for the brightening during observation period.

These criteria are used to discriminate the nova or variable star events from the observations.

It is interesting to consider the lensing case of binary stars, which occur very often (about 50% of possibility) in the universe. When the two stars (light sources) have different colors and unresolvable by a telescope, the lensing light curve should exhibit the color change. In contrast, the lensing object might be composed of binaries. In such a case, the amplification event occurs twice, so that the light curve cannot be symmetric with respect to the peak no more.

Further interesting observational proposals have been made. For example, the annual variation of about a few percent of the probability of lensing events can be expected due to the motion of the Earth around the Sun (Griest 1992). We can measure the relative motion of the solar system to the Magellanic Clouds (actually proper motion of the Magellanic Cloud stars were measured by Kroupa et al. 1994). Time variation of the spectra of the stars can be expected during the brightening phase and we can estimate the distance to the lens (hence, the mass of the lens) (Maoz and Gould 1994). The other interesting proposals are concerning the proper motion and parallax measurements of MACHO (Gould 1994ab). Various fine optical techniques will be used near future to find the mass of individual MACHOs.

It is difficult at present to estimate the total mass of MACHO in our Galaxy because of the small number of samples and uncertainty of the distance to MACHO. However, the best reliable estimate of the MACHO density (Turner 1993) leded the contribution of MACHOs to the cosmological

constant $\Omega_0 < 0.1$. This is because the upper limit of the galaxy mass deduced from the rotation-curve measurements (described in section I) does not change by the observations of frequency of MACHO events at the present level. If the microlens events occur more frequently by a factor of 10 more than the present predictions, our understanding of the present universe would fail and the exotic dark matters must be introduced.

IV. GLAMOUR

Is there any method to detect the gravitational lens events at radio frequencies? Concerning on the number of the light sources, the radio continuum sources are approximately about 10^4 . Taking into account the small probability of finding the lensing events, we have to say that it is difficult to detect such events at radio frequencies. One possibility is to use the water masers at 22.235 GHz.

Water maser emission at 22 GHz are resolved in numerous numbers of spots by VLBI measurements (for example, Gwinn et al. 1992). Emission has radial velocities spreading between -300 km s^{-1} to $+300 \text{ km s}^{-1}$. The spot sizes are approximately 10^{13} cm and spots are spread in the region of the size of about 10^{17} cm (in a few arc second). The number of spots observed by VLBI experiments are approximately 1000 ($I > \text{about } 60 \text{ Jy}$). If we extend the number-intensity relation obtained from the VLBI experiments, we can deduce that the total number of water maser spots with the intensity stronger than 1 Jy is about 10^5 in the strongest water maser source as W49N. This number is slightly uncertain because the VLBI experiment does not necessarily pick up all the maser flux which is observed by a single dish telescope.

If we can monitor the W49N water maser intensities by a single dish with an enough good sensitivity and a spectral resolution of about 0.3 km s^{-1} with a band coverage of $\pm 500 \text{ km s}^{-1}$, we can monitor about 10^5 maser spots simultaneously by a single dish radio telescope. This is an enough number of light sources to detect microlensing event in the direction of W49. The W49 N water maser sources lies in the galactic plane at the distance of about 13 kpc from the Sun. Then it is supposed that, due to the higher mass density of the disk, the microlensing events occur more frequently than the microlensing in the halo direction.

However, the water maser intensity varies in a time scale of about several months. It is difficult to discriminate the intrinsic time variation of masers from microlensing events. Deguchi and Watson (1986ab) investigated the effect of diffraction and interference between the two microlensed images. According to their results, the interference is observable when the two parameter X and Y satisfy the following conditions:

$$X = (a/a_0)^2 < 1 \quad (5)$$

$$Y = (\lambda_0 / \lambda) \approx 1 \quad (6)$$

where a and a_0 are the source size and the Einstein radius of the lens, respectively, λ_0 is the Schwartzchild radius $[2 G M/c^2 = 1720 \text{ cm } (M / 2 \cdot 10^{31} \text{ g})]$, and λ is the wavelength observed. This condition can be derived from the interference condition (see Deguchi and Watson 1986b for detailed calculations). If we suppose the lenses of the mass $0.01 M_\odot$, above conditions (3) and (4) is approximately satisfied. The interference fringe can be observed as a variation of maser intensity in the spectral line and the oscillation time scale is given as

$$t = a_0 / v / Y = 23 \text{ min. } (M/0.01 M_\odot)^{-0.5} (\lambda/1.3 \text{ cm}) \quad (7)$$

In this case, we can observe the oscillation of the intensity of the maser intensity in the time scale of about 23 minutes (see figure 2). If we can observe this oscillation of intensity in the water maser spots, we can attribute the variation of the maser intensity as a microlensing event. The Einstein ring radius of the lens event is approximately 0.3 milliarc sec and this is approximately the same as the resolution of the present earth-based VLBI experiments. So that the Einstein rings or double images may be resolvable by the present earth-based and future space VLBI experiments when they are done at the time of brightening.

We would like to propose the experiment to detect the gravitational lensing events at radio frequency using twin 30-m dishes to eliminate the false signals which may occur as a result of telescope malfunctioning. The twin telescopes are devoted to observe H_2O maser spots at 22.235 GHz with a maser receiver (or HEMT) of $T_{\text{sys}} = 50 \text{ K}$ and an acoust-optospectrometer (AOS) of 37 KHz 2000 Channels ($\Delta v = 0.55 \text{ km/s}$ and $\Delta V_{\text{LSR}} = 600 \text{ km/s}$). The integration time will be 1 minutes which gives the rms noise level of $3 \Delta T = 0.2 \text{ K}$. The signals from two telescopes are sought for the correlation of a days.

We would like to call this project as *GLAMOUR*, i. e., Gravitational Lens Amplification of Masers by Opportune Ulterior Rendezvous.

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Table 1. Comparison between the experiments

Group	MACHO	EROS	OGLE
Object	Large Magellanic Cloud	LMC	Galactic Bulge 14 fields
Telescope	Mt. Strolmo 1.27m (Great Melbourne Telescope)	ESO Schmidt 40-cm (CCD)	Las Campanas 1-m (Swope)
Method	CCD Red -Bleu (0.5 ° x 0.5 ° 2x2 mosaic of 2048x2048 pixels)	5°x5° Plate CCD 16 mosaic 579x400 pxs)	CCD I V
Image processing	SoDophot	MAMA(plate)	Dophoto
Term	1992.7-93.7	1990-1993(plate) 91.12-91.3 and 92.7-93-3 (CCD)	924-92.8 45 nights
Number of Stars	1.2 Million	4 Million	0.65 Million
Criterion	Color change <0.1 mag. single event	Monte Carlo Simulation A>1.34	Non-var. DB 3 sigma
Results	1 candidate (V=19.6, R=19.0)	no during 91-92 CCD 2 events (plate)	1 event

Fig. 1. The gravitationa lens brightning observed by Alcock et al. (1993).

The peak occurs at the day number 435. The color does not change and the profile is symmetrical with respect to the peak. This is a strong evidence if gravitational lensing.

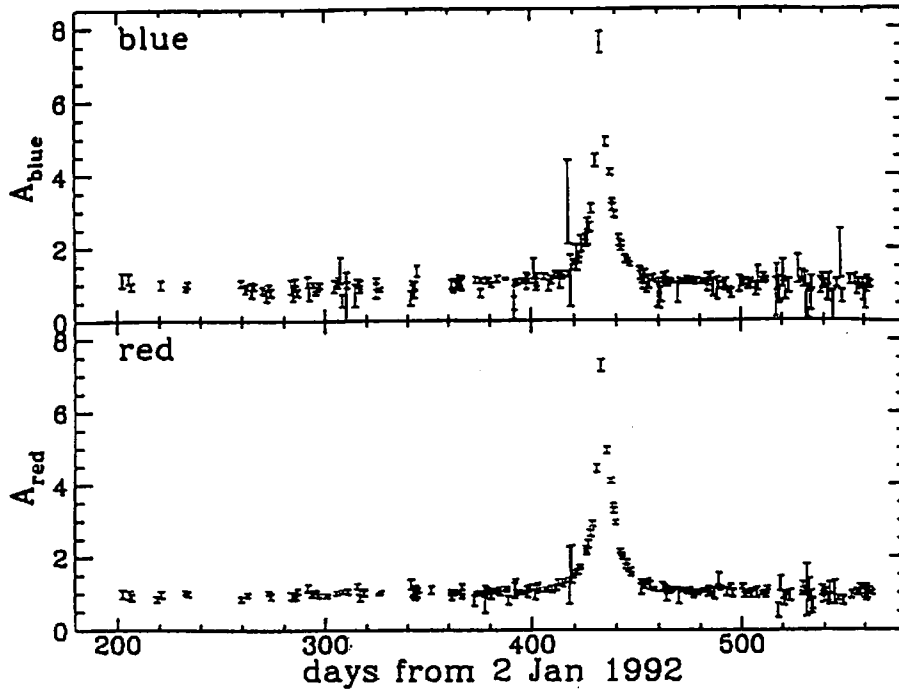
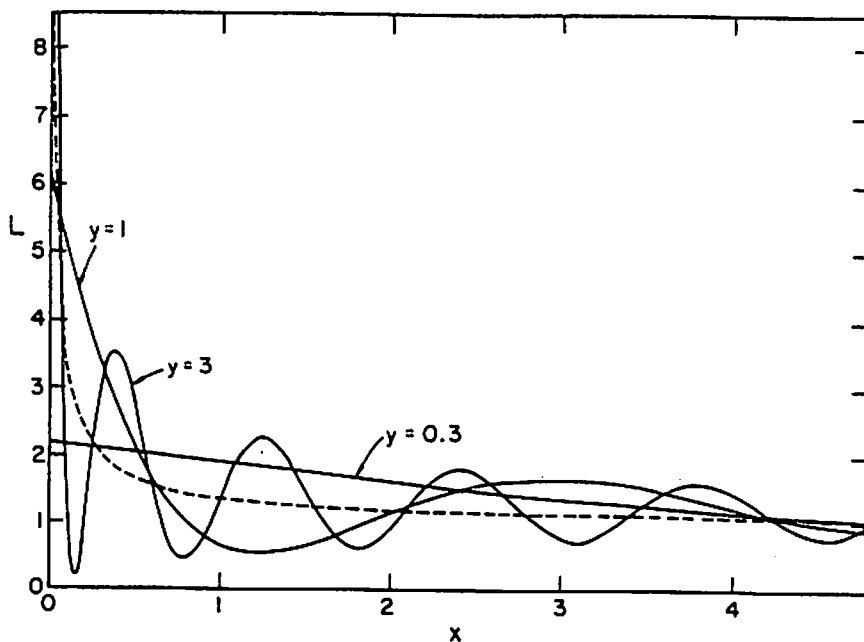


Fig. 2 The interference fringe of the microlensing of very small light source

(From Deguchi and Watson 1986a). When the source moves from the center of the lens to an infinite distance, the luminosity oscillates as shown in this figure. The broken curve is the Leibes-Refsdal solution.



Pulsar Timing and Mass of Star and MACHOs

— Gravitational Time Delay of Pulsar Timing and Measurement of Stellar Mass —

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Abstract.

An application of pulsar timing observation is proposed for measuring the mass of a single star. In the case that there is a star in front of a pulsar, the time of arrival of the pulse from the pulsar will be affected by the gravitational field of the star. Utilizing this effect, the possibility of the measurement of the stellar mass by observing the change in this effect is shown when the relative configuration of the observer, star and pulsar varies. The possibility of detecting the MACHOs is also considered by introducing the concept of "time delay depth".

1. Introduction

The mass of a star is one of the most fundamental quantities in astrophysics. However, the measurement of stellar mass, especially that of a single star, is very difficult. Recently, one kind of the dark matter candidates in Galactic halo, MACHOs, was found in microlensing experimental projects (Aubourg et.al.1993; Alcock et.al.1993; Udalski et.al.1993). These masses are supposed to be of the order of 0.1 of a Solar mass assuming certain conditions for the velocity and distance of the MACHOs. Due to these assumptions, it is not possible to determine what kind of stars these MACHOs are. Presently it is considered necessary to obtain a more accurate mass measurement of MACHOs.

A method to measure the stellar mass using Gravitational Lensing, one of the relativistic effects, was recently proposed by us (Hosokawa et.al. 1993). In this paper, we consider Gravitational Time Delay, the other relativistic effect, to measure the stellar mass. Gravitational time delay by the Sun is the well known Shapiro delay (Shapiro 1964). In pulsar timing observations, this effect by the Sun has been noted and corrections made in the subsequent analysis of pulse data (Backer & Hellings 1986).

One kind of pulsar called millisecond pulsars have long term high pulse period stability. For example, the period (P) of PSR1937+21 is 1.558 msec and its time derivative (dP/dt) is 1.0×10^{-19} sec/sec. Therefore millisecond pulsars have proved to be one of the most stable clocks of all known natural phenomenon. So, pulse timing observation of millisecond pulsars is important for time-keeping metrology

(Foster & Backer 1990). For this reason, the Communications Research Laboratory (CRL) has been taking pulsar timing observations since 1989 (Hanado et.al.1993).

In pulsar timing observations, the times of arrival (TOA) of pulses from a pulsar are measured in the proper time of the moving observer and must be related to the proper time of the pulsar. A procedure to transform the proper time t of the observer into the proper time T of the pulsar is expressed as follows, (for example Taylor and Weisberg (1989)),

$$T = t + t_{\text{Dynamical}} + t_{\text{plasma}} + t_{E\odot} + t_{s\odot} + (t_s) \quad (1.1)$$

where $t_{\text{Dynamical}}$ is the correction term that comes from the position and motion of the observer and the pulsar, t_{plasma} is the time delay correction term caused by interstellar plasma which is dependent upon frequency, $t_{E\odot}$ is the solar system's "Einstein delay" that is a combination of the gravitational redshift and the time delay due to the motion of the observer, $t_{s\odot}$ is the Shapiro delay by the Sun and t_s is the time delay caused by the interstellar star.

The TOA data has been fitted to a model that included the pulsar spin frequency ($\nu = 1/P$) and its first two derivative ($\dot{\nu}, \ddot{\nu}$), the pulsar position and proper motion, and the dispersion measured.

In this TOA data analysis, it has not been considered necessary to pay attention to the gravitational time delay caused by the star being between the observer and the pulsar because this would be approximately constant in the general case. Now consider the observer, star and pulsar as being well aligned, while the relative configuration of the observer, star and pulsar varies. In this situation, the differences in the time delay caused by the change in position of the interstellar star should be taken into account for timing observations. It is shown that there is the possibility of measuring the stellar mass by observing the change in this effect. Among many time delay effects, hereafter we will focus attention on the gravitational time delay by the star.

In section 2, we describe the theory to measure stellar mass by pulsar timing observation. Section 3, we consider the possibility of detecting the MACHOs using pulsar observation. Here we discuss the concept of "Time Delay Depth" in the time delay effect which corresponds to the "Optical Depth" in microlensing. Section 4, summarizes our conclusions and remaining problems.

2. Measurement of Stellar Mass using Gravitational Time Delay

Consider that the pulse from pulsar P is delayed by the gravitational field of a foreground star S and arrives at the observer O . Suppose that the pulsar and the star can be observed optically. Let us take D_s and D_p as the distance to the star and the pulsar respectively, M as the mass of a star and b as the impact parameter of arrived pulse path and star. In this case, the gravitational time delay by the star S is expressed as follows,

$$t_s = \left[\frac{2GM}{c^3} \right] \times \log \left[\frac{2D_s}{b} \times \frac{2(D_p - D_s)}{b} \right] \quad (2.1)$$

The magnitude of $2GM/c^3$ is 1.0×10^{-5} sec ($10\mu\text{sec}$) in the case of the Sun. Here we express the equation by using the mass ratio $m = M/M_\odot$ where $M_\odot$ is mass of the Sun, and the separation angle between the star and the pulsar; $\theta \equiv b/D_s$,

$$t_s = m\{Constant - \log \theta^2\} \times 10[\mu\text{s}] . \quad (2.2)$$

Note that in this formula, we do not need to know the distance of the star and the pulsar because the formula for the gravitational time delay is expressed by a logarithmic function, the information on the distance to the star and pulsar can be put into the constant term.

For simplification, we consider only taking into account the relative proper motion of the pulsar and the star. Because of the relative proper motion μ , the separation angle will change with time by,

$$\theta(t) = \sqrt{\theta_0^2 + (\mu t)^2} , \quad (2.3)$$

where θ_0 is the angle of separation at the time when P and S are at closest proximity, and choose this time as the time origin $t = 0$. In this case, the time variation of TOA can be expressed as

$$t_s(t) = m\{Constant - \log(\theta_0^2 + (\mu t)^2)\} \times 10[\mu\text{s}] . \quad (2.4)$$

Then, if we observe TOA at two different times, t_1 and t_2 , the difference in this effect can be written as follow.

$$\Delta t_s \equiv t_s(t_2) - t_s(t_1) = m \log \left[\frac{\theta_0^2 + (\mu t_1)^2}{\theta_0^2 + (\mu t_2)^2} \right] \times 10[\mu\text{s}] . \quad (2.5)$$

Therefore by observing Δt_s and $\theta(t)$, the mass of star can be obtained from,

$$m = \left[\frac{\Delta t_s}{10[\mu\text{s}]} \right] / \log \left[\frac{\theta_0^2 + (\mu t_1)^2}{\theta_0^2 + (\mu t_2)^2} \right] \quad (2.6)$$

Here we take $\theta = 5''$, the variation angle $\Delta\theta = 0.01''/\text{year}$ and Mass $M = M_\odot$ to estimate the values of this effect. Suppose these as typical values, then the difference in the time delay is expected to be $40\text{ns}/\text{year}$. Presently in pulsar timing observations, the uncertainty in each TOA is 200ns and is expected to be reduced to 70ns in near future at CRL (Imae 1993). Therefore this time delay difference would be detectable over a period of several years of observations.

The time dependence of the difference in the time delay is shown in Fig.1. Fig.1a shows the trajectories of the stars. Here we take the relative proper motion μ as $0.01''/\text{year}$, and for each trajectories $\theta_0 = 0.1'', 0.3''$ and $1.0''$. Fig.1b shows the time dependence of the time delay on each trajectory. The difference in the time delay between t_1 and t_2 can be see from this figure. Fig.1c shows the time dependence of the time delay difference in the case of a one year observation period. Here the detection limit of the phase shift of the pulse owing to the time delay effect is regarded as the uncertainty in each TOA. The limit is expressed λ [second], or nondimensional parameter λ_0 that is the ratio of λ to $10[\mu\text{s}]$. In a one year observation period, more than a $m \times 100\text{ns}$ time delay difference will be observed in the upper region of the dotted line.

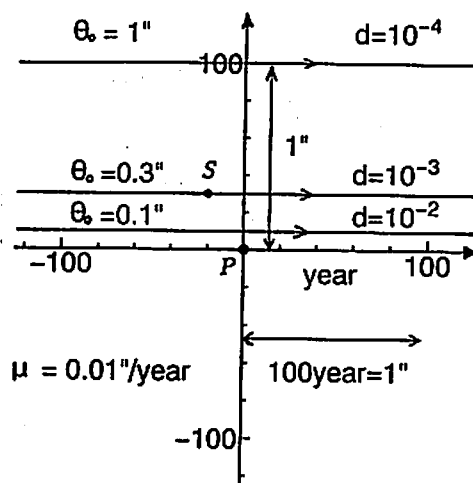


Fig.1a Trajectories of Stars

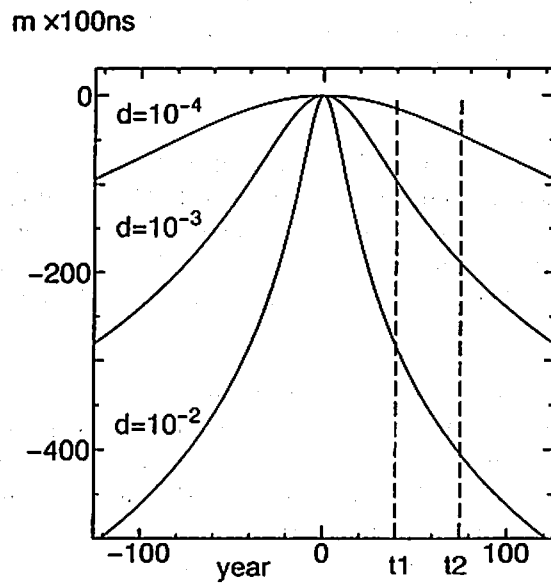


Fig.1b Time Delay

$m \times 100\text{ns}$

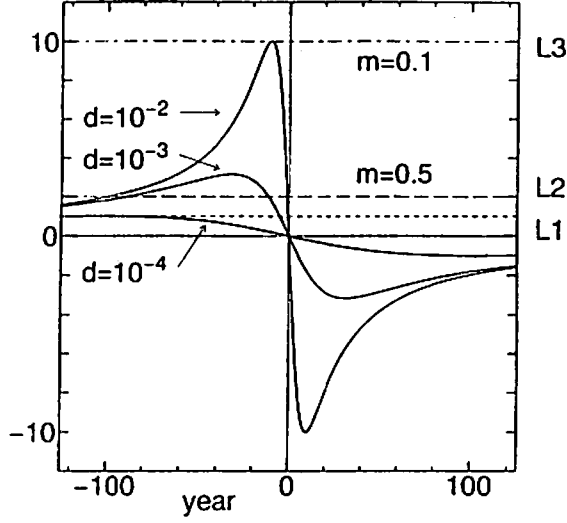


Fig.1c **Difference of the Time Delay**

From these figures, we can see that the possibility of detecting the time delay effect depends on i) the mass of star m , ii) the ratio of μ to θ_0 ($d \equiv (\mu/\theta_0)^2$), iii) the starting time of observation t_1 , iv) the period of observation $\Delta t \equiv t_2 - t_1$, and v) the detection limit λ .

In order to consider the separability of this effect from the other effects, we would like to examine the time dependence of this effect. This effect changes nonlinearly with time, but other effects are periodic (Earth rotation effect, etc.) or linear (distance variation by proper motion, etc.) with time. Therefore by subtracting these periodic and linear effects, this nonlinear and nonperiodic effect can be obtained.

In TOA data analysis, if the gravitational time delay term is not taken into account, this effect would be absorbed into the change in the magnitude of the pulse frequency ν and its time derivatives $\dot{\nu}, \ddot{\nu}$. In general, the variation ratio of $\Delta\nu$ to ν and $\Delta\dot{\nu}$ to $\dot{\nu}$ are small. But in some case, the ratio of $\Delta\ddot{\nu}$ to $\ddot{\nu}$ can be large. Theoretically, the second frequency derivative can be written $\ddot{\nu} = 3\dot{\nu}^2/\nu$ for the magnetic dipole spin-down case, so $\ddot{\nu}_{mp} \sim 10^{-32} \text{s}^{-3}$ for a millisecond pulsar (Manchester & Taylor 1977). Hence if the observed magnitude is much larger than $\ddot{\nu}_{mp}$, the gravitational time delay term must be taken into account. Note that even if S cannot be observed optically, (S is the MACHO, for example), the stellar mass can be derived from refitting the TOA data analysis with the additional new parameters $m, d = (\mu/\theta_0)^2$ and t_1 .

3. Time Delay Depth

In this section, we would like to consider the possibility of detecting the MACHOs and measuring the mass of the MACHOs in pulsar timing observation. In order to consider this, we should calculate the "event rate" of the time delay effect. The event rate is the probability of a detectable time delay occurring caused by the motion of the MACHOs. For the first step, we introduced the concept of "Time Delay Depth" in the time delay effect which correspondences to the "Optical Depth" in microlensing. The detail of the optical depth is discussed by Paczynski (1986) and Griest (1990).

Suppose a MACHO shifts its position by a , then the value of the time delay would differ by the position shift. The area over which the time delay can be measured is defined as the Time Delay Cross Section (TDCS). The integration of the TDCS of each MACHOs between the observer and the pulsar is defined as the time delay depth;

$$\tau \equiv \int_0^{D_p} dD [\text{TDCS}] \times [n(D)] \quad (3.1)$$

where D is the distance to MACHO and $n(D)$ is the number density of MACHOs.

Suppose that all MACHOs have the same mass and move in the same direction with a constant velocity. In this case, we can see the TDCS from Fig.1c. For example, consider some typical values for MACHOs; $a = 5\text{AU}$ and $D = 500\text{pc}$ (i.e. $\mu = 0.01''$), and $\lambda = 100\text{ns}$ (i.e. $\lambda_0 = 0.01$). In this case, the time delay effect is detectable in the upper region of dotted line (L_1 line) in the case for $m = 1$. Also on the L_2 line for $m = 0.5$, and broken line (L_3 line) for $m = 0.1$. Fig.2 shows the TDCS for each mass. The TDCS depends on the ratio of the detection limit λ_0 to the mass m . We express the TDCS by using the parameter $\alpha \equiv \lambda_0/2m$,

$$[\text{TDCS}] = [\text{factor}(\alpha)] \times \pi a^2 \approx \frac{1}{(\exp(\alpha) - 1)^2} \times \pi a^2 \quad (3.2)$$

Note that the TDCS is independent of the distance of the MACHOs, because the difference of the time delay is independent of the distance of the MACHOs. Then the time delay depth is expressed as follows,

$$\begin{aligned} \tau &\approx [\text{factor}(\alpha)] \times [\pi a^2 \times D_p] \times [n] \\ &\sim [\text{factor}(\alpha)] \times \frac{1}{m} \times 6 \times 10^{-10} \times a_0^2 \quad , \end{aligned} \quad (3.3)$$

where $a_0 = a/[1\text{AU}]$ and $D_p = 1\text{Kpc}$. The term $\frac{1}{m}$ comes from the number density term. Fig.3 shows the mass dependence of the time delay depth. In order to estimate the magnitude of time delay depth, let us take the value of the shift as $a = 30\text{AU}$ (i.e. it is equivalent to the shift of several year's proper motion), $\lambda \sim 100\text{ns}$ and $m = 1$. Then we obtain a value $\tau \sim 0.01$.

In the large mass condition, $\alpha \ll 1$, the mass dependence of τ is proportional to m/λ_0^2 , and in the small mass condition, $\alpha > 1$, is $\frac{1}{m} \exp(-\lambda_0/m)$. Therefore if the TOA analysis is performed using several values for the detection limit λ_0 , it is

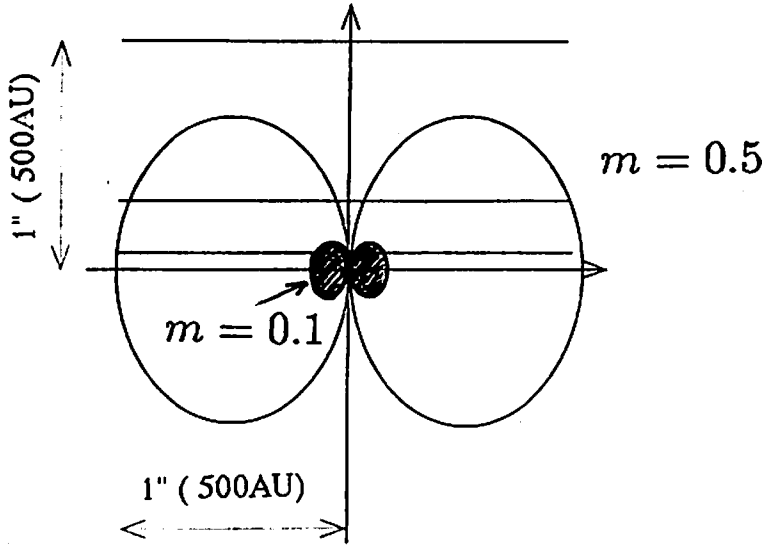


Fig.2 Time Delay Cross Section

expected to obtain the information on the mass of MACHOs because the parameter α acts as the role of the cut-off parameter.

From these results, we still cannot consider the possibility of detecting MACHOs, because we have not yet calculated the "event rate". Supposing the motion of the MACHOs (or dark matter) near the Earth are random, the event rate can be expressed as follows,

$$\Gamma \equiv \int_0^{D_p} dD \int dV [\text{TDCS}(V)] \times [F(V, D)] \times [n(D)] \quad , \quad (3.4)$$

where V is the velocity of MACHOs and $F(V, D)$ is the velocity distribution function. From this equation we can discuss the mass distribution dependence and the velocity distribution dependence of the event rate. There remains the calculation of the event rate in considering the possibility of the detection of the MACHOs.

$$[factor(\alpha)] \times \frac{1}{m}$$

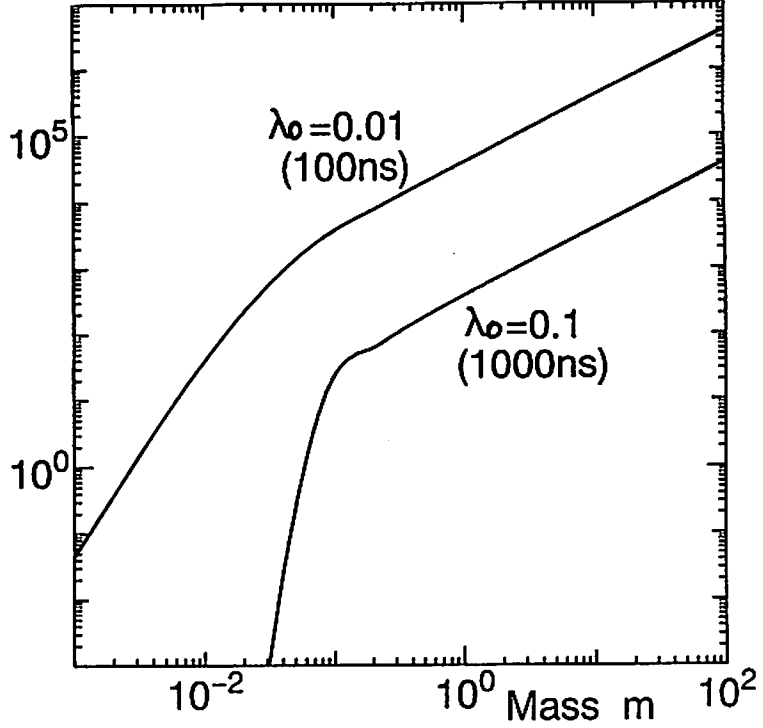


Fig.3 Mass Dependence of Time Delay Depth

4. Conclusion

We have discussed the measurement of stellar mass using pulsar timing observations. First, we have shown the time dependence of the time delay difference and the separability of this effect from the other effects in TOA data analysis. Next, we have considered the possibility of detection of MACHOs as an application of the method discussed above. As a first step, we have introduced the concept of the "Time Delay Depth". The value of the time delay depth may suggest if there is some possibility of detection. To obtain the event rate as a function of the mass distribution and the velocity distribution of MACHOs is a remaining problem.

The details of this paper will be published elsewhere in the future.

Acknowledgements

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Gravitational lens effects and the mass of MACHOs

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Abstract. There are four unknown parameters on the microlensing matter. By observing the light curve, two of them can be obtained. The ways to obtain the other two parameters are discussed. In order to obtain them, high resolution optical interferometric observation and the more accurate observation on the shape of the light curve will be needed.

1. Introduction

The possibility of microlensing by galactic dark halo objects was discussed by Paczynski and Griest (Paczynski, 1987; Griest 1991). Last year, some candidates for microlensing events were reported (Alcock et.al. 1993; Aubourg et.al. 1993).

If we believe these are true microlensing events, then there are four unknown parameters for each MACHO that acted as a gravitational microlens. These are the mass of the MACHO M , distance to the MACHO D , relative proper motion between background source and MACHO μ and the smallest impact parameter r_m . By use of the following quantities,

$$\theta_0 = \sqrt{\frac{4GM}{D}}, \text{ apparent angular radius of Einstein ring,} \quad (1.1)$$

$$\beta_m = r_m/D, \text{ smallest separation angle in the source plane,} \quad (1.2)$$

these unknown parameter can be expressed as θ_0, μ, β_m and D .

However, by observing the light curve of the microlensing event, we can obtain only two parameters; the largest amplification A_{max} and characteristic timescale t_e . The largest amplification A_{max} will tell us the ratio β_m/θ_0 and $t_e = 2\theta_0/\mu$. How can we obtain the other necessary information on the MACHO that contributes to the microlensing event ?

In this paper we consider the way to obtain the information on the MACHO other than A_{max} and t_e .

2. Primary image and secondary image of microlensing

The configuration of the source S, point mass P and observer O and the paths of the light ray are shown in Fig. 1. Here r is the distance between P and the line OS, r_1 is that distance between P and the light path of the primary image and r_2 is that distance between P and the path of the secondary image. Using these quantities let us introduce the following angles.

$$\beta = r/D, \quad \theta_1 = r_1/D, \quad \theta_2 = r_2/D. \quad (2.1)$$

The apparent angular radius of the Einstein ring θ_0 will give us the angular scale of this phenomenon. This can be expressed as follows.

$$\theta \sim \sqrt{\frac{M}{M_\odot} \frac{10 \text{ kpc}}{D}} \text{ milliarcsecond(mas)} \quad (2.2)$$

If we take M as large as $0.1M_\odot$ and D as 10 kpc , then θ_0 becomes about 0.3 mas . So for MACHOs we can expect θ_0 as $0.1 \sim 1 \text{ mas}$.

The fundamental formulae for microlensing are well known. For the apparent separation angles,

$$\theta_1 = \frac{\sqrt{\beta^2 + 4\theta_0^2} + \beta}{2} \quad (2.3)$$

$$\theta_2 = \frac{\sqrt{\beta^2 - 4\theta_0^2} + \beta}{2} \quad (2.4)$$

On luminosity amplification,

$$A_1 = \frac{\theta_1^4}{\theta_1^4 - \theta_0^4} \quad (2.5)$$

$$A_2 = \frac{\theta_2^4}{\theta_0^4 - \theta_2^4} \quad (2.6)$$

$$A = A_1 + A_2 = \frac{u^2 + 2}{u\sqrt{u^4 + 4}}, \text{ where } u = \frac{\beta}{\theta_0} \quad (2.7)$$

These formulae tell us the trajectory and the light curve of each image. An example is shown in Fig. 2. In this example minimum separation in source plane is taken as $0.2\theta_0$ and the relative motion between P and S is taken as being linear. Fig. 2-1 shows the trajectories of both the primary image and the secondary image and Fig. 2-2 is the light curves of both images. In this example it can be seen that in the time of largest amplification, separation between two images is larger than $2\theta_0$, luminosity of primary image is three times greater and that of secondary image becomes almost the same as the original one.

If the source has the original magnitudes ~ 20 and θ_0 is the order of a milliarcsecond, these two images will be resolved in the next generation optical interferometer observations that are now being planned. For example, the OSI project

is aiming to observe stars of magnitudes ~ 20 with a resolution of 0.1mas (Shao and Colavita 1992). Hence, if the real time monitoring of the light curve is adopted and optical interferometric observation becomes possible during the time of largest amplification, we can obtain a value of θ_0 , the third parameter of the lensing point mass, directly from the separation angle of the two images. Of course, the same information can be obtained by observing the variation of separation angles between primary image and other background sources. This is essentially the same method as we have discussed before (Hosokawa et.al.1993). But in most cases, it would be harder to obtain the information by this method than by the method discussed above. This is because the angular shift of the primary image is smaller than θ_0 and requires the detection of the difference of the separations.

3. Asymmetry of the light curve caused by earth revolution

One of the best ways to obtain the fourth parameter of the lensing matter is to detect the effects of earth revolution. In most observations the event duration t_e is about 30 days. If this event duration is typical for these phenomena, it would be too short to see the effect of the variation of the configuration caused by earth revolution in this period. However, when the relative velocities of S,P and O are considered, the Earth revolution plays a important role. In 30 days, variation of the Earth's velocity will amount to 15km/sec . In the case that the relative transverse velocity μD is on the order of a hundred km/sec, this velocity variation cannot be neglected. It can be expected that this velocity variation will make the light curve asymmetric because the access velocity might be different from the leaving velocity. Supposing that the compo.nent of the acceleration vector a along the direction of proper motion is constant during the duration of the event, then this effect can be expressed as ;

$$\mu D(\frac{t_e}{2} + \Delta t) - \frac{1}{2}a(\frac{t_e}{2} + \Delta t)^2 = \mu D(\frac{t_e}{2} - \Delta t) + \frac{1}{2}a(\frac{t_e}{2} - \Delta t)^2 \quad (3.1)$$

where Δt is the shift of the time of largest amplification from the middle of the event period. This equation will tell us the information on μD , the fourth parameter of the MACHO. When the acceleration of the Earth is parallel to proper motion, a will have the largest value, about 0.5km/sec/day . For example, let us take this value for a , 30 days for t_e and observed peak time shift Δt as 0.5 day, then about a hundred km/sec is obtained for μD . The value of μD is so sensitive to the peak time shift that very frequent and accurate observation of the brightness of microlensing would be required.

4. Conclusion

We discussed on the ways to obtain the other information on the MACHOs that cause the microlensing events than obtained from the peak value and the period of the light curve. High resolution narrow field observation by the space-based optical interferometer can be applied for these events. The microlensing events are so rare that it is worth to consider the possibility of applying various observation methods for these events.

Acknowledgements

We would like to thank Dr.Nishikawa in CRL Hiraiso for useful suggestions. This work was partly supported by Japanese Science and Technology Agency Fellowship.

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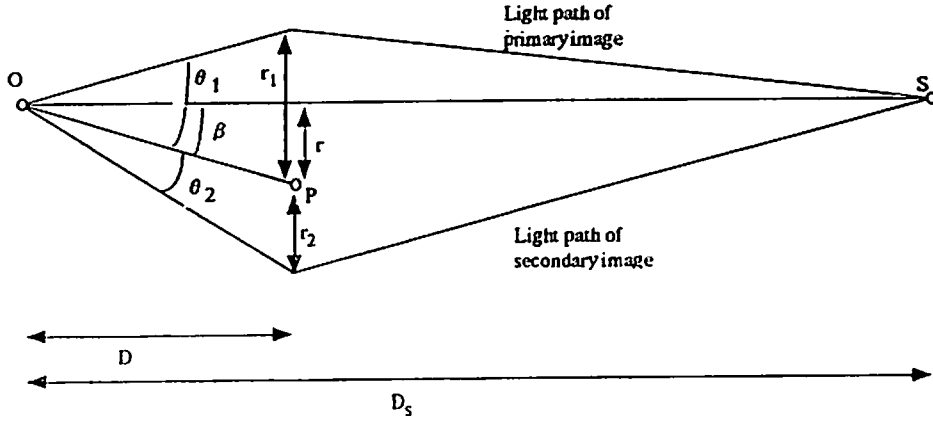


Figure 1. Configuration of source S, lensing matter P and observer O

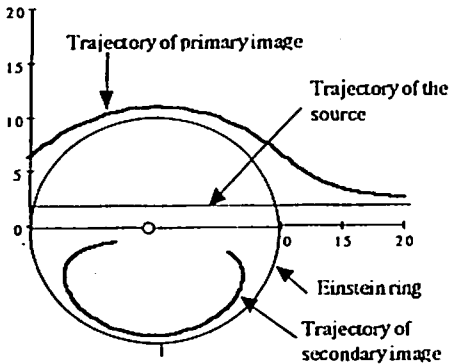


Figure 2-1. Trajectories of primary image and secondary image

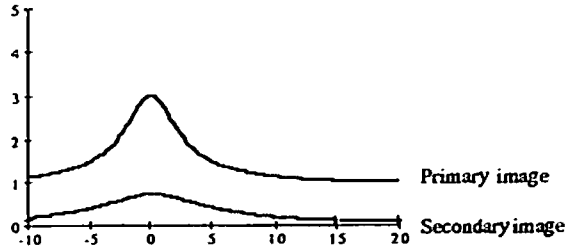


Figure 2-2. Light curves of primary image and secondary image

N - m relation based on the Press–Schechter theory

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Abstract

We estimated the N - m relation of B-band and K-band with a new luminosity function for galaxy obtained from a mass function based on Press–Schechter formalism. We found that galaxy counts for B-band favor low n (n is the power-law spectrum of the primordial density fluctuation $\langle |\delta_k|^2 \rangle \propto k^n$) and that K-band galaxy count data supports a high n . It suggests a possibility to settle the contradiction between B-band and K-band by using the fitting formula for the CDM primordial density fluctuation obtained by Bardeen *et al.*

I. Introduction

Recent technological development in observation has made it possible to observe very faint galaxies. Tyson (1988) observed very faint galaxies (~ 28 mag) in B-band and found that in the faint part ($\gtrsim 26$ mag), the number of observed galaxies is much larger than that estimated in the flat universe. How to explain the excess in the faint part theoretically became a serious problem in observational cosmology.

Several authors have investigated the problem by using many galaxy evolution models (*e.g.* Broadhurst *et al.* 1991). Among them, Fukugita *et al.* (1990) analyzed the N - m relation with $\lambda_0 \neq 0$ and some evolution models and found that, in the best fit model for the N - m relation in B-band $\Omega_0 = 0.1$ and $\lambda_0 = 0.9$. Watanabe (1991) analyzed the N - m taking into account the gravitational lens effects due to inhomogeneities in the universe. He suggested that the excess in the faint part is not attributed to the gravitational lens effect.

On the other hand, Lilly *et al.* (1991) observed very faint galaxies ($\lesssim 23$ mag) in K-band. Some authors analyzed their data by using adequate evolutionary models on luminosity functions and found that data of K-band agree with theoretical data obtained from a universe model with $\lambda_0 = 0$ (Broadhurst *et al.*, Glazebrook *et al.* 1993).

In the above analyses, the Schechter form was used as luminosity function (LF) obtained from data of nearby galaxies. For high redshift galaxies, various corrections such as luminosity evolution and merging effects of galaxies were given to the LF. Even with such corrections, it is difficult to reproduce the observations of B-band and K-band by the same LF and the same universe model. The inconsistency between B-band and K-band may originate in LF. Then we tried to construct a new LF theoretically to discuss the galaxy counts in B-band and K-band.

We first construct a mass function (MF) for galaxies by using the Press–Schechter theory. The MF is used to obtain an LF for galaxy counts with some $\langle M_*/L \rangle$ s. Next we estimate the differential galaxy counts by using our new LF (Umemura & Yoshida 1994). $H_0 = 50h_{50}[\text{km/sec/Mpc}]$, $\lambda_0 = 0$, Ω_b (density parameter for baryonic matter) = 0.05 are assumed.

II. Luminosity Function

In this section, we derive an LF for galaxy count by using the Press-Schechter theory (Press & Schechter 1974). First we obtain an MF based on the Press-Schechter theory. Next a condition is imposed on the MF to obtain MF for galaxies.

2.1 MASS FUNCTION BASED ON PRESS-SCHECchter THEORY

The Press-Schechter theory is used for the formation of the bound objects with overdensities more than some critical density δ_c which originates in the primordial density fluctuation $\langle |\delta_k|^2 \rangle$ (Press & Schechter 1974). This formalism gives the distribution function of bound objects $f(M)$ as follows:

$$f(M) = \frac{A}{M} \frac{\partial \nu}{\partial M} \exp[-\frac{1}{2}\nu^2], \quad (2.1)$$

where M is a total mass of a bound object (dark matter + baryonic matter), A is a normalization factor and

$$\nu = \frac{\delta_c}{\sigma(M)D_1(z, \Omega_0)},$$

$$\sigma^2(M) \propto \int_0^\infty d^3k W^2(kr) \langle |\delta_k|^2 \rangle,$$

where $W(y)$ is the window function, given by $3(\sin y - y \cos y)/y^3$, and $D_1(z, \Omega_0)$ is the growing factor of the density fluctuation in the linearized spherical symmetry theory. In the case of $\Omega_0 < 1$, the factor is 1 for $(1+z) < \Omega_0^{-1}$, otherwise $(\Omega_0(1+z))^{-1}$. r is the comoving radius of the bound object, related with mass M as $M = 4\pi\rho_0 r^3/3$ in the top-hat model. $\sigma(M)$ is normalized at $r = 16h_{50}^{-1}[\text{Mpc}]$, where the density fluctuation becomes non-linear, and z is the redshift of the bound object.

The redshift dependence in eq. (2.1) is due to the effect of the clustering of bound objects. At a low redshift, high mass objects increase due to the clustering effect.

2.2 MASS FUNCTION FOR GALAXY COUNT

Since eq. (2.1) is the MF which contains all the mass of bound objects, it contains large mass objects such as cluster of galaxies and also does small mass objects such as invisible dwarf galaxies. To obtain an MF for galaxy count, we impose a condition on (2.1): "for a given mass M , the bound object has a cooling time smaller than its dynamical time" (cooling criterion) i.e., $t_{\text{cool}} < t_{\text{dyn}}$ (Rees & Ostriker 1977, White & Rees 1978).

For high mass objects The condition $t_{\text{cool}} < t_{\text{dyn}}$ is equivalent to the condition that for a temperature $T(> 5 \times 10^5 \text{K})$, the density of the bound object is large enough. Scheaffer & Silk (1988) derived the cooling criterion as an analytic form,

$$\rho > 5 \times 10^{-25} \left(\frac{\Omega_0}{10\Omega_b} \right)^2 \left(\frac{T}{10^7 \text{K}} \right)^{1.3} \text{gcm}^{-3}, \quad (2.2)$$

where T is the virial temperature given by

$$T = 6.5 \times 10^5 (\Omega_0 h_{50}^2)^{1/3} (1+z) \left(\frac{M}{10^{12} M_\odot} \right)^{2/3} \text{K}.$$

The density at virial equilibrium is given by $\rho = \rho_0 \Delta(1+z)^3$ ($\Delta \sim 200$: overdensity for virialized clusters). From the above expressions, eq. (2.2) can be written as the relation between redshift and mass of bound object

IV. Conclusion

We have constructed the luminosity function applicable to the galaxy count by using the Press-Schechter theory with the cooling criterion, and estimated the galaxy count for B- and K-bands.

As mentioned above, we found that the Ω_0 -dependence of the N - m relation based on our LF is opposite to that of the N - m relation based on the Schechter LF. The reason is that, whereas the N - m relation based on the Schechter LF depends on Ω_0 only in volume element dV , our LF as well as dV depends on Ω_0 . It is due to the fact that the clustering of bound objects stops early in a low Ω_0 universe and many low mass objects remain so far.

Koo *et al.* concluded that their non-evolving LF with a low Ω_0 reproduces the observed data of galaxy counts, $B - R$ colors and redshift distribution. Our LF with a low Ω_0 does not evolve since $z \lesssim 4$ in galaxy mass scale (fig.7) because of the cooling criterion. Koo *et al.*'s suggestion supports the LF which we obtained theoretically. Although our results are still preliminary, we suggest a possibility that our LF in the $n = 1$ CDM model with a low Ω_0 settles the contradiction between B- and K-band galaxy counts.

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Figure Captions

- Fig.1 Luminosity functions given by (2.1) at some redshifts: $z = 0$ (solid line), $z = 2$ (dotted line), $z = 4$ (dashed line), and $z = 6$ (dot-dashed line). The histogram is a fitting form obtained by Koo *et al.*
 Fig.2 Luminosity functions obtained by using the cooling criterion at some redshifts. Lines are the same as in Fig.1.
 Fig.3 Differential galaxy counts for B-band with some density parameter: $\Omega_0 = 0.2$ (solid line), 0.5 (dotted line), 0.7 (dashed line), 1.0 (dot-dashed line).
 Fig.4 Differential galaxy counts for K-band. Lines are the same as in Fig.3.
 Fig.5 Power-law spectrum of the primordial density fluctuation in the CDM model.
 Fig.6 Luminosity functions given by (2.1) at some redshifts in the CDM model. Lines are the same as in Fig.1.
 Fig.7 Luminosity functions obtained by using the cooling criterion at some redshifts in the CDM model. Lines are the same as in Fig.1.
 Fig.8 Differential galaxy counts (B-band) in the CDM model. Lines are the same as in Fig.3.
 Fig.9 Differential galaxy counts (K-band) in the CDM model. Lines are the same as in Fig.3.

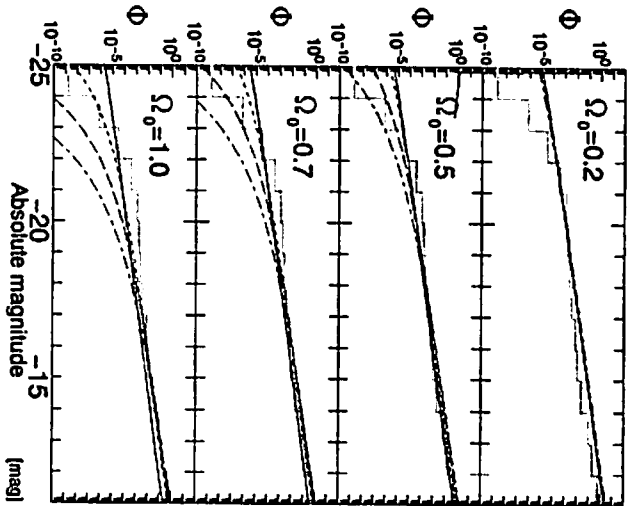


Fig.6 Luminosity function in the $r=1$ CDM

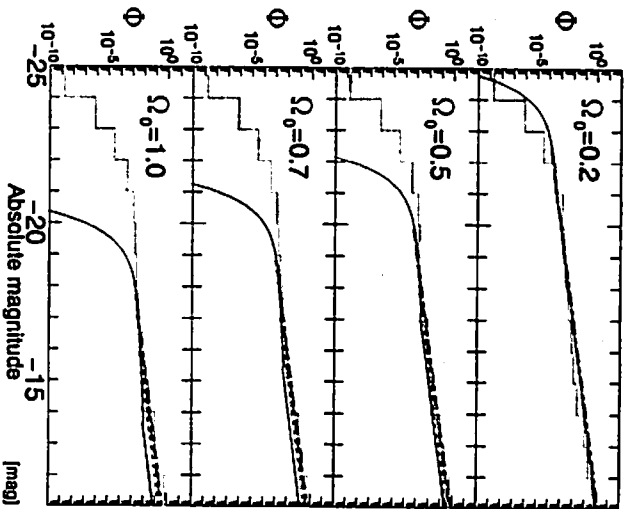


Fig.7 Luminosity function in the $r=1$ CDM

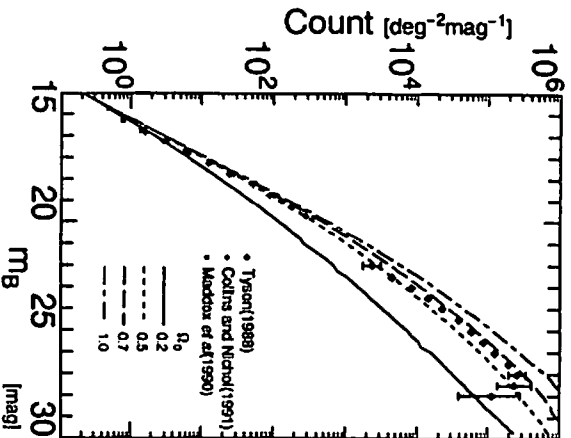


Fig.8 Differential Galaxy Count (B-band)

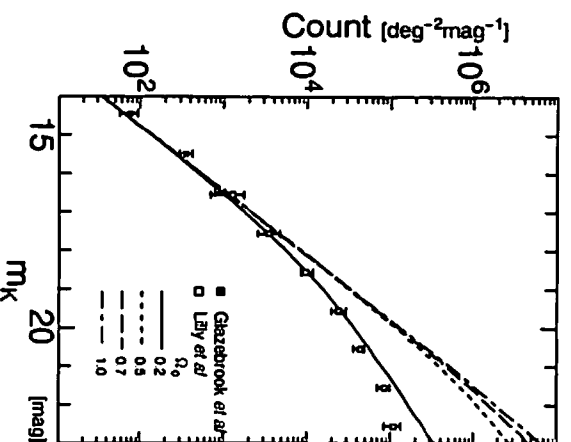


Fig.9 Differential Galaxy Count (K-band)

Steradian Telescope

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1 Introduction

We propose to build *steradian telescope*, which takes a picture of a sky of one steradian in every 0.1 second using 144 wide-field cameras. The limiting magnitude for a point source is $\sim 15^m$ for a 0.1 sec exposure and $\sim 20^m$ for a 1000 sec integration. We need two units of *steradian telescope* separated by 10 $\sim$ 100 km distances to distinguish astronomical events from background events due to cosmic rays, meteors, and artificial satellites.

Major targets of *steradian telescope* are a time-correlated search for the optical counterparts of gamma-ray bursts, microlensing events due to MACHOs (Massive Astronomical Halo Objects), a long term monitor of active galactic nuclei, variable stars, and search for new asteroids and comets.

2 Instruments and data processing

A unit of the *steradian telescope* is composed of 144 cameras with a diameter of 20 cm. The

field of view of each camera is 5 degree $\times$ 5 degree ($\sim 7.6 \times 10^{-3}$ steradian). Total field of view of 144 cameras is about one steradian. A mosaic CCD array with four CCDs (2048 $\times$ 2048 pixels) is installed in the focal plane of each camera. The angular resolution which corresponds to one pixel of CCD is 4.4 arcsec. We set the exposure time to be 0.1 second so that the motions of stars due to the rotation of the Earth are as small as 1.5 arcseconds, which is 3 times smaller than the spatial resolution of *steradian telescope*. This short exposure time allows us to use a fixed mount for installing cameras. A longer exposure image is available by the digital integration. The background events due to cosmic rays, meteors, and artificial satellite can be distinguished from astronomical flashes using trigonometric parallaxes obtained by two units of *steradian telescope* separated by a distance of $\sim$ 100 km.

The obtained image is shifted to compensate the rotation of the Earth and compared with the reference image. When a significant change is found the positions and the intensities of the events are recorded. The event data are compared with those obtained by another unit. If a similar event is found at the same

position, we recognize it as an astronomical event. The obtained image is integrated over a certain integration time, for example, one hour. The limiting magnitude is about 15^m for a 0.1 second exposure and 20^m for a 1000 seconds integration.

The total data rate from 576 CCDs is as high as 40 GB/s, which no available computer systems can receive. That would be a reason why a system such a *steradian telescope* was not feasible. However, the data rate from one CCD is about 40 MB/s, which can be handled with a custom LSI, specially designed for the data processing of CCD images. Since the data processing of the data from a CCD can be done almost independently of those from another CCD, we can perform them by a massively parallel architecture.

3 Astronomical Targets

3.1 Optical flash associated with Gamma-ray Bursts

Gamma-ray burst is a phenomena that gamma-ray from a corner of the sky explosively increases. Although 20 years has past from its discovery, there are no consensus on what is gamma-ray burst. The optical identification, which expected to give us a lot of information on gamma-ray bursts, is eagerly waited for a long time. However, optical flash associated with gamma-ray burst has not been obtained yet[6]. The current upper limit is about 4.7^m for GRB910184. This observation set a constraint on the ratio of energy in optical wavelength (E_{opt}) to that in gamma-ray (E_γ) is less than 0.04.

Steradian telescope can offer an extensive search of the optical flash associated with gamma-ray bursts. It can set a constraint to E_γ/E_{opt} to be less than 10^{-7} , if no optical counter part cannot be found. Such a small E_{opt}/E_γ kills almost all models of gamma-ray bursts that are proposed so far.

According to the Burst and Transient Source Experiment (BATSE) on board Gamma-Ray Observatory (GRO), 800 gamma-ray bursts with a fluence E_γ larger than $10^{-10} J m^{-2}$ take place the full sky per

year. We can expect the frequency, Γ_{GRB} , of BATSE-steradian simultaneous events as

$$\Gamma_{GRB} = 7 \left(\frac{\Omega_{opt}}{str.} \right) \left(\frac{\Omega_{BATSE}}{6str.} \right) \left(\frac{f_{opt}}{0.3} \right) yr^{-1}, \quad (1)$$

where Ω_{opt} and Ω_{BATSE} are the solid angles of field of view of *steradian telescope* and BATSE, respectively, and f_{opt} is the efficiency of *steradian telescope*.

3.2 MACHO (Massive Astronomical Compact Halo Objects)

Paczynski[10] pointed that massive astronomical Compact Halo Objects (MACHO) in our galaxy can be detected by the brightening of stars due to gravitational microlensing of a foreground MACHO, which is transitting near the line of sight. If the dark matter of our galaxy is composed of MACHOs, the probability that a star suffers microlensing of a MACHO is about 10^{-6} . Three group monitored $\sim 10^6$ stars in Large Magellanic Cloud and the bulge of our galaxy and reported several candidate events[1, 2, 14].

Figure 1 shows the number of microlensing event expected in a solid angle of one steradian as a function of a galactic longitude for a fixed galactic latitude. Here, we use the standard model of our galaxy [4] and take into account of interstellar absorption. The expected number of event is well above unity for the region within 30 degrees from galactic center.

3.3 QSO

One of the most distinct natures of QSO is the time variability. A long term monitor of QSO reveals that 70% of QSOs showed a significant variability[7, 8]. In particular, blazars (BL Lac Objects and Optically Violently variable quasars) shows a violent eruption with the time scale shorter than one day (figure 2)[13].

A long-term monitor of a large sample of QSOs with *steradian telescope* will reveals the

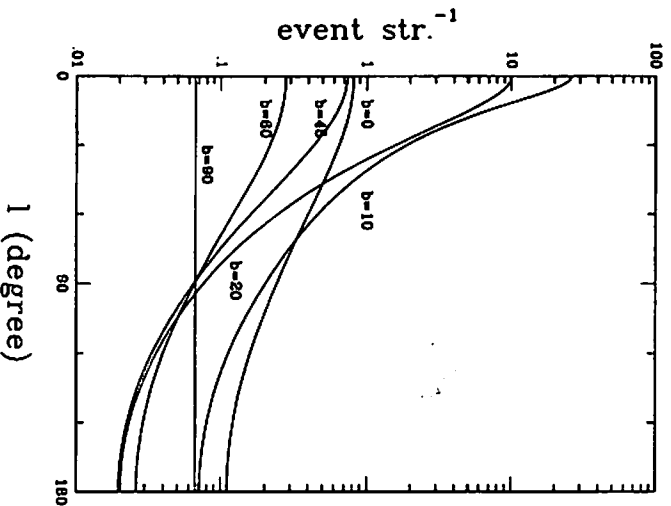


Figure 1: The expected number of microlensing event in a solid angle of one steradian are plotted against galactic longitude for a fixed latitude.

emission mechanism and the nature of central engine of QSOs. Previous surveys of QSO with a field of view of several square degree suggest that the number density of QSO brighter than 20^m is about 30 per square degree. *Steradian telescope* with a field of view of about one steradian can monitor $10^4 \sim 10^5$ QSOs simultaneously.

3.4 Supernovae

Berkeley Automated Supernova Search (hereafter BASS) uses a computer controlled 30-inch telescope and takes images of $400 \sim 600$ galaxies per clear night[11]. Their limiting magnitude for the detection of supernovae is $16 \sim 17^m$. This program found 20 supernovae from 1986 to 1990. Their almost bias-free search allows them to conclude that the

explosion frequency of type Ib SNe is several times higher than expected[9]. They also conducted an international campaign to observe SN1990E, which has a peculiar light curve and successfully obtained hour to hour spectra before its maximum[12].

Although Berkeley Automated Supernova Search is quite successful, it has two limitations. First, their limiting magnitude is as bright as $16^m \sim 17^m$ because of their short exposure time for one galaxy (less than 1 min.). The limiting magnitude of *steradian telescope* is as faint as 20^m , since we integrate images over 1000 seconds using data obtained by different cameras. It can detect type Ia SNe ($M_V = -19$) as far as 630 Mpc, which corresponds to $z=0.13$ ($H_0 = 75 \text{ km s}^{-1} \text{ Mpc}^{-1}$ is assumed). Second, they do not observe intergalactic space. According to the numerical simulations of clusters of galaxies[5], a large fraction of stars is stripped out from their parent galaxies and ejected to intracuster space. They may fail to found supernovae take place in such ejected stars.

3.5 Variable Stars

Steradian telescope can offer a bias-free survey of variable stars. We discuss pulsating stars, eclipsing binaries, and flare stars in this section.

Many types of pulsating stars such as Cepheids, RR Lyre stars, and long period variables are good distance indicators. We can make a catalogue of variable stars which includes ten thousands Cepheids, a hundred thousands RR Lyre Stars and long period variables with one year operation of *steradian telescope*. This database with a large sample of distance indicators, give us three dimensional distribution of stars in our galaxies. *Steradian telescope* also contribute to the search for rapid oscillators such as rapidly oscillating Ap stars, or δ Sct stars.

Steradian telescope which monitors ten million stars simultaneously, can perform bias-free survey of long period eclipsing binaries. In conventional telescopes, it is quite difficult to find eclipsing binaries of main sequence stars with a separation of $\sim 1AU$ since the ratio of the duration of eclipse to the orbital period is as small as 10^{-6} .

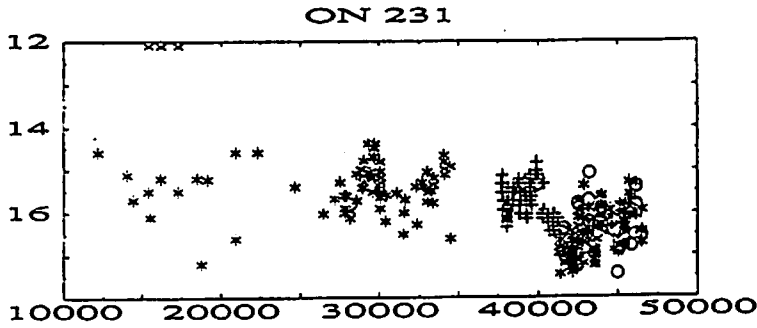


Figure 2: The time variability of QSO ON231[3].

A young ($10^8 \sim 10^9$ years old) low mass stars ($0.1 \sim 0.5 M_{\odot}$) has a strong magnetic activities. Its rapid rotation and a deep convection zone produces a strong magnetic field through dynamo process. These magnetically active stars shows intense stellar flares, which are $10^5 \sim 10^8$ times more energetic than those in the sun. A rough estimate suggests that the number of these magnetically active stars reaches a quarter of total number of stars in our galaxy.

3.6 Asteroids and Comets

The limiting magnitude of *steradian telescope* is $17^m \sim 18^m$ for asteroids with the proper motion of several arc minutes per hour. We can detect the asteroids in the asteroid belt with the absolute magnitude (corrected to $d_{\text{sun}} = 1\text{AU}$ and $d_{\text{Earth}} = 1\text{AU}$) of $15^m \sim 16^m$, which corresponds to the radius of several km. *Steradian telescope*, in particular, is effective to find near Earth asteroids, which is difficult to find by a conventional telescope with a narrow field of view (several arc min square) because of their rapid motion.

4 Discussions

The site of *steradian telescope* is required to have a high rate of clear night, since its major targets are the all-sky monitor of astronomical objects. A small amount of scintillation

and a dark sky are other important factors, since they determine the sensitivity for transient events and accuracy of photometry of *steradian telescope*. Another important factor is the easy access to the conventional telescopes to perform follow up observations of astronomical objects discovered by *steradian telescope*.

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Mass in de Sitter Space-Time

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About ten years ago, Abbott and Deser presented a reasonable definition of the conserved Killing “energy” in the space-time with a cosmological constant Λ , i.e., in the de Sitter and anti-de Sitter space-times (more strictly speaking, the asymptotically de Sitter and anti-de Sitter space-times). This is the so-called Abbott-Deser (AD) mass and in the limit of $\Lambda \rightarrow 0$, it agrees with the ADM mass, which is the conserved energy of the asymptotically flat space-time.

The positivity of the AD mass as well as the Arnowitt-Deser-Misner (ADM) mass is one of the most important issues of general relativity in connection with the stability of the maximally symmetric space-times, i.e., of the Minkowski, de Sitter and anti-de Sitter space-times. Hence, in the case of the space-time with $\Lambda > 0$, the positivity of the AD mass seems to be closely related with the *cosmic no hair conjecture*, which states that all space-times with a positive cosmological constant approach the de Sitter space-time asymptotically. By Shoen and Yau, the positivity of the ADM mass was proven and later Witten presented a simpler proof by spinor representation. While the positivity of the AD mass in the case of $\Lambda < 0$ was discussed by Gibbons et al. and that in the asymptotically de Sitter space-time ($\Lambda > 0$) has been recently investigated and proven by one of the present authors under some ansatz corresponding intuitively to “zero-momentum” condition.

However, under more general situation, the AD mass can be negative without any violation of the local energy condition. Then, in order to see the effect of the negativity of the AD mass, we shall consider the Tolman-Bondi space-time with a positive cosmological constant Λ , which describes the motion of an inhomogeneous dust sphere in the asymptotically de Sitter space-time.

First we shall give the expression of the AD mass M_{AD} . The original expression of the AD mass M_{AD} presented by Abbott and Deser is rather complicated. Hence we rewrite the definition of M_{AD} by the use of the geometrical quantities of the 3-space. The geometrical quantities of the 3-space mean the intrinsic metric γ_{ab} of the 3-space and the extrinsic curvature K_a^b which describes how the 3-space is embedded in the 4-dimensional space-time. As already mentioned, the AD mass is defined in the asymptotically de Sitter space-time which permits the one parameter family of 3-space with

$$\begin{aligned}\gamma_{ab} &\longrightarrow a^2(t)\delta_{ab}, \\ K_a^b &\longrightarrow -H\delta_a^b = -\frac{1}{a}\frac{da}{dt}\delta_a^b,\end{aligned}\tag{1}$$

at the spatial infinity, where $a(t)$ is the scale factor of de Sitter space-time $\exp(Ht)$ with $H \equiv \sqrt{\Lambda/3}$ and with the time coordinate t .

The expression of the AD mass is then given by

$$M_{AD} = a(t)(M + \Delta M),\tag{2}$$

where

$$M = \frac{1}{16\pi} \oint d\bar{S}_a (\partial_b \gamma^{ab} - \gamma^{ab} \gamma^{cd} \partial_b \gamma_{cd}),\tag{3}$$

and

$$\Delta M = \frac{H}{8\pi a^3(t)} \oint d\bar{S}_a \gamma^{1/2} x^b (K_b^a - K \delta_b^a - 2H \delta_b^a),\tag{4}$$

where γ is the determinant of the 3-metric γ_{ab} , $x^a = (x, y, z)$ in the Cartesian coordinates and D_a is the covariant derivative within the 3-space. The surface integral of Eqs.(3) and (4) is performed over a 2-surface with radius r in the limit of $r \rightarrow \infty$.

Now in order to see the effect of the sign of the AD mass on the evolution of matter, we consider the Tolman-Bondi space-time with Λ . The metric of this

space-time can be written as following form by using the co-moving coordinates,

$$ds^2 = -dt^2 + X^2(r, t)dr^2 + Y^2(r, t)(d\theta^2 + \sin^2 \theta d\varphi^2). \quad (5)$$

Then from the Einstein equations, we obtain,

$$X = \frac{Y'}{W(r)}, \quad (6)$$

$$\dot{Y}^2 = -V(Y; r) \equiv W^2 - 1 + \frac{2m(r)}{Y} + H^2 Y^2, \quad (7)$$

where $\dot{}$ and $'$ are the derivative with respect to t and r , respectively. $W(r)$ is an arbitrary function which corresponds to the binding energy. $m(r)$ is also an arbitrary function which is related with the energy density $\rho(r, t)$ of the dust sphere by

$$m(r) = 4\pi \int_0^r \rho W(r) X Y^2 dr. \quad (8)$$

Hence $m(r)$ is regarded as the effective mass within $[0, r]$. Eq.(7) is regarded as the energy equation of a zero-energy particle in the potential $V(Y; r)$ in which the radial coordinate r is merely a parameter labeling a dust fluid element. Hence we can see the motion of a dust fluid element at r by investigating the potential $V(Y; r)$.

Here we assume for the density distribution as

$$\rho = \begin{cases} > 0, & 0 \leq r \leq r_s, \\ = 0, & r > r_s. \end{cases} \quad (9)$$

Assuming the 3-space is everywhere regular at $t = t_0$, the behavior of X , Y and W near the origin is given by the regularity

$$\begin{aligned} X(r, t_0) &\longrightarrow a(t_0)[1 + \mathcal{O}(r^2)], \\ Y(r, t_0) &\longrightarrow a(t_0)r[1 + \mathcal{O}(r^2)], \quad \text{for } r \longrightarrow 0 \\ W(r) &\longrightarrow 1 + \mathcal{O}(r). \end{aligned} \quad (10)$$

On the other hand, in order that the space-time is asymptotically de Sitter, the

behavior of X , Y and W for $r \rightarrow \infty$ should be,

$$\begin{aligned} X(r, t_0) &\longrightarrow a(t_0), \\ Y(r, t_0) &\longrightarrow a(t_0)r, \quad \text{for } r \longrightarrow \infty \\ W(r) &\longrightarrow 1. \end{aligned} \quad (11)$$

Here we shall calculate the AD mass of this system. By Eq.(11), for $r \rightarrow \infty$, $m = \text{constant}$ and Eq.(7) becomes as

$$\dot{Y}^2 = \frac{2m}{Y} + H^2 Y^2. \quad (12)$$

This equation is easily integrated and the solution consistent with Eq.(11) is obtained as

$$\begin{aligned} X &= a(t) \left(1 - \frac{m}{2H^2 a^3(t)r^3}\right)^{-1/3} \left(1 + \frac{m}{2H^2 a^3(t)r^3}\right), \\ Y &= a(t)r \left(1 - \frac{m}{2H^2 a^3(t)r^3}\right)^{2/3}. \end{aligned} \quad (13)$$

The extrinsic curvature K_a^b is then obtained as

$$\begin{aligned} K_r^r &= -H \left(1 - \frac{2m}{H^2 a^3(t)r^3} + \frac{m^2}{4H^4 a^6(t)r^6}\right) \left(1 - \frac{m^2}{4H^4 a^6(t)r^6}\right)^{-1}, \\ K_\theta^\theta &= K_\varphi^\varphi = -H \left(1 + \frac{m}{2H^2 a^3(t)r^3}\right) \left(1 - \frac{m}{2H^2 a^3(t)r^3}\right)^{-1}, \end{aligned} \quad (14)$$

and other components vanish. Using Eqs.(13) and (14), we can calculate the AD mass of this system by Eqs.(2), (3) and (4). From Eq.(13), we see that $h_{ab} \rightarrow \mathcal{O}(r^{-3})$ for $r \rightarrow \infty$ in the Cartesian coordinates and hence $M = 0$ by Eq.(2). On the other hand, ΔM becomes as

$$\Delta M = \frac{H}{8\pi} \lim_{r \rightarrow \infty} \int (K_r^r - K - 2H) a^2(t) r^3 \sin \theta d\theta d\varphi = \frac{m}{a(t)}. \quad (15)$$

Then, from Eq.(1), we obtain

$$M_{AD} = m(r \rightarrow \infty) = m(r \geq r_s). \quad (16)$$

Here it is worthy to note that the region $r \geq r_s$ of this space-time neces-

sarily agrees with the Schwarzschild-de Sitter space-time by Birkhoff's theorem. Assuming $W = 1$, we perform the following coordinate transformation;

$$\begin{aligned} dt &= dT + \frac{\dot{Y} dY}{1 - \dot{Y}^2}, \\ dr &= \frac{1}{Y'} \left(\dot{Y} dT + \frac{dY}{1 - \dot{Y}^2} \right). \end{aligned} \quad (17)$$

Then, using Eq.(6), we obtain the familiar static form for the metric of the Schwarzschild-de Sitter space-time as

$$\begin{aligned} ds^2 &= - (1 - \dot{Y}^2) dT^2 + \frac{dY^2}{1 - \dot{Y}^2} + Y^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \\ &= - \left(1 - \frac{2M_{AD}}{Y} - H^2 Y^2 \right) dT^2 + \left(1 - \frac{2M_{AD}}{Y} - H^2 Y^2 \right)^{-1} dY^2 \\ &\quad + Y^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \end{aligned} \quad (18)$$

where we have used Eqs.(12) and (16) in the second equality. Now it is obvious from the above equation that the mass parameter of the Schwarzschild-de Sitter space-time is just the AD mass.

Since the local motion of the fluid element depends only on the effective mass m , the qualitative feature of the dust motion is essentially the same as that of a homogeneous dust ball, which has been investigated by one of the present author. However, $m < 0$ does never happen in the case of homogeneous dust sphere. Therefore in this paper we shall focus only on the case of $m < 0$ and investigate its effect on the motion of the dust fluid element.

First, we show in what situation m and M_{AD} become negative. In order that m is negative, W should be negative from Eq.(4.8) in some regions since ρ , X and Y are all non-negative quantities. Hence, for example, if we choose W and ρ as in the Fig.1, $m(r)$ can become negative at some point within $(r_-, r_+]$, where $W = 0$ at $r = r_{\pm}$. Here it should be noted that only the combination of Y and m (or ρ) which makes $V(Y; r)$ non-positive is allowed (see Eq.(7)) and, by this

constraint, we get an important relation. Assuming $m(r_+) \leq 0$, because of $V \leq 0$ at $r = r_+$, we obtain

$$|m(r_+)| \leq \frac{1}{2} Y(r_+, t_0) [H^2 Y^2(r_+, t_0) - 1]. \quad (19)$$

It is worthy to note that when $H = 0$, the above equation cannot be satisfied. This supports the fact that the gravitational mass cannot be negative in the asymptotically flat space-time. From Eq.(19), we can see that $Y(r_+, t_0) \geq H^{-1}$, where the equality can be satisfied only if $m = 0$. Since, $W < 0$, i.e., $Y' < 0$ in the region $(r_-, r_+]$, we obtain

$$Y > H^{-1} \quad \text{for the region of } m(r) < 0. \quad (20)$$

From the above equation, it seems that when the AD mass is negative, the extent of the matter and radiation causing AD mass beyond over the cosmological horizon scale H^{-1} also in more general case. Since the “timelike” Killing vector of the background de Sitter space-time is no longer timelike outside the cosmological horizon $a(t)r > H$, the configuration with $M_{AD} < 0$ seems to be non-stationary. Conversely the AD mass of the stationary space-time seems to be necessarily non-negative.

We can easily see that, in the case of $m < 0$, there is only one positive root $Y = Y_B$ for $V = 0$ and

$$V = \begin{cases} > 0 & Y < Y_B, \\ \leq 0 & Y \geq Y_B, \end{cases} \quad (21)$$

Hence the fluid element with $m < 0$ cannot reach the singularity $Y = 0$. Instead, it goes to $Y = \infty$ and the infinitesimal region occupied by this fluid element approaches to the de Sitter space-time ($\rho \rightarrow 0$) locally. The conformal diagram of this space-time is depicted in Fig.2 in which the central region of the dust sphere with $m > 0$ is omitted. As already mentioned, in this spherically symmetric model, the vacuum region ($\rho = 0$) agrees with the Schwarzschild-de Sitter space-time by Birkhoff’s theorem and therefore in the case of $M_{AD} = m(r \rightarrow \infty) < 0$

the naked singularity exists as can be seen from Fig.2 (right vertical line, $Y = 0$). However, even if such a naked singularity exists, $t = \text{constant}$ hypersurface which is denoted by dashed lines in Fig.2 does not intersect with this singularity. In other words, there is a regular Cauchy surface whose time development is also regular in this space-time. Hence one might regard this naked singularity as a kind of the initial singularity in the Friedmann universe model, which does not cause a serious problem in the evolution of the universe. Further, it is worthy to note that ADM mass in the asymptotically flat space-time is non-negative under the situation as the above in which there is a regular Cauchy surface and its time development is also regular.

From the analysis of the inhomogeneous dust sphere with $M_{AD} < 0$, the dust fluid in the region suffered by the negative effective mass m necessarily expands to $Y \rightarrow \infty$ and such a region approaches to the de Sitter space-time asymptotically in local sense. Hence, although the AD mass can be negative, cosmic no hair conjecture holds.

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FIGURE CAPTIONS

1. The schematic diagram of representative configuration with negative AD mass. A peak of the density exists within the region $(r_-, r_+]$ in which $W < 0$. $r = r_-$ is the maximal surface while $r = r_+$ corresponds to the minimal surface.
2. The conformal diagram of the Tolman-Bondi space-time with $M_{AD} < 0$. The central region of $m > 0$ is omitted. The dashed lines correspond to $t = \text{constant}$ hypersurfaces which do not intersect the singularity $Y = 0$.

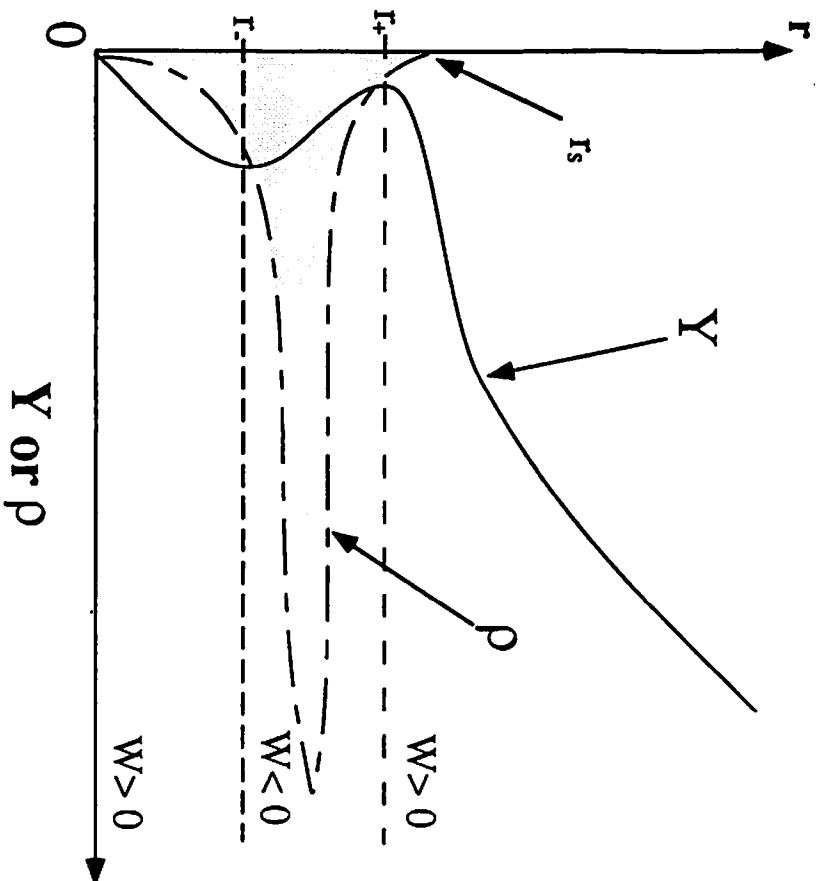


Fig. 1

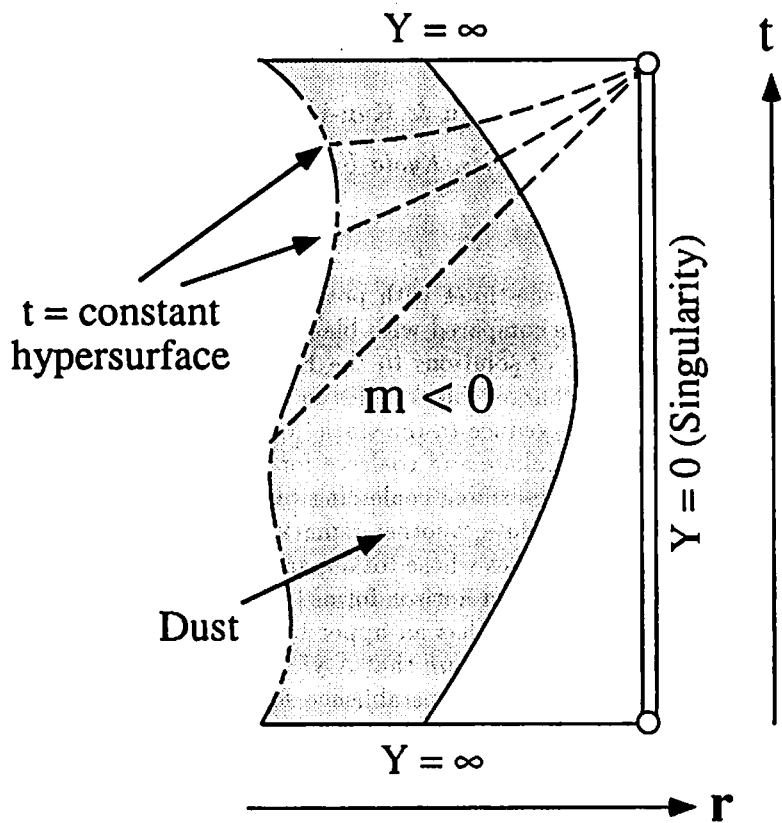


Fig.2

A Cosmological Constant Limits the Size of Black Holes

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I. Introduction

Recent studies of black holes in space-times with positive cosmological constant Λ have revealed some surprising differences compared with black holes in the zero Λ case. Kastor and Traschen have found a family of solutions to the Einstein-Maxwell equations which describe an arbitrary number of charged black holes in an otherwise closed cosmos [1]. For the zero Λ case, these solutions reduce to the static Majumdar-Papapetrou solutions, but for positive Λ , the black holes collide and coalesce for a certain range of parameters, providing the first exact solutions describing coalescing black holes. In a previous study of uncharged multi-black-hole initial data, another remarkable result was discovered, and proposed as a general feature: black holes whose total mass is larger than a certain critical value do not coalesce, *i.e.* no new apparent horizon forms [2]. Also, the area of the apparent horizon that forms in the subcritical case has an upper bound, approached as the critical case is approached. This led to the conjecture that Λ yields an upper bound on the area of a black hole, with the bound $12\pi/\Lambda$ being established for maximal slices [3]. A similar study has since been made of the Kastor-Traschen solutions, with the same conclusions: there is a critical mass beyond which the black holes do not coalesce, and when they do coalesce, there is an upper bound on the area of the new apparent horizon [4]. In this article, we show that there is a sharp bound $4\pi/\Lambda$ on the area of a black or white hole.

We emphasize that by ‘black or white hole’ we mean any region of trapped surfaces that has a boundary of the outer type, according to the classification in [5]. The boundary, referred to as the *trapping horizon* [5], is a generalization of the apparent horizon as usually defined [6], and henceforth we will consistently speak of trapping horizons rather than apparent horizons. It transpires that the area bound also applies to event horizons, but the event horizon is defined only in asymptotically flat space-times, or for positive Λ , in asymptotically deSitter space-times [3], a rather special case.

As a preliminary, we examine exact solutions and other examples in Section II. The area bound is established for outer trapping horizons in Section III, and for event horizons in Section IV. In the Conclusion, we discuss the conjectured isoperimetric inequality and implications for the cosmic censorship conjecture.

II. Examples

Carter [7] showed that the ‘Kerr-Newman’ black-hole solutions could be generalized to include a cosmological constant Λ . In particular, there is the analogue of the Reissner-

Nordström solution, parametrized by mass M , charge Q and Λ , with line-element

$$ds^2 = -F dT^2 + F^{-1} dR^2 + R^2 dS^2 \quad F(R) = \frac{Q^2}{R^2} - \frac{2M}{R} + 1 - \frac{1}{3}\Lambda R^2 \quad (1)$$

where dS^2 refers to the unit 2-sphere. The global structure of these solutions is described for the $Q = 0$ case in [8] and [2], and in general in [9]. For a certain range of (M, Q, Λ) and choice of topology, the solutions describe a charged black hole of the Wheeler wormhole type, existing in an asymptotically deSitter cosmos. The trapping horizons are given by the zeros of $F(R)$, and have area $4\pi R^2$. Consider two special cases, as follows.

- (i) The 'Schwarzschild-deSitter' case, $Q = 0$. Fixing Λ , the trapping horizons occur at $M = \frac{1}{2}R - \frac{1}{6}\Lambda R^3$. For $M < \sqrt{1/9\Lambda}$ there are two horizons, namely a black-hole horizon and a cosmological horizon, which coincide in the degenerate $M = \sqrt{1/9\Lambda}$ case. Note that the area of the black-hole horizon does not exceed $4\pi/\Lambda$, and attains $4\pi/\Lambda$ in the degenerate case.
- (ii) $|Q| = M$. The trapping horizons occur at $M = R \pm \sqrt{\Lambda/3}R^2$. For $M < \sqrt{3/16\Lambda}$ there are three horizons, namely inner and outer black-hole horizons and a cosmological horizon, with the last two coinciding in the degenerate $M = \sqrt{3/16\Lambda}$ case. The area of the outer black-hole horizon does not exceed $3\pi/\Lambda$, attained in the degenerate case. Note that adding charge has reduced the maximal area.

Checking the full range of (M, Q, Λ) confirms that the area of the outer black-hole horizon does not exceed $4\pi/\Lambda$ for any of these solutions. That the same bound should apply to any black hole is suggested on thermodynamic grounds. The entropy of a black hole is essentially its area, and one would expect entropy to be maximized in the most symmetric configuration, namely the stationary, spherically symmetric 'Schwarzschild-deSitter' solution. A similar argument motivates the conjectured isoperimetric inequality in asymptotically flat space-times [10,11].

Another test of the area bound is provided by multi-black-hole solutions where the black holes collide and coalesce. We use 'coalescence' to mean the appearance of a new region of trapped surfaces outside the original trapping horizons of the incoming black holes. Thus a double trapping horizon forms, with inner and outer horizons both enclosing the original trapping horizons, as depicted in Figure 60 of [6]. If such a coalescence occurs, one may ask whether the new outer trapping horizon satisfies the area bound. There are two known examples for $\Lambda > 0$.

- (iii) The initial data for the vacuum Einstein equations found by Nakao *et al* [2], which develop into a multi-black-hole cosmos. For the case of two equal masses, it was found that there is a critical total mass $\sqrt{1/9\Lambda}$ such that coalescence occurs in the subcritical but not supercritical case. In Figure 13 of [2], it can be seen that the area of the new horizon is bounded above by a value which appears to be $4\pi/\Lambda$. So the collision seems to miraculously preserve the area bound.
- (iv) The Kastor-Traschen multi-black-hole solutions, which are exact Einstein-Maxwell solutions describing charged $|Q| = M$ black holes [1]. Again, for the case of two equal masses, it has been found that there is a critical total mass $\sqrt{3/16\Lambda}$ such that coalescence occurs in the subcritical but not supercritical case [4]. For these solutions,

the area of the new trapping horizon is bounded above by $3\pi/\Lambda$. As for the single-mass case (ii), the lower maximum may be attributed to the charge on the holes. This is a very curious phenomenon. It seems that nature is conspiring to prevent the formation of too large a black hole. Attempting to create a large black hole by colliding smaller ones either produces a black hole satisfying the bound or does not work at all—the black holes refuse to coalesce.

III. Trapping horizons

We adopt a recent approach to trapping horizons, generalizing apparent horizons, which has elucidated some of their fundamental properties [5]. One of the main outcomes of this analysis is that it is crucial to distinguish between inner horizons and outer horizons, since they have quite different properties—for instance, outer horizons are generically spatial while inner horizons are generically Lorentzian. We shall see that another such difference is the area bound, which applies to outer but not inner horizons.

We recall the basic definitions and notation of [5]. Consider a 2-surface S with spatial metric h , and denote the area form, Ricci scalar and covariant derivative by μ , $\mathcal{R}$ and $\mathcal{D}$ respectively. The area of a compact 2-surface is

$$A = \int_S \mu. \quad (2)$$

The future-pointing null normals $n_{\pm}$ of S may be assumed closed, $\nabla_{[a} n_{b]} = 0$, corresponding to the existence of a double-null foliation with normals $n_{\pm}$. The normalization is denoted by $n_a^+ n_-^a = -e^f$. Denoting the Lie derivatives along $n_{\pm}$ by $\mathcal{L}_{\pm}$, the expansions $\theta_{\pm}$, shears $\sigma_{\pm}$, inaffinities $\nu_{\pm}$ and anholonomicity (or twist) ω are defined by

$$\theta_{\pm} = \frac{1}{2} e^{-f} h^{cd} \mathcal{L}_{\pm} h_{cd} \quad (3a)$$

$$\sigma_{ab}^{\pm} = e^{-f} (h_a^c h_b^d \mathcal{L}_{\pm} h_{cd} - \frac{1}{2} h_{ab} h^{cd} \mathcal{L}_{\pm} h_{cd}) \quad (3b)$$

$$\nu_{\pm} = e^{-f} \mathcal{L}_{\pm} f \quad (3c)$$

$$\omega_a = \frac{1}{2} e^{-f} h_{ab} \mathcal{L}_+ n_-^b. \quad (3d)$$

The normalization may be fixed by $f = 0$ on S . The (n_+, n_+) and (n_+, n_-) components of the Einstein equation, evaluated on S , are

$$\mathcal{L}_+ \theta_+ + \nu_+ \theta_+ + \frac{1}{2} \theta_+^2 + \frac{1}{4} \sigma_{ab}^+ \sigma_+^{ab} = -8\pi \phi_+ \quad (4a)$$

$$\mathcal{L}_- \theta_+ + \theta_+ \theta_- + \frac{1}{2} \mathcal{R} - \omega_a \omega^a + \mathcal{D}_a \omega^a = 8\pi \rho + \Lambda \quad (4b)$$

and are referred to as the focussing and cross-focussing equations respectively. The dominant energy condition implies

$$\phi_+ \geq 0 \quad (5a)$$

$$\rho \geq 0. \quad (5b)$$

Note that all the results of [5] also apply to the case $\Lambda > 0$, since the Λ -term in the Einstein equation, regarded as a material source, satisfies the dominant energy condition.

An *outer* (respectively *inner*) *marginal surface* is a spatial 2-surface on which, fixing an orientation for $n_{\pm}$, $\theta_+ = 0$ and $\mathcal{L}_-\theta_+ < 0$ (respectively $\mathcal{L}_-\theta_+ > 0$). The surface gravity κ of an outer marginal surface is defined by

$$4\kappa^2 = -\mathcal{L}_-\theta_+. \quad (6)$$

A *trapping horizon* is defined as a connected part of the boundary of an inextendible trapped region, namely an inextendible region for which each point lies on a trapped surface [5]. A trapping horizon foliable by marginal surfaces is referred to as a foliable trapping horizon, or simply a horizon if the context is clear. Such a horizon must be either outer or inner, apart from degenerate cases [5]. For an outer horizon, the idea is that the outgoing light rays are diverging just outside the horizon, and converging just inside. Marginal surfaces which lie in trapping horizons must be orientable [5].

An example is provided by the Carter black-hole cosmos of Section II, for which the event horizon is of the outer type, with the Cauchy horizon and cosmological horizon being of the inner type. In general, the outer horizon may be taken as the definition of the outer boundary of a black or white hole. Note that one cannot use the event horizon as a general definition of the boundary of a black hole, since it is defined only if conformal infinity exists, i.e. in asymptotically flat or deSitter space-times. Roughly speaking, a black hole has an event horizon only if it settles down to an asymptotically stable state.

Theorem 1. In a space-time with cosmological constant $\Lambda > 0$ and matter satisfying the dominant energy condition, an orientable outer marginal surface has finite area A bounded above by $A < 4\pi/\Lambda$.

Proof. On an outer marginal surface S , the cross-focussing equation (4b) reduces to

$$-4\kappa^2 + \frac{1}{2}\mathcal{R} - \omega_a\omega^a + \mathcal{D}_a\omega^a = 8\pi\rho + \Lambda. \quad (7)$$

Since $\kappa^2 > 0$, the dominant energy condition implies $0 < \Lambda < \frac{1}{2}\mathcal{R} + \mathcal{D}_a\omega^a$. Now $\int_S \mu \mathcal{D}_a\omega^a = 0$ by the Gauss theorem and $\int_S \mu \mathcal{R} \leq 4\pi\chi$ by the Cohn-Vossen inequality, where χ is the Euler-Poincaré characteristic of S , with $\chi = 2$ for a sphere, $\chi = 1$ for a plane, and $\chi \leq 0$ for any other orientable 2-manifold. Integrating the inequality over S ,

$$0 < A\Lambda < 2\pi\chi \quad (8)$$

so that the area A is finite, whence S is compact. Since $\chi > 0$, S has spherical topology. Also $A\Lambda < 4\pi$.

In particular, this means that black and white holes lie within the Hubble radius, which corresponds to an area $12\pi/\Lambda$. Actually, there is a more precise result in terms of the

irreducible mass m , quasi-local angular momentum j , quasi-local ‘charge’ q and root-mean-square surface gravity k , defined by

$$16\pi m^2 = A \quad (9a)$$

$$4\pi j^2 = m^4 \int_S \mu \omega_a \omega^a \quad (9b)$$

$$8\pi q^2 = A \int_S \mu \rho \quad (9c)$$

$$Ak^2 = \int_S \mu \kappa^2. \quad (9d)$$

Theorem 2. In a space-time with cosmological constant $\Lambda > 0$ and matter satisfying the dominant energy condition, the irreducible mass m , angular momentum j , ‘charge’ q and r.m.s. surface gravity k of an orientable outer marginal surface satisfy

$$\frac{A\Lambda}{4\pi} = 1 - \frac{j^2}{m^4} - \frac{q^2}{m^2} - 16m^2 k^2. \quad (10)$$

Proof. Integrate the cross-focussing equation (4b) using the Gauss-Bonnet theorem for S with spherical topology, $\int_S \mu \mathcal{R} = 8\pi$.

The formula shows explicitly how the area of a black hole is reduced by angular momentum, charge and surface gravity. If any one of these quantities is fixed, there is a lower effective bound on the area.

So far, we have restricted the marginal surfaces to be non-degenerate, $\kappa^2 > 0$, in which case the bound is not attained. In fact, the definition of a trapping horizon as the boundary of an inextendible trapped region leads only to $\kappa^2 \geq 0$ on an outer horizon [5]. The additional cases, namely where κ^2 vanishes somewhere, are referred to as *degenerate* marginal surfaces.

Theorem 3. In a space-time with cosmological constant $\Lambda > 0$ and matter satisfying the dominant energy condition, a degenerate outer horizon has area $4\pi/\Lambda$ if and only if it is identical to that of the degenerate ‘Schwarzschild-deSitter’ solution.

Proof. On a degenerate outer marginal surface S , the cross-focussing (4b) equation integrates to

$$\frac{A\Lambda}{4\pi} \leq \frac{1}{2}\chi - \frac{j^2}{m^4} - \frac{q^2}{m^2} - 16m^2 k^2 \quad (11)$$

using the Cohn-Vossen inequality again. Now $A\Lambda = 4\pi$ requires $\chi \geq 2$, so that S has spherical topology again, $\chi = 2$. Also j , q and k must vanish on S . Hence ω , ρ and κ also vanish on S , which in turn means that $\mathcal{R} = 2\Lambda$, so that S is metrically spherical. A trapping horizon T foliated by such constant-area spheres must be null, by the area theorem for trapping horizons [5]. Constant area also implies that the internal expansion θ_+ vanishes on T , and the focussing equation (4a) shows that σ_+ and ϕ_+ also vanish on

T . This has now exhausted the free data on a null 3-surface [12], and the data is identical to that of the horizon of the degenerate 'Schwarzschild-deSitter' solution, i.e. T can be embedded in this solution. Note that a uniqueness result for the whole space-time would require additional initial data to be specified off the horizon.

Finally, we note that it is crucial for Theorems 1–3 that the horizon or marginal surface be outer rather than inner. For inner marginal surfaces, the surface-gravity term in the cross-focussing equation has the opposite sign, and if large enough can swamp the angular momentum and charge terms to yield marginal surfaces of arbitrarily large area and genus. As an example, the cosmological horizon of the 'Schwarzschild-deSitter' solution has area between $4\pi/\Lambda$ and $12\pi/\Lambda$. It has been conjectured that in an asymptotically deSitter space-time, the cosmological horizon, defined as the boundary of the past domain of dependence of $\mathfrak{S}^+$, should have area less than or equal to $12\pi/\Lambda$, the deSitter value [13]. Although this conjecture is still open, it does not generalize to inner trapping horizons in arbitrary space-times, since an inner marginal surface can have arbitrarily large area. As an example, the innermost horizon of the 'Schwarzschild-deSitter' solution has arbitrarily large area in the case where there are naked singularities instead of black holes.

IV. Event horizons

The classical work on black holes is mainly concerned with the idealization of asymptotic flatness [6]. For $\Lambda > 0$, there is a similar class of asymptotically deSitter space-times. Conformal infinity $\mathfrak{S}^\pm$ is spatial rather than null, but many aspects of the space-times can be treated by methods which are well known in the asymptotically flat case. In particular, one may define the event horizon as the boundary of the causal past of $\mathfrak{S}^+$. Since the area of the event horizon is non-decreasing [3], and the event horizon is expected to approach the trapping horizon(s) asymptotically, it seems that the area bound should also apply to the event horizon. A preliminary step is to establish that connected sections of the event horizon approach outer marginal surfaces as $\mathfrak{S}^+$ is approached.

Lemma. Consider the event horizon H of a strongly future asymptotically predictable, asymptotically flat or asymptotically deSitter space-time. Foliate H by spatial 2-surfaces, with future-pointing null normals $n_\pm$, n_+ tangent to the event horizon, and ξ an affine parameter along n_+ . Then

$$\lim_{\xi \rightarrow \infty} \theta_+(H) = 0 \quad (12a)$$

$$\lim_{\xi \rightarrow \infty} \mathcal{L}_{n_+} \theta_+(H) \leq 0 \quad (12b)$$

assuming the weak energy condition or null convergence condition [6].

Proof. First recall that $\theta_+(H) \geq 0$ [3,6]. The focussing equation (4a) yields the inequality $\theta_+ \leq (1/\theta_0 + (\xi - \xi_0)/2)^{-1}$, from which it follows that $\lim_{\xi \rightarrow \infty} \theta_+(H) = 0$. Consider a connected component S of $\overline{H} \cap \mathfrak{S}^+$, and a neighbouring foliation of $\mathfrak{S}^+$ containing another 2-surface $S' \subset \mathfrak{S}^+$. Propagating $n_\pm$ from S to S' , construct the null 3-surface H' through S' in the n_+ direction, in some neighbourhood U of $\mathfrak{S}^+$. Suppose there is a $p \in H' \cap U$ such that $\theta_+(p) < 0$. Then there is a neighbourhood V of p in $H' \cap U$ such that $\theta_+(V) < 0$,

and a compact set $C \subset S'' \cap V$, where S'' is a spatial 2-surface of the foliation. This leads to a contradiction, since it is impossible that $C \subset \text{int}[J^-(\mathfrak{S}^+)]$, by the same argument as in the proof of the area theorem for event horizons [3,6]. Thus $\theta_+(H' \cap U) \geq 0$, and so $\lim_{\xi \rightarrow \infty} \mathcal{L}_- \theta_+(H) \leq 0$.

Theorem 4. In a strongly future asymptotically predictable, asymptotically deSitter space-time with matter satisfying the dominant energy condition, the area A of a connected section of the event horizon is bounded above by $A \leq 4\pi/\Lambda$, with equality if and only if the event horizon is identical to that of the degenerate 'Schwarzschild-deSitter' solution.

Proof. Integrating the cross-focussing equation (4b) over connected sections S_ξ of the event horizon and taking the limit, it follows from (12) and the Cohn-Vossen inequality that

$$\lim_{\xi \rightarrow \infty} A_\xi \Lambda \leq \lim_{\xi \rightarrow \infty} 2\pi\chi_\xi \quad (13)$$

whence $\lim_{\xi \rightarrow \infty} A_\xi$ is finite and so $\lim_{\xi \rightarrow \infty} S_\xi$ is compact. Since the area of the event horizon is non-decreasing [3], $A_\xi \leq \lim_{\xi \rightarrow \infty} A_\xi$, it follows that $\lim_{\xi \rightarrow \infty} A_\xi > 0$, and so $\lim_{\xi \rightarrow \infty} \chi_\xi > 0$. Since the event horizon is orientable, $\lim_{\xi \rightarrow \infty} \chi_\xi = 2$. In particular,

$$A_\xi \leq \lim_{\xi \rightarrow \infty} A_\xi \leq 4\pi/\Lambda. \quad (14)$$

Attaining the bound on the whole event horizon H requires constant area, so $\theta_+(H) = 0$. This means that the event horizon is also a trapping horizon, and Theorem 3 then shows that the bound is attained if and only if H is identical to the horizon of the degenerate 'Schwarzschild-deSitter' solution.

Corollary. Asymptotically, the event horizon consists of sections of spherical topology.

Note that the bound applies to each connected section of the event horizon. The total area of all the connected sections at a given time may exceed the bound. In such a case, there is a topological restriction on the development of the event horizon, namely that contemporaneous sections whose total area exceeds $4\pi/\Lambda$ cannot subsequently combine into a single connected section. In other words, such black holes cannot merge in this sense. This is exactly what is observed in the examples of Kastor and Traschen and Nakao *et al* (Section II).

V. Conclusion and discussion

In asymptotically flat space-times, it is well known that the event horizon cannot divide and has non-decreasing area [6]. This is also true in an asymptotically deSitter space-time [3], but we have shown that there are additional restrictions: the area of each connected section of the event horizon cannot exceed $4\pi/\Lambda$, and so disconnected components cannot merge if their total area exceeds this value.

This has an interesting consequence for the cosmic censorship conjecture. Consider initial data representing two or more black holes, defined in the sense of outer trapping horizons, such that the total area exceeds $4\pi/\Lambda$. Suppose further that these black holes

subsequently collide. Since the resulting space-time cannot contain an event horizon, one might conjecture that the outcome of such a collision would be a naked singularity [14]. This would be a blatant violation of the cosmic censorship conjecture in either weak [15] or strong [16] form. What actually happens for the examples of Kastor and Traschen and Nakao *et al* is rather remarkable [2,4]. In the so-called subcritical case, the black holes coalesce, producing a new asymptotically stable horizon whose area can be arbitrarily close to the bound. If the black holes collided to form an asymptotically stable object in the supercritical case as well, that object would have to be a naked singularity, assuming some sort of 'no-hair' theorem to the effect that a stable solution must be one of the Carter solutions. Instead, it turns out that the black holes refuse to collide in the supercritical case, and simply keep their distance. Physically, one could ascribe this phenomenon to the repulsive effect of the cosmological constant. This lends considerable moral support to the cosmic censorship conjecture in the following sense: the delicate balance between gravitational attraction and cosmic repulsion tips exactly when the black holes can no longer coalesce, and a collision would be in danger of producing a naked singularity.

The inequality $A \leq 4\pi/\Lambda$ bears a striking resemblance to the isoperimetric inequality

$$A \leq 16\pi M_0^2 \quad (15)$$

which is conjectured to relate the ADM mass M_0 and the area A of the outermost trapping horizon (or apparent horizon) T of an asymptotically flat space-time, assuming matter satisfying the dominant energy condition [10,11]. Equality is conjectured to be attained only for the Schwarzschild solution. The conjecture may be strengthened to the following chain of inequalities:

$$A \leq \lim_{T \rightarrow i^+} A \leq \lim_{\mathfrak{S} \rightarrow i^+} 16\pi M^2 \leq 16\pi M^2 \leq \lim_{\mathfrak{S} \rightarrow i^0} 16\pi M^2 \leq 16\pi M_0^2 \quad (16)$$

where M denotes the Bondi-Sachs mass. The first inequality follows from the area theorem for trapping horizons [5], and the third and fourth from the Bondi-Sachs mass-loss result. The two missing links are plausible, since M and M_0 are both limits of the same quasi-local energy E [17], and $16\pi E^2/A \rightarrow 1$ as $T \rightarrow i^+$. If this chain of inequalities is correct, and there exists an asymptotically flat initial data set on which $A > 16\pi M_0^2$, this would also be a blatant violation of the cosmic censorship conjecture. Again, one may try to construct such a counterexample by trying to create a large black hole from the collisions of smaller ones. In numerical examples, the isoperimetric inequality is preserved, providing further moral support for the cosmic censorship conjecture [11].

Although formulated for the $\Lambda = 0$ case, the isoperimetric inequality is analogous to the area bound in that it gives a limit on the size of black holes, regarding the ADM mass as fixed. It also gives a necessary condition for a surface to be marginal. This raises the question of whether a necessary and sufficient condition for the existence of a trapped surface can be given in terms of initial data. For a light-cone, this is indeed possible [18]. For a spatial 3-surface, there is no such general result, nor even a precise conjecture. At the heuristic level, there is the hoop conjecture [19]. The light-cone result is independent of Λ , but it is not clear how Λ should affect the hoop conjecture.

Both asymptotically flat and asymptotically deSitter space-times are rather special, and since the event horizon is defined only in such cases, the general study of black holes must instead be concerned with the trapping horizon. The fundamental properties of trapping horizons have only recently been established [5]. In particular, the area of a future outer horizon is non-decreasing and, as we have shown, cannot exceed $4\pi/\Lambda$. Given that a black-hole collision involves the formation of a new outer horizon, one would also like to know if the area of the new horizon is necessarily greater than the sum of the areas of the original horizons. If this is true, this yields tests of the cosmic censorship conjecture which can be formulated in arbitrary space-times. Research in this direction is in progress.

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Hoop Conjecture and Cosmological Constant

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1. Introduction

One of the most important unresolved problems in general relativity is the cosmic censorship hypothesis (CCH)^[1] to which many efforts have been devoted over years.

It is crucial to understand the aspherical collapse for the resolution of CCH. Unfortunately, at present we have little knowledge about the aspherical collapse, but there is a conjecture for it; the hoop conjecture (HC)^[2], which states that horizons form when and only when a mass M gets compacted into a region whose circumference in *every* direction is $C \lesssim 4\pi M$. Although HC was proposed two decades ago, it has only recently been studied and at present there is no considerable counterexample to HC.^[3]

If HC is true, then we would have the following picture for the final state of spacetime after the gravitational collapse; the complete gravitational collapse of a mass results in a black hole if the mass satisfies HC, while results in a naked singularity if the mass does not satisfy HC. This picture was already adopted in practice in numerical simulation of a prolate dust collapse^[4]. Therefore, the verification of HC would have great importance to CCH. Again, investigations so far strongly suggest that HC is true.

However, HC in the presence of a cosmological constant Λ is not known and even not formulated. Since there is a suggestion that CCH may be violated in the presence of a positive Λ and that there exists the maximum mass of a black hole

in the presence of a positive Λ ^[5], it is of great importance to investigate how a cosmological constant should affect HC.

In this paper, we investigate HC in the presence of a positive cosmological constant. To do this, we solve the initial value problem with a cosmological constant for a class of axisymmetric prolate and oblate spheroids containing matter following Nakamura, Shapiro, and Teukolsky (NST)^[3]. We then search for an apparent horizon to know the relations of the matter circumference $\mathcal{C}$ to a cosmological constant when an apparent horizon forms.

From these calculations, we found that when an apparent horizon forms $\mathcal{C}$ satisfies

$$\mathcal{C} \leq 4\pi M \leq 4\pi M_{crit} \quad (01)$$

where $M_{crit} = 1/\sqrt{9\Lambda}$. Here M is the Abbott-Deser (AD) mass^[6] which is equal to the ADM mass when there is no momentum density. We also found that with a positive Λ it often occurs that an apparent horizon runs into the inside of a matter, which may suggest that a ‘seminude singularity’ (according to the terminology of NST) can easily emerge. Pertabative analysis of dynamics suggests that as in the case of HC without Λ , if the circumference of a complete collapsed body is $\mathcal{C} > 4\pi M$ a naked singularity may appear.

2. Initial Data of Prolate/Oblate Dust Collapse

We consider conformally flat initial data and then the line element is written as

$$d\ell^2 = \psi^4(dR^2 + dz^2 + R^2 d\phi^2). \quad (02)$$

We adopt the constant mean curvature slice:

$$K^a_b = H\delta^a_b \quad (03)$$

where H is regarded as a Hubble parameter. Then the only Einstein equation

that the initial data must satisfy is the Hamiltonian constraint

$$-8\psi^{-5}\Delta\psi = 16\pi\rho, \quad (04)$$

where $\rho = T_{ab}n^an^b$ with n^a being the unit normal vector to the spacelike hypersurface.

We set the density profile to

$$2\pi\psi^5\rho \equiv 2\pi\rho_N \quad (05)$$

where ρ_N is defined as

$$\rho_N = \begin{cases} M/(4\pi a^2 c/3) & \text{for } R^2/a^2 + z^2/c^2 \leq 1 \\ 0 & \text{otherwise.} \end{cases} \quad (06)$$

Since at large radii $\psi \rightarrow 1 + M/2r$, M is actually the gravitational mass of a spheroid which is called the Abbott-Deser (AD) mass being equal to the ADM mass when there is no momentum density.

Given ψ , we search for an apparent horizon. In the three-metric Eq.(02) and the slicing Eq.(03), an apparent horizon is a surface $r(\theta)$ (where $r(\theta)$ is defined by $R = r(\theta) \sin \theta$, $z = r(\theta) \cos \theta$) satisfying

$$r_{,\theta\theta} + \left(\frac{r_{,\theta}^3}{r^2} + r_{,\theta}\right)\left(\frac{4\psi_{,\theta}}{\psi} + \cot \theta\right) - r_{,\theta}^2\left(\frac{3}{r} + \frac{4\psi_{,r}}{\psi}\right) - 2r - r^2\frac{4\psi_{,r}}{\psi} - \frac{2H\psi^2(r^2 + r_{,\theta}^2)^{(3/2)}}{r} = 0. \quad (07)$$

with the boundary condition

$$r_{,\theta} = 0 \quad \text{at} \quad \theta = 0, \frac{\pi}{2}. \quad (08)$$

Since we have ψ analytically, Eq.(07) is an ordinary differential equation which we can solve by the 4-th order Runge-Kutta method.

In order to assess the hoop conjecture, we must calculate the minimum circumference C outside the matter. Previously we proposed a definition of C by using closed geodesics.^[3] In axisymmetric space, C is defined by

$$C = \max(C_{eq}^{min}, C_{pol}^{min}). \quad (09)$$

Here C_{eq}^{min} and C_{pol}^{min} are the minimum equatorial circumference and minimum polar circumference, respectively which are evaluated by closed geodesics.

3. Cosmic Hoop Conjecture

We have done large survey of parameter space and searched for apparent horizons. Typical results are shown in Fig.1. As is shown in the figure, the presence of a positive Λ makes the location of an apparent horizon shrink inward in coordinate space, and sometimes the matter goes outside the apparent horizon.

We have searched for the extremal H above which no apparent horizon forms for each semimajor axis c . The results are displayed in Fig.2. It shows that the maximum value of the matter circumference C linearly depends on H , and no apparent horizon forms when $H < H_{crit} = M^{-1}/\sqrt{27}$. At $H \simeq H_{crit}$, maximum $C/4\pi M$ is nearly equal to one.

From these results we can derive a semiempirical condition for the formation of an apparent horizon; *when an apparent horizon forms, the circumference of matter C must satisfy*

$$C \leq 4\pi M \leq 4\pi M_{crit} \quad (010)$$

where $M_{crit} = 1/\sqrt{9\Lambda}$.

4. Naked Singularity and Cosmological Constant

In the presence of a cosmological constant absence of a horizon does not necessarily imply a naked singularity because there exists the background cosmic

expansion and so it is not clear that the complete gravitational collapse proceeds. A mass may get dragged up with the cosmic expansion and may expand rather than collapse.

In order to determine whether the spheroid collapses or expands, we calculate the dynamical behavior of the mass density for the time $t \ll M$. As a coordinate condition, we take the lapse function $N = 1$ and the shift vector $N^a = 0$. Furthermore we consider dust fluid. Then the evolution of density is determined by

$$\frac{\partial(\sqrt{h}\rho)}{\partial t} = 0, \quad (011)$$

$$\frac{\partial K_{ab}}{\partial t} = -{}^3R_{ab} - 3H^2 h_{ab}, \quad (012)$$

and

$$\frac{\partial h_{ab}}{\partial t} = 2K_{ab}. \quad (013)$$

Here h is the determinant of the three metric. We expand K_{ab}, h_{ab} in a power series of $t \ll M$ as^[7]

$$K_{ab} = K_{ab}^{(0)} + K_{ab}^{(1)}t + K_{ab}^{(2)}t^2 + \dots, \quad (014)$$

and

$$h_{ab} = h_{ab}^{(0)} + h_{ab}^{(1)}t + h_{ab}^{(2)}t^2 + \dots \quad (015)$$

Of course, to zeroth order K_{ab}, h_{ab} are

$$\begin{aligned} K_{ab}^{(0)} &= H\psi^4\delta_{ab}, \\ h_{ab}^{(0)} &= \psi^4\delta_{ab}. \end{aligned} \quad (016)$$

We evaluate K_{ab}, h_{ab} order by order. Then, $\rho(t)$ is evaluated as

$$\begin{aligned} \rho(t) &= \sqrt{h(0)/h(t)}\rho(0) \\ &= \rho(0)\left(1 - 3Ht + \frac{15H^2t^2}{2} - 4\psi^{-5}\psi_{,a}^{\cdot a}t^2 + O(t^3/M^3)\right). \end{aligned} \quad (017)$$

Fig.3 shows $\rho(t)/\rho(0)$ at $z = c, R = 0$ up to $t = 0.01M$ for each e and H . From

these, it is seen that when the eccentricity approaches to unity and $H \lesssim H_{crit}$ then the collapse proceeds with or without Λ , while when the eccentricity is small and Λ is nonzero the spheroid expands. Thus it is reasonable to conclude that if the collapse proceeds, the singularity forms with or without Λ .

Details will be published elsewhere.^[8]

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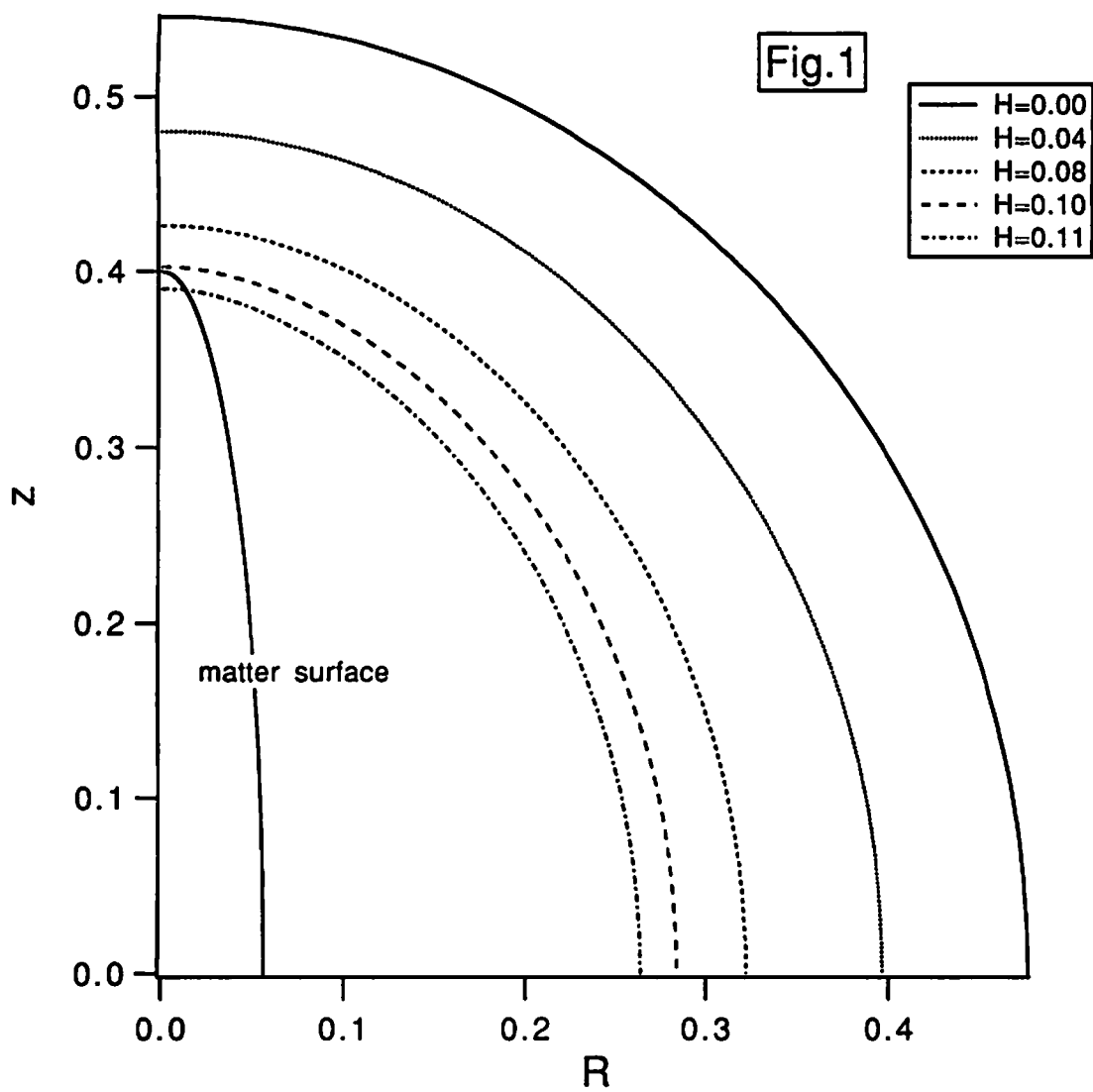
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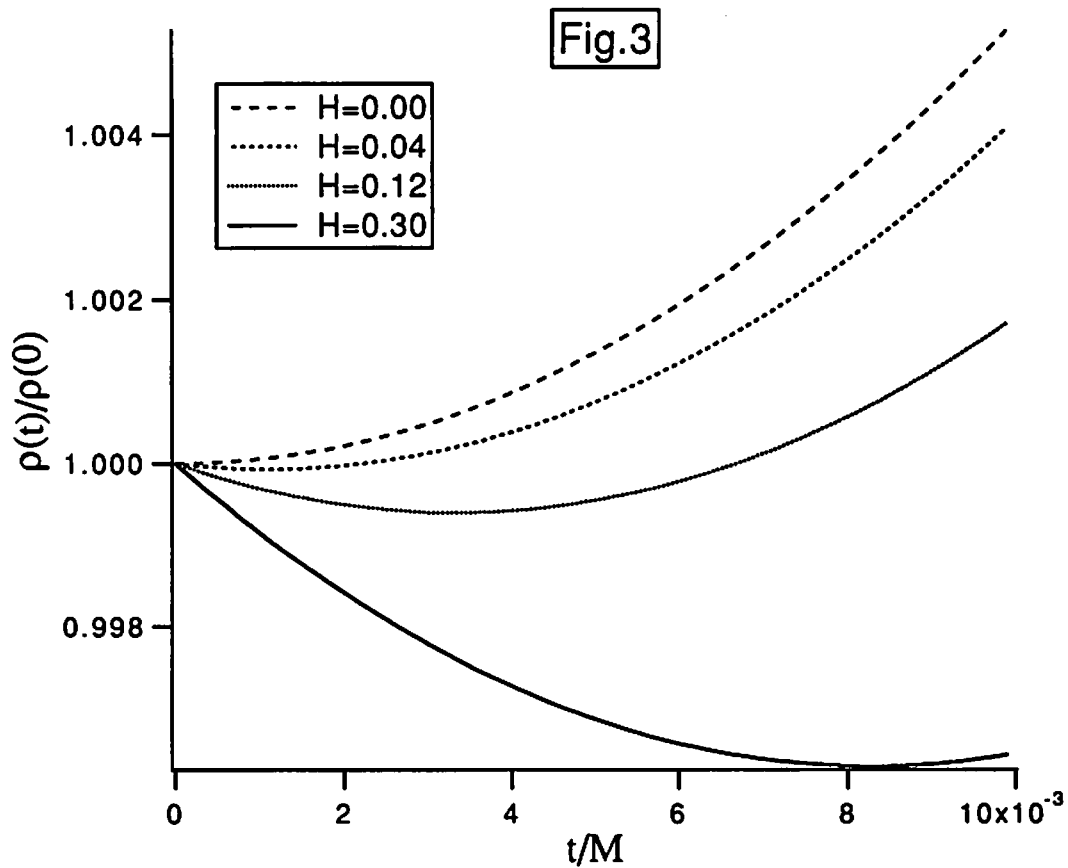
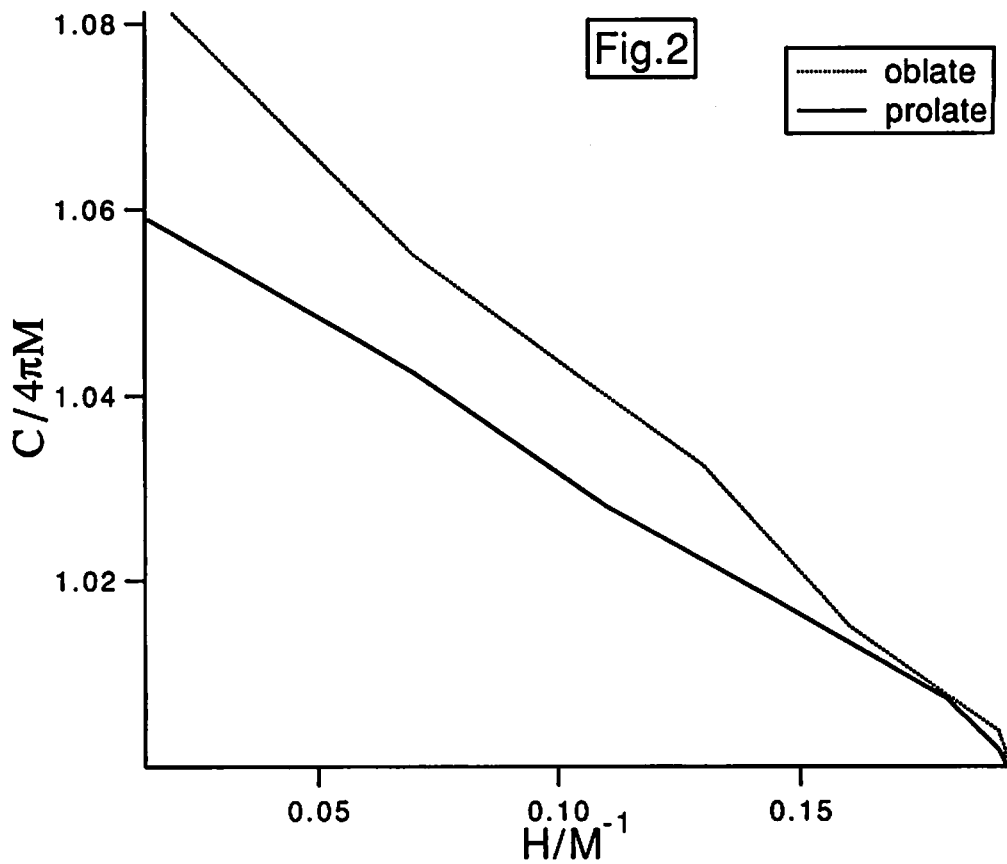
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FIGURE CAPTIONS

1. Representative shapes and location of apparent horizons for $c/M = 0.4$ and each H .
2. Maximum value of C for an apparent horizon to form for each H .
3. $\rho(t)/\rho(0)$ at $z = c = 0.4M, R = 0$ up to $t = 0.01M$ for $e = 0.99$ for each H .





Dynamics of A Scalar Field along A Flat Direction in de Sitter Space

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Abstract

We compute the one-loop effective action of a scalar field along a flat direction in de Sitter space, inflationary universe, using the derivative expansion method. Then we examine the dynamics of the scalar field. It is found that this scalar field rolls down the potential away from the origin if the value of a scalar field ϕ is much larger than Hubble constant. We also see that this dynamics will support Affleck-Dine mechanism.

It is known that a scalar potential of a global supersymmetric model often has a special direction called the flat direction [1], along which the value of the scalar potential remains unchanged. The potential energy of a supersymmetric model is given by a sum of $F_i^* F_i$ and $D^a D^a$ [2], where F_i and D^a are auxiliary fields in the scalar multiplet and the vector multiplet respectively. Therefore the flat direction is characterized by condition $F_i(\phi) = 0$ and $D^a(\phi) = 0$ under the equations of motion of the auxiliary fields. If these

conditions in terms of the scalar fields ϕ have non-trivial solutions, there exists a flat direction. The example in supersymmetric SU(5) grand unified theory (GUT) is easily found [1]. The condition of the flat direction imply the supersymmetry is not spontaneously broken [2], while the presence of the vacuum expectation value along the direction in general breaks the gauge and global symmetries. It is also possible to show that, in the supersymmetric models, the flat direction does not receive any radiative correction as the result of the non-renormalization theorem [2]. The unbroken supersymmetry and the absence of the radiative correction are the interesting characteristics of the flat direction. With a supersymmetry breaking mass m , on the other hand, the degenerated potential energy along the flat direction and the direction receives a logarithmic radiative correction proportional to the breaking mass squared m^2 .

In this paper we evaluate the one-loop level effective action along the flat direction in the de Sitter space* using the derivative expansion method. Our original interest is in the effective potential along the flat direction in the inflationary expanding universe since the analysis is relevant for the baryogenesis scenario by the Affleck-Dine mechanism [1]. In the original scenario [1], it is supposed that scalar fields along the flat direction initially have a large expectation value of the order of the GUT scale. Then after the inflation and when the Hubble parameter becomes of the order of breaking mass scale m (that is of order of the weak scale m_w), the scalar fields begin to fall down the flat direction. The motion of scalar fields (say the squark fields) generates a condensation of the baryon number density and decays into light particles.

In flat space, as mentioned above, the flat direction receives rather

*More precisely we will evaluate the effective action on Euclidean section of the de Sitter space, namely on the four-sphere S^4 [4].

small radiative corrections. However it is not clear if this is held even in curved spaces. In particular the typical mass (or curvature) scale within the inflation, namely the Hubble parameter H , may be of order of the GUT scale or the intermediate scale $\sim \sqrt{m_w m_{Pl}}$. It may then be natural to expect the flat direction receives a correction of the order of $g^2 H^2 \phi^2$ where g is typical coupling constant. If this is the case, the scalar fields along the flat direction may be dumped away within the inflation and cannot have a sufficient initial value needed for a successful Affleck-Dine scenario.

To analyze the above problem, however, we have to specify how the global supersymmetric model is coupled to gravity. Though the supergravity [2] might be a natural framework, this theory is not renormalizable. Therefore for avoiding this difficulty, we will consider general *renormalizable* models that reduce to global supersymmetric models (with soft breaking masses) in the flat space limit.. Also we will not include the radiative corrections due to the graviton.

Note that the renormalizability in curved space allows a curvature coupling term $\xi R \phi^2$ for the flat direction ϕ if it does not contradict with the gauge and the global symmetries. Since the scalar curvature R is given by $R = 12H^2$ in de Sitter space, if ξ is of the order of unity, the direction is not flat even at the tree level. In the case, the Affleck-Dine mechanism does not work.

We get the result that the one-loop level radiative correction to the potential induces a term

$$V_{eff}(\phi) \sim -\frac{3g^2}{16\pi^2} H^2 \phi^2 \ln \frac{\phi^2}{M^2} \quad (1)$$

for a large value of ϕ [3]. If the (renormalized) curvature coupling is nearly zero, $\xi \simeq 0$, and the soft breaking mass is sufficiently small as $m^2 < g^2 H^2 / \pi^2$, this correction may qualitatively modify the tree level potential and may even

give an *unbounded* potential. On our Wess-Zumino model [3] case we present global shape of the effective potential in Figures. We will argue that this asymptotic behavior is irrespective of the matter contents and the details of the model if the flat direction couples to a scalar multiplet [3]. A realistic supersymmetric model should do satisfy this requirement because a scalar field along the flat direction, say the squark field, must have a Yukawa coupling in the superpotential. So, in this paper we refer to flat direction at tree level as flat direction. Before we conclude scalar field ϕ along the flat direction keeps rolling down for a large value of ϕ , we should take into account the derivative term of the scalar field along flat direction to understand the dynamics of this field. We will calculate the quantum correction to kinetic term assuming the slow rolling of the scalar field along flat direction. But we show this correction is small compare to the tree level contribution contrary to the case of potential [5], so we conclude the scalar field along flat direction rolls down the unbounded potential as far as loop expansion is reliable.

From the above argument we give a suggestion as follows. If the value of ϕ exceeds Planck scale we expect that non-renormalizable interaction effect may make the potential bounded from below. So the scalar field along flat direction rolls down to large value in the inflationary era. And after the cosmological constant becomes small the scalar field begins to fall down slowly the tree potential with soft breaking mass to the origin. This is because as Hubble constant becomes smaller, the loop correction eq. (1) also does. After all our discussion says the Affleck-Dine mechanism works if we take into account the quantum correction in de Sitter space.

Acknowledgements

We would like to thank H. Suzuki for the arguments by effective potential. We are also grateful to M. Hotta and T. Yanagida for reading through the manuscript and giving us useful comments.

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Figure Captions

Figure 1 : The one loop level effective potential of the Wess-Zumino model with $g = 0.1$, $\xi_i = 1/4$ and $m_i^2 a^2 = 0$. The tree level potential is also depicted for a comparison.

Figure 2 : The one loop level effective potential of the Wess-Zumino model with $g = 0.1$, $\xi_i = 0$ and $m_i^2 a^2 = 0.001$. We have changed renormalization point as $gaM = 0.001, 0.01, 0.1$. The tree level potential is also depicted for a comparison.

Figure 3 : The global behavior of the effective potential. The parameters are the same as in Figure 2.

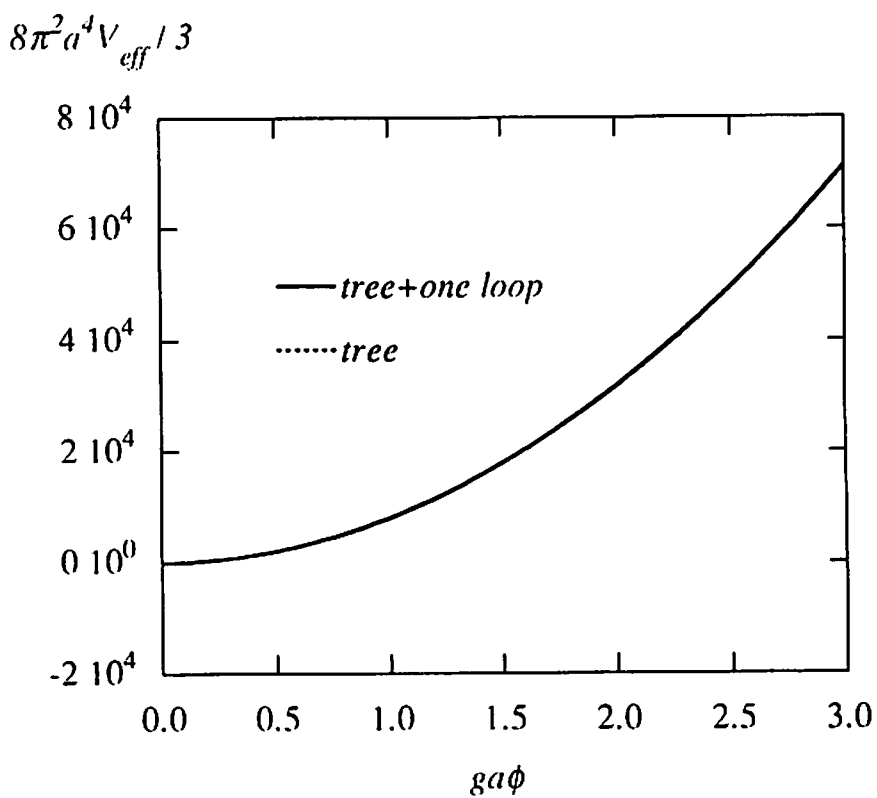


Fig.1

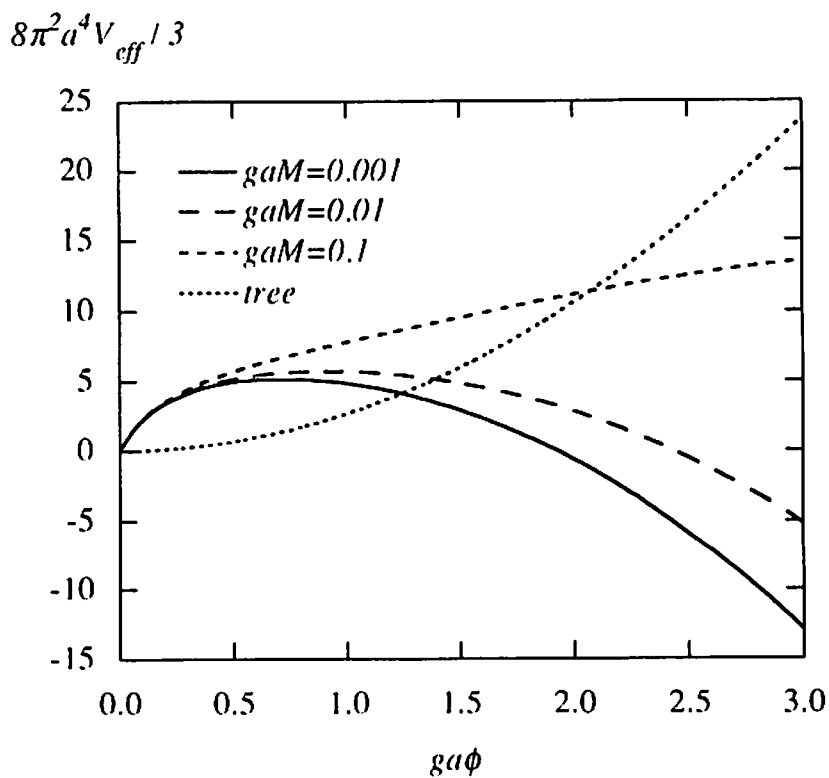


Fig.2

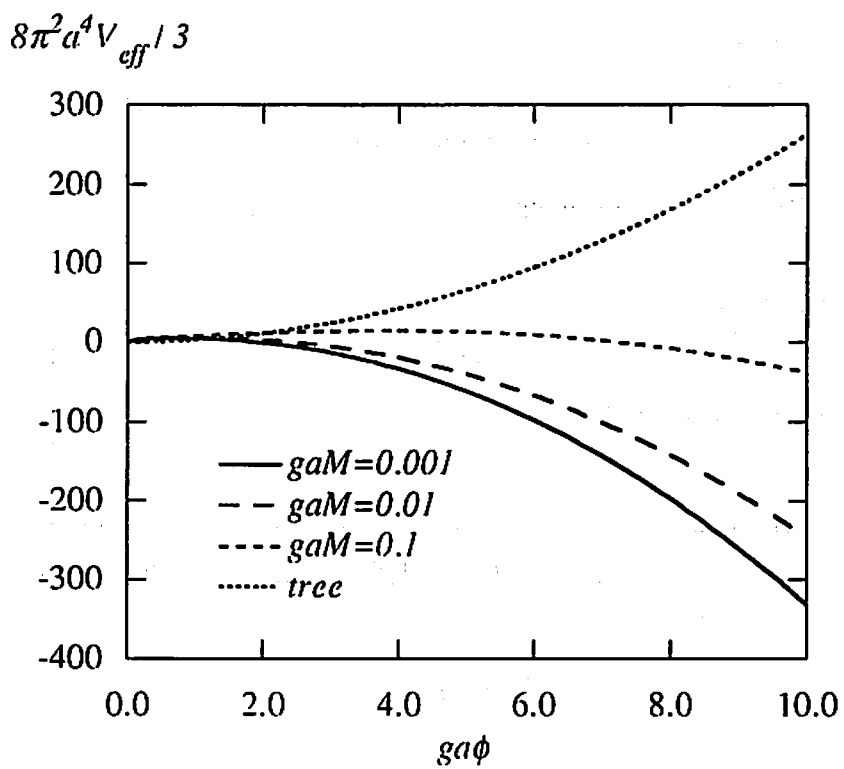


Fig.3

Domain wall interacting with gravitational waves [†]

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ABSTRACT

The behavior of a domain wall interacting with gravitational waves is investigated by the metric junction formalism. We consider a spacetime which consists of two Minkowski regions joined by a spherically symmetric domain wall as a background geometry. It is shown that in this particular background the domain wall does not emit gravitational waves spontaneously by its free oscillation in the first order perturbation, although it scatters incidental gravitational waves.

1 Introduction

By cosmological phase transitions topological defects associated with the symmetry breaking in unified theories^[2,3] are expected to be produced. Domain walls are the simplest one. Several researchers investigate the emission of the gravitational waves from the topological defects to get a proof of existing these objects.

How do we estimate the emission of the gravitational waves? Let us consider a particle emitting the gravitational wave. First, we assume the particle moves along a geodesic of a given background geometry. The particle is treated as a test particle. Next, the emission is estimated by the perturbed Einstein equation with a source term (energy-momentum tensor) by the moving particle. So far as the motion is slow enough, the backreaction from the emission to the motion may be small.

Similar treatment is adopted for topological defects which are extended objects as a cosmic string and a domain wall. The motion of the extended object in a fixed background geometry is obtained as a 'test object' by solving the equation derived from the 'Nambu-Goto action'. The emission is estimated by the Einstein equation with a source term by the moving object. However, there exists no guarantee that such the test object treatment does give a correct estimate on the gravitational wave emission rate by topological defects which have no non-relativistic counterpart.

In order to get some insights into this problem, we investigate in the general relativistic framework the behavior of perturbations of a spacetime which contains a domain wall. As

[†]This talk is based on the work^[1] by Hideo Kodama*, H.I. and Yoshihisa Fujiwara*

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the background spacetime we only consider the exact solution discussed by Vilenkin^[4] and Ipsier and Sikivie^[5] which is spherically symmetric and locally Minkowskian outside a domain wall. The domain wall is treated as an infinitesimally thin singular hypersurface with a surface energy-momentum density specific to the domain wall. The whole spacetime consists of two identical Minkowskian regions matched by the domain wall.

The reason why we adopt this background spacetime is purely technical. As systematically discussed by Gerlach and Sengupta^[6,7,8,9], the spacetime perturbations on a spherically symmetric background can be described by gauge-invariant quantities which are local combinations of the tensor harmonic expansion coefficients of the metric perturbations, and the Einstein equations can be written in terms of them. These Einstein equations can be further reduced into a single master equation for a single gauge-invariant variable corresponding to the Zerilli variable^[10,11]. In our special background this master equation can be easily solved explicitly in each Minkowskian region. The global solutions are constructed by matching these solutions along the domain wall so that they satisfy the junction condition of Israel^[12].

Throughout the paper we use the units $c = 1$ and denote $8\pi G$ by κ^2 . The signature of Lorentzian metrics is chosen to be $(-, +, +, \dots)$.

2 Background Geometry

The background geometry $(\mathcal{M}, g)$ we consider in this paper is constructed by gluing two regions $\mathcal{M}_\pm$ in the four-dimensional Minkowski spacetime along a three dimensional timelike hypersurfaces Σ . This surface Σ represents a domain wall. The possible surfaces are restricted by the junction conditions. We assume that the whole spacetime is spherically symmetric for simplicity.

The metric of a spherically symmetric spacetime is always written in $\mathcal{N} \times S^2$ form as

$$ds^2 = \gamma_{ab} dy^a dy^b + r^2 \Omega_{pq} dz^p dz^q, \quad (2.1)$$

where the metric γ_{ab} of $\mathcal{N}$ and r are functions of y^a and Ω_{pq} is the standard metric of a unit Euclidean 2-sphere S^2 . The indices $a, \dots, d$ run over 0 and 1, and $p, \dots, s$ over 2 and 3. Let D_a , $\hat{D}_p$ and ∇_μ be the covariant derivatives with respect to γ_{ab} , Ω_{pq} and the four-dimensional metric $g_{\mu\nu}$, respectively.

Let (x^μ) be a coordinate system around Σ in the whole spacetime $\mathcal{M}$ such that x^p ($p = 2, 3$) coincide with the angular coordinates z^p , and x^j ($j = 0, 2, 3$) gives a local coordinate system of Σ . Further let $(q_{jk})_\pm$ be the intrinsic metric of Σ induced from $(g_{\mu\nu})_\pm$, $(n^\mu)_\pm$ the unit normal vector to Σ directed from $\mathcal{M}_-$ to $\mathcal{M}_+$, and $(K_k^j)_\pm$ the extrinsic curvature of Σ with respect to $(n^\mu)_\pm$. Then, as shown by Israel, the junction conditions are given by the following equations:

$$[q_{jk}] = 0, \quad (2.2)$$

$$[K_k^j] = \kappa^2 \left(S_k^j - \frac{1}{2} \delta_k^j S_l^l \right), \quad (2.3)$$

where in general, for fields $Q_\pm$ on Σ , $[Q]$ represents the difference

$$[Q] := Q_+ - Q_-, \quad (2.4)$$

and S_k^j is the energy-momentum surface density of the domain wall. The first condition simply states that $(q_{jk})_{\pm}$ coincides with the intrinsic metric q_{jk} of Σ induced from $g_{\mu\nu}$.

In this paper we assume that S_k^j for a domain wall is in the form

$$\kappa^2 S_k^j = -4\lambda \delta_k^j, \quad (2.5)$$

where λ is a positive constant which characterize its surface density and tension. Then the constraint (2.3) is written as

$$[K_k^j] = 2\lambda \delta_k^j. \quad (2.6)$$

Now, we calculate K_k^j explicitly. Let τ^a be the future-directed unit tangent vector to Σ with vanishing z^p components and note that the unit normal vector n^μ has vanishing z^p components. So, τ^a and n^a are unit vectors in $\mathcal{N}$. Then the projection tensor to Σ defined by

$$q_{\mu\nu} := g_{\mu\nu} - n_\mu n_\nu \quad (2.7)$$

is expressed as

$$q_{ab} = -\tau_a \tau_b, \quad q_{pq} = r^2 \Omega_{pq}, \quad q_{ap} = 0. \quad (2.8)$$

In terms of these quantities the extrinsic curvature tensor

$$K_{\mu\nu} := -q_\mu^\lambda \nabla_\lambda n_\nu \quad (2.9)$$

is expressed as

$$K_{ab} = \tau^c D_{//} n_c q_{ab}, \quad (2.10)$$

$$K_{ap} = 0, \quad (2.11)$$

$$K_{pq} = -r D_\perp r \Omega_{pq}, \quad (2.12)$$

where

$$D_\perp := n^a D_a, \quad D_{//} := \tau^a D_a. \quad (2.13)$$

Hence junction condition (2.6) is written as

$$(\tau^c D_{//} n_c)_\pm = \pm\lambda + \mu_1, \quad (2.14)$$

$$\left(\frac{1}{r} D_\perp r\right)_\pm = \mp\lambda + \mu_2, \quad (2.15)$$

with some spherically symmetric functions μ_1 and μ_2 on Σ .

μ_1 and μ_2 cannot be arbitrary. First from the Hamiltonian constraint

$${}^3R + K_k^j K_j^k - K^2 = 0, \quad (2.16)$$

it follows that

$$\overline{K_k^j} [K_j^k] - \overline{K} [K] = 0, \quad (2.17)$$

where $\overline{Q}$ for $Q_{\pm}$ on Σ denotes the mean value

$$\overline{Q} := \frac{1}{2}(Q_+ + Q_-). \quad (2.18)$$

Inserting the above expressions of K_k^j into this equation we get $\mu_1 = \mu_2$.

Next in the coordinates $(t_{\pm}, r)$ of $\mathcal{N}_{\pm}$ for which

$$(\gamma_{ab})_{\pm} dy^a dy^b = -dt_{\pm}^2 + dr^2, \quad (2.19)$$

from Eq.(2.15) and normalization n^a is written as

$$(n^r)_{\pm} = (\mp\lambda + \mu_2)r, \quad (n^t)_{\pm} = \epsilon\sqrt{(n^r)_{\pm}^2 - 1}, \quad (2.20)$$

where $\epsilon = \pm 1$. This completely determines $\tau^a = dy^a/d\tau$ where τ is the proper time as

$$\frac{dt_{\pm}}{d\tau} = \epsilon'(n^r)_{\pm}, \quad (2.21)$$

$$\frac{dr}{d\tau} = \epsilon'(n^t)_{\pm}, \quad (2.22)$$

where ϵ' is the sign of n^r . In particular from the continuity of r and the first of Eq.(2.20) it follows that $[(n^r)^2] = 0$, which implies that μ_2 vanishes, hence

$$\mu_1 = \mu_2 = 0, \quad (2.23)$$

$$(\tau^c D_{\mu} n_c)_{\pm} = -\left(\frac{1}{r} D_{\perp} r\right)_{\pm} = \pm\lambda. \quad (2.24)$$

This boundary conditions are written in terms of K_k^j as

$$(K_k^j)_{\pm} = \pm\lambda\delta_k^j, \quad (2.25)$$

which implies that the spacetime is reflection symmetric with respect to the domain wall.

Under this condition Eqs.(2.21) and (2.22) are easily solved to yield the hyperboloid

$$r^2 - t^2 = \lambda^{-2}. \quad (2.26)$$

Thus, using the spherical Rindler coordinate:

$$t = \chi \sinh \tau, \quad r = \chi \cosh \tau, \quad (2.27)$$

the domain wall locates at $\chi = \lambda^{-1}$.

Note that Eq.(2.15) shows that $\mathcal{N}_{\pm}$ both correspond to the inside of the hyperboloid. Hence the whole spacetime is spatially compact and globally hyperbolic, but has null future and past infinities each of which have two connected components, as shown in Fig.1.

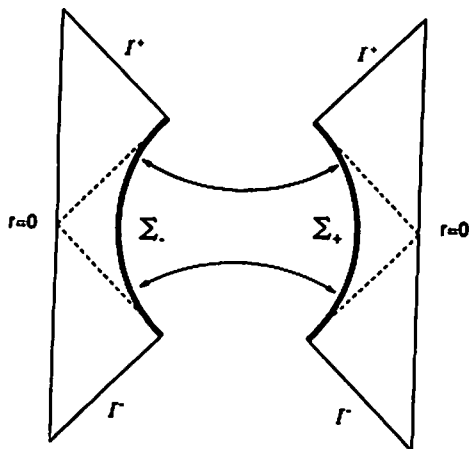


Fig.1 Conformal Diagram of the Background Spacetime.

The entire background spacetime is constructed by identifying the two boundary hyperboloids Σ_- and Σ_+ of two copies $\mathcal{M}_+$ and $\mathcal{M}_-$ of a region in the Minkowski spacetime.

3 Gauge-invariant Perturbation

Let $f_{\mu\nu}$ be the perturbation of the metric

$$f_{\mu\nu} := \delta g_{\mu\nu}. \quad (3.1)$$

Then under the infinitesimal gauge-transformation $\bar{\delta}x^\mu = \xi^\mu$, $f_{\mu\nu}$ transforms as

$$\bar{\delta}f_{\mu\nu} = -\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu. \quad (3.2)$$

In the coordinate system (y^a, z^p) this equation is written as

$$\bar{\delta}f_{ab} = -D_a \xi_b - D_b \xi_a, \quad (3.3)$$

$$\bar{\delta}f_{ap} = -r^2 D_a \left(\frac{\xi_p}{r^2} \right) - \hat{D}_p \xi_a, \quad (3.4)$$

$$\bar{\delta}f_{pq} = -\hat{D}_p \xi_q - \hat{D}_q \xi_p - 2r \xi^a D_a r \Omega_{pq}. \quad (3.5)$$

The perturbation variables f_{ab} , f_{ap} , f_{pq} , ξ_a , and ξ_p can be expanded by the scalar Y_m^l , the vector $(V_m^l)_p$ and the tensor harmonics $(T_m^l)_{pq}$ as

$$f_{ab} = \sum f_{ab} Y_m^l, \quad (3.6)$$

$$\frac{1}{r} f_p^a = \sum \left[f_{(e1)}^a (V_{(e)m}^l)_p + f_{(o1)}^a (V_{(o)m}^l)_p \right], \quad (3.7)$$

$$\frac{1}{r^2} f_{pq} = \sum \left[f_{(e0)} (T_{(e0)m}^l)_{pq} + f_{(e2)} (T_{(e2)m}^l)_{pq} + f_{(o2)} (T_{(o2)m}^l)_{pq} \right], \quad (3.8)$$

$$\xi_a = \sum \xi_a Y_m^l, \quad (3.9)$$

$$\frac{1}{r} \xi_p = \sum \left[\xi_{(e)} (V_{(e)m}^l)_p + \xi_{(o)} (V_{(o)m}^l)_p \right]. \quad (3.10)$$

In these expressions the summation is over all possible values of (l, m) , and the expansion coefficients are tensors on $\mathcal{N}$. The indices l and m for the coefficients are omitted for simplicity. The prefactors $1/r$ or $1/r^2$ on the left hand sides are put to give the expansion coefficients the same physical dimension. The symbol o and e refer to the odd and the even modes, respectively.

By inspecting the gauge transformation of the coefficients we can easily find all gauge-invariant combinations. Hereafter we restrict ourselves in the case $l \geq 2$ because $l = 0, 1$ modes have no relevance to gravitational waves. (see ref.[1])

First for the odd modes Eqs.(3.4) and (3.5) yield

$$\bar{\delta} f_{(o1)}^a = -r D^a \left(\frac{\xi_{(o)}}{r} \right), \quad (3.11)$$

$$\bar{\delta} f_{(o2)} = -\frac{2}{r} \xi_{(o)}. \quad (3.12)$$

From these equations we see that there exist two independent gauge-invariant quantities. The most natural choice is

$$F^a := f_{(o1)}^a - \frac{1}{2} r D^a f_{(o2)}. \quad (3.13)$$

Next for the even modes the gauge transformation of the expansion coefficients are expressed as

$$\bar{\delta} f_{ab} = -D_a \xi_b - D_b \xi_a, \quad (3.14)$$

$$\bar{\delta} f_{(e1)}^a = -r D^a \left(\frac{\xi_{(e)}}{r} \right) - \frac{1}{r} \xi^a, \quad (3.15)$$

$$\bar{\delta} f_{(e0)} = \frac{2l(l+1)}{r} \xi_{(e)} - \frac{4}{r} D^a r \xi_a \quad (3.16)$$

$$\bar{\delta} f_{(e2)} = -\frac{2}{r} \xi_{(e)}. \quad (3.17)$$

From these equations the variable X^a defined by

$$X^a := r f_{(e1)}^a - \frac{1}{2} r^2 D^a f_{(e2)} \quad (3.18)$$

transforms as

$$\bar{\delta} X^a = -\xi^a. \quad (3.19)$$

Hence we can easily find all the gauge-invariant combinations. The most natural independent sets are

$$F_{ab} := f_{ab} - D_a X_b - D_b X_a, \quad (3.20)$$

$$F := f_{(e0)} + l(l+1) f_{(e2)} - \frac{4}{r} D_a r X^a. \quad (3.21)$$

4 Perturbation Equations

In the flat background spacetime the perturbed vacuum Einstein equations are given by

$$2\delta G_{\mu\nu} \equiv -\square f_{\mu\nu} - \nabla_\mu \nabla_\nu f^\lambda_\lambda + \square f^\lambda_\lambda g_{\mu\nu} + \nabla_\mu \nabla_\lambda f^\lambda_\nu + \nabla_\nu \nabla_\lambda f^\lambda_\mu - g_{\mu\nu} \nabla_\lambda \nabla_\sigma f^{\lambda\sigma} = 0, \quad (4.1)$$

where $\square = \nabla_\mu \nabla^\mu$. Let us rewrite this set of equations in terms of the gauge-invariant variables introduced just above and show that they are reduced to a single master equation for each of the odd and the even modes.

From the odd tensor component we get

$$D_a(rF^a) = 0, \quad (4.2)$$

$$-\left(\square_2 - \frac{l(l+1)}{r^2}\right)(rF^a) + 2D^a r D_b r F^b = 0, \quad (4.3)$$

where $\square_2 := D_a D^a$.

Equation (4.2) implies that F_a is expressed in terms of a master variable $\Phi_{(o)}$ as

$$rF^a = \epsilon^{ab} D_b \Phi_{(o)}. \quad (4.4)$$

In terms of $\Phi_{(o)}$ Eq.(4.3) is written as

$$\epsilon^{ab} D_b \left[(r^2 \square_2 - 2r D^c r D_c + 2 - l^2 - l) \Phi_{(o)} \right] = 0. \quad (4.5)$$

This implies that the term inside the square bracket is a constant. By using the freedom of adding an arbitrary constant in the definition of $\Phi_{(o)}$ (4.4). Hence the perturbed Einstein equation is reduced to the following single equation for $\Phi_{(o)}$:

$$\left(-\square_2 + \frac{l(l+1)}{r^2} \right) \left(\frac{\Phi_{(o)}}{r} \right) = 0. \quad (4.6)$$

For the even mode variables Eq.(4.1) yields the following set of gauge-invariant equations:

$$F^a_a = 0, \quad (4.7)$$

$$D_b F^b_a = \frac{1}{2} D_a F, \quad (4.8)$$

$$\left(\square_2 + \frac{2}{r} D_a r D^a - \frac{l^2 + l - 2}{r^2} \right) F = \frac{4}{r^2} D_a r D_b r F^{ab}, \quad (4.9)$$

$$\begin{aligned} & - \left(\square_2 + \frac{2}{r} D_c r D^c - \frac{l(l+1)}{r^2} \right) F_{ab} + \frac{2}{r} D_c r (D_a F^c_b + D_b F^c_a) \\ & = \frac{1}{r} (D_a r D_b + D_b r D_a) F. \end{aligned} \quad (4.10)$$

From Eqs.(4.7) and (4.8) we can express F_b^a and F in terms of a scalar master variable $\Phi_{(e)}$ as

$$F_{ab} = D_a D_b \Phi_{(e)} - \frac{1}{2} \gamma_{ab} \square_2 \Phi_{(e)}, \quad (4.11)$$

$$F = \square_2 \Phi_{(e)}. \quad (4.12)$$

Putting these expressions into Eqs.(4.9) and (4.10), we see that

$$\left(-r^2 \square_2 + 2r \partial_a r D^a + l^2 + l - 2 \right) \Phi_{(e)}. \quad (4.13)$$

is a constant. As in the odd modes this constant can be put zero by utilizing the freedom of adding arbitrary constant to $\Phi_{(e)}$. Hence the perturbed Einstein equation reduces to the same equation as that for the odd modes:

$$\left(-\square_2 + \frac{l(l+1)}{r^2} \right) \left(\frac{\Phi_{(e)}}{r} \right) = 0. \quad (4.14)$$

This wave equation is just the flat limit of the Zerilli equation for the Schwarzschild background^[10,11,13].

5 Solutions to the master equation

We construct the general solution to the perturbed Einstein equations (4.6) and (4.14) by the method of separation of variables in the Minkowski coordinates and in the Rindler coordinates, and give the relation between the mode functions in both coordinates. Throughout this section we denote the master variable simply by Φ .

First in the Minkowski coordinates, the master equation is simply given by

$$\left(\partial_t^2 - \partial_r^2 + \frac{l(l+1)}{r^2} \right) \left(\frac{\Phi}{r} \right) = 0. \quad (5.1)$$

With the help of the Fourier transform with respect to t , the general solution to this equation that is regular at $r = 0$ and belongs to $L^2(-\infty, \infty)$ with respect to t for any fixed value of r is written as

$$\Phi = r^2 \int_{-\infty}^{\infty} d\omega C(\omega) e^{-i\omega t} j_l(\omega r), \quad (5.2)$$

where $j_l(x)$ is the spherical Bessel function. The mode expansion coefficient $C(\omega)$ is written in terms of the initial data of Φ at $t = 0$ as

$$C(\omega) = \omega \int_0^{\infty} dr j_l(\omega r) (\omega \Phi + i \partial_t \Phi)|_{t=0}. \quad (5.3)$$

Next in the Rindler coordinates Eq.(2.27) which cover the region $|t| < r$, the master equation is written as

$$\left(-\chi^2 \partial_\chi^2 - \chi \partial_\chi + \partial_\tau^2 + \frac{l(l+1)}{\cosh^2 \tau} \right) \left(\frac{\Phi}{r} \right) = 0. \quad (5.4)$$

Hence the asymptotic values of Φ/r at $\mathcal{I}^-$ and $\mathcal{I}^+$ are expressed in terms of $C(\omega)$ as

$$\alpha(v) := (\Phi/r)|_{\mathcal{I}^-} = \frac{1}{2} e^{i\frac{l+1}{2}\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} C(\omega) e^{-i\omega v}, \quad (5.8)$$

$$\beta(u) := (\Phi/r)|_{\mathcal{I}^+} = \frac{1}{2} e^{-i\frac{l+1}{2}\pi} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} C(\omega) e^{-i\omega u}. \quad (5.9)$$

By inserting the expression (5.6) into the right hand sides of these equations, we get after a short calculation

$$\begin{aligned} \alpha(v) = & \theta(-v)\pi \int_{-\infty}^{\infty} dk A(k) |v|^{ik} \frac{\Gamma(-ik)}{\Gamma(ik)} \frac{\Gamma(l+1+ik)}{\Gamma(l+1-ik)} \\ & + \theta(v)\pi \int_{-\infty}^{\infty} dk B(k) v^{-ik}, \end{aligned} \quad (5.10)$$

$$\begin{aligned} \beta(u) = & \theta(-u)(-1)^{l+1}\pi \int_{-\infty}^{\infty} dk A(k) |u|^{ik} \frac{\Gamma(-ik)}{\Gamma(ik)} \frac{\Gamma(l+1+ik)}{\Gamma(l+1-ik)} \\ & + \theta(u)\pi(-1)^{l+1} \int_{-\infty}^{\infty} dk B(k) u^{-ik}. \end{aligned} \quad (5.11)$$

These equations show that $A(k)$ and $B(k)$ describe the freedoms of the incidental gravitational waves from the part ($v < 0$) of $\mathcal{I}^-$ outside the Rindler wedge and the outgoing ones to the part ($u > 0$) of $\mathcal{I}^+$ outside the Rindler wedge, respectively. This situation is schematically depicted in Fig.2.

6 Global Solutions

We construct global solutions to the perturbed Einstein equations in $\mathcal{M}$ by matching the solutions in $\mathcal{M}_+$ and $\mathcal{M}_-$ along the domain wall.

The perturbed Einstein equations must satisfy the perturbed junction condition

$$[\delta q_{jk}] = 0, \quad (6.1)$$

$$[\delta K_k^j] = \kappa^2 \left(\delta S_k^j - \frac{1}{2} \delta_k^j \delta S_l^l \right). \quad (6.2)$$

We get the boundary conditions for the global solutions to satisfy, if we express these conditions in terms of the gauge-invariant variables and rewrite them as conditions on the boundary values of the master variables. The perturbed extrinsic curvature δK_k^j is written as:

$$(\delta K_{||})_{(o1)}^a \tau_a = \frac{1}{2} \epsilon^{ab} D_b \left(\frac{F_a}{r} \right), \quad (6.3)$$

$$(\delta K_{||})_{(o2)} = \frac{1}{r} F_{\perp}, \quad (6.4)$$

$$\begin{aligned} (\delta K_{||})_a^a = & \frac{1}{2} K_a^a F_{||||} - \frac{1}{2} D_{||} F_{||\perp} - \frac{1}{2} D_{\perp} F_a^a + \frac{1}{2} n_a D_b F^{ab} \\ & + \{-D_{||}^2 + (K_a^a)^2\} X_{\perp}, \end{aligned} \quad (6.5)$$

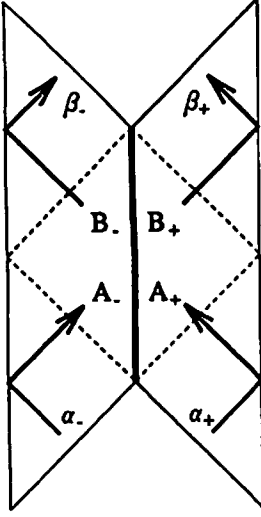


Fig.2 Relation between the Rindler modes and the Mikowskian modes.

The solution to this equation is given by

$$\begin{aligned} \frac{\Phi}{r} &= \int_{-\infty}^{\infty} dk \psi_l(\tau, k) \left(A(k) e^{ik \ln x} + B(k) e^{-ik \ln x} \right) \\ &= \int_{-\infty}^{\infty} dk G_l \left(1, 1 - ik; \frac{1 + \tanh \tau}{2} \right) \left(A(k) e^{-ik \ln(-u)} + B(k) e^{-ik \ln v} \right). \end{aligned} \quad (5.5)$$

where $u := t + r, v := t - r$ and $G_l(a, b; x)$ is the Jacobi polynomial.

In connecting the solutions in the two regions $\mathcal{M}_+$ and $\mathcal{M}_-$ along the domain wall, the expression (5.5) is more useful. However, it is not appropriate for studying the behavior of the solution in the whole spacetime region since it is restricted in the Rindler wedge $|t| < r$. Obviously the expression (5.2) should be used for that purpose. Hence we need the relation between the coefficients in these two expansions in order to construct the global solution and study its behavior. This relation is easily obtained by substituting (5.5) into Eq.(5.3):

$$\begin{aligned} C(\pm|\omega|) &= e^{-i\pi/2} \int_{-\infty}^{\infty} dk e^{\pm k\pi/2} \Gamma(1 - ik) \\ &\quad \times \left(A(k) |\omega|^{-ik} \frac{\Gamma(l + 1 + ik)}{\Gamma(l + 1 - ik)} + B(k) |\omega|^{ik} \right). \end{aligned} \quad (5.6)$$

The meaning of this relation becomes clearer if we rewrite it as the relation between the asymptotic value of Φ at the future null infinity $\mathcal{I}^+$ and the past null infinity $\mathcal{I}^-$. First note that Φ/r has a finite limit at null infinities:

$$\frac{\Phi}{r} \rightarrow \int_{-\infty}^{\infty} \frac{d\omega}{2\omega} \left(e^{-i\omega u - i\frac{l+1}{2}\pi} + e^{-i\omega v + i\frac{l+1}{2}\pi} \right) C(\omega). \quad (5.7)$$

$$(\delta K_{//})_{(e1)}^a \tau_a = \frac{1}{2r} F_{//\perp} + D_{//} \left(\frac{X_{\perp}}{r} \right), \quad (6.6)$$

$$\begin{aligned} (\delta K_{//})_{(e0)} &= -\frac{1}{2} K_p^p F_{\perp\perp} - \frac{2}{r} D_{//} r F_{//\perp} - \frac{1}{2} D_{\perp} F \\ &\quad + \left\{ -\frac{2}{r} D_{//} r D_{//} - \frac{l(l+1)}{r^2} + \frac{1}{2} (K_p^p)^2 \right\} X_{\perp}, \end{aligned} \quad (6.7)$$

$$(\delta K_{//})_{(e2)} = \frac{1}{r^2} X_{\perp}, \quad (6.8)$$

where $(\delta K_{//})$ denotes the four-dimensional extension of the tensor δK_k^j given by

$$(\delta K_{//})_{\nu}^{\mu} := q_{\lambda}^{\mu} \delta K_{\nu}^{\lambda}, \quad (6.9)$$

the symbols $V_{//}$ and $V_{\perp}$ for a vector V_a on $\mathcal{N}$ denote

$$V_{//} := \tau^a V_a, \quad V_{\perp} := n^a V_a, \quad (6.10)$$

and $T_{///}$, $T_{//\perp}$ and $T_{\perp\perp}$ for a symmetric tensor T_{ab} on $\mathcal{N}$ denote

$$T_{///} := \tau^a \tau^b T_{ab}, \quad T_{//\perp} := \tau^a n^b T_{ab}, \quad T_{\perp\perp} := n^a n^b T_{ab}. \quad (6.11)$$

We should comment on the meaning of the variable X^a at the domain wall. In the perturbation theory we pull back the perturbed geometry to the unperturbed background geometry by some embedding. The ambiguity in the choice of this embedding is the gauge freedom. When one considers spacetimes with singularities such as the domain wall, the perturbative treatment obviously breaks down unless the embedding preserves the singularity structure. So, we must restrict the embeddings so that it maps the domain wall in the background spacetimes to that in perturbed spacetimes. In terms of the gauge transformation this restriction is expressed as $\xi_{\perp} = \xi_a n^a = 0$ on the domain wall. Hence from Eq.(3.19) $X_{\perp}$ becomes gauge invariant on the domain wall, and in a sense describes the perturbation of the location of the domain wall.

In order to fix the junction condition completely, we must specify the perturbation of the surface energy-momentum density S_k^j of the domain wall. We assume that S_k^j keeps the form (2.5) under perturbation then

$$\delta S_k^j = -4\delta\lambda \delta_k^j = 0. \quad (6.12)$$

Thus the second junction condition (6.2) is written as

$$[\delta K_k^j] = 0. \quad (6.13)$$

Now let us rewrite the above junction condition in terms of the master variable and construct the global solutions. We discuss the odd and the even modes separately.

Odd modes

From Eq.(6.1) $f_{(o2)}$ and $f_{(o1)//}$ is continuous across the domain wall. Hence we get

$$[F_{//}] = \left[f_{(o1)//} - \frac{r}{2} D_{//} f_{(o2)} \right] = 0. \quad (6.14)$$

From Eqs.(6.4) and (6.13) we get

$$\left[\epsilon^{\alpha\beta} D_\beta \left(\frac{1}{r} F_\alpha \right) \right] = 0. \quad (6.15)$$

Using Eq.(4.4) and master equation (4.6) Eqs.(6.15) and (6.14) are expressed in terms of $\Phi_{(o)}$ as

$$[D_\perp \Phi_{(o)}] = 0, \quad (6.16)$$

$$[\Phi_{(o)}] = 0. \quad (6.17)$$

Since Eq.(6.16) is expressed from Eq.(2.24) as

$$\overline{\partial_x \left(\frac{\Phi_{(o)}}{r} \right)} = -\lambda \frac{\overline{\Phi_{(o)}}}{r}, \quad (6.18)$$

the junction conditions on the solution are written in terms of the mode expansion coefficients A and B in Eq.(5.5) as

$$[A] + [B] = 0, \quad (6.19)$$

$$(k-i)\bar{A} - (k+i)\bar{B} = 0. \quad (6.20)$$

These equations are easily solved to yield

$$B_\pm = -\frac{i}{k+i} A_\pm + \frac{k}{k+i} A_\mp. \quad (6.21)$$

We can express this result as a relation of the asymptotic values of $\Phi_{(o)}/r$ at the null infinities of the total spacetime, $\alpha_\pm(v)$ ($v < 0$) and $\beta_\pm(u)$ ($u > 0$), with the help of Eqs.(5.10) and (5.11):

$$\begin{aligned} \beta_\pm(u) &= (-1)^{l'+1} \left\{ (-1)^l \alpha_\mp(-1/u) \right. \\ &\quad + \sum_{n=1}^l \frac{n}{n+1} \frac{t+nC_l}{(n-1)!} u^n \int_{-1/u}^0 dv v^{n-1} \alpha_\mp(v) \\ &\quad \left. - \sum_{n=1}^l \frac{1}{n+1} \frac{t+nC_l}{(n-1)!} u^n \int_{-1/u}^0 dv v^{n-1} \alpha_\pm(v) \right\}. \end{aligned} \quad (6.22)$$

From this we find that $\beta_\pm(u)$ depends only on the values of $\alpha_\pm(v)$ at $v \geq -1/u$. Since $-1/v$ is the maximum value of u on the light cone emitted at v of $\mathcal{I}^-$, this implies that the domain wall emits gravitational waves only during the period when the incoming waves hit it, hence that the domain wall has no dynamical degree of freedom. This result is natural because there exists no odd variable describing the position of the domain wall.

Even Modes

First from Eq.(6.1) $f_{///}$, $f_{(e1)///}$, $f_{(e0)}$ and $f_{(e2)}$ are continuous across the domain wall Σ . From this it follows that

$$[X_{//}] = \left[r f_{(e1)///} - \frac{1}{2} r^2 D_{//} f_{(e2)} \right] = 0. \quad (6.23)$$

On the other hand from Eq.(6.8) and Eq.(6.13) $X_{\perp}$ is also continuous across Σ :

$$[X_{\perp}] = r^2 \left[(\delta K_{//})_{(e2)} \right] = 0. \quad (6.24)$$

Noting this continuity of X_a , we can calculate the jump of values of F and $F_{///}$ as

$$[F] = \left[f_{(e0)} + l(l+1)f_{(e2)} - \frac{4}{r} D_a r X^a \right] = 8\lambda X_{\perp}, \quad (6.25)$$

$$[F_{///}] = [f_{///} - 2D_{//} X_{//} - 2K_a^a X_{\perp}] = -4\lambda X_{\perp}. \quad (6.26)$$

Further from Eqs.(6.6) and (6.7) the condition (6.13) yields

$$[F_{\perp//}] = 0, \quad (6.27)$$

$$[D_{\perp} F] = -4\lambda \overline{F_{\perp\perp}}. \quad (6.28)$$

Using Eqs.(4.11) and (4.12), we rewrite these junction conditions Eqs.(6.25)-(6.28) as conditions on the master variable $\Phi_{(e)}$.

From Eq.(6.25) we find that $X_{\perp}$ is completely determined by the jump of $\Phi_{(e)}$ and its χ -derivative at Σ as

$$X_{\perp} = \frac{1}{4\lambda} \cosh \tau \left[\partial_{\chi} \left(\frac{\Phi_{(e)}}{r} \right) \right] - \frac{1}{4} \left(\sinh \tau \partial_{\tau} - \frac{l(l+1)}{2 \cosh \tau} \right) \left[\frac{\Phi_{(e)}}{r} \right]. \quad (6.29)$$

Further elimination of $X_{\perp}$ from Eqs.(6.25) and (6.26) yields the boundary condition on $\Phi_{(e)}$,

$$\left(\partial_{\tau}^2 + \frac{l(l+1)}{\cosh^2 \tau} \right) \left[\frac{\Phi_{(e)}}{r} \right] = -\frac{1}{\lambda} \left[\partial_{\chi} \left(\frac{\Phi_{(e)}}{r} \right) \right]. \quad (6.30)$$

The condition (6.27) is written as

$$\partial_{\tau} \left\{ \cosh \tau \partial_{\chi} \left(\frac{\Phi_{(e)}}{r} \right) \right\} = 0. \quad (6.31)$$

On the other hand the last condition (6.28) is written as

$$2 \sinh \tau \partial_{\tau} \left\{ \cosh \tau \partial_{\chi} \left(\frac{\Phi_{(e)}}{r} \right) \right\} = -(l^2 + l - 2) \partial_{\chi} \left(\frac{\Phi_{(e)}}{r} \right). \quad (6.32)$$

These two equations reduce to the single equation

$$\partial_{\chi} \left(\frac{\Phi_{(e)}}{r} \right) = 0. \quad (6.33)$$

Equations (6.30) and (6.33) yield the boundary conditions for $\Phi_{(e)}$ to satisfy on the domain wall. As in the case of odd modes, these conditions can be easily translated into conditions on the mode expansion coefficient as

$$k((k-i)[A] + (k+i)[B]) = 0, \quad (6.34)$$

$$k(\overline{A} - \overline{B}) = 0. \quad (6.35)$$

Hence for $k \neq 0$ $B_{\pm}$ is expressed in terms of $A_{\pm}$ as

$$B_{\pm} = \frac{i}{k+i} A_{\pm} + \frac{k}{k+i} A_{\mp}. \quad (6.36)$$

As in the odd modes, we can rewrite this equation as a relation between the amplitudes of the incoming and the outgoing waves. The result is given by

$$\begin{aligned} \beta_{\pm}(u) = & (-1)^{l+1} \left\{ (-1)^l \alpha_{\mp}(-1/u) \right. \\ & - \sum_{n=1}^l \frac{n}{n+1} \frac{l+n C_l}{(n-1)!} u^n \int_{-1/u}^0 dv v^{n-1} \alpha_{\mp}(v) \\ & \left. - \sum_{n=1}^l \frac{1}{n+1} \frac{l+n C_l}{(n-1)!} u^n \int_{-1/u}^0 dv v^{n-1} \alpha_{\pm}(v) \right\}. \end{aligned} \quad (6.37)$$

Since this equation is the same as Eq.(6.22) apart from the difference of the sign of the second term, we can conclude that the domain wall has no dynamical degree of freedom for the even perturbations either. Note that this does not mean that the wall behaves as a rigid surface. In fact we can easily see that the variable $X_{\perp}$, which can be interpreted as describing the displacement of the wall, take non-zero values while incidental waves go across the wall. Hence, in contrast to the odd-mode case, this result is non-trivial.

7 Discussion

We have analyzed the behavior of gravitational waves and their interaction with a domain wall by the first-order perturbation theory on a special background solution. The main conclusion is that the domain wall does not emit gravitational waves spontaneously by its free oscillations. This conclusion is quite embarrassing because in the treatment neglecting gravity the domain wall has its own dynamical freedom and its oscillation is expected to become a source of gravitational waves even in the first order with respect to its oscillation amplitude.

As shown in section 6, the domain wall immediately stops oscillation after the incidental gravitational wave packet goes through the wall. This implies that the wall behaves as if it has no inertia, hence that it has no intrinsic dynamical freedom.

Here note that the result comes from a non-trivial interaction between the domain wall and gravitational waves through the junction condition. In fact, from Eqs.(2.25) and (6.13), the perturbation of the junction condition (2.17) yields the condition

$$\delta \overline{K} = 0. \quad (7.1)$$

From Eq.(6.5), Eq.(6.7) and the field equations (4.7) and (4.8), this condition is written as

$$\left(-D_{||}^2 - \frac{2}{r}D_{||}rD_{||} - \frac{l(l+1)}{r^2} + 4\lambda^2\right)X_{\perp} = \frac{1}{2r^4}D_{||}(r^4\overline{F_{||\perp}}) + \frac{1}{4}\overline{D_{\perp}F}. \quad (7.2)$$

This equation should be compared with the perturbative equation of motion derived from ‘Nambu-Goto action’ for a domain wall in a flat background when the gravity of the wall is neglected. As shown by Garriga and Vilenkin^[14], the perpendicular displacement ϕ from a stationary configuration of the wall satisfies a Klein-Gordon type wave equation on the wall,

$$(\Box_3 + \epsilon^2)\phi = 0, \quad (7.3)$$

where $\Box_3$ is the d’Alembertian on the wall and ϵ is the difference of the vacuum energies on the two sides of the wall. Since the left hand side of Eq.(7.2) coincides with the harmonic expansion of d’Alembertian on the hyperboloid apart from the negative mass term, these two equations have quite similar structures except for the gravitational force term in our case. In particular Eq.(7.2) apparently shows that oscillations of the wall can propagate along it freely except for possible growth or decay due to the effective negative mass term if the gravitational field is fixed. However, this apparent dynamics is lost if we take all the constraints and the field equations into account because $X_{\perp}$ is completely determined by the boundary values of the master variable $\Phi_{(e)}$ representing the gravitational wave freedom through Eq.(6.29). It is easy to show that the expression (6.29) for $X_{\perp}$ automatically satisfies (7.2) if the boundary condition (6.30) is satisfied. Thus the dynamical freedom of the wall is lost due to the consistency between the gravitational perturbations of geometry and the motion of the domain wall.

Besides the spherical symmetry, our background solution has some special features such as the reflection symmetry and the thin wall nature. However, we cannot think of a physical mechanics by which these features prohibit the spontaneous emission of gravitational waves. Further we can show that for more general types of perturbations to S_k^j than Eq.(6.12) the wall can emit gravitational waves spontaneously. Hence there is a possibility that the same result holds for a more general class of background spacetimes if the perturbation of S_k^j is given by Eq.(6.12). To see whether this expectation is true or not, the case of the spherically symmetric background with a non-reflection-symmetric domain wall is under investigation.

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Creation of a Child Universe in Vaidya Spacetime with Out-going Negative Energy Radiation

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1

Since the foundation of the theory of general relativity was established, theoretical physicists encountered various non-trivial behavior of spacetimes. One of the most interesting behavior is the multiproduction of the universe discovered by Sato et al[1]. They found that the phenomenon occurs by the motion of false vacuum bubble in the process of the cosmological phase transition. After their works, several authors discussed whether the creation of a child universe occurs by producing a false vacuum bubble *in the laboratory*, that is, whether a false vacuum bubble prepared on a regular initial spacelike hypersurface can inflates permanently or not[2,3,4]. Here the regular initial spacelike hypersurface means the Cauchy surface on which the observer *in the laboratory* can manipulate the Cauchy data with no obstructions from some past singularities. In classical theory, the above possibility is forbidden by a certain kind of no-go theorem[5] at least in spherical symmetric cases if the energy-momentum tensor obeys the weak energy condition. Hence we must investigate other routes to escape from the no-go theorem. For example, the authors of [3,4] pointed out the realization of the possibility by the use of quantum mechanical tunneling of the bubble itself.

In this paper we present another scenario of the creation of a child universe. Our strategy adopted here is to break the weak energy condition by introduc-

ing the domain wall which radiate negative energy flux. Our discussion here is devoted to the behavior of the spherically symmetric false vacuum bubble surrounded by a thin wall. First, we develop the formalism[6] to treat the motion of the domain wall radiating out-going energy flux. In this place the spacetime outside the wall is assumed to be described as Vaidya's radiating Schwarzschild spacetime[7]. Next, we apply the formalism to the domain wall with the surface stress-energy tensor modified from the vacuum case. Here we assume the existence of such an unusual domain wall in the beginning and will not discuss how to generate it. We just comment that such an unusual surface stress-energy tensor may be realized as quantum expectation value in 'quantum field theory on the wall' and that one can perform the experiment if a certain kind of quantum technology is enough developed in future. Finally, in the case of a certain kind of surface stress-energy tensor, we find that a false vacuum bubble expands monotonically and at the same time the domain wall can move in the other side disconnected from *the laboratory side* through the worm-hole like the Einstein-Rosen bridge.

2

The equation of motion of a domain wall is deduced from the Einstein equation,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}. \quad (01)$$

Assuming the spherically symmetric system with a thin domain wall, the geometries of the inside and the outside of the wall are easily determined. The general form of the energy-momentum tensor is written as

$$T_{\mu\nu}(x) = S_{\mu\nu}(x^i)\delta(\eta) + \theta(-\eta)T_{\mu\nu}^-(x) + \theta(\eta)T_{\mu\nu}^+(x), \quad (02)$$

in a Gaussian normal coordinate system (x^i, η) based in the wall trajectory Σ , in which each x^i ($i = 1, 2, 3$) generates the coordinate system. Here $\eta < 0$ ($\eta > 0$)

region corresponds to the inside(outside) of the wall. The quantity $S_{\mu\nu}(x^i)$ is the surface stress-energy tensor on Σ and can be written as

$$S^{\mu\nu} = \sigma(\tau)U^\mu U^\nu - \zeta(\tau)(h^{\mu\nu} + U^\mu U^\nu), \quad (03)$$

due to the symmetry assumed here. The quantity U^μ is the unit tangent vector along the comoving observer on the wall, and $h^{\mu\nu}$ is the induced metric on the hypersurface of the wall trajectory. The quantities σ and ζ are regarded as the surface energy density and surface tension respectively.

In the inside region the energy-momentum tensor $T_{\mu\nu}^-$ is dominated by non-zero vacuum energy, $-\frac{3\chi^2}{8\pi G}g_{\mu\nu}$. The inside of the wall is described as the part of the de Sitter space,

$$ds^2 = -(1 - \chi^2 r^2)dt^2 + (1 - \chi^2 r^2)^{-1}dr^2 + r^2 d\Omega^2. \quad (04)$$

Outside the domain wall, due to the existence of the out-going null energy flux radiated by the wall, the energy-momentum tensor $T_{\mu\nu}^+$ can be written as

$$T_{\mu\nu}^+ = \frac{\partial_v m(v)}{4\pi r^2} l_\mu l_\nu, \quad (05)$$

and the outside geometry is described with Vaidya's radiating Schwarzschild metric[7],

$$ds^2 = -(1 - \frac{2Gm(v)}{r})dv^2 - 2dvdr + r^2 d\Omega^2, \quad (06)$$

where l^μ is out-going radial null vector and v is called retarded time coordinate (in fact, v coincides with the observer's time at infinity). The function $m(v)$ is a Bondi mass, and is determined by the basic equations in the following.

Following Israel[6], the most important equation of motion for the domain wall is given as a junction condition deduced from the Einstein equation (1). By the use of the extrinsic curvature tensor K_j^i , the junction condition reads

$$K_j^{i+} - K_j^{i-} = -4\pi G\sigma, \quad (07)$$

where the superscripts $+$ and $-$ denote the outside and the inside.

By evaluating the extrinsic curvature tensors on both sides of the wall, the eq.(7) has the form,

$$\beta_V - \beta_D = -4\pi G\sigma(\tau)R(\tau), \quad (08)$$

where the wall trajectory is $r = R(\tau)$, and $\beta_V \equiv \pm(\dot{R}^2 + 1 - \frac{2Gm}{R})^{1/2}$ and $\beta_D \equiv \pm(\dot{R}^2 + 1 - \chi^2 R^2)^{1/2}$. This equation is modified into the following convenient form,

$$\begin{aligned} \dot{R}^2 + V(R, m, \sigma) &= -1, \\ V &\equiv -\chi^2 R^2 - \left[\frac{2Gm + ((4\pi G\sigma)^2 - \chi^2)R^3}{8\pi G\sigma R^2} \right]^2. \end{aligned} \quad (09)$$

The above form coincides with the one by Blau et al exactly, although the situation is not the Schwarzschild vacuum case any longer.

Next another useful basic equation is obtained from the energy-momentum conservation law. After some manipulation the final expression is as follows

$$\frac{d(4\pi R^2 \sigma)}{d\tau} - \zeta \frac{d(4\pi R^2)}{d\tau} = \frac{\partial_v m}{(\beta_V + \dot{R})^2}. \quad (010)$$

The right-hand side corresponds to the energy flow which the domain wall radiate toward the outside. Here we ignore the energy flow toward the inside to simplify the discussion in the next section. The energy flow toward the inside can be included directly by the use of de Sitter-Schwarzschild version of Vaidya metric[8].

3

To determine the behavior of the domain wall, the set of the basic equations obtained in the preceding section is not complete. Usually the surface stress-energy tensor must obey the additional condition to reflect the underlying characteristics of the material of the wall. In the vacuum case expected in unified theories with first order phase transitions, the surface stress-energy tensor approximates to $S^i_j = \sigma \delta^i_j$ where $\dot{\sigma} = 0$ [2]. That is, $\sigma = \zeta = \text{constant}$ (called as

normal vacuum case in this talk). From this result, the eq.(10) reads $m = \text{const.}$ and the problem is reduced to the one-dimensional potential problem based on the eq.(9). The trajectories of the wall are described as horizontal dashed lines in $(r - m)$ diagram (Fig.1) and are classified into several types[2]. In the following, we generalize the above case to $\sigma \neq \zeta$ case in order to find the possibility of the creation of a child universe as mentioned in the first section. In contrast with the normal vacuum case, the eq.(10) becomes the non-trivial form,

$$\frac{dm}{dr} = 2\Delta r \left(\frac{dv}{d\tau} \right)^{-1}, \quad (011)$$

where the Planck units are introduced ($l_P m \rightarrow m$, $l_P^{-1} r \rightarrow r$, $4\pi l_P^3 \sigma \rightarrow \sigma, \dots$), and $\Delta = \sigma - \zeta \neq 0$. From the detailed examination of this equation, one can find that the wall trajectories are modified like the real lines shown in Fig.1 if $\Delta > 0$. Then in the case of a large enough Δ , as easily expected, the wall can move from the region of *the laboratory* side (I) to the other side (II) with its proper radius expanding monotonously, avoiding the potential barrier (the shaded region in Fig.1). One can also confirm the existence of such solutions by numerical evaluation.

For further understanding of the global structure of the spacetime, we show the conformal diagram of the outside spacetime in Fig.2 whose apparent horizons are described as the line (a) and the line (b). First of all, it should be noted that the former apparent horizon (a) is always null and coincides the future event horizon itself, while the latter one (b) becomes time-like during the existence of out-going negative energy null flux and does not coincide the past event horizon. From this diagram one can easily understand how the wall moves in the other side from *the laboratory* side.

We studied the formalism of the dynamics of a false vacuum bubble enclosed with a spherical thin wall radiating the out-going null energy flux. The formalism

applied to the wall whose surface stress-energy tensor is different from the normal vacuum one. The difference is characterized by $\Delta = \sigma - \zeta$. We showed that the gate for *the new world* is opened if Δ is large enough. As compared with the way by quantum tunneling, the treatment discussed here presents the real route to *the new world* in the real spacetime. However the surface stress-energy tensor of the wall must be well-devised (perhaps quantum mechanically).

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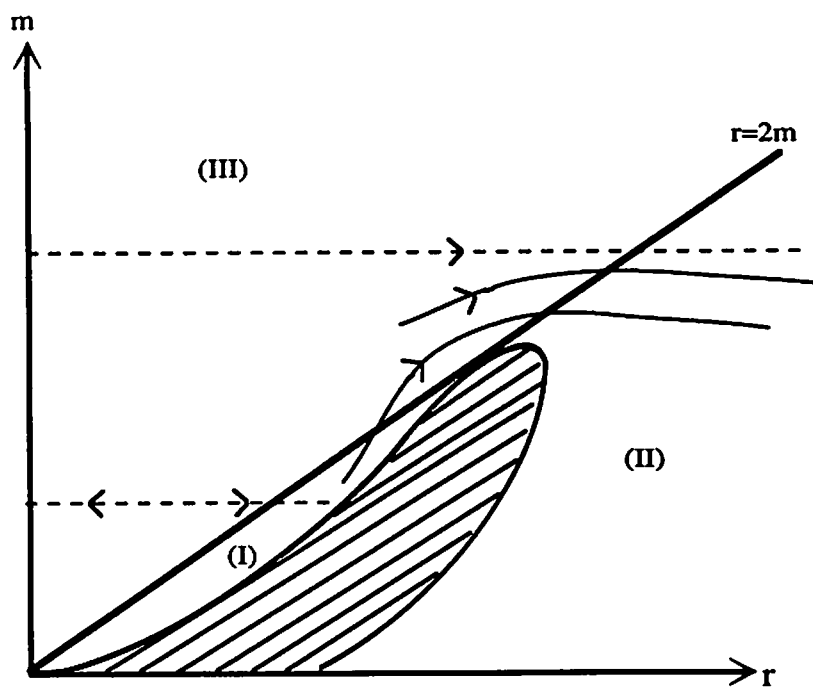


Fig.1

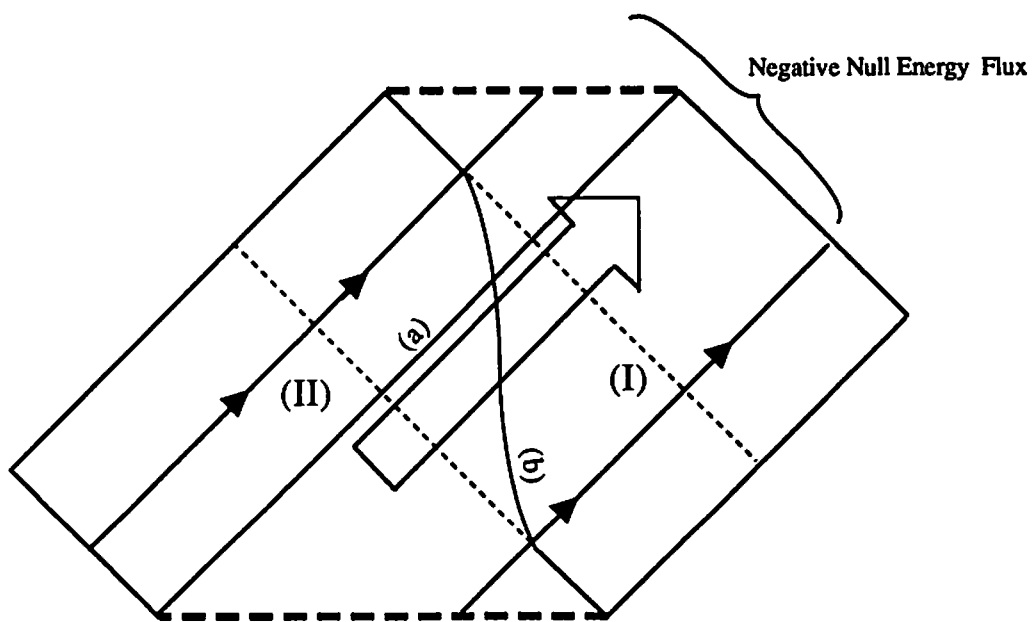


Fig.2

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**Quantum State During and After
 $O(4)$ -Symmetric Bubble Nucleation:
on Extension to the Case with Gravity**

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ABSTRACT

We perform the extension of our previous analysis of the quantum state during and after $O(4)$ -symmetric bubble nucleation to the case including gravitational effects. We find that there exists a simple relationship between the case with and without gravitational effects, and so the former reduces to the latter when we consider the quantum fluctuations of a conformally coupled scalar field which is massless before tunneling.

1. Introduction

Our universe is considered to have experienced not a few number of phase transitions in the early epoch. Those phase transitions are related to many kinds of interesting ideas about the senario of the early universe, such as inflationary universe senarios, baryogenesis and so on.

As one possibility, we proposed to revive the simple old senario of the inflationary universe by considering the particle creation during and after the phase transition [1]. Provided the problem of the entropy production can be solved somehow without considering the bubble collisions in the old senario of the inflation, it seems possible to generate our universe inside one nucleated bubble. However, since our knowledge about the quantum state of the field after $O(4)$ -symmetric bubble nucleation or the back reaction caused by the presense of the fluctuation degrees of freedom is far from sufficient, we can not judge at the moment whether this type of senario is possible or not.

Besides the inflatinary universe senario, what happens after the bubble nucleation is an interesting issue as a fundamental process relating to the problem of the matter production. Therefore it is necessary to develope a method to describe the quantum fluctuations during and after tunneling systematically. As a first step, we developed a method of multi-dimensional wave function [2](Paper 1) and applied it to the problem of $O(4)$ -symmetric bubble nucleation, neglecting the effects of gravity (*i.e.*, on flat space-time background) and assuming a simple model[3](Paper 2).

Since the esults obtained there are quite encouraging, they give us a good motivation to extend our analysis to more general cases. In this paper, we focus on how to incorrpolate the effect of gravity. It will be shown that our previous results for the flat background can be extended to the case with gravity as long as we consider the quantum fluctuations of a conformally coupled scalar field.

This paper is organized as follows. In section 2, we briefly review the method to determine the quantum state after $O(4)$ -symmetric bubble nucleation, which was discussed in paper 2. In section 3, we show that the quantum state after tunneling with and without gravitational effects can be examined quite similarly. In section 4, we give a brief conclusion.

2. False vacuum decay in flat space time

Lowest WKB order description

We consider a system whose Hamiltonian is given by

$$H_\sigma = \int d^3x \frac{1}{2} (P_\sigma^2 + (\nabla\sigma)^2) + V(\sigma) \quad (2.1)$$

where the potential of σ field, $V(\sigma)$, is assumed to have the shape shown in Fig. 1. We consider the situation where the state of the field σ is initially in the false vacuum indicated by σ_F . This state is not stable quantum mechanically. Therefore it decays to the configuration with true vacuum bubble through quantum tunneling. This process can be described by a nontrivial classical solution of the Euclidianized field equation with the boundary condition,

$$\sigma(x) \rightarrow \sigma_F \quad (|x| \rightarrow \infty). \quad (2.2)$$

We call it instanton. Assuming the $O(4)$ -symmetric configuration as

$$\sigma(x) = \sigma(\eta) \quad \eta = |x|, \quad (2.3)$$

the field equation becomes

$$\frac{d^2\sigma}{d\eta^2} + \frac{3}{\eta} \frac{d\sigma}{d\eta} = \frac{dV}{d\sigma}. \quad (2.4)$$

The equation can be solved under the boundary condition (2.2). The solution is shown in Fig. 2 schematically. We consider the case where, except for the wall region, the field configuration is essentially constant, i.e., $\sigma(x) = \sigma_F$ outside of the wall and $\sigma(x) = \sigma_T$ inside of the wall. It changes rapidly at the wall. We use the same notation $\sigma(x)$ to represent the instanton but it will not cause a confusion. This instanton can be analytically continued to the Lorentzian region at $\tau = 0 = t$. The field configuration in the Lorentzian region is given by $\sigma(\sqrt{-t^2 + \mathbf{x}^2})$ and the schematic picture is shown in Fig. 3.

Then the lowest order description of tunneling is given by the the classical solution along the path, C , in the complex plane of time t drawn in Fig. 4. In $\text{Re}(t) < 0$ region, the solution is given by the trivial false vacuum one; $\sigma = \sigma_F$, and in $\text{Re}(t) > 0$ region, it is given by the tunneling solution $\sigma(\sqrt{\tau^2 + \mathbf{x}^2})$ and its

analytic continuation to the Lorentzian region. These two solutions can be matched smoothly at a sufficient large $-\tau(= \text{Im}(t))$. Thus we can say that the quantum tunneling process, which results in a jump of the field configuration in the real time, can be smoothly connected in the complex plane of time.

Quantum fluctuation after tunneling

Next we consider the quantum fluctuation after the $O(4)$ -symmetric bubble nucleation. For simplicity, we examine a model whose Hamiltonian is given by

$$H = H_\sigma + H_\phi \quad (2.5)$$

where H_σ is the same as before and

$$H_\phi = \int d^3x \frac{1}{2} (P_\phi^2 + (\nabla\phi)^2) + \frac{1}{2} m^2(\sigma)\phi^2. \quad (2.6)$$

We do not consider the quantum fluctuations of the tunneling field σ itself. According to the results of Paper 1, the quantum state of ϕ after tunneling can be represented by a certain Bogoliubov state $|\Phi\rangle$, i.e.,

$$a_k|\Phi\rangle = 0 \quad (2.7)$$

where a_k is an annihilation operator, a set of which expands the field operator as

$$\hat{\phi}(x) = \sum_k a_k u_k(x) + a_k^* u_k^*(x), \quad (2.8)$$

with $\{u_k(x)\}$ being the corresponding set of mode functions. So the state can be determined if $\{u_k(x)\}$ is known. The mode functions for the quantum state after tunneling are obtained by the following procedure. First, we recall that the mode functions satisfy the linearized field equation. In the present case, it is

$$[\square - m^2(\sigma(x))] v_k^*(x) = 0. \quad (2.9)$$

The results obtained in Paper 1 indicate that we can get the negative frequency mode functions after tunneling by solving the Eq. (2.9) along C with the initial condition,

$$v_k^*(x) = \sqrt{\frac{k}{\pi}} e^{ikt} j_l(|k|r) Y_{lm}^*(\Omega), \quad (2.10)$$

which is the ordinary Minkowski negative frequency modes. However, the normalization of thus obtained mode functions is not guaranteed now. So we need

to multiply the mode functions by some constant factors as $u_k^*(x) = \alpha_k v_k^*(x)$, to satisfy the normalization condition; $(u_k, u_{k'}) = \delta(k - k')$.

3. Incorporation of the effects of gravity

As is discussed in the previous section, we consider the system which consists of two fields, i.e., the tunneling field σ and another field ϕ coupled to it. As we are going to include the effect of gravity, we assume the Lagrangean of the form,

$$\mathcal{L} = \mathcal{L}_{grav} + \mathcal{L}_\sigma + \mathcal{L}_\phi. \quad (3.1)$$

where

$$\begin{aligned} \mathcal{L}_{grav} &= -\sqrt{-g} \frac{1}{16\pi G} R, \\ \mathcal{L}_\sigma &= \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \nabla_\mu \sigma \nabla_\nu \sigma - V(\sigma) \right], \\ \mathcal{L}_\phi &= \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \left(m^2(\sigma) + \frac{1}{6} R \right) \phi^2 \right]. \end{aligned} \quad (3.2)$$

Here for simplicity we assume the conformal coupling for ϕ and $m^2(\sigma_F) = 0$.

First we need to construct a solution of the Euclidian equation of motion to obtain the lowest WKB order description. We also denote the instanton by $\sigma(x)$ and $g_{\mu\nu}(x)$ as before.

As for the $O(4)$ -symmetric bubble nucleation with the effect of gravity, the instanton is given in [4]. Consider the $O(4)$ -symmetry metric and field:

$$\begin{aligned} ds^2 &= n^2(\eta) d\eta^2 + a^2(\eta) d\Omega_{(3)}^2, \\ \sigma(x) &= \sigma(\eta), \quad \phi(x) = 0. \end{aligned} \quad (3.3)$$

The variation of the action with respect to σ gives the Euclidian field equation,

$$\ddot{\sigma} + \left(3\frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) \dot{\sigma} = \frac{dV}{d\sigma}, \quad (3.4)$$

where a dot means a derivative with respect to η . The variation with respect to n gives a constraint equation,

$$\dot{a}^2 - \frac{4\pi G}{3} a^2 \dot{\sigma}^2 = n^2 \left(1 - \frac{8\pi G}{3} a^2 V(\sigma) \right). \quad (3.5)$$

The variation with respect to a does not give any new equation. As a method to solve the above constrained system, we set a gauge condition. Here, we simply

choose it as $n(\eta) = a(\eta)$. As it is not necessary for the discussion below, we do not pursue the problem to get a detailed form of the solution for a given potential, $V(\sigma)$. Instead we consider the tunneling solution in the thin wall approximation. Indicating the wall position by $\eta = \bar{\eta}$, it is given by

$$\left\{ \begin{array}{l} a(\eta) = \frac{1}{H_F \cosh \eta}, \quad H_F^2 = \frac{8\pi G}{3} V(\sigma_F), \\ \sigma(\eta) = \sigma_F, \quad (\text{for } \eta < \bar{\eta}) \end{array} \right. \quad \left\{ \begin{array}{l} a(\eta) = \frac{1}{H_T \cosh(\eta - \eta_*)}, \\ H_T^2 = \frac{8\pi G}{3} V(\sigma_T), \quad \eta_* = \bar{\eta} - \text{Arccosh} \left(\frac{H_F}{H_T} \cosh \bar{\eta} \right), \quad (H_T \neq 0) \\ a(\eta) = \frac{1}{H_F e^{\bar{\eta}} \cosh(\bar{\eta})} e^{\eta}, \quad (H_T = 0) \\ \sigma(\eta) = \sigma_T, \quad (\text{for } \eta < \bar{\eta}) \end{array} \right. \quad (3.6)$$

When the thin wall approximation is valid, the spatial metric and the configuration of the tunneling field is identical to the false vacuum one in the region $\eta < 0$. We mention that when we can not neglect the wall thickness, this is not necessarily the case. We do not consider such a situation here. A schematical picture of this solution is shown in Fig. 5.

As was pointed out in [5], the coordinate η can not play the role of time because when we see the tunneling solution as a sequence of the spatial configurations assigned for each value of τ , it must connect the configuration at the false vacuum; $\sigma(\mathbf{x}; -\infty) = \sigma_F$, and that with the nucleated bubble after tunneling. The appropriate choice of the time coordinate for our problem is obtained by the following coordinate transformations;

$$\tau = e^{\eta} \cos r, \quad \rho = e^{\eta} \sin r.$$

Then the metric becomes

$$ds_{O(4)}^2 = \frac{a^2(\eta)}{\tau^2 + \rho^2} \left(d\tau^2 + d\rho^2 + \rho^2 d\Omega_{(2)}^2 \right); \quad \eta = \ln \sqrt{\tau^2 + \rho^2},$$

and the analytic continuation to the Lorentzian region is given by $t = -i\tau$. How these coordinates span the $O(4)$ -symmetric bubble is shown in Fig. 6. We can see

that the Euclidian region and the Lorentzian region is connected at the $\tau = 0 = t$ surface of these coordinates and the configuration corresponding to a sufficient large $-\tau$ is identical to the false vacuum one. So one may expect that the quantum state after tunneling can be obtained by solving the mode functions along the path C in Fig. 4 also in this case as before. It can be shown that this is the case but we have no room to explain it here. For details, see [6].

Before we solve the field equation

$$\left[g^{\mu\nu} \nabla_\mu \nabla_\nu - \left(m^2(\sigma(x)) + \frac{1}{6} R \right) \right] v_k^*(x) = 0 \quad (3.7)$$

along the path C , we point out a couple of remarkable properties of these coordinates.

The first point is that up to the conformal factor it is identical to the flat space-time metric. This means that the $O(4)$ -symmetric geometry can be transformed to the flat metric by the conformal transformation given by,

$$g_{O(4)\mu\nu} = \Omega^2 g_{f\mu\nu}, \quad (3.8)$$

where

$$\Omega(x) = \frac{a(\eta)}{e^\eta}; \quad \eta = \ln \sqrt{\tau^2 + \rho^2}, \quad (3.9)$$

which depends only on η .

The second is that the instanton is given by a function of only η just as in the case of flat background.

Now we are going to solve the field equation (3.7). Here we use the fact that Eq. (3.7) can be rewritten as

$$\left[\frac{1}{\sqrt{g_f}} \partial_\mu \sqrt{g_f} g_f^{\mu\nu} \partial_\nu - m^2(\sigma(x)) \Omega^2(x) \right] v_{fk}^*(x) = 0 \quad (3.10)$$

by the conformal transformation (3.8), with the conformal factor, (3.9), where

$$v_k^*(x) = \Omega^{-1}(x) v_{fk}^*(x).$$

Therefore we can construct a solution of Eq. (3.7) from that of Eq. (3.10), which is the same as the flat case with the substitution, $m^2(\sigma(x)) \rightarrow m^2(\sigma(x)) \Omega^2(x)$. As

the mass is assumed to be zero in the false vacuum, Eq. (3.10) reduces to that of a massless field in the flat spacetime there. Here we would like to say that the required initial condition of the negative frequency mode functions can be obtained by the conformal transformation of the Minkowski mode functions, $u_{Mk}^*(x)$, as

$$v_k^*(x) = \Omega(x)^{-1} u_{Mk}^*(x). \quad (3.11)$$

To show it, we need to check if the vacuum state defined by this set of mode functions agrees with the so-called Euclidian vacuum state.

We can see it easily that thus obtained mode functions at false vacuum side are regular everywhere on the Euclidian region corresponding to $-\infty < \tau < 0$ using the fact that the conformal factor reduces to

$$\Omega(x) = \frac{2}{H(1 + (\tau^2 + \rho^2))}, \quad (3.12)$$

in the false vacuum. Therefore this set of mode functions indeed represents the Euclidian vacuum state.

To summarize, the effect of in the present case of a conformally coupled scalar field is to replace the mode functions for the flat background with those multiplied by the inverse of the conformal factor (3.8) and $m^2(x)$ with $m^2(x)\Omega^2(x)$. Especially if we consider the case where the vacuum energy of the true vacuum is zero, the conformal factor reduces to $1/H_F e^{\eta} \cosh \bar{\eta}$.

4. Conclusion

We considered an extension of our previous analysis of the quantum state after $O(4)$ -symmetric bubble nucleation to the case with the effect of gravity. In the previous work (Paper 2), we considered the quantum fluctuations of a massless scalar field which couples to the tunneling field on the bubble wall. For this model, we got the expression for the energy momentum tensor and it behaves like a radiation field which decays proportional to the inverse fourth power of the distance from the wall.

Our conclusion of this paper is that the inclusion of gravity does not affect the nature of the quantum state after bubble nucleation as long as we consider a conformally coupled scalar field. This result is, of course, non-trivial but can be expected because our system is almost completely conformally trivial. So if we consider the case in which the conformal triviality is largely broken, we may observe some other interesting phenomena. Further research on this direction is required.

ACKNOWLEDGEMENTS

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Figure Captions

Fig. 1: The potential form of the tunneling field.

Fig. 2: The instanton in the flat spacetime.

Fig. 3: The analytic continuation to the Lorentzian region of the instanton.

Fig. 4: The tunneling path in the complex plane of time.

Fig. 5: The instanton with gravity and its analytic continuation.

Fig. 6: The way how the τ constant hypersurfaces span.

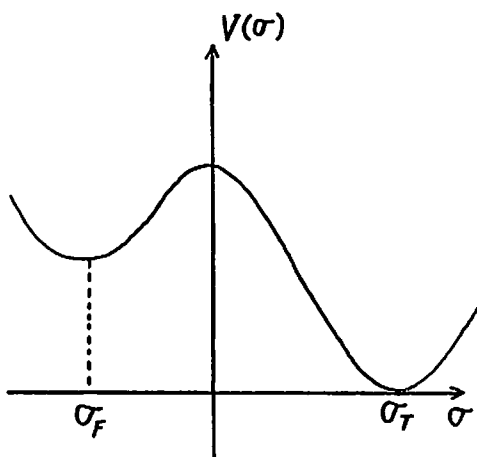


Fig.1

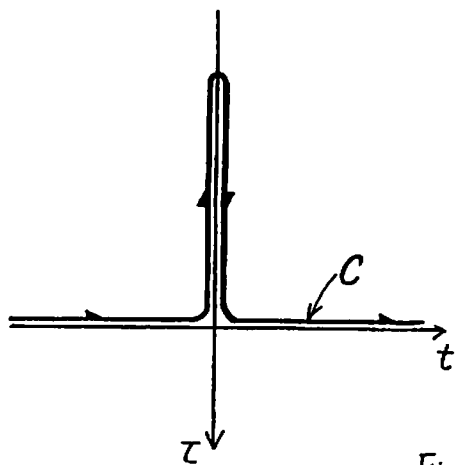


Fig.4

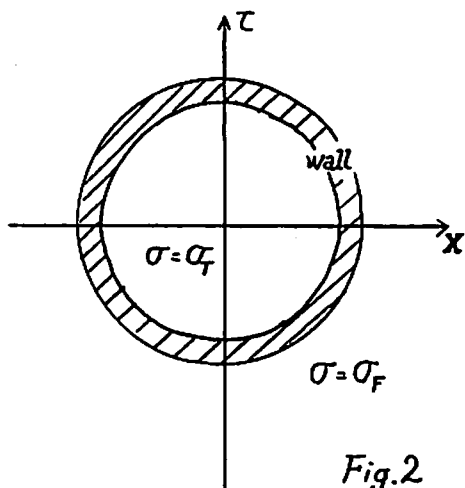


Fig.2

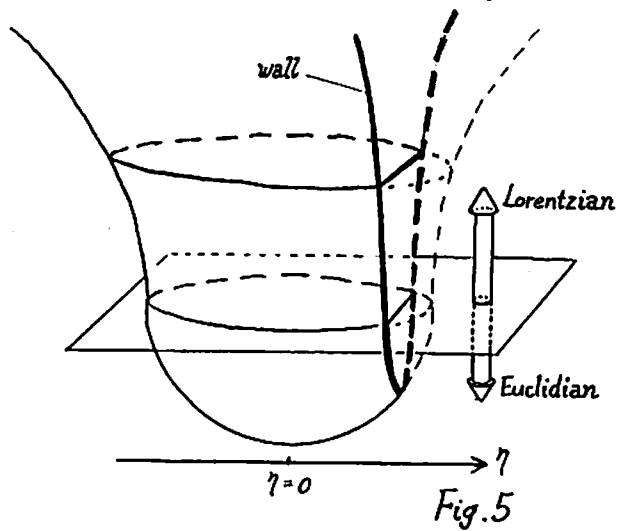


Fig.5

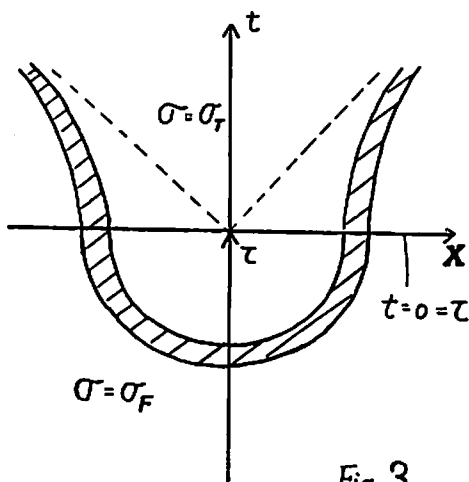


Fig.3

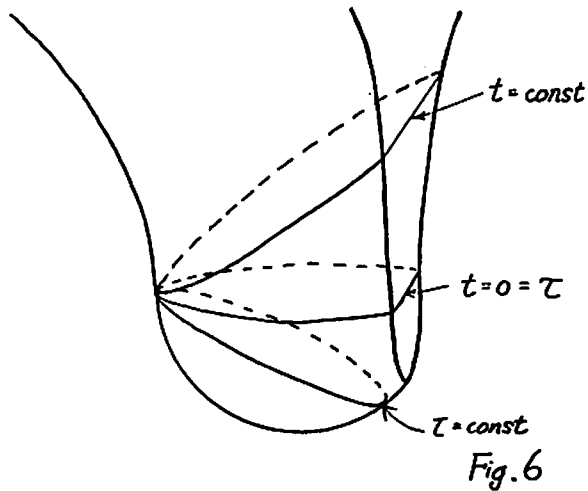


Fig.6

Self-dual Vortices in Curved Space

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Abstract

Self-dual vortex-type solitons both in Abelian Higgs model and Chern-Simons Higgs model coupled to gravity are reviewed. The Bogomol'nyi bound is derived and the soliton solutions for the systems are analyzed as well as the global structure of spacetime geometry of the solutions.

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Abelian gauge theories coupled to a complex scalar field in (2+1) spacetime dimensions produce solitonic excitations and, for the specific form of scalar potential, they saturate so-called Bogomol'nyi limit which shows interesting aspects in both theory and application. Examples are Nielsen-Olesen vortices in Abelian Higgs model [1,2], and topological and notopological Chern-Simons solitons in Chern-Simons Higgs model [3,4]. They have been examined also with the inclusion of gravity [5,6,7,8]. The role of such excitations in Abelian Higgs model have been studied in dual-transformed lattice model with the Higgs degrees frozen out [9]. The dual transformation of both Abelian Higgs model and Chern-Simons Higgs model was derived in continuum [10].

In this note, we will discuss the physics related to the self-dual solitons in (2+1) dimensional scalar electrodynamics coupled to Einstein gravity. We select two models described by the Lagrange densities

$$\mathcal{L}_{Max} = -\frac{1}{16\pi G}R - \frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + \frac{1}{2}g^{\mu\nu}\overline{D_\mu\phi}D_\nu\phi - V_{Max}(|\phi|) \quad (1)$$

$$\mathcal{L}_{CS} = -\frac{1}{16\pi G}R + \frac{\kappa}{2}\frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}}A_\mu\partial_\nu A_\rho + \frac{1}{2}g^{\mu\nu}\overline{D_\mu\phi}D_\nu\phi - V_{CS}(|\phi|), \quad (2)$$

where $\phi = |\phi|e^{i\Omega}$, the gauge covariant derivative is $D_\mu = \partial_\mu - ieA_\mu$. The form of scalar potential will be fixed later so as to give suitable Bogomol'nyi equations.

From now on let us derive the Bogomol'nyi bound and explore the classical self-dual soliton solutions Einstein Maxwell Higgs theory in Eq.(1) and Einstein Chern-Simons Higgs theory in Eq.(2). We take the Euler invariant defined by $E = \frac{1}{16\pi G} \int d^2x \sqrt{\gamma} {}^2R$ as the total energy of the system constituted by static matter fields and the stationary metric

$$ds^2 = N^2(dt + K_i dx^i)^2 - \gamma_{ij} dx^i dx^j, \quad (3)$$

where the components of the metric, N, K_i , and γ_{ij} , ($i, j = 1, 2$), are functions of spatial coordinates only.

After arranging the terms of energy, we obtain the Bogomol'nyi-type bound for Einstein Maxwell Higgs theory under $A_0 = 0$ gauge

$$\begin{aligned} E_{Max} = \int d^2x \sqrt{\gamma} \left\{ \frac{1}{4} \gamma^{ik} \gamma^{jl} (F_{ij} \mp \sqrt{\gamma} \epsilon_{ij} \sqrt{2V_{Max}}) (F_{kl} \mp \sqrt{\gamma} \epsilon_{kl} \sqrt{2V_{Max}}) \right. \\ \left. + \frac{1}{4} (\overline{D_i\phi} \mp i\sqrt{\gamma} \epsilon_{0ik} \gamma^{kl} D_l\phi) (D_j\phi \mp i\sqrt{\gamma} \epsilon_{0jm} \gamma^{mn} D_n\phi) \right. \\ \left. \pm \frac{\epsilon^{ij}}{2\sqrt{\gamma}} F_{ij} \left[\sqrt{2V_{Max}} - \frac{e}{2} (|\phi|^2 - v^2) \right] \right\} \quad (4) \end{aligned}$$

$$\pm \frac{ev^2}{2} \Phi$$

$$\pm \frac{i}{4} \int d^2x \epsilon^{ij} \partial_i (\bar{\phi} D_j \phi - \overline{D_j \phi} \phi),$$

and that for Einstein Chern-Simons Higgs theory

$$E_{CS} = \int d^2x \sqrt{\gamma} \left\{ \frac{e^2}{2} |\phi|^2 \left(A_0 \pm \frac{U}{e|\phi|} \right)^2 \right.$$

$$+ \frac{1}{4} \gamma^{ij} (\bar{D}_i \phi \mp i \sqrt{\gamma} \epsilon_{0ik} \gamma^{kl} \tilde{D}_l \phi) (\tilde{D}_j \phi \mp i \sqrt{\gamma} \epsilon_{0jm} \gamma^{mn} \tilde{D}_n \phi)$$

$$\mp \frac{e}{2} (|\phi|^2 - v^2) \left[\frac{\epsilon^{0ij}}{\sqrt{\gamma}} \partial_i (A^j - K_j A_0) + K A_0 + \frac{e^2}{\kappa} A_0 |\phi|^2 \right]$$

$$\mp e A_0 |\phi| \left[U - \frac{e^2}{2\kappa} |\phi| (|\phi|^2 - v^2) \right] \quad (5)$$

$$+ \left[W + \frac{ev^2}{2} K \left(\pm \frac{A_0}{v^2} (|\phi|^2 - v^2) + \frac{ev^2}{2\kappa} (1 - \phi_\infty^2) \right) - \frac{3}{32\pi G} K^2 \right] \Big\}$$

$$\pm \frac{ev^2}{2} \Phi$$

$$+ \int d^2x \partial_i \left[\epsilon^{0ij} \left(\pm \frac{ev^2}{2} K_j (A_0 \mp \frac{ev^2}{2\kappa} (1 - \phi_\infty^2)) \pm \frac{i}{4} (\bar{\phi} \tilde{D}_j \phi - \phi \overline{\tilde{D}_j \phi}) \right) \right],$$

where $\tilde{D}_i \phi = (\partial_i - ieA_i + ieK_i A_0) \phi$, $K = \frac{\epsilon^{ij}}{\sqrt{\gamma}} \partial_i K_j$, $V_{CS} = \frac{1}{2} U^2 + W$, Φ is the magnetic flux defined by

$$\Phi = -\frac{1}{2} \int d^2x \epsilon^{ij} F_{ij}, \quad (6)$$

and ϕ_∞ is a constant which is identified with the boundary value of scalar amplitude at spatial infinity divided by the vacuum expectation value v . Hence, for the specific scalar potentials

$$V_{Max} = \frac{e^2}{8} (|\phi|^2 - v^2)^2, \quad (7)$$

$$V_{CS} = \frac{e^4}{8\kappa^2} \left\{ |\phi|^2 (|\phi|^2 - v^2)^2 - \pi G (|\phi|^4 - 2v^2 |\phi|^2 + v^4 \phi_\infty^2)^2 \right\}, \quad (8)$$

the energy is proportional to the magnetic flux Φ and then multi-vortex solutions characterized by magnetic flux are supported in the static limit of these systems. In case of $U(1)$ Chern-Simons gauge theory, the flux carrying objects are charged because of the Gauss' law ($U(1)$ charge) $= -\frac{\kappa}{e} \Phi$. When the energy is not larger than meson mass times charge, this Chern-Simons Higgs model can support nontopological solitons. Another characteristic to distinguish Chern-Simons solitons from Nielsen-Olesen vortices is that they carry non-zero spin; the former carry the spin $J = \frac{1}{8\pi G} \oint_{|x| \rightarrow \infty} dx^i K_i$ and the latter are spinless.

From now on we fix the gauge for γ_{ij} as the conformal gauge

$$\gamma_{ij} = \delta_{ij} b(x^i) \quad (9)$$

and that for K_i as the Coulomb gauge $\nabla_i K^i = 0$ which enables K_i to be expressed by a function ψ

$$K^i = \frac{\kappa}{e^2 v^2} \frac{\epsilon^{0ij}}{\sqrt{\gamma}} \partial_j \ln \psi. \quad (10)$$

Solving the equations of motion, we obtain the Bogomol'nyi equation for the Maxwell system

$$\partial^2 \ln \frac{|\phi|^2}{v^2} = e^2 v^4 e^{h(z)+\bar{h}(\bar{z})} \left(\frac{\frac{|\phi|^2}{v^2} e^{-\frac{|\phi|^2}{v^2}}}{\prod_{p=1}^n |z - z_p|^2} \right)^{4\pi G v^2} \left(\frac{|\phi|^2}{v^2} - 1 \right) \mp 2\epsilon^{0ij} \partial_i \partial_j \Omega, \quad (11)$$

and two remaining equations for Chern-Simons system

$$\begin{aligned} \partial^2 \ln \frac{|\phi|^2}{v^2} &= \frac{e^4 v^4}{\kappa^2} e^{h(z)+\bar{h}(\bar{z})} \left(\frac{\frac{|\phi|^2}{v^2} e^{-\frac{|\phi|^2}{v^2}} \psi^{\phi_\infty^2-1}}{\prod_{p=1}^n |z - z_p|^2} \right)^{4\pi G v^2} \left(\frac{|\phi|^2}{v^2} - 1 \right) \\ &\times \left(\frac{|\phi|^2}{v^2} - 2\pi G v^2 \left(\frac{|\phi|^4}{v^4} - 2 \frac{|\phi|^2}{v^2} + \phi_\infty^2 \right) \right) \mp 2\epsilon^{0ij} \partial_i \partial_j \Omega, \end{aligned} \quad (12)$$

$$\partial^2 \ln \psi = -2\pi G \frac{e^4 v^6}{\kappa^2} e^{h(z)+\bar{h}(\bar{z})} \left(\frac{\frac{|\phi|^2}{v^2} e^{-\frac{|\phi|^2}{v^2}} \psi^{\phi_\infty^2-1}}{\prod_{p=1}^n |z - z_p|^2} \right)^{4\pi G v^2} \left(\frac{|\phi|^4}{v^4} - 2 \frac{|\phi|^2}{v^2} + \phi_\infty^2 \right) \quad (13)$$

where ∂^2 is the flat-space Laplacian.

Suppose that the following conditions hold: 1. There exist nonsingular and finite-energy solutions 2. the base manifold defined by the solution is smooth. We look for the rotationally symmetric solutions. The results are given as follows*. The former system described by Eq.(11) supports the following configurations carrying quantized magnetic flux $\Phi = \pm \frac{2\pi n}{e}$: (i) when $4\pi G v^2 n < 1$, the boundary value of scalar field takes a Higgs vacuum one and the underlying spatial manifold is asymptotically a cone with deficit angle $\delta = 8\pi^2 G v^2 n$, (ii) when $4\pi G v^2 n = 1$, Eq.(11) is reduced to a first-order equation such as

$$\frac{1}{n} \int \frac{d|\phi|}{|\phi|} \frac{1}{\sqrt{1 - e^{h+\bar{h}} \left(\frac{|\phi|^2}{v^2} e^{|\phi|^2/v^2} \right)^{4\pi G v^2}}} = \int \frac{dr}{r}, \quad (14)$$

and it includes an asymptotic cylinder with circumference $\frac{\sqrt{8\pi^3 G}}{e} n$ when an arbitrary number $e^{h+\bar{h}} = 8\pi G v^2 n^2 e^{-4\pi G v^2}$ and a two sphere with the circumference along the tropical line

*For the detailed analysis see our papers in Refs[6,8]

$\frac{\sqrt{8\pi^3 G}}{e} n$ when $e^{h+\bar{h}} > 8\pi G v^2 n^2 e^{-4\pi G v^2}$. The latter system described by Eq.(12) and Eq.(13) include the solutions only when ϕ_∞ is 1 or 0. When $\phi_\infty = 1$ and $4\pi G v^2 n < 1$, these spinning vortices carry quantized flux (or charge), $\Phi = -\frac{Q}{e} = \pm \frac{2\pi n}{e}$, and the corresponding asymptotic manifold is a cone with deficit angle $\delta = 8\pi^2 G v^2 n$. When $\phi_\infty = 0$ specifically $|\phi| \sim r^{-\epsilon}$ ($\epsilon > 0$) for large r , the model supports the nontopological solitons with spin J and charge (or flux) $Q = -\kappa\Phi = \pm \frac{2\pi\kappa}{e} (n + \epsilon + \frac{2Ge^2 v^2}{\kappa} J)$. In this case, (i) if $\left| \frac{2Ge v^2}{\kappa} \right| < 1$, these solutions constitute asymptotic cone with deficit angle $\delta = 8\pi^2 G v^2 (n + \epsilon + \frac{2Ge^2 v^2}{\kappa} J)$, (ii) if $\left| \frac{2Ge v^2}{\kappa} \right| = 1$, both asymptotic cylinder and two sphere are constituted according to the value of $e^{h+\bar{h}}$ similar to the Einstein Maxwell Higgs case. When we consider spinning cosmic string parallel to along a spatial axis, an issue attracting attention is whether closed timelike curves are forbidden or not [15]. Examining the asymptotic form of metric components for large r , we notice that, for the cases of open space, self-dual Chern-Simons solitons with no overlapping are not those which show such acausal behavior.

Another way of understanding the role of vortices in $U(1)$ gauge theory is to recapitulate it in dual formulation and elicit the physically relevant aspects such as phase transitions induced by vortices and classical dynamics of vortices [9,10]. From now on we reformulate the theories of our interest to dual transformed versions. Let us begin with taking into account the path integral

$$Z = \int [dg_{\mu\nu}] [dA_\mu] [|\phi| d|\phi|] [d\Omega] e^{i \int d^3x \sqrt{g} \mathcal{L}}, \quad (15)$$

where the Lagrange density $\mathcal{L}$ is either $\mathcal{L}_{Max}$ or $\mathcal{L}_{CS}$. The procedure to achieve the dual transformation for scalar electrodynamics is divided into three Lagrangian-independent steps and one Lagrangian-dependent Gaussian integraton. The first step is to linearize the interaction between scalar field and gauge field by introducing an auxiliary field C_μ

$$\begin{aligned} & \exp \left\{ i \int d^3x \sqrt{g} \frac{1}{2} g^{\mu\nu} |\phi|^2 (\partial_\mu \Omega - e A_\mu) (\partial_\nu \Omega - e A_\nu) \right\} \\ &= \int [dC_\mu] \prod_x \frac{g^{\frac{1}{4}}}{|\phi|^3} \exp \left\{ i \int d^3x \sqrt{g} \left[-\frac{g^{\mu\nu}}{2|\phi|^2} C_\mu C_\nu + g^{\mu\nu} C_\mu (\partial_\nu \Omega - e A_\nu) \right] \right\}. \end{aligned} \quad (16)$$

The second step is to divide the path integral measure for the phase of the scalar field into two contributions

$$[d\Omega] = [d\Theta][d\eta]. \quad (17)$$

Here the first term Θ , expressed by a multi-valued function such as

$$\Theta(t, \vec{x}) = \sum_p (-1)^a \tan^{-1} \frac{x^2 - x_p^2}{x^1 - x_p^1}, \quad (18)$$

describes the configuration of vortices and antivortices, and the second term η , a single-valued function, represents the fluctuation around a given vortex sector. Then η -integration gives a restriction for C_μ

$$\int [d\eta] \exp \left\{ i \int d^3x \sqrt{g} g^{\mu\nu} C_\mu \partial_\nu \eta \right\} \approx \frac{1}{\sqrt{g}} \delta(\nabla_\mu C^\mu). \quad (19)$$

If we note that the auxiliary field C_μ is classically nothing but the $U(1)$ current, the above condition can be regarded as depicting a quantum conservation of current. Particularly in Chern-Simons gauge theories ($\mathcal{L} = \mathcal{L}_{CS}$), it is related to the Bianchi identity. The third step is the introduction of the dual gauge field H_μ by rewriting a part of path integral as

$$\int [dC_\mu] \frac{1}{\sqrt{g}} \delta(\nabla_\mu C^\mu) \cdots = \int [dH_\mu] [dC_\mu] \delta(\sqrt{g} C^\mu - C \epsilon^{\mu\nu\rho} \partial_\nu H_\rho) \cdots, \quad (20)$$

here the constant C is chosen to be 1 for the Maxwell Higgs theory, Eq.(1), and to be $\frac{\kappa}{e}$ for the Chern-Simons Higgs theory, Eq.(2), for later convenience. Now that the action at this stage is quadratic both in the auxiliary field C_μ and in the gauge field A_μ , the path integrals for those fields are Gaussians and can be carried out in closed form. We now obtain the dual-transformed version of Einstein Maxwell Higgs theory

$$\begin{aligned} Z = & \int [g dg_{\mu\nu}] [dH_\mu] [|\phi|^{-2} d|\phi|] [d\Theta] [d\chi] \\ & \exp i \int d^3x \sqrt{g} \left\{ -\frac{1}{16\pi G} R + \frac{1}{2} g^{\mu\nu} \partial_\mu |\phi| \partial_\nu |\phi| - V(|\phi|) \right. \\ & \left. - \frac{1}{4|\phi|^2} g^{\mu\rho} g^{\nu\sigma} H_{\mu\nu} H_{\rho\sigma} + \frac{1}{2} \frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}} H_{\mu\nu} \partial_\rho \Theta + \frac{e^2}{2} g^{\mu\nu} (H_\mu - \frac{1}{e} \partial_\mu \chi) (H_\nu - \frac{1}{e} \partial_\nu \chi) \right\}, \end{aligned} \quad (21)$$

and that of Einstein Chern-Simons Higgs theory

$$\begin{aligned} Z = & \int [g^{\frac{3}{2}} dg_{\mu\nu}] [dH_\mu] [|\phi|^{-2} d|\phi|] [d\Theta] \\ & \exp i \int d^3x \sqrt{g} \left\{ -\frac{1}{16\pi G} R + \frac{1}{2} g^{\mu\nu} \partial_\mu |\phi| \partial_\nu |\phi| - V(|\phi|) \right. \\ & \left. - \frac{\kappa^2}{4e^2|\phi|^2} g^{\mu\rho} g^{\nu\sigma} H_{\mu\nu} H_{\rho\sigma} - \frac{\kappa}{2} \frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}} H_\mu \partial_\nu H_\rho + \frac{\kappa}{2e} \frac{\epsilon^{\mu\nu\rho}}{\sqrt{g}} H_{\mu\nu} \partial_\rho \Theta \right\}, \end{aligned} \quad (22)$$

where a single-valued field χ is introduced to guarantee the gauge invariance of Maxwell Higgs theory, and $H_{\mu\nu} = \partial_\mu H_\nu - \partial_\nu H_\mu$. Note that this transformation can be generalized to almost

any Abelian theory in any number of dimensions. In $(D + 1)$ dimensions we have only to replace dual vector field to an antisymmetric tensor field of rank $(D - 1)$ $H_{\mu_1\mu_2\cdots\mu_{D-1}}$ defined by

$$C^\mu = \frac{\epsilon^{\mu\nu_1\cdots\nu_D}}{\sqrt{g}} \partial_{\nu_1} H_{\nu_2\nu_3\cdots\nu_D}, \quad (23)$$

and, for the theory with couplings to an arbitrary number of Abelian gauge fields, we may appropriately introduce several dual fields of which the number is the same as that of conserved currents.

From the dual-transformed actions in Eq.(21) and Eq.(22), we easily read the contents of gauge bosons in Higgs phase; an even-parity photon of mass ev in massive vector theory (Eq.(21)) and an odd-parity helicity-one photon of mass $\frac{e^2 v^2}{\kappa}$ in topologically massive gauge theory from (Eq.(22)) [12]. Gauge coupling e is inversely multiplied to the interaction term between the gauge field and the Higgs field, which looks like the strong coupling expansion being done. However, when Higgs effects are important, one must take into account the nonpolynomial interaction in the Maxwell-like term and the Jacobian in Higgs measure. Note that the path integral measure of gravitational field has nontrivial Jacobian compared with that of original theory, and it depends on the form of gauge interaction.

A final comment is in order. While we obtain all possible rotationally symmetric solutions of Bogomol'nyi equations, the stability of those solutions, multi-soliton solutions with zero-mode counting, the existence of such solutions away from the Bogomol'nyi limit [13] remain to be clarified. Some aspects of classical self-dual solitons were discussed using the dual transformed form of the model and the quantum field theoretic issues such as the phase transition structure need further study. As the particular choice we made for scalar potentials in Eq.(7) and Eq.(8) was understood by a consideration of supersymmetric models in flat spacetime [14] a supergravity version of the model may be interesting to investigate.

Acknowledgments

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Effects of the Back Reaction in the Brans-Dicke-Dilaton Cosmology*

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1 Introduction

In the Einstein gravity, quantum corrections of conformally-invariant quantum fields change the field equations drastically in the 4-dimensional space time. However, behaviours of the real universe are very well approximated by the classical field equations. We think that some symmetry and/or mechanism, which soften effects of the quantum back reaction, must exist in unified theory of elementary particles and gravitation. Though to search such symmetry is a very interesting problem, many technical preparations are required in order to answer it. Therefore before considering this problem, we should examine whether the simple semi-classical approach is improved in other models of gravity and some modification of the Einstein gravity would not settle the defect of the semi-classical approach.

In this talk, we shall study effects of the back reaction in the Brans-Dicke-Dilaton (BDD) gravity. (The Brans-Dicke gravity is very similar to the dilaton gravity, therefore we call them the BDD gravity.) In the framework of the BDD gravity, the extended inflationary cosmology was proposed [1]. Contrary to the old inflation model, it predicts a power-law inflation and gives a natural solution to the "graceful exit" problem. Therefore, we shall examine the extended inflationary model in the presence of the back reaction and see if the desirable behaviour of the classical solution remain unchanged on some conditions which enable the semi-classical solutions to behave like classical ones in the Einstein gravity.

2 The BDD theory

Action of the BDD theory [2] is given by

$$S = \int d^4x \sqrt{-g} \left[\Phi R + \omega \left(\frac{\partial_\mu \Phi \partial^\mu \Phi}{\Phi} \right) \right],$$

where Φ is the BDD field and ω is a dimensionless constant which parametrizes the theory. From the action, field equations are obtained in the semi-classical approximation as,

$$\begin{aligned} -\nabla_{(\mu} \nabla_{\nu)} \Phi + g_{\mu\nu} \nabla_\gamma \nabla^\gamma \Phi + \omega \left\{ \frac{g_{\mu\nu} (\partial_\gamma \Phi \partial^\gamma \Phi)}{2\Phi} - \frac{\partial_\mu \Phi \partial_\nu \Phi}{\Phi} \right\} \\ - \Phi \left(R_{\mu\nu} - \frac{g_{\mu\nu}}{2} R \right) = 8\pi (T_{\mu\nu}^{class} + \langle T_{\mu\nu} \rangle_{ren}), \\ 2\omega \nabla_\gamma \nabla^\gamma \Phi - \omega \frac{\partial_\gamma \Phi \partial^\gamma \Phi}{\Phi} - R\Phi = 0, \end{aligned}$$

where the convention of Birrel and Davise [3] is used, and $T_{\mu\nu}^{class}$ is the classical part of the stress-energy tensor, and $\langle T_{\mu\nu} \rangle_{ren}$ is the renormalized stress-energy tensor of quantum fields. If we consider conformally-invariant quantum fields in the conformally flat spacetime, $\langle T_{\mu\nu} \rangle_{ren}$ have well-defined expressions. The symbolical form of $\langle T_{\mu\nu} \rangle_{ren}$ is the following

$$\langle T_{\mu\nu} \rangle_{ren} = -\frac{\alpha}{6} {}^{(1)}H_{\mu\nu}(\partial^2 R, R^2) + \beta {}^{(3)}H_{\mu\nu}(R^2),$$

where ${}^{(1)}H_{\mu\nu}(\partial^2 R, R^2)$ is a polynomial composed of 2-derivatives of curvature tensors and of squares of curvature tensors, and ${}^{(3)}H_{\mu\nu}(R^2)$ is a polynomial composed only of squares of curvature tensors, and α, β are parameters which depend on the spin of quantum fields.

In the rest of the discussion, we shall restrict ourselves to the homogeneous, isotropic and spatially flat spacetime. Then the spacetime is described by the following metric,

$$ds^2 = dt^2 - a^2(t) \sum_{i=1}^3 (dx^i)^2.$$

The BDD field Φ has the dimension $(energy)^2$. It is convenient to use the following parametrization,

$$\Phi = G^{-1} e^{-2\phi}, \quad a = e^h.$$

We use the "Planck mass" $M_p^2 = \frac{3}{8\pi G}$ as unit for the energy density ρ and the pressure p in the classical part of the stress-energy tensor,

$$\rho = T^{class}_{00} = M_p^4 \tilde{\rho}, \quad p = -T^{class}_{ii} = M_p^4 \tilde{p}.$$

We also use unit $M_p^{-1}T$ for time t ,

$$t = (M_p^{-1}T)\tau,$$

where T is a numerical constant and we take $T=10^4$ in the later numerical

calculations.

At the classical level, i.e., by ignoring the contribution from $\langle T_{\mu\nu} \rangle_{ren}$, we need the boundary conditions, e.g. $h(\tau_0)$, $\phi(\tau_0)$, $\phi'(\tau_0)$ and $\tilde{\rho}(\tau_0)$. We take $h(\tau_0) = \phi(\tau_0) = 0$, without loss of generality.

In the universe with non-vanishing cosmological constant $\tilde{\rho} = -\tilde{p} = \tilde{\rho}_0$, the solution to the classical field equation is

$$G\Phi = e^{-2\phi} = \left[1 + \frac{T\sqrt{\tilde{\rho}_0}}{\Omega}(\tau - \tau_0) \right]^2, \quad a = e^h = \left[1 + \frac{T\sqrt{\tilde{\rho}_0}}{\Omega}(\tau - \tau_0) \right]^{\omega + \frac{1}{2}},$$

where $\Omega^2 = \frac{(3+2\omega)(5+6\omega)}{12}$. This solution gives the extended inflationary senario in the BDD gravity, and the scale factor increase in power-law for large τ , in contrast with the exponential increase in the old inflation in the Einstein gravity. We shall examine if the desirable behaviour of the classical solution remains unchanged in the presence of the quantum back reaction.

3 Qualitative study of the field equations

Including effects of the quantum back reaction, we cannot solve the equations exactly. However, we can examine the qualitative behaviours of the solutions. In Ref. [4], it was shown that the Einstein equations can be mapped to a potential problem with (anti-)dissipation. We found that this argument is useful also for the qualitative study of the cosmological solutions in the BDD gravity.

The semi-classical equations can be written as,

$$\begin{aligned} x &\equiv \left| \frac{a'}{a} \right|^{\frac{1}{2}} = |h'|^{\frac{1}{2}}, & \varepsilon &\equiv \text{sgn}(a'), \\ m &= \frac{3}{T^2} \sum_{particle} \alpha, & l &= \frac{3}{T^2} \sum_{particle} \beta, \\ \phi' - (\phi')^2 + 2\varepsilon x^2 \phi' + \frac{2}{\omega}(\varepsilon x x' + x^4) &= 0, \\ \frac{d}{d\tau} \left\{ \frac{1}{2}(x')^2 + V(x) \right\} &= -3\varepsilon x(x')^2 + \frac{3}{8m} \varepsilon(1+\kappa) T^2 \tilde{\rho} \\ &\quad + \frac{e^{-2\phi}}{4m} \left\{ \phi' x^2 + \frac{2\omega}{3x^3} (\phi')^2 x' + \frac{2\varepsilon}{x} \phi' x' \right\}, \\ V(x) &\equiv \frac{1}{8m} \left\{ -\frac{T^2 \tilde{\rho}}{x^2} - e^{-2\phi} x^2 + \frac{l}{3} x^6 \right\}, \end{aligned}$$

where $\kappa = \frac{1}{3}$ and $\tilde{\rho} = \frac{c}{a^4}$ for radiation dominat universe, and $\kappa = -1$ and $\tilde{\rho} = \tilde{\rho}_0$ for the universe with the cosmological constant. We assume that the energy density of (classical part of) the matters is sufficiently low

compared with the energy density given by the Planck mass, i.e., $1 > 4lT^2\tilde{\rho}_0$. Then the potential $V(x)$ have the form in Fig. 1, and there are extrema A and B.

In the Einstein limit ($\omega \rightarrow \infty$), it is easily seen that the extremum A essentially corresponds to the classical solution [4]. On the other hand, in the BDD gravity, it is not obvious if the system staying at A behaves like the classical one, because the potential depends on ϕ . Furthermore, the (anti-)dissipative effect is not explicit because of the terms depending on ϕ' . These points will be examined by the numerical method. However, we can expect from the form of the potential that the solution which corresponds to the extremum A are unstable for $m > 0$ and stable for $m < 0$. The equation of ϕ dose not contain higher-derivative terms, and therefore stable behaviour of ϕ is expected for stable x .

4 Numerical calculations

Then we shall give the solutions obtained by the numerical method. As the boundary condition, we input the values of $\phi'(\tau_0)$, $h'(\tau_0)$ and $h''(\tau_0)$, and fix the value of the cosmological constant as $\tilde{\rho}_0 = \frac{1}{T^2}$.

$$4.1 \quad m > 0, \quad \phi' = -\frac{1}{\Omega}, \quad h' = \frac{\omega + \frac{1}{2}}{\Omega}, \quad h'' = -\frac{\omega + \frac{1}{2}}{\Omega^2} \quad (\text{Fig. 2})$$

We show the result for $960\pi^2 T^2 m = 18$, $960\pi^2 T^2 l = 62$ and $\omega = 5$. The value of the boundary condition is that from the classical solution. The solution exhibit singular behaviour within the time $t < \frac{1}{M_p}$, even for the classical

boundary condition. This result agrees with the qualitative discussion.

4.2 $m < 0$

We show the results for $960\pi^2 T^2 m = -18$, $960\pi^2 T^2 l = 62$ and $\omega = 5$.

$$4.2.1 \quad \phi' = -\frac{1}{\Omega}, \quad h' = \frac{\omega + \frac{1}{2}}{\Omega}, \quad h'' = -\frac{\omega + \frac{1}{2}}{\Omega^2} T \quad (\text{Fig. 3})$$

This boundary condition corresponds to small oscillation around the extremum A. Expansion rate is close to the classical one, and we observe $\langle h'/h'_{\text{class}} \rangle \sim 1$. However both a and ϕ oscillate with the period $\sim \frac{T}{M_p} 2\pi\sqrt{-m}$.

$$4.2.2 \quad \phi' = -\frac{1}{\Omega}, \quad h' = \frac{\omega + \frac{1}{2}}{\Omega}, \quad h'' = -\frac{\omega + \frac{1}{2}}{\Omega^2} T^2 \quad (\text{Fig. 4})$$

This boundary condition corresponds to large oscillation around the extremum A. The solution behaves quite differently from the classical solution. Strong effect of dissipation is observed.

5 Stability of the Minkowski spacetime

In the previous sections, we showed that the sign of the higher-derivative term in the stress-energy tensor is important for the stability problem of the cosmological solutions. Stability of the Minkowski spacetime has been also discussed in the presence of the quantum back reaction in the Einstein gravity [5, 4]. It is important to see if the above regular or singular behaviour of the cosmological solutions is related with the stability of the Minkowski spacetime in the BDD gravity.

To this end we shall follow the discussion by Horowitz and Wald [5]. We parameterize the fields $g_{\mu\nu}$ and Φ as

$$g_{\mu\nu} = (1+w)\eta_{\mu\nu}, \quad \Phi = G^{-1}(1-2\varphi).$$

It is not difficult to show that the general solution to the linearized field equations is given as follows,

$$w = r_\mu x^\mu + r + \frac{1}{2\omega+3}g(x) + f(x),$$

$$\varphi = q_\mu x^\mu + q + \left\{1 - \frac{3}{2\omega(2\omega+3)}\right\}g(x) - \frac{3}{2\omega}f(x),$$

where r_μ , r , q_μ and q are constants, and functions $f(x)$ and $g(x)$ satisfy the following equations,

$$\left[\eta^{\mu\nu}\partial_\mu\partial_\nu - \frac{2\omega+3}{2\omega K}\right]f(x) = 0, \quad K = \frac{8\pi T^2}{3}m,$$

$$\eta^{\mu\nu}\partial_\mu\partial_\nu g(x) = 0.$$

In the solution of the metric, the term $r_\mu x^\mu + r$ can be eliminated by a gauge transformation. For $m > 0$, the mass squared of $f(x)$, $K = \frac{8\pi T^2}{3}m$, is negative, and because of that the Minkowski spacetime is unstable. On the other hand for $m < 0$, $f(x)$ is massive and it describes stable fluctuation. However, in contrast with the Einstein gravity, there is always massless fluctuation which is described by $g(x)$. Its appearance is naturally expected because there is no potential term for the BDD field. To see whether $g(x)$ causes unstability of the Minkowski spacetime, we have to examine higher-order terms in an inhomogenous model. Also the linear term in the solution of the BDD field, $q_\mu x^\mu + q$, has physical meaning, and the Minkowski spacetime is generally unstable to the "linear-dilaton" spacetime. We have checked this observation by numerical method.

6 Conclusion

In this talk, we studied effects of the back reaction in the BDD cosmology. Main motivation is to see if quantum back reaction changes (desirable) behaviours of the classical solutions in models of gravity other than the Einstein gravity.

According to the sign of the parameter in the stress-energy tensor, there appears instability in the cosmological solutions and even stable solutions oscillate with the period $\sim \frac{T}{M_p} 2\pi\sqrt{-m}$. However, by fine tuning of the boundary condition, we can get a solution that behaves like the classical solution. This conclusion is similar to that of the Einstein gravity.

The stability of the Minkowski spacetime was also discussed. For $m > 0$, there is an unstable fluctuation and the Minkowski spacetime is quite unstable to singular spacetime. For $m < 0$, there are massless and massive fluctuation, and in the most cases the Minkowski spacetime is unstable to the regular "linear-dilaton" spacetime.

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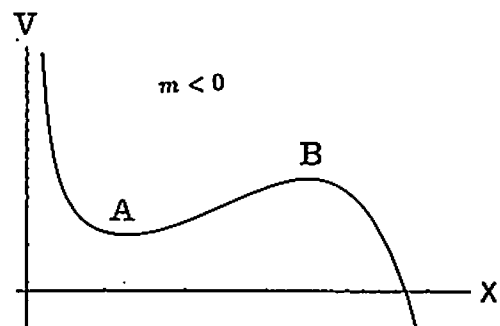
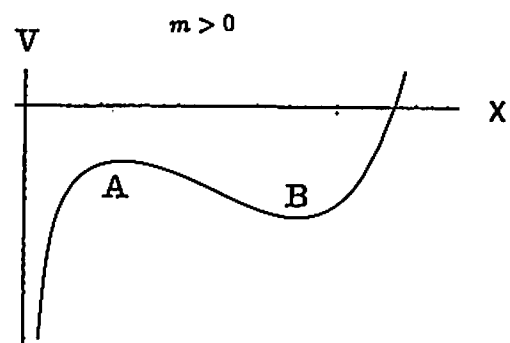


Fig.1

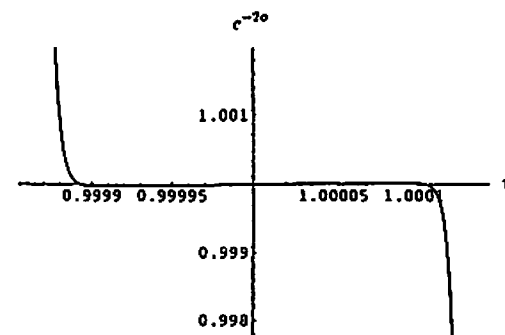
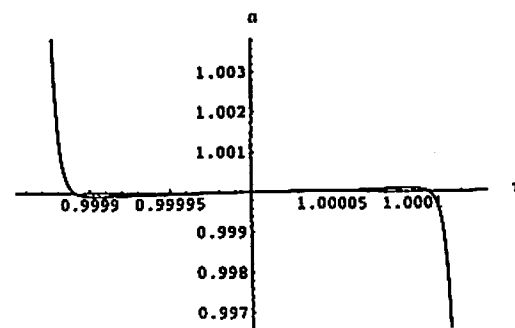


Fig.2

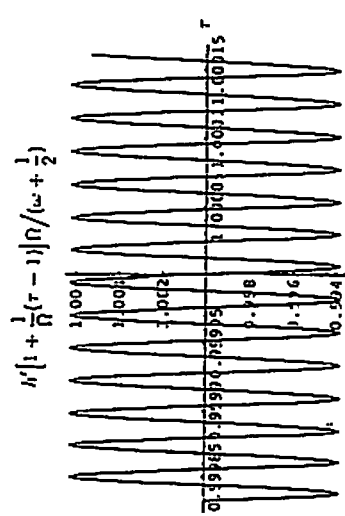
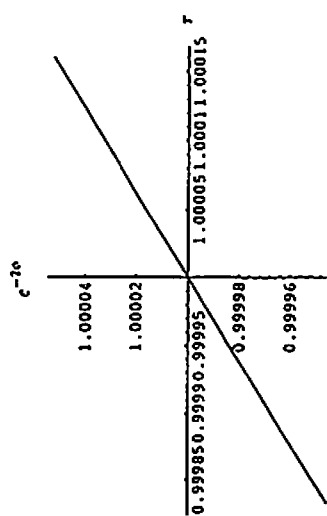
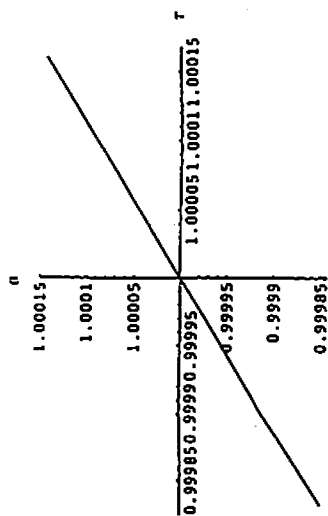


Fig. 3

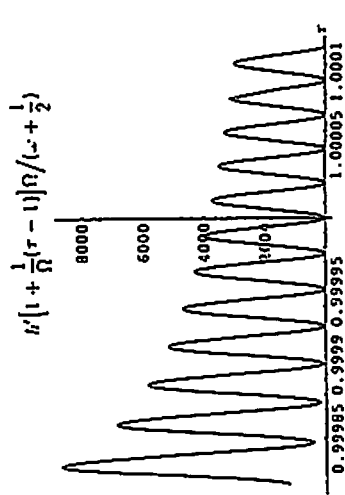
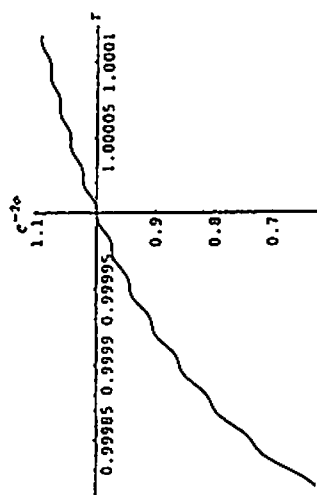
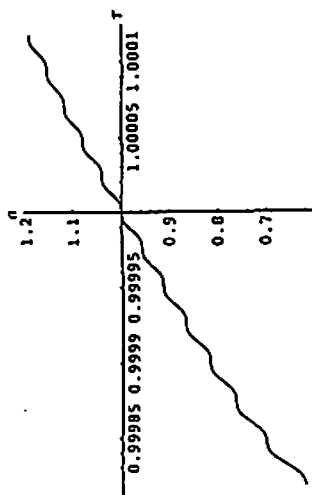


Fig. 4

Space-time Geometry in Exactly Solvable Quantum Dilaton Gravity

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Abstract

A concise description is given of our recent work on an exactly solvable model of two dimensional quantum dilaton gravity of the CGHS type. The main achievement is to show how one can extract space-time geometries in a fully quantum mechanical manner. In particular, we demonstrate that generation of black hole geometries with and without naked singularity by an in-falling matter shock wave can be explicitly described in our framework.

1. Introduction

Recent progress in the matrix model and the Liouville approaches to two dimensional quantum gravity has made it possible to incorporate full quantum fluctuations of the gravitational field in the calculation of the partition functions and the correlation functions. In spite of this success, more intuitive information, for example that of the structure of the space-time, has yet eluded us in these approaches and the understanding of how various events take place due to the gravitational interaction is still lacking.

The advent of the CGHS model [1], a classically solvable model of dilaton gravity in two dimensions, has given us an exciting opportunity to improve on this situation. This model contains such an interesting classical solution as the one describing the formation of a black hole by an in-falling matter shock wave. Moreover, semi-classical analysis incorporating the one loop anomaly in the matter sector suggested the existence of Hawking radiation. These properties indicate that this model is an excellent laboratory for the study of quantum gravity.

Full understanding of this intriguing model, however, is not an easy task. It turned out that the semi-classical approximation breaks down when the mass of the black hole becomes small and attempts at full-fledged quantum analysis have met with a number of difficulties.

The aim of our work is to go beyond such existing analyses both technically and conceptually. Specifically, we wish (i) to perform an exact operator quantization of a model

of the CGHS type, (ii) to show how one can extract space-time structure in such a framework, and eventually (iii) to understand Hawking radiation fully quantum mechanically (*i.e.* not semi-classically). In this article, we shall focus upon the item (ii). For details, see [2]-[4].

2. The Model

The Action

The original action of the CGHS model [1] is given by

$$S_{CGHS} = \frac{1}{\gamma^2} \int d^2\xi \sqrt{-g} \left\{ e^{-2\varphi} \left[-4g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi - R_g + 4\lambda^2 \right] + \frac{1}{2} g^{\alpha\beta} \sum_{i=1}^N \partial_\alpha f^i \partial_\beta f^i \right\} \quad (1)$$

where g, R_g, φ and f^i are the metric, the scalar curvature, the dilaton and N massless matter fields respectively. λ is the dilatonic cosmological constant, and γ is a constant.

The model we consider is the one discussed by Russo and Tseytlin [5], which is classically equivalent to the CGHS model. Its action is given by

$$S^{cl} = \frac{1}{\gamma^2} \int d^2\xi \sqrt{-h} \left[-h^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi - R_h \Phi + 4\lambda^2 e^\Phi + \frac{1}{2} h^{\alpha\beta} \sum_{i=1}^N \partial_\alpha f^i \partial_\beta f^i \right]. \quad (2)$$

These two actions are connected by the field transformations

$$\Phi = e^{-2\varphi}, \quad h_{\alpha\beta} = e^{2\omega} g_{\alpha\beta}, \quad \omega = \frac{1}{2} (\ln \Phi - \Phi). \quad (3)$$

We adopt the conformal gauge $h_{\alpha\beta} = e^{2\rho} \hat{h}_{\alpha\beta}$ and call ρ the Liouville mode. To make the analysis rigorous, we put the system in an interval of length L and impose periodic boundary conditions on the fields appearing in the action. For this purpose the following dimensionless quantities will be useful:

$$x^\mu = (t, \sigma) = \xi^\mu / L, \quad \mu = \lambda L. \quad (4)$$

The fields will then be 2π periodic in σ .

Quantum Theory

Now let us move on to the quantum action. A viable choice of the functional measure and its effect has been discussed by Hamada and Tsuchiya [6] following the method of DDK [7]. Their analysis leads to the quantum action

$$S = \frac{1}{\gamma^2} \int d^2x \sqrt{-\hat{h}} \left[-\hat{h}^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi - 2\hat{h}^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \rho - R_{\hat{h}} \Phi + 4\mu^2 e^{\Phi+2\rho} \right. \\ \left. + \kappa \left(\hat{h}^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho + R_{\hat{h}} \rho \right) + \frac{1}{2} \hat{h}^{\alpha\beta} \partial_\alpha \vec{f} \cdot \partial_\beta \vec{f} \right] + S^{gh}(\hat{h}, b, c), \quad (5)$$

where S^g is the usual action of the b - c ghost and $\kappa/\hbar\gamma^2 = (N - 24)/24\pi$.

Hereafter, we set the reference metric to be flat, i.e. $\hat{h}_{\alpha\beta} = \eta_{\alpha\beta}$, and deal exclusively with the special case where the number of matter fields N equals 24. Then the equations of motion can be solved exactly and the general solutions are readily obtained. The matter fields simply become periodic free fields, while the dilaton and the Liouville fields are expressed in terms of periodic free fields ψ and χ in the form

$$\Phi = -\chi - AB, \quad \rho = \frac{1}{2}(\psi - \Phi). \quad (6)$$

A and B are determined by the equations $\partial_+ A = \mu e^{\psi^{+1/2}}$, $\partial_- B = \mu e^{\psi^{-1/2}}$. Using the light-cone variables $x^\pm = x^0 \pm x^1$ and $\tilde{\gamma} = \gamma/\sqrt{4\pi}$, the mode expansion of $\psi = \psi^{+1/2}(x^+) + \psi^{-1/2}(x^-)$ is given by

$$\psi^{+1/2}(x^+)/\tilde{\gamma} = q^+/2 + p^+x^+ + i \sum_{n \neq 0} \frac{\alpha_n^+}{n} e^{-in x^+}, \quad (7)$$

and a corresponding expression $\psi^{-1/2}(x^-) = \psi^{+1/2}(x^+, \alpha_n^+ \leftrightarrow x^-, \tilde{\alpha}_n^+)$. Similarly, the mode expansions for $\chi/\tilde{\gamma}$ and $f^i/\tilde{\gamma}$ are obtained by replacing $(q^+, p^+, \alpha_n^+, \tilde{\alpha}_n^+)$ by $(q^-, p^-, \alpha_n^-, \tilde{\alpha}_n^-)$ and $(q_f^i, p_f^i, \alpha_{f,n}^i, \tilde{\alpha}_{f,n}^i)$ respectively.

As was demonstrated in [2], we can map the original interacting fields Ψ and ρ into the free fields ψ and χ by a *quantum canonical transformation*. This means that our model is rigorously quantized by adopting the usual commutators for the free fields.

Physical States

Classically, the constraints following from general covariance are expressed in terms of the energy-momentum tensor as $T_{\mu\nu} = 0$. In the quantum case they are imposed on states. The condition $T_{+-} = 0$ is easily checked to be satisfied through the operator equation of motion, and thus the quantized theory becomes a conformal field theory. The remaining constraints are classically expressed in terms of the free fields as

$$T_{\pm\pm}(x^\pm) = \frac{1}{2}(\partial_\pm \tilde{f})^2 + \partial_\pm \chi \partial_\pm \psi - \partial_\pm^2 \chi = 0. \quad (8)$$

In the BRST framework to be adopted, they turn into

$$d|\text{phys}\rangle = \bar{d}|\text{phys}\rangle = 0, \quad (9)$$

where d and $\bar{d}$ are the left- and right- BRST charges respectively, which are constructed through the usual procedure. By making use of the techniques developed in the non-critical string theory, we can obtain all the physical states as the BRST cohomology classes [8].

Except for a few special states (which will be insignificant for our subsequent analysis), all these physical states can be generated by a set of BRST invariant oscillators, called the DDF operators. The most convenient set turned out to be

$$\tilde{A}_{-n}^i \equiv e^{-i(n/\tilde{\gamma}p^+) \ln(\tilde{\gamma}p^+)} \int_0^{2\pi} \frac{dy^+}{2\pi\tilde{\gamma}} e^{-in\eta^+/\tilde{\gamma}p^+} \partial_+ f^i(y^+), \quad (10)$$

$$\eta^+ \equiv \ln \left(e^{\tilde{\gamma}q^+/2} A(x^+)/\mu \right), \quad (11)$$

where η^+ is a conformal scalar. Indeed one can easily check

$$[\tilde{A}_m^i, \tilde{A}_n^j] = \hbar m \delta_{i,j} \delta_{m+n,0}, \quad (12)$$

$$d(\text{ a polynomial of } \tilde{A}_{-n}^i \text{ 's}) |\vec{P}\rangle_l = 0, \quad (13)$$

where $|\vec{P}\rangle_l$ is a down-vacuum. $\tilde{A}_n^i$'s are essentially the matter oscillators dressed by the dilation-Liouville modes.

It should be kept in mind that a general physical state has the form

$$|\Psi\rangle = |\Psi_0\rangle + |\Lambda\rangle, \quad (14)$$

where $|\Psi_0\rangle$ is a representative of a physical state and $|\Lambda\rangle$ is an arbitrary BRST exact state (i.e. a “gauge part”). Choices of $|\Psi_0\rangle$ and $|\Lambda\rangle$ will be elaborated in the next section.

3. Extraction of Space-time Structure

How Can One Extract Space-time Structure ?

So far we have been able to quantize our model rigorously and solved it completely. However, physical picture of space-time structure is still nowhere to be seen. Thus the remaining crucial question is “how do we extract physical information out of our model ?”

It is instructive to recall what we do in the case of classical general relativity. First, we solve the Einstein's equation for a given matter distribution and obtain the metric in a coordinate system corresponding to an observer. Then from this metric and the associated quantities such as the curvature we understand the space-time structure and the physics such as the formation of a black hole, the bending of light and so on as seen by that observer.

This suggests that we should compute the mean values of appropriate physics-probing operators in a physical state and interpret them physically. Specifically, we deal with the (original inverse) metric and the left going energy momentum tensor of the matter part.

These operators are expressed as

$$g^{\alpha\beta} = \Phi e^\psi \eta^{\alpha\beta} = - : (\chi + AB) e^\psi : \eta^{\alpha\beta}, \quad (15)$$

$$T_{++}^f(\xi^+) = \frac{1}{2} : \partial_{\xi^+} \tilde{f} \partial_{\xi^+} \tilde{f} : . \quad (16)$$

A word of explanation on the use of mean values may be necessary. Due to the lack of probabilistic interpretation, mean value in quantum gravity does not directly carry the usual meaning of expectation value. However, as is particularly clear in the path integral formulation, mean value of an operator does have the meaning of the average over all possible quantum fluctuations. Thus our stand point is to regard the mean value as an important quantity which is intimately related to the value registered by a macroscopic measuring apparatus. The result of our calculation will turn out to support this point of view.

Now since $g^{\alpha\beta}$ and T_{++}^f are gauge variant, the mean values of these operators depend on the choice of the "gauge part" $|\Lambda\rangle$. Thus we need to fix the gauge within the conformal gauge by choosing a $|\Lambda\rangle$. It is not difficult to see that such a choice indeed correspond to a choice of gauge: For example, if we set $|\Lambda\rangle = d|\Psi_{-1}\rangle$, it follows that

$$\delta\langle g^{\alpha\beta} \rangle \equiv \langle \Psi | g^{\alpha\beta}(x) | \Psi \rangle - \langle \Psi_0 | g^{\alpha\beta}(x) | \Psi_0 \rangle = \frac{\hbar}{i} \partial^\alpha \epsilon^\beta + \frac{\hbar}{i} \partial^\beta \epsilon^\alpha + \mathcal{O}(\hbar). \quad (17)$$

In this case, $\epsilon^- = 0$ and $\epsilon^+ = \text{Re}\langle \Psi_0 | c g^{\alpha\beta} | \Psi_{-1} \rangle - (1/2) \langle \Psi_{-1} | c g^{\alpha\beta} d | \Psi_{-1} \rangle$, where c is the right-going ghost. This is an ordinary gauge transformation within the conformal gauge up to an insignificant correction of order $\hbar$.

At this juncture, let us make two important comments: First, despite the use of gauge-dependent operators, the gauge invariance conditions $d|\Psi\rangle = \bar{d}|\Psi\rangle = 0$ on the physical states are absolutely necessary. They correspond classically to $T_{\pm\pm} = 0$ as mentioned before and arise from the variation of the action with respect to the metric. Hence they represent the very dynamics of the gravitational interaction. Gauge invariance of the physical state and that of the physics-probing operators $g^{\alpha\beta}$ and T_{++}^f are conceptually quite different. Related to this is our second comment that the mean value of a gauge-invariant operator $\mathcal{O}$ can be easily checked to be a constant and cannot have any coordinate dependence. This shows the difficulty in getting physical information by dealing only with gauge invariant operators.

Choice of Physical States

Now we come to the question of how we should select specific physical states of interest among the infinitely many candidates of the form $|\Psi\rangle = |\Psi_0\rangle + |\Lambda\rangle$. A number of observations will help us set the criteria: First, we can show, by a simple argument, that the

mean value of an operator $\mathcal{O}(x)$ exhibits non-trivial coordinate dependence only when $|\Lambda\rangle$ is composed of a superposition of states with various Virasoro weights. Second, we obtain useful informations by writing out the mean value of $\mathcal{O}(x)$ in the form

$$\begin{aligned} \langle \Psi | \mathcal{O}(x) | \Psi \rangle &= \langle \Psi_0 | \mathcal{O}(x) | \Psi_0 \rangle + \langle \Psi_0 | \mathcal{O}(x) | \Lambda \rangle \\ &\quad + \langle \Lambda | \mathcal{O}(x) | \Psi_0 \rangle + \langle \Lambda | \mathcal{O}(x) | \Lambda \rangle. \end{aligned} \quad (18)$$

On the right hand side, the first term turns out to be always a constant and the fourth term consists only of gauge parts. On the other hand the second and the third terms have the structure that the physical information contained in $|\Psi_0\rangle$ is extracted through the inner product with the gauge part. This suggests that we should arrange $|\Psi_0\rangle$ and $|\Lambda\rangle$ so that the main coordinate dependence of the mean values of $g^{\alpha\beta}$ and T_{++}^f comes from the second and third terms. Third, we will be interested in the limit $L \rightarrow \infty$ (see Eq.(4)), in which the “parameter size” of the universe becomes infinite. Thus we want to make sure that the relevant mean values will be finite in this limit.

Furthermore, we wish to realize the following features. As we intend to obtain, as interesting examples, macroscopic black hole geometries like the one discovered by CGHS, we expect that appropriate coherent states must be considered. As for the gauge choice, it is desirable to realize the gauge $\psi = 0$ adopted by CGHS [1] for ease of comparison. (In [1], this gauge was expressed as $w = 0$.) Finally we also wish that in the absence of matter excitations the mean value of the metric will describe the linear dilaton vacuum represented by $g^{\alpha\beta} = -(1/2)\lambda^2\xi^+\xi^-\eta^{\alpha\beta}$.

Taking into account all the criteria above, we have made the the following choice of states:

$$|\Psi\rangle = |\Psi_0\rangle + |\Lambda\rangle, \quad (19)$$

$$|\Lambda\rangle = \frac{1}{\hbar^2} (db_{-M} + \bar{d}\bar{b}_{-M}) |\Omega\rangle, \quad |\Omega\rangle \equiv \sum_{k=-\infty}^{\infty} \omega_k \sum_{l=\pm 1,0} |\tilde{P}(k,l)\rangle_l, \quad (20)$$

$$\begin{aligned} |\tilde{P}(k,l)\rangle &\equiv e^{-ig_0 p^+ / \hbar \gamma} \gamma^2 \int_{-\infty}^{\infty} dp^+ \int_{-\infty}^{\infty} dp^- W(p^+) |p^+, p^-(k,l), \tilde{p}_f\rangle_l \\ &\quad \times \delta(p^- - \frac{1}{p^+}(\hbar - \frac{1}{2}p_f^2)), \end{aligned} \quad (21)$$

$$p^-(k,l) \equiv p^- - \frac{\hbar}{p^+}k - i\hbar\tilde{\gamma}l, \quad (22)$$

$$|\Psi_0\rangle \equiv e^G |\tilde{P}\rangle, \quad |\tilde{P}\rangle \equiv |P(\tilde{0}, 0)\rangle, \quad (23)$$

$$G \equiv \frac{1}{\hbar} \sum_{n \geq 1} \frac{\tilde{\nu}_n}{n} \tilde{A}_{-n}, \quad \tilde{\nu}_n = \nu_n e^{in x_0^+}, \quad (24)$$

$$g_0, x_0^+ = \text{real constants}, \quad p_f^2 \equiv \tilde{p}_f \cdot \tilde{p}_f. \quad (25)$$

In the above, $b_{-M}(\bar{b}_{-M})$ is the mode of the left-(right-) going anti-ghost $b(\bar{b})$ at level M , and we have omitted the superscript i in the DDF operators $\tilde{A}_{-n}^i$ as we consider the excitations of only one kind of the matter field. Note that the gauge part $|\Lambda\rangle$ contains no non-zero-mode excitations. Various parameters appearing in these expressions are given by

$$\nu_n = \nu(Lu) = \nu u^d e^{-au^2}, \quad \omega_n = \omega(Lu) = \frac{-\omega}{Lu} \quad (n \neq 0), \quad (26)$$

$$a, \omega = \text{positive constants}, \quad \omega_0 = \text{a constant to be adjusted}, \quad (27)$$

$$W(p^+) = p^+ e^{-\beta(p^+ - p_0^+)^2/2}, \quad (28)$$

and ν, β and p_0^+ are some constants. It turns out that the parameter d in ν_n determines the asymptotic behavior of the mean value of T_{++}^f for large ξ^+ .

One can check that the physical state above enjoys the properties

$$\tilde{A}_n |\Psi_0\rangle = \tilde{\nu}_n |\Psi_0\rangle, \quad \langle \Psi | \psi | \Psi \rangle \xrightarrow{L \rightarrow \infty} 0, \quad (29)$$

showing that $|\Psi_0\rangle$ is a coherent state and that $\psi = 0$ gauge is realized in the mean value in the large L limit as desired. Furthermore, when the physical state contains no excitations of the matter field, *i.e.*, $\nu_n = 0$, the mean value of the metric, with appropriate choices of parameters, describes the linear dilaton vacuum :

$$\langle \Psi | g^{\alpha\beta} | \Psi \rangle \xrightarrow{L \rightarrow \infty} -(1/2) \lambda_g^2 \xi^+ \xi^- \eta^{\alpha\beta} \quad (\lambda_g : \text{a constant}). \quad (30)$$

This allows us to identify our choice of coordinate system and will facilitate the comparison of our result with the CGHS configuration.

It can be checked that the remaining criteria are also met and hence the class of physical states chosen above should be quite appropriate for our purposes.

4. Emergence of Black Hole Geometries

Large L Limit

With the physical state and the physics-probing operators given above, we can perform the calculations of the mean values without approximation. The results so obtained contain some complicated infinite sums and are still difficult to interpret. Fortunately, in the large L limit that we are interested in, the expressions for the mean values $T_{++}^f(\xi^+)$, $\langle g^{\alpha\beta} \rangle$ simplify considerably. They can be expressed in the form

$$\langle T_{++}^f(\xi^+) \rangle \xrightarrow{L \rightarrow \infty} A \{ B + I_T(\xi^+ - \xi_0^+) \}, \quad (31)$$

$$\langle g^{\alpha\beta} \rangle \xrightarrow{L \rightarrow \infty} C \{ I_X(\xi^+ - \xi_0^+) - D\xi^+ + E - \lambda_g^2 \xi^+ \xi^- \}, \quad (32)$$

where $A \sim E$ and ξ_0^+ are some constants and I_T and I_χ are functions defined by

$$I_\chi(\xi) = \int_{1/L}^{\infty} du \frac{\cos u\xi}{u^2} \int_0^u dv v (u-v)^d e^{-av^2(u-v)^2}, \quad (33)$$

$$I_T(\xi) = -\partial_\xi^2 I_\chi(\xi). \quad (34)$$

It should be pointed out that the part $-\lambda_g^2 \xi^+ \xi^-$ in (32) is the only coordinate dependence coming from the gauge part, namely $\langle \Lambda | \mathcal{O}(x) | \Lambda \rangle$ in (18). We also wish to emphasize that Eq.(34) represents the important relation

$$\partial_{\xi^+}^2 \langle \chi \rangle \xrightarrow{L \rightarrow \infty} \frac{1}{2} \langle : \partial_{\xi^+} f \partial_{\xi^+} f : \rangle. \quad (35)$$

This arises from the constraint $\langle T_{++} \rangle = 0$ and corresponds to the energy balance seen in [1] between the matter and the dilaton-Liouville sectors in the $\psi = 0$ gauge.

In this way, we obtain the explicit functional forms of the mean values. By utilizing these forms and the scalar curvature calculated from $\langle g^{\alpha\beta} \rangle$, we are able to understand the space-time geometries described by our model.

Black Hole Geometries

Within our choice of the physical state, the essential properties of the resulting geometries are controlled by the value of the parameter d . (We will only discuss cases with $d > -1/2$ since for $d < -1/2$ mean values contain various divergent terms in the large L limit.)

In the case for $d = -1/2$, the function $I_T(\xi)$ becomes essentially a Gaussian, the width of which is proportional to the parameter a . Thus for small a , $\langle T_{++}^f \rangle$ represents a narrow shock wave. As for $I_\chi(\xi)$, it behaves like $\sim |\xi|$ for large $|\xi|$. These behaviors, together with an appropriate choice of the parameters, tell us that we have a black hole geometry produced by an in-falling matter shock wave. In Fig. 1, we show the line of curvature singularity for this configuration. One recognizes that it is essentially the same as the one discovered by CGHS. By a simple calculation, we can show that if we require $\langle T_{++}^f \rangle$ to be finite, then there is no right-going energy flow i.e. $\langle T_{--}^f \rangle = 0$. This may be related to the fact that for 24 matter fields there is no anomalous contribution from the measure, as discussed in Sec. 2.

For cases with $d > -1/2$, the geometries obtained exhibit essentially the same features. So, as an example, we discuss the case with $d = 1/2$. In this case, $I_T(\xi)$ behaves like a Gaussian times a polynomial of ξ , while I_χ becomes essentially a Gaussian. A typical geometry is shown in Fig.2. We see that at first a space-like singularity with a horizon appears, but subsequently a naked singularity is formed. Finally the geometry tends to that of the linear dilaton vacuum for large ξ^+ and the black hole fades away. This geometry resembles the one that occurs in the RST model [9], but with an important

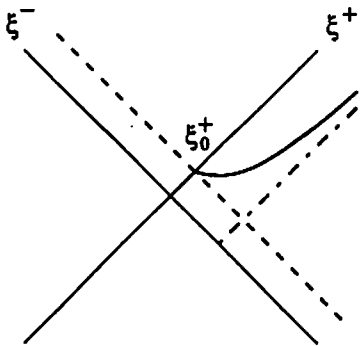


Fig.1 : The line of curvature singularity (solid line) for $d = -1/2$ produced by a left-going smeared shock wave along $\xi^+ = \xi_0^+$ (dotted line). The dot-dashed line represents the event horizon and the space-time quickly approaches the linear dilaton vacuum to the left of $\xi^+ = \xi_0^+$.

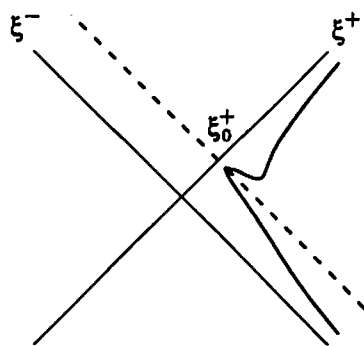


Fig.2 : The line of curvature singularity (solid line) for $d = 1/2$ produced by a left-going smeared shock wave along $\xi^+ = \xi_0^+$ (dotted line). Note the appearance of naked singularity. The space-time approaches the linear dilaton vacuum for large ξ^+ .

difference. In contrast to the RST model, we have no such pathological phenomena as the existence of right-going negative energy flow or the infinite curvature as $\xi^+ \rightarrow \infty$ in [9]. In fact, positive $\langle T_{++}^f \rangle$ leads to a positive $\langle T_{--}^f \rangle$ in our case. Existence of a positive right-going energy flow in the configuration with an evanescent black hole is quite suggestive of Hawking radiation. However, in short of a reliable notion of the S -matrix, we shall refrain from further discussion on this point.

By now it should be clear that one can construct various geometries by different choices of the representative and the gauge part of physical states. This well corresponds to the classical situation, where many solutions are possible and can be represented in various coordinate systems.

4. Conclusion

In this article, we have given a concise description of how one can extract space-time geometries in exactly solvable two dimensional quantum dilaton gravity. With an appropriate choice of physical states , we have been able to witness the emergence of black hole geometries. One of the examples is the geometry describing the formation of a stable

black hole similar to that of CGHS. The other is the one describing the formation and subsequent evanescence of a black hole with a naked singularity. We emphasize that we have been able to extract direct physical information such as the space-time geometry *in a fully quantum mechanical manner* in a model of quantum gravity.

In principle, our method should be applicable to various models of quantum gravity and cosmology besides the one considered in this work. These applications are left for future work.

As has already been mentioned, for further understanding of Hawking radiation in a fully quantum manner we need a proper definition of the S -matrix. This appears to be a difficult task since we do not a priori have the notion of background space-time nor that of asymptotic states. Furthermore there is no inherent “time evolution” in quantum gravity. Despite these obstacles, we hope to be able to make progress on this issue as it is the key to deeper understanding of quantum gravity.

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Physical Scaling and Renormalization Group in Two-Dimensional Gravity

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Recently a gravitational dressing of the renormalization group in two spacetime dimensions was studied in ref. [1]. It was shown that one-loop beta functions for marginal perturbations of a conformal field theory are multiplicatively renormalized due to quantum effects of gravity. The beta function in the presence of quantum gravity $\tilde{\beta}$ is related to the beta function without quantum gravity β as

$$\tilde{\beta} = \frac{k+2}{k+1} \beta, \quad (1)$$

where k is the level of the gravitational $SL(2, R)$ Kac-Moody algebra and is given in terms of the matter central charge c by

$$k = \frac{1}{12} \left(c - 37 - \sqrt{(1-c)(25-c)} \right). \quad (2)$$

This result was obtained in ref. [1] mainly using the light-cone gauge for gravity [2].

The purpose of our paper is to study this relation further and clarify the origin of the factor $\frac{k+2}{k+1}$ in eq. (1). We will use the $(2+\epsilon)$ -dimensional approach for quantum gravity [3]–[6] to compute quantum gravitational effects on the beta functions. We will find that gravitational radiative corrections to one-loop divergence in the matter effective action cancel out at the lowest order of the gravitational constant. Therefore one would obtain exactly the same beta functions as those in theories without

quantum gravity. We argue that the factor in eq. (1) is due to a proper definition of the physical scale transformation in the presence of quantum gravity. We obtain that factor by using the dimensionless renormalized cosmological constant to define the physical scale transformation. We will discuss the relation (1) in the light-cone gauge [2] from this point of view.

We shall consider a nonlinear sigma model coupled to gravity in $d = 2 + \epsilon$ dimensions. The action is

$$S = \int d^d x \sqrt{-g} \left[\frac{1}{16\pi G_0} R^{(d)} - \frac{1}{4\pi\alpha'} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^j G_{0ij}(X) \right], \quad (3)$$

where G_0 is the bare gravitational constant and G_{0ij} is the bare target space metric. Using the normal coordinate expansion [7], which we will discuss below, we can separate this action into two parts. The first part is the Einstein gravity coupled to free scalar fields. The second part consists of interaction terms due to the presence of nontrivial background fields G_{ij} in the target space.

Quantization of the first part was discussed in refs. [4], [5]. One-loop renormalization of the gravitational constant is

$$\frac{1}{G_0} = \mu^\epsilon \left(\frac{1}{G} - \frac{2}{3} \frac{25 - c}{\epsilon} \right), \quad (4)$$

where G is the renormalized gravitational constant and μ is the renormalization scale. It was shown in ref. [5] that the two-dimensional gravity [2], [8] can be obtained as a limit $\epsilon \rightarrow 0$ in the strong coupling region $|G| \gg |\epsilon|$. In this region the bare gravitational constant is given by

$$G_0 = -\frac{3}{2} \mu^{-\epsilon} \frac{\epsilon}{25 - c}. \quad (5)$$

Using the Einstein action with this bare gravitational constant, anomalous dimensions of physical operators were obtained and were shown to be consistent with the previous results using other methods [2], [8].

Using this approach for quantum gravity we can now study effects of quantum gravity on the beta functions of matter self-couplings by taking the second part of the action into account. There are two parameters for loop expansions. The parameter for matter loops is α' and that for graviton loops is G_0 in eq. (5). Effects

of quantum gravity become small for large $-c$. In the limit $c \rightarrow -\infty$ we obtain a theory without quantum gravity.

To obtain the beta functions at the first order in α' we compute divergences in the effective action using the matter action in eq. (3) with G_{0ij} replaced by $\mu' G_{ij}$, where G_{ij} is the dimensionless renormalized coupling. We use the background field method [9] and the normal coordinate expansion [7]. The matter fields are expanded as $X^i = \hat{X}^i + \xi^i + O(\xi^2)$, where $\hat{X}^i$ are background fields and ξ^i are quantum fluctuations (normal coordinates). It is more convenient to use $\xi^a = \xi^i E_i^a(\hat{X})$ as quantum fields, where E_i^a is the target space vielbein: $G_{ij} = E_i^a E_j^b \eta_{ab}$. The matter action can be expanded as [7]

$$\begin{aligned} S_{\text{matter}} = & -\frac{\mu^\epsilon}{4\pi\alpha'} \int d^d x \sqrt{-g} g^{\mu\nu} \left[\partial_\mu \hat{X}^i \partial_\nu \hat{X}^j G_{ij}(\hat{X}) + 2D_\mu \xi^a \partial_\nu \hat{X}^i E_{ia}(\hat{X}) \right. \\ & + D_\mu \xi^a D_\nu \xi^b \eta_{ab} - \partial_\mu \hat{X}^i \partial_\nu \hat{X}^j R_{iajb}(\hat{X}) \xi^a \xi^b \\ & - \frac{4}{3} \partial_\mu \hat{X}^i \xi^a \xi^b D_\nu \xi^c R_{caib}(\hat{X}) - \frac{1}{3} \partial_\mu \hat{X}^i \partial_\nu \hat{X}^j \xi^a \xi^b \xi^c D_c R_{iajb}(\hat{X}) \\ & - \frac{1}{3} D_\mu \xi^a D_\nu \xi^b \xi^c \xi^d R_{acbd}(\hat{X}) - \frac{1}{2} \xi^a \xi^b \xi^c D_\mu \xi^d \partial_\nu \hat{X}^i D_c R_{daib}(\hat{X}) \\ & \left. - \frac{1}{12} \partial_\mu \hat{X}^i \partial_\nu \hat{X}^j \xi^a \xi^b \xi^c \xi^d (D_a D_b R_{icjd} - 4R^k{}_{aib} R_{kejd}) (\hat{X}) + O(\xi^5) \right]. \quad (6) \end{aligned}$$

The covariant derivative is defined as

$$D_\mu \xi^a = \partial_\mu \xi^a + \partial_\mu \hat{X}^i \Omega_i{}^a{}_b(\hat{X}) \xi^b, \quad (7)$$

where $\Omega_i{}^a{}_b$ is the target space spin connection.

The metric is parametrized by a background field $\hat{g}_{\mu\nu}$ and quantum fields $h^\mu{}_\nu$, ϕ as [5]

$$g_{\mu\nu} = \hat{g}_{\mu\rho} (e^{\kappa_0 h})^\rho{}_\nu e^{-\kappa_0 \phi}, \quad h_{\mu\nu} \equiv \hat{g}_{\mu\rho} h^\rho{}_\nu = h_{\nu\mu}, \quad h^\mu{}_\mu = 0, \quad (8)$$

where $\kappa_0^2 = 16\pi G_0$. The metric dependence of the matter action is

$$\sqrt{-g} g^{\mu\nu} = \sqrt{-\hat{g}} (e^{-\kappa_0 h})^\mu{}_\rho \hat{g}^{\rho\nu} e^{-\frac{1}{2}\epsilon \kappa_0 \phi}. \quad (9)$$

Notice that the conformal mode ϕ is multiplied by a factor ϵ . This corresponds to the fact that the perturbation due to the nontrivial background G_{ij} is a marginal

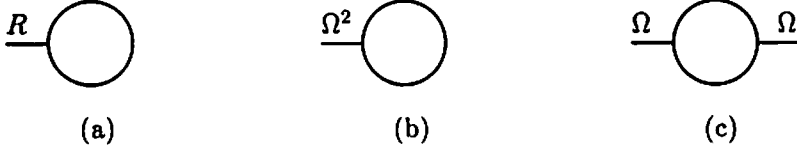


Figure 1: One-loop divergent diagrams without quantum gravity.

operator in two dimensions. We use the gauge fixing term of ref. [5] for the general coordinate symmetry

$$S_{\text{GF}} = -\frac{1}{2} \int d^d x \sqrt{-\hat{g}} \left(\hat{D}^\nu h_{\nu\mu} + \frac{1}{2} \epsilon \hat{D}_\mu \phi \right)^2. \quad (10)$$

In this gauge the propagators of the gravitational fields are

$$\begin{aligned} \langle h_{\mu\nu}(x) h_{\rho\sigma}(y) \rangle &= -i \left(\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \frac{2}{d} \eta_{\mu\nu} \eta_{\rho\sigma} \right) \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2} e^{ip \cdot (x-y)}, \\ \langle \phi(x) \phi(y) \rangle &= \frac{4i}{\epsilon d} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2} e^{ip \cdot (x-y)}, \quad \langle h_{\mu\nu}(x) \phi(y) \rangle = 0. \end{aligned} \quad (11)$$

First let us recall the one-loop beta function when gravity is not quantized [10], [7]. We are interested in order $O(\alpha')$ terms in the effective action Γ . As shown in Fig. 1, there are three divergent diagrams of this order. However, the divergence of the diagram (b) is canceled by that of the diagram (c). They should cancel since they are not allowed by the covariance and the local Lorentz symmetry in the target space. Only the diagram (a) contributes to one-loop divergence. We obtain

$$\Gamma = -\frac{1}{4\pi\alpha'} \frac{\alpha' \zeta}{\epsilon} \int d^d x \sqrt{-\hat{g}} \hat{g}^{\mu\nu} \partial_\mu \hat{X}^i \partial_\nu \hat{X}^j R_{ij}(\hat{X}) + O(\epsilon^0), \quad (12)$$

where $\zeta = 1$. The divergence can be removed by adding a counter term

$$S_{\text{c.t.}} = \frac{\mu^\epsilon}{4\pi\alpha'} \frac{\alpha'}{\epsilon} \int d^d x \sqrt{-g} g^{\mu\nu} \partial_\mu X^i \partial_\nu X^j R_{ij}(X) \quad (13)$$

to the matter action. This counter term corresponds to choosing the bare coupling in eq. (3) as

$$G_{0ij} = \mu^\epsilon \left(G_{ij} - \frac{\alpha'}{\epsilon} R_{ij} \right). \quad (14)$$

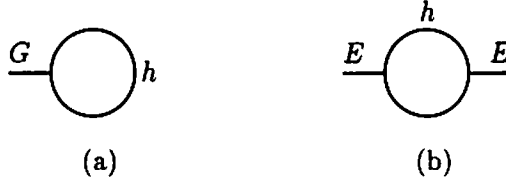


Figure 2: $O(\kappa_0^2)$ divergent diagrams. Internal lines with h are graviton propagators. Other internal lines are matter propagators.

From the μ -independence of the bare coupling G_{0ij} we obtain the one-loop beta function in the limit $\epsilon \rightarrow 0$ as

$$\beta_{ij} \equiv \mu \frac{\partial}{\partial \mu} G_{ij} = \alpha' R_{ij}. \quad (15)$$

Now we shall consider quantum effects of gravity. At order $O(\kappa_0^2)$ there are two divergent diagrams as shown in Fig. 2. In these diagrams the internal lines are $h_{\mu\nu}$ or ξ^a . Diagrams with internal ϕ lines do not contribute to divergences because the ϕ -vertices always contain a factor ϵ as shown in eq. (9). Their divergences have the form

$$\Gamma = -\frac{1}{4\pi\alpha'} \frac{\eta}{\epsilon} \int d^d x \sqrt{-\hat{g}} \hat{g}^{\mu\nu} \partial_\mu \hat{X}^i \partial_\nu \hat{X}^j G_{ij}(\hat{X}). \quad (16)$$

The coefficient η for each diagram is given by

$$\eta_{(a)} = -\frac{\kappa_0^2}{2\pi}, \quad \eta_{(b)} = \frac{\kappa_0^2}{2\pi}. \quad (17)$$

Therefore the divergences of these two diagrams cancel each other and there is no divergence of order $O(\kappa_0^2)$. Actually, the coefficients in eq. (17) are of order $O(\epsilon)$ since $\kappa_0^2 = O(\epsilon)$, and eq. (16) for each diagram is finite.

Next let us consider order $O(\alpha'\kappa_0^2)$ divergences. We have to compute divergences in two-loop diagrams. Since $\kappa_0^2 = O(\epsilon)$, we only need to know ϵ^{-2} singularities of Feynman integrals. We have to take into account the vertices in the one-loop counter term (13). The normal coordinate expansion of this counter term is

$$S_{\text{c.t.}} = \frac{\mu^\epsilon}{4\pi\alpha'} \frac{\alpha'}{\epsilon} \int d^d x \sqrt{-g} g^{\mu\nu} \left[\partial_\mu \hat{X}^i \partial_\nu \hat{X}^j R_{ij}(\hat{X}) + \partial_\mu \hat{X}^i \partial_\nu \hat{X}^j \xi^a D_a R_{ij}(\hat{X}) \right]$$

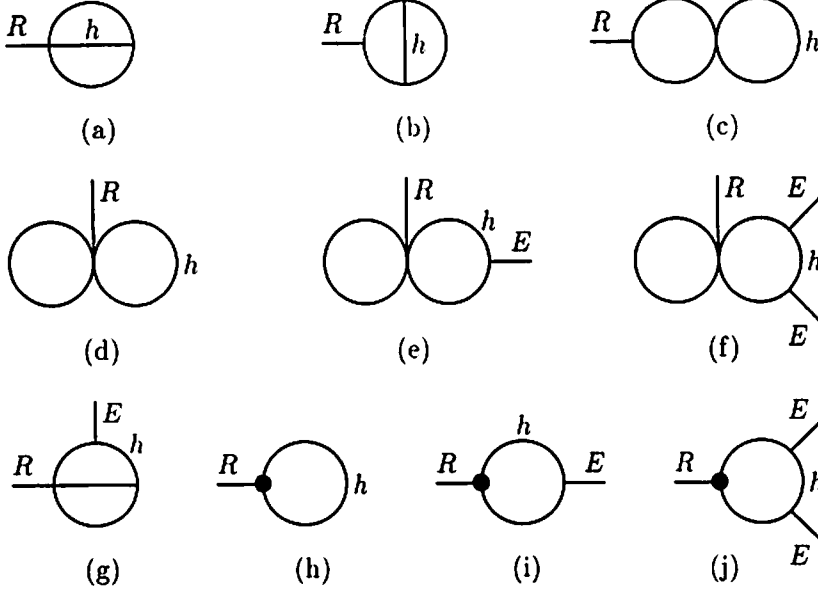


Figure 3: $O(\alpha'\kappa_0^2)$ divergent diagrams. Internal lines with h are graviton propagators. Other internal lines are matter propagators. Vertices with a dot are those of the one-loop counter term.

$$\begin{aligned}
& +2\partial_\mu \hat{X}^i D_\nu \xi^a R_{ia}(\hat{X}) + \frac{1}{2}\partial_\mu \hat{X}^i \partial_\nu \hat{X}^j \xi^a \xi^b (D_a D_b R_{ij} - 2R^k{}_{aib} R_{kj})(\hat{X}) \\
& +2\partial_\mu \hat{X}^i D_\nu \xi^a \xi^b D_b R_{ia}(\hat{X}) + D_\mu \xi^a D_\nu \xi^b R_{ab}(\hat{X}) + O(\xi^3) \Big]. \quad (18)
\end{aligned}$$

Divergent diagrams of order $O(\alpha'\kappa_0^2)$ are shown in Fig. 3. In addition to the diagrams in Fig. 3 there are divergent diagrams with the Ω vertices. However, they should cancel each other by the covariance and the local Lorentz symmetry in the target space as in Fig. 1. Divergence in each diagram in Fig. 3 has the same form as eq. (12). The coefficient ζ for each diagram is

$$\begin{aligned}
\zeta_{(a)} &= 0, \quad \zeta_{(b)} = -\frac{\kappa_0^2}{2\pi\epsilon}, \quad \zeta_{(c)} = \frac{\kappa_0^2}{2\pi\epsilon}, \quad \zeta_{(d)} = -\frac{\kappa_0^2}{2\pi\epsilon}, \quad \zeta_{(e)} = \frac{2\kappa_0^2}{3\pi\epsilon}, \\
\zeta_{(f)} &= -\frac{\kappa_0^2}{6\pi\epsilon}, \quad \zeta_{(g)} = 0, \quad \zeta_{(h)} = \frac{\kappa_0^2}{2\pi\epsilon}, \quad \zeta_{(i)} = -\frac{\kappa_0^2}{\pi\epsilon}, \quad \zeta_{(j)} = \frac{\kappa_0^2}{2\pi\epsilon}. \quad (19)
\end{aligned}$$

Since $\kappa_0^2 = O(\epsilon)$, these coefficients are finite for $\epsilon \rightarrow 0$. We see that the sum of divergences in the diagrams (a)–(j) cancel out. This cancellation is similar to

the observation in ref. [5] that the diagrams with $h_{\mu\nu}$ lines do not contribute to the renormalization of the operator $e^{-\frac{\Delta}{2}(1-\Delta_0)\phi}$ to two-loop order. In the present case of marginal operators there is no ϕ contribution either. It can happen that gravitational corrections to divergences completely cancel also at higher orders of the loop expansions in κ_0^2 . This may not be so surprising because $h_{\mu\nu}$ is a gauge degree of freedom in two dimensions and can be set to zero by a general coordinate transformation.

Thus we have seen that the one-loop divergence does not receive gravitational radiative correction. Therefore we obtain the same beta function (15) as in the theory without quantum gravity. This is consistent with the result in ref. [1] in the sense that the beta functions are essentially the same in theories with and without quantum gravity. The factor $\frac{k+2}{k+1}$ in eq. (1) can be understood by considering a definition of the physical scale transformation.

Without gravitational interactions, the beta function (15) is defined as a response of the renormalized quantities to the change of the renormalization scale μ . On the other hand, the characteristic feature of gravity is to provide a metric which can determine the physical scale in the theory. Therefore we should measure the response of the renormalized quantities to the change of the physical scale. It has been pointed out that the renormalization of the gravitational constant G can be made meaningful only after fixing the scale of the metric by renormalizing a reference operator O [4]. Since the gravitational constant becomes dimensionless in two-dimensions, it is most natural to consider the reference operator which also has dimensionless coupling in two dimensions. Namely we choose the reference operator to be spinless and with a conformal dimension $(1, 1)$, such as the Thirring interaction. After fixing the scale of the metric, we can absorb divergences of the type $\sqrt{-g}R$ into a renormalization of the gravitational constant G , and the divergences of the type $\sqrt{-g}$ into a renormalization of the cosmological constant Λ . Instead of the gravitational constant, the cosmological constant Λ naturally provides the physical scale in the presence of the gravitational interaction in two dimensions, since it is the only dimensionful parameter in the gravity theory in two dimensions.

To use the cosmological term in defining the physical scale transformation, let us introduce the cosmological term into the action (3)

$$\Lambda_0 \int d^d x \sqrt{-g}, \quad (20)$$

where Λ_0 is the bare cosmological constant. Renormalization of the cosmological term was discussed in ref. [5]. To remove divergences, the bare cosmological constant is expressed as

$$\Lambda_0 = \mu^d \Lambda Z, \quad (21)$$

where Z is a divergent renormalization factor and Λ is a dimensionless renormalized cosmological constant. From the μ -independence of Λ_0 we obtain

$$\mu \frac{\partial \Lambda}{\partial \mu} = -(d + \gamma) \Lambda, \quad \gamma = \mu \frac{\partial}{\partial \mu} \ln Z, \quad (22)$$

where γ is the anomalous dimension. In the two-dimensional limit the exact form of γ was obtained in ref. [5] by considering divergences coming from the conformal mode to arbitrary orders in the loop expansion

$$\gamma = -2 - \alpha Q, \quad (23)$$

where α and Q are defined by

$$Q = \sqrt{\frac{25 - c}{3}}, \quad \alpha = -\frac{1}{2\sqrt{3}} (\sqrt{25 - c} - \sqrt{1 - c}). \quad (24)$$

In this case the solution of eq. (22) is

$$\Lambda \sim \mu^{-\frac{2}{x}}, \quad \frac{1}{x} = 1 + \frac{\gamma}{2} = -\frac{\alpha Q}{2}. \quad (25)$$

In the limit of no quantum gravity $c \rightarrow -\infty$ the anomalous dimension vanishes $\gamma \rightarrow 0$.

Now let us consider the beta function in the presence of gravity. Instead of the renormalization scale μ , we use the physical scale derived from the cosmological constant to define the beta function. Since the renormalized cosmological constant Λ is defined by eq. (21), the inverse square root of the cosmological constant $\sqrt{\Lambda^{-1}}$ should be used in place of μ in $d = 2$ dimensions. Without gravitational interactions, the beta function is usually defined as a change of the renormalized coupling G_{ij} in response to the change of the renormalization scale

$$\beta_{ij} = \frac{\partial G_{ij}(\mu)}{\partial \ln \mu}. \quad (26)$$

In the presence of gravity, we should consider the beta function as a response to the physical scale $\sqrt{\Lambda^{-1}}$

$$\tilde{\beta}_{ij} = \frac{\partial G_{ij}}{\partial \ln \sqrt{\Lambda(\mu)^{-1}}} = \frac{\partial \ln \mu}{\partial \ln \sqrt{\Lambda(\mu)^{-1}}} \frac{\partial G_{ij}}{\partial \ln \mu} = x \beta_{ij}. \quad (27)$$

In the limit of no quantum gravity $c \rightarrow -\infty$, we have $x \rightarrow 1$ and $\tilde{\beta}_{ij} \rightarrow \beta_{ij}$. Using eqs. (25) and (24) we see that the relation between $\tilde{\beta}_{ij}$ and β_{ij} in eq. (27) is exactly the same as eq. (1).

Another way to state this scale transformation is the following. We can consider a generalized renormalization scale transformation $\mu \rightarrow \lambda^y \mu$ with a power y of the scaling parameter λ . We would like to require the renormalized cosmological constant Λ in eq. (21) to have the classical response in two dimensions

$$\Lambda \rightarrow \lambda^{-2} \Lambda \quad (28)$$

even in the presence of quantum effects. Therefore the renormalization scale dependence in eq. (25) dictates $y = x$. Clearly this transformation assures that the scale transformation of the cosmological constant be independent of the matter content of the theory. We can now define the beta function in the presence of quantum gravity as a response to this physical scale transformation

$$\tilde{\beta}_{ij} = \lim_{\lambda \rightarrow 1} \frac{1}{\ln \lambda} [G_{ij}(\lambda^x \mu) - G_{ij}(\mu)] = x \beta_{ij}, \quad (29)$$

where β_{ij} is defined in eq. (26). Thus we obtain the same $\tilde{\beta}_{ij}$ as in eq. (27).

Let us comment on modifications if one chooses to use other physical operators instead of the cosmological term to define a physical scale transformation of μ . We can repeat the above analysis using the result on the renormalization of physical operators in ref. [5]. If we use a physical operator corresponding to a matter operator with a conformal dimension (Δ_0, Δ_0) , we obtain a relation

$$\tilde{\beta}_{ij} = -\frac{2(1 - \Delta_0)}{\beta_Q} \beta_{ij}, \quad (30)$$

where

$$\beta = -\frac{1}{2\sqrt{3}} \left(\sqrt{25 - c} - \sqrt{1 - c + 24\Delta_0} \right). \quad (31)$$

However, we would like to emphasize that the usual scaling arguments in the presence of gravity are in fact based upon the use of the physical scale defined in terms of the cosmological constant Λ [8]. We are here advocating a similar argument to define the beta function in the presence of gravity.

To confirm the above interpretation of the factor in eq. (1) we shall examine the physical scaling in the light-cone gauge [2]. As in ref. [1] the beta functions can be obtained from coefficients of the operator product expansion (OPE), which in turn can be obtained from a ratio of two- and three-point correlation functions. To see how the cutoff and the renormalization scale are introduced in this method let us recall the relation between the OPE coefficients and the beta functions. As in ref. [1] we consider an action for a conformal field theory perturbed by marginal (conformal weight one) operators $O_n(x)$

$$S = S_0 + \sum_n \lambda_n O_n, \quad O_n = \int d^2x O_n(x). \quad (32)$$

The vacuum expectation value can be expanded as

$$\langle \cdots \rangle = \left\langle (\cdots) \left(1 + i \sum_n \lambda_n O_n + \frac{1}{2} i^2 \sum_{n,m} \lambda_n \lambda_m O_n O_m + \cdots \right) \right\rangle_0. \quad (33)$$

Let us consider the $O(\lambda^2)$ terms. The x -integrations in O_1 and O_2 give short distance divergences. The short distance behavior of the product of the integrands is given by the OPE

$$O_n(x_1) O_m(x_2) \sim \frac{1}{(x_1 - x_2)^2} \sum_l g_l^{nm} O_l(x_2). \quad (34)$$

Therefore the x -integrations give

$$\begin{aligned} O_n O_m &= \int d^2x_1 d^2x_2 \frac{1}{(x_1 - x_2)^2 + \epsilon^2} \sum_l g_l^{nm} O_l(x_2) \\ &= 2\pi i \ln \epsilon \sum_l g_l^{nm} O_l + \cdots, \end{aligned} \quad (35)$$

where we have regularized the integral by making a Wick rotation $x^0 = -ix^2$ and introducing a cutoff ϵ in the coordinate space. This short distance divergence can be removed by introducing the bare coupling constants

$$\lambda_{0n} = \lambda_n + \pi \ln(\epsilon\mu) \sum_{l,m} \lambda_l \lambda_m g_n^{lm} + O(\lambda^3), \quad (36)$$

where we have introduced a renormalization scale μ . Then the beta functions are given by

$$\beta_n = \mu \frac{\partial \lambda_n}{\partial \mu} = -\pi \sum_{l,m} \lambda_l \lambda_m g_n^{lm} + O(\lambda^3). \quad (37)$$

Thus we can obtain the beta functions from the OPE coefficients g_n^{lm} .

In the light-cone gauge the metric has the form

$$g_{\mu\nu} dx^\mu dx^\nu = -2dx^+ dx^- + h_{++} dx^+ dx^+ \quad (38)$$

and the cosmological term is a c-number

$$\Lambda_0 \int d^2x \sqrt{-g} = \Lambda_0 \int d^2x. \quad (39)$$

There is no short distance divergence. Therefore the relation between the bare and dimensionless renormalized cosmological constants is $\Lambda_0 = \mu^2 \Lambda$ and we obtain

$$\Lambda \sim \mu^{-2}. \quad (40)$$

We see that the usual scaling $\mu \rightarrow \lambda\mu$ is physical, i.e., it changes the renormalized cosmological constant Λ as in eq. (28). We should obtain the factor in eq. (1) for the usual beta functions. Indeed it was obtained in ref. [1] in this way. Therefore our interpretation of the factor is consistent with the analysis in the light-cone gauge.

The gravitational dressing of the beta function in the conformal gauge has already been discussed in ref. [1] using the cosmological constant as the physical scale similarly to our treatment. The renormalization group equation in the conformal gauge was also discussed in ref. [11].

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Wheeler-DeWitt Equation for the Discrete Space-time

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Abstract

We propose a model for the space-time under the assumption that the space-time is discrete by nature. As the concept of space-time has no meaning in quantum theory, we can not accept the discreteness of the space-time literally. In quantum gravity, the dynamical quantity is the 3-geometry. Hence, it is the superspace that should be discretized. The discrete version of the Heisenberg algebra is derived by using the central difference. Finally, we obtain the discrete Wheeler-DeWitt equation and discuss its consequence briefly.

The theory of general relativity put forth a revolutionary new viewpoint on space-time structure and gravitation. No one doubt that it shows an aspect of the truth of the nature. However, the singularity theorem indicates the limit of the applicability of the general relativity. Moreover, the quantum theory of gravitation contains extraordinary difficulties. In order to circumvent this situation, let us consider the subject from more fundamental point of view. Up to now, we have considered the space-time as a continuous physical reality. It is our starting point to doubt this prejudice. Now, we would like to stress that the space-time might be discrete by nature.

To make the discussion precise, we shall begin to explain the canonical approach to the quantum theory of gravity. From the ADM analysis, [1] we know that the following parametrization is convenient for the Hamiltonian approach to the general relativity:

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \quad (1)$$

where h_{ij} is the metric of 3-geometry, N and N^i are the lapse function and the shift function, respectively. Using the above ADM formalism, one obtains the so-called

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the superspace. Of course, the scheme of the discretization is not unique. However, we expect that a reasonable discretization scheme reveal the general features of the discrete space-time.

Now, we propose the discretization scheme as

$$\Delta f = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x} . \quad (5)$$

This difference form is inspired by the theory of the quantum group, [3] although there is no connection between our theory and the quantum group. In the limit $q \rightarrow 1$, of course, it reduces to the ordinary derivative operator,

$$\lim_{q \rightarrow 1} \Delta f(x) = \frac{df}{dx} . \quad (6)$$

In the continuum theory, the canonical method for quantization is replacing p with $-i\hbar\partial/\partial x$. By analogy, one put

$$p = -i\hbar\Delta .$$

However, in the present case, the operators x and p does not lead to the closed algebra. To close the algebra we must add the auxially operator

$$u^\dagger f(x) = f(qx) . \quad (7)$$

We call the above operator $u^\dagger$ the step operator. Assuming the formal hermiticity of x and p , one can derive the discrete Heisenberg algebra

$$px - \frac{1}{q}xp = -i\hbar u^\dagger , \quad (8)$$

$$pu^\dagger - qu^\dagger p = 0 , \quad (9)$$

$$xu^\dagger - \frac{1}{q}u^\dagger x = 0 . \quad (10)$$

As for the check, take the limit $q \rightarrow 1$, then one can obtain the familiar Heisenberg algebra

$$px - xp = -i\hbar .$$

Once, we obtain the discrete Heisenberg algebra in the quantum mechanical case, it is straightforward to write down the discrete Wheeler-DeWitt equation

$$H(h_{ij}, \pi^{kl})(\vec{n})\psi = 0 ,$$

Wheeler-DeWitt equation as [2]

$$\hat{H}(h_{ij}(x), \pi^{kl}(x))\psi = \hat{H}^i(h_{ij}(x), \pi^{kl}(x))\psi = 0 ,$$

where

$$\hat{H} = 16\pi G G_{ijkl} \hat{\pi}^{ij} \hat{\pi}^{kl} - \frac{1}{16\pi G} \sqrt{\hat{h}} (R^{(3)} - 2\Lambda) , \quad (2)$$

$$\hat{H}^i = -2D_j \hat{\pi}^{ji} . \quad (3)$$

Here, the 3-metric $\hat{h}_{ij}$ and its conjugate momentum $\hat{\pi}^{kl}$ must satisfy the following commutation relation

$$[\hat{h}_{ij}(x), \hat{\pi}^{kl}(x')] = \frac{1}{2} i \hbar (\delta_i^k \delta_j^l + \delta_i^l \delta_j^k) \delta(x - x') . \quad (4)$$

It is impossible to solve the Wheeler-DeWitt equation in general. Even in the solvable case, it is difficult to extract directly the physical information from the wave function. Hence, usually, we use the WKB approximation to obtain the semi-classical information of the system. It necessarily loses the information of the quantum geometry.

Now, we take an alternative attitude to attack the quantum gravity. The Wheeler-DeWitt equation is formulated based on the manifold picture. However, we have no reason to assume that the space-time is represented by the continuous manifold, because we have no continuous observation of the time and space. It may be natural to assume that the space-time is discrete by nature. If so, there is a question "what equation is replaced for the Wheeler-DeWitt equation?". This is the theme of this paper.

To settle this problem, one must clarify that what object should be discretized. Is it space-time as we imagined intuitively. No! Because, in quantum theory, there is no space-time. In particular, there is no time. The arguments of the wave function is not the space-time but 3-geometry. The set of 3-geometries is called the superspace. In the classical theory, a trajectory in the superspace represents a space-time. In the classical picture, the discretization of the time is equivalent to the discretization of the trajectory. Although the concept of the space-time loses its meaning in the quantum theory, the superspace has the meaning. Then, what we must discretize is

$$\pi^{kl}(\vec{n})h_{ij}(\vec{m}) - R_{\frac{1}{q}}(\vec{n}, \vec{m})h_{ij}(\vec{m})\pi^{kl}(\vec{n}) = -\frac{i}{2}\hbar(\delta_i^k\delta_j^l + \delta_i^l\delta_j^k)u^\dagger(\vec{n})\delta_{\vec{n}, \vec{m}}, \quad (11)$$

$$\pi^{kl}(\vec{n})u^\dagger(\vec{m}) - R_q(\vec{n}, \vec{m})u^\dagger(\vec{m})\pi^{kl}(\vec{n}) = 0, \quad (12)$$

$$h_{ij}(\vec{n})u^\dagger(\vec{m}) - R_{\frac{1}{q}}(\vec{n}, \vec{m})u^\dagger(\vec{m})h_{ij}(\vec{n}) = 0, \quad (13)$$

where $R_q(\vec{n}, \vec{m}) = 1 + (q-1)\delta_{\vec{n}, \vec{m}}$. In the above formalism, we also discretize the 3-space and the lattice point is represented by $\vec{n}, \vec{m}$.

As for the simplest example, one can consider the mini-superspace model. The simplest one is the homogeneous and isotropic universe with a cosmological constant.

$$ds^2 = -\frac{N^2(t)}{a(t)}dt^2 + a(t)d\Omega_3^2, \quad (14)$$

where $d\Omega_3$ is a line element of the 3-sphere.

A simple calculation leads to the hamiltonian,

$$H = \frac{1}{2}(-4p^2 + \lambda a - 1).$$

The discrete Wheeler-DeWitt equation becomes

$$[\frac{1}{2}(-4p^2 + \lambda a - 1)]\psi = 0, \quad (15)$$

with the commutation relations:

$$pa - \frac{1}{q}ap = -i\hbar u^\dagger, \quad (16)$$

$$pu^\dagger - qu^\dagger p = 0, \quad (17)$$

$$au^\dagger - \frac{1}{q}u^\dagger a = 0. \quad (18)$$

Analyzing this model, one can deduce the physical properties of the discrete mini-superspace, i.e.

$$\Delta a \Delta p = 0. \quad (19)$$

As we discretize the space-time, the uncertainty of the canonical variables disappears.

In conclusion, we have proposed the discrete Wheeler-DeWitt equation which is formulated based on the picture that the space-time is discrete in the fundamental sense. To uncover the physics behind the discrete Wheeler-DeWitt equation, we must invent the WKB analysis for this case. There might be a chance to resolve the singularity problem in the way that there is no singular point in the configuration space.

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QUANTUM POTENTIAL INTERPRETATION AND QUANTUM COSMOLOGY¹

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Abstract

We apply Bohm's quantum potential interpretation to quantum cosmology or quantum geometrodynamics. In quantum geometrodynamics based on the Arnowitt-Deser-Misner canonical formalism, we discuss how to obtain the trajectories of the universe acted on not only by classical potentials, but also by quantum potentials. We study the de Sitter minisuperspace model by introducing "extended" Robertson-Walker time which exists not only in classically allowed region but also in classically forbidden region.

¹This report is based on preprint KOBE-TH-93-04

1. In quantum cosmology, the whole universe is described by a wave function Ψ , which is defined on the superspace (the space of all three metrics and matter field configurations). The wave function Ψ is a solution to the Wheeler-DeWitt(WDW) equation [1], $H\Psi = 0$. Since superspace is of infinite dimensions, the full formalism of quantum cosmology is very difficult to deal with in practice. In addition to the technical difficulties in solving the WDW equation, we also encounter a nontrivial interpretational problem about the wave function of the universe. This problem has been discussed by many authors [2]. It seems that the “Copenhagen” interpretation for the quantum mechanics is inapplicable to quantum cosmology from the following several reasons. Firstly, the “Copenhagen” formalism assumes a possible division of the world into “observer” and “observed”. However, in quantum theory of the universe, there can be no fundamental division into observer and observed. Also, measurement and observer can not be fundamental at early universes, since both can not exist at the early stage of the evolution of the universe. Secondly, the “collapse” of the wave function in the “Copenhagen” scenarios requires a description involving time evolution. It is not usually easy to introduce “time” in the WDW equation, since the WDW equation is a “timeless” equation.

An alternative interpretation of the quantum theory was proposed by Bohm [3], based on the notion of a particle acted on, not only by classical potential but also by “quantum potential”. The quantum potential is determined by a solution to Schrödinger equation. In this quantum potential interpretation of quantum theory, for example, electron is a certain kind of particle following continuous and well-defined trajectories. In quantum potential interpretation, it was explained that all sorts of quantum processes, such as transition between states, quantum interference in the two-slit experiment are able to take place, in principle, without the need for an observer and also for collapse of the wave function [4]. Therefore, we think that Bohm’s quantum potential interpretation is very suitable for quantum theory of the universe.

We apply this quantum potential interpretation to quantum geometrodynamics based on the ADM canonical formalism [5]. We discuss how to obtain the trajectories of the universe subject not only to classical potentials, but also to quantum potentials. We study the de Sitter minisuperspace model² by introducing “extended” Robertson-Walker time, which describes well-defined trajectories of the evolution of the universe not only in classically allowed region but also in classically forbidden region. It is shown that the universe can start at $a \approx 0$, where

²For the discussion about quantum potential interpretation of the Friedman-Robertson-Walker universe model with a scalar field, see preprint KOBE-TH-93-04

a is the scale factor, and “extended” Robertson-Walker time is identified with the usual one in classically allowed region by an appropriate shift of this time (axis). It is also shown that, in a simple minisuperspace model, probability density is always positive (even in classically forbidden region) if the universe is expanding.

2. We apply the quantum potential interpretation to the quantum geometrodynamics. We recapitulate the ADM canonical formalism. The $(3+1)$ decomposition of the metric reads

$$ds^2 = -(Ndt)^2 + (dx^i + N^i dt)(dx^j + N^j dt)h_{ij}, \quad (1)$$

where N, N^i and h_{ij} are the lapse, shift functions and spatial metrics. The action for the Einstein gravity plus a real scalar field in the unit $16\pi G = 1$, is cast in the form,

$$\begin{aligned} S &= \int d^4x \sqrt{-g} R^{(4)} - \frac{1}{2} \int d^4x \sqrt{-g} [g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi)] + \text{surface term} \\ &= \int d^4x N \sqrt{h} (K_{ij} K^{ij} - K^2 + R^{(3)}) + \text{surface term} \\ &\quad + \int d^4x \sqrt{h} \left[\frac{1}{2N} \dot{\phi}^2 - \frac{N^i}{N} \partial_i \phi \dot{\phi} - \frac{N}{2} h^{ij} \partial_i \phi \partial_j \phi + \frac{1}{2N} (N^i \partial_i \phi)^2 - \frac{N}{2} V(\phi) \right], \end{aligned} \quad (2)$$

where $K_{ij} = \frac{1}{2N} (\dot{h}_{ij} - N_{i|j} - N_{j|i})$ is the extrinsic curvature, $K = h_{ij} K^{ij}$ is its trace, a dot stands for the differentiation with respect to time and $R^{(3)}$ denotes the three dimensional scalar curvature. The stroke indicates the covariant derivative defined by the spatial metric h_{ij} . The canonical conjugate momentum π^{ij} to h_{ij} is given by

$$\pi^{ij} = \sqrt{h} (K^{ij} - h^{ij} K). \quad (3)$$

It is straight forward to obtain the action in the phase space,

$$S = \int d^4x [\pi^{ij} \dot{h}_{ij} - NH - N_i H^i] + \text{surface term}, \quad (4)$$

$$\begin{aligned} H &= \frac{1}{\sqrt{h}} (\pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2) - \sqrt{h} R^{(3)} \\ &\quad + \frac{1}{\sqrt{h}} \frac{\pi_\phi^2}{2} + \frac{1}{2} \sqrt{h} h^{ij} \partial_i \phi \partial_j \phi + \frac{\sqrt{h}}{2} V(\phi), \end{aligned} \quad (5)$$

$$H^i = -2\pi_{ij}^i + \pi_\phi \partial^i \phi, \quad (6)$$

where $\pi = h_{ij} \pi^{ij}$ and $\pi_\phi = \frac{\sqrt{h}}{N} \dot{\phi} - \sqrt{h} \frac{N^i}{N} \partial_i \phi$ is the canonical conjugate momentum to ϕ .

The wave function of the universe $\Psi(h_{ij}(x), \phi(x))$ satisfies the equations (assuming that the Universe is closed) [1]

$$[h^{-q} \frac{\delta}{\delta h_{ij}} h^q G_{ijkl} \frac{\delta}{\delta h_{kl}} + \sqrt{h} R^{(3)} + \frac{1}{2\sqrt{h}} \frac{\delta^2}{\delta \phi^2} - \frac{1}{2} \sqrt{h} h^{ij} \partial_i \phi \partial_j \phi - \frac{1}{2} \sqrt{h} V(\phi)] \Psi = 0, \quad (7)$$

$$i[2D_j \frac{\delta}{\delta h_{ij}} - h^{ij} \partial_j \phi \frac{\delta}{\delta \phi}] \Psi = 0, \quad (8)$$

where $h = \det(h_{ij})$, the parameter q represents the factor-ordering ambiguity, D_j is a co-variant derivative in the metric h_{ij} , and $G_{ijkl} = \frac{1}{2} h^{-1/2} (h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl})$. Eq.(7) is the Wheeler-DeWitt (WDW) equation. We write Ψ in a polar decomposition as $\Psi(h_{ij}, \phi) = A(h_{ij}, \phi) \exp(iS(h_{ij}, \phi))$, in terms of which the WDW equation (7) reads

$$G_{ijkl} \frac{\delta S}{\delta h_{ij}} \frac{\delta S}{\delta h_{kl}} + \frac{1}{2} \frac{1}{\sqrt{h}} \left(\frac{\delta S}{\delta \phi} \right)^2 - \sqrt{h} (R^{(3)} - Q_G) + \frac{1}{2} \sqrt{h} h^{ij} \partial_i \phi \partial_j \phi + \frac{1}{2} \sqrt{h} (V(\phi) + Q_M) = 0, \quad (9)$$

$$Q_G(h_{ij}, \phi) = -h^{-1/2} \frac{1}{A} \left(G_{ijkl} \frac{\delta^2 A}{\delta h_{ij} \delta h_{kl}} + h^{-q} \frac{\delta h^q G_{ijkl}}{\delta h_{ij}} \frac{\delta A}{\delta h_{kl}} \right), \quad (10)$$

$$Q_M(h_{ij}, \phi) = -h^{-1} \frac{1}{A} \left(\frac{\delta^2 A}{\delta \phi^2} \right), \quad (11)$$

$$\frac{\delta}{\delta h_{ij}} [2h^q G_{ijkl} \frac{\delta S}{\delta h_{kl}} A^2] + \frac{\delta}{\delta \phi} \left[\frac{h^q}{\sqrt{h}} \frac{\delta S}{\delta \phi} A^2 \right] = 0. \quad (12)$$

These two equations (9)(12) come from the real and imaginary part of the WDW equation(7) respectively. In the same way, eq.(8) reads

$$2D_j \frac{\delta A}{\delta h_{ij}} - h^{ij} \partial_j \phi \frac{\delta A}{\delta \phi} = 0, \quad (13)$$

$$2D_j \frac{\delta S}{\delta h_{ij}} - h^{ij} \partial_j \phi \frac{\delta S}{\delta \phi} = 0. \quad (14)$$

We shall call Q_G and Q_M the gravitational quantum potential and the matter quantum potential. Clearly eq.(9) resembles the Hamilton-Jacobi equation except for additional terms Q_G, Q_M . This suggests that we may regard the quantum universe as a universe with momentum $\pi^{ij} = \frac{\delta S}{\delta h_{ij}}, \pi_\phi = \frac{\delta S}{\delta \phi}$ subject not only to the classical potentials, but also to the quantum potentials Q_G, Q_M . Following quantum potential interpretation, we assume that

$$\frac{\delta S}{\delta h_{kl}} = \pi^{kl} \equiv \sqrt{h} (K^{kl} - h^{kl} K), \quad \frac{\delta S}{\delta \phi} = \pi_\phi \equiv \frac{\sqrt{h}}{N} \dot{\phi} - \sqrt{h} \frac{N^i}{N} \partial_i \phi. \quad (15)$$

Then eq.(9), eq.(12) and eq.(14) become

$$G_{ijkl}\pi^{ij}\pi^{kl} + \frac{1}{2}\frac{1}{\sqrt{h}}\pi_\phi^2 - \sqrt{h}(R^{(3)} - Q_G) + \frac{1}{2}\sqrt{h}h^{ij}\partial_i\phi\partial_j\phi + \frac{1}{2}\sqrt{h}(V(\phi) + Q_M) = 0, \quad (16)$$

$$\frac{\delta}{\delta h_{ij}}[2PK_{ij}] + \frac{\delta}{\delta\phi}[P\frac{1}{\sqrt{h}}\pi_\phi] = 0, \quad \text{with } P \equiv h^q A^2, \quad (17)$$

$$-2\pi_{|j}^{ij} + \pi_\phi\partial^i\phi = 0, \quad (18)$$

where we have used $G_{ijkl}\pi^{kl} = K_{ij}$. Since, roughly speaking, K_{ij} represents the rate of change of the metric h_{ij} with time, it seems to be reasonable that $\tilde{J}_{ij} \equiv 2PK_{ij}$, $\tilde{J}_\phi \equiv P\frac{1}{\sqrt{h}}\pi_\phi$ are regarded as the probability flux of the universe in a statistical ensemble of well-defined trajectories. This becomes clearer if we adopt the gauge $N = 1, N_i = 0$. Then eq.(17) becomes

$$\frac{\delta}{\delta h_{ij}}(P\dot{h}_{ij}) + \frac{\delta}{\delta\phi}(P\dot{\phi}) = 0, \quad \text{with } P \equiv h^q A^2. \quad (19)$$

Eq.(19) is regarded as the continuity equation, and $\tilde{J}_{ij} \equiv P\dot{h}_{ij}$, $\tilde{J}_\phi \equiv P\dot{\phi}$ are the probability flux. The trajectories of the universe acted on not only by classical potentials, but also by quantum potentials Q_G, Q_M can be, in principle, calculated by integrating the equations(15) under consideration of "constraints"(16) and (18), where $S(h_{ij}, \phi)$ and $A(h_{ij}, \phi)$ are determined by a solution to the WDW equation.

The equations of motion of a universe are

$$\dot{h}_{ij} = \frac{2N}{\sqrt{h}}\pi_{ij} - \frac{N}{\sqrt{h}}h_{ij}h_{mn}\pi^{mn} + N_{i|j} + N_{j|i}, \quad (20)$$

$$\begin{aligned} \dot{\pi}^{ij} = & -N\sqrt{h}\{R^{ij} - \frac{1}{2}h^{ij}(R^{(3)} - Q_G) + \frac{\delta Q_G}{\delta h_{ij}}\} \\ & + \frac{N}{2}\frac{h^{ij}}{\sqrt{h}}(\pi^{mn}\pi_{mn} - \frac{1}{2}\pi^2) - \frac{2N}{\sqrt{h}}(\pi^{im}\pi_m^j - \frac{1}{2}h_{mn}\pi^{mn}\pi^{ij}) \\ & - N_{|m}^i\pi^{mj} - N_{|m}^j\pi^{mi} + (\pi^{ij}N^m)_{|m} + \sqrt{h}(N^{ij} - h^{ij}N_{|m}^{lm}) \\ & + \frac{N}{2}[\frac{1}{2}\frac{h^{ij}}{\sqrt{h}}\pi_\phi^2 - \frac{1}{2}h^{ij}h^{mn}\partial_m\phi\partial_n\phi - \sqrt{h}\partial^i\phi\partial^j\phi \\ & - \frac{1}{2}\sqrt{h}h^{ij}(V(\phi) + Q_M) - \sqrt{h}\frac{\delta Q_M}{\delta h_{ij}}], \end{aligned} \quad (21)$$

$$\dot{\phi} = N \frac{\pi_\phi}{\sqrt{h}} + N^i \partial_i \phi, \quad (22)$$

$$\dot{\pi}_\phi = \partial_j (N \sqrt{h} h^{ij} \partial_i \phi) - N \frac{\sqrt{h}}{2} \left(\frac{\delta V(\phi)}{\delta \phi} + \frac{\delta Q_M}{\delta \phi} \right) - N \sqrt{h} \frac{\delta Q_G}{\delta \phi} + \partial_i (N^i \pi_\phi). \quad (23)$$

These equations can be obtained by the variation of the following “action” containing quantum potentials:

$$S_Q = \int d^4x [\pi^{ij} \dot{h}_{ij} + \pi_\phi \dot{\phi} - N H_Q(\pi^{ij}, h_{ij}) - N_i H^i(\pi^{ij}, h_{ij})], \quad (24)$$

$$\begin{aligned} H_Q &= \frac{1}{\sqrt{h}} (\pi^{ij} \pi_{ij} - \frac{1}{2} \pi^2) - \sqrt{h} (R^{(3)} - Q_G) \\ &+ \frac{1}{\sqrt{h}} \frac{\pi_\phi^2}{2} + \sqrt{h} \frac{1}{2} h^{ij} \partial_i \phi \partial_j \phi + \frac{\sqrt{h}}{2} (V(\phi) + Q_M), \end{aligned} \quad (25)$$

$$H^i = -2\pi^{ij}_{|j} + \pi_\phi \partial^i \phi. \quad (26)$$

To solve eq.(20) ~ eq.(23) is very difficult, since we must solve the WDW equation (7) first and obtain the quantum potentials Q_G, Q_M . Hereafter, our attention will be concentrated on minisuperspace models in which the infinite number of degrees of freedom of the gravitational and matter fields are restricted to a finite number.

3. We shall consider the de Sitter minisuperspace model [6][9]. It is described by the closed Robertson-Walker metric

$$ds^2 = \sigma^2 [-N^2 dt^2 + a(t)^2 d\Omega_3^2]. \quad (27)$$

where $\sigma^2 = \frac{2G}{3\pi}$, $a(t)$ is the scale factor and $d\Omega_3^2$ is the metric on the unit three-sphere. The Einstein action with (rescaled) cosmological constant for this metric is

$$S = \frac{1}{2} \int dt N \left[-\frac{a\dot{a}^2}{N^2} + a - \Lambda a^3 \right]. \quad (28)$$

The WDW equation in this model is given by [9]

$$\left\{ \frac{\partial^2}{\partial a^2} + \frac{p}{a} \frac{\partial}{\partial a} - a^2 [1 - a^2 \Lambda] \right\} \Psi(a) = 0, \quad (29)$$

where the parameter p represents the factor-ordering ambiguity. We write Ψ in a polar decomposition as $\Psi(a) = A(a) \exp\{iS(a)\}$, in terms of which eq.(29) can be rewritten as

$$- \left(\frac{\partial S}{\partial a} \right)^2 - a^2 + a^4 (\Lambda + Q_G(a)) = 0, \quad (30)$$

$$\partial_a (a^p A^2 \partial_a S) = 0. \quad (31)$$

These two equations come from the real and imaginary part of eq.(29), respectively, and the gravitational quantum potential is

$$Q_G(a) = a^{-4} \frac{1}{A} \left(\frac{\partial^2 A}{\partial a^2} + \frac{p}{a} \frac{\partial A}{\partial a} \right). \quad (32)$$

Following quantum potential interpretation, we identify $\partial_a S$ with the canonical momentum p_a to a :

$$-a\dot{a} \equiv p_a = \frac{\partial S}{\partial a}. \quad (33)$$

Then, eq.(30) and eq.(31) become

$$\dot{a}^2 = -1 + a^2(\Lambda + Q_G(a)), \quad (34)$$

$$\partial_a(P\dot{a}) = 0, \quad \text{with } P \equiv a^{p+1}A^2, \quad (35)$$

where a dot stands for the differentiation with respect to “extended” Robertson-Walker time, which describes causally deterministic trajectory of the universe not only in classically allowed region ($\Lambda a^2 > 1$), but also in classically forbidden region ($\Lambda a^2 < 1$). In the region of minisuperspace where $|Q_G| \ll \Lambda$, eq.(34) approaches to the classical equation of motion for the de Sitter space. We note that the probability density $\tilde{J}_a \equiv P\dot{a}$ is conserved if $a(t)$ is a solution to eq.(33). The continuity equation (35) implies conservation of probability density. The choice $p = 1$ corresponds to the minisuperspace model (with a scalar field) dealt with by Vink [7], from the viewpoint of quantum potential interpretation. We will keep the value of p unspecified. With the choice $p = -1$, eq.(29) can be solved exactly [8]. The solution corresponding to the Vilenkin boundary condition is (disregarding the normalization constant)

$$\Psi_V = \frac{\text{Ai}(z) + i\text{Bi}(z)}{\text{Ai}(z_0) + i\text{Bi}(z_0)}, \quad (36)$$

where

$$z = (2\Lambda)^{-\frac{1}{3}}(1 - a^2\Lambda), \quad (37)$$

$z_0 = z(a = 0)$, and $\text{Ai}(z)$, $\text{Bi}(z)$ are Airy functions. This wave function(36) corresponds to a positive frequency out-going mode. In classically allowed region, the positive frequency mode corresponds to the expanding universe. From (36), gravitational quantum potential (32) becomes

$$\begin{aligned} Q_G(a) = & (2V)^{\frac{2}{3}} a^{-2} \{ (\text{Ai}'^2 + z\text{Ai}^2 + \text{Bi}'^2 + z\text{Bi}^2)(\text{Ai}^2 + \text{Bi}^2)^{-1} \\ & - (\text{AiAi}' + \text{BiBi}')^2 (\text{Ai}^2 + \text{Bi}^2)^{-2} \}, \end{aligned} \quad (38)$$

where dash stands for the differentiation with respect to z . “Total potential” $W(a) \equiv 1 - a^2(\Lambda + Q_G(a))$ is shown in Fig.1. Fig.1 shows that the universe can start at $a \approx 0$. The asymptotic forms of the Airy functions at large value of z ($z \rightarrow +\infty$) are :

$$Ai(z) \sim \frac{1}{2}\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}\exp(-\zeta), \quad (39)$$

$$Bi(z) \sim \pi^{-\frac{1}{2}}z^{-\frac{1}{4}}\exp(\zeta), \quad (40)$$

$$Ai(-z) \sim \pi^{-\frac{1}{2}}z^{-\frac{1}{4}}\sin(\zeta + \pi/4), \quad (41)$$

$$Bi(-z) \sim \pi^{-\frac{1}{2}}z^{-\frac{1}{4}}\cos(\zeta + \pi/4), \quad (42)$$

where $\zeta = \frac{2}{3}z^{\frac{3}{2}}$. We assume that $0 < \Lambda \ll 1$ in Planck unit. Using these asymptotic forms, gravitational quantum potential (38) for small scales ($a \rightarrow 0$) and large scales ($a \rightarrow \infty$) is given by

$$Q_G(a) \approx a^{-2} - \Lambda, \quad \text{for } a \ll \frac{1}{\sqrt{\Lambda}}, \quad (43)$$

$$Q_G(a) \approx 0, \quad \text{for } a \gg \frac{1}{\sqrt{\Lambda}}. \quad (44)$$

Then, eq.(34) becomes

$$\dot{a}^2 \approx 0, \quad \text{for } a \ll \frac{1}{\sqrt{\Lambda}}, \quad (45)$$

$$\dot{a}^2 \approx -1 + a^2\Lambda, \quad \text{for } a \gg \frac{1}{\sqrt{\Lambda}}. \quad (46)$$

For the exact gravitational quantum potential (38), we have solved eq.(34) numerically (see Fig.2). Fig.2 and eq.(46) shows explicitly that “extended” Robertson-Walker time is identified with the usual one at large scales ($\Lambda a^2 \gg 1$) in classically allowed region ($\Lambda a^2 > 1$) by an appropriate shift of this time (axis).

4. We have applied quantum potential interpretation to quantum geometrodynamics based on the ADM canonical formalism, and discussed how to obtain the trajectories of the universe acted on not only by classical potentials, but also by quantum potentials. It was pointed out that the trajectories of the universe can be, in principle, calculated by integrating the equations(15) under consideration of “constraints”(16) and (18). We have studied the de Sitter minisuperspace model from the viewpoint of the quantum potential interpretation, by introducing the “extended” Robertson-Walker time which exists even in classically forbidden region.

It was shown that the universe described by Vilenkin wave function can start at $a \approx 0$, with well-defined trajectories, and “extended” Robertson-Walker time is identified with usual one at large scales ($\Lambda a^2 \gg 1$) in classically allowed region ($\Lambda a^2 > 1$) by an appropriate shift of this time (axis). It was also shown that, in quantum potential interpretation, the probability density $\rho_a(\phi) = \tilde{J}_a = P\dot{a}$ is always positive (even in classically forbidden region) if the universe is expanding ($\dot{a} > 0$).

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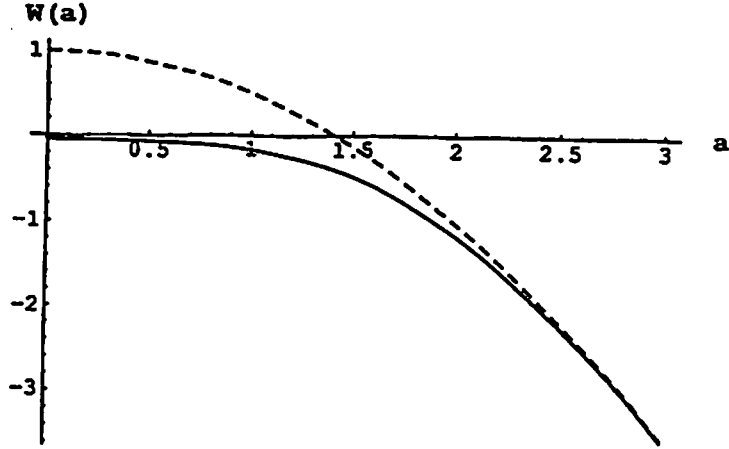


Fig.1 "Total potential" $W(a) \equiv 1 - a^2(\Lambda + Q_G(a))$ for $Q_G(a)$ given by eq.(38) (solid curve), $Q_G(a) = 0$ (dashed curve), with $\Lambda = 0.5$ in Planck unit.

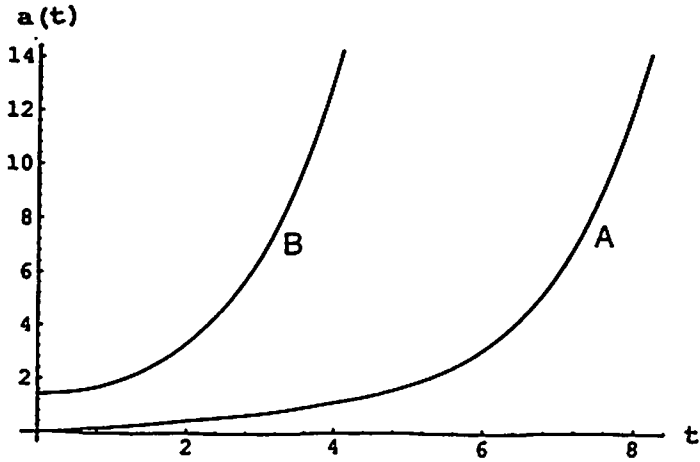


Fig.2 Solutions $a(t)$ to eq.(34) for $Q_G(a)$ given by eq.(38)(label A), $Q_G(a) = 0$ (label B), with $\Lambda = 0.5$ in Planck unit. The equation (34) for $Q_G(a)$ given by eq.(38) was integrated numerically, by starting from $t = 0$ at $a \approx 0$ (in this example, we have actually used $t = 0$ at $a = 0.01$). The curve labeled by B represents the de Sitter space $a(t) = \frac{1}{\sqrt{\Lambda}} \cosh(\sqrt{\Lambda}t)$ for $t \geq 0$.

Graphical Representation of Invariants in General Relativity and Quantum Gravity

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We present a graphical way to describe invariants and covariants in the (4 dim) general relativity. This makes us free from the complexity in the treatment of many suffixes . Two new off-shell relations between $(\text{mass})^{* * 6}$ invariants are obtained. These are important for 2-loop off-shell calculation in the perturbative quantum gravity. We list up all independent invariants with dimensions of $(\text{mass})^{* * 4}$ and $(\text{mass})^{* * 6}$. Furthermore the 6 dim Gauss-Bonnet identity is expressed in terms of the independent invariants. For further detail, see Ref.[1].

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Quantized Topological 2-form Gravity

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Abstract

We develop a perturbation theory of four-dimensional topological 2-form gravity without cosmological constant. A 2-form and an $SU(2)$ connection 1-form are used as fundamental variables instead of metric. There is no quantum correction from two-loop and higher orders in covariant gauge, in Landau gauge and in background covariant gauge. We improve naive dimensional regularization and calculate an exact quantum correction at one-loop order. We observe that the renormalizability depends on gauge choice in topological 2-form gravity.

1 Introduction

How to describe the four dimensional quantum gravity is a longstanding problem in theoretical physics. To construct a renormalizable gravity theory in four dimensions, it seems that we need new variables or new actions instead of metric or the Einstein action. One of the candidates of the new variable, self-dual 2-form, was introduced in in the early 1960s [1] [2]. We will refer to it as 2-form Einstein gravity. However the unrenormalizability of the theory still remains as a major problem. To avoid renormalizability problem, topological gravity has studied, which is apparently free from divergence. Recently a close connection between 2-form Einstein gravity and topological gravity was shown [3] [4]. The action of topological 2-form gravity is obtained by dropping a certain constraint term in the action of 2-form Einstein gravity. This constraint term is proportional to the Weyl tensor, which governs the modes of the gravitational waves. Thus topological 2-form gravity is interpreted as a theory which can describe the vacuum state (i.e. the state of no gravitational waves) of the Einstein gravity. Since we would like to construct a renormalizable theory of gravity and demonstrate its finiteness in a simpler model, we consider topological 2-form gravity in this paper. In the near future, we will be able to attack the 2-form Einstein gravity to make good use of this experience.

2 Review of 2-form Einstein Gravity

The action of the Euclidean¹ 2-form Einstein gravity is given in terms of a 2-form Σ^k and an $SU(2)$ spin connection 1-form ω_k in the presence of the cosmological constant λ ,

$$S = \int \Sigma^k \wedge R_k + \lambda \Sigma^k \wedge \Sigma_k + \frac{1}{2} \psi_{ij} \Sigma^i \wedge \Sigma^j, \quad (1)$$

where $R_k \equiv d\omega_k + (\omega \times \omega)_k$ and ψ_{ij} is a symmetric trace-free Lagrange multiplier field. Here we use the notation for $SU(2)$ indices $(F \times G)_i = \epsilon_{ijk} F^j G^k$ and ϵ_{ijk} is the structure constant of $SU(2)$. The $SU(2)$ indices $i, j, k, \dots = 1, 2, 3$ imply that they transform under *chiral* local-Lorentz representation (1, 0) of $SO(4) = SU(2) \times SU(2)/\mathbb{Z}_2$. In this formulation, the metric field $g_{\mu\nu}$ is defined in terms of the 2-form Σ^k as

$$\begin{aligned} g^{\frac{1}{2}} g_{\mu\nu} &= \frac{1}{12} \epsilon^{\alpha\beta\gamma\delta} \Sigma_{\mu\alpha}^k (\Sigma_{\beta\gamma} \times \Sigma_{\delta\nu})_k, \\ g &\equiv \det(g_{\mu\nu}). \end{aligned} \quad (2)$$

The constraint equation, which is obtained by varying eq.(1) with respect to ψ_{ij} , implies that Σ^k is composed as a wedge product of 1-forms e^k :

$$(\text{trace-free}) \Sigma^i \wedge \Sigma^j = 0 \iff \Sigma_{\mu\nu}^k = -\epsilon_{\mu\nu\lambda\rho} e^{\lambda a} e^{\rho b} \eta_{ab}^k \quad (3)$$

here η_{ab}^k is t'Hooft's η symbol, which connects $SU(2)$ index k and $SO(4)$ indices a, b and self-dual for $SO(4)$ indices [5]. Using this result and translating the $SU(2)$ spinor indices into the $SO(4)$ (the local-Lorentz) indices, the action (1) becomes identical with the chiral decomposition of the first-order Palatini action in which the usual spin connection is replaced by its anti-self-dual part with respect to the $SO(4)$ indices. As is discussed in [6], this chiral action also gives the Einstein equation. With the help of the equations of motion derived from the action (1), this ψ_{ij} is determined to be proportional to the anti-self-dual part of the Weyl tensor which just governs the modes of the gravitational waves.

Since the action (1) describes general relativity, it is invariant under the local-Lorentz transformation and diffeomorphism,

$$\begin{aligned} \delta\omega^i &= D\phi^i + \mathcal{L}_\xi \omega^i, \\ \delta\Sigma^i &= \epsilon_{ijk} \Sigma^j \phi^k + \mathcal{L}_\xi \Sigma^i, \end{aligned} \quad (4)$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to a vector field ξ^μ , and the chiral local-Lorentz transformation corresponds to the $SU(2)$ gauge transformation with a parameter ϕ^k . We use the following notation for spin-covariant exterior derivative D : $D\phi^i \equiv d\phi^i + 2\epsilon_{ijk}\omega^j\phi^k$, in this and the next section.

¹In case of Lorentzian signature, we need reality condition to recover real metric. Therefore in this paper we will work in Euclidean signature for simplicity.

3 Classical Topological 2-form Gravity

Let us consider the situation when we drop the last constraint term (3) in the action (1). Then the action becomes

$$S_{classical} = \int \Sigma^k \wedge R_k + \lambda \Sigma^k \wedge \Sigma_k . \quad (5)$$

In this particular case, a new symmetry with a parameter 1-form θ^k emerges in addition to the chiral local Lorentz symmetry, which we will call as θ_1 -symmetry,

$$\begin{aligned} \delta\omega^i &= D\phi^i - 2\lambda\theta^i , \\ \delta\Sigma^i &= 2\epsilon_{ijk}\Sigma^j\phi^k + D\theta^i . \end{aligned} \quad (6)$$

Although the theory remains invariant under diffeomorphism just as in the previous section, there is no need to add it to eq.(6) because, modulo the equations of motion derived from (5), diffeomorphism can be generated by the combination of the above chiral local-Lorentz and θ_1 -transformations (6) with $\phi^k = \xi^\nu\omega_\nu^k$ and $\theta_\mu^k = 2\xi^\nu\Sigma_{\nu\mu}^k$. With the appearance of the θ_1 -symmetry, the theory turns out to be on-shell reducible in the sense that the classical gauge transformation laws (6) are invariant under

$$\begin{aligned} \delta\phi^k &= 2\lambda v^k , \\ \delta\theta^k &= Dv^k , \end{aligned} \quad (7)$$

if the equations of motion are satisfied.

4 An Improved Dimensional Regularization

We have to calculate some superficially divergent Feynman integrals in order to demonstrate that topological 2-form gravity theory is finite. To obtain finite quantum corrections in a perturbation theory, we need some kind of regularization. It seems that the best regularization scheme is the Pauli-Villars regularization [7],[8], in which the gauge invariance is manifest. However in our model, owing to the existence of unusual 2-form $\Sigma_{\mu\nu}^k$, we do not know how to introduce regulator fields. We use an improved dimensional regularization scheme in this paper. The point is that the dimensions of the symbol $\epsilon^{\mu\nu\lambda\rho}$ is fixed to be four.

We are confronted by difficulty: In the quantum action, the field $\Sigma_{\mu\nu}^k$ in the classical term remains four-dimensional, while $\Sigma_{\mu\nu}^k$ in the gauge-fixing term is extended to be n -dimensional. The same situation occurs in the field ω_μ^k . Then we cannot obtain the propagators in the extended dimensions.

Thus our improved regularization scheme is not justified in the extended dimensions.

Putting aside the above difficulty, we will proceed to carry out our calculation by our improved dimensional regularization.

5 Topological 2-form Gravity in Covariant Gauge

In the topological theories, metric is needed only for the gauge fixing. We choose the flat Euclidean metric as a background metric. Owing to the reducibility, we use powerful BFV Hamiltonian formalism² for the topological 2-form gravity, which was carried out in Ref.[4].

So far we have obtained quantum tree action and BRST transformations in the presence of cosmological constant λ . Although we will be able to proceed in this general model, we would like to proceed in a simpler model. Setting $\lambda = 0$, now the classical symmetry becomes

$$\begin{aligned}\delta\omega^i &= D\phi^i, \\ \delta\Sigma^i &= 2\epsilon_{ijk}\Sigma^j\phi^k + D\theta^i.\end{aligned}\tag{8}$$

The on-shell reducibility becomes

$$\begin{aligned}\delta\phi^k &= 0, \\ \delta\theta^k &= Dv^k.\end{aligned}\tag{9}$$

We have chosen the gauge fixing conditions for chiral local-Lorentz, θ_1 -symmetry and on-shell reducible symmetries³, respectively:

$$\begin{aligned}\partial^\mu\omega_{\mu i} &= 0, \\ D^\nu\Sigma_{\mu\nu i} &= 0, \\ D^\mu C_{\mu i} &= D^\mu\bar{C}_{\mu i} = 0,\end{aligned}\tag{10}$$

where C^μ_i and $\bar{C}^\mu_i$ are fermionic ghost and anti-ghost, respectively, for θ_μ^k parameter in θ_1 -symmetry. According to BFV quantization method we enlarge the phase space as follows. We introduce fermionic ghost C^k and anti-ghost $\bar{C}^k$ for ϕ^k parameter in chiral local Lorentz symmetry and bosonic ghost-for-ghost C_1^k and its anti-field $\bar{C}_1^k$ for v^k parameter in the reducible symmetry. The subscript 1 means the class of the first stage in the gauge

²We also used BV Lagrangian formalism [9] for the check of our calculation; both results agree with each other.

³Should we fix such the on-shell symmetry? The answer is 'Yes'. If we do not fix it, we cannot construct propagators.

structure. We also introduce Lagrange multiplier fields: bosonic π^k for the gauge fixing of chiral local Lorentz, bosonic $\hat{\pi}_\mu^k$ for that of θ_1 -symmetry, and fermionic $\hat{\pi}_1^k$ and τ^k for that of reducible symmetries. Besides these fields, we need bosonic extra-ghost ρ^k in order to make the theory covariant completely.

Using the BRST invariant action, propagators can be written down. All the fields in this theory cannot freely propagate; they necessarily change into different fields in the course of propagation.

There has been an expectation that effects of higher order loops do not contribute in topological field theories in general. As a matter of fact in the three-dimensional non-Abelian BF theory and Chern-Simons theory, it is proven that there exist at most one-loop corrections [10]. In this analogy we are naturally led to the following conjecture: There exist at most one-loop quantum corrections in Topological 2-form Gravity in the covariant gauge.

We have proved that this is the truth and obtained the following relation:

$$L - 1 + E_\omega + E_{\hat{\pi}_\mu} + E_{\hat{\pi}_1} + E_{\hat{C}_\mu} + E_{\hat{C}_1} + E_C = 0. \quad (11)$$

Here we denote the number of Y -field external lines in a connected Feynman diagram H as E_Y , the number of loops in H as L . Since the number of each external line is not less than 0, there is no connected diagram whose number of loops is greater than one. We can get the exact result from at most one-loop calculation.

Now we study the one-particle-irreducible (1PI) connected diagrams with external lines amputated. Since we are interested in the vertex functions, the important fields are connected with the amputated external lines. Now only ω_μ^i and $\hat{\pi}_\mu^i$ are such fields at one loop order. After integrating out dynamical fields in the effective action, there remain only these two fields which appear in quantum corrections to the action. We denote the number of these fields as X_ω and $X_{\hat{\pi}_\mu}$, respectively in a 1PI diagram, say H . The superficially divergent graphs are

$$(X_\omega, X_{\hat{\pi}_\mu}) = 4 - (X_\omega + X_{\hat{\pi}_\mu}) \quad (12)$$

We obtain one-loop quantum corrections of superficially divergent graphs. We use 'background field method' for this calculation. In the 'background field method', we decompose amputated external fields into background c-number fields plus quantum q-number fields:

$$\begin{aligned} \omega_\mu^k &\rightarrow \omega_\mu^{k(q)} + \widetilde{\omega_\mu^k}, \\ \hat{\pi}_\mu^k &\rightarrow \hat{\pi}_\mu^{k(q)} + \widetilde{\hat{\pi}_\mu^k}. \end{aligned} \quad (13)$$

Here the superscript (q) in the right hand sides means quantum fields. The tilde over the fields in the right hand sides means that they are the background fields. We will omit the quantum label (q) from now on.

In the background field method, we can calculate the one-loop quantum corrections using only the terms which are bilinear in the quantum components. Combining these terms, we construct the superficially divergent diagrams.

First we consider the graph $(X_\omega, X_{\hat{\pi}_\mu}) = (0, 2)$.

Its quantum correction is,

$$-\Gamma_{\hat{\pi}_\mu \hat{\pi}_\mu} = -2\widetilde{\hat{\pi}}_\mu^i \widetilde{\hat{\pi}}_\rho^i \int \frac{d^4 p}{(2\pi)^4} \frac{\delta_{\mu\rho} p_\alpha (p+k)^\alpha - p^\rho (p+k)^\mu}{p^2 (p+k)^2}. \quad (14)$$

This quantum correction yields quadratic divergence. In order to cancel this divergence, we can think of adding the bilinear term of $\hat{\pi}_\mu$, which is constructed from adding this field to the gauge fixing condition for θ_1 -symmetry. However this term cannot be arranged to cancel eq.(14). Thus we conclude that we cannot renormalize this divergence. We consider that this unrenormalizable divergence is because of our gauge choice, and investigate this problem further in the next section.

6 Topological 2-form Gravity in Landau Gauge

Why there appeared unremovable divergence in four-dimensional topological 2-form gravity in the previous chapter? So far it has been believed that all the topological field theories are finite. In fact in the case of Schwartz type topological theories one can prove the finiteness of the theory from the fact that the space of solutions to the field equations, modulo the gauge symmetries, is finite dimensional. However the proof of finiteness applies only when the theory is restricted to an appropriate moduli space that the finiteness of the model is manifest. As we will see in this section, this restriction to a finite dimensional moduli space is nothing but a gauge choice.

Therefore we must choose alternative gauge fixing conditions to avoid the divergence in the two-point function of the Lagrange multiplier field $\hat{\pi}_\mu$. The essential observation is that in the Landau gauge, this field cannot enter in the three-point vertex. This fact strongly suggests us to use the Landau gauge. There is another reason to choose the Landau gauge. In the course of the calculations in the covariant gauge, we noticed the papers [11] [12] in which the finiteness of four dimensional BF theory in the Landau gauge was shown by using Slavnov identity.

From now on we use the following definition of covariant derivative and curvature for simplicity,

$$\begin{aligned} (D_\mu \theta)^i &= \partial_\mu \theta^i + \epsilon^{ijk} \omega_\mu^j \theta^k \\ R_{\mu\nu}^i &= \partial_\mu \omega_\nu^i - \partial_\nu \omega_\mu^i + \epsilon^{ijk} \omega_\mu^j \omega_\nu^k \end{aligned} \quad (15)$$

Now we choose the gauge fixing conditions for chiral local-Lorentz, θ_1 -symmetry and reducible symmetries, respectively, in the following Landau gauge:

$$\begin{aligned}\partial^\mu \omega_{\mu i} &= 0, \\ \partial^\nu \Sigma_{\mu\nu i} &= 0, \\ \partial^\mu C_{\mu i} &= \partial^\mu \bar{C}_{\mu i} = 0.\end{aligned}\tag{16}$$

The quantization of this model, owing to its reducibility, is not straightforward and requires the BFV or BV procedure in the same way as in the previous section. This quantization was carried out in [13]. Since we would like to use the same propagators in the previous section, we re-define some fields so that the kinetic terms are the same as in "covariant gauge". This action is also off-shell invariant under simpler BRST transformations.

The loop-structures in the theories are expected to be independent of the gauge-choices. Therefore in this gauge, we are also led to the conjecture: There is no quantum correction from two-loop and higher orders in Topological 2-form Gravity in the Landau gauge. In this gauge, we have also proved that this is the case. The following equation is obtained:

$$L - 1 + E_\omega + E_{\tilde{\pi}_\mu} + E_{\tilde{\pi}_1} + E_{\tilde{C}_\mu} + E_{\tilde{C}_1} + E_C = 0.\tag{17}$$

The superficial degree of divergence for the diagram H in the Landau gauge is,

$$\begin{aligned}\omega(H) &= 4 - (X_\omega + X_{\tilde{C}_\mu}) - 2X_{\tilde{C}_1} \\ &= 4 - (X_\omega + X_C).\end{aligned}\tag{18}$$

The exact quantum corrections in the Landau gauge are the following:

$$\begin{aligned}-\Gamma &= -\frac{1}{96\pi^2} \int d^4x [(\partial^\alpha \widetilde{\omega^{\beta i}})(\partial_\alpha \widetilde{\omega_{\beta i}}) + 2(\partial^\alpha \widetilde{\omega_{\alpha}^i})(\partial^\beta \widetilde{\omega_{\beta}^i}) \\ &\quad + \epsilon^{ijk}(\partial^\alpha \widetilde{\omega_{\alpha}^i})\widetilde{\omega_{\alpha j}^{\beta}}\widetilde{\omega_{\beta k}^{\alpha}} \\ &\quad - \frac{1}{4}\widetilde{\omega^{\alpha i}}\widetilde{\omega_{\alpha i}}\widetilde{\omega^{\beta j}}\widetilde{\omega_{\beta j}} - \frac{1}{2}\widetilde{\omega^{\alpha i}}\widetilde{\omega_{\alpha}^{\beta}}\widetilde{\omega_{\alpha}^j}\widetilde{\omega_{\beta j}}].\end{aligned}\tag{19}$$

Recall that the field ω_μ^i is invariant under the θ_1 -transformation, when the cosmological constant is 0. Therefore our result eq.(19) is invariant under the classical θ_1 -symmetry. However, this result is not covariant under the classical $SU(2)$ gauge transformation. Thus we observe that the classical $SU(2)$ symmetry breaks at the quantum level.

7 Topological 2-form Gravity in Background Covariant Gauge

In this section, we proceed in the background covariant gauge. We use an $SU(2)$ connection as the background field. It seems that this gauge is the most appropriate choice for the perturbation theory in the background field method. We choose the gauge fixing conditions for chiral local Lorentz symmetry, θ_1 -symmetry and reducible symmetries, respectively:

$$\begin{aligned} (\widetilde{D}^\mu \omega_\mu^{(q)})^i &= 0, \\ (\widetilde{D}^\nu \Sigma_{\mu\nu})^i &= 0, \\ (\widetilde{D}^\mu C_\mu)^i &= (\widetilde{D}^\mu \bar{C}_\mu)^i = 0. \end{aligned} \quad (20)$$

Here the tilde over the covariant derivative means that the connection of this covariant derivative $\widetilde{D}_\mu$ is the background $SU(2)$ connection $\widetilde{\omega}_\mu^i$.

We have also proved that there is no quantum correction from two-loop and higher orders in this gauge. We obtain the following equation:

$$\begin{aligned} L - 1 + E_{\widetilde{\omega}} + E_{\widetilde{\pi}_\mu} + E_{\widetilde{\pi}_1} + E_{\widetilde{C}_\mu} + E_{\widetilde{C}_1} + E_C \\ + \frac{1}{2}(V_{\Sigma\widetilde{\omega}\widetilde{\omega}} + V_{\widetilde{\omega}\widetilde{\omega}\widetilde{C}C} + V_{\widetilde{\omega}\widetilde{\omega}\widetilde{C}_\mu C_\nu} + V_{\widetilde{\omega}\widetilde{\omega}\widetilde{C}_1 C_1} + V_{\widetilde{\omega}\widetilde{\omega}\widetilde{C}_\mu \widetilde{C}_\nu C_1}) = 0. \end{aligned} \quad (21)$$

Here we denote the total number of vertices $\Sigma\widetilde{\omega}\widetilde{\omega}$ in a connected Feynman diagram as $V_{\Sigma\widetilde{\omega}\widetilde{\omega}}$.

At one loop order, the superficial degree of divergence for the diagram H in this gauge is,

$$\begin{aligned} \omega(H) &= 4 - (X_{\widetilde{\pi}_\mu} + 2X_{\widetilde{C}_1} + X_{\widetilde{C}_\mu} + 2X_{\widetilde{\pi}_1} + X_{\widetilde{\omega}}) \\ &= 4 - (X_{\widetilde{\pi}_\mu} + X_C + X_{\widetilde{\pi}_1} + X_{\widetilde{\omega}}). \end{aligned} \quad (22)$$

The quantum corrections in terms of the background $SU(2)$ connection $\widetilde{\omega}_\mu^i$ are:

$$\begin{aligned} -\Gamma &= -\lim_{n \rightarrow 4} \frac{1}{(n-4)192\pi^2} \int d^4x [56(\partial^\alpha \widetilde{\omega}^{\beta i})(\partial_\alpha \widetilde{\omega}_{\beta i}) \\ &\quad - 8(\partial^\alpha \widetilde{\omega}_\alpha^i)(\partial^\beta \widetilde{\omega}_{\beta i}) + 132\epsilon^{ijk}(\partial^\alpha \widetilde{\omega}_\alpha^i)\widetilde{\omega}_{\alpha j}\widetilde{\omega}_{\beta k} + 73\widetilde{\omega}^{\alpha i}\widetilde{\omega}_{\alpha i}\widetilde{\omega}^{\beta j}\widetilde{\omega}_{\beta j} \\ &\quad - 46\widetilde{\omega}^{\alpha i}\widetilde{\omega}^{\beta j}\widetilde{\omega}_{\alpha j}\widetilde{\omega}_{\beta i}] + \text{finite parts}. \end{aligned} \quad (23)$$

Unfortunately, this divergence cannot be removed. Thus we conclude that the background covariant gauge is not appropriate in our topological 2-form gravity.

8 Discussions

We have learned that the renormalizability inevitably depends on gauge choice in our topological 2-form gravity. Such a gauge-dependence often appears in topological field theories [14]. Using different gauge, but with the same regulator, yields different results. Therefore in our topological 2-form gravity, we also have to choose an appropriate gauge-fixing conditions in order to restrict the theory to an appropriate moduli space so that the theory becomes finite.

The Vilkovisky-De Witt effective action program gives a notion of a unique effective action [15], [14]. It is interesting to apply this program to our model. Then we may be able to show that the unique effective action in our model is the one for the Landau gauge. Moreover we would like to investigate the geometrical meaning of the Landau gauge.

Our improved dimensional regularization is invariant under the $SU(2)$ transformation. However our regularization is justified only in four dimensions. Therefore one of the possible origins of $SU(2)$ non-covariance in the quantum corrections is our dimensional regularization.

The details of this talk will be reported in Ref. [16].

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Reheating in the Kaluza-Klein Cosmology and the Quantum Group¹

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Abstract

In Kaluza-Klein (K-K) type cosmology, problems of reheating after inflation and collapsing of the internal spaces to singularity have to be resolved. These problems suggest that some changes of physical laws occur when the internal spaces approach the singularity. We propose to change the canonical commutation relations for the inflaton i.e. the scale factors of the internal spaces and show that, for a simplified model, the collapse is avoided. The change of the commutation relations can naturally be interpreted from the quantum group theoretical point of view.

§1. Introduction

Among the unified theory of elementary particles, K-K type theories are most popular. They have advantages in explaining internal degrees of freedom of elementary particles and inclusion of gravity. There are some important problems with these theories. Aside from the problem of how the whole spacetime has been decomposed into internal and external spaces, there are two important problems that should be explained. One is that no effects have been known which indicate the existence of the internal spaces. The other is the large asymmetry between the internal and external spaces. The inflation of the universe is the process that causes the asymmetry to occur. In addition, in K-K theories, inflation is caused by the scale factors of the internal spaces. Therefore investigation of inflation is important in order to support K-K theories. The process of inflation cools the

¹§2 is based on the results of collaboration with T.Yano (Dept. of Material Sciences, Faculty of Engineering, Ehime University) and T.Watanabe (Matsuyama City Office).

universe far below the temperature at which nucleosynthesis can occur. Thus reheating of the universe is necessary. We proposed a model which explains sufficient inflation¹⁾ and a simplified version of the model which can explain the reheating of the universe²⁾. In this work, we give an argument that the mechanism introduced in an ad hoc manner into the latter model can naturally be interpreted from the quantum group theoretical point of view.

§2. K-K Inflation

In this section we explain the mechanism of reheating in K-K cosmology. As inflationary models³⁾, K-K cosmological ones have an advantage that the nature of the inflaton field is known well.

Since the process of inflation is treated as a classical one, the action describes uniquely how the inflation proceeds. There are essentially two types of actions which have been considered so far:

- (i) Einstein-Hilbert action in higher dimensional spacetime.
- (ii) Action with additional higher order curvature terms.

In type (ii) models, additional terms are introduced in order to take into account the quantum effects. They give a minimum to the inflaton potential. This is desirable since reheating is usually assumed to be caused by the damping oscillation of the inflaton around the minimum of the potential. However the parameters of the model have not been chosen for the model to be successful⁴⁾. Furthermore only particular solutions have been found which give inflation. This is because the field equations derived from the action of type(ii) are so complicated that it is difficult to investigate how natural is the inflation⁵⁾.

Here we investigate reheating using type (i) action. Although sufficient inflation is possible with this action¹⁾, it is difficult to reheat the universe in the classical treatment. However the process of inflation is essentially a quantum mechanical one, since matters are produced during this process for the universe to evolve from the big bang. Therefore we treat the inflaton quantum mechanically for the reheating process²⁾. We assume that if the inflaton has the ground state, it plays the role of the minimum of the potential.

The inflaton potential is somewhat complicated, so we simplify it and adopt the following Hamiltonian;

$$H = \frac{1}{2} (x p)^2 + V(x), \quad V(x) = -\frac{k}{x^\alpha}. \quad (2.1)$$

Here x corresponds to the scale factor of the internal space and p is the momentum canonically conjugate to it. k and $\alpha(> 2)$ are positive constants.

If we quantize the system canonically, i.e. $[x, p] = i\hbar$, the solutions of the energy eigenvalue equation, $H\psi(x) = E\psi(x)$, are given in terms of Bessel functions: $\psi_E(x) =$

$N J_\epsilon(2\sqrt{2k}/\alpha x^{\alpha/2})$, where $\epsilon = \sqrt{-8E}$ and N is the normalization constant. The energy E can take arbitrary negative values. Therefore the system is unstable. This means that the internal space collapses and the universe is not reheated. We can interpret this due to the lack of the quantum effects or the lack of the uncertainty, Δp .

When the internal space continues to collapse, it approaches the singularity. So it is natural to think that some physical laws are modified and the singularity is avoided. The uncertainty relation is determined by the commutation relations. Therefore we assume that the commutation relations are modified as follows:

$$[x, p] = i\hbar(x), \quad \hbar(x) = \hbar(1 + \lambda x^{-\rho}), \quad (2.2)$$

i.e. we replace the Planck constant $\hbar$ by a "Planck function" $\hbar(x)$. Correspondingly the Schrödinger equation should be modified as;

$$i\hbar(x) \frac{\partial \Psi}{\partial t} = E \Psi(x, t), \quad (2.3)$$

then the "energy eigenvalue" equation is given by;

$$H \psi(x) = E[\hbar(x)/\hbar] \psi(x), \quad (2.4)$$

where the Hamiltonian is given by

$$\begin{aligned} H = & -\frac{1}{2} \left[x^2 \hbar^2(x) \frac{d^2}{dx^2} + x \hbar(x) \left\{ \hbar(x) + 2x \frac{d\hbar}{dx} \right\} \frac{d}{dx} \right. \\ & \left. + \frac{1}{2} x \hbar(x) \frac{d\hbar}{dx} + \frac{1}{4} \left(x \frac{d\hbar}{dx} \right)^2 + \frac{1}{2} x^2 \hbar(x) \frac{d^2 \hbar}{dx^2} \right] + V(x). \end{aligned} \quad (2.5)$$

It is difficult to solve this equation explicitly. So we investigate whether the ground state exists or not.

Putting

$$\psi(x) = u(x) \phi(x), \quad u(x) = c/\sqrt{x} \hbar(x), \quad c = \text{const.} \quad (2.6)$$

we obtain

$$\left[-\frac{1}{2} \frac{d^2}{dx^2} + U(x) \right] \phi(x) = 0, \quad (2.7)$$

where

$$\begin{aligned} U(x) = & \frac{1}{8} \frac{1}{x^2(1 + \lambda x^{-\rho})^2} \left[\lambda^2(\rho^2 - 1)x^{-2\rho} + 2\lambda(\rho^{-2} - 1 + 4|E|)x^{-\rho} \right. \\ & \left. - 8kx^{-\alpha} + 8|E| - 1 \right]. \end{aligned} \quad (2.8)$$

Equation (2.7) is thought of as the zero - eigenvalue equation for $\phi(x)$ with the potential $U(x)$. The potential goes to infinity as x approaches zero and approaches $+0$ as x goes

to infinity. The necessary and sufficient condition for the existence of the zero energy eigenfunction is that the potential becomes negative for some value of x . It is shown that the condition is satisfied if the $|E|$ is bounded. This means that the ground state of the inflaton exists.

The modification of the Planck constant to the Planck function is somewhat artificial. It is desirable to find some situation which supports the modification, especially geometrical one. As one of such supports, we introduce quantum group theoretical results obtained by A. Kempf in the next section.

§3. Quantum Group Theoretical Deformation and Uncertainty Relation

Recently A.Kempf obtained⁶⁾ very interesting results concerning the modification of the commutation relations and the uncertainty relations using quantum group theoretical method.

The starting point is the deformed commutation relation between the creation and annihilation operators:

$$a a^\dagger - q a^\dagger a = 1. \quad (3.1)$$

Then the coordinate x and the momentum p are assumed to be expressed as

$$x \equiv L(a^\dagger + a), \quad p \equiv iK(a^\dagger - a), \quad (3.2)$$

where constants L, K satisfy

$$KL = \hbar(q^2 + 1)/4. \quad (3.3)$$

The commutation relation of x and p is calculated to be

$$[x, p] = i\hbar \left[1 + (q^2 - 1) \left(\frac{x^2}{4L^2} + \frac{p^2}{4K^2} \right) \right]. \quad (3.4)$$

The uncertainty relation is also calculated by the standard method:

$$\begin{aligned} \Delta x \Delta p &\geq \frac{1}{2} |<[x, p]>| \\ &= \frac{\hbar}{2} \left[1 + (q^2 - 1) \left(\frac{(\Delta x)^2 + <x>^2}{4L^2} + \frac{(\Delta p)^2 + <p>^2}{4K^2} \right) \right]. \end{aligned} \quad (3.5)$$

The region in Δx - Δp plain that satisfies this relation is bounded by a hyperbola as in the undeformed case. But the asymptotes and the hyperbola itself are deformed so that, for $q > 1$, both Δx and Δp have the minima;

$$\begin{aligned} \Delta x_{\min} &= L\sqrt{A} \sqrt{\frac{q^2 - 1}{q^2}} \geq L\sqrt{\frac{q^2 - 1}{q^2}}, \\ \Delta p_{\min} &= K\sqrt{A} \sqrt{\frac{q^2 - 1}{q^2}} \geq K\sqrt{\frac{q^2 - 1}{q^2}}, \end{aligned} \quad (3.6)$$

where

$$A = 1 + (q^2 - 1) \left(\frac{\langle x \rangle^2}{4L^2} + \frac{\langle p \rangle^2}{4K^2} \right). \quad (3.7)$$

These results are general for systems with one degree of freedom.

§4. Discussion

We apply the general results described in the previous section to the simplified model investigated in §2. We can interpret the existence of $\Delta x_{\min}$ as the indication that the internal space does not collapse. The situation would be visualized as follows: The internal space is discretized and there exists a non-singular ground state. This situation is analogous to the one in the Bohr theory of atoms.

From eq.(3.5) we have

$$\Delta x \Delta p \geq \frac{1}{4} \hbar A (q^2 + 1) \geq \frac{1}{4} \hbar (q^2 + 1). \quad (4.1)$$

This equation is apparently independent of scale. So the problem is from what scale does the deformation becomes effective. Intuitively, the scale would be the Planck scale. It is desirable that the scale appears explicitly in the formulation. We assume that the deformation parameter q itself involves this information: $q = q(\ell)$, where ℓ is the scale parameter. Comparing the above equation with the Planck function, $\hbar(x)$, given in (2.2), we can assume the following form for $q(\ell)$;

$$q(\ell) = 1 + \lambda (\ell/\ell_P)^{-\rho}. \quad (4.2)$$

With this assumption, it may be that the quantum theory with modified commutation relations described in §2 as the effective theory of the theory based on the deformed commutation relation.

The discussion up to this point concerns only the kinematical aspects. It is interesting to investigate the dynamical aspects in order to confirm the above discussion.

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Quantum Theory of (2+1)-Dimensional Black Hole[†]

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1. Introduction

Two decades ago it was discovered by Stephen Hawking that a black hole can evaporate by radiating particles. The spectrum of emitted particles is thermal with a temperature inversely proportional to the black hole mass. The emission is statistical due to the existence of the event horizon, whose area yields the entropy.

This evaporation process is a quantum phenomenon of particle creation on a fixed curved background space-time. The back-reaction of the emitted particles to the space-time geometry has been studied only in the semi-classical approximation. However, the semi-classical picture of the black hole evaporation is not valid in the final stage, since the temperature becomes the Planck scale. One would expect that full quantum gravity plays a significant role in determining the final state of black hole.

There have been many attempts to attack this problem. One of them is (1+1)-dimensional black hole with dilaton field. Another is minisuperspace models of black hole in which the mass is a dynamical degree of freedom. They met some technical difficulties and cannot answer to the above question decisively.

On the other hand, there are many related, but rather conceptual and more fundamental questions in quantizing black hole space-time. For example, event horizon is a classical concept which is important in many respects of black hole physics as well as in the Hawking evaporation. How does such a classical concept play a role in quantum theory of gravity, if any? What is the observable to be quantized, horizon or its area or the mass of black hole? In what way is the quantum theory constructed in a specific way related with choice of observer? How would other

[†] This talk is based on the work [1].

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classical aspects such as “thermodynamical” properties of black hole be related to quantum theory of black hole?

In this talk, I investigate the classical and quantum aspects of (2+1)-dimensional black hole without matter fields. The space-time was recently found by Bañados, Teitelboim and Zanelli [2][3] as a solution of (2+1)-dimensional Einstein gravity with negative cosmological constant. Since it is a solvable dynamical system as I shall show, there is no technical difficulties such as non-renormalizability in four-dimensional Einstein gravity. More importantly, it would serve as a toy model that is useful in order to answer the problems such as the above conceptual ones.

The plan of this talk is as follows. In section 2, I present a brief description of three-dimensional black hole solution and their properties. Hopefully it may serve as an introduction to the subsequent talks on the related topics by M. Siino, T. Koike, and K. Shiraishi. Nevertheless, the main purpose of the description is to see that the three-dimensional black hole possesses a similar global causal structure as that of Schwarzschild space-time. In section 3, making an ansatz in which the mass of black hole is regarded as a dynamical variable, I solve the classical equations of motion in the so-called mean curvature slice (York gauge). Such solutions are shown to cover a portion of the entire black hole space-time. In section 4, I reduce this system in the ADM canonical theory in the York gauge. Then it is shown that appropriate treatment of the boundary terms arising from inner boundary and outer boundary is necessary and crucial to quantize the system. This point is discussed in section 5, as well as several interesting conceptual problems in quantizing black hole.

2. Black Hole in (2+1)-Dimensions

Recently, black hole solutions have been found in the (2+1)-dimensional Einstein gravity with negative cosmological constant[†] by Bañados, Teitelboim and Zanelli [2][3]. Note that since the space-time is three-dimensional, any solution of the Einstein equation with negative cosmological constant is locally isometric to the anti-de Sitter space-time. As is described below, the black hole space-time is a quotient space of (the universal covering space of) the anti-de Sitter space-time by a discrete subgroup of the isometry group of the anti-de Sitter space-time.

[†] Black hole solution does not exist if weak energy condition is satisfied, as follows from the Einstein equation.

First, recall the anti-de Sitter space-time. Let the line element of a four-dimensional flat pseudo-Riemannian space with signature $(- - ++)$ be

$$dS^2 = -dX_0^2 - dX_1^2 + dX_2^2 + dX_3^2.$$

Three-dimensional anti-de Sitter space-time AdS^3 can be expressed as an embedded (one-leaf) hyperboloid in it:

$$-X_0^2 - X_1^2 + X_2^2 + X_3^2 = -\ell^2,$$

which is a constant negative curvature space-time with Riemann curvature given by

$$R_{\mu\nu\lambda\sigma} = -\frac{1}{\ell^2} (g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda}). \quad (1)$$

The space-time trivially satisfies the Einstein equation with negative cosmological constant. The cosmological constant is read from (1) to be $\Lambda = -(1/\ell^2)$. AdS^3 is topologically $S^1 \times R^2$. AdS^3 has trivial timelike curves in the S^1 direction, which can be unwrapped by taking the universal covering space $\widetilde{\text{AdS}^3}$ of it.

It is clear from the embedding that AdS^3 space-time has a manifest symmetry, namely $\text{SO}(2, 2)$ isometry. The so-called “static chart” is obtained by the following parametrization (using X_0 - X_1 and X_2 - X_3 rotation)

$$\begin{aligned} X_0 &= \sqrt{R^2 + \ell^2} \sin T/\ell, & X_2 &= R \sin \Theta, \\ X_1 &= \sqrt{R^2 + \ell^2} \cos T/\ell, & X_3 &= R \cos \Theta, \end{aligned}$$

where $-\pi \leq T/\ell \leq \pi$, $0 \leq R < \infty$ and $0 \leq \Theta \leq 2\pi$. Then the metric of AdS^3 is in this chart

$$ds^2 = -(1 + R^2/\ell^2) dT^2 + \frac{dR^2}{1 + R^2/\ell^2} + R^2 d\Theta^2. \quad (2)$$

This is a complete chart of AdS^3 and is still a complete chart of the universal covering $\widetilde{\text{AdS}^3}$ if one takes the range $-\infty < T < \infty$. Peculiar property of this space-time is that it is not globally hyperbolic since spacelike infinity is timelike (see Figure 1) [4].

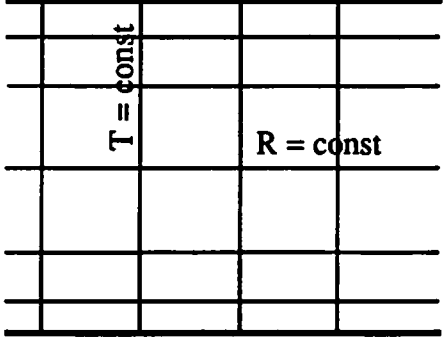


Figure 1

Penrose diagram of $\widetilde{\text{AdS}^3}$. The static coordinate (2) is drawn in it. AdS^3 is obtained by identifying with a certain period in T . The space-time is not globally hyperbolic.

Now three-dimensional black hole solution is obtained by identifying AdS^3 in the following way. Utilizing X_0 - X_3 and X_1 - X_2 Lorentz boost, introduce a coordinate system on AdS^3

$$\begin{aligned} X_0 &= \sqrt{\ell^2 - \tilde{r}^2} \cosh \tilde{t}/\ell, & X_1 &= \tilde{r} \cosh \tilde{\theta}, \\ X_3 &= \sqrt{\ell^2 - \tilde{r}^2} \sinh \tilde{t}/\ell, & X_2 &= \tilde{r} \sinh \tilde{\theta}, \end{aligned} \quad (3)$$

for $-\infty < \tilde{t} < \infty$, $-\infty < \tilde{\theta} < \infty$ and $0 \leq \tilde{r} \leq \ell$, and

$$\begin{aligned} X_0 &= \sqrt{\tilde{r}^2 - \ell^2} \sinh \tilde{t}/\ell, & X_1 &= \tilde{r} \cosh \tilde{\theta}, \\ X_3 &= \sqrt{\tilde{r}^2 - \ell^2} \cosh \tilde{t}/\ell, & X_2 &= \tilde{r} \sinh \tilde{\theta}, \end{aligned} \quad (4)$$

for $\ell < \tilde{r} < \infty$. The metric reads

$$ds^2 = -(\tilde{r}^2/\ell^2 - 1) d\tilde{t}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2/\ell^2 - 1} + \tilde{r}^2 d\tilde{\theta}^2. \quad (5)$$

Note that $\tilde{\theta}$ is a boost parameter, which can take any value between $-\infty$ and ∞ . Identify the points in AdS^3 , represented by $\tilde{\theta}$ and $\tilde{\theta} + 2\pi\sqrt{M}$ where M is a positive constant. Then by doing an appropriate rescaling of coordinate, $\theta \equiv \tilde{\theta}/\sqrt{M}$, $r \equiv \tilde{r}\sqrt{M}$, $t \equiv \tilde{t}/\sqrt{M}$, so that the angular coordinate θ has the usual range 2π , (5) can be cast into

$$ds^2 = -(r^2/\ell^2 - M) dt^2 + \frac{dr^2}{r^2/\ell^2 - M} + r^2 d\theta^2. \quad (6)$$

This is the three-dimensional static black hole solution. The space-time possesses a similar global structure as the Schwarzschild space-time in four-dimension does (see Figure 2). The

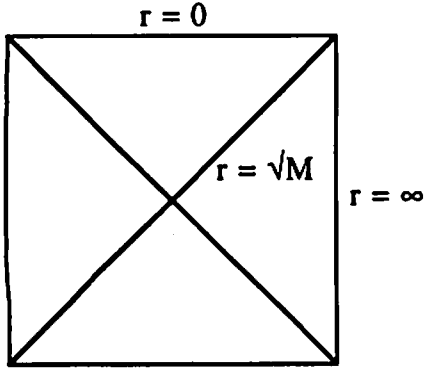


Figure 2
Penrose diagram of three-dimensional black hole. The event horizon is given by $r = \sqrt{M}$. One cannot smoothly extend the space-time to the region $r^2 < 0$.

event horizon is given by $r = \sqrt{M}$ (in the unit of $\ell = 1$). It is shown in [3] that the identification parameter M represents the mass of the black hole by studying the Hamiltonian theory of this system.

Another family of solutions including a parameter of angular momentum of black hole as well as the mass parameter, or “Kerr” space-time can be also constructed similarly by identifying a linear combination of $\tilde{t}$ and $\tilde{\theta}$. I will consider only the case of vanishing angular momentum throughout this talk.

Let us summarize some properties of the three-dimensional black hole and make it clear in what sense its global structure resembles with that of four-dimensional Schwarzschild space-time.

1. It is *not* globally hyperbolic just as AdS^3 is not.
2. Note that, by the boost $\tilde{\theta}$ symmetry in (3), only the portion of AdS^3 given by $X_1^2 - X_2^2 = \tilde{r}^2 > 0$ is identified. One may consider an analytic extension of the black hole space-time to the region corresponding to $\tilde{r}^2 < 0$. If it is possible, the space-time could smoothly continue beyond the boundary $r = 0$ so that the event horizon $r = \sqrt{M}$ would not exist. However, since there exist fixed points $(X_1 = 0, X_2 = 0)$ for this action of identification, one must remove them from AdS^3 in order to obtain a smooth manifold as a quotient space. Therefore, the resulting space-time is *geodesically incomplete* at $r = 0$. The situation here resembles the so-called Misner space-time (see [4], Taub-NUT space-time).[†] Indeed, the space-time is *non-Hausdorff* near $r = 0$. Moreover, there exist closed timelike

[†] The situation is different in the case of nonvanishing angular momentum. In that case, there is no fixed point in the identification and the space-time can be extended to be completely nonsingular.

curves if one extends the space-time beyond $r = 0$ to the region corresponding to $\tilde{r}^2 < 0$ not smoothly.

3. Since the metric on the quotient space-time naturally induced from AdS^3 is that of constant negative curvature, the $r = 0$ is not a curvature singularity. It can be also observed from (6) that it is not a conical singularity.

3. Classical Theory

Now, in this section, I shall treat the mass parameter as a dynamical variable and solve the Einstein equation by adopting a certain time slicing. The reader who is not familiar with the Hamiltonian formulation of general relativity should consult such a book as [5]. For the York's procedure for solving the initial-value problem, see the references cited in [6][7].

The (2+1) decomposition of the space-time metric g is

$$ds^2 = -(N^2 - N_a N^a) dt^2 + 2N_a dt dx^a + h_{ab} dx^a dx^b,$$

where N is lapse function, N^a is shift vector and h_{ab} is a spatial metric. Then the canonical formulation is given by the action of the form:

$$S = \int d^3x \left[\pi^{ab} \dot{h}_{ab} - NH - N_a H^a \right] + \text{surface terms}, \quad (7)$$

where π^{ab} is a tensor density canonically conjugate to h_{ab} , which geometrically represents the extrinsic curvature K^{ab} of each spacelike hypersurface $t=\text{constant}$. H and H^a are

$$\begin{aligned} H &= \sqrt{h} (K^{ab} K_{ab} - K^2 - {}^{(2)}R + 2\Lambda), \\ H^a &= -2\sqrt{h} D_b (K^{ab} - h^{ab} K), \end{aligned} \quad (8)$$

where D_a is the covariant derivative with respect to the spatial metric h_{ab} .

The surface terms in the action (7) will be explained in more detail later, but their existence is necessary, at least, in order to derive the canonical equations of motion that are equivalent to the Einstein equation. On the other hand, by varying the action with respect to N and N^a , one obtains the Hamiltonian and momentum constraints $H(x) = 0$ and $H_a(x) = 0$ which are first class constraints corresponding to the invariance under time reparametrization and spatial coordinate transformation, respectively.

I make an ansatz for the spatial metric in which the mass parameter is regarded as a dynamical variable. Since in general a metric of a two-dimensional space is conformally flat (locally), I assume the spatial metric of the circular symmetric form:

$$h = h_{ab} dx^a dx^b = e^{2\lambda} (d\xi^2 + \alpha^2 d\theta^2). \quad (9)$$

Here $e^{2\lambda}$ is a conformal factor which is a function of a spatial coordinate ξ and time. α is a function of time and represents a moduli parameter. The range of angular coordinate θ is defined as $0 \leq \theta \leq 2\pi$. The range of ξ will be determined below.

I fix the time slicing by adopting the so-called York time in which each spacelike hypersurface has constant mean curvature, namely the trace of the extrinsic curvature $K \equiv h_{ab} K^{ab}$ is spatially constant. The advantage of this York gauge is that one can reduce the dynamics of gravity in a tractable form. I will use K as a time parameter as $\tau \equiv -K$. (In the following, the unit of $\Lambda = -1$ is chosen.)

Let us solve the constraint equations. According to the York procedure, the Hamiltonian and momentum constraints (8) decouple with each other due to the York gauge condition. This is done by introducing the conformal class

$$\begin{aligned} h_{ab} &= e^{2\lambda} \bar{h}_{ab}, \\ \tilde{K}^{ab} &= e^{-4\lambda} \bar{\tilde{K}}^{ab}, \end{aligned}$$

where $\tilde{K}^{ab}$ is the traceless part of the extrinsic curvature, $\tilde{K}^{ab} \equiv K^{ab} - \frac{1}{2} h^{ab} K$. $\bar{h}_{ab}$ is given by (9), and $\bar{\tilde{K}}^{ab}$ is the transverse tensor with respect to $\bar{h}_{ab}$, which can be written as

$$\bar{\tilde{K}}_{\xi\xi} = -p(\tau), \quad \bar{\tilde{K}}_{\theta\theta} = p(\tau) \alpha(\tau)^2,$$

where $p(\tau)$ is a function of time τ only.

The remaining constraint is reduced to a second-order differential equation that determines the conformal factor. After integrating it, one obtains

$$e^{-\lambda} = \sqrt{c} \cdot \text{sn} \left(\frac{q}{\sqrt{c}} \xi \right), \quad (10)$$

where $q \equiv \sqrt{\tau^2 + 4}/2$, $c(\tau)$ is one of the integration constants, and the other integration constant is already fixed so that $\lambda = \infty$ when $\xi = 0$. Here sn is the Jacobi elliptic function with the

modulus given by k :

$$\operatorname{sn}^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}.$$

The quarter period of $\operatorname{sn} x$ is the complete elliptic integral of the first kind:

$$K(k) = \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}.$$

k is related to c by $c = kq/p$. It is observed from (10) that the conformal factor is a periodic function of ξ with the quarter period $\xi_{1/4}$ given by the relation $(q/\sqrt{c})\xi_{1/4} = K(k)$. I fix this periodicity in order to make the dynamical variable $\alpha(\tau)$ have a definite meaning of moduli in the spatial metric (9) by letting $\xi_{1/4} = \pi/2$. Then the relation yields

$$p = \frac{k}{q} \left(\frac{2}{\pi} K(k) \right)^2, \quad (11)$$

which means that $k(\tau)$ is essentially the traceless part of the extrinsic curvature $p(\tau)$. Thus I fix the range of ξ by $0 < \xi < \pi/2$. Since the spatial metric is also put as

$$d\sigma^2 = e^{2\lambda} \left[\left(\frac{d\lambda}{\partial_\xi \lambda} \right)^2 + \alpha^2 d\theta^2 \right],$$

$\xi = 0$ corresponds to the spatial infinity. $\xi = \pi/2$ is an apparent singularity which will be seen to be the horizon of the black hole.

After solving the constraint equations, it is a straightforward computation to solve the equations of motion for dynamical evolution. In particular, the lapse function is given by

$$N = -\frac{\lambda_\xi}{2} \int_{\xi_0(\tau)}^\xi d\xi' \frac{e^{2\lambda'} + f(\tau)}{\lambda_\xi'^2}, \quad (12)$$

where the subscript ξ denotes a partial derivative with respect to ξ . $f(\tau)$ and $\xi_0(\tau)$ are integration constants. Since the lapse function determines the behavior of the York time slicing, these constants should be specified by an appropriate boundary condition of the time slice. It can be shown that in order to cover the outside of the horizon by the York time slices one must have $f = -1/c$.

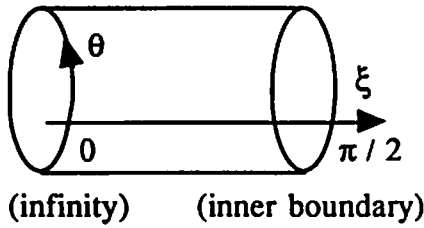


Figure 3
Spatial section with $0 < \xi < \pi/2$, $0 \leq \theta \leq 2\pi$. The inner boundary corresponds to the event horizon in the classical theory.

The remaining evolution equations yield the equations of motion for the variables $\alpha(\tau)$ and $p(\tau)$, or equivalently $k(\tau)$. They are

$$(qk)' = \frac{1}{2}, \quad (13)$$

$$\frac{\dot{\alpha}}{\alpha} = -\frac{\dot{K}(k)}{K(k)} + \frac{\dot{q}}{q}. \quad (14)$$

The solutions of these equations are easily obtained as

$$k = \frac{\tau - \tau_0}{\sqrt{\tau^2 + 4}}, \quad (15)$$

$$\alpha = \alpha_0 q \left(\frac{2}{\pi} K(k) \right)^{-1}. \quad (16)$$

They are the classical solutions for α and k . When $\tau = \tau_0$, the traceless part of extrinsic curvature p vanishes. It can be shown that the behavior of the lapse function at the inner boundary $\xi = \pi/2$ is fixed above, one must have $\tau_0 = 0$. The other constant α_0 is related to the horizon of the black hole as is seen below.

Now that I have obtained the explicit expression for the space-time metric, it should be verified that the solution is actually locally isometric to the black hole solution (6). By finding a coordinate transformation from (τ, ξ) to (t, r) , one can see that the York time slices cover the whole of the first quadrant outside the horizon ($\sqrt{M} < r < \infty$) in Figure 2. The slice asymptotically approaches the future and past event horizons as $\tau \rightarrow \pm\infty$. Note that by comparing the angular part of the metric in (6) and (9) it follows that $r = \alpha e^\lambda$. One can then see from (10) and (16) that the inner boundary $\xi = \pi/2$ corresponds to the horizon $r = \sqrt{M}$ and that $\alpha_0 = \sqrt{M}$. Thus the spatial section is schematically like Figure 3.

4. Reduced Action

4.1. REDUCED ACTION IN THE YORK GAUGE

Let us now reduce the action in the York gauge. Ignoring the surface terms which will be discussed below, the action (7) can be rewritten in the following form:

$$S = \int d^3x \left[\tilde{K}^{ab} h \cdot \dot{\tilde{h}}_{ab} - K \sqrt{h} - NH - N_a H^a \right]. \quad (17)$$

Here $\tilde{h}_{ab} \equiv h_{ab}/\sqrt{h}$ and $\tilde{K}^{ab}$ is the traceless part of the extrinsic curvature. After one solves the constraint equations $H = 0 = H^a$ by adopting the York gauge, K =spatially constant, and by parametrizing the time as $t = \tau \equiv -K$, the above (17) is reduced to be

$$S_{red} = \int d\tau d^2x \left[h \tilde{K}^{ab} \cdot \frac{\partial \tilde{h}_{ab}}{\partial \tau} - \sqrt{h} \right]. \quad (18)$$

The momentum conjugate to $\tilde{h}_{ab}$ is $h \tilde{K}^{ab}$, and the total Hamiltonian is given by the spatial area as was first shown in [7].

The first term of the reduced action (18) is put as

$$\int d^2x h \tilde{K}^{ab} \cdot \frac{\partial \tilde{h}_{ab}}{\partial \tau} = 2\pi^2 p \dot{\alpha} = \frac{8}{q} k K(k)^2 \dot{\alpha},$$

due to (11). The Hamiltonian term becomes

$$\int d^2x \sqrt{h} = 2\pi\alpha \int_0^{\pi/2} d\xi e^{2\lambda} = 2\pi\alpha \cdot \frac{1}{q\sqrt{c}} \int_0^{K(k)} du \frac{1}{\text{sn}^2 u}. \quad (19)$$

(Recall that $c = kq/p$.) The integral is divergent since the space is unbounded at infinity. By using a formula:

$$\text{Ns } u \equiv \int_0^u dt \left(\frac{1}{\text{sn}^2 t} - \frac{1}{t^2} \right) - \frac{1}{u} = u - E(u; k) - \sqrt{(1 - \text{sn}^2 u)(1 - k^2 \text{sn}^2 u)},$$

where $E(u; k)$ is the incomplete elliptic integral of the second kind with the modulus k , on has

the relation

$$\lim_{u_0 \rightarrow 0} \int_{u_0}^u dt \frac{1}{\operatorname{sn}^2 t} = Ns u + \lim_{u_0 \rightarrow 0} \frac{1}{u_0}.$$

The divergent part in the Hamiltonian (19) can be regularized by using this, and the reduced action becomes

$$S_{red} = 4 \int d\tau \left[\frac{2}{q} k K(k)^2 \dot{\alpha} - \frac{1}{q^2} K(k)(K(k) - E(k))\alpha \right], \quad (20)$$

where $E(k)$ is the complete elliptic integral of the second kind:

$$E(k) = \int_0^1 dt \sqrt{\frac{1 - k^2 t^2}{1 - t^2}}.$$

However, the variation of (20) with respect to α and k does not correctly give the equations of motion (13) (14) that were derived from the Einstein equations. This is because some surface terms are necessary to the action.

In general, according to Regge and Teitelboim [8], one must add some surface terms at spatial boundary to the Hamiltonian in order that the variation of the action gives correct equations of motion. Such terms are necessary if the spatial section has boundary as the black hole space-time.

Take the “naive” Hamiltonian in (7)

$$H_0 = \int d^2x [NH + N_a H^a], \quad (21)$$

with H and H^a given by (8). The variation of (21) with respect to h_{ab} and π^{ab} is

$$\begin{aligned} \delta H_0 = & \int d^2x [\mathcal{A}^{ab} \delta h_{ab} + \mathcal{B}_{ab} \delta \pi^{ab}] \\ & - \oint d^1 \Sigma_d N (h^{ac} h^{bd} - h^{ab} h^{cd}) D_c (\delta h_{ab}) + \oint d^1 \Sigma_d (D_c N) (h^{ac} h^{bd} - h^{ab} h^{cd}) \delta h_{ab} \\ & - \oint d^1 \Sigma_b \frac{1}{\sqrt{h}} (2N^c \pi^{bd} - N^b \pi^{cd}) \delta h_{cd} - \oint d^1 \Sigma_b \left(\frac{2}{\sqrt{h}} \right) N_a \delta \pi^{ab}. \end{aligned} \quad (22)$$

The four surface terms arise after the partial integration of the variation of (21) is done. The first term including $\mathcal{A}^{ab}$ and $\mathcal{B}_{ab}$ yields the usual canonical equations of motion that are equivalent to

the Einstein equation. One cannot abandon these surface terms a priori because the boundary conditions on the field variables, such as asymptotic flatness, may not allow vanishment of them. It means that one has to limit the class of field configurations in which the field variations are taken in the variational principle, by imposing appropriate boundary conditions. In order for the variational principle to be well defined, one has to ammend our Hamiltonian so that

$$H = H_0 + H_S,$$

where H_S is (minus of) the four surface terms in (22). For example, in a four-dimensional asymptotically flat space-time, the second term (involving $D(\delta h)$) in (22) at spatial infinity survives whose numerical value is the ADM mass.

Returning to the reduced action, some surface terms should be added from spatial boundary, which consists of the spatial infinity $\xi = 0$ and the inner boundary $\xi = \pi/2$ (see Figure 3). In the previous section, I have imposed some conditions on the behavior of the fields near $\xi = \pi/2$ so that the York time slices cover the outside of the horizon. By assuming the behavior of fields at the inner boundary, its contribution to the surface terms is found to be

$$\begin{aligned} H_S|_{inner} &= - \oint d\tau d\theta (D_\xi N)(h^{a\xi} h^{b\xi} - h^{ab} h^{\xi\xi}) \delta h_{ab} \\ &= - \int d\tau \frac{4}{q^2} E(\bar{k}) K(k) \alpha. \end{aligned} \tag{23}$$

Here $\bar{k}$ is the solution of the classical equations of motion, (15). On the other hand, at spatial infinity $\xi = 0$, I have not fixed the asymptotic behavior of the fields completely. For example, the lapse function (12) has an unfixed integration constant ξ_0 which can be shown to determine the asymptotic behavior of the York slice at infinity $\xi = \pi/2$. One has to do a careful treatment of this surface term at infinity since it represent what is meant by mass in this system. I will discuss this point in more detail in the next section. Unfortunately, detailed analysis of it is not exploited yet and is under consideration now.

5. Concluding Remarks

After reducing the action in the York gauge and adding appropriate surface terms to the action, one obtains the total Hamiltonian $H(k, \alpha, \tau)$ consisting of the corresponding terms in (20), (23) and the one at infinity. Then the quantization is straightforward, and the Schrödinger equation takes the form

$$i \frac{\partial}{\partial \tau} \psi(k, \tau) = H(k, \frac{\partial}{\partial k}, \tau) \psi(k, \tau), \quad (24)$$

in the k representation of wavefunction. There are some technical problems such as operator ordering, hermiticity, etc. though. With an explicit form of the Hamiltonian, one would solve these problems and can construct a quantum theory of the (2+1)-dimensional black hole.

I would like to point out several questions to be answered. They are not intrinsic to the (2+1)-dimensional black hole, but applicable to the quantum theory of black hole in general. They also involve rather conceptual problems in quantum gravity.

1. *Role of boundary:*

In the Hamiltonian formulation, the surface terms coming from the boundary of the space-time manifold have significant roles when one considers such a space-time as black hole. The situation is in contrast with that of quantum cosmology in which there exists no spatial boundary, or the presence of such terms can be neglected somehow.

First, such terms give a true total Hamiltonian of the system. One does not have a well-defined variational principle without them. One has to restrict a class of field configurations in the variational principle by setting a boundary of the spatial section and imposing boundary conditions on field variables. There is no a priori way to specify how to do that. One may freely fix a boundary and impose some boundary condition as far as the theory is self-consistent. However, if one wants to know what a quantum theory constructed in that specific way means, one would have to resort to some classical concepts for observables. This point sheds some light onto the role of “horizon” in quantum gravity. Horizon is an important concept of classical general relativity in many respects. The “horizon” would serve as one of the classical concepts in terms of which one can tell the meaning of the theory since it fixes the boundary of space-time manifold in a classically reasonable way. This is the viewpoint from which I fix the inner boundary at the classical horizon.

On an equal footing, one can consider another theory *inside* the boundary, or another with no inner boundary (treating the entire black hole space-time). What is the relation between this theory and that of outside the boundary? Surface terms at the boundary may throw a bridge between them.

As for the other boundary at infinity, the specification of boundary conditions determines the meaning of mass of this system. The mass is a kind of global charge in gauge theories, which is a Noether charge with respect to the asymptotic symmetry of space-time at infinity. (See [8][9] for more details.) A different restriction of asymptotic structure may yield a different mass variable as a canonical variable.

2. *Role of York time:*

In the Schrödinger equation (24), the wavefunction evolves in the York time. What is the meaning of this choice of time and the corresponding quantum theory? Because there exists no natural choice of “time” in general relativity, one has to extract it from dynamical variables of space-time. The choice would correspond to a certain observer of this black hole. Presumably it is related with the proper time of a certain observer at infinity. The precise meaning of it may set some restriction on the initial value of the wavefunction at, say $\tau = 0$, in (24).

3. An elegant application of the surface terms in the Hamiltonian formulation is recently given by Wald [10] in the context of the first law of black hole. What can be said about the relation to the “thermodynamical” properties of black hole?

Such questions as well as the Hawking evaporation process (by including matter field) could be studied in this toy model of (2+1)-dimensional black hole. By doing so, one will obtain a deal of insights into the quantum nature of black hole space-time.

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Compact Three Dimensional Black Hole: Topology Change and Closed Timelike Curve

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Abstract

We present a compactified version of the 3-dimensional black hole recently found by considering extra identifications and determine the analytical continuation of the solution beyond its coordinate singularity by extending the identifications to the extended region of the spacetime. In the extended region of the spacetime, we find a topology change both in the ordinary 3-dimensional black hole and in the compactified one. Especially, in the case of the compactified 3-dimensional black hole, we show an example of topology change from one double torus to eight spheres with three punctures.

1. Introduction

The topology change of the universe is a fascinating subject in general relativity. Fujiwara, Higuchi, Hosoya, Mishima and the present author[1] have recently found various quantum topology change solutions. Their quantum topology change solutions are tunneling manifolds with the Euclidean signature. Such quantum effects occur only on the Planck mass scale.

In this paper we present a new topology change solution with the Lorentzian signature. It means that the topology change will occur in classical gravity. There are some works treating the possibility of topology change in classical gravity[2]. Especially, Geroch showed that in the case of a compact universe there must exist a closed timelike curve in the spacetime representing a topology changing process. In the same case, Tipler also pointed out that spacetime singularity necessarily appears in the spacetime if matter field satisfies the weak energy condition.

To study the topology change we must treat the complicated topology of spacetime. If we go down to the (2+1)-dimensional gravity, the problem becomes much simpler while the topology change is still non-trivial. Further, the (2+1)-dimensional gravity is useful for studying roles of global structure of spacetime in classical and quantum gravity since there

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are no local dynamical degrees of freedom in the (2+1)-dimensional spacetime. Therefore we adopt the (2+1)-dimensional gravity as a good toy-model to study the Lorentzian topology change.

To formulate the (2+1)-dimensional gravity, we consider the Teichmüller parameters or moduli parameters as dynamical variables. Hosoya and Nakao, and Moncrief[3] investigated a quantum dynamics of the Teichmüller parameters of a torus universe in the ADM-formalism with the York timeslice. The Teichmüller parameters represent the identifications of points in the universal covering space of the torus and determines the shape of the torus. In the case of point particles, Deser, Jackiw and 'tHooft[4] treated the mass of a point particle as a deficit angle of a Minkowski spacetime, which is determined by the change of period for the angular coordinate. We can see that in the (2+1)-dimensional gravity the identification of points of spacetime is crucial in the study of the dynamics of global structure.

Recently Bañados, Teitelboim and Zanelli[5] have found a (2+1)-dimensional black hole solution with a negative cosmological constant. Though the circular symmetric (2+1)-dimensional vacuum solution has no event horizon, their black hole solution constructed also by making certain identifications in the anti-de Sitter space has an event horizon. Since the Hawking temperature of the 3-dimensional black hole $T_H \propto \sqrt{M}$ becomes zero in the mass zero limit of the 3-dimensional black hole, it is expected that the Hawking radiation dies away when the mass of black hole becomes zero while it blows up in the 4-dimensional black hole. This suggests that the black hole evaporation can be treated semi-classically in (2+1)-dimensions. A 3-spacetime with a negative cosmological constant has been studied since the variety of topology is rich in that space. Further, the violation of energy condition due to the negative cosmological constant allows topology change[1][2].

In this paper, we will investigate the 3-dimensional black hole solution and its compactification, where the shape of the space is parametrized by a set of global parameters which becomes extra degrees of freedom. These 3-dimensional black hole solutions will be analytically extended. We analyze topology and the global causal structure of the whole of these spacetimes by studying the identification of points in the 3-anti-de Sitter spacetime. The similarity with the Misner solution suggests existence of closed timelike curves.

In the section 2, the 3-dimensional black hole solution is compactified to give a double torus space. The section 3 shows their analytical extensions. The topology is discussed there. The closed curves are analyzed in section 4. They make the global causal structure clear. The final section is devoted to summary and discussions.

2. Compactified 3-dimensional Black Hole

As shown by Bañados, Teitelboim and Zanelli[5], there exists a black hole solution in the (2+1)-dimensional pure gravity with a negative cosmological constant. The metric of non-rotating black hole is given by

$$ds^2 = -(r^2/l^2 - M)dt^2 + \frac{dr^2}{(r^2/l^2 - M)} + r^2 d\phi^2, \quad (1)$$

where M is the mass of the black hole and $-1/l^2$ is equal to the cosmological constant. This solution is a maximally symmetric spacetime with a constant curvature $-2/l^2$ (3-anti-de Sitter) with certain identifications of points in spacetime (originally investigated by [5]). In this sense, this solution is regarded as the topological one. More precisely, we consider the identifications of points by a discrete subgroup of the isometry group $SO(2, 2)$ acting on the 3-pseudo-sphere $x^2 + y^2 - z^2 - w^2 = l^2$, which is a 3-anti-de Sitter spacetime embedded into the flat spacetime with a signature $(- - ++)$. The coordinate parametrization for the 3-dimensional black hole

(3D-BH) in the external region ($r \geq l$) is given as

$$\begin{aligned} ds^2 &= -dx^2 - dy^2 + dz^2 + dw^2, \\ x &= \sqrt{r^2/M - l^2} \sinh \frac{\sqrt{M}}{l} t, \\ y &= \frac{r}{\sqrt{M}} \cosh \sqrt{M} \phi, \\ z &= \sqrt{r^2/M - l^2} \cosh \frac{\sqrt{M}}{l} t, \\ w &= \frac{r}{\sqrt{M}} \sinh \sqrt{M} \phi. \end{aligned}$$

The internal solution ($r < l$) is given by an analytical continuation of the expression written above as

$$\begin{aligned} x &= \sqrt{l^2 - r^2/M} \cosh \frac{\sqrt{M}}{l} t, \\ y &= \frac{r}{\sqrt{M}} \cosh \sqrt{M} \phi, \\ z &= \sqrt{l^2 - r^2/M} \sinh \frac{\sqrt{M}}{l} t, \\ w &= \frac{r}{\sqrt{M}} \sinh \sqrt{M} \phi. \end{aligned}$$

This chart of the 3D-BH solution covers the part of the pseudo-sphere in which $y^2 - w^2 > 0$. In this region the periodic boundary condition of the angular coordinate, $(t, r, \phi) \leftrightarrow (t, r, \phi + 2\pi)$, defines an identification of points and restricts its fundamental region. This identification is written in terms of (x, y, z, w) as

$$\begin{pmatrix} x \\ z \end{pmatrix} \longleftrightarrow \begin{pmatrix} x \\ z \end{pmatrix}, \quad \begin{pmatrix} y \\ w \end{pmatrix} \longleftrightarrow \begin{pmatrix} \cosh 2\pi\sqrt{M} & \sinh 2\pi\sqrt{M} \\ \sinh 2\pi\sqrt{M} & \cosh 2\pi\sqrt{M} \end{pmatrix} \begin{pmatrix} y \\ w \end{pmatrix}. \quad (2)$$

This transformation is an element of the $SO(2, 1)$ group, the subgroup of the isometry group $SO(2, 2)$ of the 3-anti-de Sitter spacetime. We quotient the 3-anti-de Sitter spacetime by the discrete subgroup of $SO(2, 2)$ generated by (2) to obtain the 3D-BH solution. Considering also the coordinate parametrization for the Robertson-Walker chart of the anti-de Sitter spacetime, we find that this element of the group preserves not only the timeslice of the 3D-BH but also the timeslice of the RW-chart. Since each spatial hypersurface of the RW-chart of the 3-anti-de Sitter spacetime is a 2-hyperbolic space, we can construct various handle-body universes by certain tessellations on them so that the tessellation of the handle-body becomes consistent with the 3D-BH solution. For example, we consider a double torus universe with a discrete subgroup of $SO(2, 1) = \text{Isom}(H_2)$ generated by

$$\begin{aligned} T_1 &= T, \\ T_2 &= R^{-1}TR, \\ T_3 &= R^{-2}TR^2, \\ T_4 &= RTR^{-1}, \end{aligned} \quad (3)$$

where T is a Lorentz boost and R is a rotation by $\pi/4$. The actions of these transformations on

H_2 are shown in Fig.1. In terms of (x, y, z, w) , these T and R are represented by the matrices,

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh \alpha & 0 & \sinh \alpha \\ 0 & 0 & 1 & 0 \\ 0 & \sinh \alpha & 0 & \cosh \alpha \end{pmatrix}, \quad (4)$$

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \pi/4 & -\sin \pi/4 \\ 0 & 0 & \sin \pi/4 & \cos \pi/4 \end{pmatrix}. \quad (5)$$

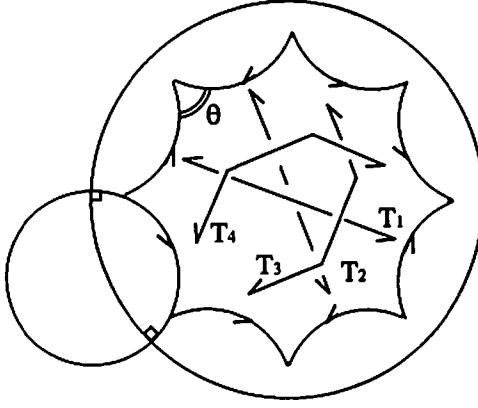


Fig. 1. An edge is identified with an edge in the opposite side.

Compared with (2) this double torus universe is consistent with the 3D-BH solution provided that α is equal to $2\pi\sqrt{M}$. To get a regular double torus (the angle θ in Fig.1 is $\pi/4$), however, α should be $2 \tanh^{-1} \sqrt{(2\sqrt{2}-2)/3}$; otherwise the double torus has a conical singularity. From now on, we suppose that

$$2\pi\sqrt{M} = \alpha = 2 \tanh^{-1} \sqrt{\frac{2\sqrt{2}-2}{3}}, \quad (6)$$

for the regular double torus. We call this regular double torus spacetime with a negative cosmological constant as a *compactified 3D-BH*. Because the double torus is constructed on the RW-chart, the foliation by the 2-hyperbolic spaces which are the universal covering spaces of the double tori covers only the region $y^2 - z^2 - w^2 \geq 0$. The compactified 3D-BH is the quotient of this region by the discrete subgroup. The compactified 3D-BH is singular only on a cone $y^2 - z^2 - w^2 = 0$ where the RW-chart is singular. The property of these singularities will be discussed in the next section.

3. Singularity and Misner-like Extension

In this section, we consider the causal structure of the 3D-BH solution which is similar to that of the Misner spacetime. We extend the 3D-BH spacetime in the same way as we normally treat the Misner solution[6].

3.1. Ordinary 3D-BH

In the Misner solution the apparent singularity is merely a coordinate singularity. We can extend the solution to the region beyond the coordinate singularity. It is well known that there are two possible extensions of the solution corresponding to the two directions along the spatial axis. To make all geodesics in these directions complete we should do both of the extensions. The resultant spacetime, however, becomes non-Hausdorff at the boundary between the original region and the extended region as shown in[7].

Now, we extend the 3D-BH by analogy with the Misner solution. By a coordinate transformation $\rho = r^2$, the metric of the ordinary 3D-BH (1) changes to the following form which has the Misner-like section:

$$ds^2 = \left\{ - (M - \rho/l^2)^{-1} \frac{d\rho^2}{4\rho} + \rho d\phi^2 \right\} + (M - \rho/l^2) dt^2 . \quad (7)$$

When one approaches the singularity ($\rho = 0$), this section behaves like the Misner solution. There are two extensions of spacetime, corresponding to the two directions along the ϕ -axis, which are similar to the extensions of the Misner solution. We consider the following two coordinate transformations corresponding to the two extensions,

$$\phi' = \phi \pm \frac{1}{2\sqrt{M}} \log \frac{\sqrt{M} + \sqrt{M - \rho/l^2}}{\sqrt{M} - \sqrt{M - \rho/l^2}} .$$

They give the two expressions for the metric,

$$ds^2 = \pm \frac{d\phi' d\rho}{\sqrt{M - \rho/l^2}} + \rho d\phi'^2 + (M - \rho/l^2) dt^2 , \quad (8)$$

respectively. Since these metrics are not degenerate on $\rho = 0$, we can extend the spacetime from the region with $\rho > 0$ to the region with $\rho < 0$. By one of the two extensions a certain set of incomplete geodesics become complete, while the other extension makes the other set of incomplete geodesics complete.

The identification ($\phi \mapsto \phi + 2\pi$) should exist also in the extended region. The action of the transformation T (see (2)) in the extended region generates this identification. Fig.2 shows this identification in the original region and also in the extended region for a $x, z = \text{constant}$ section of the pseudo-sphere $x^2 + y^2 - z^2 - w^2 = l^2$. Thus we find that the topology of the space changes from a single $R \times S_1$ (the original region) to two $R \times S_1$'s (the extended region). It is noted that the spacetime is non-Hausdorff on the boundary between the original region and the extended region $y^2 - w^2 = 0$, if we do both of the extensions.

3.2. Compactified 3D-BH

We extend also the compactified 3D-BH by the above mentioned method. The original region is the region with $x^2 < l^2$; it is easily confirmed that this region coincides with the region covered by the RW-chart. The coordinate extension is carried out in a similar way as the ordinary 3D-BH. This time there are eight extensions corresponding to the eight directions of the identification. By considering the covering space of the RW-anti-de Sitter spacetime, we find that this extended region is the region $x^2 - l^2 = z^2 + w^2 - y^2 > 0$ of the 3-anti-de Sitter not covered by the RW-chart (see Fig.3). The original region is foliated by a

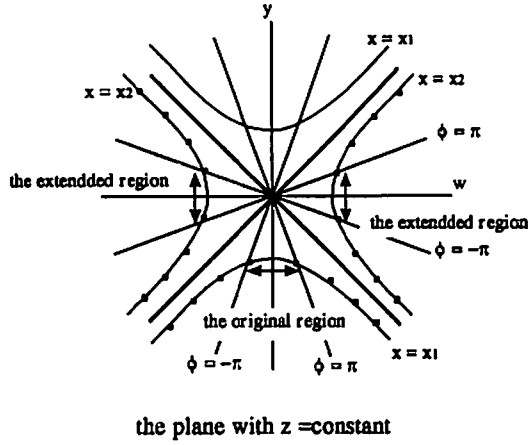


Fig. 2. A section of the pseudo-sphere with $x, z = \text{constant}$ is a hyperbola parametrized by $\phi = 1/\sqrt{M} \tanh^{-1}(w/y)$. A point $\phi = \phi_0$ is identified with a point $\phi = \phi_0 + 2\pi$.

one parameter family of 2-hyperbolic surfaces:

$$H_2(x_0^2) = \{y^2 - z^2 - w^2 = \lambda^2 \equiv l^2 - x_0^2 > 0\},$$

with $x_0^2 (< l^2)$ being a parameter. The identifications shown in Fig.1 produce a double torus at each surface. As stated in the previous section, these identifications are defined by a discrete subgroup of $SO(2, 2) = \text{Isom}(3\text{-anti-de Sitter})$ generated by T_i in (3).

We also make such identifications in the extended region in the following way. The extended region is foliated by the (1+1)-de Sitter spacetimes:

$$dS_2(x_0^2) = \{z^2 + w^2 - y^2 = \lambda'^2 \equiv x_0^2 - l^2 > 0\};$$

parametrized by $x_0^2 (> l^2)$. Since the discrete subgroup preserves each $dS_2(x_0^2)$, the identifications are induced for each hypersurface. We can find the topology of this region by determining the fundamental region in the hypersurface $dS_2(x_0^2)$. The action of T_1 on the surfaces $dS_2(x_0^2)$ is shown in Fig.4. The other actions of transformations are given by a rotation of T_i by $\pi(1-i)/4$ with $i = 1 \sim 4$. The fundamental regions are shown in Fig.5 both for $H_2(x_0^2)$ and for $dS_2(x_0^2)$. The fundamental region of the $dS_2(x_0^2)$ consists of eight quadrilaterals. Fig.6 shows that such a quadrilateral is topologically equivalent to a sphere with three conical singularities.

To summarize, the analytical continuation of the compactified 3D-BH produces the topology changing geometry from a double torus to eight spheres with three punctures. This topology change is quite different from the quantum topology change investigated in Ref.[1]. Previously the quantum topology change was described by a Euclidean tunneling manifold whereas the present topology changing solution has the Lorentzian signature. The topology change presented in this paper happens in classical gravity. We also note that the spacetime is non-Hausdorff, and it is discussed in detail in [9]. As stated in [7], this kind of non-Hausdorff topology is not so uncomfortable as the case shown in Fig.7, since we have no continuous curve which bifurcates.

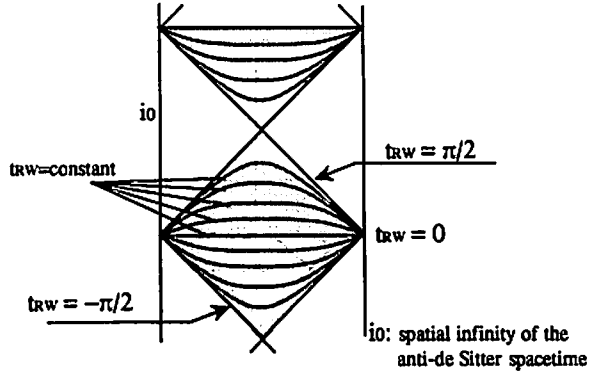


Fig. 3. The conformal diagram of the anti-de Sitter spacetime. The RW-chart covers only a part of the whole anti-de Sitter spacetime, as shown by the shadowed region.

4. Summary and Discussions

In this paper, we have investigated the analytical continuations of the ordinary 3D-BH and of the compactified 3D-BH. The compactified 3D-BH is isometric to a regular double torus universe with a negative cosmological constant if the mass of the black hole is equal to $m_c = (1/\pi \tanh^{-1} \sqrt{(2\sqrt{2} - 2)/3})^2$. If the mass of the black hole were different from m_c , however, the double torus would contain a conical singularity. Even if so, all the results of this paper will not be affected.

One might argue that the compactified 3D-BH would not be a proper black hole. Any compactification would remove the spatial infinity and the null infinity from the spacetime. Then, as the argument may go, we could not define any event horizon. In this paper, however, we treat the compactified 3D-BH solution as a black hole because the double torus universe can be regarded as an open 2-hyperbolic space which possesses a certain periodicity. The horizon can be defined in the universal covering spacetime. To summarize we can define a compactified space as a black hole if its universal covering space is a black hole spacetime.

We have shown that the analytical continuations of the 3D-BH's produce a topology changing spacetime geometry. Especially the topology change of the ordinary 3D-BH is easily attributed to the singularity where the spacetime is non-Hausdorff. A well-known simple example of non-Hausdorff topological space is shown in Fig.7[7], and is the simplest example of a topology change, from one point to two points, if one imagines that the x and y axes are the time axes. The non-Hausdorff topology causes the branching of the space which is one type of the topology change. In the ordinary 3D-BH, the topology of a hypersurface changes from a single $R \times S_1$ to two $R \times S_1$'s. This topology change belongs to the branch-type topology change corresponding to the change of the number of connected components. However, the topology change of the compactified 3D-BH (a double torus universe) is not so trivial, since the Euler number changes from -2 (a double torus) to $8 \times (-1)$ (eight spheres with three punctures). This can be seen by considering the geodesics running from the original region to the extended region[10]. Anyway these topology change is closely related to non-Hausdorff topologies. We may speculate that all Lorentzian topology changes are related to the non-Hausdorff topology.

As for the problem of the topology change in a Lorentzian spacetime, some authors have discussed its possibility or the condition for it to occur[2][8]. These works, however,

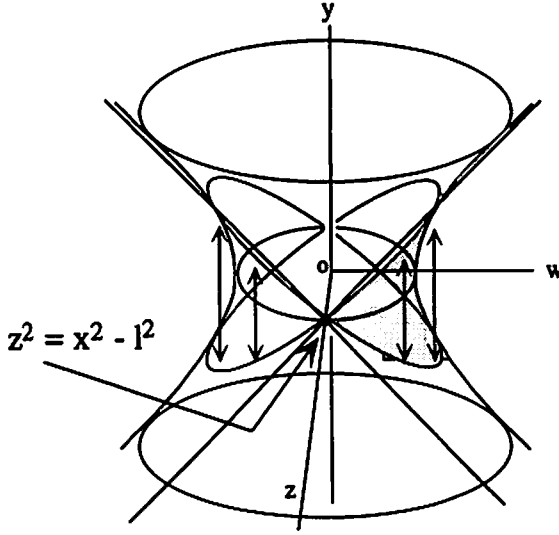


Fig. 4. The action of T_1 on a $dS_2(x_0^2)$ is shown by the arrows.

considered spacelike hypersurfaces as an initial and a final states. As shown in this paper there is a new type of topology change if we consider timelike hypersurfaces.

The investigation of this paper about a double torus is also applicable to a 2-surface with higher genus. When we consider the case of genus $n > 2$, the mass of the black hole should be $(1/\pi \tanh^{-1} \sqrt{2/(1 + \sec(\pi/2n))})^2$ to form a regular spatial hypersurface with a genus n . In this case the topology changes from a single n -torus to $4n$ spheres with three conical singularities. The causal structure of the spacetimes investigated in this paper are shown in [11].

For the open chart of 3-de Sitter spacetime or the 3-Milne universe, the same compactification and extension can be done. In these cases we will also find a topology change and a closed timelike curve. It is our future plan to study the cosmological implication in the Friedmann spacetimes with compactified hyperbolic spatial hypersurface. Furthermore we can study the Teichmüller deformation and its quantum theory of a further simplified model, for example a torus with a conical singularity in the 3D-BH[10].

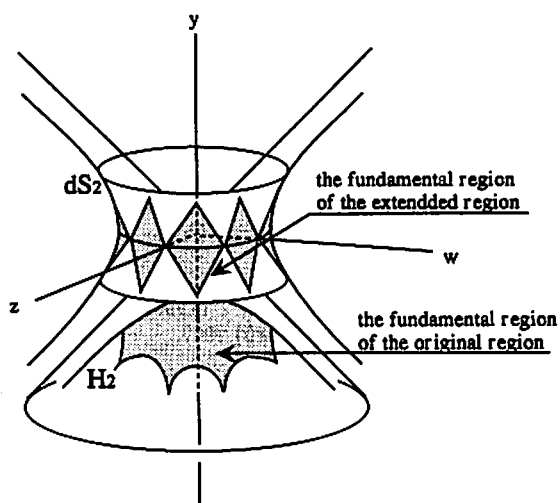


Fig. 5. While the fundamental region is an octagon, in the $H_2(x_0^2)$ region of spacetime it changes to eight quadrilaterals in $dS_2(x_0^2)$.

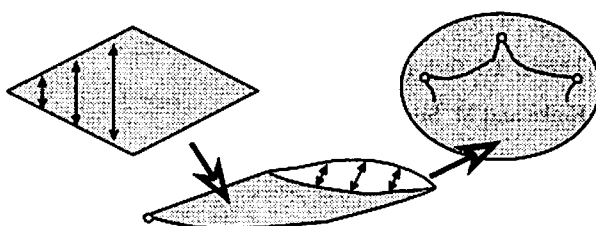


Fig. 6. The quadrilateral becomes a sphere with three punctures by the identifications.

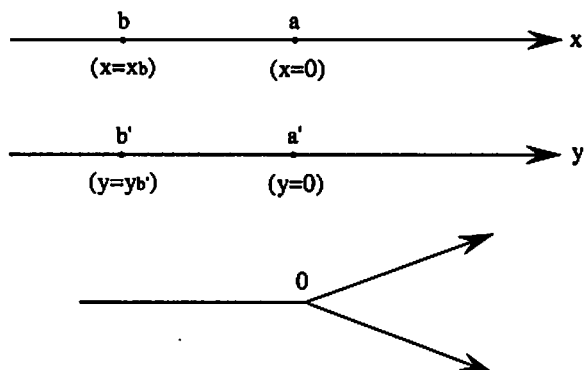


Fig. 7. An example of a non-Hausdorff manifold. The two lines above are identical for $x = y < 0$. However, the two points a ($x = 0$) and a' ($y = 0$) are not identified.

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Comments on dynamics of three-dimensional Lorentzian topology-changing spacetimes

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Abstract

A few comments are given on preceding Siino's talk [1], in particular, concerning which spacetimes admit analytic extension to topology-changing spacetimes.

One of the authors [2] constructed a three-dimensional spacetime in which the topology of the spatial section changes. It is similar to the Misner solution (*e.g.* [3]) in many points; in particular, it is a non-Hausdorff manifold and the causality is violated in the extended region. The 3-anti-de Sitter spacetime is a 3-pseudosphere

$$-x^2 - y^2 + z^2 + w^2 = -l^2$$

embedded in the 4-manifold with metric

$$ds^2 = -dx^2 - dy^2 + dz^2 + dw^2,$$

The spacetime is constructed by quotienting the 3-anti-de Sitter spacetime by a discrete subgroup Γ of $SO(2,2)$ generated by

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$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh \alpha & 0 & \sinh \alpha \\ 0 & 0 & 1 & 0 \\ 0 & \sinh \alpha & 0 & \cosh \alpha \end{pmatrix}, \quad \alpha = 2 \tanh^{-1} \sqrt{\frac{2\sqrt{2}-2}{3}}, \quad (0.1)$$

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \pi/4 & -\sin \pi/4 \\ 0 & 0 & \sin \pi/4 & \cos \pi/4 \end{pmatrix}. \quad (0.2)$$

It represents a universe whose topology changes from a regular double torus, through eight spheres with three punctures on each of them, and again into a double torus. One can think of the region $|x| < l$ as a vacuum, hyperbolic Friedmann–Robertson–Walker (FRW) universe with negative cosmological constant. This shows that an ordinary (spatially compact) FRW universe may admit analytic continuation to some larger spacetime *i.e.* it may come from some universe via topology change.

The following questions naturally arise. Which spacetimes admit analytic extension into topology-changing spacetimes? What are the implications to our real Universe? We comment on the first question in this talk. The latter question will be considered elsewhere [4].

Let us consider what is the essence of the above construction of the topology-changing spacetime. First, as in the Misner spacetime, the identification for the universal covering spacetime is compatible with the hyperbolic FRW slicing, *i.e.* Γ is a discrete subgroup of $SO(2,1)$ which is the symmetry group of 2-hyperboloid. The FRW chart covers the region which is an incomplete spacetime whose boundary is null surface. This causes the non-Hausdorffness and topology changes of the extended spacetime. Second, the analytic extension is carried out by extending the universal cover and the action of Γ on the extended spacetime. The resulting spacetime is obtained by identifying the points which are mapped by the action of Γ .

Having these in mind we can easily construct some other examples with different universal

covering spacetime.

The *3-de Sitter spacetime* is an embedded 3-pseudosphere

$$-x^2 + y^2 + z^2 + w^2 = l^2$$

in 4-Minkowski spacetime whose metric is

$$ds^2 = -dx^2 + dy^2 + dz^2 + dw^2$$

An hyperbolic FRW slicing is given by the foliations of cross-sections by constant w hyperplanes; they are the orbits of $SO(2,1)$, a subgroup of the whole symmetry group $SO(3,1)$. They are 2-hyperboloids, a light cone and 2-de Sitter spacetimes, which are expressed in bold lines in Fig. 1 (one dimension is suppressed). The foliation is the same as in the 3-anti-de Sitter case. So a regular double torus obtained by identification experiences the same topology change. This may indicate that the Universe may have emerged through a topology change if we are living in a hyperbolic FRW universe which was inflating in an early stage.

The *3-Minkowski spacetime* is a spacetime with metric

$$ds^2 = -dt^2 + dx^2 + dy^2$$

(Fig. 2). The future light cone of the origin with the hyperbolic FRW chart is called the *Milne universe*. The symmetry group of the whole Minkowski spacetime is the Poincaré group $ISO(2,1)$. The orbit of its subgroup $SO(2,1)$ are the same as the above case. So, topologically, the fate of a regular double torus is also the same.

These examples generalize to the four spacetime dimensions in a straightforward manner, in which case the spatial sections are 3-hyperboloid and the simplest compact three-dimensional space obtained by identifications is called the Seifert-Weber manifold, whose fundamental region is a dodecahedron.

Up to now our discussion on three-dimensional universes was concentrated on the *isotropic* double torus, i.e. on non-dynamical universes. What is the fate of dynamical spacetimes? In

general a dynamical double torus universe do not shrink to a point at the 'initial singularity'; the universe collapses in some direction faster than another. The dynamical variables are the Teichmüller parameters in vacuum three-dimensional gravity (possibly with the cosmological constant). Unfortunately the dynamics of a double torus universe is not completely solved. Waelbroeck [5] proposed a treatment of vacuum three-dimensional gravity with point particles. It is a canonical formalism of finite dynamical variables which carries the information of identifications of the flat *three-dimensional* Minkowski spacetime, rather than those with respect to some time slicings. This formalism seems to admit direct geometrical insight into the whole (*i.e.* three-dimensional) spacetime in connection with its dynamics. It may be best way to investigate the extendibility of dynamical spacetimes.

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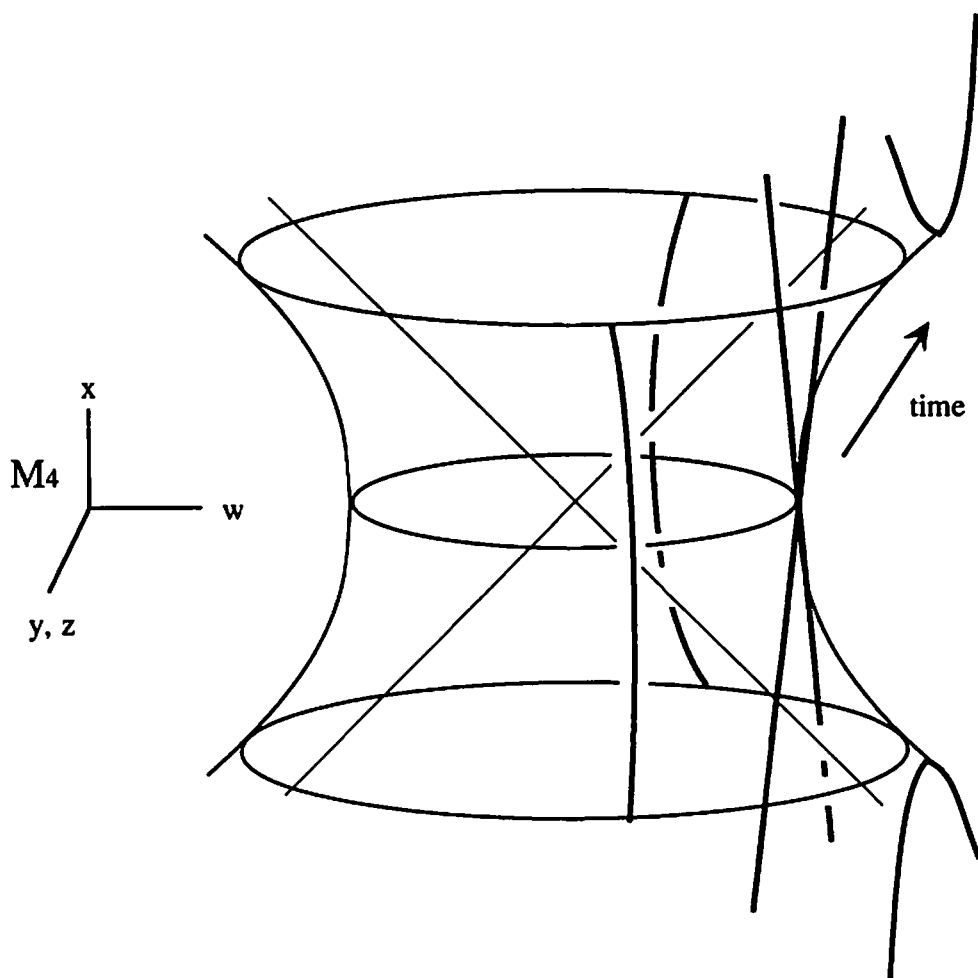


Fig. 1: the 3-de Sitter spacetime

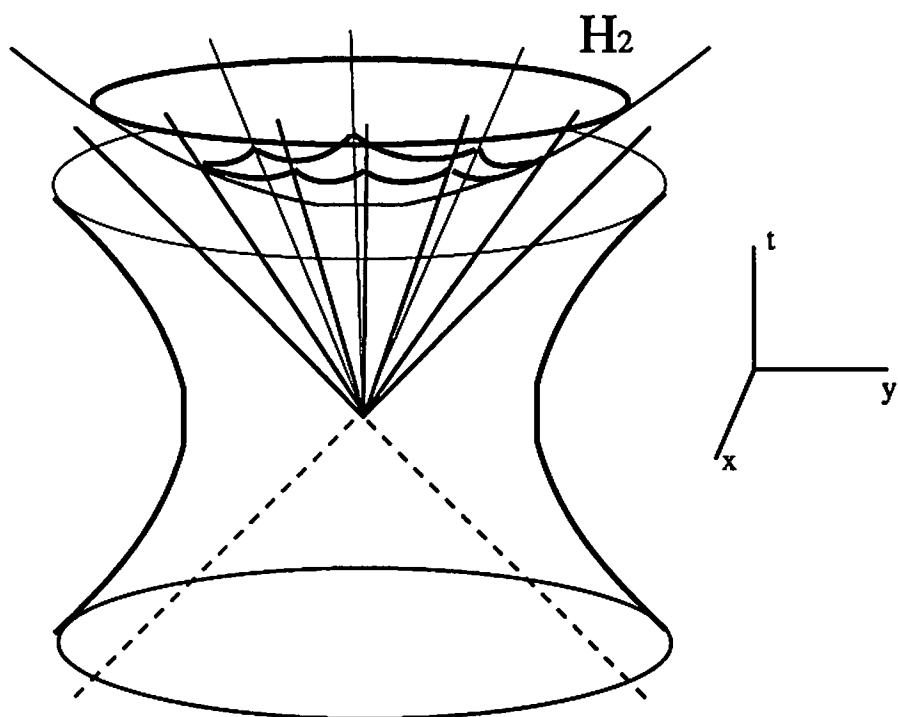


Fig.2: the 3-Milne spacetime

Quantum Effects near 3D Black Holes^{1,2}

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Recently, the black hole solution to the three-dimensional Einstein equations with a negative cosmological constant has been found [1] and various aspects on the black hole (BH) has been analyzed by authors [1,2,3].

In this talk, we will examine the vacuum polarization of stress-energy tensor for conformally-coupled massless scalar fields in the non-rotating BH background in three dimensions and discuss the back-reaction to the metric.

The authors of ref. [1] have found the BH solution to the three-dimensional Einstein equation including a negative cosmological constant ($\lambda > 0$):

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \lambda g_{\mu\nu} .$$

Their solution can be written by the metric

$$ds^2 = -(\lambda r^2 - M) dt^2 + \frac{dr^2}{\lambda r^2 - M} + r^2 d\theta^2 ,$$

for the non-rotating case. M is the mass of the 3DBH [1]. The horizon length is $r_H = (M/\lambda)^{1/2}$.

A new coordinate ρ defined by $r = r_H \sec \rho$ ($0 \leq \rho \leq \pi/2$) simplifies the

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metric; the Euclideanized metric then takes the form

$$ds_E^2 = \lambda^{-1} (\sec \rho)^2 [d\rho^2 + \kappa^2 \sin^2 \rho d\tau^2 + M d\theta^2] ,$$

where we set $\kappa^2 = M\lambda$.

We consider a conformally-coupled, massless scalar field in this background space-time. The propagator G_E for this scalar field in the Euclidean geometry is the solution to the following equation:

$$(\square - \frac{1}{8}R)G_E(x, x') = -\frac{1}{\sqrt{g}}\delta(x, x') .$$

We can use the mode sum method to obtain an exact form of the propagator [3]. Here we derive the exact form of the propagator by using the knowledge of the field theory in an anti-de Sitter space [4].

If one replaces $\sqrt{M}\theta$ in the metric by a new angular coordinate $\tilde{\theta}$, he finds that the metric seems to be the one for the Euclideanized anti-de Sitter space (EAdS). Thus we find that the Euclidean 3DBH space-time has the same local structure as the one of EAdS.

The propagator in the background of EAdS, whose metric is obtained by the substitution $\sqrt{M}\theta \rightarrow \tilde{\theta}$, can be written as [4]

$$G_E^{\text{EAdS}}(\rho, \tau, \tilde{\theta}; \rho', \tau', \tilde{\theta}') = \frac{\sqrt{\lambda} (\cos \rho)^{1/2} (\cos \rho')^{1/2}}{4\sqrt{2}\pi \sqrt{\cosh(\tilde{\theta} - \tilde{\theta}') - \cos \rho \cos \rho' - \sin \rho \sin \rho' \cos \kappa(\tau - \tau')}} \\ - \alpha \frac{\sqrt{\lambda} (\cos \rho)^{1/2} (\cos \rho')^{1/2}}{4\sqrt{2}\pi \sqrt{\cosh(\tilde{\theta} - \tilde{\theta}') + \cos \rho \cos \rho' - \sin \rho \sin \rho' \cos \kappa(\tau - \tau')}} ,$$

where α is a constant. (We adopt the "mixed" boundary condition including a parameter α .)

To construct the propagator in the 3DBH space-time from, we must take the periodicity in the angular variable into consideration. The construction can be just carried out by means of the image method [5]. The result is:

$$G_E^{\text{BH}}(\rho, \tau, \theta; \rho', \tau', \theta') \\ = \sum_{k=-\infty}^{\infty} \frac{\sqrt{\lambda} (\cos \rho)^{1/2} (\cos \rho')^{1/2}}{4\sqrt{2}\pi \sqrt{\cosh \sqrt{M}(\theta - \theta' + 2\pi k) - \cos \rho \cos \rho' - \sin \rho \sin \rho' \cos \kappa(\tau - \tau')}} \\ - \alpha \sum_{k=-\infty}^{\infty} \frac{\sqrt{\lambda} (\cos \rho)^{1/2} (\cos \rho')^{1/2}}{4\sqrt{2}\pi \sqrt{\cosh \sqrt{M}(\theta - \theta' + 2\pi k) + \cos \rho \cos \rho' - \sin \rho \sin \rho' \cos \kappa(\tau - \tau')}} .$$

This result agrees with the expression derived by the mode sum method.

Now we calculate the quantum fluctuations of the conformally-coupled

scalar field in the 3DBH space-time. The quantum effects around the 3DBH can be computed from the propagator.

We can regard the coincidence limit of the propagator $G_E(x,x)$ after appropriate regularizations as the vacuum polarization $\langle \varphi^2 \rangle$. We find that the vacuum value $\langle \varphi^2 \rangle$ in the 3DBH space-time takes the form as a function of ρ :

$$\langle \varphi^2 \rangle(\rho) = -\alpha \frac{\sqrt{\lambda}}{8\pi} + \sum_{k=1}^{\infty} \left[\frac{\sqrt{\lambda} \cos \rho}{2\sqrt{2}\pi \sqrt{\cosh 2\pi\sqrt{M}k - 1}} - \alpha \frac{\sqrt{\lambda} \cos \rho}{2\sqrt{2}\pi \sqrt{\cosh 2\pi\sqrt{M}k + \cos 2\rho}} \right].$$

For $\alpha=0$, the dependence on ρ becomes very simple. In terms of the original coordinate r , $\langle \varphi^2 \rangle$ is inversely proportional to the radial coordinate r . $\langle \varphi^2 \rangle$ approaches zero in the limit of $r \rightarrow \infty$ if and only if $\alpha=0$. The value of $\langle \varphi^2 \rangle$ at spatial infinity is a constant $-\alpha\sqrt{\lambda}/(8\pi)$. The value of $\langle \varphi^2 \rangle$ at the horizon approaches a constant $-\alpha\sqrt{\lambda}/(8\pi)$ in the large mass limit.

Next we turn to the vacuum stress tensor. The expectation value for the stress tensor can be obtained by the following coincidence limit with regularization:

$$\langle T_{\nu}^{\mu}(x) \rangle = \lim_{x' \rightarrow x} \frac{1}{4} \left[3 \nabla^{\mu} \nabla_{\nu} - \delta_{\nu}^{\mu} \nabla^{\sigma} \nabla_{\sigma} - \nabla^{\mu} \nabla_{\nu} - \frac{\lambda \delta_{\nu}^{\mu}}{4} \right] G_E^{BH}(x, x').$$

After some calculation, we find the simple form:

$$\langle T_{\nu}^{\mu} \rangle(\rho) = F(\rho) \text{diag.}(1, 1, -2) + H(\rho) \text{diag.}(1, 0, -1),$$

where

$F(\rho)$

$$= \frac{\lambda^{3/2} \cos^3 \rho}{16\sqrt{2}\pi} \sum_{k=1}^{\infty} \left[\frac{1}{(\cosh 2\pi\sqrt{M}k - 1)^{1/2}} + \frac{4}{(\cosh 2\pi\sqrt{M}k - 1)^{3/2}} - \alpha \frac{\cosh 2\pi\sqrt{M}k - 1}{(\cosh 2\pi\sqrt{M}k + \cos 2\rho)^{3/2}} \right],$$

$$H(\rho) = -\alpha \frac{3\lambda^{3/2} \sin^2 \rho \cos^3 \rho}{8\sqrt{2}\pi} \sum_{k=1}^{\infty} \frac{\cosh 2\pi\sqrt{M}k - 1}{(\cosh 2\pi\sqrt{M}k + \cos 2\rho)^{5/2}}.$$

Obviously the vacuum stress tensor is traceless. One can also check the conservation law $\nabla_{\mu} \langle T_{\nu}^{\mu} \rangle = 0$ in the 3DBH background.

The values of $\langle T_{\nu}^{\mu} \rangle$ at the horizon approaches zero in the large mass limit. On the other hand, in the limit of $M \rightarrow 0$, all components of $\langle T_{\nu}^{\mu} \rangle$ diverge. Note also that $\langle T_t^t \rangle = \langle T_r^r \rangle$ is satisfied on the horizon. In the limit of $\rho = \pi/2$ ($r \rightarrow \infty$), all the components of $\langle T_{\nu}^{\mu} \rangle$ for any case become zero. One can regard that this is due to the "redshift" factor $(g_{tt})^{-1/2}$ which goes to

zero at spatial infinity. Thus thermal equilibrium is not attained between the BH and the radiation in this system. It turns out that the value of $\langle T_{\nu}^{\mu} \rangle$ is not so sensitive to the value of α for all cases.

Now we investigate the effect of back reaction due to the quantum effects. The back reaction of quantum fields to the metric has been studied in the Schwarzschild metric in four dimensions [6]. The Einstein equation is modified by the vacuum stress tensor and then becomes

$$R_{\nu}^{\mu} - \frac{1}{2} R \delta_{\nu}^{\mu} - \lambda \delta_{\nu}^{\mu} = \pi \langle T_{\nu}^{\mu} \rangle$$

We must choose the small parameter which denotes the perturbation. The perturbation parameter, ϵ , should be proportional to the Planck constant, which is set by unity here. Hence we will take $\epsilon=1/r_H$.

We consider the following metric which suffers the quantum effect:

$$ds^2 = - e^{2\psi(r)} (\lambda r^2 - m(r)) dt^2 + \frac{dr^2}{\lambda r^2 - m(r)} + r^2 d\theta^2,$$

where $m(r)$ and $\psi(r)$ is the function of r which are to be solved to satisfy the Einstein equations with quantum correction.

Using the above metric, the Einstein equation reads

$$\begin{aligned} -\frac{1}{2r} \frac{dm(r)}{dr} &= \pi \langle T_t^t \rangle, \\ -\frac{\lambda r^2 - m(r)}{r} \frac{d\psi(r)}{dr} &= \pi (\langle T_t^t \rangle - \langle T_r^r \rangle). \end{aligned}$$

The equations are solved to the first order perturbation in the form

$$m(r) = M(1 + \epsilon \mu(r)),$$

with

$$\mu(r) = -\frac{2\pi r_H}{M} \int_{r_H}^r \langle T_t^t \rangle \bar{r} d\bar{r},$$

where the mass of the BH is "renormalized" so that $m(r_H)=M$ [6], and

$$\psi(r) = \epsilon (\rho(r) - \rho(r_0)),$$

with

$$\rho(r) = -\pi r_H \int_{r_H}^r \frac{\langle T_t^t \rangle - \langle T_r^r \rangle}{\lambda \bar{r}^2 - M} \bar{r} d\bar{r},$$

where we assume the boundary condition $\psi(r_0)=0$ [6].

The mass correction μ take a finite value in the large r limit in each

case. In the large M limit:

$$m(\infty) = M - \varepsilon \frac{(1-\alpha)M^{3/2}}{8} e^{-\pi\sqrt{M}}, (\alpha \neq 1)$$

$$m(\infty) = M - \varepsilon \frac{M^{3/2}}{4} e^{-3\pi\sqrt{M}}, (\alpha = 1)$$

where we leave the parameter ε which manifestly displays the perturbation. For the scalar field, the mass correction is negative (positive) for $\alpha \leq 1$ ($\alpha > 1$) in the asymptotic region.

One can find the quantum effect diminishes the attractive force in the region $r > r_c$. The value for r_c is given by:

$$r_c \approx 1.618 r_H \text{ for } \alpha \neq 1,$$

$$r_c \approx 2.806 r_H \text{ for } \alpha = 1.$$

To summarize, we have obtained the propagator for a conformally-coupled massless scalar field in 3DBH space-time and computed the vacuum expectation value $\langle \varphi^2 \rangle$ and stress tensor for scalar fields.

The quantum correction to the gravitational force on the static test particle has been estimated. It is found that there is a critical radius outside the BH, which divides the regions where the positive and negative correction arise.

One may wish to study the thermodynamics of the 3DBH including the quantum-field back reaction. Such analyses have been carried out in refs. [6] for four-dimensional BHs. The approach used in ref. [6] is equally effective for the three-dimensional case. However, there exist some subtleties in the 3DBH case, since the quantum effects do not have the scaling property, as the four dimensional case. (In four dimensions, the quantum quantities can be well approximated by functions of M/r .) We leave the discussion on the thermodynamics with quantum fields for the future work.

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Poincaré gauge theory of (2+1)-dimensional gravity

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Abstract

A Poincaré gauge theory of (2+1)-dimensional gravity is developed. Fundamental gravitational field variables are dreibein fields and Lorentz gauge potentials, and the theory is underlain with the Riemann-Cartan space-time. The most general gravitational Lagrangian density, which is at most quadratic in curvature and torsion tensors and invariant under local Lorentz transformations and under general coordinate transformations, is given. We find, among other things, the following. (1) Solutions of the vacuum Einstein equation satisfy gravitational field equations in the vacuum in this theory. (2) For a class of the parameters in the gravitational Lagrangian density, the torsion is “frozen” at the place where “spin” density of the source field is not vanishing. In this case, the field equation actually agrees with the Einstein equation, when the source field is “spin”less. (3) A teleparallel theory developed in a previous paper is “included as a solution” in a limiting case. (4) A Newtonian limit is obtainable, if the parameters in the Lagrangian density satisfy certain conditions.

I. INTRODUCTION

Recently, lower dimensional gravity has been attracting considerable attentions. The (2+1)-dimensional Einstein theory has no Newtonian limit and no dynamical degrees of

freedom, but it has non-trivial global structures. This theory has been studied mainly because of the local triviality and of the global non-triviality [1]. For the (3+1)-dimensional gravity, there have been proposed various theories alternative to the Einstein theory, among which we have a teleparallel theory [2] and Poincaré gauge theories [3-5]. A Poincaré gauge theory has been examined [6] also for the (1+1)-dimensional case.

It would be significant to develop various theories of gravity also for (2+1)-dimensional case, which will bring us toy models useful to examine basic concepts in theories of gravity. In a previous paper [7], the present author has proposed a teleparallel theory of (2+1)-dimensional gravity having a Newtonian limit and black hole solutions.

The purpose of this paper is to develop a Poincaré gauge theory of (2+1)-dimensional gravity, in a limiting case of which a teleparallel theory given in Ref. [7] is "included as a solution".

II. DREIBEINS, THREE-DIMENSIONAL LORENTZ GAUGE POTENTIAL AND RIEMANN-CARTAN SPACE-TIME

The three-dimensional space-time is assumed to be a differentiable manifold endowed with the Lorentzian metric $g_{\mu\nu}dx^\mu \otimes dx^\nu$ ($\mu, \nu = 0, 1, 2$) related to the fields $e^k = e^k_\mu dx^\mu$ ($k = 0, 1, 2$) through the relation $g_{\mu\nu} = e^k_\mu \eta_{kl} e^l_\nu$ with $(\eta_{kl}) \stackrel{\text{def}}{=} \text{diag}(-1, 1, 1)$. Here, $\{x^\mu; \mu = 0, 1, 2\}$ is a local coordinate of the space-time. The fields $e_k = e^\mu_k \partial/\partial x^\mu$, which are dual to e^k , are the dreibein fields.

The field strengths of e^k_μ and of A^{kl}_μ are given by [8]

$$T^k_{lm} \stackrel{\text{def}}{=} e^\mu_l e^\nu_m (\partial_\mu e^k_\nu - \partial_\nu e^k_\mu) + e^\mu_l A^k_{m\mu} - e^\mu_m A^k_{l\mu} \quad (2.1)$$

and

$$R^{kl}_{mn} \stackrel{\text{def}}{=} e^\mu_m e^\nu_n (\partial_\mu A^{kl}_\nu - \partial_\nu A^{kl}_\mu - A^k_{r\mu} A^{lr}_\nu + A^k_{r\nu} A^{lr}_\mu), \quad (2.2)$$

respectively.

The covariant derivative $D_k\varphi$ of the field φ belonging to a representation of the three-dimensional Lorentz group is introduced by the standard method in gauge theories. For the world vector field $V = V^\mu\partial/\partial x^\mu$, the covariant derivative with respect to the affine connection $\Gamma_{\lambda\nu}^\mu$ is given by $D_\nu V^\mu = \partial_\nu V^\mu + \Gamma_{\lambda\nu}^\mu V^\lambda$. We make the requirement, $D_l V^k = e^\nu_l e^k_\mu D_\nu V^\mu$ for the Lorentz vector field V^k , where $V^\mu \stackrel{\text{def}}{=} e^\mu_k V^k$. Then the relation, $A^k_{l\mu} \equiv \Gamma_{\lambda\mu}^\nu e^k_\nu e^\lambda_l + e^k_\nu \partial_\mu e^\nu_l$, follows, and we have

$$T^k_{\mu\nu} \equiv e^k_\lambda T^\lambda_{\mu\nu}, \quad (2.3)$$

$$R^k_{l\mu\nu} \equiv e^k_\lambda e^\rho_l R^\lambda_{\rho\mu\nu}, \quad (2.4)$$

$$D_\lambda g_{\mu\nu} \stackrel{\text{def}}{=} \partial_\lambda g_{\mu\nu} - \Gamma_{\mu\lambda}^\rho g_{\rho\nu} - \Gamma_{\nu\lambda}^\rho g_{\mu\rho} \equiv 0 \quad (2.5)$$

with

$$T^\mu_{\nu\lambda} \stackrel{\text{def}}{=} \Gamma_{\lambda\nu}^\mu - \Gamma_{\nu\lambda}^\mu, \quad (2.6)$$

$$R^\mu_{\nu\lambda\rho} \stackrel{\text{def}}{=} \partial_\lambda \Gamma_{\nu\rho}^\mu - \partial_\rho \Gamma_{\nu\lambda}^\mu + \Gamma_{\tau\lambda}^\mu \Gamma_{\nu\rho}^\tau - \Gamma_{\tau\rho}^\mu \Gamma_{\nu\lambda}^\tau. \quad (2.7)$$

The components $T^\mu_{\nu\lambda}$ and $R^\mu_{\nu\lambda\rho}$ are those of the torsion tensor and of the curvature tensor, respectively, and they are both non-vanishing in general. Thus, the space-time is of the Riemann-Cartan type. The field components e^k_μ and e^μ_k will be used, as for the case of V^μ and V^k in the above, to convert Latin and Greek indices.

III. LAGRANGIAN DENSITIES AND GRAVITATIONAL FIELD EQUATIONS

For the matter field φ , $L_M(\varphi, D_k\varphi)$ is a Lagrangian [9] invariant under three-dimensional local Lorentz transformations and under general coordinate transformations, if $L_M(\varphi, \partial_k\varphi)$ is an invariant Lagrangian on the three-dimensional Minkowski space-time.

For the fields e^k_μ and A^{kl}_μ , Lagrangians, which are invariant under local Lorentz transformations including also inversions and under general coordinate transformations and at most quadratic in torsion and curvature tensors, are given by

$$L_T = \alpha t^{klm} t_{klm} + \beta v^k v_k + \gamma a^{klm} a_{klm} + \delta , \quad (3.1)$$

$$L_R = a_1 E^{kl} E_{kl} + a_2 I^{kl} I_{kl} + a_3 R^2 + a R . \quad (3.2)$$

Here, t_{klm} , v_k and a_{klm} are the irreducible components of T_{klm} defined by

$$t_{klm} \stackrel{\text{def}}{=} \frac{1}{2}(T_{klm} + T_{lkm}) + \frac{1}{4}(\eta_{mk} v_l + \eta_{ml} v_k) - \frac{1}{2} \eta_{kl} v_m , \quad (3.3)$$

$$v_k \stackrel{\text{def}}{=} T^l{}_{lk} \quad (3.4)$$

and

$$a_{klm} \stackrel{\text{def}}{=} \frac{1}{3}(T_{klm} + T_{mkl} + T_{lmk}) , \quad (3.5)$$

respectively, and E_{kl} and I_{kl} are the irreducible components of R_{klmn} defined by

$$E_{kl} \stackrel{\text{def}}{=} \frac{1}{2}(R_{kl} - R_{lk}) \quad (3.6)$$

and

$$I_{kl} \stackrel{\text{def}}{=} \frac{1}{2}(R_{kl} + R_{lk}) - \frac{1}{3} \eta_{kl} R , \quad (3.7)$$

respectively, where we have defined $R_{kl} \stackrel{\text{def}}{=} R^m{}_{klm}$ and $R \stackrel{\text{def}}{=} R^k{}_k$. Also, α , β , γ , δ , a_1 , a_2 , a_3 and a are real constant parameters. Then, $\mathbf{L} \stackrel{\text{def}}{=} \sqrt{-g}(L_G + L_M(\varphi, D_k \varphi))$ with $L_G \stackrel{\text{def}}{=} L_T + L_R$ and $g \stackrel{\text{def}}{=} \det(g_{\mu\nu})$ is the total Lagrangian of the system. For the case with $a_1 = a_2 = a_3 = 0$, $a \neq 0$, $\alpha = \beta = \gamma = 0$ and with $\delta = 0$, the Lagrangian L_G reduces to the Einstein-Cartan Lagrangian [10].

The field equation $\delta \mathbf{L} / \delta e^i{}_\mu = 0$ reads [11]

$$\begin{aligned} & 2a R_{ji} + 4J_{[i]k[j]l]} R^{kl} + 4J^{[kl]}{}_{[j]l]} R_{ki} - 2J_{[i}{}^{,k]}{}_{[j]k]} R \\ & - 2D^k F_{ijk} + 2v^k F_{ijk} + 2H_{ij} - \eta_{ij} L_G = T_{ij} , \end{aligned} \quad (3.8)$$

where we have defined

$$J_{ijkl} \stackrel{\text{def}}{=} 2a_3 R \eta_{ik} \eta_{jl} + 2\eta_{ik} (a_1 E_{jl} + a_2 I_{jl}) , \quad (3.9)$$

$$D^k F_{ijk} \stackrel{\text{def}}{=} e^{\mu k} (\partial_\mu F_{ijk} + A_i^m{}_\mu F_{mjk} + A_j^m{}_\mu F_{imk} + A_k^m{}_\mu F_{ijm}) , \quad (3.10)$$

$$F_{ijk} \stackrel{\text{def}}{=} \alpha (t_{ijk} - t_{ikj}) + \beta (\eta_{ij} v_k - \eta_{ik} v_j) + 2\gamma a_{ijk} , \quad (3.11)$$

$$H_{ij} \stackrel{\text{def}}{=} T_{kli} F^{kl}{}_j - \frac{1}{2} T_{jkl} F_i{}^{kl} . \quad (3.12)$$

Also, T_{ij} denotes the energy-momentum density of the field φ defined by $\sqrt{-g} T_{ij} \stackrel{\text{def}}{=} e_{j\mu} \delta L_M / \delta e^i{}_\mu$ with $L_M \stackrel{\text{def}}{=} \sqrt{-g} L_M$. The field equation $\delta L / \delta A^{ij}{}_\mu = 0$ reads

$$2D^l J_{[ij][kl]} + \left(\frac{4}{3} t_k{}^{[lm]} - \delta_k{}^{[l} v^{m]} + a_k{}^{lm} \right) J_{[ij][lm]} - H_{ijk} = S_{ijk} \quad (3.13)$$

with

$$\begin{aligned} D^l J_{[ij][kl]} \stackrel{\text{def}}{=} e^{\mu l} (\partial_\mu J_{[ij][kl]} + A_i^m{}_\mu J_{[mj][kl]} + A_j^m{}_\mu J_{[im][kl]} + A_k^m{}_\mu J_{[ij][ml]} \\ + A_l^m{}_\mu J_{[ij][km]}) , \end{aligned} \quad (3.14)$$

$$H_{ijk} \stackrel{\text{def}}{=} - \left(\alpha + \frac{2a}{3} \right) (t_{kij} - t_{kji}) - \left(\beta - \frac{a}{2} \right) (\eta_{ki} v_j - \eta_{kj} v_i) + (4\gamma - a) a_{ijk} . \quad (3.15)$$

Here, S_{ijk} is the "spin" [12] density of φ defined by $\sqrt{-g} S_{ijk} \stackrel{\text{def}}{=} -e_{k\mu} \delta L_M / \delta A^{ij}{}_\mu$.

The gravitational field equations (3.8) and (3.13) can also be expressed in terms of the Riemann-Christoffel curvature tensor and of torsion tensor, and the following is known from them.

(1) Solutions of the vacuum Einstein equation with the cosmological constant Λ satisfy the vacuum gravitational field equations with $\delta = 2\Lambda(a + 6a_3\Lambda)$.

(2) For the case with $a_1 = a_2 = a_3 = 0$ and with $(3\alpha + 2a)(2\beta - a)(4\gamma - a) \neq 0$, the torsion is "frozen" at the place where the "spin" density S_{ijk} does not vanish. If $a = 1/(2\kappa)$ with κ being the "Einstein gravitational constant in addition and if the source field is "spin"less, field equations for the gravitational and source fields agree with those in the Einstein theory.

IV. A LIMITING CASE

Suppose that the parameters $a_i (i = 1, 2, 3)$ have the expression $a_i = \bar{a}_i / f^2, (i = 1, 2, 3)$, where f is a parameter characterizing the magnitude of a_i 's and it can be regarded as standing for the coupling strength between the Lorentz gauge field and the matter field φ . Multiplying both sides of (3.13) by f^2 and taking the limit $f \rightarrow 0$, we get

$$2D^l \bar{J}_{[ij][kl]} + \left(\frac{4}{3} t_k^{[lm]} - \delta_k^{[l} v^{m]} + a_k^{lm} \right) \bar{J}_{[ij][lm]} = 0, \quad (4.1)$$

where

$$\bar{J}_{ijkl} \stackrel{\text{def}}{=} 2\bar{a}_3 R\eta_{ik}\eta_{jl} + 2\eta_{ik}(\bar{a}_1 E_{jl} + \bar{a}_2 I_{jl}). \quad (4.2)$$

By substituting $R_{ijkl} = 0$, which is a solution of (4.1), into (3.8), we obtain

$$-2D^k F_{ijk} + 2v^k F_{ijk} + 2H_{ij} - \eta_{ij} L_T = T_{ij}. \quad (4.3)$$

Specifically, if we set $A^{\eta}_{\mu} = 0$, (4.3) with $\delta = 0$ reduces to (3.7) of Ref. [7], which is the gravitational field equation in a teleparallel theory. To put it briefly, the teleparallel theory developed in Ref. [7] is "included as a solution" in the limiting case with $a_i \rightarrow \infty, (i = 1, 2, 3)$ and with $\delta = 0$.

V. LINEARIZED GRAVITATIONAL FIELD EQUATIONS AND RELATION TO THE NEWTON THEORY

A. Linearized equations

We examine the gravitational field equations in the weak field situations in which

$$a^i_{\mu} \stackrel{\text{def}}{=} e^i_{\mu} - \delta^i_{\mu} \quad (5.1)$$

and A^{η}_{μ} are so small that it is sufficient to keep only terms linear in a^i_{μ} and in A^{η}_{μ} . In this approximation, Greek and Latin indices need not be distinguished with each other,

and the components $g_{\mu\nu}$ of the metric tensor has the expression $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu} \stackrel{\text{def}}{=} a_{\mu\nu} + a_{\nu\mu}$. Considering the case with $\delta = 0$ and employing $a_{\mu\nu}$ and the torsion tensor $T_{\lambda\mu\nu}$ as independent field variables, we can find the linearized forms of (3.8) and (3.13). Also, it is known that we can put the harmonic condition, $\partial^\nu \bar{h}_{\mu\nu} = 0$ with $\bar{h}_{\mu\nu} \stackrel{\text{def}}{=} h_{\mu\nu} - (1/2)\eta_{\mu\nu}h$, which we shall assume from now on. From the linearized forms of (3.8) and of (3.13), we can obtain the fourth-order field equation for $\bar{h}_{\mu\nu}$,

$$A\Box\bar{h}_{\mu\nu} + B\Box^2\bar{h}_{\mu\nu} + C(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\Box\bar{h} = T_{\mu\nu}^{(\text{eff})} \quad (5.2)$$

with

$$A \stackrel{\text{def}}{=} -a, \quad (5.3)$$

$$B \stackrel{\text{def}}{=} -\frac{3\alpha a_2}{3\alpha + 2a}, \quad (5.4)$$

$$C \stackrel{\text{def}}{=} \frac{1}{6(2\beta - a)}\{8\beta(a_2 + 6a_3) - 3aa_2\} - \frac{aa_2}{3\alpha + 2a} \quad (5.5)$$

and

$$\begin{aligned} T_{\mu\nu}^{(\text{eff})} \stackrel{\text{def}}{=} & T_{(\mu\nu)} - 2\partial^\lambda S_{\lambda(\mu\nu)} - \frac{a_2 + 24a_3}{6(2\beta - a)}(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)T \\ & - \frac{2a_2}{3\alpha + 2a}\left(\Box T_{(\mu\nu)} - \frac{1}{2}(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)T \right. \\ & \left. + 2\partial^\lambda\partial^\rho\partial_{(\mu}S_{\nu)\lambda\rho}\right). \end{aligned} \quad (5.6)$$

B. h_{00} due to a static, "spin"less point like source

We consider the gravitational field produced by a static, "spin"less source located at the origin, for which $S_{\lambda\mu\nu}$ is vanishing and $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \begin{cases} Mc^2\delta^2(\vec{r}), & \mu = \nu = 0, \\ 0, & \text{otherwise} \end{cases} \quad (5.7)$$

with $\vec{r} \stackrel{\text{def}}{=} (x^1, x^2)$. The time-independent circularly symmetric potentials h_{00} , for (a) the case with $a = 0$, and $\alpha\beta a_2(a_2 + 24a_3) \neq 0$ and for (b) the case with $a_i = \bar{a}_i/f^2$, ($i = 2, 3$), $f \rightarrow 0$, and $\alpha\beta \neq 0$, are given by

$$h_{00}(r) = \frac{(3\alpha + 4\beta)}{24\pi\alpha\beta} Mc^2 \ln r - \frac{Mc^2(a_2 + 6a_3)}{4\pi a_2(a_2 + 24a_3)} r^2 \ln r + C_1 r^2 + C_2 \quad (5.8)$$

and

$$h_{00}(r) = \frac{3\alpha + 4\beta}{24\pi\alpha\beta} Mc^2 \ln r + C_3 r^2 + C_4, \quad (5.9)$$

respectively, where C_i , ($i = 1, 2, 3, 4$), are integration constants and $r \stackrel{\text{def}}{=} |\vec{r}|$.

C. Relation to the Newton theory

The world line of free motion of a macroscopic body, for which effects due to the “spin”s of the fundamental constituent particles can be ignored, is shown to be the geodesic line of the metric $g_{\mu\nu} dx^\mu \otimes dx^\nu$. Hence, this body obeys the Newton equation of motion, $d^2\vec{r}/dt^2 = -\partial U/\partial\vec{r}$ with $U \stackrel{\text{def}}{=} -c^2 h_{00}/2$, when the motion is sufficiently slow and the gravitational field is weak. The potential U given by the solution (5.8) satisfies the Newton equation for the gravitational potential, $\Delta U = 4\pi G M \delta(\vec{r})$, if the conditions

$$3\alpha + 4\beta = -\frac{96\alpha\beta\pi G}{c^4}, \quad (5.10)$$

$$a_2 + 6a_3 = 0, \quad (5.11)$$

are both satisfied *and when the integration constants C_1 is chosen to be zero: $C_1 = 0$* . Here, G stands for “Newton gravitational constant” and c is the speed of light in the vacuum. Also the solution (5.9) gives the Newton potential, if the condition (5.10) is satisfied *and if $C_3 = 0$* . Thus, each of the case (a) with the conditions (5.10) and (5.11) and of the case (b) with the condition (5.10) can give a Newtonian limit by a suitable choice of the integration constant. But, it should be noted that *the field equations for $h_{\mu\nu}$ are fourth-order differential equations for both cases.*

VI. COMMENTS

The following is worth mentioning.

(A) Even for the case with $a_1 = a_2 = a_3 = 0$, $a = 1/(2\kappa)$ and $(3\alpha+2a)(2\beta-a)(4\gamma-a) \neq 0$, field equations are different from those in the Einstein theory, if the source field has non-vanishing “spin”. For this case, space-times in the vacuum regions are locally the same as those in the Einstein theory, but the quantized theory and non-local properties such as the gravitational Aharonov-Bohm effect [13] due to the “spin”ning source fields are presumably different from those of the latter theory.

(B) The condition (5.10) agrees with the condition (5.11) in Ref. [7] in its form.

(C) As is known from (1) of the Sec. III, black hole solutions [14, 15] of the ordinary three-dimensional vacuum Einstein equation with a negative cosmological constant satisfy the vacuum gravitational field equations in our theory. Also, it is worth adding that these solutions are independent of the black holes in a teleparallel theory discussed in Ref. [7].

Details of the present work and additional descriptions are given in Ref. [16].

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A TEST OF A KIND OF THE EQUIVALENCE PRINCIPLE BY LONG-BASELINE NEUTRINO-OSCILLATION EXPERIMENTS

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In this talk I discuss neutrino oscillations due to possible nonuniversality of the gravitational coupling constants. It has been pointed out [1] [2] that a breakdown of the universality of the gravitational couplings to different neutrino flavors could lead to neutrino oscillations. In particular the authors of [2] studied the possibility in which the solar neutrinos can be used to test this kind of breakdown of the universality. Since the flux of the solar neutrinos is relatively small and the energy spectrum is beyond our control, the utility of the solar neutrino for this purpose is limited. Here we would like to propose a possible long-baseline experiments of neutrino oscillations to test the breakdown of the universality of the gravitational couplings to neutrinos [3] [4] [5]. As we will see, the breakdown of the universality of the gravitational couplings to neutrinos of different flavors leads to a violation of Einstein's equivalence principle (see, e.g., [6]) which states that all the laws of physics must take on their familiar special-relativistic forms in any and every local Lorentz frame, anywhere and any time in the universe. No consistent theory is known to predict such nonuniversality of the gravitational coupling constants, so our motivation is to give an upper bound on such nonuniversality. In the present case, it turns out that we can probe the magnitude of the breakdown of Einstein's equivalence principle to the order of 10^{-15} , assuming that there are neutrino mixings. Among various experiments to test the equivalence principle (see e.g., Ref. [7] for a review), there have been few tests of Einstein's equivalence principle for neutrinos [8]. The universality of the gravitational couplings that I discuss in this talk is of different type from these experiments in the past, so our discussions here are complementary to them.

In this talk we assume that there are two neutrino flavors which have different couplings to gravity and that the eigenstates of these different gravitational couplings do not coincide with those of the electroweak flavors. Throughout the present discussions we consider neutrino oscillations between two flavors for simplicity. Let us suppose that two kinds of neutrinos are in a certain background field with the different gravitational couplings, and that they are described by the following Lagrangian

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$$\begin{aligned}
\mathcal{L} = & e(G_1)\bar{\nu}_1 [ie^{a\mu}(G_1)\gamma_a D_\mu(G_1) - m_1] \nu_1 \\
& + e(G_2)\bar{\nu}_2 [ie^{a\mu}(G_2)\gamma_a D_\mu(G_2) - m_2] \nu_2 \\
& + (\text{interaction terms with electroweak gauge bosons}),
\end{aligned} \tag{1}$$

where we have included mass terms to keep generality, $e^{a\mu}(G_i)$ ($i = 1, 2$) are the vierbein fields of some background metric with different Newton constants G_i ($i = 1, 2$), and $e(G_i) \equiv \det e_\mu^a(G_i)$. For simplicity we assume that the eigenstates of the gravitational couplings coincide with those of the masses. Notice that even if these neutrinos are massless, we cannot rotate these two fields so that these are the eigenstates of the electroweak theory, since the gravitational coupling terms are not invariant under the rotation in the flavor space. Since the gravitational couplings for these two kinds of neutrinos are different, even if we choose a coordinate system in which the Dirac equation for ν_1 in (1) becomes the one in a flat space-time, the Dirac equation for ν_2 in the same coordinate system does not necessarily do so. Thus Einstein's equivalence principle is violated in (1).

The configuration of the long-baseline experiment we will discuss is depicted in Fig. 1, and the neutrino beams go underneath the ground. The density of the Earth is not constant, so in principle we have to regard the density as a function of the radius [9]. When we consider the case like the longbase-line experiment in the DUMAND project [10], where the neutrino beam goes deep in the Earth ($L \equiv (\text{length of the trajectory of the neutrino beam}) \sim 6,000$ km), this is actually the case. However, in the case like the one in the SOUDAN2 project [10], where the neutrino beam goes just slightly under the surface of the Earth ($L \sim 800$ km), we can assume that the density is approximately constant. In this talk we will take this assumption for simplicity, and the analysis without this approximation is given in [4].

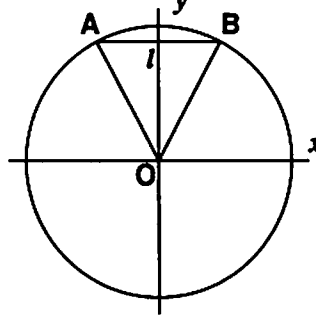


Fig. 1

First let us consider the Dirac equation of left-handed neutrinos without any flavor in the interior Schwarzschild background [11]:

$$(ie^{a\mu}\gamma_a D_\mu - m)\psi = 0, \tag{2}$$

where $e_{a\mu}$ is the vierbein of the interior Schwarzschild metric

$$ds^2 = (e_t^0)^2 dt^2 - (e_r^1)^2 dr^2 - (e_\theta^2)^2 d\theta^2 - (e_\phi^3)^2 d\phi^2 \tag{3}$$

and is given by

$$e_t^0 = \frac{3}{2}\sqrt{1 - \frac{\alpha}{R}} - \frac{1}{2}\sqrt{1 - \frac{\alpha r^2}{R^3}}, \quad e_r^1 = \frac{1}{\sqrt{1 - \frac{\alpha r^2}{R^3}}}, \quad e_\theta^2 = r, \quad e_\phi^3 = r \sin \theta. \quad (4)$$

$D_\mu \psi \equiv (\partial_\mu - \frac{1}{2}\omega_{\mu ab}\sigma^{ab})\psi$ is the covariant derivative acting on a spinor ψ , $\omega_{\mu ab}$ is the spin connection given by $e_{[\nu}^b \omega_{\mu]}^a = \partial_{[\mu} e_{\nu]}^a$, and α in (4) is the Schwarzschild radius.

One characteristic dimensionless parameter in our case is ER , where E is the energy of the neutrino, and R is the radius of the Earth. For $E=10$ GeV and $R=6,400$ Km, $ER \sim 3 \times 10^{23}$, and derivative terms in the spin connections are all of the order of $1/ER$, so we will neglect them throughout this talk. α in eq. (4) is the Schwarzschild radius of the Earth which is about 9 mm, so we also expand (2) to the first order in α/r . In this approximation the positive energy part of the Dirac equation finally becomes

$$\frac{d\nu}{du} = i \left[1 - \frac{m^2}{2E^2} + \frac{3\alpha}{4R} - \frac{\alpha \ell^2}{4R^3} \left(1 - \frac{u^2}{B^2} \right) \right] \nu,$$

where $u \equiv Ex$ is a dimensionless coordinate along the x -axis, $\ell \equiv \sqrt{R^2 - (L/2)^2}$ is the distance of the trajectory of the neutrino beam from the center of the Earth, and $B \equiv E\ell$ is a very large number.

In case of two kinds of neutrinos described by the Lagrangian (1), it is straightforward to see that the Dirac equation for (1) is given by

$$\frac{d}{du} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = i \left[1 - \frac{m_1^2 + m_2^2}{4E^2} + \left(-3 + \frac{\ell^2}{R^2} \left(1 - \frac{u^2}{B^2} \right) \right) \frac{f_1 + f_2}{4} \Phi + \Delta(u)\sigma_3 \right] \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}, \quad (5)$$

where

$$\Delta(u) \equiv \frac{\Delta m^2}{4E^2} + \left(3 - \frac{\ell^2}{R^2} \left(1 - \frac{u^2}{B^2} \right) \right) \frac{\Delta f}{4} \Phi. \quad (6)$$

Here $\Delta m^2 \equiv m_2^2 - m_1^2$ is the difference of the masses, we have defined the Newton potential $\Phi \equiv -GM/R$ on the surface of the Earth, and we have also defined the difference $\Delta f = f_2 - f_1$ of the dimensionless gravitational couplings of the two neutrino species

$$\begin{pmatrix} f_1 \Phi \\ f_2 \Phi \end{pmatrix} = - \begin{pmatrix} \alpha_1/2R \\ \alpha_2/2R \end{pmatrix} = - \begin{pmatrix} G_1 M/R \\ G_2 M/R \end{pmatrix}.$$

The equation (5) can be easily integrated from $u = -EL/2$ to $u = EL/2$.

Now let us introduce the flavor eigenstates ν_a , ν_b of the weak interaction by

$$\begin{pmatrix} \nu_a \\ \nu_b \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}.$$

Then the probability of detecting a different flavor ν_b at a distance L after producing one neutrino flavor ν_a is given by

$$P(\nu_a \rightarrow \nu_b) = \sin^2 2\theta \sin^2 \left[\left(\frac{\Delta m^2}{4E^2} + \left(1 + \frac{L^2}{6R^2} \right) \frac{\Delta f \Phi}{2} \right) EL \right]. \quad (7)$$

This formula applies to the transition between ν_μ and ν_τ , where no MSW effect [12] is expected to occur.

In case of the transition between ν_e and ν_μ , we have to take the MSW effect [12] into consideration, and the Dirac equation is modified as

$$\frac{d}{du} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = i \begin{pmatrix} \Delta(u) \cos 2\theta - \frac{G_F N_e}{\sqrt{2}E} & \Delta(u) \sin 2\theta \\ \Delta(u) \sin 2\theta & -\Delta(u) \cos 2\theta + \frac{G_F N_e}{\sqrt{2}E} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix}, \quad (8)$$

where G_F is the Fermi coupling constant, N_e is the density of electrons in the Earth. (8) can be solved in the same way as before by introducing the variables

$$\begin{aligned} \Delta_N(u) \cos 2\theta_N &= \Delta(u) \cos 2\theta - \frac{G_F N_e}{\sqrt{2}E} \\ \Delta_N(u) \sin 2\theta_N &= \Delta(u) \sin 2\theta. \end{aligned} \quad (9)$$

Note that θ_N does depend on the variable u in this case. It is easy to integrate (9), and we have the transition probability of detecting ν_e at a distance L from the source of ν_μ beams

$$P(\nu_\mu \rightarrow \nu_e) = \sin^2 2\theta_N \left(u = \frac{EL}{2} \right) \sin^2 \left(\int_{-EL/2}^{EL/2} du \Delta_N(u) \right), \quad (10)$$

where we have used the fact $\theta_N(u = \frac{EL}{2}) = \theta_N(u = -\frac{EL}{2})$, and Δ_N, θ_N are defined through (9). The integration in the exponent in (10) can be performed numerically [3].

We have studied the quantity $P(\nu_a \rightarrow \nu_b)$ for various cases [3] [4]. As in the case of neutrino oscillations due to masses, we can exclude certain regions in the $(\sin^2 2\theta, \Delta f)$ plot. A typical figure is given in Fig. 2, which shows the region $(\sin^2 2\theta, \Delta f)$ for ν_μ - ν_τ oscillations which can be excluded by the proposed longbase-line experiment in the SOUDAN2 project. From Fig. 2 we see that we can give an upper limit as small as 10^{-14} on the nonuniversality of the gravitational coupling constants of neutrinos. In case of the DUMAND project this upper limit could reach 10^{-15} [4].

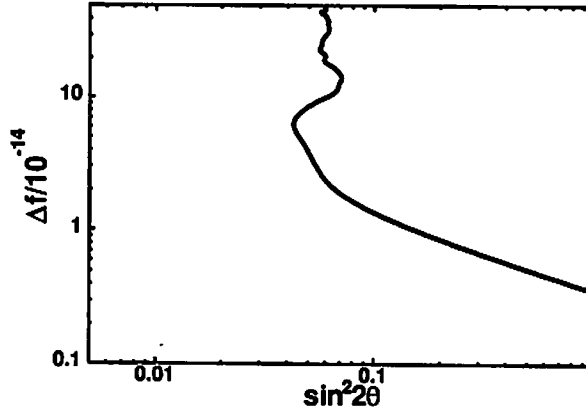


Fig.2

One important feature of this experiment is the dependence of the probability on the detection threshold energy of muons, which is, roughly speaking, proportional to the energy of the incident neutrinos. To emphasize this aspect, let us make comparison of the energy dependence of the probability for various cases. So far we have considered the case where the particle which mediates the force between neutrinos is a spin-2 particle, i.e., graviton. In this case the energy defined in the Dirac equation

$$i\frac{\partial\psi}{\partial t} = E\psi \quad (11)$$

is given by

$$E_{J=2} = \left(1 + \frac{1}{2}\Phi_{J=2}\right) \sqrt{\vec{p}^2 + m^2} \quad (12)$$

On the other hand, if the force is mediated by a scalar or a vector particle, then the energy would be given by

$$\begin{aligned} E_{J=0} &= \sqrt{\vec{p}^2 + (m + \Phi_{J=0})^2} \\ E_{J=1} &= \sqrt{\vec{p}^2 + m^2} + \Phi_{J=1}, \end{aligned} \quad (13)$$

respectively. Here Φ_J ($J = 0, 1, 2$) denotes certain potentials for spin- J forces. Examples for such forces are a scalar field in [13] which couples only to tau neutrinos and a torsion tensor field (i.e., a dual of an axial vector field) in [14] whose eigenstates are different from those of the electroweak interaction. It is well-known that the nonrelativistic behavior of scalar forces is similar to that of tensor forces, i.e., gravity. In the ultrarelativistic limit, however, the situation changes drastically. In this limit we have from (12) and (13)

$$\begin{aligned} E_{J=0} - |\vec{p}| - \frac{m^2}{2|\vec{p}|} &\simeq \frac{1}{2|\vec{p}|}\Phi_{J=0}^2 + \frac{m}{|\vec{p}|}\Phi_{J=0} \\ E_{J=1} - |\vec{p}| - \frac{m^2}{2|\vec{p}|} &\simeq \Phi_{J=1} \\ E_{J=2} - |\vec{p}| - \frac{m^2}{2|\vec{p}|} &\simeq \frac{|\vec{p}|}{2}\Phi_{J=2}, \end{aligned} \quad (14)$$

where only terms proportional to m and Φ_J could be relevant to neutrino oscillations. So if all neutrinos are massless and if neutrino oscillations occur solely due to the presence of Φ_J , then the probability of neutrino oscillations would be given by

$$\begin{aligned} \text{Prob}(J=0) &\sim \sin^2 2\theta \sin^2 \left(\frac{\Phi_{J=0}^2 L}{|\vec{p}|} \right) \\ \text{Prob}(J=1) &\sim \sin^2 2\theta \sin^2 (\Phi_{J=1} L) \\ \text{Prob}(J=2) &\sim \sin^2 2\theta \sin^2 (|\vec{p}| \Phi_{J=2} L). \end{aligned} \quad (15)$$

In Fig. 3 we give the energy dependence of the probability in case of gravity. In case of a scalar force and a vector force the energy dependence would look like Figs. 4 and 5, respectively. From these figures we conclude that the shape of the energy spectrum depends on the spin of the particle which mediates the force, and hence it should be possible to identify the spin of the particle which mediates a possible new force by looking at the energy dependence. (15) also explains why we have obtained such a severe constraint on the universality of the gravitational coupling constants of neutrinos, as the argument of sine in $\text{Prob}(J=2)$ in (15) is proportional to the energy of the neutrino. We note in passing that ordinary neutrino oscillations due to masses is analogous to the case of a scalar force, as far as the energy dependence is concerned. In accelerator experiments like those we have proposed in this talk, the length L is much smaller and the energy of neutrinos $|\vec{p}|$ is much larger than typical quantities in astrophysical observations discussed in [7]. In longbase-line experiments, therefore, while it is easy to test gravity, it would be more difficult to detect a new scalar force or a new vector force such as those in [13] and [14].

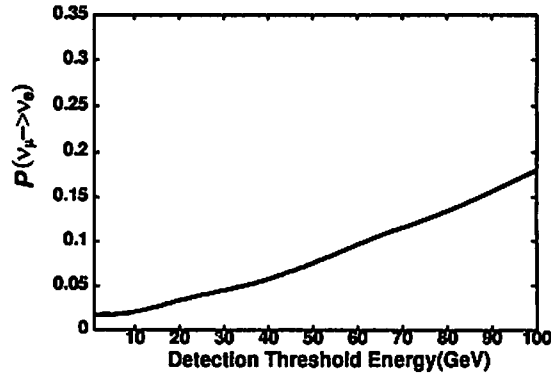


Fig.3 (J=2)

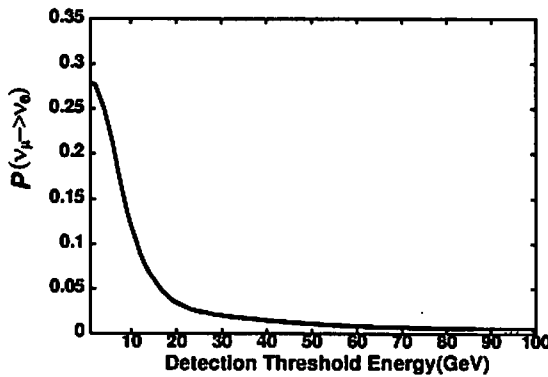


Fig.4 (J=0)

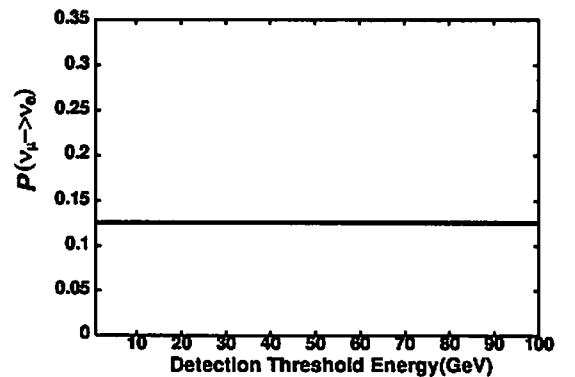


Fig.5 (J=1)

In this talk we have proposed long-baseline experiments to test the universality of the gravitational couplings of neutrinos, and we found that we could probe the dimensionless parameter Δf as small as 10^{-14} or 10^{-15} which is smaller by a few orders of magnitudes than the upper limit on a breakdown of the equivalence principle from different types of experiments. Although we have not evaluated systematic errors in detail, we hope our analysis will stimulate and motivate long-baseline experiments in the near future.

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